# Stability of facets of self-similar motion of a crystal

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**Abstract**

We are concerned with a quasi-steady Stefan type problem with Gibbs-Thomson relation and the mobility term which is a model for a crystal growing from supersaturated vapor. The evolving crystal and the Wulff shape of the interfacial energy are assumed to be (right-circular) cylinders.

In pattern formation deciding what are the conditions which guarantee that the speed in the normal direction is constant over each facet, so that the facet does not break, is an important question. We formulate such a condition with an aid of a convex variational problem with a convex obstacle type constraint.

We derive necessary and sufficient conditions for the non-breaking of facets in terms of the size and the supersaturation at space infinity when the motion is self-similar.

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## 1 Introduction

We are motivated by physical experiments which show that one can grow large, regular, elongated ice crystals, (see [GoG]). In our earlier work we presented a model of evolution of crystals growing from vapor and we showed existence of solutions, (see [GR1]). We also studied some properties of solutions, see [GR2], [GR3]. In this paper we address the issue of stability of facets. The question is: are the velocities of facets in the normal direction really constant over each facet? A facet must break if its velocity is not a constant. We use a variational principle to formulate these conditions. We present a general method and give sharp estimates which are valid only for the self-similar evolution. In order to provide details, we first present the evolution model (a Stefan type problem). We stress that we deal only with special interfaces, namely right-circular cylinders that serve as an approximation to real crystals.

In [GR1] we justified the model presented below which dates back to the work of Seeger, (see [Se]) on planar polygonal crystals. That is, we assumed that the process is so slow that the quasi-steady approximation of the diffusion equation for mass transport is justified. In other words the supersaturation $\sigma$ satisfies the equation

$$\Delta\sigma = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \Omega(t) \tag{1.1}$$

outside crystal $\Omega(t)$. It is also physically reasonable to assume that $\sigma$ has a specific value at infinity, *i.e.*

$$\lim_{|x|\to\infty} \sigma(t,x) = \sigma^\infty. \tag{1.2}$$

The velocity of the growing crystal is determined by the normal derivative of $\sigma$ at the surface,

$$\frac{\partial\sigma}{\partial\mathbf{n}} = V \quad \text{on} \quad \partial\Omega(t) \tag{1.3}$$

where $\mathbf{n}$ is the outer normal. This equation expresses the mass conservation.

The value of $\sigma$ at the surface is coupled to the surface velocity and its curvature through the Gibbs-Thomson relation

$$-\sigma = -\mathrm{div}\,\xi - \beta V, \tag{1.4}$$

where $\beta$ is the kinetic coefficient and $\xi$ is a Cahn-Hoffman vector. A relation of this sort is quite natural, see Gurtin [G, Chapter 8].

Let us comment on the Cahn-Hoffman vector $\xi$ appearing in (1.4). For a smooth surface $S$ and a smooth energy density function $\gamma$ we have

$$\xi(x) = (\nabla\gamma)(\mathbf{n}(x)),$$

which is a well-defined quantity. However, it is our intention to consider energy density functions $\gamma$ which are only Lipschitz continuous and (positively) 1-homogeneous, and surfaces $S$ with edges. In that case some care is necessary while defining $\xi$, (see [GPR] for a related study). We shall assume that $\gamma$ is convex. For a convex function $\gamma$ its subdifferential $\partial\gamma$ is a well-defined nonempty convex set. But in general $\partial\gamma$ is not a singleton, thus we have only that

$$\xi(x) \in (\partial\gamma)(\mathbf{n}(x)) \tag{1.5}$$

which should be considered as an additional constraint for (1.1)–(1.4).

This condition gives us, in fact, some freedom of choice of $\xi$. However, it turns out that the averages of $\mathrm{div}_S\xi$ on each facet $S_i$ (*i.e.* the top, the bottom and the lateral part of $\partial\Omega$, the notation will be explained in §2) are independent of the choice of $\xi$,

$$\int_{S_i(t)} \mathrm{div}_S\xi(t,x)\, d\mathcal{H}^2(x) = -\kappa_i(t)\mathcal{H}^2(S_i(t)).$$

Farther, we shall write $|S_i|$ for $\mathcal{H}^2(S_i)$. This is explained in Propositions 2.1 and 2.2, here $\kappa_i$ denotes the crystalline curvature of $S_i$. Thus, we may consider system (1.1)–(1.3) with condition (1.4) replaced with its average

$$\int_{S_i(t)} \sigma(t,x)\, d\mathcal{H}^2(x) = (-\kappa_i(t) + \beta_i V_i(t))|S_i(t)|, \tag{1.6}$$

where $V(t)|_{S_i(t)} = \text{const}(t)$ is enforced not deduced, provided that $\beta_i$ is the constant value of $\beta$ on $S_i$. In fact, we established in [GR1] existence and uniqueness of (local-in-time) solutions to (1.1)–(1.3), (1.6) augmented with initial data, $\Omega(0) = \Omega_0$.

We show here that these two kinds of evolution are equivalent provided that we can find for the latter a proper selection of the Cahn-Hoffmann vector $\xi$, (*i.e.* $\xi$ satisfies (1.5)), for which

$$\sigma - \text{div}_S \xi = \text{const} \tag{1.7}$$

holds; this is the content of Theorem 2.3.

We shall say that a facet $S_i$ is *stable* or *does not break*, as long as (1.7) holds on it for $\xi$ satisfying (1.5). We shall make it more precise in §4.2. Finding conditions guaranteeing stability of facets is the main purpose of this paper. For example we prove for a self-similar evolution, see [GR3], that a facet is stable for a small crystal but not for a large one if the crystal is growing. In the case of shrinking motion, the facets are stable if the evolving crystal is near the equilibrium or it is very small. There might be an intermediate size for which the facets are unstable. This will be discussed in §4.

In order to achieve our goal we need a rule for the selection of $\xi$. Following the idea of [GG1] and [FG] we propose a variational principle. That is, the correct choice of $\xi$ should be a minimizer of

$$\min_{\xi \in \mathcal{D}_i} \int_{S_i} \frac{1}{2} \, |\text{div}_S \xi - \sigma|^2 \tag{1.8}$$

where the set $\mathcal{D}_i$ is properly chosen, in particular it incorporates the constraints. (The set $\mathcal{D}_i$ is a convex set). This will be explained in detail in §2.2 and §3. There, in §3, we will present general consequences of this variational principle which will permit us to select $\xi$ enjoying the symmetries of the problem.

The main effort of studying stability is presented in §4. We comment first on relations between minimizers of (1.8) and solutions to (1.7). Then, we present general necessary and sufficient conditions for (1.7) to hold, provided that $\sigma$ is given. We do not have enough knowledge about $\sigma$ for the coupled problem (1.1)–(1.5) unless the evolution is self-similar, [GR3]. We are able to give a necessary and sufficient condition for stability of facets for a self-similar motion. It is expressed in terms of the size of the crystal and the supersaturation at space infinity. The cases of growing and shrinking crystals are different, that is why we consider them separately. The main results are Theorems 4.8 and Theorem 4.14 which state the non-breaking conditions in terms of the scale factor $a$, supersaturation at infinity $\sigma^\infty$ and a number of universal constants depending only on the Wulff shape.

Stability of facets for a singular interfacial energy $\gamma$ recently attracts several mathematicians. For the interface controlled model (*i.e.* the $\sigma$ in (1.4) is given) it is known that facets are always stable for a planar evolution, provided that $\sigma$ is spatially constant, see [FG], [GG1], [GG2]. However, for surface evolution problems in three-dimensional space a facet may break even if $\sigma$ is a constant (see e.g. [BNP1], [BNP2], [BNP3]). The bibliography of [GG2] includes several related references, that is why we do not repeat them here. Our result is the first one for a coupled system (1.1)–(1.5) discussing stability of facets. In fact, we give a necessary condition on the size of the evolving crystal for stability of facets of self-similar solutions.

Stability of facets is also an important problem for the theory of crystal growth. In fact, Kuroda, Irisawa and Ookawa, [KIO], discuss this problem from a physical point of view.

However, their model for polyhedral crystals is (1.1)–(1.3) accompanied with

$$\sigma = \beta V$$

instead of (1.4). They explain stability of facets by allowing the $\sigma$-dependence of $\beta$. However, in case of a small growing crystal, the stability in question is not well-explained by their theory, while such a phenomenon is covered by our result. They claim that the facets of a medium size growing crystal are also stable, while other sizes are unstable which is inconsistent with our result. Discrepancy is partly due to the presence of the div$\xi$ term and no $\sigma$-dependence of $\beta$.

As far as the authors know, there is no mathematical literature on one-phase Stefan type problems with (1.4) even for smooth $\gamma$, (including both div$\xi$ and $\beta V$ terms). The bibliography of [GR1], [GR3] includes several related works, so that we do not repeat them here.

On the other hand there is a literature on numerical treatment of the stability issue for a Stefan type problems with (1.4). One of the first examples is the paper by Roosen and Taylor, [RT]. The authors concentrate on a related, but different question of shattering facets during growth. Their paper, however, does not present stability condition similar to our variational principle. We shall not pursue the direction of numerical experiments.

A separate question is the behavior of the system at the onset of instability. It is an interesting issue which will be studied in a forthcoming paper.

# 2 Setting up the problem

In this section we set up the problem. Subsequently we make a reduction of the system to a known set of equations. We also present some structure of solutions and exhibit a scaling law.

## 2.1 Preliminaries

Our evolving crystal $\Omega(t)$ is assumed to be, as it is done in the physics literature, see [Ne], [YSF], a straight cylinder,

$$\Omega(t) = \{(x, x_2, x_3) \in \mathbb{R}^3, x_1^2 + x_2^2 \leq R^2(t), |x_3| \leq L(t)\}.$$

In other words, we only have to know $R(t)$ and $L(t)$ to describe its evolution. We shall call by *facets* the following subsets of $\partial\Omega(t)$

$$\begin{array}{l} S_\Lambda = \{x \in \partial\Omega(t) : \quad x_1^2 + x_2^2 = R^2\}, \\ S_T = \{x \in \partial\Omega(t) : \quad x_3 = L\}, \quad S_B = \{x \in \partial\Omega(t) : x_3 = -L\}. \end{array}$$

We shall call them the lateral side, top and bottom. We also define the set of indices $I = \{\Lambda, T, B\}$. We shall specify the initial data $\Omega(0) = \Omega_0$. We denote by $V_i$ the velocity of facet $S_i$, $i \in I$, in the direction of $\mathbf{n}$, the outer normal to $\partial\Omega(t)$.

We explicitly assume that, at time $t$, $\sigma(t)$ enjoys the symmetry of $\Omega(t)$, *i.e.* $\sigma$ is axisymmetric and symmetric with respect to the plane $x_3 = 0$:

$$\sigma(t) = \bar{\sigma}(t, \sqrt{x_1^2 + x_2^2}, |x_3|).$$

We want to consider a surface energy density $\gamma$ which is consistent with our $\Omega(t)$. To be specific we take

$$\gamma(x_1, x_2, x_3) = r\gamma_\Lambda + |x_3|\gamma_{TB}, \quad \gamma_\Lambda, \gamma_{TB} > 0, \tag{2.1}$$

where $r^2 = x_1^2 + x_2^2$ and $\gamma_\Lambda$, $\gamma_{TB}$ are positive constants. Hence, its Frank diagram $F_\gamma$ defined as

$$F_\gamma = \{p \in \mathbb{R}^3 : \ \gamma(p) \le 1\}$$

consists of two straight cones with common base, which is the disk $\{(x_1, x_2, 0), x_1^2 + x_2^2 \le 1/\gamma_\Lambda\}$, same height and the vertices at

$$(0, 0, \pm 1/\gamma_{TB}).$$

Now, the Wulff shape of $\gamma$ is defined by

$$W_\gamma = \{x \in \mathbb{R}^3 : \ \forall \mathbf{n} \in \mathbb{R}^3, |\mathbf{n}| = 1, \quad x \cdot \mathbf{n} \le \gamma(\mathbf{n})\}.$$

In our setting $W_\gamma$ is a cylinder of radius $R$ equal to $\gamma_\Lambda$ and half-height $L$ equal to $\gamma_{TB}$. Hence, all cylinders like $\Omega(t)$ above are *admissible*, in the sense that normal $\mathbf{n}$ to the top facet of $\Omega(t)$ (respectively: bottom, lateral surface of $\Omega(t)$) is the normal to top facet of $W_\gamma$ (respectively: bottom, lateral surface of $W_\gamma$).

