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Special values of the standard zeta functions for elliptic modular forms

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Abstract

We give an algorithm for computing the special values of twisted standard zeta functions of elliptic modular forms by using the pullback formula for the Siegel Eisenstein series of degree 2.

1 Introduction

Let M and k be positive integers, and ϕ a Dirichlet character modulo M . For a normalized cuspidal Hecke eigenform f of weight k and Nebentypus ϕ with respect to $\Gamma_0(M)$, and a Dirichlet character χ modulo N , let $L(f, s, \chi)$ be the standard zeta function of f twisted by χ . (For the precise definition of the standard zeta function, see Section 3.) The standard zeta function of a modular form is an important subject in number theory, and it is related to many other areas, especially to the Galois representations. As for this, see, for examples, [Doi, Hida and Ishii, 1998] and [Dummigan, 2001]. The special values of the standard zeta function are particularly important. To be more precise, assume that k is even, and put

$$L^*(f, m, \chi) = \frac{L(f, m, \chi)}{\pi^{k+2m} \langle f, f \rangle}$$

for an odd positive integer $m \leq k-1$, where $\langle *, * \rangle$ is the normalized Petersson product. As is well known, these values are algebraic numbers and their qualitative natures have been fully investigated by many people (cf. [Sturm, 1980], [Shimura, 2000], [Böcherer and Schmidt, 2000]). To investigate various problems related to these values, it is important to compute these values exactly. Several people have considered algorithms for computing these values and have carried out the computations. Zagier [Zagier, 1977] gave an explicit formula expressing $L^*(f, m, \chi)$ in the case where M is a squarefree positive integer congruent to 1 modulo 4, ϕ is the Kronecker character $(\frac{M}{*})$ corresponding to the extension $\mathbf{Q}(\sqrt{M})/\mathbf{Q}$, and χ is trivial. Sturm [Sturm, 1980] gave an algorithm for computing these values for a general χ . However, it seems difficult to give exact values

by a direct use of his method. Stopple [Stopple, 1996] gave an explicit formula expressing $L^*(f, m, \chi)$ in case $M = 1$ and χ is a quadratic character.

In [Katsurada, 2003], we have announced some formulas which seem useful for the computation of $L^*(f, m, \chi)$ in the case where $M = 1$ or a prime number congruent to 1 modulo 4, $\phi = \left(\frac{M}{*}\right)$, and χ is not necessarily quadratic character of prime conductor p such that $\chi(-1) = 1$. In this paper, we give a complete proof to these formulas under more general setting. The main tool is the pullback formula of the Siegel Eisenstein series of degree 2 due to Böcherer and Schmidt [Böcherer and Schmidt, 2000], Shimura [Shimura, 2000]. Such a formula has been used to study a qualitative nature of the special values of the standard zeta function. However, as far as the author knows, no one has used the formula to give its exact values. In this paper, we carry out such a computation.

To explain our method briefly, for simplicity $M \neq p$. Let k and l be even positive integers such that $l \leq k$. Then we define a certain Siegel Eisenstein series $E_{2,l}^*(Z, Mp^2, \phi\bar{\chi}, s)$ in Section 2. Write $\mathbf{e}(u) = \exp(2\pi\sqrt{-1}u)$ for a complex number u . Then, as is well known, if $l \geq 4$, $E_{2,l}^*(Z, Mp^2, \phi\bar{\chi}, 0)$ becomes a holomorphic modular form of weight l and of Nebentypus $\phi\bar{\chi}$, and has the following Fourier expansion:

$$E_{2,l}^*(Z; Mp^2, \phi\bar{\chi}, 0) = \sum_A c_{n,l}(A, Mp^2, \phi\bar{\chi}, 0)\mathbf{e}(\text{tr}(AZ)),$$

where A runs over all positive definite half-integral matrices of degree 2, and $\text{tr}(\ast)$ denotes the trace of a matrix. Put

$$\tilde{c}_{2,l}(A, 0) = \tilde{c}_{2,l}(A, Mp^2, \phi\bar{\chi}, 0) = A(l, 0)^{-1}c_{2,l}(A, Mp^2, \phi\bar{\chi}, 0)$$

with a suitable normalizing factor $A(l, 0)^{-1}$ (cf. Theorem 2.1), and for two positive integers m_1, m_2 put

$$\epsilon(m_1, m_2; l, 0) = \sum_{r^2 \leq 4m_1m_2} \tilde{c}_{2,l}\left(\begin{pmatrix} m_1 & r/2 \\ r/2 & m_2 \end{pmatrix}, 0\right)G_l^{k-l}(m_1m_2, r)\chi(r)\tau(\bar{\chi}),$$

where $G_l^{k-l}(u, v)$ is the polynomial introduced by Zagier [Zagier, 1977], and $\tau(\bar{\chi})$ is the Gauss sum (cf. Section 3). Furthermore, put $t(m; l, 0) = \epsilon(p, p^2m; l, 0) - \phi(p)p^{k-2}\epsilon(p, m; l, 0)$, and

$$\mathcal{F}_{p,p}(z) = \sum_{m=1}^{\infty} t(m; l, 0)\mathbf{e}(mz).$$

Then by the holomorphy of the Eisenstein series and the theory of differential operators on modular forms due to Ibukiyama [Ibukiyama, 1999], $\mathcal{F}_{p,p}(z)$ belongs

to $S_k(\Gamma_0(Mp), \phi)$ (cf. Sections 3 and 4.) Now take a basis $\{f_i\}_{i=1}^{d_1}$ of $S_k(\Gamma_0(M), \phi)$ consisting of primitive forms, and write

$$f_i(z) = \sum_{m=1}^{\infty} a_i(m) \mathbf{e}(mz)$$

with $a_i(1) = 1$. Then by the pullback formula due to Böcherer and Schmidt [Böcherer and Schmidt, 2000], we have

$$\mathcal{F}_{p,p}(z) = \gamma_{k,l,p,M} \sum_{i=1}^{d_1} L^*(f_i, l-1, \chi) \bar{c}_i^2 \tilde{f}_i(z),$$

where $\gamma_{k,l,p,M}$ is a rational number explicitly determined by k, l, p, M , and c_i is a certain algebraic number with absolute norm 1, and

$$\tilde{f}_i(z) = \sum_{m=1}^{\infty} a_i(pm) \mathbf{e}(mz)$$

(cf. (2) of Theorem 4.2.) We note that an explicit form of $\tilde{c}_{2,l}(A, 0)$ is given (cf. Theorem 2.1.) Thus, by the above formula combined with the trace formula of Hecke operators, we can compute the norm $N_{K_{f,\chi}}(L^*(f, m, \chi))$ for a primitive form $f \in S_k(\Gamma_0(M), \phi)$ and for an odd integer m such that $3 \leq m \leq k-1$. Here $K_{f,\chi}$ is the field generated over \mathbf{Q} by all the eigenvalues of Hecke operators relative to f and all the values of χ (cf. Theorem 4.4.) If χ^2 is not trivial, $E_{2,2}^*(Z; Mp^2, \phi\bar{\chi}, 0)$ becomes also holomorphic, and by the same procedure, we obtain an exact value for $N_{K_{f,\chi}}(L^*(f, 1, \chi))$. On the contrary, if χ^2 is trivial, $E_{2,2}^*(Z; Mp^2, \phi\bar{\chi}, 0)$ is not holomorphic. However, $E_{2,2}^*(Z; Mp^2, \phi\bar{\chi}, -1/2)$ is holomorphic, and by the same procedure, we obtain an exact value of $N_{K_{f,\chi}}(L^*(f, 0, \chi))$, and by the functional equation due to Li [Li, 1979], we can finally compute $N_{K_{f,\chi}}(L^*(f, 1, \chi))$ also in this case (cf. (2) of Proposition 4.5) In case $M = p$ we obtain similar results (cf. (1) of Theorem 4.2, (1) of Theorem 4.4, and (1) of Proposition 4.4.) In Section 5, we give some numerical examples, and discuss some related topics.

Besides such a practical computation, as an application of Theorem 4.2, we show that a prime factor of the denominator of $L^*(f, m, \chi)$ gives a congruence between f and another primitive form (cf. Theorem 4.8.)

By using the method in this paper, we expect more fruitful results about the special values of standard zeta functions of other modular forms, for examples, of Siegel modular forms and of Hilbert modular forms. We will discuss these topics in subsequent papers.

2 Fourier coefficients of Siegel Eisenstein series

Let $GSp_n^+(\mathbf{R})$ be the group of proper symplectic similitudes of degree n , and \mathbf{H}_n Siegel's upper half space of degree n . As usual we write $\gamma(Z) = (AZ + B)(CZ +$

$D)^{-1}$ and $j(\gamma, Z) = \det(CZ + D)$ for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_n^+(\mathbf{R})$. We write $f|_k\gamma(z) = (\det \gamma)^{k/2} j(\gamma, z)^{-k} f(\gamma(z))$ for $\gamma \in GSp_n^+(\mathbf{R})$ and a C^∞ -function f on \mathbf{H}_n . We simply write $f|\gamma$ for $f|_k\gamma$ if there is no confusion. Let $Sp_n(\mathbf{Z})$ be the Siegel modular group of degree n . For a positive integer M , we denote by $\Gamma_0^{(n)}(M)$ (resp. $\Gamma_0^{\prime(n)}(M)$) the subgroup of $Sp_n(\mathbf{Z})$ consisting of matrices whose lower left $n \times n$ block (resp. upper right $n \times n$ block) is congruent to O modulo M . For a Dirichlet character ϕ modulo M , we denote by $\tilde{\phi}$ (resp. $\tilde{\phi}'$) the character of $\Gamma_0^{(n)}(M)$ (resp. $\Gamma_0^{\prime(n)}(M)$) defined by $\tilde{\phi}(\gamma) = \phi(\det D)$ (resp. $\tilde{\phi}'(\gamma) = \phi(\det A)$) for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. We denote by $\mathbf{1}_M$ the trivial character modulo M , and in particular put $\mathbf{1} = \mathbf{1}_1$. For a Dirichlet character ϕ modulo M , we denote by $M_k(\Gamma_0^{(n)}(M), \phi)$ (resp. $M_k^\infty(\Gamma_0^{(n)}(M), \phi)$) the space of holomorphic (resp. C^∞ -) modular forms of weight k and Nebentypus ϕ with respect to $\Gamma_0^{(n)}(M)$, and by $S_k(\Gamma_0^{(n)}(M), \phi)$ the subspace of $M_k(\Gamma_0^{(n)}(M), \phi)$ consisting of cusp forms. In particular if $\phi = \mathbf{1}_M$, we write $S_k(\Gamma_0^{(n)}(M))$ for $S_k(\Gamma_0^{(n)}(M), \phi)$ and the others. Furthermore, for a subgroup Γ of $Sp_n(\mathbf{Z})$ we denote by Γ_∞ the subgroup of Γ consisting of matrices whose lower left $n \times n$ block is O . For a function f on \mathbf{H}_n we write $f^c(Z) = \overline{f(-\bar{Z})}$. Let dv denote the invariant volume element on \mathbf{H}_n given by $dv = \det(\text{Im}(Z))^{-n-1} \wedge_{1 \leq j \leq l \leq n} (dx_{jl} \wedge dy_{jl})$. Here for $Z \in \mathbf{H}_n$ we write $Z = (x_{jl}) + \sqrt{-1}(y_{jl})$ with real matrices (x_{jl}) and (y_{jl}) . For two C^∞ -modular forms f and g of weight k and Nebentypus ϕ with respect to $\Gamma_0^{(n)}(M)$, we define the Petersson scalar product $\langle f, g \rangle_{\Gamma_0^{(n)}(M)}$ by

$$\langle f, g \rangle_{\Gamma_0^{(n)}(M)} = \int_{\Gamma_0^{(n)}(M) \backslash \mathbf{H}_n} f(Z) \overline{g(Z)} \det(\text{Im}(Z))^k dv,$$

provided the integral converges. Furthermore, we define the normalized Petersson scalar product

$$\langle f, g \rangle = m(\Phi_{\Gamma_0^{(n)}(M)})^{-1} \langle f, g \rangle_{\Gamma_0^{(n)}(M)},$$

where $\Phi_{\Gamma_0^{(n)}(M)}$ is the fundamental domain for \mathbf{H}_n modulo $\Gamma_0^{(n)}(M)$, and $m(\Phi_{\Gamma_0^{(n)}(M)}) = \int_{\Gamma_0^{(n)}(M) \backslash \mathbf{H}_n} dv$. For a Dirichlet character ψ we denote by $L(s, \psi)$ the Dirichlet L -function associated to ψ . Let n, l and M be positive integers. For a Dirichlet character ϕ modulo M such that $\phi(-1) = (-1)^l$, we define the Eisenstein series $E'_{n,l}(Z; M, \phi, s)$ by

$$\begin{aligned}
 E'_{n,l}(Z; M, \phi, s) &= \det \text{Im}(Z)^s L(l + 2s, \phi) \prod_{i=1}^{[n/2]} L(2l + 4s - 2i, \phi^2) \\
 &\times \sum_{\gamma \in \Gamma_0^{\prime(n)}(M) \backslash \Gamma_0^{\prime(n)}(M)} \tilde{\phi}'(\gamma) j(\gamma, Z)^{-l} |j(\gamma, Z)|^{-2s}.
 \end{aligned}$$

