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# STABILITY FOR ASYMPTOTICALLY PERIODIC MULTIVALUED DYNAMICAL SYSTEMS GENERATED BY DOUBLE OBSTACLE PROBLEMS

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**Abstract.** In this paper let us consider double obstacle problems, which includes regional economic growth models. By prescribed time-dependent obstacles, our problems are non-autonomous systems and it is impossible to show the uniqueness of solutions. Therefore the associated dynamical systems are multivalued. In this paper from the viewpoint of attractors we shall consider the periodic stability for the double obstacle problem with asymptotically periodic data. Namely, assuming that time-dependent data converges to time-periodic ones as time goes to infinity, we shall construct the global attractor for the asymptotically periodic multivalued dynamical system. Moreover we shall discuss the relationship to the attractor for the limiting periodic problem.

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# 1 Introduction

Let  $\Omega$  be a bounded domain in  $R^N$  ( $1 \leq N < +\infty$ ) with smooth boundary  $\Gamma := \partial\Omega$  and  $q$  be a fixed number with  $2 \leq q < +\infty$ . Then, for each  $s \geq 0$  let us consider the following double obstacle problem  $(P)_s$ : Find functions  $u \in C([s, +\infty); L^2(\Omega))$  and  $\theta \in L^2_{loc}((s, +\infty); L^2(\Omega))$  such that

$$(P)_s \begin{cases} u'(t) - \mathbf{div}(|\nabla u(t)|^{q-2} \nabla u(t)) - g(u(t)) = \theta(t, x) & \text{in } Q_s := [s, +\infty) \times \Omega; \\ 0 \leq \theta(t, x) \leq h(t, u(t, x)) & \text{a.e. on } (s, +\infty) \times \Omega; \\ u(t) = l(t) & \text{a.e. on } (s, +\infty) \times \Gamma; \\ u(s) = u_0 & \text{in } \Omega. \end{cases}$$

Where  $g(\cdot)$ ,  $h(\cdot, \cdot)$ ,  $l(\cdot)$  are given functions. Here we note that  $(P)_s$  with  $q = 2$  is a regional economic growth model, in which the unknown function  $u$  represents a stock of available capital, the unknown function  $\theta$  is a rate of investment and  $-g(u)$  is a recursive depreciation of capital.

In the case that  $q = 2$  and the boundary condition  $l(t) \equiv 0$  for any  $t > 0$ , the existence of solution for  $(P)_s$  was proved in [2, 9] and Papageorgiou [9] studied the optimal control problem. Unfortunately, by given double obstacles,  $(P)_s$  loses the uniqueness of solutions for a given initial value. Recently, from the viewpoint of attractors Kapustian and Valero [6] considered the asymptotic behaviour of solutions for  $(P)_s$  without uniqueness in the case that  $q = 2$  and time-independent given functions  $h(t, \cdot) \equiv h(\cdot)$ ,  $l(t) \equiv 0$  for any  $t \geq 0$ . Namely they constructed the global attractor for the multivalued autonomous dynamical system associated with  $(P)_s$ .

In the general case  $2 \leq q < +\infty$ , the existence of solution for  $(P)_s$  was proved in [12]. Moreover, assuming that the given functions  $h(t, \cdot)$  and  $l(t)$  converge to time-independent ones  $h^\infty(\cdot)$  and  $l^\infty$  as  $t \rightarrow +\infty$  in appropriate senses, the author [12] constructed the global attractor for  $(P)_s$  and discussed the relationship to the one for the limiting autonomous system of  $(P)_s$ .

In this paper for a given period  $T_0 > 0$  let us consider an asymptotically  $T_0$ -periodic problem  $(AP)_s$  for  $(P)_s$ . Namely we assume that  $h(t, \cdot) - h_p(t, \cdot) \rightarrow 0$ ,  $l(t, \cdot) - l_p(t, \cdot) \rightarrow 0$  in appropriate senses as  $t \rightarrow +\infty$ , where  $h_p(t, \cdot)$  and  $l_p(t)$  are  $T_0$ -periodic in time, i.e.

$$h_p(t, \cdot) = h_p(t + T_0, \cdot), \quad l_p(t) = l_p(t + T_0), \quad \forall t \in R_+ := [0, +\infty).$$

Then, by the above asymptotically  $T_0$ -periodic stability conditions we have a limiting non-autonomous  $T_0$ -periodic double obstacle problem  $(PP)_{T_0}$  of  $(AP)_s$  as follows: Find functions  $u \in C([0, +\infty); L^2(\Omega))$  and  $\theta \in L^2_{loc}((0, +\infty); L^2(\Omega))$  such that

$$(PP)_{T_0} \begin{cases} u'(t) - \mathbf{div}(|\nabla u(t)|^{q-2} \nabla u(t)) - g(u(t)) = \theta(t, x) & \text{in } Q_0 = [0, +\infty) \times \Omega; \\ 0 \leq \theta(t, x) \leq h_p(t, u(t, x)) & \text{a.e. on } (0, +\infty) \times \Omega; \\ u(t) = l_p(t) & \text{a.e. on } (0, +\infty) \times \Gamma; \\ u(0) = u_0 & \text{in } \Omega. \end{cases}$$

The main object of this paper is to investigate the large-time behaviour of solutions for  $(AP)_s$  and  $(PP)_{T_0}$  without uniqueness from the viewpoint of attractors. In fact, we

shall show the existence of attractors for  $(AP)_s$  and  $(PP)_{T_0}$  and discuss the relationship between them.

Throughout this paper,  $|\cdot|_{L^q(\Omega)}$  (resp.  $|\cdot|_{W^{1,q}(\Omega)}$ ) is a standard norm of  $L^q(\Omega)$  (resp.  $W^{1,q}(\Omega)$ ) for each  $q \geq 2$ . For the subset  $A$  of  $L^2(\Omega)$ ,  $\overline{A}$  denotes the closure of  $A$  in  $L^2(\Omega)$ . For two sets  $A$  and  $B$  in  $L^2(\Omega)$ , we define the so-called Hausdorff semi-distance

$$\text{dist}_{L^2(\Omega)}(A, B) := \sup_{x \in A} \inf_{y \in B} |x - y|_{L^2(\Omega)}.$$

## 2 Assumptions and weak formulation

In this paper we consider the asymptotically  $T_0$ -periodic double obstacle problem  $(AP)_s$  under the following assumptions:

**(A1)**  $g(\cdot)$  is a Lipschitz continuous function on  $R$  satisfying the following property:

$$\min \left\{ \liminf_{z \rightarrow -\infty} \frac{-g(z)}{z}, \liminf_{z \rightarrow +\infty} \frac{-g(z)}{z} \right\} =: g_0 > 0;$$

**(A2)**  $h(\cdot, \cdot)$  and  $h_p(\cdot, \cdot)$  are non-negative continuous functions on  $R_+ \times R$ .  $h_p(t, z)$  is  $T_0$ -periodic in  $t$  for each  $z \in R$ . And there exists a positive constant  $L$  with  $0 < L < \frac{g_0}{2}$  such that

$$\begin{aligned} |h(t, z_1) - h(t, z_2)| &\leq L|z_1 - z_2|, \quad \forall t \in R_+, z_i \in R \ (i = 1, 2), \\ |h_p(t, z_1) - h_p(t, z_2)| &\leq L|z_1 - z_2|, \quad \forall t \in R_+, z_i \in R \ (i = 1, 2). \end{aligned}$$

Moreover, for any  $z \in R$ ,  $\sup_{t \in [0, T_0]} |h(mT_0 + t, z) - h_p(t, z)| \rightarrow 0$  as  $m \rightarrow +\infty$ ;

**(A3)**  $l, l_p \in L^\infty(R_+; W^{1,q}(\Omega))$  with  $\sup_{t \in R_+} |l'|_{L^2(t, t+1; W^{1,q}(\Omega))} + \sup_{t \in R_+} |l'_p|_{L^2(t, t+1; W^{1,q}(\Omega))} < +\infty$ .