The Gibbs-Thomson relation is expressed by (1.4) and (1.5). In these formulas the argument of $\gamma$ is the outer normal to $\partial\Omega$. However, we prefer to assume that $\gamma : \mathbb{R}^3 \to \mathbb{R}$ is 1-homogenous, convex and Lipschitz continuous. There is just one such extension. As we mentioned earlier we have some freedom while choosing $\xi$. Thus, we should be a bit careful about the meaning of surface divergence $\mathrm{div}_S \xi$. At this point we recall that for $\xi$ defined in $U$ a neighborhood of $S_i$

$$\mathrm{div}_S \xi = \mathrm{trace}\, (\mathrm{Id} - \mathbf{n} \otimes \mathbf{n})\nabla\xi, \quad \text{for } x \in S_i, \tag{2.2}$$

where $\mathbf{n}$ is an outer normal to the surface. This definition is independent of the extension of $\xi$ to $U$ (see [Si], [GPR]).

Due to convexity of $\gamma$ and its lack of smoothness only the subdifferential $\partial\gamma(\mathbf{n})$ is well-defined for each $\mathbf{n}$ normal to $\partial\Omega(t)$. Thus, a Cahn-Hoffman vector $x \to \xi(x)$ is just a section of the subdifferential.

## 2.2 Crystalline curvature and $\mathrm{div}_S \xi$

We will specify now the minimal assumptions on $\xi$, which enable us to proceed. Finer properties will be deduced in a later section. Namely we require that

$$\xi(t, \cdot)|_{S_i(t)} \in L^2(S_i(t)) \quad \text{and} \quad \mathrm{div}_S(\xi(t, \cdot)|_{S_i}(t)) \in L^2(S_i(t)), \qquad i \in \{T, B, \Lambda\}, \tag{2.3}$$

and in addition

$$\xi \in \partial\gamma(\mathbf{n}), \quad \mathcal{H}^2\text{-a.e.} \tag{2.4}$$

Since the subdifferential is a bounded set, one may have an impression that is it more appropriate to consider

$$\xi(t, \cdot)|_{S_i(t)} \in L^\infty(S_i(t)) \quad \text{and} \quad \mathrm{div}_S(\xi(t, \cdot)|_{S_i}(t)) \in L^\infty(S_i(t)).$$

This approach is indeed followed by Belletini, Novaga and Paolini in [BNP1–3]. However, we depend so much on the $L^2$ topology, that we prefer the function spaces in (2.3). At the end however, $\xi$ will turn out to be quite smooth.

We will now see what are the consequences of (2.3) and (2.4) for the existence of traces. At this point we note that if $x \in S_i \cap S_j, i \neq j$, then

$$\partial\gamma(\mathbf{n}_\Lambda(x)) \cap \partial\gamma(\mathbf{n}_i(x)) = \{\gamma_{BT}\mathbf{n}_i + \gamma_\Lambda \mathbf{n}_\Lambda\}, \qquad i = T, B. \tag{2.5}$$

Since $S_T$, $S_B$ are balls in a plane, then (2.3) implies that only the trace of the normal component of $\xi$ at $\partial S_i$, $i = T, B$, is well-defined. More precisely, if $\nu_i$ is the vector field tangent to $TS_i$ at $\partial S_i$ and normal to $\partial S_i$, then $\xi \cdot \nu_i$ is defined as an element of $W^{-1/2,2}(\partial S_i)$. Basically, this fact is well-established. For more details see [FM] and [T]. We conclude that (2.5) implies that

$$\xi \cdot \nu_i = \gamma_\Lambda \qquad \text{in } W^{-1/2,2}(\partial S_i), \quad i = T, B. \tag{2.6}$$

On $S_\Lambda$ the structure of $\xi$ is simpler. Namely, the condition $\xi \in \partial\gamma(\mathbf{n}_\Lambda)$ implies that

$$\xi = \gamma_\Lambda \mathbf{n}_\Lambda + \xi_3 \mathbf{n}_T,$$

where $\xi_3 \in [-\gamma_{TB}, \gamma_{TB}]$. Thus, (2.2) and (2.3) imply that

$$\text{div}_S \xi = \frac{\gamma_\Lambda}{R} + \frac{\partial \xi_3}{\partial x_3} \in L^2(S_i).$$

Hence we infer that

$$\xi \cdot \nu_\Lambda = \gamma_{TB} \qquad \text{in } L^2(\partial S_\Lambda). \tag{2.7}$$

By definition a solution to (1.1)–(1.5) is a triple $(\Omega(t), \sigma(t), \xi(t))$. However, at the moment we abstain from making the notion of solution precise apart from the requirement that (1.1)–(1.5) hold for all $t \geq 0$.

We are interested in such kinds of evolution that initial cylinder $\Omega(0)$ retains its form all time instances, that is we want that $\Omega(t)$ be another cylinder, possibly of different aspect ratio $L(t)/R(t)$. This is possible only if

$$V_i(t) \text{ does not depend upon on the point } x \in S_i,\ i \in I. \tag{2.8}$$

But of course $V_\Lambda$ may be different from $V_T$. However, $V_T = V_B$ due to assumed symmetry with respect to the plane $\{x_3 = 0\}$.

We would like to reduce the number of unknown in system (1.1)–(1.5). This is in fact possible for evolution conforming to (2.8) due to the following fact. It was stated as [GR1, Proposition 1], however the assumption (2.4) was missing. Here we provide the correct proof.

**Proposition 2.1.** *Let us suppose that $\gamma$ is defined by (2.1), $\Omega$ is an admissible cylinder. Then:*
(1) *There exists $\xi_0 \in C^\infty(\mathbb{R}^3; \mathbb{R}^3)$ satisfying (2.4), (2.6), (2.7) and such that*

$$\text{div}_S \xi_0|_{S_i} = -\kappa_i, \quad i \in \{T, \Lambda, B\}. \tag{2.9}$$

*Namely, $\xi_0$ is given by the formula*

$$\xi_0 = \left( \frac{\gamma_\Lambda}{R} x_1, \frac{\gamma_\Lambda}{R} x_2, \frac{\gamma_{TB}}{L} x_3 \right). \tag{2.10}$$

(2) *If $\xi$ satisfies (2.3), (2.4), (2.6) and (2.7), then*

$$\int_{S_i} \mathrm{div}_S \xi \, d\mathcal{H}^2 = \int_{S_i} \mathrm{div}_S \xi_0 \, d\mathcal{H}^2 = -\kappa_i |S_i|,$$

*where the numbers $\kappa_T = \kappa_B, \kappa_\Lambda$ are called* crystalline curvatures *of the top, bottom, and the lateral surfaces, (cf. [GR1, Proposition 1]) and*

$$\kappa_\Lambda = -\frac{\gamma_\Lambda}{R} - \frac{\gamma_{TB}}{L}, \qquad \kappa_T = -2\frac{\gamma_\Lambda}{R}. \tag{2.11}$$

**Remark.** We use the symbol $\mathcal{H}^k$ to denote the $k$-dimensional Hausdorff measure, $k = 1, 2$. However, for the sake of simplicity we shall write $|S_i|$ instead of $\mathcal{H}^2(S_i)$.

*Proof of Proposition 2.1.* (1) Obviously, (2.4), (2.6), (2.7) are fulfilled by $\xi_0$. It is a matter of easy calculations using (2.2) to check that (2.9) holds.

(2) Let us consider first the case of $S_T$, $S_B$. We denote by $\nu$ the outer unit normal to $\partial S_i$ in the tangent plane $TS_i$ of $S_i$, $i = T, B$. Since $\xi \cdot \nu$ is just an element of $W^{-1/2,2}(\partial S_i)$, it may be evaluated on $1 \in W^{1/2,2}(\partial S_i)$. This yields,

$$\int_{S_i} \mathrm{div}_S \xi \, d\mathcal{H}^2 = \int_{\partial S_i} \xi \cdot \nu \, d\mathcal{H}^1.$$

Subsequently, due to (2.7) and part (1) of this proposition we are in a position to conclude

$$\int_{S_i} \mathrm{div}_S \xi \, d\mathcal{H}^2 = \int_{\partial S_i} \xi_0 \cdot \nu \, d\mathcal{H}^1 = \int_{S_i} \mathrm{div}_S \xi_0 \, d\mathcal{H}^2 = -\kappa_i |S_i|.$$

For $S_\Lambda$ the trace $\xi \cdot \nu$ is a an element of $L^2(\partial S_\Lambda)$. However, the main point is we have to use the Gauss formula for surface divergence, see [Si],

$$\int_{S_\Lambda} \mathrm{div}_S \xi \, d\mathcal{H}^2 = \int_{\partial S_\Lambda} \xi \cdot \nu \, d\mathcal{H}^1 + \int_{S_\Lambda} H \cdot \xi \, d\mathcal{H}^2, \tag{2.12}$$

where $H$ denotes the mean curvature vector of the surface. Due to assumption (2.7)

$$\int_{\partial S_\Lambda} \xi \cdot \nu \, d\mathcal{H}^1 = \int_{\partial S_\Lambda} \xi_0 \cdot \nu \, d\mathcal{H}^1.$$

Moreover, due to the structure of $\partial\gamma(\mathbf{n}_\Lambda)$ we see that $H \cdot \xi = H \cdot \xi_0$ holds $\mathcal{H}^2$ a.e. on $S_\Lambda$. Hence, our claim follows. $\square$

This fact allows us to simplify equations (1.1)–(1.5) if the following condition holds

$$(\sigma - \mathrm{div}_S \xi)|_{S_i} = \text{ const}_i. \tag{2.13}$$

Namely, integrating (1.4) over $S_i$ yields

$$\begin{aligned} \int_{S_i(t)} \sigma(t,x) \, d\mathcal{H}^2(x) + \kappa_i(t)|S_i(t)| \qquad &= \int_{S_i(t)} (\sigma(t,x) - \mathrm{div}_S \xi(t,x)) \, d\mathcal{H}^2(x) \\ &= \int_{S_i(t)} \beta_i V_i(t) \, d\mathcal{H}^2(x) = \beta_i V_i(t) |S_i(t)|, \end{aligned}$$

*i.e.* we arrive at

$$-\int_{S_i(t)} \sigma(t,x) \, d\mathcal{H}^2(x) = (\kappa_i(t) - \beta_i V_i(t))|S_i(t)|. \tag{2.14}$$

In fact the system (1.1)–(1.3), (2.14) augmented with initial cylinder $\Omega(0) = \Omega_0$ has a unique local in time solution $(R, L, \sigma)$:

**Proposition 2.2.** ([GR1, Theorem 1]) *There exists $(R, L, \sigma)$ a unique weak solution to*

$$\Delta\sigma(t) = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \Omega(t), \qquad \lim_{|x|\to+\infty} \sigma(t,x) = \sigma^\infty \tag{2.15}$$

$$\frac{\partial\sigma(t)}{\partial\mathbf{n}} = V(t) \quad \text{on} \quad \partial\Omega(t) \tag{2.16}$$

$$-\!\!\!\!\!\!\int_{S_i(t)} \sigma(t)\, d\mathcal{H}^2 = (\kappa_i(t) - \beta_i V_i(t))|S_i(t)| \tag{2.17}$$

*augmented with an initial condition $\Omega(0) = \Omega_0$, where $\Omega_0$ is an admissible cylinder. Moreover,*

$$R, L \in C^{1,1}([0,T))$$
$$\nabla\sigma \in C^{0,1}([0,T); L^2(\mathbb{R}^3 \setminus \Omega(t))).$$

In order to make the notation more concise we shall write $(\Omega(t), \sigma(t))$ in place of $(R(t), L(t), \sigma(t))$, (but sometimes we suppress the argument $t$).

