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# Uniformly ultimate boundedness of solutions for some 3-dimensional systems

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## Abstract

For some 3-dimensional system of electrical circuits, the uniformly ultimate boundedness of solutions is proved, and consequently existing unstable manifolds around equilibrium points is bounded.

**Key words:** boundedness of solutions ; 3-dimensional system ; unstable manifold.

## 1 Introduction.

The boundedness of solutions has been studied variously for 2-dimensional systems [6],[7,chapter VII] ; however the 3-dimensional case has been dealt only in a few types of systems [8] in spite the significance of applications. Our concern of this paper is in the following 3-dimensional system (1), which describes the dynamics of some electrical circuits [5, p.19].

$$\begin{aligned} \dot{x} &= a(y-x) - g(x) \\ \dot{y} &= b(x-y) + z \\ \dot{z} &= -cy \end{aligned} \quad \left( \cdot = \frac{d}{dt} \right) \quad (1)$$

where  $a$ ,  $b$  and  $c$  are positive constants,  $g(x)$  is Lipschitz-continuous and  $g(-x) \equiv -g(x)$ . When  $(x(t), y(t), z(t))$  denotes an arbitrary solution of (1), our results may be stated in Theorem 1 and 2:

### Theorem 1

Assume that  $g(x)x > 0$  for large  $|x|$  and  $4c > ab$ . Then solutions of (1) are uniformly bounded and uniformly ultimate-bounded, namely if  $|x(0)| +$

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$|y(0)| + |z(0)| < \alpha$  for positive number  $\alpha$ , then there exist positive constants  $\beta(\alpha), T(\alpha)$  and  $B$  such that  $|x(t)| + |y(t)| + |z(t)| < \beta(\alpha)$  for  $t \geq 0$  and  $|x(t)| + |y(t)| + |z(t)| < B$  for  $t \geq T(\alpha)$ , where  $B$  is independent of  $\alpha$ . Consequently existing unstable manifolds around equilibrium points are bounded.

Theorem 1 will be proved by construction of strictly positive, invariant sets  $D_e$  for  $e > e_0$ , where  $e_0$  is a positive constant, such that  $D_{e'} > D_e$  for  $e' > e$  and  $\bigcup_{e > e_0} D_e = R^3$ ;  $D_e$  is said to be strictly positive-invariant if  $(x(0), y(0), z(0)) \in \partial D_e$ , then  $(x(t), y(t), z(t)) \in D_e^o$  for  $t > 0$ , where  $\partial D_e$  and  $D_e^o$  denote the boundary of  $D_e$  and the inner set of  $D_e$ , respectively. The positive invariance of  $D_e$  implies the existence of compact attractors for solutions of (1), and as suggested by the computer used results [5], this attractor is almost unstable manifolds around equilibrium points. Since  $g(0) = 0$ , the origin is an equilibrium point. The following result may describe the deformation of attractor with respect to the parameter  $k = -g'(0)$ .

### Theorem 2

All of assumption of Theorem 1 is assumed. Furthermore, let  $g(x)$  be continuously once-differentiable with respect to  $x$ . Then the following hold :

- (i) If  $k < a$  and  $k^2 - (a + b)k + c > 0$ , then the origin is asymptotically stable, and consequently there exists no unstable manifold around origin,
- (ii) If  $4c < (a + b)^2$  and a solution of the equation  $k^2 - (a + b)k + c = 0$ , say  $k = k_0 > 0$ , satisfies that  $k_0 < \frac{c}{b}$ , then there exists a positive number  $\varepsilon_0$  and  $C^1$ -functions  $k(\varepsilon)$  and  $\omega(\varepsilon)$  for  $|\varepsilon| < \varepsilon_0$ , where  $k(0) = k_0$  and  $\omega(0) = \frac{2\pi}{\sqrt{c - bk_0}}$ , such that if  $g'(0) = -k(\varepsilon)$ , then (1) has the periodic solution  $(x(t, \varepsilon), y(t, \varepsilon), z(t, \varepsilon))$  of period  $\omega(\varepsilon)$ , which is continuous for  $\varepsilon$ ,  $x(t, 0) \equiv y(t, 0) \equiv z(t, 0) \equiv 0$  and is not constant for  $\varepsilon \neq 0$ ,
- (iii) If  $k < a$ ,  $4c < (a + b)^2$  and  $k^2 - (a + b)k + c < 0$ , then there exists a 2-dimensional, bounded, invariant, unstable manifold around origin,
- (iv) If  $k > a$ , then there exists a 1-dimensional, bounded, invariant, unstable manifold around origin.