Moreover  $l_p$  is  $T_0$ -periodic in time and

$$J_m := \sup_{t \in [0, T_0]} |l(mT_0 + t) - l_p(t)|_{W^{1,q}(\Omega)} \rightarrow 0 \quad \text{as } m \rightarrow +\infty;$$

Now we give weak formulations of  $(AP)_s$  and  $(PP)_{T_0}$ . To do so, we define a closed convex subset  $K(t)$  of  $W^{1,q}(\Omega)$  for each  $t \in R_+$  by

$$K(t) := \{z \in W^{1,q}(\Omega) ; z = l(t) \text{ a.e. on } \Gamma \}. \quad (2.1)$$

Also the set  $K_p(t)$  is also defined by replacing  $l$  by  $l_p(t)$  in (2.1).

**Definition 2.1.** (i) For each  $s \geq 0$  and  $u_0 \in L^2(\Omega)$ , a couple of functions  $\{u, \theta\}$  is called a solution of  $(AP)_s$  if  $u \in C([s, +\infty); L^2(\Omega)) \cap L^2_{loc}((s, +\infty); W^{1,q}(\Omega))$ ,  $u' \in L^2_{loc}((s, +\infty); L^2(\Omega))$ ,  $\theta \in L^2_{loc}((s, +\infty); L^2(\Omega))$ ,  $u(0) = u_0$  in  $L^2(\Omega)$ ,

$$u(t) \in K(t) \quad \text{for a.e. } t \geq s,$$

$$0 \leq \theta(t, x) \leq h(t, u(t, x)) \quad \text{a.e. on } (s, +\infty) \times \Omega,$$

and

$$\begin{aligned} & \int_{\Omega} (u'(t, x) - \theta(t, x) - g(u(t, x)))(u(t, x) - z(x)) dx \\ & + \int_{\Omega} |\nabla u(t, x)|^{q-2} \nabla u(t, x) \cdot (\nabla u(t, x) - \nabla z(x)) dx = 0 \\ & \text{for any } z \in K(t) \text{ and a.e. } t \geq s. \end{aligned}$$

(ii) A solution of  $(PP)_{T_0}$  is similarly defined by replacing  $h(t), l(t), K(t)$  by  $h_p(t), l_p(t), K_p(t)$  in (i).

### 3 Existence of global solutions

In this section we shall show the existence and global boundedness of solutions for  $(AP)_s$  and  $(PP)_{T_0}$ .

By the same argument in [11, 12], we can get the following the result.

**Theorem 3.1.** (cf. [11, 12]) *Assume that (A1)-(A3) hold. Then, for each  $s \geq 0$  and  $u_0 \in \overline{K(s)}$  the double obstacle problem  $(AP)_s$  has at least one solution  $\{u, \theta\}$  with initial value  $u(s) = u_0$ . Moreover, for each  $\delta > 0$  and the bounded set  $B \subset L^2(\Omega)$  there is a positive constant  $N_\delta$  such that*

$$\begin{aligned} & \sup_{t \geq s} |u(t)|_{L^2(\Omega)}^2 + \sup_{t \geq 0} \int_t^{t+1} |\nabla u(\tau)|_{L^q(\Omega)}^q d\tau \\ & + \sup_{t \geq s+\delta} |u'|_{L^2(t, t+1; L^2(\Omega))}^2 + \sup_{t \geq s+\delta} |\nabla u(t)|_{L^q(\Omega)}^q \leq N_\delta \end{aligned}$$

for all  $s \geq 0$  and  $u_0 \in \overline{K(s)} \cap B$ .

In fact, by applying the abstract theory of nonlinear evolution equations governed by time-dependent subdifferential of convex functions, we can prove Theorem 3.1. For detail proofs, see [11, 12].

Here note that the limiting  $T_0$ -periodic double obstacle problem  $(PP)_{T_0}$  can be considered as the special case of  $(AP)_s$  by taking  $h_p(t, \cdot)$  and  $l_p(t)$  as  $h(t, \cdot)$  and  $l(t)$ . Therefore, by Theorem 3.1 we can get the similar result on the existence and global boundedness of solutions for  $(PP)_{T_0}$  on  $[0, +\infty)$ .

### 4 Attractor for the limiting periodic problem

In this section we shall construct a global attractor for the limiting  $T_0$ -periodic double obstacle problem  $(PP)_{T_0}$ . To do so, let us define a solution operator for  $(PP)_{T_0}$ . In fact, by Theorem 3.1 we can define a family  $\{U(t, s); 0 \leq s \leq t < +\infty\}$  of solution operators. But we cannot get the uniqueness of solution for  $(PP)_{T_0}$ . Hence the solution operator  $U(t, s)$  from  $\overline{K_p(s)}$  into  $\overline{K_p(t)}$  is multivalued. Namely, for each  $s, t \in R_+$  with  $s \leq t$ ,  $U(t, s)$  assigns to any  $u_0 \in \overline{K_p(s)}$  the set

$$U(t, s)u_0 := \left\{ z \in \overline{K_p(t)} \left| \begin{array}{l} \text{There is a solution } \{u, \theta\} \text{ of } (PP)_{T_0} \\ \text{such that} \\ u(s) = u_0 \text{ and } u(t) = z. \end{array} \right. \right\}.$$

Then, it is easy to check the following properties of  $\{U(t, s)\}$ :

- (U1)  $U(s, s) = I$  on  $\overline{K_p(s)}$  for any  $s \in R_+$ ;
- (U2)  $U(t_2, s)z = U(t_2, t_1)U(t_1, s)z$  for any  $0 \leq s \leq t_1 \leq t_2 < +\infty$  and  $z \in \overline{K_p(s)}$ ;
- (U3)  $U(t + T_0, s + T_0) = U(t, s)$  for any  $0 \leq s \leq t < +\infty$ , that is,  $U$  is  $T_0$ -periodic.
- (U4)  $\{U(t, s)\}$  has the following demiclosedness:
  - If  $0 \leq s_n \leq t_n < +\infty$ ,  $s_n \rightarrow s$ ,  $t_n \rightarrow t$ ,  $z_n \in \overline{K_p(s_n)}$ ,  $z \in \overline{K_p(s)}$ ,  $z_n \rightarrow z$  in  $L^2(\Omega)$  and a element  $w_n \in U(t_n, s_n)z_n$  converges to some element  $w \in L^2(\Omega)$  as  $n \rightarrow +\infty$ , then  $w \in U(t, s)z$

Therefore  $\{U(t, s)\}$  forms a multivalued  $T_0$ -periodic dynamical process. For some properties of the multivalued mapping, see [1], for instance.