Let us suppose now that $(\Omega(t), \sigma(t))$ is a solution to (2.15)–(2.17). In order to obtain a solution to (1.1)–(1.5) we have to select a Cahn-Hoffman vector $\xi$. If we can prove existence of sufficiently regular section

$$\mathbb{R} \times \Omega \ni (t,x) \to \xi(t,x) \in \partial\gamma(\mathbf{n}(x))$$

such that (2.13) holds, then the triple $(\Omega(t), \sigma(t), \xi(t))$ is a solution to (1.1)–(1.5). It is clear that the constants in (2.13) coincide with $\beta_i V_i$.

In this way we showed some sort of equivalence of solutions to (1.1)–(1.5) and (2.15)–(2.17):

**Theorem 2.3.** *(1) Let us suppose that $(\Omega(t), \sigma(t), \xi(t))$ is such that equations (1.1)–(1.5) are satisfied for each $t \geq 0$ and $\nabla\sigma(t) \in L^2(\mathbb{R}^3 \setminus \Omega(t))$, $\Omega(t)$ is an admissible cylinder, $\xi|_{S_i}(t,\cdot) \in L^2(S_i)$, div$_S\xi|_{S_i}(t,\cdot) \in L^2(S_i)$, $i = B, \Lambda, T$ and the initial position $\Omega_0$ of $\Omega(t)$ is given. Finally, we assume that (2.13) holds. Then $(\Omega(t), \sigma(t))$ satisfies (2.15)–(2.17) with crystalline curvatures $\kappa_i$ given by (2.11).*

*(2) Let us suppose that $(\Omega(t), \sigma(t))$ is a solution to (2.15)–(2.17) constructed in Proposition 2.2, i.e. in [GR1, Theorem 1]. If there exists $\xi$ such that $\xi|_{S_i} \in L^2(S_i)$, div$_S\xi|_{S_i}(t,\cdot) \in L^2(S_i)$, $i = T, \Lambda, B$, $\xi(x) \in \partial\gamma(\mathbf{n}(x))$, $\mathcal{H}^2$-a.e., and satisfying (2.13), then the triple $(\Omega(t), \sigma(t), \xi(t))$ is a solution to (1.1)–(1.5).* □

Subsequently we will concentrate on guaranteeing feasibility of construction of $\xi$ fulfilling

$$\xi(t,x) \in \partial\gamma(\mathbf{n}(x)).$$

We will outline general tools. However, sharp non-breaking results will be only for self-similar solutions constructed in [GR3].

Carrying out the above program requires a detailed knowledge of the structure of $\sigma$. This is presented below.

## 2.3 The structure of $\sigma$ and the scalings

In order to present the useful structure of $\sigma$ we have to introduce some additional objects. Namely, we need $f_i$ which is a unique solution to

$$-\Delta f_i \quad = 0, \qquad \text{in } \mathbb{R}^3 \setminus \Omega, \tag{2.18}$$

$$\frac{\partial f_i}{\partial \nu} \quad = \delta_{ij}, \qquad \text{on } S_j, \quad j \in I, \tag{2.19}$$

such that $\nabla f_i \in L^2(\mathbb{R}^3 \setminus \Omega)$ and $\lim_{x\to\infty} f_i(x) = 0$. Here, $\delta_{ij}$ is the Kroneker delta and $\nu$ denotes the inner normal to $\partial\Omega$. For functions $f, g$ such that $\nabla f$, $\nabla g \in L^2(\mathbb{R}^3 \setminus \Omega)$ we also define the following quantities

$$(f, g) := \int_{\mathbb{R}^3\setminus\Omega} \nabla f(x) \cdot \nabla g(x)\, dx, \qquad \|f\|^2 := (f, f).$$

Let us mention that the equation above takes the following weak form

$$\int_{\mathbb{R}^3\setminus\Omega} \nabla f_i(x) \cdot \nabla h(x)\, dx = \int_{S_i} h(x)\, d\mathcal{H}^2(x) \tag{2.20}$$

for all $h$ such that $\nabla h \in L^2(\mathbb{R}^3 \setminus \Omega)$.

We showed in [GR1] that (2.15)–(2.17) can be reduced to a following system of ODE's

$$(\mathcal{A} + \mathcal{D})\mathbf{V} = \mathbf{B} \tag{2.21}$$

where

$$\mathbf{V} = (V_\Lambda, V_T, V_B), \quad \mathbf{B} = (|S_\Lambda|(\sigma^\infty + \kappa_\Lambda), |S_T|(\sigma^\infty + \kappa_T), |S_B|(\sigma^\infty + \kappa_B))$$
$$\mathcal{A} = \{(f_i, f_j)\}_{i,j=\Lambda,T,B}, \quad \mathcal{D} = \text{diag}\,\{\beta_\Lambda|S_\Lambda|, \beta_T|S_T|, \beta_B|S_B|\}.$$

Moreover, $\sigma$ is given by (see [GR1])

$$\sigma(t) = -\sum_{i\in I} V_i(t) f_j(t) + \sigma^\infty. \tag{2.22}$$

The studies of non-breaking of facet in the case of self-similar solutions require clarifying the behavior of our system under scaling of domains. Suppose we define a new variable $y$ by formula

$$y = ax$$

where $a > 0$, thus $\Omega$ is transformed to $a\Omega = \tilde{\Omega}$, $S_i$ goes to $aS_i = \tilde{S}_i$. If $h$ is defined on $\Omega$, then we transform it to $\tilde{h} : \tilde{\Omega} \to \mathbb{R}$, by setting

$$\tilde{h}(y) = h(\frac{y}{a}).$$

We also define

$$f^a(y) = af(\frac{y}{a}). \tag{2.23}$$

We recall from [GR3] the Proposition below clarifying the role of definition of $f_i^a$.

**Proposition 2.4.** [GR3, Proposition 2.2] *Let us suppose that $f_i$ satisfies (2.20) on $\mathbb{R}^3 \setminus \Omega$, then*

$$\int_{\mathbb{R}^3\setminus a\Omega} \nabla_y f_i^a(y) \nabla_y \tilde{h}(y)\, dy = \int_{aS_i} \tilde{h}(y)\, d\mathcal{H}^2(y)$$

*for all $\tilde{h}$ with $\nabla\tilde{h} \in L^2(\mathbb{R}^3 \setminus a\Omega)$.* □

## 2.4 Critical size crystals

In [GR3] we studied self-similar solutions. This task required information about steady states of (2.15)–(2.17). We recall it here since it is useful for our purposes.

**Proposition 2.5.** ([GR3, Proposition 3.1]) *Let us suppose that $\sigma^\infty > 0$ is given and $\gamma$ is defined by (2.1). Then $(\Omega(t), \sigma(t))$ is a stationary solution to (2.15)–(2.17) if and only if*

$$\Omega(0) = aW_\gamma \quad \textit{and} \quad \sigma(t) = \sigma^\infty,$$

*where $a = 2/\sigma^\infty$.* □

**Remark.** According to Theorem 2.3 the stationary states of (2.15)–(2.17) will be also solutions to (1.1)–(1.5), provided that we specify a Cahn-Hoffman vector $\xi$. Actually, we may take $\xi_0$ defined by (2.10) for $\Omega = aW_\gamma$, *i.e.* $(\frac{2}{\sigma^\infty}W_\gamma, \sigma^\infty, \xi_0)$ provides a solution to (1.1)–(1.5).

We may now say that for given $\sigma^\infty$, then $\frac{2}{\sigma^\infty}W_\gamma$ is of *critical size*. We might expect that $\Omega(0)$ (not necessarily a scaled Wulff shape) contained in $\frac{2}{\sigma^\infty}W_\gamma$ will have the tendency to shrink, while those $\Omega$ containing $\frac{2}{\sigma^\infty}W_\gamma$ will grow. We express it below.

**Proposition 2.6.** ([GR3, Proposition 3.2]) *Let us suppose that $\sigma^\infty > 0$ and a solution $(\Omega(t), \sigma(t))$ to (2.15)–(2.17) is given and $\kappa_i(t)$ are crystalline curvature of $S_i(t)$, $i \in I$. We also assume that $V_\Lambda(t) \cdot V_T(t) > 0$.*

*(a) If $\sigma^\infty + \kappa_i(t) > 0$, for all $i \in I$, then $V_i(t) > 0$, for all $i \in I$.*

*(b) If $\sigma^\infty + \kappa_i(t) < 0$, for all $i \in I$, then $V_i(t) < 0$, for all $i \in I$.* □

**Remark.** The condition $V_i \cdot V_j > 0$ looks strange at the first glance, but it may be violated in general. It is nonetheless satisfied for self-similar solutions.

# 3 Cahn-Hoffman vector $\xi$

In this section we will collect facts on $\xi$ which are important in the studies of stability. They will be general conclusions from a variation principle, regularity (2.3), the constraint (2.4) and boundary conditions (2.6), (2.7). Our starting point is the observation that the constraint (1.5), which is of the form,

$$\xi(x) \in \mathcal{S}(x) \subset \mathbb{R}^3 \quad \mathcal{H}^2\text{-a.e.}$$

is invariant with respect to rotations about the vertical axis. That is, if $Q_\alpha \in SO(3)$ is a rotation by the angle $\alpha$, then

$$\mathcal{S}(Q_\alpha x) \subset Q_\alpha \mathcal{S}(x). \tag{3.1}$$

It is easy to see that (3.1) holds for $\mathcal{S}(x) = \partial\gamma(\mathbf{n}_i(x))$, $i \in \{\Lambda, B, T\}$, $x \in \partial\Omega$.

We state here a variational principle, but we will not elaborate upon it until Section 4. We define three functionals

$$\mathcal{E}_i(\xi) = \frac{1}{2}\int_{S_i} |\text{div}_S \xi - \sigma|^2 d\mathcal{H}^2.$$

We postulate that the correct choice of $\xi(t, \cdot)$ is such that it is a solution to the minimization problems,

$$\mathcal{E}_i(\xi) = \min\{\mathcal{E}_i(\zeta) : \zeta \in \mathcal{D}_i\}, \qquad i = \Lambda, T, B. \tag{3.2}$$

where

$$\mathcal{D}_i \quad = \quad \{\xi \in L^2(S_i) : \ \mathrm{div}_S \xi \in L^2(S_i), \ (2.4), (2.6), (2.7) \text{ hold}\}. \tag{3.3}$$

We note that the sets $\mathcal{D}_i$ and $\mathrm{div}_S \mathcal{D}_i$ are closed, convex subsets of $L^2(S_i)$ and $\mathcal{E}_i(\xi)$ is one half of the squared distance from $\sigma \in L^2(S_i)$ to $\mathrm{div}_S \mathcal{D}_i$. Hence, there is a unique $\mathrm{div}_S \xi$ minimizing each $\mathcal{E}_i$. The data of the problem *i.e.* $\sigma$ and $\partial\Omega$ enjoy some symmetries. These two observations combined with the freedom of choosing $\xi$ will lead us to the conclusion that $\xi$ may be selected so that it shares the symmetries of $\partial\Omega$. Moreover, due to fixed boundary conditions on $S_i \cap S_j$, $i \neq j$, we may modify $\xi|_{S_i}$ independently of any modification of $\xi|_{S_j}$.