We then define $E_{n,l}^*(Z; M, \phi, s)$ by

$$E_{n,l}^*(Z; M, \phi, s) = j(\iota, Z)^{-l} E'_{n,l}(\iota(Z); M, \phi, s),$$

where $\iota = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$. Let $\mathcal{H}_n(\mathbf{Z})$ denote the set of half-integral matrices of degree n over \mathbf{Z} , and we denote by $\mathcal{H}_n(\mathbf{Z})_{>0}$ (resp. $\mathcal{H}_n(\mathbf{Z})_{\geq 0}$) the subset of $\mathcal{H}_n(\mathbf{Z})$ consisting of positive definite (resp. semi-positive definite) matrices. Then it is well known that $E_{n,l}^*(Z; M, \phi, s)$ belongs to $M_l^\infty(\Gamma_0^{(n)}(M), \phi)$, and has the following Fourier expansion:

$$E_{n,l}^*(X + \sqrt{-1}Y; M, \phi, s) = \sum_{A \in \mathcal{H}_n(\mathbf{Z})} c_{n,l}(A, Y, M, \phi, s) \mathbf{e}(\text{tr}(AX)).$$

In particular, if $E_{n,l}^*(Z; M, \phi, s)$ belongs to $M_l(\Gamma_0^{(n)}(M), \phi)$, it has the following Fourier expansion:

$$E_{n,l}^*(Z; M, \phi, s) = \sum_{A \in \mathcal{H}_n(\mathbf{Z})_{\geq 0}} c_{n,l}(A, M, \phi, s) \mathbf{e}(\text{tr}(AZ)).$$

Throughout the rest of this paper, we exclusively consider the case $n = 2$. Let l be an even positive integer. Let $M > 1$ be an integer, and let ϕ be a Dirichlet character modulo M such that $\phi(-1) = 1$. Then $E_{2,l}^*(Z; M, \phi, 0)$ belongs to $M_l(\Gamma_0^{(2)}(M), \phi)$ in case $l \geq 4$. Furthermore $E_{2,2}^*(Z; M, \phi, 0)$ belongs to $M_2(\Gamma_0^{(2)}(M), \phi)$ if $\phi^2 \neq \mathbf{1}_M$. We remark that $E_{2,2}^*(Z; M, \phi, 0)$ is neither holomorphic nor nearly holomorphic in the sense of [Shimura, 2000] if $\phi^2 = \mathbf{1}_M$. However, $E_{2,l}^*(Z; M, \phi, -1/2)$ belongs to $M_2(\Gamma_0^{(2)}(M), \phi)$ in this case.

Now to see the Fourier coefficient of the Eisenstein series, for an element $A = \begin{pmatrix} a_{11} & a_{12}/2 \\ a_{12}/2 & a_{22} \end{pmatrix} \in \mathcal{H}_2(\mathbf{Z})$, put $e = e_A = \text{GCD}(a_{11}, a_{12}, a_{22})$. For an element $A \in \mathcal{H}_2(\mathbf{Z})$ such that $\text{rank } A = 1$ and for each prime number p define a polynomial $F_p(A, X)$ as

$$F_p(A, X) = \sum_{i=0}^{\text{ord}_p(\epsilon_A)} (pX)^i,$$

where ord_p denotes the normalized additive valuation on the field of p -adic numbers. For an element $A = \begin{pmatrix} a_{11} & a_{12}/2 \\ a_{12}/2 & a_{22} \end{pmatrix} \in \mathcal{H}_2(\mathbf{Z})_{>0}$ write $-4\det A = \mathfrak{d}_A \mathfrak{f}_A^2$ with \mathfrak{d}_A the fundamental discriminant of $\mathbf{Q}(\sqrt{-\det A})$ and \mathfrak{f}_A a positive integer. Furthermore, let $\chi_A = \left(\frac{\mathfrak{d}_A}{*}\right)$ be the Kronecker character corresponding to $\mathbf{Q}(\sqrt{-\det A})/\mathbf{Q}$. For a prime number p define a polynomial $F_p(A, X)$ as

$$F_p(A, X)$$

$$= \sum_{i=0}^{\text{ord}_p(e_A)} (p^2 X)^i \sum_{j=0}^{\text{ord}_p(f_A)-i} (p^3 X^2)^j - \chi_A(p) p X \sum_{i=0}^{\text{ord}_p(e_A)} (p^2 X)^i \sum_{j=0}^{\text{ord}_p(f_A)-i-1} (p^3 X^2)^j.$$

For a Dirichlet character ψ , let m_ψ denote its conductor, and $\psi^{(0)}$ the associated primitive character. Furthermore, let $B_{m,\psi}$ be the m -th generalized Bernoulli number associated with ψ , and let $\tau(\psi)$ be the Gauss sum defined by

$$\tau(\psi) = \sum_{X \bmod L} \psi(X) \mathbf{e}(X/L).$$

Let l be an even positive integer, and $s = 0$ or $-1/2$. Let ϕ be a Dirichlet character such that $\phi(-1) = 1$. Now assume that the triple (l, s, ϕ) satisfies one of the following conditions (h-1), (h-2), (h-3):

- (h-1) $l \geq 4$ and $s = 0$,
- (h-2) $l = 2$, $s = 0$ and ϕ^2 is not trivial,
- (h-3) $l = 2$ and $s = -1/2$.

Theorem 2.1 (cf. [Katsurada, 1999], [Shimura, 2000]) *Let $M > 1$ be an integer, and ϕ a Dirichlet character modulo M such that $\phi(-1) = 1$. Let l be an even positive integer, and $s = 0$ or $-1/2$. Assume that the triple (l, s, ϕ) satisfies one of the conditions (h-1), (h-2), (h-3). First assume that (l, s, ϕ) satisfies either the condition (h-1) or (h-2). Then for $A \in \mathcal{H}_2(\mathbf{Z})_{\geq 0}$ put*

$$\begin{aligned} \tilde{c}_{2,l}(A, 0) &= \tilde{c}_{2,l}(A; M, \phi, 0) \\ &= \begin{cases} (4 \det A)^{l-3/2} \prod_{p|f_A} F_p(A, \phi(p)p^{-l}) B_{l-1, (\phi\chi_A)^{(0)}} \overline{\tau((\phi\chi_A)^{(0)})} (-\sqrt{-1}) m_{(\phi\chi_A)^{(0)}}^{1-l} \prod_{p|M} (1 - (\phi\chi_A)^{(0)}(p)p^{1-l}) & A > 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Next assume that (l, s, ϕ) satisfies the condition (h-3). Then for $A \in \mathcal{H}_2(\mathbf{Z})_{\geq 0}$ put

$$\begin{aligned} \tilde{c}_{2,2}(A, -1/2) &= \tilde{c}_{2,2}(A; M, \phi, -1/2) \\ &= \begin{cases} -\prod_{p|f_A} F_p(A, \phi(p)p^{-1}) B_{1, (\phi\chi_A)^{(0)}} \prod_{p|M} (1 - (\phi\chi_A)^{(0)}(p)) & A > 0 \\ -1/2 \prod_{p|e_A} F_p(A, \phi(p)p^{-1}) \prod_{p|M} (1 - (\phi^2)^{(0)}(p)p) B_{2, (\phi^2)^{(0)}} & \text{rank } A = 1 \\ 1/8 \prod_{p|M} \{(1 - (\phi^2)^{(0)}(p)p)(1 - (\phi)^{(0)}(p)p)\} B_{2, (\phi^2)^{(0)}} B_{2, (\phi)^{(0)}} & A = O. \end{cases} \end{aligned}$$

Let

$$A(l, s) = \frac{8\pi^{5/2}}{\Gamma(3/2)} \text{ or } \frac{(-1)^{l/2} 2^l \pi^{3l-3/2}}{\Gamma(l)^2 \Gamma(l-1/2)}$$

according as $l = 2$ and $s = -1/2$, or $l \geq 4$ and $s = 0$, where $\Gamma(*)$ is Gamma function. Then we have

$$c_{2,l}(A; M, \phi, s) = A(l, s) \tilde{c}_{2,l}(A; M, \phi, s).$$

Remark 2.1 Assume that (l, s, ϕ) satisfies the condition (h-1) or (h-2). Let m be the conductor of ϕ , and write $f_A = f'_A \tilde{f}_A$ with $f'_A = \prod_{p|m} p^{\text{ord}_p(f_A)}$ and $(\tilde{f}_A, m) = 1$. Then by the functional equation of $F_p(A, X)$ (cf. [Katsurada, 1999]), we can rewrite $\tilde{c}_{2,l}(A, 0)$ as

$$\begin{aligned} \tilde{c}_{2,l}(A, 0) &= (|\mathfrak{d}_A| f'_A)^{l-3/2} \phi(\tilde{f}_A)^2 \prod_{p|\tilde{f}_A} F_p(A, \bar{\phi}(p) p^{l-3}) \\ &\times B_{l-1, \overline{(\phi\chi_A)^{(0)}}}(-\sqrt{-1}) \tau((\phi\chi_A)^{(0)}) m_{(\phi\chi_A)^{(0)}}^{1-l} \prod_{p|M} (1 - (\phi\chi_A)^{(0)}(p)). \end{aligned} \quad (2.1)$$

Let ϕ be a Dirichlet character modulo M with conductor m such that $\phi(-1) = 1$. Put $m' = M/m$. If $|\mathfrak{d}_A|$ is prime to m , we have

$$\begin{aligned} \tau((\phi\chi_A)^{(0)}) &= \sqrt{-1} \phi(\mathfrak{d}_A) \chi_A(m) \tau(\phi) |\mathfrak{d}_A|^{1/2}, \\ m_{(\phi\chi_A)^{(0)}} &= m |\mathfrak{d}_A|, \end{aligned}$$

and

$$\prod_{p|M} (1 - (\phi\chi_A)^{(0)}(p) p^{1-l}) = \prod_{p|m'} (1 - \chi_A(p) p^{1-l}).$$

Thus if $4 \det A$ is prime to M , we have

$$\begin{aligned} \tilde{c}_{2,l}(A, 0) &= \phi(-4 \det A) \prod_{p|\tilde{f}_A} F_p(A, \bar{\phi}(p) p^{l-3}) B_{l-1, \overline{(\phi\chi_A)^{(0)}}} \chi_A(m) \tau(\phi) m^{1-l} \\ &\times \prod_{p|m'} (1 - (\phi\chi_A)^{(0)}(p) p^{1-l}). \end{aligned} \quad (2.2)$$

In particular if M is a squarefree odd positive integer dividing $m_1 m_2$ and r is an integer prime to $m_1 m_2$, we have

$$\begin{aligned} \tilde{c}_{2,l}(A, 0) &= \phi(r)^2 \prod_{p|\tilde{f}_A} F_p(A, \bar{\phi}(p) p^{l-3}) B_{l-1, \overline{(\phi\chi_A)^{(0)}}} \\ &\times \tau(\phi) m^{1-l} \prod_{p|m'} (1 - (\phi\chi_A)^{(0)}(p) p^{1-l}) \end{aligned} \quad (2.3)$$

for $A = \begin{pmatrix} m_1 & r/2 \\ r/2 & m_2 \end{pmatrix}$. On the other hand, if $\phi^2 = \mathbf{1}_M$, we have

$$\begin{aligned} \tilde{c}_{2,l}(A, 0) &= \prod_{p|\tilde{f}_A} F_p(A, \phi(p) p^{l-3}) (|\mathfrak{d}_A| f'_A)^{l-3/2} B_{l-1, \overline{(\phi\chi_A)^{(0)}}} \\ &\times m_{(\phi\chi_A)^{(0)}}^{3/2-l} \prod_{p|M} (1 - (\phi\chi_A)^{(0)}(p) p^{1-l}). \end{aligned} \quad (2.4)$$

Thus if $\phi^2 = \mathbf{1}_M$ and $|\mathfrak{d}_A|$ is prime to m , we have

$$\tilde{c}_{2,l}(A, 0) = \prod_{p|\mathfrak{f}_A} F_p(A, \phi(p)p^{l-3}) B_{l-1, \overline{(\phi\chi_A)^{(0)}}} \times m^{3/2-l} \prod_{p|m'} (1 - (\phi\chi_A)^{(0)}(p)p^{1-l}) \quad (2.5)$$

(2) Assume that M is a squarefree odd positive integer and that $|\mathfrak{d}_A|$ is prime to m . If ϕ is primitive,

$$\prod_{p|M} (1 - (\phi\chi_A)^{(0)}(p)p^{1-l}) = 1.$$

On the other hand, let $\phi = \mathbf{1}_M$. Let $A = \begin{pmatrix} m_1 & r/2 \\ r/2 & m_2 \end{pmatrix} \in \mathcal{H}_2(\mathbf{Z})_{>0}$ with $m_1 m_2$ divided by M and r prime to $m_1 m_2$. Then

$$\tilde{c}_{2,2}(A; M, \mathbf{1}_M, -1/2) = 0. \quad (2.6)$$

Furthermore, for any positive integer $l \geq 2$, we have

$$\tilde{c}_{2,l}(A; M, \mathbf{1}_M, 0) = - \prod_{p|\mathfrak{f}_A} F_p(A, p^{l-3}) B_{1, \chi_A^{(0)}} \prod_{p|M} (1 - p^{1-l}) \quad (2.7)$$

provided (l, s, ϕ) satisfies the condition (h-1) or (h-2). On the other hand, if $\phi^2 = \mathbf{1}_M$ but $\phi \neq \mathbf{1}_M$, we have

$$\tilde{c}_{2,2}(A; M, \phi, -1/2) = \begin{cases} -1/12 \prod_{p|e_A} F_p(A, \phi(p)p^{-1}) \prod_{p|M} (1-p) & \text{rank } A = 1 \\ 1/48 \prod_{p|M} \{(1-p)(1 - (\phi)^{(0)}(p)p)\} B_{2, (\phi)^{(0)}} & A = O. \end{cases} \quad (2.8)$$