We shall prove Theorem 2 by investigating the characteristic exponents of the variational system of (1) with respect to origin and by applying [2, p.330], [4, p.244] and Hopf bifurcation theorem [1]. Examples will be stated in Section 3.

Next we shall consider the perturbed case of (1) by external forces :

$$\dot{x} = a(y - x) - g(x) + p(t),$$

$$\begin{aligned}\dot{y} &= b(x - y) + z + q(t), \\ \dot{z} &= -cy,\end{aligned}\tag{2}$$

where  $a, b, c$  and  $g(x)$  are the same as in (1) and  $p(t), q(t)$  are continuous and bounded for  $t \in R$ . This system is equivalent to the system :

$$\dot{x} + \frac{a}{c}\dot{z} + ax + g(x) = p(t), \quad \ddot{z} + b\dot{z} + cz + bcx + cq(t) = 0.\tag{3}$$

By a slight modification of the proof of Theorem 1 we shall show that the following holds :

### Corollary 1

Assume that there exists positive constants  $d$  and  $H$  such that  $g(x) \operatorname{sgn} x > d|x|$  for  $|x| \geq H$ . Then solutions of (2) are uniformly bounded and uniformly ultimately bounded. Moreover, if  $p(t)$  and  $q(t)$  has the same period in  $t$ , say  $2\pi$ , then there exists at least one  $2\pi$ -periodic solution.

The proof will be given in Section 2 in addition to the proof of Theorem 1.

## 2 Proof of Theorem 1 and Corollary 1

First of all we shall construct a simply connected domain in  $R^3$ .  $e$  denotes a large, positive number throughout this section,  $S_0$  the ellipsoid :

$$\frac{b}{a}x^2 + y^2 + \frac{z^2}{c} = \frac{(e+1)^2}{c},$$

$D$  the closed inside of  $S_0$  and  $S_i$  ( $1 \leq i \leq 4$ ) the surfaces in  $D$  such that

$$S_1 = \{(x, y, z) \in D ; y \geq 0 \text{ and } z = e\},$$

$$S_2 = \{(x, y, z) \in D ; y \leq 0 \text{ and } z = -\lambda y^2 + e\},$$

where

$$\lambda = \frac{h}{e} \quad \text{and} \quad h = \frac{b^2}{4} + \frac{c}{2},$$

and  $S_3$  and  $S_4$  are symmetric with respect to origin to  $S_1$  and  $S_2$ , respectively. Setting  $C_i = S_i \cap S_0$  for  $1 \leq i \leq 4$ , we shall find that  $C_1 \cup C_2$  is simply closed as well as  $C_3 \cup C_4$  and that

$$(C_1 \cup C_2) \cap (C_3 \cup C_4) = \emptyset \quad \text{for large } e.\tag{4}$$

In fact, (4) will be verified in the following. Every point  $(x, y, z)$  of  $C_2$  satisfies

$$\frac{b}{a}x^2 + \frac{e-z}{\lambda} + \frac{z^2}{c} = \frac{(e+1)^2}{c},$$

because  $y^2 = \frac{e-z}{\lambda}$ , which yields by  $\lambda = \frac{h}{e}$  that

$$z \geq \frac{e}{2} \left\{ \frac{c}{h} - \sqrt{\frac{c^2}{h^2} - \frac{4c}{h} + 4 \left(1 + \frac{1}{e}\right)^2} \right\},$$

and hence

$$\lim_{e \rightarrow \infty} \frac{z}{e} = \frac{c}{h} - 1 > -1.$$

Therefore  $C_2$  is above  $z = -e$  for large  $e$ , immediately  $C_4$  is below  $z = e$ , and hence (4) is proved.