For the sake of presentation we introduce the notation

$$\mathbf{e}_r = (x_1, x_2, 0)/r, \quad \mathbf{e}_\varphi = (-x_2, x_1, 0)/r,$$

where $r^2 = x_1^2 + x_2^2$. We also denote by $Q_\alpha$ a rotation about the vertical axis by angle $\alpha$.

**Proposition 3.1.** *Let us assume that $\sigma \in L^2(S_i)$ and $\xi \in \mathcal{D}_i$, $i \in I$, is a solution to (3.2). Then:*
*(a) The vector field $\bar{\xi} \in \mathcal{D}_i$ given by the formula*

$$\bar{\xi}(x) = \frac{1}{2\pi} \int_0^{2\pi} Q_{-\alpha} \xi(Q_\alpha x) d\alpha,$$

*is also a minimizer of $\mathcal{E}_i$, $i = T, \Lambda, B$. Moreover, $\bar{\xi}$ is rotationally invariant i.e. for any $Q_\alpha$, $\alpha \in (0, 2\pi)$*

$$Q_{-\alpha} \bar{\xi}(Q_\alpha x) = \bar{\xi}(x) \tag{3.4}$$

*and*

$$\mathrm{div}_S \bar{\xi} = \mathrm{div}_S \xi.$$

*(b) The vector field $\tilde{\xi} \in \mathcal{D}_i$ given by the formula*

$$\tilde{\xi}(x) = \frac{1}{2}(\xi(x_1, x_2, -x_3) + \xi(x_1, x_2, x_3))$$

*is also a minimizer of $\mathcal{E}_i$, $i = T, \Lambda, B$. It satisfies*

$$\tilde{\xi}(x_1, x_2, -x_3) = \tilde{\xi}(x_1, x_2, x_3)$$

*and*

$$\mathrm{div}_S \tilde{\xi} = \mathrm{div}_S \xi.$$

*Proof.* **(a)** Let us notice first that if $\xi$ is a minimizer then $Q_{-\alpha} \circ \xi \circ Q_\alpha$ is in $\mathcal{D}_i$. This is so due to invariance of the boundary condition with respect to rotations about the vertical axis expressed in (3.1).

We claim that $Q_{-\alpha} \circ \xi \circ Q_\alpha$ is a minimizer. Indeed, by formula (2.2) we see that

$$\mathrm{div}_S Q_{-\alpha} \circ \xi \circ Q_\alpha(x) = \mathrm{div}_S \xi(y)|_{y = Q_\alpha x}.$$

Thus, due to rotational symmetry of $\sigma$ and $S_i$

$$\begin{aligned} \mathcal{E}_i(Q_{-\alpha} \circ \xi \circ Q_\alpha) &= \frac{1}{2} \int_{Q_\alpha S_i} |\mathrm{div}_S \xi(Q_\alpha x) - \sigma(Q_\alpha x)|^2 \, d\mathcal{H}^2(x) \\ &= \frac{1}{2} \int_{S_i} |\mathrm{div}_S \xi(y) - \sigma(y)|^2 \, d\mathcal{H}^2(y) = \mathcal{E}_i(\xi). \end{aligned}$$

In addition $Q_{-\alpha}\circ\xi\circ Q_\alpha$ satisfies the boundary conditions due to (3.1). Hence it is indeed a minimizer.

We set
$$\bar\xi(x)=\frac{1}{2\pi}\int_0^{2\pi}Q_{-\alpha}\circ\xi\circ Q_\alpha(x)d\alpha.$$
Due to (3.1) $\bar\xi$ is in $\mathcal{D}_i$ too for $i\in I$. We now check that $\bar\xi(x)$ is a minimizer of $\mathcal{E}_i$
$$\mathcal{E}_i(\bar\xi)=\frac12\int_{S_i}|\frac{1}{2\pi}\int_0^{2\pi}\mathrm{div}_S Q_{-\alpha}\circ\xi\circ Q_\alpha(x)d\alpha-\sigma(x)|^2\,d\mathcal{H}^2(x).$$
Due to Young's inequality we see
$$\begin{aligned}\mathcal{E}_i(\bar\xi)\le &\quad \frac{1}{2\pi}\int_0^{2\pi}\frac12\int_{S_i}|\mathrm{div}_S Q_{-\alpha}\circ\xi\circ Q_\alpha(x)-\sigma(x)|^2 d\alpha\,d\mathcal{H}^2(x)\\ = &\quad \frac{1}{2\pi}\int_0^{2\pi}\mathcal{E}_i(Q_{-\alpha}\circ\xi\circ Q_\alpha)d\alpha=\mathcal{E}_i(\xi).\end{aligned}$$
The rotational invariance of $\bar\xi$ follows from the very definition of $\bar\xi$, namely
$$\begin{aligned}Q_{-\alpha}\circ\bar\xi\circ Q_\alpha(x) &= Q_{-\alpha}\left(\frac{1}{2\pi}\int_0^{2\pi}Q_{-\varphi}\circ\xi\circ Q_\varphi(Q_\alpha x)d\varphi\right)\\ &= \frac{1}{2\pi}\int_\alpha^{2\pi+\alpha}Q_{-\theta}\circ\xi\circ Q_\theta(x)d\theta=\bar\xi(x),\end{aligned}$$
for $\theta=\varphi+\alpha$.

The equality
$$\mathrm{div}_S\xi=\mathrm{div}_S\bar\xi$$
follows from the fact that due to strict convexity of the integrand in $\mathcal{E}_i$, there is a unique $\mathrm{div}_S\xi$ minimizing $\mathcal{E}_i$.

(**b**) It is obtained by a similar, simpler, arguments, however, we skip details. □

We may combine (a) and (b) of previous proposition, in this way we deduce.

**Corollary 3.2.** *There exists $\xi$, belonging to all $\mathcal{D}_i$, $i\in I$, which is a solution to (3.2), which is rotationally invariant, i.e. (3.4) holds, and symmetric with respect to the plane $\{x_3=0\}$.* □

We notice that in fact the functionals $\mathcal{E}_T$ and $\mathcal{E}_B$ coincide and subsequently we will frequently take advantage of this fact.

Let us suppose that $\bar\xi$ is given by Proposition 3.1 (a). If we write $\bar\xi(x)=\xi_1\mathbf{e}_r+\xi_2\mathbf{e}_\varphi+\xi_3\mathbf{e}_3$, where $\mathbf{e}_3=(0,0,1)$, then the rotational invariance implies
$$\xi_i=\xi_i(r,x_3),\quad i\in I. \tag{3.5}$$

Let us calculate $\mathrm{div}_S\xi$ on $S_\Lambda, S_T$, for $\xi$ given by Corollary 3.2. If we take into account (3.5), then formula (2.2) yields
$$\mathrm{div}_S\xi=\frac{\partial\xi_1}{\partial r}+\xi_1\frac1r+\frac{\partial\xi_2}{\partial\varphi}\frac1r=\frac{\partial}{\partial r}\xi_1(r,L)+\frac1r\xi_1(r,L)\quad\text{on}\quad S_T$$
and
$$\mathrm{div}_S\xi=\xi_1\frac1R+\frac{\partial\xi_2}{\partial\varphi}\frac1R+\frac{\partial\xi_3}{\partial x_3}=\xi_1(R,x_3)\frac1R+\frac{\partial\xi_3}{\partial x_3}(R,x_3)\quad\text{on}\quad S_\Lambda.$$

We immediately see that because of (2.3) these formulas imply that

$$\xi_1 \in H^1(S_T) \qquad \text{and} \qquad \xi_3 \in H^1(S_\Lambda). \tag{3.6}$$

The structure of the above formulas suggests another possible simplification of $\xi$.

**Proposition 3.3.** *Let us suppose that $\xi \in \mathcal{D}_i$ is a minimizer of $\mathcal{E}_i$, $i \in I$, as in Corollary 3.2. Then, there exists $\phi : \Omega \to \mathbb{R}$, $\phi|_{S_i} \in H^2(S_i)$, such that $\nabla\phi \in \mathcal{D}_i$, $i \in I$, and*

$$\tilde{\xi} = \nabla\phi \quad \text{is a minimizer of} \quad \mathcal{E}_i, \quad i \in I.$$

*Proof.* Let us define $\varphi : (0,R) \to \mathbb{R}$ by relations

$$\frac{d\varphi}{dr}(r) = \xi_1(r,L) \quad \text{and} \quad \varphi(0) = 0.$$

Next we define $\psi : [0,L] \to \mathbb{R}$ by

$$\frac{d\psi}{dz}(z) = \xi_3(R,z) \quad \text{and} \quad \psi(0) = 0.$$

We set

$$\phi(r,x_3) = \varphi(r) + \psi(|x_3|).$$

Because of $\xi_3(R,-z) = \xi_3(R,z)$ and (3.6) we deduce that $\psi \in H^2(S_\Lambda)$. Furthermore, we have to make sure that $\tilde{\xi} \in \mathcal{D}_i$, $i \in I$. By definition

$$\tilde{\xi} = \xi_1(r,L)\mathbf{e}_r + \xi_3(R,L)\mathbf{e}_3 \quad \text{on } S_T, \qquad \tilde{\xi} = \xi_1(R,L)\mathbf{e}_r + \xi_3(R,x_3)\mathbf{e}_3 \quad \text{on } S_\Lambda.$$

Now, by the very structure of $\partial\gamma(\mathbf{n}_T)$ and $\partial\gamma(\mathbf{n}_\Lambda)$, we conclude that $\tilde{\xi} \in \mathcal{D}_i$, $i \in I$.

It is obvious that $\tilde{\xi}$ satisfies

$$\mathrm{div}_S\tilde{\xi} = \mathrm{div}_S\xi \quad \text{on} \quad S_\Lambda, S_T.$$

Hence $\tilde{\xi}$ is another minimizer, as desired. □

We shall frequently use this proposition in our further discussion of minimizers of $\mathcal{E}_i$.

# 4 Stability of facets

The stability analysis is performed at a fixed particular $t \geq 0$. For this reason we shall drop in this section the time argument in $\Omega$ and $\sigma$. However, the time regularity of $\xi$ is important to us. We shall comment on that at the very end of this section. Before turning to our main topic we recall two important facts. The first one is our main analytical tool.