3 Pullback formula

Now we define Boecherer's differential operator. For the detail, see [Böcherer and Schmidt, 2000]. First we define the differential operator \mathcal{D}_α on the module $C^\infty(\mathbf{H}_2)$ of C^∞ -functions on \mathbf{H}_2 by

$$\mathcal{D}_\alpha(f) = -(\alpha - 1/2)\partial f / \partial z_{12} + z_{12}(\partial^2 f / \partial z_{11} \partial z_{22} - 1/4 \partial^2 f / \partial z_{12}^2)$$

for $f \in C^\infty(\mathbf{H}_2)$ and $Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix} \in \mathbf{H}_2$. For a non-negative integer ν define the differential operator \mathcal{D}_α^ν and $\tilde{\mathcal{D}}_\alpha^\nu$ by

$$\mathcal{D}_\alpha^\nu = \mathcal{D}_{\alpha+\nu-1} \dots \mathcal{D}_\alpha,$$

and

$$\tilde{\mathcal{D}}_\alpha^\nu = \mathcal{D}_\alpha^\nu|_{z_{12}=0}.$$

Furthermore, for $s \in \mathbf{C}$ we define $\mathcal{D}_{\alpha,s}^\nu$ and $f \in C^\infty(\mathbf{H}_2)$ by

$$\tilde{\mathcal{D}}_{\alpha,s}^\nu(f)(z_{11}, z_{22}) = (y_{11}y_{22})^s \tilde{\mathcal{D}}_{\alpha+s}^\nu(\det Y^{-s} f(Z)),$$

where $Z = X + \sqrt{-1}Y \in \mathbf{H}_2$ and $Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{12} & y_{22} \end{pmatrix}$. It is well known that $\tilde{\mathcal{D}}_l^\nu$ and $\tilde{\mathcal{D}}_{l,s}^\nu$ map $M_l^\infty(\Gamma_0^{(2)}(M), \phi)$ into $M_{l+\nu}^\infty(\Gamma_0^{(1)}(M), \phi) \otimes M_{l+\nu}^\infty(\Gamma_0^{(1)}(M), \phi)$. Furthermore $\tilde{\mathcal{D}}_l^\nu$ maps $M_l(\Gamma_0^{(2)}(M), \phi)$ into $M_{l+\nu}(\Gamma_0^{(1)}(M), \phi) \otimes M_{l+\nu}(\Gamma_0^{(1)}(M), \phi)$, and in particular if $\nu > 0$, its image is contained in $S_{l+\nu}(\Gamma_0^{(1)}(M), \phi) \otimes S_{l+\nu}(\Gamma_0^{(1)}(M), \phi)$. Clearly these two operators $\tilde{\mathcal{D}}_l^\nu$ and $\tilde{\mathcal{D}}_{l,s}^\nu$ coincides with each other if $s = 0$. Furthermore, for $F(z_1, z_2) \in M_l^\infty(\Gamma_0^{(2)}(M), \phi)$ and $g(z_1) \in S_{l+\nu}(\Gamma_0^{(1)}(M), \phi)$ we have the following identity as functions of z_2 :

$$\langle \tilde{\mathcal{D}}_l^\nu F(*, z_2), g \rangle = d_{l,\nu,s} \langle \tilde{\mathcal{D}}_{l,s}^\nu F(*, z_2), g \rangle \quad (3.1)$$

Here we take the inner product as functions of z_1 , and

$$d_{l,\nu,s} = \prod_{\mu=1}^{\nu} \frac{l-1+\nu-\mu/2}{l+s-1+\nu-\mu/2}.$$

In addition to the above notation, let $N \geq 1$ be a positive integer, and χ a Dirichlet character modulo N . Assume that N^2 divides M . For positive even integers l, k such that $l \leq k$ we define a function $\mathfrak{E}(z_1, z_2) = \mathfrak{E}_{2,k}(z_1, z_2; l, M, \phi, \chi, s)$ on $\mathbf{H}_1 \times \mathbf{H}_1$ by

$$\mathfrak{E}_{2,k}(z_1, z_2; l, M, \phi, \chi, s) = \tilde{\mathcal{D}}_{l,s}^{k-l} \left(\sum_{x \in \mathbf{Z}/N\mathbf{Z}} \bar{\chi}(x) E_{2,l}^*(*; M, \phi \bar{\chi}, s)|_k R(x/N) \right) \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix},$$

where $R(x) = \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & x & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Now to see an explicit form of $\tilde{\mathcal{D}}_l^\nu$, for an even

positive integer l and non-negative integer ν we define a polynomial $G_l^{2\nu}(u, v)$ in u, v by

$$G_l^{2\nu}(u, v) = \sum_{\mu=0}^{\nu} (-1)^\mu \frac{(l+2\nu-\mu-2)!}{(2\nu-2\mu)! \mu!} u^\mu v^{2\nu-2\mu}.$$

This polynomial was introduced by Zagier [Zagier, 1977].

We define Ibukiyama's differential operator $\mathcal{G}_l^{2\nu}$ on $C^\infty(\mathbf{H}_2)$ by

$$\mathcal{G}_l^{2\nu} = G_l^{2\nu}(\partial^2/\partial z_{11}\partial z_{22}, \partial/\partial z_{12})|_{z_{12}=0}.$$

We note that

$$\mathcal{G}_l^{2\nu}(\mathbf{e}(\text{tr}(AZ))) = G_l^{2\nu}(a_{11}a_{22}, a_{12})(2\pi\sqrt{-1})^{2\nu} \mathbf{e}(a_{11}z_{11} + a_{22}z_{22}) \quad (3.2)$$

for $A = \begin{pmatrix} a_{11} & a_{12}/2 \\ a_{12}/2 & a_{22} \end{pmatrix}$ and $Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix}$. It is well known that $\mathcal{G}_l^{2\nu}$ is a constant multiple of $\tilde{\mathcal{D}}_l^{2\nu}$ (cf. [Ibukiyama, 1999]), and more precisely, by comparing $\mathcal{G}_l^{2\nu}(z_{12}^{2\nu})$ and $\tilde{\mathcal{D}}_l^{2\nu}(z_{12}^{2\nu})$ for $Z = \begin{pmatrix} z_1 & z_{12} \\ z_{12} & z_2 \end{pmatrix} \in \mathbf{H}_2$, we have

$$\mathcal{G}_l^{2\nu} = \frac{(l + 2\nu - 2)!}{\prod_{\mu=1}^{2\nu} (\mu/2)(l - 1 + 2\nu - \mu/2)} \tilde{\mathcal{D}}_l^{2\nu}. \quad (3.3)$$

By (3.3) we have for $F(Z) \in M_l^\infty(\Gamma_0^{(2)}(M), \phi)$ and $g(z_1) \in S_{l+\nu}(\Gamma_0^{(1)}(M), \phi)$ we have

$$\langle \mathcal{G}_l^{2\nu} F(*, z_2), g \rangle = e_{l,2\nu,s} \langle \tilde{\mathcal{D}}_{l,s}^\nu F(*, z_2), g \rangle, \quad (3.4)$$

where

$$e_{l,2\nu,s} = \frac{(l + 2\nu - 2)!}{\prod_{\mu=1}^{2\nu} (\mu/2)(l - 1 + 2\nu - s - \mu/2)}.$$

Now put

$$\begin{aligned} & \mathcal{E}_{2,k}(z_1, z_2; l, M, \phi, \chi, s) \\ &= (2\pi\sqrt{-1})^{l-k} \mathcal{G}_l^{k-l} \left(\sum_{x \in \mathbf{Z}/N\mathbf{Z}} \bar{\chi}(x) E_{2,l}^*(*; M, \phi\bar{\chi}, s) |_{k,R}(x/N) \right) \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}. \end{aligned}$$

Then $\mathcal{E}_{2,k}(z_1, z_2; l, M, \phi, \chi, s)$ belongs to $M_k(\Gamma_0(M), \phi) \otimes M_k(\Gamma_0(M), \phi)$. Furthermore, as to the cuspidality of $\mathcal{E}_{2,k}(z_1, z_2; l, M, \phi, \chi, s)$, by a careful examination of the behavior at cusps, we have :

Proposition 3.1 *Let k and l be positive even integers such that $k \leq l$, and $s = 0$ or $-1/2$. Let ϕ and χ be Dirichlet characters modulo M and N , respectively satisfying the above conditions. Assume that the triple $(l, s, \phi\bar{\chi})$ satisfies one of the conditions (h-1), (h-2), (h-3) in Section 2. Furthermore assume $k > l$, or $N > 1$. Then $\tilde{\mathcal{E}}_{2,k}(z_1, z_2; l, M, \phi, \chi, s)$ belongs to $S_k(\Gamma_0(M), \phi) \otimes S_k(\Gamma_0(M), \phi)$*

Remark 3.1 We remark that in case $N > 1$, $\tilde{\mathcal{E}}_{2,k}(z_1, z_2; k, M, \phi, \chi, s)$ belongs to $S_k(\Gamma_0(M), \phi) \otimes S_k(\Gamma_0(M), \phi)$ even if $\chi = \mathbf{1}_N$.

Now by (3.4) we have

$$\mathcal{E}_{2,k}(z_1, z_2; l, M, \phi, \chi, s) = (2\pi\sqrt{-1})^{l-k} e_{l,k-l,s} \tilde{\mathcal{E}}_{2,k}(z_1, z_2; l, M, \phi, \chi, s).$$

Furthermore, by (3.2) and [Böcherer and Schmidt, 2000, (6.11)], we have

$$\begin{aligned} & \mathcal{E}_{2,k}(z_1, z_2; l, M, \phi, \chi, s) \\ = & \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{r^2 \leq 4m_1m_2} c_{2,l} \left(\begin{pmatrix} m_1 & r/2 \\ r/2 & m_2 \end{pmatrix}, \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}, M, \phi\bar{\chi}, s \right) G_l^{k-l}(m_1m_2, r) \\ & \times T(r, \bar{\chi}) \mathbf{e}(m_1x_1) \mathbf{e}(m_2x_2), \end{aligned}$$

where we write $z_1 = x_1 + \sqrt{-1}y_1, z_2 = x_1 + \sqrt{-1}y_2$, and

$$T(r, \phi) = \sum_{x \bmod N} \phi(x) \mathbf{e}(rx/N)$$

for a Dirichlet character ϕ modulo N .

From now on put $\Gamma_0(N) = \Gamma_0^{(1)}(N)$. Let M and k be positive integers, and ϕ a Dirichlet character modulo M such that $\phi(-1) = (-1)^k$. Let

$$f(z) = \sum_{m=1}^{\infty} a(m) \mathbf{e}(mz)$$

be a normalized cuspidal Hecke eigenform of weight k and Nebentypus ϕ with respect to $\Gamma_0(M)$. Then, for a Dirichlet character χ modulo N , we define the standard zeta function $L(f, s, \chi)$ twisted by χ as

$$L(f, s, \chi) = \prod_p \{(1 - \chi(p)\alpha_p\beta_p p^{-s-k+1})(1 - \chi(p)\alpha_p^2 p^{-s-k+1})(1 - \chi(p)\beta_p^2 p^{-s-k+1})\}^{-1},$$

where α_p, β_p are complex numbers such that

$$\alpha_p + \beta_p = a(p), \alpha_p\beta_p = \phi(p)p^{k-1} \quad (3.5)$$

for each prime number p . Then by [Böcherer and Schmidt, 2000, Theorem 3.1], we have

Theorem 3.2 *In addition to the notation and the assumption as above, assume that $M > 1$, N^2 divides M , $\phi^2 = \mathbf{1}_M$ and $\chi(-1) = 1$. Let $f \in S_k(\Gamma_0(M), \phi)$ be a common eigenfunction of all Hecke operators. Then we have*

$$\begin{aligned} & \langle f, \mathcal{E}_{2,k}(*, -\bar{z}; l, M, \phi, \chi, \bar{s}) \rangle_{\Gamma_0(M)} \\ & = \kappa_{l,k}(s) N^{k+l+2s-2} M^{1-k/2} L(f|W_M, l+2s-1, \chi) f|W_M |T(M/N^2)(z), \end{aligned}$$

where

$$\kappa_{l,k}(s) = \frac{(-1)^{l/2}}{2^{-3+2k-l+2s} \pi^{k-l-1}} \frac{\Gamma(k+s-1/2)\Gamma(k+s-1)}{\Gamma(l+s)\Gamma(l+s-1/2)} \frac{\Gamma(k-1)}{\prod_{\mu=1}^{k-l} (\mu/2)(k-1-s-\mu/2)},$$

$T(M/N^2)$ is the Hecke operator, and $W_M = \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix}$.

Remark 3.2 We slightly change the notation in [Böcherer and Schmidt, 2000]. There is a minor misprint in [Böcherer and Schmidt, 2000, Theorem 3.1]. In page 1339, line 9, " $2^{1+n(n+1)/2-2ns}$ " should read " $2^{1-nl+n(n+3)/2-2ns}$," and this correction has been done in [Theorem 3.1, Katsurada, 2003].