Furthermore set  $S_5$  be the connected component of  $S_0$  bounded by  $C_1 \cup C_2$  and  $C_3 \cup C_4$ , and  $D_e$  the closed inside of  $\cup_{i=1}^5 S_i$ , which is simply connected. We shall prove two properties of  $D_e$ . First an arbitrary ball  $B = \{(x, y, z) \in R^3; x^2 + y^2 + z^2 \leq n^2\}$  for positive number  $n$  is contained in  $D_e$  if  $e > \left(1 + h + \sqrt{1 + c + \frac{bc}{a}}\right) n$ . In fact, for each point  $(x, y, z) \notin D_e$  we may assume at least one of the four cases such that  $\frac{b}{a}x^2 + y^2 + \frac{z^2}{c} > \frac{(e+1)^2}{c}$ ,  $|z| > e$ ,  $z > -\lambda y^2 + e$  and  $z < \lambda y^2 - e$ . Since  $|x| \leq n$ ,  $|y| \leq n$  and  $|z| \leq n$ , these yield that  $\left(\frac{b}{a} + 1 + \frac{1}{c}\right) n^2 > \frac{(e+1)^2}{c}$ ,  $n > e$ , and  $n + \lambda n^2 > e$ . Clearly the first two inequalities never hold. The third one is equal to that  $n + \frac{hn^2}{e} > e$ , which implies that  $2n > e$ , because  $e > hn$ , and this also never hold.

Secondly every point  $(x, y, z)$  of  $S_5$  for large  $e$  satisfies that

$$(x - y)^2 + \frac{1}{a}xg(x) > 0. \quad (5)$$

In fact, letting  $l$  and  $m$  be positive numbers such that  $g(x)x > 0$  for  $|x| \geq l$  and  $m = \max\{|g(x)x|; |x| \leq l\}$ , and  $N$  be a positive number such that  $\sqrt{N^2 - l^2} > l + \sqrt{\frac{m}{a}}$ , we can varify that if  $x^2 + y^2 \geq N^2$ , then (5) holds. Therefore we shall show that every point  $(x, y, z)$  of  $S_5$  for large  $e$  satisfies  $(x, y) \notin E$ , where  $E$  is the disk :  $x^2 + y^2 \leq N^2$ . In order to prove this assertion, since  $D_e$  is convex and the boundary of  $S_5$  in  $S_0$  is  $\cup_{i=1}^4 C_i$ , it is sufficient to prove that the projection of  $C_1 \cup C_3$  into the  $(x, y)$ -plane, say  $\Gamma_1$ , and the projection of  $C_2 \cup C_4$ , say  $\Gamma_2$ , contain  $E$  in their insides, respectively, for large  $e$ .

$\Gamma_1$  and  $\Gamma_2$  may be represented as the following :

$$\begin{aligned} \Gamma_1 & : \frac{b}{a}x^2 + y^2 = \frac{2e+1}{c}, \\ \Gamma_2 & : \frac{b}{a}x^2 + y^2 + \frac{1}{c}(e - \lambda y^2)^2 = \frac{(e+1)^2}{c}. \end{aligned}$$

Clearly  $\Gamma_1$  contains  $E$  in its inside for large  $e$ . Also  $\Gamma_2$  is verified to be a simple closed curve, which is symmetric to origin. Since this formula of  $\Gamma_2$  is reduced to

$$\frac{b}{a}x^2 + y^2 + \frac{h^2}{ce^2}y^4 - \frac{2h}{c}y^2 = \frac{2e+1}{c},$$

any point of  $\Gamma_2$  cannot remain in  $E$  as  $e$  goes to  $+\infty$ . We may easily prove that  $D_e \subset D_{e'}$  for  $e' > e$ . Thus our assertion about (5) holds.

Now we shall prove the strictly positive invariance of  $D_e$  for large  $e$ , namely that if  $P = (x(0), y(0), z(0)) \in \cup_{i=1}^5 S_i$ , then  $(x(t), y(t), z(t)) \in D_e^o$  for  $0 < t < \varepsilon$  and for some positive number  $\varepsilon$ . The position of  $P$  may be classified into the four cases :

- (i)  $P \in S_5 \setminus \cup_{i=1}^4 C_i$ ,
- (ii)  $P \in S_1 \cup S_3 \setminus C_1 \cup C_3$ ,
- (iii)  $P \in S_2 \cup S_4 \setminus C_2 \cup C_4$ ,
- (iv)  $P \in \cup_{i=1}^4 C_i$ .