**Proposition 4.1.** (Berg's effect, [GR2, Theorem 1]) *Suppose that $\sigma$ is a unique solution to*

$$\Delta\sigma = 0 \quad \textit{in} \quad \mathbb{R}^3 \setminus \Omega, \quad \sigma(\infty) = \sigma^\infty$$
$$\frac{\partial\sigma}{\partial\mathbf{n}} = V_i \textit{ on } S_i, \quad i = \Lambda, T, B$$

*where $\sigma = \sigma(r,x_3)$, $\sigma(r,-x_3) = \sigma(r,x_3)$, $\mathbf{n}$ is the outer normal to $\Omega$ and $V_i$, $i = \Lambda, T, B$ are constants, moreover $V_T = V_B$.*

*(a) If $V_T > 0$, then $\frac{\partial\sigma}{\partial x_3} > 0$ for $x_3 > 0$ and $\frac{\partial\sigma}{\partial x_3} < 0$ for $x_3 < 0$ on $S_\Lambda$.*

*(b) If $V_\Lambda > 0$, then $\frac{\partial\sigma}{\partial r} > 0$ on $S_T \cup S_B$.*

**Remark.** A similar statement holds if we reverse the inequality signs.

Some parts of the facet stability analysis will be general, however, the sharp results will be obtained for a special class of evolution, *i.e.* self-similar motion. We recall that a solution $(\Omega, \sigma)$ to (2.15)–(2.17) is known as self-similar, provided that $\Omega(t) = a(t)\Omega(0)$, where $a(t)$ is a scale function. We remark that the scale $a$ equals to $\dfrac{R}{R_0} = \dfrac{L}{L_0}$, where $R_0$ is the radius of $W_\gamma$ and $L_0$ is the half-height of $W_\gamma$.

What is more interesting is that self-similar solutions are possible only for some special choice of $\gamma$ (hence $W_\gamma$) and $\beta$'s. Namely, we have shown in [GR3]

**Proposition 4.2.** [GR3, Theorem 4.7, Theorem 4.8]) *There is a choice of $\gamma_\Lambda$, $\gamma_{TB} \equiv \gamma_T = \gamma_B$ appearing in (2.1) and $\beta_T = \beta_B$, $\beta_\Lambda$ satisfying*

$$\gamma_i\beta_i = \textit{const}, \qquad i \in I$$

*or equivalently*

$$\rho_0 = \frac{L}{R} = \frac{\beta_T}{\beta_\Lambda}$$

*such that system (2.15)–(2.17) with $\Omega(0) = aW_\gamma$ and $\sigma^\infty > 0$, $\sigma^\infty \neq 2$ has a self-similar solution. In fact, $\gamma_\Lambda$, $\gamma_{TB}$, $\beta_T = \beta_B$, $\beta_\Lambda$ fulfill the above property if and only if $\rho_0$ is a solution to (see equation (4.8) in [GR3])*

$$4\rho^2(f_T, f_T + f_B) + 6\rho(f_T, f_\Lambda) = ||f_\Lambda||^2.$$

Our goal is to find a proper selection of $\xi$ as suggested by Theorem 2.3. We want to prove that for such a $\xi$

$$\sigma - \mathrm{div}_S\xi = \mathrm{const} \tag{4.1}$$

holds on each facet and that $\xi$ belongs to $\mathcal{D}_i$ for each $i \in I$. For each $i = T, B, \Lambda$ we defined a functional $\mathcal{E}_i$ on $H^1(S_i)$ by formula

$$\mathcal{E}_i(\xi) = \frac{1}{2}\int_{S_i} |\sigma - \mathrm{div}_S\xi|^2\, d\mathcal{H}^2, \qquad i = \Lambda,\ T,\ B,$$

for $\xi \in \mathcal{D}_i$. By definition the sets $\mathcal{D}_i$, $i \in I$, incorporate the constraint (2.4) and the boundary conditions (2.6), (2.7). We **postulate** that the correct $\xi$ must be such that $\xi|_{S_i}$ is a minimizer of $\mathcal{E}_i$ for each $i = \Lambda, T, B$. The idea that the proper selection of the Cahn-Hoffman vector comes from a minimization process is not new. For surfaces it has been introduced in [GGM, Section 9], [BNP2] and for evolution of curves in [FG], [GG1]. A similar idea is also present in [GPR].

## 4.1 Euler-Lagrange equations

We shall now investigate the relationship between solutions to (4.1) belonging to $\mathcal{D}_i$ and minimizers of $\mathcal{E}_i$. We will see that equation (4.1) is the Euler-Langrange equation for the functional $\mathcal{E}_i$.

**Proposition 4.3.** *Let us suppose that $\bar\xi$ is a solution to the minimization problem*

$$\mathcal{E}_j(\bar\xi) = \inf_{\mathcal{D}_j} \mathcal{E}_j(\xi),$$

*such that*

$$\bar\xi(x) \in \mathit{Int}\, \partial\gamma(\mathbf{n}_i) \qquad \textit{for all } x \in S_i \setminus S_j,\ j \neq i. \tag{4.2}$$

*Here, $\mathit{Int}\,\partial\gamma(\mathbf{n})$ denotes the relative interior of the set $\partial\gamma(\mathbf{n})$. In case of $i = \Lambda$ we additionally assume that $\bar\xi$ is rotationally symmetric. Then,*

$$\sigma - \mathrm{div}_S = \mathrm{const} \qquad \text{on } S_j.$$

**Remark.** An argument similar to the one used here can be found in [GGM, Lemma 9.5] for crystalline evolution of polyhedra for constant $\sigma$.

*Proof of Proposition 4.3.* Let us take $h \in C_0^\infty(S_j;\mathbb{R}^3)$, $h = h_1\mathbf{e}_r + h_2\mathbf{e}_\varphi + h_3\mathbf{e}_3$. The condition $\bar\xi + h \in \partial\gamma(\mathbf{n})$ implies:

$$\begin{aligned} &h_3 \equiv 0 \quad \text{if} \quad i = T, B,\\ &h_1 = h_2 \equiv 0 \quad \text{if} \quad i = \Lambda, \end{aligned}$$

and the norms $\|h_1\mathbf{e}_r + h_2\mathbf{e}_\varphi\|_{C^0(S_i)}$, $i = T, B$, $\|h_3\|_{C^0(S_\Lambda)}$ are small. Otherwise $h$ is arbitrary.

Then we obviously have

$$\begin{aligned} 0 &\le \mathcal{E}_j(\bar\xi + h) - \mathcal{E}_j(\bar\xi)\\ &= \frac{1}{2}\int\limits_{S_j} \left(|\sigma - \mathrm{div}_S\,\bar\xi|^2 - 2(\sigma - \mathrm{div}_S\,\bar\xi)\mathrm{div}_S\, h + (\mathrm{div}_S\, h)^2\right) d\mathcal{H}^2 - \mathcal{E}_j(\bar\xi)\\ &= \int\limits_{S_j} \left(-(\sigma - \mathrm{div}_S\,\bar\xi)\mathrm{div}_S\, h + \frac{1}{2}(\mathrm{div}_S\, h)^2\right) d\mathcal{H}^2. \end{aligned}$$

Since $\bar\xi$ is in the interior of $\partial\gamma$ and the support of $h$ is separated away from $S_j \cap S_i$, $j \neq i$, then for an arbitrary $h$ with sufficiently small norm $||h||_{C^0}$, we may consider $-h$ in place of $h$. Thus, we conclude that

$$\int\limits_{S_j} (\sigma - \mathrm{div}_S\bar\xi)\mathrm{div}_S h\, d\mathcal{H}^2 = 0, \qquad \forall h \in C_0^\infty(S_j;\mathbb{R}^3), \quad \bar\xi + h \in \partial\gamma(\mathbf{n}).$$

We consider first the case $i = \Lambda$, then $\mathrm{div}_S h = \frac{\partial h_3}{\partial x_3}$, hence the above identity takes the form

$$\int_{-L}^{L} (\sigma - \mathrm{div}_S\bar\xi) \int_0^{2\pi} \frac{\partial h_3}{\partial x_3}\, d\varphi dx_3 = 0.$$

Since the function $h_3$ is arbitrary we deduce that

$$(\sigma - \mathrm{div}_S\bar\xi) = \mathrm{const}.$$

If $i = T, B$, then we need an auxiliary result below, whose application will immediately yield the Lemma. $\square$

**Lemma 4.4.** *Let us suppose that $f \in L^2(S_i)$, where $i = T, B$. If for each $\xi \in C_0^\infty(S_i, \mathbb{R}^2)$ it is true that*

$$\int_{S_i} f \operatorname{div} \xi \; d\mathcal{H}^2 = 0,$$

*then*

$$f \equiv \text{const}.$$

Although this result is rather known we decided to include the proof for the sake of completeness.

*Proof.* *Step 1.* Let us assume first that $f \in C^1(\bar{S}_i) \cap L^2(S_i)$, then integration by parts yields

$$\begin{aligned} 0 &= -\int_{S_i} \nabla f(x) \cdot \xi(x) \; d\mathcal{H}^2(x) + \int_{\partial S_i} f(x) \cdot (\xi(x) \cdot \nu) \, d\mathcal{H}^1(x) \\ &= -\int_{S_i} \nabla f(x) \cdot \xi(x) \; d\mathcal{H}^2(x), \end{aligned}$$

where $\nu$ is the outer normal vector to $\partial S_i$ in $TS_i$.

Thus, the claim follows as a consequence of the du Bois – Raymond Lemma.

*Step 2.* We assume that $f \in L^2(S_i)$. We take the standard mollifier kernel $\rho_\epsilon(x) = \epsilon^{-n}\rho(x/\epsilon)$, where $\rho \in C_0^\infty(B(0,1))$ and $\int_{B(0,1)} \rho(x)\, dx = 1$. We set

$$f_\epsilon(x) = \int_{S_T} f(y) \rho_\epsilon(x - y) \, d\mathcal{H}^2(y).$$

A similar definition is valid on $S_B$.

We consider $\xi$ such that

$$\operatorname{dist}(\operatorname{supp} \xi, \partial S_i) \geq \delta > 0.$$

We can easily check that

$$\int_{S_i} f_\epsilon(x) \operatorname{div}\xi(x) \, d\mathcal{H}^2(x) = \int_{S_i} f(x) \operatorname{div}(\xi * \rho_\epsilon)(x) \, d\mathcal{H}^2(x) = 0$$

and $\operatorname{supp}(\xi * \rho_\epsilon) \subset S_i$ for $\epsilon < \delta$.

Thus, by the first step $f_\epsilon \equiv C_\epsilon$ in $\{x \in S_i : \operatorname{dist}(x, \partial S_i) > \delta\}$. But $f_\epsilon$ converges in $L^2(S_i)$ to $f$. Hence, $f$ itself must be a constant. □

We will prove a converse statement. Due to Corollary 3.2, we may restrict our attention to $\xi$ satisfying the symmetries of the problem, *i.e.* in the form of (3.5), however this is not necessary.

**Proposition 4.5.** *Let us suppose that $\xi \in \mathcal{D}_i$ is a solution to (4.1). Then, $\xi$ is a minimizer of $\mathcal{E}_i$.*

*Proof.* Let us take any $\bar{\xi} \in \mathcal{D}_i$. Then, $\bar{\xi} = \xi + h$, where $h$ satisfies (2.3) and $h \cdot \nu_i = 0$ in a sense explained in (2.6), (2.7). We will see that $\mathcal{E}_i(\bar{\xi}) \geq \mathcal{E}_i(\xi)$. Indeed,

$$\mathcal{E}_i(\xi + h) = \mathcal{E}_i(\xi) - \int_{S_i} (\sigma - \operatorname{div}_S \xi) \operatorname{div}_S h \, d\mathcal{H}^2 + \frac{1}{2} \int_{S_i} (\operatorname{div}_S h)^2 \, d\mathcal{H}^2.$$

Now, we recall $\sigma - \mathrm{div}\,\xi = V_i\beta_i$. First we will consider $i = T, B$. The integration by parts, *i.e.* the evaluation of $h \cdot \nu_i$ on $1 \in W^{1/2,2}(\partial S_i)$, yields

$$\int_{S_i} (\sigma - \mathrm{div}_S\,\xi)\mathrm{div}_S\,h\,d\mathcal{H}^2 = V_i\beta_i \int_{\partial S_i} h \cdot \nu\,d\mathcal{H}^1 = 0.$$

The Proposition follows for $i = T, B$.