Now let ϕ be as in Theorem 3.2, and assume that the triple $(l, s, \phi\bar{\chi})$ satisfies one of the conditions (h-1),(h-2),(h-3) in Section 2. Then we define a function $\tilde{\mathcal{E}}_{2,k}(z_1, z_2; l, M, \phi, \chi, s)$ on $\mathbf{H}_1 \times \mathbf{H}_1$ by

$$\begin{aligned} & \tilde{\mathcal{E}}_{2,k}(z_1, z_2; l, M, \phi, \chi, s) \\ &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{r^2 \leq 4m_1m_2} \tilde{c}_{2,l} \left(\begin{pmatrix} m_1 & r/2 \\ r/2 & m_2 \end{pmatrix}, M, \phi\bar{\chi}, s \right) G_l^{k-l}(m_1m_2, r) T(r, \bar{\chi}) \mathbf{e}(m_1z_1) \mathbf{e}(m_2z_2), \end{aligned}$$

where $\tilde{c}_{2,l} \left(\begin{pmatrix} m_1 & r/2 \\ r/2 & m_2 \end{pmatrix}, M, \phi\bar{\chi}, s \right)$ is the one defined in Theorem 2.1. Then by Theorem 2.1 we have

$$\mathcal{E}_{2,k}(z_1, z_2; l, M, \phi, \chi, s) = A(l, s) \tilde{\mathcal{E}}_{2,k}(z_1, z_2; l, M, \phi, \chi, s).$$

From now on, for a Dirichlet character ψ modulo M_0 we use the same symbol ψ to denote the character modulo M induced from ψ if M_0 divides M . For a positive integer r put $\delta_r = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$, and let $S_k(\Gamma_0(M), \phi)^{(r)} = \{f|\delta_r; f \in S_k(\Gamma_0(M), \phi)\}$, and $S_k(\Gamma_0(M), \phi)^{new}$ the space of new forms in $S_k(\Gamma_0(M), \phi)$. Furthermore for a primitive form f in $S_k(\Gamma_0(M), \phi)$ let c_f be the complex number such that $f|W_M = c_f f^c$. Let $\lambda_f(m)$ be the eigenvalue of the Hecke operator $T(m)$ for a positive integer m . For an odd positive integer $m \leq k-1$, let

$$\Lambda(f, m, \chi) = \frac{\Gamma(k-1)\Gamma(k+m-1)\Gamma(m+1)}{\Gamma(k-m)} \frac{L(f, m, \chi)}{2^{2k+2m-4} \pi^{k+2m} \langle f, f \rangle},$$

and

$$\Lambda(f, 0, \chi) = \Gamma(k-1) \frac{L(f, 0, \chi)}{2^{2k-3} \pi^k \langle f, f \rangle}.$$

We note that $m(\Phi_{\Gamma_0(N)}) = \pi/3[\Gamma : \Gamma_0(N)]$. Thus by Theorem 3.2 we obtain the following two theorems:

Theorem 3.3 *Let p be a prime number such that $p \equiv 1 \pmod{4}$, $\phi = \left(\frac{\cdot}{p}\right)$, and χ a Dirichlet character modulo p such that $\chi(-1) = 1$.*

(1) *Let f be a primitive form in $S_k(\Gamma_0(p^2), \phi)^{new}$. Then*

$$\begin{aligned} & \langle f, \tilde{\mathcal{E}}_{2,k}(*, -\bar{z}; l, p^2, \phi, \chi, s) \rangle \\ &= 3[\Gamma : \Gamma_0(p^2)]^{-1} p^{l+2s} \Lambda(f^c, l+2s-1, \chi) \langle f, f \rangle c_f f^c(z). \end{aligned}$$

(2) Let f be a primitive form in $S_k(\Gamma_0(p), \phi)$. Then we have

$$\begin{aligned} & \langle f, \tilde{\mathcal{E}}_{2,k}(*, -\bar{z}; l, p^2, \phi, \chi, s) \rangle \\ &= 3[\Gamma : \Gamma_0(p^2)]^{-1} p^{l+2s} \Lambda(f^c | \delta_p, l+2s-1, \chi) \langle f | \delta_p, f | \delta_p \rangle c_f f^c | \delta_p(z), \end{aligned}$$

and

$$\begin{aligned} & \langle f | \delta_p, \tilde{\mathcal{E}}_{2,k}(*, -\bar{z}; l, p_0 p^2, \phi, \chi, s) \rangle \\ &= 3[\Gamma : \Gamma_0(p^2)]^{-1} p^{l+2s} \Lambda(f^c, l+2s-1, \chi) \langle f, f \rangle c_f f^c(z). \end{aligned}$$

Theorem 3.4 Let $p_0 = 1$ or a prime number such that $p_0 \equiv 1 \pmod{4}$, and $\phi = (\frac{p_0}{*})$. Furthermore, let p be a prime number different from p_0 , and χ a Dirichlet character modulo p such that $\chi(-1) = 1$.

(1) Let f be a primitive form in $S_k(\Gamma_0(p_0 p^2), \phi)^{new}$. Then

$$\begin{aligned} & \langle f, \tilde{\mathcal{E}}_{2,k}(*, -\bar{z}; l, p_0 p^2, \phi, \chi, s) \rangle \\ &= 3[\Gamma : \Gamma_0(p_0 p^2)]^{-1} p_0^{1-k/2} p^{l+2s} \Lambda(f^c, l+2s-1, \chi) \langle f, f \rangle c_f \overline{\lambda_f(p_0)} f^c(z). \end{aligned}$$

(2) Let f be a primitive form in $S_k(\Gamma_0(p_0 p), \phi)^{new}$. Then we have

$$\begin{aligned} & \langle f, \tilde{\mathcal{E}}_{2,k}(*, -\bar{z}; l, p_0 p^2, \phi, \chi, s) \rangle \\ &= 3[\Gamma : \Gamma_0(p_0 p^2)]^{-1} p_0^{1-k/2} p^{l+2s} \Lambda(f^c | \delta_p, l+2s-1, \chi) \langle f | \delta_p, f | \delta_p \rangle c_f \overline{\lambda_f(p_0)} f^c | \delta_p(z), \end{aligned}$$

and

$$\begin{aligned} & \langle f | \delta_p, \tilde{\mathcal{E}}_{2,k}(*, -\bar{z}; l, p_0 p^2, \phi, \chi, s) \rangle \\ &= 3[\Gamma : \Gamma_0(p_0 p^2)]^{-1} p_0^{1-k/2} p^{l+2s} \Lambda(f^c, l+2s-1, \chi) \langle f, f \rangle c_f \overline{\lambda_f(p_0)} f^c(z). \end{aligned}$$

(3) Let f be a primitive form in $S_k(\Gamma_0(p_0), \phi)$. Then we have

$$\begin{aligned} & \langle f, \tilde{\mathcal{E}}_{2,k}(*, -\bar{z}; l, p_0 p^2, \phi, \chi, s) \rangle \\ &= 3[\Gamma : \Gamma_0(p_0 p^2)]^{-1} p_0^{1-k/2} p^{l+2s} \Lambda(f^c | \delta_{p^2}, l+2s-1, \chi) \langle f | \delta_{p^2}, f | \delta_{p^2} \rangle c_f \overline{\lambda_f(p_0)} f^c | \delta_{p^2}(z), \\ & \quad \langle f | \delta_p, \tilde{\mathcal{E}}_{2,k}(*, -\bar{z}; l, p_0 p^2, \phi, \chi, s) \rangle \\ &= 3[\Gamma : \Gamma_0(p_0 p^2)]^{-1} p_0^{1-k/2} p^{l+2s} \Lambda(f^c | \delta_p, l+2s-1, \chi) \langle f | \delta_p, f | \delta_p \rangle c_f \overline{\lambda_f(p_0)} f^c | \delta_p(z), \end{aligned}$$

and

$$\begin{aligned} & \langle f | \delta_{p^2}, \tilde{\mathcal{E}}_{2,k}(*, -\bar{z}; l, p_0 p^2, \phi, \chi, s) \rangle \\ &= 3[\Gamma : \Gamma_0(p_0 p^2)]^{-1} p_0^{1-k/2} p^{l+2s} \Lambda(f^c, l+2s-1, \chi) \langle f, f \rangle c_f \overline{\lambda_f(p_0)} f^c(z). \end{aligned}$$

4 Computation of $L(f, l, \chi)$

In this section we give some formulas to compute $L(f, m, \chi)$ for a primitive form $f \in S_k(\Gamma_0(N), \psi)$ in the following two cases:

(1) N is a prime number p such that $p \equiv 1 \pmod{4}$, $\psi = \left(\frac{p}{*}\right)$, and χ is a Dirichlet character modulo p .

(2) N is 1 or a prime number p_0 such that $p_0 \equiv 1 \pmod{4}$, $\psi = \left(\frac{p_0}{*}\right)$, and χ is a Dirichlet character modulo p , where p is a prime number different from p_0 .

In any case, χ is a primitive character modulo p , or $\mathbf{1}_p$. Put $M = p^2$ or p_0p^2 according as the case (1) or (2). Let l and k be even integers such that $2 \leq l \leq k$. Assume that the triple $(l, s, \psi\bar{\chi})$ satisfies one of the conditions (h-1),(h-2),(h-3) in Section 2. For two positive integers m_1, m_2 put

$$\begin{aligned} \epsilon(m_1, m_2; l, s) &= \epsilon(m_1, m_2; l, M, \psi, \chi, s) \\ &= \sum_{r^2 \leq 4m_1m_2} \tilde{c}_{2,l} \left(\begin{pmatrix} m_1 & r/2 \\ r/2 & m_2 \end{pmatrix}, M, \psi\bar{\chi}, s \right) G_l^{k-l}(m_1m_2, r) T(r, \bar{\chi}). \end{aligned}$$

We note that $T(r, \bar{\chi}) = p - 1$ or $\chi(r)\tau(\bar{\chi})$ if $\chi = \mathbf{1}_p$ and $r \equiv 0 \pmod{p}$, or not. Furthermore, for each positive integer m_1 put

$$\mathcal{F}_{m_1}(z_2) = \sum_{m_2=1}^{\infty} \epsilon(m_1, m_2; l, s) \mathbf{e}(m_2 z_2),$$

and for a prime number p put

$$\mathcal{F}_{m_1, p}(z_2) = \sum_{m_2=1}^{\infty} (\epsilon(m_1, p^2 m_2; l, s) - \psi(p)p^{k-2} \epsilon(m_1, m_2; l, s) \mathbf{e}(m_2 z_2)).$$

We note that

$$\tilde{\mathcal{E}}_{2,k}(z_1, z_2; l, M, \psi, \chi, s) = \sum_{m_1=1}^{\infty} \mathcal{F}_{m_1}(z_2) \mathbf{e}(m_1 z_1).$$

Take a basis $\{f_i\}_{i=1}^{d_1}$ of $S_k(\Gamma_0(N), \psi)$ consisting of primitive forms. Let $f_i|W_N = c_i f_i^c$ with constant c_i , and write

$$f_i(z) = \sum_{m=1}^{\infty} a_i(m) \mathbf{e}(mz)$$

with $a_i(1) = 1$.

First we have

Lemma 4.1 *Let N be a positive integer and ψ a Dirichlet character modulo N .*

(1) *Let f and g be Hecke eigenforms in $S_k(\Gamma_0(N), \psi)$, and let p be a prime number. Then we have*

$$\langle f | \delta_p, g \rangle = p^{-k/2} \overline{\lambda_g(p)} \langle f, g \rangle - \alpha(N, p) \bar{\psi}(p) p^{-1} \overline{\langle g | \delta_p, f \rangle},$$

where $\lambda_g(p)$ denotes the eigenvalue of the Hecke operator $T(p)$ with respect to g , and $\alpha(N, p)$ is 0 or 1 according as p divides N , or not.

(2) *Let $g \in S_k(\Gamma_0(N), \psi)$ be a Hecke eigenform. Let p be a prime number dividing N . Then we have*

$$\langle g | \delta_p, g \rangle = p^{-k/2} \overline{\lambda_g(p)} \langle g, g \rangle.$$

(3) *Let $f \in S_k(\Gamma_0(N), \psi)$ be a primitive form. Let p be a prime number not dividing N . Then we have*

$$\langle f | \delta_p, f \rangle = \frac{p^{-k/2} \bar{\psi}(p) \lambda_f(p)}{1 + p^{-1}} \langle f, f \rangle,$$

and

$$\langle f | \delta_{p^2}, f \rangle = \frac{p^{-k} \bar{\psi}(p)^2 \lambda_f(p)^2 - \bar{\psi}(p) p^{-1} (1 + p^{-1})}{1 + p^{-1}} \langle f, f \rangle.$$

Proof The assertions follows immediately from [Shimura, 1976, (2.5)], and [Shimura, 1976, Lemma 1]. \square

Now we compute the value $\Lambda(f, l, \chi)$.

Theorem 4.2 *Let the notation and the assumption be as above.*

(1) *In case (1), for any even positive integer $l \leq k$, we have*

$$\mathcal{F}_p(z_2) = 3p^{k/2+l+2s-1} (p+1)^{-1} \sum_{i=1}^{d_1} \Lambda(f_i, l+2s-1, \chi) \bar{c}_i f_i(z_2).$$

(2) *Put $t_{p_0} = p_0 + 1$ or 1 according as p_0 is a prime number or 1. Then, in case (2), for any even positive integer $l \leq k$, we have*

$$\mathcal{F}_{p,p}(z_2) = 3t_{p_0}^{-1} p_0^{1/2} p^{k+l+2s-2} \sum_{i=1}^{d_1} \bar{c}_i^2 \tilde{f}_i(z_2),$$

where we write $\tilde{f}(z) = \sum_{m=1}^{\infty} a(pm) \mathbf{e}(mz)$ for a modular form $f(z) = \sum_{m=1}^{\infty} a(m) \mathbf{e}(mz)$.