Clearly, the limiting  $T_0$ -periodic obstacle problem  $(PP)_{T_0}$  can be reformulated as an evolution equation

$$(E)_{T_0} \quad u'(t) + \partial\varphi_p^t(u(t)) + G_p(t, u(t)) \ni 0 \quad \text{in } L^2(\Omega), \quad t > s,$$

where  $\varphi_p^t$  is a  $T_0$ -periodic proper lower semicontinuous convex functions on  $L^2(\Omega)$  defined by

$$\varphi_p^t(z) = \begin{cases} \frac{1}{q} \int_{\Omega} |\nabla z|^q dx & \text{if } z \in K_p(t), \\ +\infty & \text{if } z \in L^2(\Omega) \setminus K_p(t). \end{cases}$$

Also,  $G_p(t, \cdot)$  is a  $T_0$ -periodic multivalued operator in  $L^2(\Omega)$  defined by

$$G_p(t, z) := \left\{ w \in L^2(\Omega); \quad \begin{array}{l} w = -g(z) - v \quad \text{in } L^2(\Omega) \\ 0 \leq v(x) \leq h_p(t, z(x)) \quad \text{a.e. on } \Omega \end{array} \right\}.$$

The author [13] showed the existence of  $T_0$ -periodic attractor for  $(E)_{T_0}$ . So, by applying the abstract results to  $(PP)_{T_0}$ , we can get the  $T_0$ -periodic stability results for  $(PP)_{T_0}$  as follows:

**Theorem 4.1.** (cf. [13]) *Suppose (A1)-(A3). For each  $\tau \geq 0$ , we define the  $T_0$ -step mapping  $U_\tau := U(\tau + T_0, \tau)$  and  $U_\tau^k := U(\tau + kT_0, \tau)$  for each  $k \in N$ . Then, there exists a subset  $\mathcal{A}_\tau$  of  $K_p(\tau)$  such that*

- (i)  $\mathcal{A}_\tau$  is non-empty and compact in  $L^2(\Omega)$ ;
- (ii) for each bounded set  $B$  in  $L^2(\Omega)$  and each number  $\epsilon > 0$  there exists a positive number  $N_{B, \epsilon} \in N$  such that

$$\text{dist}_{L^2(\Omega)}(U_\tau^k z, \mathcal{A}_\tau) < \epsilon \quad \text{for all } z \in \overline{K_p(\tau)} \cap B \text{ and all } k \geq N_{B, \epsilon};$$

- (iii)  $U_\tau^k \mathcal{A}_\tau = \mathcal{A}_\tau$  for any  $k \in N$ .

In fact, we can construct the compact absorbing set  $B_{0,\tau}$  for the discrete multivalued dynamical system  $U_\tau$ . Here, we define the set  $\mathcal{A}_\tau := \bigcap_{n \in Z_+} \overline{\bigcup_{k \geq n} U_\tau^k B_{0,\tau}}$  where  $Z_+ := N \cup \{0\}$ . Then we see that the set  $\mathcal{A}_\tau$  has the properties (i)-(iii) in Theorem 4.1. For detail proofs, see [13].

**Theorem 4.2.** (cf. [13]) *Suppose (A1)-(A3). Let  $\mathcal{A}_s$  and  $\mathcal{A}_\tau$  be the global attractors of  $U_s$  and  $U_\tau$ , with  $0 \leq s \leq \tau \leq T_0$ , respectively. Then, we have  $\mathcal{A}_\tau = U(\tau, s)\mathcal{A}_s$ .*

**Theorem 4.3.** (cf. [13]) *Under the assumptions (A1)-(A3), let  $\mathcal{A}_\tau$  be the global attractor of  $U_\tau$  for each  $\tau \geq 0$ . We put the set  $\mathcal{A} := \bigcup_{0 \leq \tau \leq T_0} \mathcal{A}_\tau$ . Then,  $\mathcal{A}$  has the following properties:*

- (i)  $\mathcal{A}$  is non-empty and compact in  $L^2(\Omega)$ ;  
(ii) for each bounded set  $B$  in  $L^2(\Omega)$  and each number  $\epsilon > 0$  there exists a finite time  $T_{B,\epsilon} > 0$  such that

$$\text{dist}_{L^2(\Omega)}(U(t + \tau, \tau)z, \mathcal{A}) < \epsilon$$

for all  $\tau \in R_+$ , all  $z \in \overline{K_p(\tau)} \cap B$  and all  $t \geq T_{B,\epsilon}$ .

## 5 Attractor of asymptotically periodic problems

In this section we shall construct a global attractor for the asymptotically  $T_0$ -periodic double obstacle problems  $(\text{AP})_s$ .

In section 3 we see that  $(\text{AP})_s$  has at least one solution on  $[s, +\infty)$ . So we can define a solution operator  $E(t, s)$  ( $0 \leq s \leq t < +\infty$ ) for  $(\text{AP})_s$ . But we cannot show the uniqueness of solutions for  $(\text{AP})_s$  on  $[s, +\infty)$ . Therefore  $E(t, s)$  is multivalued, that is,  $E(t, s)$  ( $0 \leq s \leq t < +\infty$ ) is the operator from  $\overline{K(s)}$  into  $\overline{K(t)}$  which assigns to each  $u_0 \in \overline{K(s)}$  the set

$$E(t, s)u_0 := \left\{ z \in L^2(\Omega) \left| \begin{array}{l} \text{There is a solution } \{u, \theta\} \text{ of } (\text{AP})_s \text{ on } [s, +\infty) \\ \text{such that} \\ u(s) = u_0 \text{ and } u(t) = z. \end{array} \right. \right\}.$$

Then we easily see that  $\{E(t, s) ; 0 \leq s \leq t < +\infty\}$  satisfies the following evolution properties :

(E1)  $E(s, s) = I$  on  $\overline{K(s)}$  for any  $s \geq 0$ .

(E2)  $E(t_2, s)z = E(t_2, t_1)E(t_1, s)z$  for any  $0 \leq s \leq t_1 \leq t_2 < +\infty$  and  $z \in \overline{K(s)}$ .

(E3)  $E(t, s)$  has the following demiclosedness:

- Assume that  $s_n, s, t_n, t \in R_+$  with  $s_n \rightarrow s$  and  $t_n \rightarrow t$ ,  $u_{0n} \in \overline{K(s_n)}$ ,  $u_0 \in \overline{K(s)}$  with  $u_{0n} \rightarrow u_0$  in  $L^2(\Omega)$  and a element  $z_n \in E(t_n + s_n, s_n)u_{0n}$  converges to some element  $z$  in  $L^2(\Omega)$  as  $n \rightarrow +\infty$ . Then,  $z \in E(t + s, s)u_0$ .

In order to construct a global attractor for  $\{E(t, s) ; 0 \leq s \leq t < +\infty\}$  associated with  $(AP)_s$ , we give a definition of a discrete  $\omega$ -limit set under  $E(t, s)$ .

**Definition 5.1.** (Discrete  $\omega$ -limit set for  $E(\cdot, \cdot)$ ) Let  $\mathcal{B}(L^2(\Omega))$  be a family of bounded subsets of  $L^2(\Omega)$ . Let  $\tau \in R_+$  be fixed. Then for each  $B \in \mathcal{B}(L^2(\Omega))$ , the set

$$\omega_\tau(B) := \bigcap_{n \in Z_+} \overline{\bigcup_{k \geq n, m \in Z_+} E(kT_0 + mT_0 + \tau, mT_0 + \tau) \overline{K(mT_0 + \tau)} \cap B}$$

is called the discrete  $\omega$ -limit set of  $B$  under  $E(\cdot, \cdot)$ .

**Remark 5.1.** By definition of the discrete  $\omega$ -limit set  $\omega_\tau(B)$ , it is easy to see that  $x \in \omega_\tau(B)$  if and only if there exist sequences  $\{k_n\} \subset Z_+$  with  $k_n \uparrow +\infty$ ,  $\{m_n\} \subset Z_+$ ,  $\{z_n\} \subset B$  with  $z_n \in \overline{K(m_nT_0 + \tau)}$  and  $\{x_n\} \subset L^2(\Omega)$  with  $x_n \in E(k_nT_0 + m_nT_0 + \tau, m_nT_0 + \tau)z_n$  such that  $x_n \rightarrow x$  in  $L^2(\Omega)$  as  $n \rightarrow +\infty$ .

Now let us mention main theorems in this paper.