If $i = \Lambda$, then we have to use the Gauss formula appropriate for surface divergence, (2.12), we see

$$\int_{S_\Lambda} (\sigma - \mathrm{div}_S\,\xi)\mathrm{div}_S\,h\,d\mathcal{H}^2 = V_i\beta_i \int_{\partial S_\Lambda} h \cdot \nu\,d\mathcal{H}^1 + (\sigma - \mathrm{div}_S\,\xi)\int_{S_\Lambda} H \cdot h\,d\mathcal{H}^2.$$

However, as at the end of the proof of Proposition 2.1, the constraint (2.4) implies that $H \cdot h = 0$ $\mathcal{H}^2$ – a.e. and the Proposition follows. □

## 4.2 Solutions to Euler-Lagrange equations

We begin our construction of solutions to (4.1). In order to make our analysis applicable to all admissible cylinders we consider the two of $S_\Lambda$ and $S_T, S_B$ separately. However, the final sharp qualitative conclusion will be about self-similar motion we constructed in [GR3].

We shall construct solutions to (4.1) in the form permitted by Corollary 3.2 and Proposition 3.3, *i.e.*

$$\xi = \varphi_r(r)\mathbf{e}_r + \psi_{x_3}(x_3)\mathbf{e}_3 \tag{4.3}$$

and satisfying the boundary condition (2.6), (2.7). For the sake of simplicity of notation we will write $\sigma(r)$ instead of $\sigma(r, L)$ and $\sigma(z)$ in place of $\sigma(R, z)$ when this does not lead to ambiguity. We stress that $\sigma \in L^2(S_i)$ is given here.

Let us consider facet $S_T$. We notice that on $S_T$ we have, (see (2.2)),

$$\mathrm{div}_S\xi = \mathrm{div}_S\nabla\varphi = \varphi_{rr} + \frac{1}{r}\varphi_r.$$

The condition $\sigma - \mathrm{div}_S\xi = \beta_T V_T \equiv$ const. implies

$$\sigma - \beta_T V_T = \frac{1}{r}(r\varphi_r)_r.$$

After multiplication by $r$ and integration we see

$$\varphi_r(r) = \frac{1}{r}\int_0^r s\sigma(s)\,ds - \frac{r}{2}\,\beta_T V_T.$$

The condition $\xi \in \partial\gamma(\mathbf{n}_\Lambda) \cap \partial\gamma(\mathbf{n}_T)$ on $S_\Lambda \cap S_T$ implies (2.5) that yields

$$\varphi_r(R) = \gamma(\mathbf{n}_\Lambda).$$

Hence,

$$\beta_T V_T = \left(\frac{1}{R}\int_0^R s\sigma(s)\,ds - \gamma(\mathbf{n}_\Lambda)\right)\frac{2}{R}.$$

We notice that this expression coincides with (1.6) on $S_T$. Finally, we reach a formula for solutions to (4.1) on $S_T$.

$$\varphi_r(r) = \frac{1}{r}\int_0^r s\sigma(s)\,ds + \frac{r}{R}\left(\gamma(\mathbf{n}_\Lambda) - \frac{1}{R}\int_0^R s\sigma(s)\,ds\right). \tag{4.4}$$

Now, we will solve the Euler-Lagrange equation (4.1) on $S_\Lambda$. Let us calculate $\mathrm{div}_S\xi$ on $S_\Lambda$ for $\xi$ of the form (4.3). By formula (2.2) we can see

$$\mathrm{div}_S\,\xi = \frac{\varphi_r(R)}{R} + \psi_{x_3x_3}$$

and condition (4.1) takes the form

$$\psi_{zz}(z) = \sigma(R,z) - \left(\frac{\varphi_r(R)}{R} + \beta_\Lambda V_\Lambda\right),$$

where $\varphi_r(R) = \gamma(\mathbf{n}_\Lambda) = \gamma_\Lambda$ is constant. Integration of the above equality yields

$$\psi_z(z) = \int_0^z \sigma(s)ds - \left(\frac{\gamma_\Lambda}{R} + \beta_\Lambda V_\Lambda\right)z,$$

where due to the symmetry of the problem with respect to $\{x_3 = 0\}$ we have $\psi_z(0) = 0$. Moreover at $z = L$ the value of $\psi_z(L)$ is imposed by the constraint $\nabla\psi \in \partial\gamma(\mathbf{n}_\Lambda)\cap\partial\gamma(\mathbf{n}_T)$ yielding (2.5), *i.e.*

$$\psi_z(L) = \gamma(\mathbf{n}_T).$$

It follows that

$$0 < \gamma(\mathbf{n}_T) = \int_0^L \sigma(s)ds - L\left(\frac{\gamma_\Lambda}{R} + \beta_\Lambda V_\Lambda\right).$$

Hence

$$\beta_\Lambda V_\Lambda = \frac{1}{L}\int_0^L \sigma(s)\,ds - \frac{\gamma_\Lambda}{R} - \frac{\gamma(\mathbf{n}_T)}{L}$$

in agreement with (1.6) for $S_\Lambda$. We conclude that $\psi_z$ the solution to (4.1) on $S_\Lambda$ takes the form

$$\psi_z(z) = \int_0^z \sigma(s)\,ds - \frac{z}{L}\int_0^L \sigma(s)\,ds + \frac{\gamma(\mathbf{n}_T)}{L}z, \tag{4.5}$$

Once we obtain the formulas for $\xi$, then the question of stability is reduced to a relatively simple matter, as explained below. To be precise, we say that a facet $S_i$ is *stable* if there exists a minimizer $\xi \in \mathcal{D}_i$ of (3.2) such that

$$\mathrm{div}_S\xi - \sigma = \mathrm{const} \qquad \text{on } S_i.$$

As we have observed, we may assume that $\xi$ is of the form given in Proposition 3.3.

**Theorem 4.6.** **(Necessary and sufficient conditions)** *Let us suppose that $\sigma$ is given by Proposition 2.2, thus in particular $\sigma|_{S_i} \in L^2(S_i)$. If $\xi \in \mathcal{D}_i$ is a solution to (3.2), then there exists $\bar\xi \in \mathcal{D}_i$ another minimizer of $\mathcal{E}_i$, which is of the form (4.3) and*

$$\mathit{div}_S\xi = \mathit{div}_S\bar\xi.$$

*Moreover,*
**(i)** *Facet $S_T$ (and $S_B$) is stable if and only if $\varphi$ is given by (4.4) and*

$$\varphi_r(r) \in [-\gamma(\mathbf{n}_\Lambda), \gamma(\mathbf{n}_\Lambda)], \quad \text{for } r \in [0, R], \qquad \varphi_r(0) = 0 \quad \text{and} \quad \varphi_r(R) = \gamma(\mathbf{n}_\Lambda).$$

**(ii)** *Facet $S_\Lambda$ is stable if and only if $\psi$ is given by (4.5) and*

$$\psi_{x_3}(x_3) \in [-\gamma(\mathbf{n}_T), \gamma(\mathbf{n}_T)], \quad \text{for } r \in [-L, L], \qquad \psi_{x_3}(0) = 0, \quad \psi_{x_3}(L) = \gamma(\mathbf{n}_T).$$

*Proof.* By Proposition 3.3 there exists $\phi(r, x_3) = \varphi(r) + \psi(x_3)$, such that $\bar{\xi} = \nabla\phi$ is a minimizer of $\mathcal{E}_i$, $i \in I$, and $\mathrm{div}_S\bar{\xi} = \mathrm{div}_S\xi$.

(i) *Necessity.* The stability implies that $\mathrm{div}_S\bar{\xi} - \sigma = \beta_T V_T$ and we can solve (4.1). Its only solution is given by formula (4.4). Since $\bar{\xi} \in \mathcal{D}_i$ we obviously have that $\varphi_r(r) \in [-\gamma(\mathbf{n}_\Lambda), \gamma(\mathbf{n}_\Lambda)]$, $\varphi_r(R) = \gamma(\mathbf{n}_\Lambda)$, while $\varphi_r(0) = 0$ is a consequence of smoothness of $\varphi$.

(i) *Sufficiency.* This is the content of Proposition 4.5.

The remaining case (ii) is treated in a similar way. The details are omitted.

□

The above statement is general but not satisfactory because we would expect sufficient and necessary stability conditions expressed in terms of $\kappa_i$ and $\sigma^\infty$. Such results will be obtained for self-similar solutions. We shall treat separately the cases $V_i > 0$ and $V_i < 0$ since the results differ.

Let us close with remarks on time regularity of $\xi$. The construction of solutions to (4.1) which we provided in formulas (4.4), (4.5) show that $R$, $L$ are involved in an algebraic and integral way. Hence $\xi$ will share the smoothness of $R$, $L$. We stress that, in fact, $\xi$ is well-defined not only on $\partial\Omega(t)$ but also on $\mathbb{R}^3$. We can express it as follows.

**Corollary 4.7.** *Let us suppose that the motion of $\Omega(t)$ is such that the facet $S_B$, $S_T$ and $S_\Lambda$ are stable. Then, there exists a minimizer of $\mathcal{E}_i$, $i \in I$, such that the functions*

$$t \mapsto \xi(t, x) \in \mathbb{R}$$

*are of class $C^{1,1}$ in time for fixed $x$.*

## 4.3 Stability of growing self-similar solutions

Our goal is to prove transparent necessary and sufficient stability conditions for growing solutions. Let us note that Proposition 2.6 implies that if $\sigma^\infty + \kappa > 0$, then the self-similar solution grows, we set here $\kappa = \kappa_T = \kappa_B = \kappa_\Lambda$, as $\Omega(t)$ is a scaled Wulff shape. Below, we state our main result.

**Theorem 4.8.** *Let us suppose that $\gamma$, $\beta$ are such that there exists a self-similar motion of $\Omega(0)$, i.e $\Omega(t) = a(t)W_\gamma$ and the assumption of Proposition 4.2 are satisfied. Moreover, we assume that $\sigma^\infty + \kappa > 0$, where $\kappa$ is the curvature of $\Omega(0)$.*

**(i)** *The stability of motion of $S_T$ at time $t$ is equivalent to*

$$\frac{a(t)(\sigma^\infty a(t) - 2)c_T}{\beta_T + a(t)c_T} \le \min_{\vartheta \in (0,1)} \frac{1 + \vartheta}{\vartheta \bar{d}_T(\vartheta)}. \tag{4.6}$$

*The constant $c_T$, and the function $\bar{d}_T$ depend only on $W_\gamma$, the aspect ratio $\varrho_0$. Their definitions are given below, see formulas (4.10) and (4.19).*

**(ii)** *The stability of motion of $S_\Lambda$ at time $t$ is equivalent to*

$$\frac{a(t)(\sigma^\infty a(t)-2)c_\Lambda}{\beta_\Lambda + a(t)c_\Lambda} \le \min_{\vartheta\in(0,1)} \frac{1+\vartheta}{\vartheta \bar{d}_\Lambda(\vartheta)} \tag{4.7}$$

*The constant $c_\Lambda$, and the function $\bar{d}_\Lambda$ depend only on $W_\gamma, \varrho_0$. Their definitions are given below, see (4.10) and (4.20)*

We will divide the proof in a series of Lemmas. We begin with general remarks on $\varphi_r$.