Proof (1) Put $\tilde{\mathcal{E}}(z_1, z_2) = \tilde{\mathcal{E}}_{2,k}(z_1, z_2; l, p^2, \psi, \chi, s)$. Then by Proposition 3.1, $\tilde{\mathcal{E}}(z_1, z_2)$ belongs to $S_k(\Gamma_0(p^2), \psi) \otimes S_k(\Gamma_0(p^2), \psi)$. As is well known,

$$S_k(\Gamma_0(p^2), \psi) = S_k(\Gamma_0(p), \psi) \oplus S_k(\Gamma_0(p), \psi)^{(p)} \perp S_k(\Gamma_0(p^2), \psi)^{new}.$$

Put $d_1 = \dim S_k(\Gamma_0(p), \psi)$ and $d_2 = \dim S_k(\Gamma_0(p^2), \psi)^{new}$. Take a basis $\{g_i\}_{i=1}^{d_2}$ of $S_k(\Gamma_0(p^2), \psi)^{new}$ which are common eigen-function of Hecke operators. Then $\{f_i \ (i = 1, 2, \dots, d_1), f_i|_{\delta_p} \ (i = 1, 2, \dots, d_1), g_i \ (i = 1, 2, \dots, d_2)\}$ forms a basis of $S_k(\Gamma_0(p^2), \psi)$. Let c_i be as above. Then we have $f_i|W_p = c_i f_i^c, f_i|W_{p^2} = c_i f_i^c|_{\delta_p}$. Furthermore we have $g_i|W_{p^2} = c'_i g_i^c$ with constant c'_i . From this we have $f_i|_{\delta_p}|W_{p^2} = c_i f_i^c$. We note that $\langle g_i, g_j \rangle = 0$ for any $1 \leq i \neq j \leq d_2$, and $\langle f_i, g_j \rangle = \langle g_j, f_i \rangle = 0$ for any $1 \leq i \leq d_1$ and $1 \leq j \leq d_2$. Thus, by Theorem 3.3, we have

$$\begin{aligned} \tilde{\mathcal{E}}(z_1, z_2) &= \sum_{i,j=1}^{d_2} b_{ij} g_i(z_1) g_j(z_2) \\ &+ \sum_{i,j=1}^{d_1} a_{ij}^{(0,0)} f_i(z_1) f_j(z_2) + \sum_{i,j=1}^{d_1} a_{ij}^{(0,1)} f_i(z_1) f_j|_{\delta_p}(z_2) \\ &+ \sum_{i,j=1}^{d_1} a_{ij}^{(1,0)} f_i|_{\delta_p}(z_1) f_j(z_2) + \sum_{i,j=1}^{d_1} a_{ij}^{(1,1)} f_i|_{\delta_p}(z_1) f_j|_{\delta_p}(z_2). \end{aligned}$$

Now let

$$g_i(z) = \sum_{m=1}^{\infty} b_i(m) \mathbf{e}(mz) \quad (i = 1, 2, \dots, d_2)$$

with $b_i(1) = 1$. Then by (1) of Theorem 3.3, we have

$$\begin{aligned} \langle g_i, \tilde{\mathcal{E}}(*, -\bar{z}_2) \rangle &= 3(p+1)^{-1} p^{l+2s} \Lambda(g_i^c, l+2s-1, \chi) \langle g_i, g_i \rangle c_i g_i^c(z_2) \\ &= \sum_{j=1}^{d_2} \overline{b_{ij}} \langle g_i, g_j \rangle g_j^c(z_2). \end{aligned}$$

Since g_1, \dots, g_{d_2} are orthogonal with each other, we have $b_{ij} = p^{l+2s} \Lambda(g_i, l+2s-1, \chi) \overline{c'_i}$ or 0 according as $i = j$ or not. Furthermore, by (2) of Theorem 3.3, we have

$$\begin{aligned} &\langle f_i, \tilde{\mathcal{E}}(*, -\bar{z}_2) \rangle \\ &= 3p^{-1} (p+1)^{-1} p^{l+2s} \Lambda(f_i^c|_{\delta_p}, l+2s-1, \chi) \langle f_i|_{\delta_p}, f_i|_{\delta_p} \rangle c_i f_i^c|_{\delta_p}(z_2) \\ &= \sum_{j=1}^{d_1} \overline{a_{ij}^{(0,0)}} \langle f_i, f_i \rangle f_j^c(z_2) + \sum_{j=1}^{d_1} \overline{a_{ij}^{(0,1)}} \langle f_i, f_i \rangle f_j^c|_{\delta_p}(z_2) \end{aligned}$$

$$+ \sum_{j=1}^{d_1} \overline{a_{ij}^{(1,0)}} \langle f_i, f_i | \delta_p \rangle + f_j^c(z_2) + \sum_{j=1}^{d_1} \overline{a_{ij}^{(1,1)}} \langle f_i, f_i | \delta_p \rangle + f_j^c | \delta_p(z_2).$$

We note that

$$\Lambda(f_i^c | \delta_p, l + 2s - 1, \chi) = \overline{\Lambda(f_i, l + 2s - 1, \chi)},$$

and

$$\langle f_i, f_i | \delta_p \rangle = \lambda_i(p) p^{-k/2} \langle f_i, f_i \rangle, \langle f_i | \delta_p, f_i | \delta_p \rangle = \langle f_i, f_i \rangle.$$

Thus for any $1 \leq i \leq d_1$ we have

$$a_{ij}^{(0,1)} + \overline{\lambda_i(p)} p^{-k/2} a_{ij}^{(1,1)} = 3p^{-1} (p+1)^{-1} p^{l+2s} \Lambda(f_i, l + 2s - 1, \chi) \overline{c_i} \text{ or } 0$$

according as $i = j$ or not. Similarly, we have

$$a_{ij}^{(0,0)} + \overline{\lambda_i(p)} p^{-k/2} a_{ij}^{(1,0)} = 0$$

for any $1 \leq i, j \leq d_1$. Similarly, by taking the inner product of $f_i | \delta_p(z_1)$ against $\tilde{\mathcal{E}}(z_1, -\overline{z_2})$, we have

$$\lambda_i(p) p^{-k/2} a_{ij}^{(0,0)} + a_{ij}^{(1,0)} = 3p^{-1} (p+1)^{-1} p^{l+2s} \Lambda(f_i, l + 2s - 1, \chi) \overline{c_i} \text{ or } 0$$

according as $i = j$ or not, and

$$\lambda_i(p) p^{-k/2} a_{ij}^{(0,1)} + a_{ij}^{(1,1)} = 0$$

for any $1 \leq i, j \leq d_1$. Thus we have

$$a_{ii}^{(0,0)} = \frac{-3p^{-1} (p+1)^{-1} p^{l+2s-k/2} \overline{\lambda_i(p)} \Lambda(f_i, l + 2s - 1, \chi) \overline{c_i}}{1 - p^{-k} |\lambda_i(p)|^2},$$

and

$$a_{ii}^{(1,0)} = a_{ii}^{(0,1)} = -p^{1-k/2} \lambda_i(p) a_{ii}^{(0,0)}, a_{ii}^{(1,1)} = p^{1-k} \lambda_i(p)^2 a_{ii}^{(0,0)}$$

for any $i = 1, \dots, d_1$ and

$$a_{ij}^{(0,0)} = a_{ij}^{(1,0)} = a_{ij}^{(0,1)} = a_{ij}^{(1,1)} = 0$$

for any $1 \leq i \neq j \leq d_1$. Thus we have

$$\begin{aligned} \tilde{\mathcal{E}}(z_1, z_2) &= \sum_i^{d_2} b_{ii} g_i(z_1) g_i(z_2) \\ &+ \sum_{i=1}^{d_1} a_{ii}^{(0,0)} \{f_i(z_1) f_i(z_2) - p^{1-k/2} \lambda_i(p) f_i(z_1) f_i | \delta_p(z_2) \\ &- p^{1-k/2} \lambda_i(p) f_i | \delta_p(z_1) f_i(z_2) + p^{1-k} \lambda_i(p)^2 f_i | \delta_p(z_1) f_i | \delta_p(z_2)\}. \end{aligned}$$

We note that $b_i(pm) = 0$ and $a_i(pm) = \lambda_i(p)a_i(m)$. Thus we have

$$\begin{aligned}\mathcal{F}_p(z_2) &= \sum_{i=1}^{d_1} a_{ii}^{(0,0)} \{ \lambda_i(p) f_i(z_2) - p^{1-k/2} \lambda_i(p)^2 f_i | \delta_p(z_2) \\ &\quad - p^{1-k/2} \lambda_i(p) p^{k/2} f_i(z_2) + p^{1-k} \lambda_i(p)^2 p^{k/2} f_i | \delta_p(z_2) \} \\ &= \sum_{i=1}^{d_1} (1-p) a_{ii}^{(0,0)} \lambda_i(p) f_i(z_2).\end{aligned}$$

We note that $|\lambda_i(p)|^2 = p^{k-1}$. This proves the assertion (1).

(2) Put $\tilde{\mathcal{E}}(z_1, z_2) = \tilde{\mathcal{E}}_{2,k}(z_1, z_2; l, p^2, \psi, \chi, s)$. As is well known,

$$\begin{aligned}S_k(\Gamma_0(p_0 p^2), \psi) &= S_k(\Gamma_0(p_0), \psi) \oplus S_k(\Gamma_0(p_0), \psi)^{(p)} \oplus S_k(\Gamma_0(p_0), \psi)^{(p^2)} \\ &\quad \perp S_k(\Gamma_0(p_0 p), \psi)^{new} \perp S_k(\Gamma_0(p_0 p), \psi)^{new(p)} \perp S_k(\Gamma_0(p_0 p^2), \psi)^{new}.\end{aligned}$$

Take bases $\{g_i\}_{i=1}^{d_2}$ of $S_k(\Gamma_0(p_0 p), \psi)^{new}$, and $\{h_i\}_{i=1}^{d_3}$ of $S_k(\Gamma_0(p_0 p^2), \psi)^{new}$ consisting of primitive forms. Then $\{f_i (i = 1, 2, \dots, d_1), f_i | \delta_p (i = 1, 2, \dots, d_1), f_i | \delta_{p^2} (i = 1, 2, \dots, d_1), g_i (i = 1, 2, \dots, d_2), g_i | \delta_p (i = 1, 2, \dots, d_2), h_i (i = 1, 2, \dots, d_3)\}$ forms a basis of $S_k(\Gamma_0(p_0 p^2), \psi)$. Thus we have

$$\begin{aligned}\tilde{\mathcal{E}}(z_1, z_2) &= \sum_{i,j=1}^{d_3} c_{ij}^{(0,0)} h_i(z_1) h_j(z_2) \\ &\quad + \sum_{\alpha, \beta=0}^1 \sum_{i,j=1}^{d_2} b_{ij}^{(\alpha, \beta)} g_i | \delta_{p^\alpha}(z_1) g_j | \delta_{p^\beta}(z_2) + \sum_{\alpha, \beta=0}^2 \sum_{i,j=1}^{d_1} a_{ij}^{(\alpha, \beta)} f_i | \delta_{p^\alpha}(z_1) f_j | \delta_{p^\beta}(z_2)\end{aligned}$$

with $c_{ij}^{(0,0)}, b_{ij}^{(\alpha, \beta)}, c_{ij}^{(\alpha, \beta)} \in \mathbf{C}$. Now let $f_i | W_{p_0} = c_i f_i^c, g_i | W_{p_0 p} = c'_i g_i^c$ and $h_i | W_{p_0 p^2} = c''_i h_i^c$ with constant c''_i . Then we have $f_i | W_{p_0 p^2} = c_i f_i^c | \delta_{p^2}, f_i | \delta_p | W_{p_0 p^2} = c_i f_i^c | \delta_p$, and $g_i | \delta_p | W_{p_0 p^2} = \overline{c'_i} g_i^c$. For a positive integer write $\lambda_i(m) = \lambda_{f_i}(m), \lambda_i(m)' = \lambda_{g_i}(m)$, and $\lambda_i(m)'' = \lambda_{h_i}(m)$. We have $\overline{\lambda_i(p)} = \psi(p) \lambda_i(p)$. Then by a direct computation combined with Theorem 3.4 and Lemma 4.1 we have

$$\begin{aligned}\tilde{\mathcal{E}}(z_1, z_2) &= \sum_i^{d_3} c_{ii} h_i(z_1) h_i(z_2) \\ &\quad + \sum_{i=1}^{d_2} b_{ii} \{ -p^{-k/2} \lambda_i(p) g_i(z_1) g_i(z_2) \\ &\quad + g_i(z_1) g_i | \delta_p(z_2) + g_i | \delta_p(z_1) g_i(z_2) - p^{-k/2} \lambda_i(p) g_i | \delta_p(z_1) g_i | \delta_p(z_2) \} \\ &\quad + \sum_{i=1}^{d_1} a_{ii} \{ p^{-1} f_i | \delta_{p^2}(z_1) f_i | \delta_{p^2}(z_2) - p^{-k/2} \lambda_i(p) f_i | \delta_p(z_1) f_i | \delta_{p^2}(z_2) + f_i(z_1) f_i | \delta_{p^2}(z_2) \}\end{aligned}$$

$$\begin{aligned}
& -p^{-k/2}\lambda_i(p)f_i|\delta_{p^2}(z_1)f_i|\delta_p(z_2) + (1 + \psi(p)\lambda_i(p)^2p^{-k} - p^{-2})f_i|\delta_p(z_1)f_i|\delta_p(z_2) \\
& \quad - \psi(p)\lambda_i(p)p^{-k/2}f_i(z_1)f_i|\delta_p(z_2) \\
& + f_i|\delta_{p^2}(z_1)f_i(z_2) - \psi(p)p^{-k/2}\lambda_i(p)f_i|\delta_p(z_1)f_i(z_2) + \psi(p)p^{-1}f_i(z_1)f_i(z_2)\},
\end{aligned}$$

where

$$\begin{aligned}
c_{ii} &= 3p^{-1}(p+1)^{-1}t_{p_0}^{-1}p^{l+2s}p_0^{1-k/2}\Lambda(h_i, l+2s-1, \chi)\overline{c_i''}\lambda_i(p_0)'', \\
b_{ii} &= 3p^{-1}(p+1)^{-1}t_{p_0}^{-1}\frac{p^{l+2s}p_0^{1-k/2}\Lambda(g_i, l+2s-1, \chi)\overline{c_i'}\lambda_i(p_0)'}{1-p^{-2}},
\end{aligned}$$

and

$$a_{ii} = 3p^{-1}(p+1)^{-1}t_{p_0}^{-1}\frac{(1+p^{-1})p^{l+2s}p_0^{1-k/2}\Lambda(f_i, l+2s-1, \chi)\overline{c_i}\lambda_i(p_0)}{(1-p^{-1})((1+p^{-1})^2 - \psi(p)p^{-k}\lambda_i(p)^2)}.$$