**Theorem 5.1.** (Discrete attractors of  $(AP)_\tau$ ) *Suppose the conditions (A1)-(A3). For each  $\tau \in R_+$ , let  $\mathcal{A}_\tau$  be the global attractor of  $T_0$ -periodic dynamical systems  $U_\tau$ , which is obtained in section 4. Here we put*

$$\mathcal{A}_\tau^* := \overline{\bigcup_{B \in \mathcal{B}(L^2(\Omega))} \omega_\tau(B)}. \quad (5.1)$$

Then, we have

- (i)  $\mathcal{A}_\tau^* \subset K_p(\tau)$  is non-empty and compact in  $L^2(\Omega)$ ;
- (ii) for each bounded set  $B \in \mathcal{B}(L^2(\Omega))$  and each number  $\epsilon > 0$  there exists a positive number  $N_{B, \epsilon} \in N$  such that

$$\text{dist}_{L^2(\Omega)}(E(kT_0 + \tau, \tau)z, \mathcal{A}_\tau^*) < \epsilon$$

for all  $z \in \overline{K(\tau)} \cap B$  and all  $k \geq N_{B, \epsilon}$ ;

- (iii)  $\mathcal{A}_\tau^* \subset U_\tau^i \mathcal{A}_\tau^* \subset \mathcal{A}_\tau$  for any  $i \in N$ .

**Remark 5.2.** By the definition of  $\omega_\tau(B)$  and  $\mathcal{A}_\tau^*$ , we easily see that  $\mathcal{A}_\tau^* = \mathcal{A}_{\tau+nT_0}^*$  for any number  $n \in N$ . Hence  $\mathcal{A}_\tau^*$  is  $T_0$ -periodic in time.

Our second main theorem gives a relationship between global attractors  $\mathcal{A}_s^*$  and  $\mathcal{A}_\tau^*$ .

**Theorem 5.2.** *Suppose the conditions (A1)-(A3). Let  $\mathcal{A}_s^*$  and  $\mathcal{A}_\tau^*$  be discrete attractors for  $E(\cdot, s)$  and  $E(\cdot, \tau)$  with  $0 \leq s \leq \tau < +\infty$ , respectively. Then, we have  $\mathcal{A}_\tau^* \subset U(\tau, s)\mathcal{A}_s^*$ , where  $U(\tau, s)$  is the  $T_0$ -periodic process given in section 4.*

By Theorems 5.1 and 5.2, we can construct the attractor for asymptotic  $T_0$ -periodic problems  $(AP)_\tau$ .



**Theorem 5.3.** (Global attractor of  $(\text{AP})_\tau$ ) *Assume (A1)-(A3). For any  $\tau \in R_+$ , let  $\mathcal{A}_\tau^*$  be the discrete attractors for  $E(\cdot, \tau)$  obtained in Theorem 5.1. Here we put*

$$\mathcal{A}^* := \bigcup_{\tau \in [0, T_0]} \mathcal{A}_\tau^*. \quad (5.2)$$

Then, for any bounded set  $B \in \mathcal{B}(H)$ ,

$$\omega_E(B) := \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s, \tau \in R_+} E(t + \tau, \tau)(\overline{K(\tau)} \cap B)} \subset \mathcal{A}^*. \quad (5.3)$$

By Theorem 5.3, we can say that the set  $\mathcal{A}^*$  can be called the attractor of  $(\text{AP})_\tau$ .

In order to prove Theorems 5.1-5.3, we prepare some lemmas.

**Lemma 5.1.** *If  $\{s_n\} \subset R_+$ ,  $\{\tau_n\} \subset R_+$ ,  $s \in R_+$ ,  $\tau \in R_+$ ,  $s_n \rightarrow s$ ,  $\tau_n \rightarrow \tau$ ,  $\{m_n\} \subset Z_+$  with  $m_n \rightarrow +\infty$ ,  $z_n \in \overline{K(m_n T_0 + s_n)}$ ,  $z \in \overline{K_p(s)}$ ,  $z_n \rightarrow z$  in  $L^2(\Omega)$  and a element  $w_n \in E(m_n T_0 + \tau_n + s_n, m_n T_0 + s_n) z_n$  converges to some element  $w \in L^2(\Omega)$  as  $n \rightarrow +\infty$ , then  $w \in U(\tau + s, s)z$*

**Proof.** Since  $\tau_n \rightarrow \tau$ , we may assume that  $\{\tau_n\} \subset [0, T]$  for some  $T > 0$ .

By  $w_n \in E(m_n T_0 + \tau_n + s_n, m_n T_0 + s_n) z_n$ , there exists a solution  $\{v_n, \theta_n\}$  of  $(\text{AP})_{m_n T_0 + s_n}$  such that

$$v_n(m_n T_0 + \tau_n + s_n) = w_n \text{ and } v_n(m_n T_0 + s_n) = z_n.$$

Here we put  $u_n(t) := v_n(t + m_n T_0 + s_n)$ . Then, we easily see that the function  $u_n$  is the solution for

$$(\text{AP})_0 \left\{ \begin{array}{l} u_n'(t) - \mathbf{div}(|\nabla u_n(t)|^{q-2} \nabla u_n(t)) - g(u_n(t)) = \theta_n(t + m_n T_0 + s_n, x) \quad \text{in } Q_0; \\ 0 \leq \theta_n(t + m_n T_0 + s_n, x) \leq h(t + m_n T_0 + s_n, u_n(t, x)) \quad \text{a.e. on } (0, +\infty) \times \Omega; \\ u_n(t) = l(t + m_n T_0 + s_n) \quad \text{a.e. on } (0, +\infty) \times \Gamma; \\ u_n(0) = z_n \quad \text{in } \Omega. \end{array} \right.$$

Let  $\delta \in (0, 1)$  be fixed. Since  $z_n \rightarrow z$  in  $L^2(\Omega)$  as  $n \rightarrow +\infty$ ,  $\{z_n\}$  is bounded in  $L^2(\Omega)$ . Therefore, by Theorem 3.1 there exists a positive constant  $N_\delta > 0$  such that

$$\begin{aligned} & \sup_{t \geq \delta} |u_n(t)|_{L^2(\Omega)}^2 + \sup_{t \geq \delta} \int_t^{t+1} |\nabla u_n(\tau)|_{L^q(\Omega)}^q d\tau \\ & + \sup_{t \geq \delta} |u_n'|_{L^2(t, t+1; L^2(\Omega))}^2 + \sup_{t \geq \delta} |\nabla u_n(t)|_{L^q(\Omega)}^q \leq N_\delta. \end{aligned} \quad (5.4)$$

Here it follows from the convergence assumption (A2), (A3) and (5.4) that (by taking a subsequence of  $\{n\}$ , if necessary) there are functions  $u_\delta$  and  $\theta_\delta$  such that

$$\left\{ \begin{array}{l} u_\delta'(t) - \mathbf{div}(|\nabla u_\delta(t)|^{q-2} \nabla u_\delta(t)) - g(u_\delta(t)) = \theta_\delta(t + s, x) \quad \text{in } [\delta, +\infty) \times \Omega; \\ 0 \leq \theta_\delta(t + s, x) \leq h_p(t + s, u_\delta(t, x)) \quad \text{a.e. on } (\delta, +\infty) \times \Omega; \\ u_\delta(t) = l_p(t + s) \quad \text{a.e. on } (\delta, +\infty) \times \Gamma. \end{array} \right.$$

By the standard diagonal process, we can get the solution  $\{u, \theta\}$  for  $(PP)_{T_0}$  such that

$$(PP)_{T_0} \begin{cases} u'(t) - \mathbf{div}(|\nabla u(t)|^{q-2} \nabla u(t)) - g(u(t)) = \theta(t+s, x) & \text{in } Q_0; \\ 0 \leq \theta(t+s, x) \leq h_p(t+s, u(t, x)) & \text{a.e. on } (0, +\infty) \times \Omega; \\ u(t) = l_p(t+s) & \text{a.e. on } (0, +\infty) \times \Gamma; \\ u(0) = z \end{cases}$$

and

$$u_n \longrightarrow u \text{ in } C([0, T]; H) \text{ as } n \rightarrow +\infty. \quad (5.5)$$