First we shall treat (i) by setting  $V(x, y, z) = \frac{b}{a}x^2 + y^2 + \frac{z^2}{c}$  and  $W(t) = V(x(t), y(t), z(t))$  for solution  $(x(t), y(t), z(t))$  of (1). Since

$$\dot{W}(t) = -2b \left\{ (x - y)^2 + \frac{1}{a}xg(x) \right\}$$

for  $x = x(t)$  and  $y = y(t)$ , (5) implies that  $\dot{W}(0) < 0$ , and hence  $W(t) < W(0)$  for  $0 < t < \varepsilon$  and for a positive number  $\varepsilon$ . Moreover, since  $P \notin \cup_{i=1}^4 S_i$ , it follows from the continuity argument that  $(x(t), y(t), z(t)) \notin \cup_{i=1}^4 S_i$  for  $0 < t < \varepsilon$ , where  $\varepsilon$  is taken to be furthermore small, if necessary. These conclude  $(x(t), y(t), z(t)) \in D_e^o$  for  $0 < t < \varepsilon$ .

Secondly we shall treat (ii), and only the case where  $P \in S_1 \setminus C_1$ , while the remaining case where  $P \in S_3 \setminus C_3$  follows from this case, because the right-hand side of (1) is symmetric to origin. Since  $P \notin \cup_{i=2}^5 S_i$ , the continuity implies that  $(x(t), y(t), z(t)) \notin \cup_{i=2}^5 S_i$  for  $0 \leq t \leq \varepsilon$  and for some positive number  $\varepsilon$ . We shall consider the two cases :  $y(0) > 0$  and  $y(0) = 0$ , respectively. If  $y(0) > 0$ , then  $y(t) > 0$  for  $0 < t < \varepsilon$ , and  $\dot{z}(0) = -cy(0) < 0$ . Therefore  $z(t) < z(0) = e$  for  $0 < t < \varepsilon$ . These conclude  $(x(t), y(t), z(t)) \in D_e^o$  for  $0 < t < \varepsilon$ . If  $y(0) = 0$ , then  $\dot{y} = bx(0) + e$ , and since  $\frac{b}{a}x^2(0) + \frac{e^2}{c} \leq \frac{(e+1)^2}{c}$ , we get

$$bx(0) > -\sqrt{\frac{ab}{c}(2e+1)},$$

which implies

$$\dot{y}(0) > e - \sqrt{\frac{ab}{c}(2e+1)}.$$

Therefore,  $\dot{y}(0) > 0$  for large  $e$ , and hence  $y(t) > 0$  for  $0 < t < \varepsilon$ . Moreover, Since  $\dot{z}(0) = -cy(0) = 0$  and  $\ddot{z}(0) = -cj(0) < 0$ , it follows that  $z(t) < e$  for  $0 < t < \varepsilon$ . These also conclude  $(x(t), y(t), z(t)) \in D_e^o$  for  $0 < t < \varepsilon$ .

We shall treat (iii), and only the case where  $P \in S_2 \setminus C_2$ , while the remaining case where  $P \in S_4 \setminus C_4$  also follows from this case by symmetry.

In order to prove that  $(x(t), y(t), z(t))$  strictly crosses  $S_2$  from outside into inside as  $t$  increases, we shall verify that

$$\dot{y}(0) > 0$$

and

$$\frac{\dot{z}(0)}{\dot{y}(0)} < \frac{\partial}{\partial y}(-\lambda y^2 + e) \quad \text{for } y = y(0).$$

From (1), these are equivalent to that

$$b(x - y) + z > 0, \quad (6)$$

and that

$$\frac{c}{b(x - y) + z} < 2\lambda, \quad (7)$$

because  $y < 0$ , where  $x = x(0)$ ,  $y = y(0)$  and  $z = z(0)$ . Above all (6) and (7) are equivalent to that

$$\frac{c}{2\lambda} < b(x - y) + z. \quad (8)$$

Substituting  $z = -\lambda y^2 + e$  into (8), we get

$$bx > \frac{c}{2\lambda} + \lambda y^2 - e + by. \quad (9)$$

On the other hand, since  $\frac{b}{a}x^2 + y^2 + \frac{z^2}{c} \leq \frac{(e+1)^2}{c}$ , it follows that

$$bx \geq -\alpha \sqrt{2e + 1 + (2e\lambda - c)y^2 - \lambda^2 y^4}, \quad (10)$$

where  $\alpha = \sqrt{\frac{ab}{c}}$ .