Now let

$$g_i(z) = \sum_{m=1}^{\infty} b_i(m)\mathbf{e}(mz) \quad (i = 1, 2, \dots, d_2)$$

and

$$h_i(z) = \sum_{m=1}^{\infty} c_i(m)\mathbf{e}(mz) \quad (i = 1, 2, \dots, d_3)$$

with $b_i(1) = c_i(1) = 1$. We note that $c_i(pm) = 0$ and $b_i(pm) = \lambda_i(p)b_i(m)$. Thus we have

$$\begin{aligned}
\mathcal{F}_p(z_2) &= \sum_{i=1}^{d_1} a_{ii}\{p^{k/2}(1-p^{-2})f_i|\delta_p(z_2) - \psi(p)(1-p^{-1})\lambda_i(p)f_i(z_2)\} \\
&\quad + \sum_{i=1}^{d_2} p^{k/2}(1-p^{-2})b_{ii}g_i(z_2).
\end{aligned}$$

We note that $b_i(p^2m) = \psi(p)p^{k-2}b_i(m)$. (cf. [Miyake, 1989, Theorem 4.6.17]). Thus we have

$$\begin{aligned}
& \epsilon(p, p^2m; l, s) - \psi(p)p^{k-2}\epsilon(p, m; l, s) \\
&= \sum_{i=1}^{d_1} a_{ii}\{p^k(1-p^{-2})a_i(pm) - \psi(p)(1-p^{-1})\lambda_i(p)a_i(p^2m)\} \\
&\quad - \psi(p)p^{k-2}\{p^k(1-p^{-2})a_i(p^{-1}m) - \psi(p)(1-p^{-1})\lambda_i(p)a_i(m)\} \\
&= \sum_{i=1}^{d_1} a_{ii}p^k(1-p^{-1})((1+p^{-1})^2 - \psi(p)\lambda_i(p)^2p^{-k})a_i(pm) \\
&= 3t_{p_0}^{-1}p^{k+l+2s-2}p_0^{1-k/2}\sum_{i=1}^{d_1}\Lambda(f_i, l+2s-1, \chi)\overline{c_i}\lambda_i(p_0)a_i(pm)
\end{aligned}$$

for any positive integer m . This proves the assertion (2). \square

Corollary *Let the notation and the assumption be as above. Furthermore, put*

$$t(m; l, s) = \begin{cases} 3^{-1}(p+1)p^{-k/2-l-2s+1}\epsilon(p, m; l, s) & \text{case (1)} \\ 3^{-1}t_{p_0}p_0^{-1/2}p^{-k-l-2s+2}(\epsilon(p, p^2m; l, s) - \psi(p)p^{k-2}\epsilon(p, m; l, s)) & \text{case (2)}. \end{cases}$$

Then we have

(1) *In case (1), for any positive integer m we have*

$$t(m; l, s) = \sum_{i=1}^{d_1} \Lambda(f_i, l + 2s - 1, \psi)\overline{c_i}a_i(m).$$

(2) *In case (2), for any positive integer m we have*

$$t(m; l, s) = \sum_{i=1}^{d_1} \Lambda(f_i, l + 2s - 1, \chi)\overline{c_i}^2a_i(pm).$$

The above corollary is a certain generalization of [Katsurada, 2003, Theorem 4.1]. Namely, in that theorem, we have restricted ourselves to the case where $l \leq k-2$. Furthermore, in (3) of that theorem, we have restricted ourselves to the case where $(\frac{p_0}{p}) = 1$ and m is prime to p_0p , and in the above corollary such conditions have been removed. We also note that $\Lambda(f, m, \mathbf{1}_p) = \Lambda(f, m, \mathbf{1})(1-p^{-m-k+1}a(p)^2)$ for a primitive form f in $S_k(\Gamma_0(p), (\frac{p}{*}))$, where $a(p)$ denotes the p -th Fourier coefficient of f . Thus (1) of that theorem is essentially included in (1) of the above corollary as a special case. However, for a practical computation, we here record the statement, which can be easily proved in a way similar to Theorem 4.2.

Proposition 4.3 *Let f_i ($i = 1, \dots, d_1$) and the others be as in (1) of Theorem 4.2. Then for any even positive integer $l \leq k-2$, we have*

$$\mathcal{F}_1(z_2) = 3(p+1)^{-1}p^{1/2} \sum_{i=1}^{d_1} \Lambda(f_i, l + 2s - 1, \mathbf{1})\overline{c_i}^2 f_i(z_2).$$

Corollary *In addition to the above notation, put*

$$t(m; l, s) = 3^{-1}(p+1)p^{-1/2}\epsilon(1, m; l, s).$$

Then, for any positive integer m we have

$$t(m; l, s) = \sum_{i=1}^{d_1} \Lambda(f_i, l + 2s - 1, 1)\overline{c_i}^2a_i(m).$$

In the above case, $\tilde{\mathcal{E}}_{2,k}(z_1, z_2; k, p, \psi, \mathbf{1}, 0)$ does not belong to $S_k(\Gamma_0(p), \psi) \otimes S_k(\Gamma_0(p), \psi)$ but belongs to $M_k(\Gamma_0(p), \psi) \otimes M_k(\Gamma_0(p), \psi)$. Thus by modifying the above method, we obtain a similar formula for the value of $\Lambda(f, k-1, \mathbf{1})$. Now for a prime number q not dividing $p_0 p$ put

$$\beta_{i+1} = \beta(i+1, q; l, s) = \sum_{r=0}^{\lfloor i/2 \rfloor} ({}_i C_r - {}_i C_{r-1}) q^{r(k-1)} t(q^{i-2r}; l, s)$$

for $i = 0, 1, \dots, d_1 - 1$, where ${}_i C_r = \frac{i!}{r!(i-r)!}$. We understand that ${}_i C_{-1} = 0$. For a Hecke eigenform f let K_f be the field generated over \mathbf{Q} by all the eigenvalues of the Hecke operators. Furthermore, for a character χ , let $K_{f,\chi}$ be the field generated over K_f by all the values of χ . Put $e_f = [K_f : \mathbf{Q}]$, and we denote by $N_{K_f}(\alpha)$ the norm of α over \mathbf{Q} for $\alpha \in K_f$. Similarly we define $e_{f,\chi}$ and $N_{K_{f,\chi}}(\alpha)$ for $\alpha \in K_{f,\chi}$. Let $\{f_i\}_{i=1}^d$ be the basis of $S_k(\Gamma_0(N), \psi)$ as above, and write $K_i = K_{f_i}$ and $e_i = e_{f_i}$. Let $\Phi(X) = \Phi_{T(m)}(X)$ be the characteristic polynomial of $T(m)$ on $S_k(\Gamma_0(N), \psi)$. We note that $N_{K_i}(c_i) = 1$ in Theorem 4.2 and Proposition 4.3. Thus by [Goto, 1998, Lemma 2.2] we obtain

Theorem 4.4 *Let the notation and the assumption be as in Theorem 4.2 and Proposition 4.3. Let f be a primitive form in $S_k(\Gamma_0(N), \psi)$, and $a(q)$ be the q -th Fourier coefficient of f . Assume that $\Phi'_{T(q)}(a(q)) \neq 0$. Write $\Phi_{T(q)}(X) = \sum_{i=0}^{d_1} b_{d_1-i} X^i$, $K = K_{f,\chi}$ and $e = e_{f,\chi}$.*

(1) *In (1) of Theorem 4.2, or Proposition 4.3, we have*

$$N_K(\Lambda(f, l+2s-1, \chi)) = N_K\left(\frac{\sum_{i=0}^{d_1-1} \sum_{j=i}^{d_1-1} \beta_{d_1-j} b_{j-i} a(q)^i}{\Phi'_{T(q)}(a(q))}\right).$$

(2) *In (2) of Theorem 4.2, if $N(a(p)) \neq 0$, we have*

$$N_K(\Lambda(f, l+2s-1, \chi)) = N_K\left(\frac{\sum_{i=0}^{d_1-1} \sum_{j=i}^{d_1-1} \beta_{d_1-j} b_{j-i} a(q)^i}{a(p) \Phi'_{T(q)}(a(q))}\right).$$

Now we give an exact value of $\Lambda(f, 1, \chi)$ for a Dirichlet character such that χ^2 is trivial. For a Hecke eigenform $f \in S_k(\Gamma_0(M), \psi)$,

$$\tilde{L}(f, s, \mathbf{1}) = \frac{L(f, s, \mathbf{1})}{\prod_{p|M} (1 - \frac{1}{a(p)} p^{-s-k+1})},$$

and

$$\tilde{L}(f, s, \psi) = \frac{L(f, s, \psi)}{\prod_{p|M} (1 - p^{-s})}.$$

Put for an odd positive integer $l \leq k - 1$

$$\tilde{\Lambda}(f, l, \chi) = \frac{\Gamma(k-1)\Gamma(k+l-1)\Gamma(l+1)}{\Gamma(k-l)} \frac{\tilde{L}(f, l, \chi)}{2^{2k+2l-4}\pi^{k+2l} \langle f, f \rangle},$$

and

$$\tilde{\Lambda}(f, 0, \chi) = \Gamma(k-1) \frac{\tilde{L}(f, 0, \chi)}{2^{2k-3}\pi^k \langle f, f \rangle}$$

for $\chi = \mathbf{1}$ or ψ . If χ^2 is trivial, $E_{2,2}^*(Z; M, \psi\chi, 0)$ does not belong to $M_2(\Gamma_0(M), \psi\bar{\chi})$, and thus we cannot give an exact value of $\Lambda(f, 1, \chi)$ by a direct use of the above method. However, we can relate the value $\Lambda(f, 1, \chi)$ to $\Lambda(f, 0, \chi)$ by using the functional equation. We explain this in the following three cases:

- (1) M is a prime number p , and $\chi = \mathbf{1}$
- (2) M is a prime number p , and $\chi = \psi = \left(\frac{\cdot}{*}\right)$
- (3) $M = 1$, and $\chi = \left(\frac{\cdot}{*}\right)$ with p a prime number such that $p \equiv 1 \pmod{4}$.

First in case (1), put

$$\tilde{R}(f, s, \mathbf{1}) = p^{(s+k-1)/2} \pi^{-3/2(s+k-1)} \Gamma((s+k-1)/2) \Gamma((s+k)/2) \Gamma((s+1)/2) \tilde{L}(f, s, \mathbf{1}).$$

Next in case (3), for $f \in S_k(\Gamma_0(1))$, and the character χ modulo p , put

$$R(f, s, \chi) = p^{3(s+k-1)/2} \pi^{-3/2(s+k-1)} \Gamma((s+k-1)/2) \Gamma((s+k)/2) \Gamma((s+1)/2) L(f, s, \chi).$$

Then by Li [Li, 1979], we have the following functional equation:

Proposition 4.5 (1) *In case (1),*

$$\tilde{R}(f, 1-s, \mathbf{1}) = \tilde{R}(f, s, \mathbf{1}).$$

In particular

$$\tilde{\Lambda}(f, 1, \mathbf{1}) = p^{-1/2} \tilde{\Lambda}(f, 0, \mathbf{1}).$$

- (2) *In case (3), under the above notation and the assumption, we have*

$$R(f, 1-s, \chi) = R(f, s, \chi).$$

In particular, in case (3), we have

$$\Lambda(f, 1, \chi) = p^{-3/2} \Lambda(f, 0, \chi).$$

In case (2), the value $\Lambda(f, 1, \phi)$ can be given by a different method (cf. [Zagier, 1977].)

Proposition 4.6 *In case (2), we have*

$$\Lambda(f, 1, \phi) = 2/3(1 - p^{-2}).$$

Now we discuss congruence among modular forms. Let K be an algebraic field, and $\mathfrak{D} = \mathfrak{D}_K$ the ring of integers in K . Let \mathfrak{q} be a prime ideal of \mathfrak{D} , and $\mathfrak{D}_{\mathfrak{q}}$ the localization of \mathfrak{D} at \mathfrak{q} . Let $f(z) = \sum_{n=1}^{\infty} a(n)\mathbf{e}(nz)$, $g(z) = \sum_{n=1}^{\infty} b(n)\mathbf{e}(nz)$ be elements of $S_k(\Gamma_0(M), \phi)$ whose Fourier coefficients belong to $\mathfrak{D}_{\mathfrak{q}}$. Then we write $f \equiv g \pmod{\mathfrak{q}}$ if $a(n) \equiv b(n) \pmod{\mathfrak{q}}$ for any positive integer n . We give the following lemma which is essentially the same as Lemma 1.4 of [Doi, Hida and Ishii, 1998].