Therefore, it follows from (5.5) and  $u_n(\tau_n) = w_n$  that  $u(\tau) = w$ . Hence we have  $w \in U(\tau+s, s)z$ .  $\diamond$

**Lemma 5.2.** *Let  $\tau \in R_+$  and  $B_{0,\tau}$  be the compact absorbing set for  $U_\tau$ . Then*

$$\omega_\tau(B) \subset B_{0,\tau}, \quad \forall B \in \mathcal{B}(L^2(\Omega)). \quad (5.6)$$

**Proof.** For simplicity, at first let us consider the case of  $\tau \in [0, T_0]$ . Let us fix a bounded subset  $B \in \mathcal{B}(L^2(\Omega))$ . By the global boundedness result obtained in Theorem 3.1, there is a constant  $N_B > 0$  such that

$$\begin{aligned} & \sup_{t \geq s} |u(t)|_{L^2(\Omega)}^2 + \sup_{t \geq 0} \int_t^{t+1} |\nabla u(\tau)|_{L^q(\Omega)}^q d\tau \\ & + \sup_{t \geq s+T_0} |u'|_{L^2(t, t+1; L^2(\Omega))}^2 + \sup_{t \geq s+T_0} |\nabla u(t)|_{L^q(\Omega)}^q \leq N_B, \end{aligned} \quad (5.7)$$

for the solution  $u$  of  $(AP)_s$  on  $[s, +\infty)$  with initial value  $z$  as long as  $s \geq 0$  and  $z \in \overline{K(s)} \cap B$ .

Here for each  $m \in Z_+$ ,  $\tau \in [0, T_0]$ ,  $n \in N$ ,  $z \in \overline{K(mT_0 + \tau)} \cap B$  and  $w \in E(nT_0 + mT_0 + \tau, mT_0 + \tau)z$ , we put  $\tilde{w} := w - l(nT_0 + mT_0 + \tau) + l_p(\tau)$ . Then  $\tilde{w} \in K_p(\tau)$  and

$$\begin{aligned} |\tilde{w} - w|_{L^2(\Omega)} & \leq C_1 J_{m+n}, \\ (\text{hence } |\tilde{w}|_{L^2(\Omega)}) & \leq \sqrt{N_B} + C_1 J_{m+n} \end{aligned} \quad (5.8)$$

and

$$|\nabla \tilde{w}|_{L^q(\Omega)} \leq N_B^{\frac{1}{q}} + J_{m+n}, \quad (5.9)$$

where  $C_1 := \text{meas.}(\Omega)^{\frac{q-2}{2q}}$ .

Since  $J_k$  converges to 0 as  $k \rightarrow +\infty$ , there exists a positive number  $N_0 \in N$  such that

$$J_k \leq 1, \quad \forall k > N_0.$$

Here we put  $J_0 := 1 + \sup_{1 \leq k \leq N_0} J_k < +\infty$ .

Now, we denote the set  $\widetilde{B}_\tau$  by

$$\widetilde{B}_\tau := \{z \in L^2(\Omega); |z|_{L^2(\Omega)} \leq \sqrt{N_B} + C_1 J_0\} \cap \overline{K_p(\tau)} \quad (5.10)$$

Since  $B_{0,\tau}$  is the absorbing set for  $U_\tau$ , there is a positive number  $\widetilde{N} \in N$  such that

$$U_\tau^l \widetilde{B}_\tau \subset B_{0,\tau}, \quad \forall l \geq \widetilde{N}. \quad (5.11)$$

Now, let us prove (5.6). Let  $x$  be any element of  $\omega_\tau(B)$ . Then, by Remark 5.1 we see that there exist sequences  $\{k_n\} \subset Z_+$  with  $k_n \uparrow +\infty$ ,  $\{m_n\} \subset Z_+$ ,  $\{z_n\} \subset B$  with  $z_n \in \overline{K(m_n T_0 + \tau)}$  and  $\{x_n\} \subset L^2(\Omega)$  with  $x_n \in E(k_n T_0 + m_n T_0 + \tau, m_n T_0 + \tau)z_n$  such that

$$x_n \longrightarrow x \text{ in } L^2(\Omega) \quad \text{as } n \rightarrow +\infty. \quad (5.12)$$

Let  $\widetilde{N}$  be the positive number obtained in (5.11). It follows from (E2) that

$$\begin{aligned} x_n &\in E(k_n T_0 + m_n T_0 + \tau, k_n T_0 + m_n T_0 + \tau - \widetilde{N}T_0) \\ &\circ E(k_n T_0 + m_n T_0 + \tau - \widetilde{N}T_0, m_n T_0 + \tau)z_n \\ &\text{for any } n \text{ with } k_n \geq \widetilde{N} + 1. \end{aligned} \quad (5.13)$$

By (5.13) there is a element  $y_n \in E(k_n T_0 + m_n T_0 + \tau - \widetilde{N}T_0, m_n T_0 + \tau)z_n$  such that

$$x_n \in E(k_n T_0 + m_n T_0 + \tau, k_n T_0 + m_n T_0 + \tau - \widetilde{N}T_0)y_n. \quad (5.14)$$

Here we note that

$$|y_n|_{L^2(\Omega)}^2 \leq N_B \quad \text{and} \quad |\nabla y_n|_{L^q(\Omega)}^q \leq N_B \quad \text{for any } n \text{ with } k_n \geq \widetilde{N} + 1,$$

where  $N_B$  is the same positive constant in (5.7).

It follows from (5.8)-(5.9) that for  $y_n \in E(k_n T_0 + m_n T_0 + \tau - \widetilde{N}T_0, m_n T_0 + \tau)z_n$  we can take  $\widetilde{y}_n := y_n - l(k_n T_0 + m_n T_0 + \tau - \widetilde{N}T_0) + l_p(\tau) \in K_p(\tau)$  satisfying

$$|\widetilde{y}_n|_{L^2(\Omega)} \leq \sqrt{N_B} + C_1 J_{k_n+m_n-\widetilde{N}} \quad \text{and} \quad |\nabla \widetilde{y}_n|_{L^q(\Omega)} \leq N_B^{\frac{1}{q}} + J_{k_n+m_n-\widetilde{N}}.$$

Clearly,  $\{\widetilde{y}_n \in K_p(\tau) ; n \in N \text{ with } k_n \geq \widetilde{N} + 1\} \subset \widetilde{B}_\tau$  is relatively compact in  $L^2(\Omega)$ , hence we may assume that

$$\widetilde{y}_n \longrightarrow \widetilde{y}_\infty \text{ in } L^2(\Omega) \quad \text{as } n \rightarrow +\infty$$

for some  $\widetilde{y}_\infty \in L^2(\Omega)$ ; it is easily see that  $\widetilde{y}_\infty \in \widetilde{B}_\tau$  and

$$y_n \longrightarrow \widetilde{y}_\infty \text{ in } L^2(\Omega) \quad \text{as } n \rightarrow +\infty. \quad (5.15)$$

Here, applying Lemma 5.1, it follows from (5.12)-(5.15) that

$$x \in U(\widetilde{N}T_0 + \tau, \tau)\widetilde{y}_\infty \subset U(\widetilde{N}T_0 + \tau, \tau)\widetilde{B}_\tau = U_\tau^{\widetilde{N}}\widetilde{B}_\tau \subset B_{0,\tau}.$$

Therefore we observe that  $\omega_\tau(B) \subset B_{0,\tau}$ .