Therefore, in order to prove (9) it is sufficient to show that

$$\frac{c}{2\lambda} + \lambda y^2 - e + by < -\alpha \sqrt{2e + 1 + (2e\lambda - c)y^2 - \lambda^2 y^4}, \quad (11)$$

where

$$2e + 1 + (2e\lambda - c)y^2 - \lambda^2 y^4 \geq 0. \quad (12)$$

Substituting that  $\lambda = \frac{h}{e}$  and  $y = ev$  into (11), and then dividing the both sides of the resulting inequality by  $e$ , we get

$$\frac{c}{2h} + hv^2 - 1 + bv < -\alpha \sqrt{\frac{2e + 1}{e^2} + (2h - c)v^2 - h^2 v^4}.$$

Furthermore, by setting  $e = \infty$ , this inequality is reduced to

$$\frac{c}{2h} + hv^2 - 1 + bv \leq -\alpha \sqrt{(2h - c)v^2 - h^2 v^4}. \quad (13)$$

Similarly, (12) is also reduced to that

$$(2h - c) - h^2v^2 \geq 0. \quad (14)$$

Therefore, from the continuity arguments it is sufficient to show that (13) holds without the equality under (14). Multiplying the both sides of (13) by  $h$  and substituting  $hv = \sqrt{2h - cs}$  for  $-1 \leq s \leq 0$  into the resulting inequality, we shall obtain as our aiming inequality

$$\frac{c}{2} + (2h - c)s^2 - h + b\sqrt{2h - cs} < -\alpha(2h - c)\sqrt{s^2 - s^4}. \quad (15)$$

Furthermore, substituting  $h = \frac{b^2}{4} + \frac{c}{2}$  into (15), and then dividing the both sides of the resulting inequality by  $\frac{b^2}{2}$ , we get

$$g(s) < \frac{1}{2} \quad \text{for } -1 \leq s \leq 0, \quad (16)$$

where  $g(s) = s^2 + \sqrt{2}s + \alpha\sqrt{s^2 - s^4}$ . In the case where  $0 < \alpha < 1$ , (16) clearly holds. In fact, since  $s^2 + \sqrt{2}s \leq 0$  and  $\sqrt{s^2 - s^4} \leq \frac{1}{2}$  for  $-1 \leq s \leq 0$ , it follows that  $g(s) < \frac{\alpha}{2} < \frac{1}{2}$ . Next we shall treat the remaining case where  $1 \leq \alpha < 2$ . We find

$$g'(s) = (\sqrt{2}s + 1) \left\{ \sqrt{2} + \frac{\alpha(\sqrt{2}s - 1)}{\sqrt{1 - s^2}} \right\},$$

and that the equation :  $g'(s) = 0$  has at most two solutions in  $[-1, 0]$  such that  $s_1 = -\frac{1}{\sqrt{2}}$  and  $s_2 = \frac{\sqrt{2}\alpha^2 - \sqrt{2\alpha^2 + 4}}{2(\alpha^2 + 1)}$ . Since  $g''(s_1) = 2(1 - 2\alpha) < 0$ ,  $g(s)$  takes a local maximal value for  $s = s_1$  such that  $g(s_1) = \frac{1}{2}(\alpha - 1) < \frac{1}{2}$ . Furthermore, since  $g(0) = 0$  and  $g'(-1) = +\infty$ ,  $g(s_1)$  is the maximal value on  $[-1, 0]$ . These conclude our assertion.

Finally we shall treat (iv), and first the case where  $P \in C_1$ , while the remaining case where  $P \in \cup_{i=2}^4 C_i$ , may be treated in the same way. Because of continuity, we may assume that  $(x(t), y(t), z(t))$  remains in a small neighbourhood of  $C_1$  for  $0 \leq t \leq \varepsilon$  and for a small positive number  $\varepsilon$ . Since  $P \in S_1$ , it follows from the argument about (ii) that  $y(t) > 0$  and  $z(t) < e$  for  $0 < t < \varepsilon$ . Moreover, since  $P \in S_5$ , from the argument about (i),  $(x(t), y(t), z(t)) \in D^o$  for  $0 < t < \varepsilon$ . Above all  $(x(t), y(t), z(t)) \in D_\varepsilon^o$  for  $0 < t < \varepsilon$ . The proof of Theorem 1 is completed.