Lemma 4.7 *Let f_1, \dots, f_r be Hecke eigenforms in $S_k(\Gamma_0(M), \phi)$. Let K be the composite field of all K_{f_1}, \dots, K_{f_r} , and \mathfrak{D} the ring of integers in K . Let \mathfrak{q} be a prime ideal of \mathfrak{D} . Assume that all the Fourier coefficients and eigenvalues of f_i ($i = 1, \dots, r$) belong to $\mathfrak{D}_{\mathfrak{q}}$. Let h be an element of $S_k(\Gamma_0(M), \phi)$ whose Fourier coefficients belong to $\mathfrak{D}_{\mathfrak{q}}$. Let*

$$h = \sum_{i=1}^r l_i f_i$$

with $l_i \in K$. Assume that $f_1 \not\equiv 0 \pmod{\mathfrak{q}}$ and $\text{ord}_{\mathfrak{q}}(l_1) < 0$. Then there exists $2 \leq i \leq r$ such that

$$f_i \equiv f_1 \pmod{\mathfrak{q}}.$$

Now by Theorem 4.2, combined with Lemma 4.7, we have the following.

Theorem 4.8 *Let N, p, ψ and χ be as in Theorem 4.2. Let f be a primitive form in $S_k(\Gamma_0(N), \psi)$. Let \mathfrak{D}_{K_f} be the ring of integers in K_f , and \mathfrak{q} a prime ideal of \mathfrak{D}_{K_f} dividing the denominator of $N_{K_f, \chi/K_f}(\Lambda(f, l+2s-1, \chi))$ but not dividing Npr , where $r = 6$ or 1 according as the case (2), or not. Then there exists a primitive form g in $S_k(\Gamma_0(N), \psi)$ different from f such that*

$$g \equiv f \pmod{\mathfrak{q}'},$$

where \mathfrak{q}' is a prime ideal of $\mathfrak{D}_{K_f K_g}$ lying above \mathfrak{q} .

Proof We note that the generalized Bernoulli number associated to a Dirichlet character ϕ is an algebraic integer if the conductor of ϕ is not a power of a prime number (cf. [Carlitz, 1959], [Leopoldt, 1958]). Thus by (2.4) and (2.5), all the Fourier coefficients of $\mathcal{F}_p(z)$ belong to $\mathfrak{D}_{\mathfrak{q}}$ in case (1). Thus the assertion in the case (1) follows directly from (1) of theorem 4.2 and Lemma 4.7. In case (2), $\mathcal{F}_{p,p}$ belongs to $S_k(\Gamma_0(pN), \psi)$, and its eigenvalues and Fourier coefficients belong to $\mathfrak{D}_{\mathfrak{q}}$. In this case, we remark that \mathfrak{q} does not divide both the p -th and the p^2 -th Fourier coefficients of f . Thus we have $\tilde{f} \not\equiv 0 \pmod{\mathfrak{q}}$. Thus again by (1) of Theorem 4.2 and Lemma 4.7, we can show that there exists a primitive form g in $S_k(\Gamma_0(N), \psi)$ different from f such that

$$\tilde{g} \equiv \tilde{f} \pmod{\mathfrak{q}'},$$

where \tilde{f} (resp. \tilde{g}) is the modular form in (2) of Theorem 2 for f (resp. g .) Thus the assertion can be proved by the above remark. \square

5 Numerical examples and comments

By Theorem 4.4 combined with Proposition 4.5 we can compute the values $\Lambda(f, m, \chi)$. A subspace S of $S_k(\Gamma_0(N), \psi)$ is called non-splitting if it is spanned by all Galois conjugates of a primitive form in S . Take a primitive form f of $S_k(\Gamma_0(1))$. Assume that $S_k(\Gamma_0(1))$ is non-splitting. Then, $N_{K_f}(\Lambda(f, l, (\frac{q}{*})))$ is independent of f . Thus, in this case, we denote this value by $\mathbf{L}(k; l, q)$. Similarly, in case $S_k(\Gamma_0(q), (\frac{q}{*}))$ is non-splitting, we define $\tilde{\mathbf{L}}(k, q; l, 1)$ and $\tilde{\mathbf{L}}(k, q; l, q)$ as $N_{K_f}(\tilde{\Lambda}(f, l, \mathbf{1}))$ and $N_{K_f}(\tilde{\Lambda}(f, l, (\frac{q}{*})))$, respectively, for a primitive form f of $S_k(\Gamma_0(q), (\frac{q}{*}))$. We have computed some values by using *Mathematica*.

(1) It is conjectured by Maeda that $S_k(\Gamma_0(1))$ is non-splitting, and so far, this conjecture has been verified at least for $k \leq 2000$ (cf. [Hida and Maeda, 1997], [Farmer and James, 2002]).

We show some examples of $\mathbf{L}(k; l, q)$ for various k, l, q . From now on put $[a_1, a_2, a_3] = \begin{pmatrix} a_1 & a_3/2 \\ a_3/2 & a_2 \end{pmatrix}$. To make a computation smooth, for an odd positive integer $l \leq k - 1$ and a positive integer m prime to q , put $t(m; l) = q^{-3/2}t(m, 2, -1/2)$ or $t(m; l + 1, 0)$ according as $l = 1$ or not. We note that the Gauss sum $\tau(\chi)$ for χ is $q^{1/2}$. Thus by (2.5) we have

$$t(m; l) = 2/3q^{-k-2l+2}$$

$$\begin{aligned} & \times \left\{ \sum_{r=1}^{[2q\sqrt{qm}]} \prod_{p \parallel [q, q^2m, r]} F_p([q, q^2m, r], \chi(p)p^{l-2}) B_{l, (\chi\chi_{[q, q^2m, r]})^{(0)}} G^{k-l-1}(q^3m, r) \chi(r) \right. \\ & \left. - q^{k-2} \sum_{r=1}^{[2\sqrt{qm}]} \prod_{p \parallel [q, m, r]} F_p([q, m, r], \chi(p)p^{l-2}) B_{l, (\chi\chi_{[q, m, r]})^{(0)}} G^{k-l-1}(qm, r) \chi(r) \right\}. \end{aligned}$$

(1.1) We compute $\mathbf{L}(12; l, q)$ for $1 \leq l \leq k - 1$ and some prime numbers q . In this case $\dim S_{12}(\Gamma_0(1)) = 1$. Take a unique primitive form $f(z) = \sum_{m=1}^{\infty} a(m)\mathbf{e}(mz)$ in $S_{12}(\Gamma_0(1))$. Thus by (2) of Theorem 4.4, and (2) of Proposition 4.5, we have

$$\mathbf{L}(12; l, q) = a(q)^{-1}t(1; l + 1)$$

if $a(q) \neq 0$. Numerical examples are as follows:

l	$\mathbf{L}(12; l, 5)$
1	$2^{14} \cdot 3^5 \cdot 7 / 5^{10}$
3	$2^{14} \cdot 3^5 \cdot 7 \cdot 2851 / 5^{13}$
5	$2^{19} \cdot 3^5 \cdot 7 \cdot 1511599 / 5^{16}$
7	$2^{19} \cdot 3^8 \cdot 7^3 \cdot 521 \cdot 295387 / 5^{20}$
9	$2^{26} \cdot 3^{10} \cdot 7^2 \cdot 110308273279 / 5^{24}$
11	$2^{21} \cdot 3^{10} \cdot 7^2 \cdot 11 \cdot 2963 \cdot 5523341 / 5^{29}$

l	$\mathbf{L}(12; l, 13)$
1	$2^{14} \cdot 3^8 \cdot 5^3 \cdot 7 \cdot 563 / 13^{12}$
3	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 41177 \cdot 1445419 / 13^{16}$
5	$2^{19} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 299696968678699 / 13^{20}$
7	$2^{19} \cdot 3^7 \cdot 5^3 \cdot 7^3 \cdot 31^2 \cdot 5479 \cdot 306945156059 / 13^{23}$
9	$2^{26} \cdot 3^{10} \cdot 5^5 \cdot 7^2 \cdot 547 \cdot 10267 \cdot 1634679978646831 / 13^{28}$
11	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 17 \cdot 29 \cdot 131 \cdot 3331 \cdot 868032338256361 / 13^{32}$

l	$\mathbf{L}(12; l, 17)$
1	$2^{21} \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 2389 / 17^{12}$
3	$2^{17} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 477200018623 / 17^{14}$
5	$2^{20} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 23 \cdot 29 \cdot 997 \cdot 46316422211 / 17^{20}$
7	$2^{22} \cdot 3^8 \cdot 5^3 \cdot 7^3 \cdot 167 \cdot 11003 \cdot 322079 \cdot 915248119 / 17^{24}$
9	$2^{30} \cdot 3^{10} \cdot 5^5 \cdot 7^2 \cdot 43 \cdot 892028959 \cdot 1604767911433 / 17^{27}$
11	$2^{20} \cdot 3^{10} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 20790457101154865002708553 / 17^{21}$

The above values have been already computed in [Katsurada, 2003] except for $l = 11$. We note that the value $\mathbf{L}(12; l, q)$ has been obtained by Stopple [Sto] in the case where $q = 5$ and $l = 1, 3, 5, 7$, or $q = 13, 17, 29, 37, 41$ and $l = 1$. We note that relatively large prime numbers appear in the numerator of $\mathbf{L}(k; l, 5)$ contrary to the untwisted case in [Dummigan, 2001]. At present, the author does not know whether these primes relate to any arithmetic algebraic geometry. We note that the numerator of $\Lambda(f, l, \mathbf{1})$ is related to the order of the Shafarevich-Tate group (cf. [Dummigan, 2001].)

(1.2) The values of the standard zeta functions at $s = 1$ are particularly important. To explain this, let q be a prime number congruent to 1 modulo 4, and let $\mathfrak{D}_{\mathbf{Q}(\sqrt{q})}$ be the ring of integers in $\mathbf{Q}(\sqrt{q})$. Assume that the class number of $\mathbf{Q}(\sqrt{q})$ is one in the narrow sense. Let $S_{k,k}(SL_2(\mathfrak{D}_{\mathbf{Q}(\sqrt{q})}))$ be the space of cusp forms of weight (k, k) with respect to $SL_2(\mathfrak{D}_{\mathbf{Q}(\sqrt{q})})$. Then $S_{k,k}(SL_2(\mathfrak{D}_{\mathbf{Q}(\sqrt{q})}))$ has the following decomposition:

$$S_{k,k}(SL_2(\mathfrak{D}_{\mathbf{Q}(\sqrt{q})})) = \hat{S}_k(\Gamma_0(1)) \perp \hat{S}_k(\Gamma_0(q), \left(\frac{q}{*}\right)) \perp S_{k,k}^0,$$

where $\hat{S}_k(\Gamma_0(1))$ (resp. $\hat{S}_k(\Gamma_0(q), \left(\frac{q}{*}\right))$) is the image of $S_k(\Gamma_0(1))$ (resp. $S_k(\Gamma_0(q), \left(\frac{q}{*}\right))$) under the Doi-Naganuma map, and $S_{k,k}^0$ the orthogonal complement of $\hat{S}_k(\Gamma_0(1)) \perp \hat{S}_k(\Gamma_0(q), \left(\frac{q}{*}\right))$ in $S_{k,k}(SL_2(\mathfrak{D}_{\mathbf{Q}(\sqrt{q})}))$ with respect to the Petersson product. Take a primitive form $g \in S_{k,k}^0$, and for an integral ideal \mathfrak{A} in $\mathbf{Q}(\sqrt{q})$ let $c(\mathfrak{A}; g)$ be the

\mathfrak{A} -th Fourier coefficient of g . Let K_g be the field generated over \mathbf{Q} by all $c(\mathfrak{A}; g)$'s and K_g^+ the subfield of K_g generated by $c((p); g)$ for all rational primes p . We denote by $D(K_g/K_g^+)$ the relative discriminant of K_g/K_g^+ . Assume that $S_{k,k}^0$ is non-splitting, that is, it is spanned by all Galois conjugates of a primitive form in $S_{k,k}^0$. Then $D(K_g/K_g^+)$ does not depend on g . Hence we denote this value by $D_{k,q}$. Then in [Doi, Hida and Ishii, 1998] Doi, Hida, and Ishii, among others, conjectured the following:

”Any prime factor of $D_{k,q}$ divides either the numerator of $N_{K_f}(\Lambda(f, 1, (\frac{q}{*})))$ for some primitive form f in $S_k(\Gamma_0(1))$ or the numerator of $N_{K_f}(\tilde{\Lambda}(f, 1, \mathbf{1}))$ for some primitive form f in $S_k(\Gamma_0(q), (\frac{q}{*}))$.”

Doi, Hida and Ishii [Doi, Hida and Ishii, 1998] computed an exact value of $\tilde{\mathbf{L}}(k, q; 1, 1)$, and verified the above conjecture in some cases. Goto [Goto, 1998] computed the value $\mathbf{L}(20, 1, 5)$, and Hiraoka [Hiraoka, 2000] computed the values $\mathbf{L}(22; 1, 5)$ and $\mathbf{L}(24; 1, 5)$. Then, combining the result of [Doi, Hida and Ishii, 1998], they verified the conjecture for $(k, q) = (20, 5), (22, 5)$, and $(24, 5)$. Now let $k = 12$ and $q = 13$. In this case, $S_{12,12}^0$ and $S_k(\Gamma_0(13), (\frac{13}{*}))$ are non-splitting. Furthermore, we have $D_{12,13} = 13 \cdot 563 \cdot 6205151$ and the numerator of $\tilde{\mathbf{L}}(12, 13; 1, 1)$ is $5 \cdot 7 \cdot 13^{29} \cdot 6205151$. Thus it is expected that 563 appears in the numerator of $\mathbf{L}(12; 1, 13)$. The example in (1) shows that this is true.

Now we compute $\mathbf{L}(k; 1, 5)$ for $16 \leq k \leq 38$. As for the other numerical examples, see [Stopple, 1996].