For the general case of  $\tau \in R_+$  there are positive numbers  $i_\tau \in N$  and  $\tau_0 \in [0, T_0]$  such that  $\tau = \tau_0 + i_\tau T_0$ . Therefore, by the same argument as above, we can prove (5.6).  $\diamond$

**Proof of Theorem 5.1.** By Lemma 5.2 we easily see that  $\mathcal{A}_\tau^* \subset B_{0,\tau}$ , hence, Theorem 5.1 (i) holds. Also, it follows from (5.1) and Remark 5.1 that Theorem 5.1 (ii) holds.

Now, let us prove Theorem 5.1 (iii). At first, we show that  $\mathcal{A}_\tau^* \subset U_\tau^i \mathcal{A}_\tau^*$  for any  $i \in N$ . To do so, let  $x$  be any element of  $\mathcal{A}_\tau^*$ . By the definition of  $\mathcal{A}_\tau^*$ , we may assume that there exist sequences  $\{B_n\} \subset \mathcal{B}(L^2(\Omega))$  and  $\{x_n\} \subset L^2(\Omega)$  with  $x_n \in \omega_\tau(B_n)$  such that

$$x_n \longrightarrow x \text{ in } L^2(\Omega) \quad \text{as } n \rightarrow +\infty. \quad (5.16)$$

It follows from Remark 5.1 that for each  $n$ , there exist sequences  $\{k_{n,j}\} \subset Z_+$  with  $k_{n,j} \rightarrow +\infty$ ,  $\{m_{n,j}\} \subset Z_+$ ,  $\{z_{n,j}\} \subset B_n$  with  $z_{n,j} \in \overline{K(m_{n,j}T_0 + \tau)}$  and  $\{v_{n,j}\} \subset L^2(\Omega)$  with  $v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j}$  such that

$$v_{n,j} \longrightarrow x_n \text{ in } L^2(\Omega) \quad \text{as } j \rightarrow +\infty. \quad (5.17)$$

Let  $i$  be any number in  $N$ . We note that

$$\begin{aligned} v_{n,j} &\in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 + m_{n,j}T_0 + \tau - iT_0) \\ &\circ E(k_{n,j}T_0 + m_{n,j}T_0 + \tau - iT_0, m_{n,j}T_0 + \tau)z_{n,j} \end{aligned}$$

for  $j$  with  $k_{n,j} \geq i+1$ . Hence there is a  $w_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau - iT_0, m_{n,j}T_0 + \tau)z_{n,j}$  such that

$$v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 + m_{n,j}T_0 + \tau - iT_0)w_{n,j}. \quad (5.18)$$

For each  $n$ , by Theorem 3.1, the set  $\{w_{n,j} \in L^2(\Omega) ; j \in N \text{ with } k_{n,j} \geq i+1\}$  is relatively compact in  $L^2(\Omega)$ . So, we may assume that the element  $w_{n,j}$  converges to some element  $\tilde{w}_{n,\infty} \in L^2(\Omega)$  as  $j \rightarrow +\infty$ . Clearly,  $\tilde{w}_{n,\infty} \in \omega_\tau(B_n)$ . Moreover, from Lemma 5.1 and (5.17)-(5.18), we observe that

$$x_n \in U(iT_0 + \tau, \tau)\tilde{w}_{n,\infty} \subset U(iT_0 + \tau, \tau)\omega_\tau(B_n),$$

which implies that

$$x_n \in \bigcup_{n \geq 1} U_\tau^i \omega_\tau(B_n), \quad \forall n \geq 1. \quad (5.19)$$

Moreover, by the closedness of  $U(\cdot, \cdot)$ , we observe that for each subset  $X$  of  $B_{0,\tau}$ ,

$$\overline{U_\tau^i X} \subset U_\tau^i \overline{X}, \quad \forall i \in N. \quad (5.20)$$

Since Lemma 5.2, (5.16), (5.19) and (5.20), we see that

$$x \in \overline{\bigcup_{n \geq 1} U_\tau^i \omega_\tau(B_n)} = \overline{U_\tau^i \bigcup_{n \geq 1} \omega_\tau(B_n)} \subset U_\tau^i \overline{\bigcup_{n \geq 1} \omega_\tau(B_n)} \subset U_\tau^i \mathcal{A}_\tau^*.$$

Hence we observe that  $\mathcal{A}_\tau^*$  is semi-invariant under the  $T_0$ -periodic dynamical systems  $U_\tau$ , namely

$$\mathcal{A}_\tau^* \subset U_\tau^i \mathcal{A}_\tau^*, \quad \forall i \in N. \quad (5.21)$$

Next we shall show that  $U_\tau^i \mathcal{A}_\tau^* \subset \mathcal{A}_\tau$  for any  $i \in N$ . By (5.21), for each  $i \in N$

$$U_\tau^i \mathcal{A}_\tau^* \subset U_\tau^i U_\tau^n \mathcal{A}_\tau^* = U_\tau^{i+n} \mathcal{A}_\tau^*, \quad \forall n \in N. \quad (5.22)$$

Since  $\mathcal{A}_\tau^* \subset B_{0,\tau}$ , from (5.22) and the attractive property of  $\mathcal{A}_\tau$  it follows that

$$U_\tau^i \mathcal{A}_\tau^* \subset \mathcal{A}_\tau, \quad \forall i \in N,$$

hence we conclude that  $\mathcal{A}_\tau^* \subset U_\tau^i \mathcal{A}_\tau^* \subset \mathcal{A}_\tau$  for any  $i \in N$ . ◇

**Proof of Theorem 5.2.** Let  $x$  be any element of  $\mathcal{A}_\tau^*$ . Then by (5.1), we see that there exist sequences  $\{B_n\} \subset \mathcal{B}(L^2(\Omega))$  and  $\{x_n\} \subset L^2(\Omega)$  with  $x_n \in \omega_\tau(B_n)$  such that

$$x_n \longrightarrow x \text{ in } L^2(\Omega) \quad \text{as } n \rightarrow +\infty. \quad (5.23)$$

It follows from Remark 5.1 that for each  $n$ , there exist sequences  $\{k_{n,j}\} \subset Z_+$  with  $k_{n,j} \rightarrow +\infty$ ,  $\{m_{n,j}\} \subset Z_+$ ,  $\{z_{n,j}\} \subset B_n$  with  $z_{n,j} \in \overline{K(m_{n,j}T_0 + \tau)}$  and  $\{v_{n,j}\} \subset L^2(\Omega)$  with  $v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j}$  such that

$$v_{n,j} \longrightarrow x_n \text{ in } L^2(\Omega) \quad \text{as } j \rightarrow +\infty. \quad (5.24)$$

Note that for given  $s, \tau \in R_+$  with  $s \leq \tau$ , we can take a positive number  $i_s \in N$  satisfying

$$s \leq \tau \leq s + i_s T_0.$$

From (E2) it follows that

$$\begin{aligned} v_{n,j} &\in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 + m_{n,j}T_0 + s) \\ &\quad \circ E(k_{n,j}T_0 + m_{n,j}T_0 + s, i_s T_0 + m_{n,j}T_0 + s + T_0) \\ &\quad \circ E(i_s T_0 + m_{n,j}T_0 + s + T_0, m_{n,j}T_0 + \tau)z_{n,j} \end{aligned}$$

for any  $j \in Z_+$  with  $k_{n,j} \geq i_s + 2$ . So, there are element  $w_{n,j} \in L^2(\Omega)$  and  $y_{n,j} \in L^2(\Omega)$  such that

$$v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 + m_{n,j}T_0 + s)w_{n,j}, \quad (5.25)$$

$$w_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + s, i_s T_0 + m_{n,j}T_0 + s + T_0)y_{n,j} \quad (5.26)$$

and

$$y_{n,j} \in E(i_s T_0 + m_{n,j}T_0 + s + T_0, m_{n,j}T_0 + \tau)z_{n,j}. \quad (5.27)$$