**Lemma 4.9.** *Let us suppose that $\varphi_r$ is given by (4.4). Then it always satisfies $\varphi_r(r) < \gamma(\mathbf{n}_\Lambda)$ for all $r < R$.*

*Proof.* Let us notice that $\varphi_r(r)$ has the form

$$\varphi_r(r) = \frac{r}{R}\gamma(\mathbf{n}_\Lambda) + rg(r),$$

where we set

$$g(r) = \frac{1}{r^2}\int_0^r s\sigma(s)ds - \frac{1}{R^2}\int_0^R s\sigma(s)ds.$$

The function $g : [0,R] \to \mathbb{R}$ is continuous and differentiable on $(0,R)$. It is easy to see that $g(R) = 0$ and

$$\lim_{r\to 0^+} g(r) = \frac{\sigma(0)}{2} - \frac{1}{R^2}\int_0^R s\sigma(s)ds.$$

We deal with the growing crystal only *i.e.* $V_i > 0$. Then it is obvious that

$$g(0) = \frac{1}{R^2}\int_0^R s(\sigma(0)-\sigma(s))ds$$

and due to Berg's effect (see Proposition 4.1) we conclude that

$$g(0) < 0.$$

We calculate the derivative of $g(r)$,

$$\begin{aligned}\frac{d}{dr}g(r) &= -\frac{2}{r^3}\int_0^r s\sigma(s)ds + \frac{1}{r}\sigma(r)\\ &= \frac{2}{r^3}\int_0^r s(\sigma(r)-\sigma(s))ds.\end{aligned}$$

Another application of Berg's effect implies that $g'(r) > 0$, for $r \in (0,R)$, thus we conclude that $rg(r) < 0$ for $r \in (0,R)$ and it follows that

$$\varphi_r < \frac{r}{R}\gamma(\mathbf{n}_\Lambda) \quad \text{for} \quad r \in (0,R),$$

as desired. $\square$

**Lemma 4.10.** *Let us suppose that $\psi_{x_3}$ given by (4.5). Then it always satisfies $\psi_{x_3}(z) < \gamma(\mathbf{n}_T)$ for all $-L < z < L$.*

*Proof.* Because of the symmetry of the problem, it is sufficient to consider $z > 0$. We note that $\psi_z$ may be written as

$$\psi_z(z) = zh(z) + \frac{z}{L}\gamma(\mathbf{n}_T),$$

where

$$h(z) = \frac{1}{z}\int_0^z \sigma(s)\,ds - \frac{1}{L}\int_0^L \sigma(s)\,ds.$$

It is clear that $h : [0, L] \to \mathbb{R}$ is continuous and differentiable on $(0, L)$. Moreover,

$$\begin{aligned}\lim_{z\to 0+} h(z) &= \sigma(0) - \frac{1}{L}\int_0^L \sigma(s)\,ds \\ &= \frac{1}{L}\int_0^L (\sigma(0) - \sigma(s))\,ds < 0;\end{aligned}$$

when the last inequality follows from Berg's effect. It is also obvious that $h(L) = 0$.

We may calculate

$$\begin{aligned}h'(z) &= -\frac{1}{z^2}\int_0^z \sigma(s)\,ds + \frac{\sigma(z)}{z} \\ &= \frac{1}{z^2}\int_0^z (\sigma(z) - \sigma(s))\,ds > 0.\end{aligned}$$

This inequality follows again from Berg's effect implying $\sigma(z) > \sigma(s)$ for $z > s > 0$. We conclude that

$$zh(z) < 0,$$

hence

$$\psi_z(z) < \frac{z}{L}\gamma(\mathbf{n}_L) < \gamma(\mathbf{n}_L), \text{ for } z \in (0, L).$$

$\square$

To study the implications of the inequality $\varphi_r(r) \geq -\gamma(\mathbf{n}_\Lambda)$ and to find conditions implying $\varphi_r(r) > -\gamma(\mathbf{n}_\Lambda)$ we need good estimates on speeds $V_T$, $V_\Lambda$.

In order to state next Lemma we recall or introduce some notation. Namely, we recall that we denoted by $f_i^1$ the unique solution to (2.18)–(2.19) with $\Omega = W_\gamma$. Subsequently, we will use two auxiliary functions defined on $\mathbb{R}^3 \setminus W_\gamma$:

$$w_\Lambda = \sum_{i\in I} \mu_i f_i^1, \quad w_T = \sum_{i\in I} \alpha_i f_i^1, \tag{4.8}$$

where

$$\alpha_i = \begin{cases} 1 & i = T, B \\ \varrho_0^{-1} & i = \Lambda \end{cases} \qquad \mu_i = \begin{cases} 1 & i = \Lambda \\ \varrho_0 & i = T, B \end{cases} \tag{4.9}$$

By $S_T(W_\gamma)$, $S_\Lambda(W_\gamma)$ we denote, respectively, the top and lateral surfaces of $W_\gamma$. Finally, we set

$$c_T = \frac{1}{|S_T(W_\gamma)|}\int_{S_T(W_\gamma)} w_T\, d\mathcal{H}^2, \quad c_\Lambda = \frac{1}{|S_\Lambda(W_\gamma)|}\int_{S_\Lambda(W_\gamma)} w_\Lambda\, d\mathcal{H}^2. \tag{4.10}$$

We denote by $a$ the scale factor. We know that

$$a = \frac{R}{R_0} = \frac{L}{L_0},$$

and we recall that $R_0 = \gamma(\mathbf{n}_\Lambda)$, $L_0 = \gamma(\mathbf{n}_T)$.

We now state the estimates on the velocities.

**Lemma 4.11.** *Let us assume that $W_\gamma$. and, $\beta$'s admit self-similar evolution and $\Omega(t)$ is a self-similar solution such that $\Omega(0) = W_\gamma$ (see Proposition 4.2). Then,*
*(a)*

$$V_T = \frac{\sigma^\infty - 2/a}{\beta_T + ac_T}. \tag{4.11}$$

*(b)*

$$V_\Lambda = \frac{\sigma^\infty - 2/a}{\beta_\Lambda + ac_\Lambda}. \tag{4.12}$$

*The constant $c_\Lambda$ is defined above, see (4.10).*

It is convenient to use the notation

$$\fint_G f(x)\,d\mu(x) = \frac{1}{\mu(G)}\int_G f(x)\,d\mu(x),$$

where $\mu$ is a measure. We will write so from now on.

*Proof of Lemma.* (a) We will consider the case of $S_T$, a similar reasoning will be valid for $S_\Lambda$.

Our starting point is the equation

$$\fint_{S_T} \sigma\, d\mathcal{H}^2 = \kappa_T - \beta_T V_T. \tag{4.13}$$

We recall the representation formula for $\sigma$, i.e.

$$\sigma = \sigma^\infty - \sum_{i\in I} f_i V_i.$$

This applied to (4.13) yields

$$\fint_{S_T} \sum_{i\in I} f_i V_i + \beta_T V_T = \sigma^\infty + \kappa,$$

where we write $\kappa$ for $\kappa_T = \kappa_B$.

For all time instances of the self-similar evolution we have $V_T/V_\Lambda = \rho_0$, where $\rho_0 = L/R$ is the aspect ratio of $W_\gamma$. Hence, by (4.9) and Proposition 2.4

$$V_T\left(\beta_T + \fint_{S_T}\sum_{i\in I} f_i^a \alpha_i\, d\mathcal{H}^2\right) = \sigma^\infty + \kappa.$$

We recall here that for $\Omega = aW_\gamma$ the $f_i^a$'s are given by scaling (see (2.23))

$$f_i^a(x) = af^1(\frac{x}{a}),$$

where $f_i^1$'s are described above. Thus, if we write $S_T(W_\gamma)$ for the top of $W_\gamma$, we arrive at

$$V_T\left(\beta_T + a\fint_{S_T(W_\gamma)} w_T\, d\mathcal{H}^2\right) = \sigma^\infty - \frac{2\gamma(\mathbf{n}_\Lambda)}{R}$$

or (4.11) as desired.

Part (b) is proved along the same line. The details are omitted.

□

We are now ready to state the crucial results.

**Lemma 4.12.**
(a) *The condition*

$$\varphi_r \geq -\gamma(\mathbf{n}_\Lambda) \tag{4.14}$$

*is equivalent to*

$$\frac{a(t)(\sigma^\infty a(t)-2)c_T}{\beta_T + a(t)c_T} \leq 2 \min_{\vartheta\in(0,1)} \frac{1+\vartheta}{\vartheta \bar{d}_T(\vartheta)}. \tag{4.15}$$

*Function* $\bar{d}_T$ *is defined below.*
(b) *The condition*

$$\psi_{x_3} \geq -\gamma(\mathbf{n}_T) \tag{4.16}$$

*is equivalent to*

$$\frac{a(t)(\sigma^\infty a(t)-2)c_\Lambda}{\beta_\Lambda + a(t)c_\Lambda} \leq 2 \min_{\vartheta\in(0,1)} \frac{1+\vartheta}{\vartheta \bar{d}_\Lambda(\vartheta)}. \tag{4.17}$$

*Function* $\bar{d}_\Lambda$ *is defined in (4.20).*
*Proof.* (a) Our starting point is the observation

$$\begin{aligned}\frac{1}{r^2}\int_0^r s\sigma(s,L)ds &= \frac{1}{2} ⨍_{S_T\cap\{x_1^2+x_2^2\leq r^2\}} \sigma(x)\, d\mathcal{H}^2(x) \\ &=: \frac{1}{2}\bar{\sigma}_r\end{aligned}$$

i.e. the integral on the left hand side is one half of the average of $\sigma$ over the 2-dimensional ball with radius $r$ on $S_T$ concentric with $S_T$. Thus taking into account the definition of $\varphi_r$, the condition (4.14) may be rewritten as

$$\frac{1}{2}r(\bar{\sigma}_R - \bar{\sigma}_r) < \gamma(\mathbf{n}_\Lambda)\left(1+\frac{r}{R}\right), \quad \text{for} \quad r\in(0,R). \tag{4.18}$$

For the derivation of the necessary condition on $S_T$ we look at $\bar{\sigma}_R - \bar{\sigma}_r$. We scale those averages so that we deal with averages over the Wulff shape. But before that we use the representation formula (2.22) for $\sigma$ and we notice that $\sigma^\infty$ drops out

$$\bar{\sigma}_R - \bar{\sigma}_r = a\left(⨍_{S_T(W_\gamma)}\sum_{i\in I} V_i f_i^1(x)\, d\mathcal{H}^2(x) - ⨍_{S_T(W_\gamma)\cap\{x_1^2+x_2^2\leq \frac{r}{R}R_0\}}\sum_{i\in I} V_i f_i^1(x)\, d\mathcal{H}^2(x)\right).$$

We may now pull out $V_T$ in front and introduce $\vartheta = \frac{r}{R}$, $\vartheta\in(0,1)$. Then we see that

$$\begin{aligned}\bar{\sigma}_R - \bar{\sigma}_r &= aV_T\left(⨍_{S_T(W_\gamma)} w_T(x)\, d\mathcal{H}^2(x) - ⨍_{S_T(W_\gamma)\cap\{x_1^2+x_2^2\leq\vartheta R_0\}} w_T(x)\, d\mathcal{H}^2(x)\right) \\ &= aV_T\bar{d}_T(\vartheta).\end{aligned} \tag{4.19}$$

We denote by $\bar{d}_T(\vartheta)$ the factor in the parentheses, because it depends only on $W_\gamma$, $\varrho_0$ and $\vartheta$. Thus the necessary condition reads

$$\frac{r}{2}aV_T\bar{d}_T(\vartheta) \leq \gamma(\mathbf{n}_\Lambda)(1+\frac{r}{R}).$$

or after dividing it by $R$ and recalling that $\gamma(\mathbf{n}_\Lambda) = R_0$ we conclude that

$$a^2 V_T \le 2 \min_{\vartheta\in(0,1)} \frac{1+\vartheta}{\vartheta \bar{d}_T(\vartheta)}.$$