**Remark 1**

The assumption for  $g(x)$  to be odd is only used to short the argument of the proof, and  $e_0$  may depend on  $l$  and  $m$  such that  $g(x)x > 0$  for  $|x| \geq l$  and  $m = \max\{|g(x)x|; |x| \leq l\}$ .



Next we shall prove Corollary 1 by the same method as above. The only different parts from the above are that

$$\dot{W}(t) = -2b\{(x - y)^2 + \frac{1}{a}xg(x)\} + \frac{2b}{a}p(t)x + 2q(t)y$$

for  $x = x(t)$  and  $y = y(t)$  and that (8) is replaced by

$$\frac{c}{2\lambda} < b(x - y) + z + q(t). \quad (17)$$

In order to prove that  $\dot{W}(t) < 0$  for  $P = (x, y, z)$  in the case (i), it is sufficient to show that

$$(x - y)^2 + \frac{1}{a}xg(x) > K(|x| + |y|) \quad (18)$$

where  $K = \sup\{\frac{1}{a}|p(t)| + \frac{1}{b}|q(t)|; t \in R\}$ . Since  $xg(x)$  is almost quadratic for large  $|x|$  by our assumption, we may see that (18) holds for  $x^2 + y^2 \geq N^2$  and for large positive number  $N$ . Moreover, since we know that  $\frac{c}{2\lambda} < b(x - y) + z$  for large  $|x| + |y| + |z|$ , and since the term  $b(x - y) + z$  is linear with respect to  $x, y$  and  $z$ , (17) holds for large  $|x| + |y| + |z|$  compared with  $\sup\{|q(t)|; t \in R\}$ . These comments guarantee that the conclusion of the boundedness of solutions holds for (2).

Now we shall consider the Poincare mapping  $\varphi(u, v, w) : (u, v, w) \rightarrow (x(2\pi, u, v, w), y(2\pi, u, v, w), z(2\pi, u, v, w))$ , where  $(x(t, u, v, w), y(t, u, v, w), z(t, u, v, w))$  is the solution of (2) through  $(u, v, w)$  for  $t = 0$ . Since  $\varphi(D_e) \subset D_e$  for a positive number  $e$ , it follows from Brouwer's fixed point theorem that  $\varphi$  has one fixed point in  $D_e$ , which implies the existence of  $2\pi$ -periodic solutions. The proof of Corollary 1 is completed.

### 3 The proof of Theorem 2 and examples

For (1), the origin is an equilibrium point ; we shall find that the variational system of (1) with respect to origin is the following :

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta} \end{pmatrix} = \begin{pmatrix} k - a & a & 0 \\ b & -b & 1 \\ 0 & -c & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \quad (19)$$

where  $k = -g'(0)$ , and that the characteristic equation of the coefficient matrix of this system is the following :

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \quad (20)$$

where  $a_1 = a + b - k$ ,  $a_2 = c - bk$  and  $a_3 = (a - k)c$ , and hence

$$a_1a_2 - a_3 = b\{k^2 - (a + b)k + c\}.$$

Now we shall prove (i). A sufficient condition for the asymptotic stability of the origin is that every solution of (20) has negative real parts, which is equivalent to that  $a_1 > 0$ ,  $a_3 > 0$  and  $a_1 a_2 - a_3 > 0$  by Routh-Hurwitz theorem [3, p.158]. These conditions are guaranteed by our assumption in (i).

We shall prove (ii). Under our conditions, (20) has pure imaginary solutions  $\pm\beta i$ , where  $\beta = \sqrt{c - bk_0}$  and  $i = \sqrt{-1}$ . Considering the solution of (20) in term of  $k$  as parameter, say  $\lambda(k)$ , where  $\lambda(k_0) = \beta i$ , we can get that

$$\frac{\partial \lambda}{\partial k} = \frac{\lambda^2 + b\lambda + c}{3\lambda^2 + 2a_1\lambda + a_2},$$

whose real part is not zero, in fact the absolute value of this real part is equal to  $\frac{b\sqrt{(a+b)^2 - 4c}}{2(a_2 + a_1^2)}$ . Therefore the conclusion follows from Hopf bifurcation theorem [1].