First let $k = 16, 18, 20, 22, 26$. Then we have $\dim S_k(\Gamma_0(1)) = 1$. Take a unique primitive form $f(z) = \sum_{m=1}^{\infty} a(m)e(mz)$ in $S_k(\Gamma_0(1))$. Then we have

$$\mathbf{L}(12; l, 5) = a(5)^{-1}t(1; 2).$$

Thus we have the following table:

k	$\mathbf{L}(k; 1, 5)$
16	$2^{13} \cdot 3^4 \cdot 7^3 \cdot 11/5^{14}$
18	$2^{13} \cdot 3^4 \cdot 7 \cdot 11 \cdot 13/5^{13}$
20	$2^{15} \cdot 3^4 \cdot 7 \cdot 11 \cdot 13 \cdot 977/5^{18}$
22	$2^{15} \cdot 3^5 \cdot 7^2 \cdot 13 \cdot 17 \cdot 71/5^{18}$
26	$2^{13} \cdot 3^4 \cdot 7 \cdot 11 \cdot 17 \cdot 19 \cdot 337 \cdot 1409/5^{22}$

Next let $k = 24, 28, 30, 32, 34, 38$. Then we have $\dim S_k(\Gamma_0(1)) = 2$. Take a basis f_1, f_2 of $S_k(\Gamma_0(1))$ consisting of primitive forms. Then $K_{f_1} = K_{f_2}$ is a real quadratic field, and thus this field can be expressed as $K = \mathbf{Q}(\sqrt{D})$ with D a non-square positive integer. Let $\Phi_{T(2)}(X) = X^2 + b_1X + b_2$. Then by (2) of Theorem 4.4, we have

$$\mathbf{L}(k; 1, 5) = \frac{t(1, 2)t(2, 2)b_1 + t(2, 2)^2 + t(1, 2)^2b_2}{(-b_1^2 + 4b_2)N_K(a(5))},$$

where $a(5)$ is the fifth Fourier coefficient of f_1 . The polynomial $\Phi(X)$ and the

value $N_K(a(5))$ can easily be computed by the trace formula (cf. [Miyake, 1989].) Thus we have the following table:

k	D	$\mathbf{L}(k; 1, 5)$
24	144169	$2^{29} \cdot 3^9 \cdot 7^4 \cdot 11^4 \cdot 13 \cdot 17 \cdot 19 \cdot 109 \cdot 54449/5^{39} \cdot 144169$
28	131 · 139	$2^{26} \cdot 3^8 \cdot 7^3 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 66876860429/5^{44} \cdot 131 \cdot 139$
30	51349	$2^{26} \cdot 3^{10} \cdot 7^4 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 3253 \cdot 20017939/5^{43} \cdot 51349$
32	67 · 273067	$2^{26} \cdot 3^{10} \cdot 7^4 \cdot 11 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 37 \cdot 157 \cdot p_1/5^{54} \cdot 67 \cdot 273067$
34	479 · 4919	$2^{26} \cdot 3^{10} \cdot 7^3 \cdot 11^3 \cdot 13^2 \cdot 19 \cdot 23 \cdot 29 \cdot 191 \cdot 3191 \cdot p_2/5^{51} \cdot 479 \cdot 4919$
38	181 · 349 · 1009	$2^{28} \cdot 3^8 \cdot 7^4 \cdot 11^2 \cdot 13 \cdot 17 \cdot 23 \cdot 29 \cdot 31 \cdot p_3/5^{60} \cdot 181 \cdot 349 \cdot 1009$

$$p_1 = 222142617427425679, p_2 = 121120620073, p_3 = 24539630352019799615221087$$

(2) We compute $\tilde{\mathbf{L}}(k, q; l, 1)$ and $\tilde{\mathbf{L}}(k, q; l, q)$ when $S_k(\Gamma_0(q), (\frac{q}{*}))$ is non-splitting.

In Proposition 4.3, put $t(m; l) = p^{-1/2}t(m; 2, -1/2)$ or $t(m; l+1, 0)$ according as $l = 1$ or not. Let $\chi = (\frac{q}{*})$. Then for $l \geq 1$ we have

$$t(m; l) = 1/3(q+1)q^{-l} \left\{ 2 \sum_{1 \leq r < 2\sqrt{m}} \prod_{p|f_{[1,m,r]}} F_p([1, m, r], \chi(p)p^{l-2}) B_{l, (\chi\chi_{[1,m,r]})^{(0)}} G^{k-l-1}(m, r) \right. \\ \left. \times q^{(l-3/2)(\text{ord}_q(4 \det[1,m,r]) + \text{ord}_q(\mathfrak{b}_{[1,m,r]}))} (1 - (\chi\chi_{[1,m,r]})^{(0)}(q)q^{-l}) \right. \\ \left. - \delta(l; m, r)(1 - q)/12 \right\},$$

where $\delta(l; m, r) = 1$ or 0 according as $l = 1$ and $4m - r^2 = 0$, or not. Furthermore in case (1) of Theorem 4.4, for $l \geq 3$ put $\tilde{t}(m; l) = (1 - q^{-l})^{-1}t(m; l+1, 0)$. Then by (2.7), we have

$$\tilde{t}(m; l) = 2/3(q+1)q^{-k/2-l+1/2} \\ \times \sum_{1 \leq r \leq 2\sqrt{qm}} \prod_{p|f_{[q,m,r]}} F_p([q, m, r], p^{l-2}) B_{l, \chi_{[q,m,r]}^{(0)}} G^{k-l-1}(qm, r) \chi(r).$$

Assume that $S_k(\Gamma_0(q), (\frac{q}{*}))$ is non-splitting. Take a primitive form $f(z) = \sum_{m=1}^{\infty} a(m)\mathbf{e}(mz) \in S_k(\Gamma_0(q), (\frac{q}{*}))$ and put $K = K_f$. Let $\alpha = a(2)$ and assume that $\Phi'_{T(2)}(\alpha) \neq 0$. Write $\Phi_{T(2)}(X) = \sum_{i=0}^e b_{e-i}X^i$, and

$$\beta_{i+1} = \sum_{r=0}^{[i/2]} ({}_i C_r - {}_i C_{r-1}) q^{r(k-1)} t(q^{i-2r}; l)$$

as in Section 4. We also define $\tilde{\beta}_{i+1}$ by replacing $t(q^{i-2r}; l)$ with $\tilde{t}(q^{i-2r}; l)$. Then we have

$$\tilde{\mathbf{L}}(k, q, l, 1) = \frac{N_K(\sum_{i=0}^{e-1} \sum_{j=i}^{e-1} \beta_{e-j} b_{j-i} \alpha^i)}{\Phi_{T(q)}(q^{(k-1+l)/2}) \Phi_{T(q)}(-q^{(k-1+l)/2}) N_K(\Phi'_{T(2)}(\alpha))}$$

for $l \geq 3$ and

$$\tilde{\mathbf{L}}(k, q; 1, 1) = \frac{N_K(\sum_{i=0}^{e-1} \sum_{j=i}^{e-1} \beta_{e-j} b_{j-i} \alpha^i)}{\Phi_{T(q)}(q^{(k-1)/2}) \Phi_{T(q)}(-q^{(k-1)/2}) N_K(\Phi'_{T(2)}(\alpha))}.$$

Furthermore,

$$\tilde{\mathbf{L}}(k, q; l, q) = \frac{N_K(\sum_{i=0}^{\epsilon-1} \sum_{j=i}^{\epsilon-1} \tilde{\beta}_{\epsilon-j} b_{j-i} \alpha^i)}{N_K(\Phi'_{T(2)}(\alpha))}.$$

Here we mention the special values of the standard zeta function of the Doi-Naganuma lift $\hat{f} \in S_{k,k}(SL_2(\mathfrak{D}_{\mathbf{Q}(\sqrt{q})}))$ for a primitive form $f \in S_k(\Gamma_0(q), (\frac{q}{*}))$. For a prime number p , let α_p, β_p be the complex numbers in (3.5). For a prime ideal \mathfrak{p} of $\mathfrak{D}_{\mathbf{Q}(\sqrt{q})}$, let $A_{\mathfrak{p}}$ and $B_{\mathfrak{p}}$ such that

$$A_{\mathfrak{p}} = \alpha_p^m, B_{\mathfrak{p}} = \beta_p^m$$

if $N(\mathfrak{p}) = p^m$, where $N(\mathfrak{p}) = N_{\mathbf{Q}(\sqrt{q})/\mathbf{Q}}(\mathfrak{p})$. Then we define the standard zeta function $\mathbf{L}(\hat{f}, s)$ of \hat{f} as

$$L(\hat{f}, s) = \prod_{\mathfrak{p}} \{(1 - A_{\mathfrak{p}} B_{\mathfrak{p}} N(\mathfrak{p})^{-s-k+1})(1 - A_{\mathfrak{p}}^2 N(\mathfrak{p})^{-s-k+1})(1 - B_{\mathfrak{p}}^2 N(\mathfrak{p})^{-s-k+1})\}^{-1}.$$

As for the precise definition of the standard zeta function of a general Hilbert modular form, see [Zagier, 1977]. Put

$$\Lambda(\hat{f}, l) = \left(\frac{\Gamma(k-1)\Gamma(k+l-1)\Gamma(l+1)}{2^{2k-2l-4}\pi^{k+2l}\Gamma(k-l)} \right)^2 \frac{q^k(q+1)L(\hat{f}, l)}{B_{2,(\frac{q}{*})} \langle \hat{f}, \hat{f} \rangle},$$

where $\langle \hat{f}, \hat{f} \rangle$ denotes the normalized Petersson product of \hat{f} in $S_{k,k}(SL_2(\mathfrak{D}_{\mathbf{Q}(\sqrt{q})}))$. By [(97), Zagier, 1977], we have

$$L(\hat{f}, s) = \tilde{L}(f, s, \mathbf{1}) \tilde{L}(f, s, (\frac{q}{*})).$$

Thus we have

$$\Lambda(\hat{f}, l) = \frac{\tilde{\Lambda}(f, l, \mathbf{1}) \tilde{\Lambda}(f, l, (\frac{q}{*})) \langle f, f \rangle^2}{\tilde{\Lambda}(f, 1, \mathbf{1}) \langle \hat{f}, \hat{f} \rangle}.$$

On the other hand, we have

$$\tilde{\Lambda}(f, 1, \mathbf{1}) = \frac{\langle \hat{f}, \hat{f} \rangle}{(q+1) \langle f, f \rangle^2}$$

(cf. Page 152 in [Zagier, 1977]). Thus we have

$$\Lambda(\hat{f}, l) = \frac{\tilde{\Lambda}(f, l, \mathbf{1}) \tilde{\Lambda}(f, l, (\frac{q}{*}))}{\tilde{\Lambda}(f, 1, \mathbf{1})}. \quad (5.1)$$

Let $q = 13, k = 8$. Then $S_8(\Gamma_0(13), (\frac{13}{*}))$ is non-splitting and $\dim S_8(\Gamma_0(13), (\frac{13}{*})) = 6$. Then by [Doi and Goto, 1993] we have

$$\Phi_{T(13)}(X) = X^6 + 2 \cdot 13 \cdot 193X^5 + 7^2 \cdot 13^3 \cdot 29 \cdot 31X^4 + 2^2 \cdot 5 \cdot 13^6 \cdot 47 \cdot 179X^3$$

$$+7^2 \cdot 13^{10} \cdot 29 \cdot 31X^2 + 2 \cdot 13^{15} \cdot 193X + 13^{21},$$

$$\Phi_{T(2)}(X) = X^6 + 449X^4 + 37224X^2 + 205776$$

and we have

$$N_{K/\mathbf{Q}}(\Phi'_{T(2)}(a(2))) = 2^4 \cdot 3^2 \cdot 5^4 \cdot 41^2 \cdot 1429 \cdot 25104281^2.$$

Thus we have

$$\tilde{\mathbf{L}}(8, 13; 1, 1) = \frac{2^2 \cdot 7^6 \cdot 13^{14} \cdot 4357^2}{3^{12} \cdot 41^2 \cdot 1429^2 \cdot 25104281^2}$$

(cf. Table 1 of [Doi, Hida and Ishii, 1998]). Furthermore,

$$\tilde{\mathbf{L}}(8, 13; 3, 1) = \frac{2^{30} \cdot 3^4 \cdot 7^6 \cdot 13^4 \cdot 5^2}{41^2 \cdot 25104281^2},$$

$$\tilde{\mathbf{L}}(8, 13; 5, 1) = \frac{2^{22} \cdot 5^{18} \cdot 7^6 \cdot 313^2}{41^2 \cdot 25104281^2},$$

and

$$\tilde{\mathbf{L}}(8, 13; 3, 13) = \frac{2^{30} \cdot 3^{10} \cdot 5^8 \cdot 7^6 \cdot 4583^2 \cdot 10079^2}{13^{16} \cdot 41^2 \cdot 25104281^2},$$

$$\tilde{\mathbf{L}}(8, 13; 5, 13) = \frac{2^{38} \cdot 3^{10} \cdot 5^{18} \cdot 7^6 \cdot 4583^2 \cdot 10079^2 \cdot 20447687895923^2}{13^{28} \cdot 41^2 \cdot 25104281^2}.$$

Put

$$\hat{\mathbf{L}}(8, 13; l) = N_{K/\mathbf{Q}}(\Lambda(\hat{f}, l)).$$

Then by (5.1), we see that the prime number 4357, a prime factor of the numerator of $\tilde{\mathbf{L}}(8, 13; 1, 1)$, appears in the denominator of $\hat{\mathbf{L}}(8, 13; 3)$ and $\hat{\mathbf{L}}(8, 13; 5)$. This phenomenon is closely related to the above conjecture. We will discuss this topic more precisely in a subsequent paper.

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