After we recall the formula for $V_T$ (see Lemma 4.11) we obtain (4.15), *i.e.*

$$\frac{a(\sigma^\infty a - 2)c_T}{\beta_T + ac_T} \le 2 \min_{\vartheta\in(0,1)} \frac{1+\vartheta}{\vartheta \bar{d}_T(\vartheta)}.$$

Part (b) is proved along the same lines, we have only to provide the definition of $\bar{d}_\Lambda$,

$$\bar{d}_\Lambda(\vartheta) = (\fint_{S_\Lambda(W_\gamma)} w_\Lambda(x)\, d\mathcal{H}^2(x) - \fint_{S_\Lambda(W_\gamma)\cap\{|x_3|\le\vartheta L_0\}} w_\Lambda(x)\, d\mathcal{H}^2(x). \tag{4.20}$$

□

*Proof of Theorem 4.8* now comes as a simple combination these Lemmas. □

We close this subsection with easy to check sufficient conditions. If we drop the last summation term in (4.19), then we obtain a sufficient condition which is harder to fulfill, but it is simpler. Namely, we come to

$$\frac{r}{2} V_T a c_T \le \gamma(\mathbf{n}_\Lambda)(1 + \frac{r}{R}) \quad \text{for all } \; r \in (0, R).$$

This in turn is equivalent to

$$\frac{R}{2} V_T a c_T \le 2\gamma(\mathbf{n}_\Lambda)$$

or

$$\frac{a^2}{2} \frac{(\sigma^\infty - 2/a)c_T}{\beta_T + a\cdot c_T} < 2 \tag{4.21}$$

obviously (4.21) is satisfied if $a\sigma^\infty - 2 \le 4$, i.e.

$$a\sigma^\infty \le 6$$

is the sufficient condition we have in mind.

The same argument performed for $S_\Lambda$ yields the following sufficient condition for stability

$$a\sigma^\infty \le 4.$$

We have thus proved,

**Corollary 4.13.**
*(a) If $a\sigma^\infty \le 6$, then facets $S_T$, $S_B$ are stable. (b) If $a\sigma^\infty \le 4$, then facet $S_\Lambda$ is stable.*

## 4.4 Stability of shrinking self-similar solutions

The basic argument is similar to that used in previous subsection, but it requires some modifications and the conclusion is different because we get possibly two regions of stability: for $\Omega$ close to the Wulff shape and for $\Omega$ close to a point $\{0\}$.

**Theorem 4.14.** *Let us suppose that $W_\gamma$ and $\beta$'s are such that a self-similar motion is admissible, $\rho_0$ is the corresponding aspect ratio of $W_\gamma$, (see Proposition 4.2). We assume that $\Omega(0) = W_\gamma$ and $\sigma^\infty + \kappa < 0$.*

(i) *The motion of $S_T$ is stable if and only if*

$$\frac{a(2 - a\sigma^\infty)}{2(\beta_T + ac_T)} \leq \delta_T, \tag{4.22}$$

*where $a$ is the scale factor and the definition of $\delta_T$ is given below in (4.25).*

(ii) *The motion of $S_\Lambda$ is stable if and only if*

$$\frac{a(2 - a\sigma^\infty)}{\beta_\Lambda + ac_\Lambda} \leq \delta_\Lambda, \tag{4.23}$$

*where the definition of $\delta_\Lambda$ is given below in (4.26).*

We shall deal with (i) only, because (ii) is handled in a similar way.

Basically in order to prove this theorem we may go along the lines of Theorem 4.8, however many important details change. Hence we will sketch the argument. It will be divided in a number of steps.

**Lemma 4.15.** *Let us suppose that the assumptions of Theorem 4.14 are fulfilled.*

*(a) If $\varphi$ is provided by (4.4), then*

$$\varphi_r(r) > -\gamma(\mathbf{n}_\Lambda), \quad \textit{for all } r \in (0, R).$$

*(b) If $\psi$ is given by (4.5), then*

$$\psi_{x_3}(x_3) > -\gamma(\mathbf{n}_T), \quad \text{for all } x_3 \in (0, L).$$

*Proof.* Let us write

$$\varphi_r(r) = \frac{r}{R}\gamma(\mathbf{n}_\Lambda) + rg(r),$$

where as before

$$g(r) = \frac{1}{r^2}\int_0^r s\sigma(s, L)\,ds - \frac{1}{R^2}\int_0^R s\sigma(s, L)\,ds.$$

We can see that $g(r) = \frac{1}{2}(\bar{\sigma}_r - \bar{\sigma}_R)$ and by Berg's effect (with $V_i < 0$) we conclude that

$$g(r) > 0 \quad \text{for all } r \in (0, R).$$

Hence,

$$\varphi_r(r) > \frac{r}{R}\gamma(\mathbf{n}_\Lambda) > -\gamma(\mathbf{n}_\Lambda),$$

as desired.

Part (b) is proved exactly in the same way. The inequality for $x_3 < 0$ is obtained by the symmetry of the problem. □

**Lemma 4.16.** *Let us suppose that the assumptions of Theorem 4.14 are fulfilled. The Cahn-Hoffman vector is in the form $\xi = \nabla\varphi(r) + \nabla\psi(x_3)$. Then:*

(i) *The condition $\varphi_r(r) \leq \gamma(\mathbf{n}_\Lambda)$ is equivalent to*

$$\frac{a(2 - a\sigma^\infty)}{2(\beta_T + ac_T)} \leq \delta_T,$$

*where $a$ is the scale factor, $a = R/R_0$, and $\delta_T$ is given by (4.25) below.*

(ii) *The condition $\psi_{x_3}(x_3) \le \gamma(\mathbf{n}_T)$ is equivalent to*

$$\frac{a(2 - a\sigma^\infty)}{\beta_\Lambda + ac_\Lambda} \le \delta_\Lambda.$$

*The definition of $\delta_\Lambda$ is provided by (4.26) below.*

*Proof.* (i) By previous argument condition $\varphi_r(r) \le \gamma(\mathbf{n}_\Lambda)$ is equivalent to

$$\frac{r}{R}\gamma(\mathbf{n}_\Lambda) + \frac{r}{2}(\bar\sigma_r - \bar\sigma_R) \le \gamma(\mathbf{n}_\Lambda). \tag{4.24}$$

The definition of $\bar\sigma_r$ is as it was before. We repeat the argument and we see

$$\begin{aligned}\bar\sigma_r - \bar\sigma_R \qquad &= aV_T\left(\fint_{S_T(W_\gamma)} w_T(x)\,d\mathcal{H}^2(x) - \fint_{S_T(W_\gamma)\cap\{r\le\theta R_0\}} w_T(x)\,d\mathcal{H}^2(x)\right)\\ &\equiv aV_T\underline{d}_T(\theta),\end{aligned}$$

where $r^2 = x_1^2 + x_2^2$, $\theta = r/R$ and $w_T$ is defined by (4.8). It is obvious that $\underline{d}_T(1) = 0$. By Berg's effect (Proposition 4.1 with negative normal derivatives) we can deduce that $\underline{d}_T(\theta) < 0$ for all $\theta \in (0,1)$.

Thus, (4.24) takes the form

$$\theta + \frac{\theta}{2}a^2V_T\underline{d}_T(\theta) \le 1, \quad \text{for all } \theta \in (0,1)$$

or after recalling the form of $V_T$, see equation (4.11),

$$\frac{a(2 - a\sigma^\infty)}{2(\beta_T + ac_T)} \le \frac{1-\theta}{\theta(\underline{d}_T(1) - \underline{d}_T(\theta))}, \quad \text{for all } \theta \in (0,1).$$

We have to know that the right-hand-side is strictly positive for all $\theta \in (0,1)$. Because of the structure of the right-hand-side of the above inequality, it is sufficient to calculate its limit as $\theta$ goes to 1. We have

$$\lim_{\theta\to1^-}\frac{\underline{d}_T(1) - \underline{d}_T(\theta)}{1-\theta} = 2\left(w(R,L) - \fint_{S_T(W_\gamma)} w(x)\,d\mathcal{H}^2(x)\right) > 0.$$

Finally (4.24) is equivalent to

$$\frac{a(2 - a\sigma^\infty)}{2(\beta_T + ac_T)} \le \min_{\theta\in(0,1)}\frac{1-\theta}{\theta(\underline{d}_T(1) - \underline{d}_T(\theta))} =: \delta_T. \tag{4.25}$$

The argument for part (b) is analogous with obvious changes of definition. We reach that stability of $S_\Lambda$ is equivalent to

$$\frac{a(2 - a\sigma^\infty)}{\beta_\Lambda + ac_\Lambda} \le \min_{\theta\in(0,1)}\frac{1-\theta}{\theta(\underline{d}_\Lambda(1) - \underline{d}_\Lambda(\theta))} =: \delta_\Lambda, \tag{4.26}$$

where $w_\Lambda$ is defined by (4.8) and

$$\underline{d}_\Lambda(\theta) = \fint_{S_\Lambda(W_\gamma)} w_\Lambda(x)\,d\mathcal{H}^2(x) - \fint_{S_\Lambda(W_\gamma)\cap\{r\le\theta R_0\}} w_\Lambda(x)\,d\mathcal{H}^2(x).$$

*Proof of Theorem 4.14.* It is a straightforward combination of these two Lemmas. □

# 5 Concluding remarks

We may summarize the main results in the following way

**Corollary 5.1.** *Let us suppose that $\Omega(0) = aW_\gamma$, hence $\kappa = -\frac{2}{a}$. Let us assume that $\gamma$ and $\beta$ are chosen so that the self-similar solutions exist. If $\sigma^\infty + \kappa$ is positive, then $V_i > 0$, i.e. the crystal grows. Moreover, the facets are stable, i.e.*

$$\sigma - \mathit{div}_S \xi = \mathit{const.} \quad \mathit{on}\ S_i \quad i \in I$$

*as long as (4.6) and (4.7) hold.* □

We note that at some point of evolution (4.6) and (4.7) certainly will be violated. After that, the behavior of the system requires further research. It will be presented elsewhere.

Let us comment on Theorem 4.14. For simplicity we assume that $\Omega(0) = W_\gamma$, hence (4.22), (4.23) get a bit simplified. We notice that (4.22) or (4.23) are satisfied either for $a$ close to 0 or 1. In order to see what may happen we calculate the maximum with respect to $a$ of the left-hand-side of, say, (4.22). We can see that

$$\max_{a\in(0,1)} \frac{a(1-a)}{b+a} = \frac{\sqrt{b^2+b}(1+b-b^2) - 2b + b^3}{\sqrt{b^2+b}}$$

Hence it depends upon $b$. Thus, we may expect that the left-hand-side of the inequality in Lemma 4.16 (i) (respectively, Lemma 4.16 (ii)) exceeds $\delta_T$ (respectively, $\delta_\Lambda$). If this is so we have an interval $(a_1^T, a_2^T)$ (respectively, $(a_1^\Lambda, a_2^\Lambda)$) of values of $a$ for which the stability condition on $S_B$, $S_T$ (respectively, $S_\Lambda$) is violated. We cannot say if this really happens as long as we do not know any estimates of numerical value of $\delta_T$, $\delta_\Lambda$. In other words, we may have as much as two regions of stability depending upon the choice of parameters: namely when the scale factor $a$ is close to 0 and when $\Omega(0)$ is close to the equilibrium configuration.

Similarly as in the case of growing crystal, we do not know what happens after the stability condition is violated. This topic is a subject of further research.

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