Since  $\{z_{n,j}\} \subset B_n$ , it follows from the global boundedness results in Theorem 3.1 that there is a positive constant  $C_n := C_n(B_n) > 0$  satisfying

$$|y_{n,j}|_{L^2(\Omega)} \leq C_n, \quad \forall y_{n,j} \in E(i_s T_0 + m_{n,j}T_0 + s + T_0, m_{n,j}T_0 + \tau)z_{n,j}. \quad (5.28)$$

By (5.28) and Theorem 3.1, the set

$$\left\{ \begin{array}{l} w_{n,j} \in L^2(\Omega); \\ \quad \quad \quad \text{for any } j \in Z_+ \text{ with } k_{n,j} \geq i_s + 2 \end{array} \right\}$$

is relatively compact in  $L^2(\Omega)$ . So, we may assume that the element  $w_{n,j}$  converges to some element  $\tilde{w}_{n,\infty} \in L^2(\Omega)$  as  $j \rightarrow +\infty$ . Clearly,  $\tilde{w}_{n,\infty} \in \omega_s(B_{C_n})$ , where  $B_{C_n} := \{b \in L^2(\Omega) ; |b|_{L^2(\Omega)} \leq C_n\}$ . Moreover, by Lemma 5.2, we see that

$$\omega_s(B_{C_n}) \subset B_{0,s} \subset \overline{K_p(s)},$$

where  $B_{0,s}$  is the compact absorbing set for  $U_s$ . Also, by Lemma 5.1 and (5.24)-(5.25) we have

$$x_n \in U(\tau, s)\tilde{w}_{n,\infty} \subset U(\tau, s)\omega_s(B_{C_n}), \quad \forall n \geq 1,$$

which implies that

$$x_n \in \bigcup_{n \geq 1} U(\tau, s)\omega_s(B_{C_n}), \quad \forall n \geq 1. \quad (5.29)$$

Moreover, by the closedness of  $U(\cdot, \cdot)$ , we observe that for each subset  $X$  of  $B_{0,s}$ ,

$$\overline{U(\tau, s)X} \subset U(\tau, s)\overline{X}. \quad (5.30)$$

Since Lemma 5.2, (5.23), (5.29) and (5.30), we see that

$$x \in \overline{\bigcup_{n \geq 1} U(\tau, s)\omega_s(B_{C_n})} = \overline{U(\tau, s) \bigcup_{n \geq 1} \omega_s(B_{C_n})} \subset U(\tau, s) \overline{\bigcup_{n \geq 1} \omega_s(B_{C_n})} \subset U(\tau, s)\mathcal{A}_s^*.$$

Hence we observe that  $\mathcal{A}_\tau^*$  is the subset of  $U(\tau, s)\mathcal{A}_s^*$ , namely  $\mathcal{A}_\tau^* \subset U(\tau, s)\mathcal{A}_s^*$ .  $\diamond$

**Proof of Theorem 5.3.** For any  $B \in \mathcal{B}(L^2(\Omega))$ , let  $z_0$  be any element of  $\omega_E(B)$ . Then there exist sequences  $\{t_n\} \subset R_+$  with  $t_n \uparrow +\infty$ ,  $\{\tau_n\} \subset R_+$ ,  $\{y_n\} \subset B$  with  $y_n \in \overline{K(\tau_n)}$  and  $\{z_n\} \subset L^2(\Omega)$  with  $z_n \in E(t_n + \tau_n, \tau_n)y_n$  such that

$$\begin{aligned} t_n &:= k_n T_0 + t'_n, \quad k_n \in \mathbb{Z}_+, \quad k_n \nearrow +\infty, \quad t'_n \in [T_0, 2T_0], \quad t'_n \rightarrow t'_0, \\ \tau_n &:= i_n T_0 + \tau'_n, \quad i_n \in \mathbb{Z}_+, \quad \tau'_n \in [0, T_0], \quad \tau'_n \rightarrow \tau'_0 \end{aligned}$$

and

$$z_n \longrightarrow z_0 \quad \text{in } L^2(\Omega) \quad (5.31)$$

as  $n \rightarrow +\infty$ ; we may assume further that

$$(a) \quad t'_n + \tau'_n \nearrow t'_0 + \tau'_0 \quad \text{or} \quad (b) \quad t'_n + \tau'_n \searrow t'_0 + \tau'_0.$$

Assume that (a) holds. Let us consider the semiflow

$$v_n \in E(1 + k_n T_0 + i_n T_0 + t'_0 + \tau'_0, k_n T_0 + i_n T_0 + t'_n + \tau'_n)z_n. \quad (5.32)$$

Then, there exists functions  $u_n$  and  $\theta_n$  such that

$$\left\{ \begin{array}{l} u'_n(t) - \mathbf{div}(|\nabla u_n(t)|^{q-2} \nabla u_n(t)) - g(u_n(t)) = \theta_n(t, x) \quad \text{in } [0, +\infty) \times \Omega, \\ 0 \leq \theta_n(t, x) \leq h(t + k_n T_0 + i_n T_0 + t'_n + \tau'_n, u_n(t, x)) \quad \text{a.e. on } (0, +\infty) \times \Omega, \\ u_n(t) = l(t + k_n T_0 + i_n T_0 + t'_n + \tau'_n) \quad \text{a.e. on } (0, +\infty) \times \Gamma, \\ u_n(0) = z_n \quad \text{in } \Omega, \\ u_n(1 + t'_0 + \tau'_0 - t'_n - \tau'_n) = v_n. \end{array} \right.$$

Since  $z_n \rightarrow z_0$  in  $L^2(\Omega)$ ,  $\{z_n\}$  is bounded in  $L^2(\Omega)$ , hence we see that

$$\left\{ \begin{array}{l} v_n \in L^2(\Omega); \\ v_n \in E(1 + k_n T_0 + i_n T_0 + t'_0 + \tau'_0, k_n T_0 + i_n T_0 + t'_n + \tau'_n)z_n \end{array} \right\} \quad \text{for any } n \in N$$

is relatively compact in  $L^2(\Omega)$ . So we may assume that

$$v_n \longrightarrow v \text{ in } L^2(\Omega) \text{ for some } v \in L^2(\Omega). \quad (5.33)$$

Therefore, by Lemma 5.1 and (5.31)-(5.33), we have

$$v \in U(1 + t'_0 + \tau'_0, t'_0 + \tau'_0)z_0,$$

more precisely, (taking the subsequence of  $\{n\}$  if necessary) there are functions  $u$  and  $\theta$  such that

$$(PP)_{T_0} \left\{ \begin{array}{l} u'(t) - \mathbf{div}(|\nabla u(t)|^{q-2} \nabla u(t)) - g(u(t)) = \theta(t, x) \quad \text{in } [0, +\infty) \times \Omega, \\ 0 \leq \theta(t, x) \leq h_p(t + t'_0 + \tau'_0, u(t, x)) \quad \text{a.e. on } (0, +\infty) \times \Omega, \\ u(t) = l_p(t + t'_0 + \tau'_0) \quad \text{a.e. on } (0, +\infty) \times \Gamma, \\ u(0) = z_0 \quad \text{in } \Omega, \\ u(1) = v. \end{array} \right.$$

and

$$u_n \longrightarrow u \quad \text{in } C([0, 2]; L^2(\Omega)) \quad \text{as } n \rightarrow +\infty. \quad (5.34)$$

By (5.34), we easily observe that

$$u_n(t'_0 + \tau'_0 - t'_n - \tau'_n) \longrightarrow z_0 \quad \text{as } n \rightarrow +\infty. \quad (5.35)$$