We shall prove (iii) by applying [2, p.330], and hence it is sufficient to show that (20) has one negative solution and two solutions with positive real parts. From the term of Routh-Hurwitz theorem, a sufficient condition is that  $a_1 > 0$ ,  $a_3 > 0$  and  $a_3 > a_1 a_2$ , which are guaranteed by our assumption in (iii).

We shall prove (iv). It is sufficient to show that (20) has one positive solution and two solutions with negative real parts. One sufficient condition from the term of Routh-Hurwitz theorem is that  $a_1 < 0$ ,  $a_3 < 0$  and  $a_1 a_2 > a_3$ , and hence we require that  $k > a + b$ . On the other hand, another sufficient condition is that  $a_1 \geq 0$  and  $a_3 < 0$ , which is equal to that  $a < k \leq a + b$ . Above all combining these two sufficient conditions, we get our claim in (iv).

## Remark 2

If  $-g'(x) < a + b$  for  $x \in R$ , then the mapping :  $(x(0), y(0), z(0)) \rightarrow (x(t), y(t), z(t))$  ( $t > 0$ ) is measure-decreasing, and hence the maximal compact, connected, invariant set is a null set with respect to Lebesgue measure.

We shall illustrate the form of the attractor for (1) by the following examples.

## Example 1

We shall consider the deformation of the attractor around origin in term of  $k = -g'(0)$ . As the first case we shall assume that  $\frac{ab}{4} < c < ab$  and  $a > b$ . Setting  $k_- = \frac{a+b-\sqrt{D}}{2}$  and  $k_+ = \frac{a+b+\sqrt{D}}{2}$ , where  $D = (a+b)^2 - 4c > 0$ , we can derive that  $k_- < a < k_+$  and  $k_- < \frac{c}{b}$ . Because of (ii) of Theorem 2 there arises Hopf's bifurcation for  $k = k_-$ , and hence (1) has nonconstant periodic solutions for some  $k$  in a neighbourhood of  $k_-$ . Now if  $k < k_-$ , then the origin is asymptotically stable by (i) of Theorem 2, if  $k_- < k < a$ , then it has the 2-dimensional bounded, invariant, unstable manifold around itself

by (iii) of Theorem 2 and if  $k > a$ , then it has the 1-dimensional bounded, invariant, unstable manifold around itself.

Secondly we shall treat the case where  $ab < c < \frac{(a+b)^2}{4}$  and  $a > b$ . We can derive that  $k_+ < a$  and  $k_+ < \frac{c}{b}$ , the latter of which implies that Hopf's bifurcation arises for the both of  $k = k_-$  and  $k = k_+$ , and hence (1) has nonconstant periodic solutions for some  $k$  in neighbourhoods of  $k_-$  and  $k_+$ , respectively. If  $k < k_-$ , then the origin is asymptotically stable, if  $k_- < k < k_+$ , it has the 2-dimensional bounded, invariant, unstable manifold around itself, if  $k_+ < k < a$ , then it is again asymptotically stable, and if  $k > a$ , then it has the 1-dimensional bounded, invariant, unstable manifold around itself.

### Example 2

In (1), we shall treat the case where  $a = 2$ ,  $b = \frac{1}{3}$ ,  $c = 1$  and  $g(x) = -2.3x + x^3$ . Then (1) has three equilibrium points  $O(0, 0, 0)$ ,  $P_1\left(\sqrt{0.3}, 0, -\frac{1}{\sqrt{30}}\right)$  and  $P_2\left(-\sqrt{0.3}, 0, \frac{1}{\sqrt{30}}\right)$ , and we can verify that solutions of (20) with respect to  $O$  consists of one positive number and two numbers with negative real parts and that solutions of (20) with respect to  $P_1$  and  $P_2$  are  $-1$  and  $\frac{1}{30} \pm i\frac{\sqrt{539}}{30}$ , where  $i = \sqrt{-1}$ . Therefore  $O$  has a 1-dimensional, bounded, invariant, unstable manifold around itself, and  $P_1$  and  $P_2$  has 2-dimensional, bounded, unstable manifold around themselves, respectively, on which solutions leaves  $P_1$  and  $P_2$  rotating around themselves as  $t$  increases, respectively. Moreover, since  $-g'(x) = 2.3 - 3x^2 < a + b = \frac{7}{3}$  for  $x \in R$ , the maximal compact, invariant set is a null set.

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