Here, we note that

$$\begin{aligned} & u_n(t'_0 + \tau'_0 - t'_n - \tau'_n) \\ & \in E(k_n T_0 + i_n T_0 + t'_0 + \tau'_0, k_n T_0 + i_n T_0 + t'_n + \tau'_n) z_n \\ & = E(k_n T_0 + i_n T_0 + t'_0 + \tau'_0, i_n T_0 + t'_0 + \tau'_0) E(i_n T_0 + t'_0 + \tau'_0, i_n T_0 + \tau'_n) y_n, \end{aligned}$$

hence there is a element  $x_n \in E(i_n T_0 + t'_0 + \tau'_0, i_n T_0 + \tau'_n) y_n$  such that

$$u_n(t'_0 + \tau'_0 - t'_n - \tau'_n) \in E(k_n T_0 + i_n T_0 + t'_0 + \tau'_0, i_n T_0 + t'_0 + \tau'_0) x_n. \quad (5.36)$$

Clearly, by the global estimate of solutions,  $\{x_n\}$  is bounded, i.e.

$$\{x_n\} \subset \tilde{B} \text{ for some } \tilde{B} \in \mathcal{B}(L^2(\Omega)). \quad (5.37)$$

Hence it follows from (5.35)-(5.37) and Remark 5.2 that

$$z_0 \in \omega_{t'_0 + \tau'_0}(\tilde{B}) \subset \mathcal{A}_{t'_0 + \tau'_0}^* \subset \mathcal{A}^*.$$

Thus (5.3) holds. Assuming that (b) holds, we similarly get (5.3).  $\diamond$

Theorem 5.1 says that the attracting set  $\mathcal{A}_\tau^*$  for  $(AP)_s$  is semi-invariant under  $U_\tau$  associated with the limiting  $T_0$ -periodic problem  $(PP)_{T_0}$ , in general. Moreover, in Theorem 5.2 we see that  $\mathcal{A}_\tau^* \subset U(\tau, s)\mathcal{A}_s^*$ .

In order to get the invariance of  $\mathcal{A}_\tau^*$  under  $U_\tau$  and  $\mathcal{A}_\tau^* = U(\tau, s)\mathcal{A}_s^*$ , we have to assume the additional conditions for  $l$  and  $h$ .

**Theorem 5.4** *Suppose all conditions (A1)-(A3). Let  $\mathcal{A}_s^*$  and  $\mathcal{A}_\tau^*$  be discrete attractors*

for  $E(\cdot, s)$  and  $E(\cdot, \tau)$ , with  $0 \leq s \leq \tau < +\infty$ , respectively. Furthermore we assume that the boundary condition  $l(t)$  for  $(AP)_s$  coincides with  $l_p(t)$ , namely  $l(t) \equiv l_p(t)$  on  $\Gamma$  for any  $t \geq 0$ . And we suppose that  $h_p(t, z) \leq h(t, z)$  for any  $0 \leq t < +\infty$  and  $z \in R$ . Then,

(i)  $\mathcal{A}_\tau^* = U(\tau, s)\mathcal{A}_s^*$  for any  $0 \leq s \leq \tau < +\infty$ .

(ii)  $\mathcal{A}_\tau^* = \mathcal{A}_\tau$  for any  $\tau \in R_+$ , where  $\mathcal{A}_\tau$  is the discrete attractor of  $U_\tau$  for  $(PP)_{T_0}$ .

**Proof.** Let us show (i). By taking account of Theorem 5.2, we have only to show that  $U(\tau, s)\mathcal{A}_s^* \subset \mathcal{A}_\tau^*$ . To do so, let  $x$  be any element of  $U(\tau, s)\mathcal{A}_s^*$ .

At first, we note that for each  $n \in N$

$$\begin{aligned} U_\tau^n U(\tau, s)\mathcal{A}_s^* &= U(nT_0 + \tau, \tau)U(\tau, s)\mathcal{A}_s^* \\ &= U(nT_0 + \tau, nT_0 + s)U(nT_0 + s, s)\mathcal{A}_s^* \\ &= U(\tau, s)U_s^n \mathcal{A}_s^* \\ &\supset U(\tau, s)\mathcal{A}_s^*. \end{aligned} \tag{5.38}$$

By (5.38), there is a element  $y_n \in \mathcal{A}_s^*$  such that

$$x \in U_\tau^n U(\tau, s)y_n = U(nT_0 + \tau, s)y_n.$$

Therefore, there is a solution  $\{u, \theta\}$  of  $(PP)_{T_0}$  on  $[s, +\infty)$  such that  $u(nT_0 + \tau) = x$  and  $u(s) = y_n$ .

Let  $\{k_n\} \subset N$  be a sequence with  $k_n \rightarrow +\infty$  as  $n \rightarrow +\infty$ . Here, we put

$$u_n(\sigma, \cdot) := u(\sigma - k_n T_0, \cdot) \text{ and } \theta_n(\sigma, \cdot) := \theta(\sigma - k_n T_0, \cdot)$$

for any  $\sigma \geq k_n T_0 + s$ . Then, by the assumptions of Theorem 5.4 we see that

$$u_n(\sigma) = u(\sigma - k_n T_0) = l_p(\sigma - k_n T_0) = l_p(\sigma) = l(\sigma) \text{ on } \Gamma$$

and

$$\begin{aligned} 0 \leq \theta_n(\sigma, x) = \theta(\sigma - k_n T_0, x) &\leq h_p(\sigma - k_n T_0, u(\sigma - k_n T_0, x)) \\ &= h_p(\sigma, u_n(\sigma, x)) \leq h(\sigma, u_n(\sigma, x)) \end{aligned}$$

for any  $\sigma \geq k_n T_0 + s$  and  $x \in \Omega$ . Therefore, the pair of functions  $\{u_n, \theta_n\}$  is the solution of  $(AP)_{k_n T_0 + s}$  such that  $u_n(nT_0 + k_n T_0 + \tau) = u(nT_0 + \tau) = x$  and  $u_n(k_n T_0 + s) = u(s) = y_n$ , which implies that  $x \in E(nT_0 + k_n T_0 + \tau, k_n T_0 + s)y_n$  for any  $n \geq 1$ . By (E2), we see that

$$\begin{aligned} x &\in E(nT_0 + k_n T_0 + \tau, k_n T_0 + s)y_n \\ &= E(nT_0 + k_n T_0 + \tau, T_0 + k_n T_0 + \tau)E(T_0 + k_n T_0 + \tau, k_n T_0 + s)y_n. \end{aligned}$$

Hence there is an element  $z_n \in E(T_0 + k_n T_0 + \tau, k_n T_0 + s)y_n$  such that

$$x \in E(nT_0 + k_n T_0 + \tau, T_0 + k_n T_0 + \tau)z_n. \tag{5.39}$$

Since  $\{y_n\} \subset \mathcal{A}_s^*$  and the global estimate obtained in Theorem 3.1, we see that  $\{z_n\}$  is bounded in  $L^2(\Omega)$ , namely  $\{z_n\} \subset \tilde{B}$  for some  $\tilde{B} \in \mathcal{B}(L^2(\Omega))$ . The above fact (5.39) implies (cf. Remark 5.1) that  $x \in \omega_\tau(\tilde{B}) \subset \mathcal{A}_\tau^*$ . Thus  $U(\tau, s)\mathcal{A}_s^* \subset \mathcal{A}_\tau^*$ , which implies that (i) of Theorem 5.4 holds.

Since  $\mathcal{A}_\tau$  is invariant under  $U_\tau$  (cf. Theorem 4.1 (iii)), by the same argument in (i), we can show (ii). Therefore, Theorem 5.4 has been completed.  $\diamond$



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