Brown-Halmos Type Theorems Of Weighted Toeplitz Operators II

By

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Abstract. The spectra of the Toeplitz operators on the weighted Hardy space $H^p(Wd\theta/2\pi)$ are studied. For example, the theorems of Brown-Halmos type and Hartman-Wintner type are proved. These generalize results in the previous paper which were proved for $p = 2$. 
§1. Introduction

Let $m = d\theta/2\pi$ be the normalized Lebesgue measure on the unit circle $T$ and let $W$ be a non-negative integrable function on $T$ which does not vanish identically. Suppose $1 \leq p \leq \infty$. Let $L^p(W) = L^p(W dm)$ and $L^p(W) = L^p$ when $W \equiv 1$. Let $H^p(W)$ denote the closure in $L^p(W)$ of the set $\mathcal{P}$ of all analytic polynomials when $p \neq \infty$. We will write $H^p(W) = H^p$ when $W \equiv 1$, and then this is a usual Hardy space. $H^\infty$ denotes the weak $\ast$ closure of $\mathcal{P}$ in $L^\infty$. $P$ denotes the projection from the set $\mathcal{C}$ of all trigonometric polynomials to $\mathcal{P}$. For $1 < p < \infty$, $P$ can be extended to a bounded map of $L^p(W)$ onto $H^p(W)$ if and only if $W$ satisfies the condition $(A_p)$ (see [3, Theorem 6.2 of Chapter VII]). This is the well known theorem of Hunt, Muckenhoupt and Wheeden, which is a generalization of the theorem of Helson and Szegö (see [3, Theorem 3.2 of Chapter IV]).

Assuming that a weight $W$ satisfies the condition $(A_p)$ for $1 < p < \infty$, we define a Toeplitz operator $T^W_p$ on $H^p(W)$ as follows. For $\phi$ in $L^\infty$, suppose that

$$T^W_p \phi f = P(\phi f) \quad (f \in H^p(W)).$$

If $W \equiv 1$, we will write $T^W_p = T^p$.

In this paper, we study the spectrum $\sigma(T^W_p)$ of a Toeplitz operator $T^W_p$. For any weights $W$ in $(A_p)$ and for any $\phi$ in $L^\infty$, the symbol $\phi$ for invertible $T^W_p$ was completely described by H. Widom, A. Devinatz and R. Rochberg (see Theorem WDR in this section). This is one of our main tools. In the previous paper [7, (1) of Theorem 1], for $p = 2$ we gave a generalization of a theorem of Brown and Halmo [2, Proposition 7.19] to arbitrary weight in $(A_2)$. In §2 we generalize this theorem for arbitrary $p$. I. Spitkovsky [10] showed that the set of all weights $W$ for which $\sigma(T^W_p) = \sigma(T^p)$ for all $\phi$ in $L^\infty$ does not depend on $p$. In §2 we give another proof of this result. In fact we describe such a set of weights by using [4, Theorem 2.12]. This also generalizes (1) of Theorem 2 of the previous paper [7].

When $\phi$ is a continuous function and $W \equiv 1$, the spectrum of $T^p$ was completely described (cf. [2, Corollary 7.28]). In §3 we prove $\sigma(T^W_p) = \sigma(T^p)$ for any continuous function $\phi$ whenever $W$ satisfies the condition $(A_p)$. In the previous paper [7, (2) of Theorem 1], for $p = 2$ we gave a generalization of a theorem of Hartman and Wintner (cf. [2, Theorem 7.20]) to arbitrary weight in $(A_2)$. In §3 we improve this theorem for $p = 2$ and we generalize this theorem for arbitrary $p$ and arbitrary weight in $(A_p)$. For each inner function $q$, sing $q$ denotes the subset of $\partial D$ on which $q$ can not be analytically extended. For two inner functions $q_1$ and $q_2$, M. Lee and D. Sarason [5] showed that $\sigma(T_\phi) = D$ if $\phi = q_1\bar{q}_2$ and sing $q_1 \neq$ sing $q_2$.

For $\alpha = \alpha_1 + i\alpha_2 \in \mathcal{C}$ and $\beta = \beta_1 + i\beta_2 \in \mathcal{C}$, put $\langle \alpha, \beta \rangle = \alpha_1\beta_1 + \alpha_2\beta_2$ and $\theta(\alpha, \beta) = \text{arccos}(\langle \alpha, \beta \rangle / ||\alpha|| ||\beta||)$ for $\alpha \neq 0$ and $\beta \neq 0$. Set

$$\ell_\alpha^+ = \{ z \in \mathcal{C} : \langle z, \alpha \rangle \geq 1 \} \quad \text{and} \quad \ell_\alpha^- = \{ z \in \mathcal{C} : \langle z, \alpha \rangle \leq 1 \}$$

and let $\mathcal{E}_{\alpha\beta}$ denote $\ell_\alpha^+ \cap \ell_\beta^-$. For each pair $(\alpha, \beta)$

$$\mathcal{C} = \mathcal{E}_{\alpha\beta}^+ \cup \mathcal{E}_{\alpha\beta}^- \cup \mathcal{E}_{\alpha\beta}^+ \cup \mathcal{E}_{\alpha\beta}^-$$
and if \( \ell = -i \) and \( m = -j \), then
\[
(\mathcal{E}^\text{cm}_{a\beta})^c = \mathcal{C} \setminus \mathcal{E}^\text{cm}_{a\beta} \supset \mathcal{E}^{ij}_{a\beta}.
\]
For any bounded subset \( E \) in \( \mathcal{C} \), there exists a pair \((\alpha, \beta)\) such that \( \mathcal{E}^{ij}_{a\beta} \supset E \) for some \((i, j)\). When \( 0 < t < \pi/2 \), put
\[
h^I(E) = \bigcap \left\{ (\mathcal{E}^\text{cm}_{a\beta})^c ; \mathcal{E}^{ij}_{a\beta} \supset E \text{ and } \ell = -i, \ m = -j, |\theta(\alpha, \beta)| = \pi - 2t \right\}
\]
for a subset \( E \) in \( \mathcal{C} \). If \( t = 0 \), then \( h^0(E) \) is the closed convex hull of \( E \). If \( E \) is a simple set such that \( E = [a, b] \) or \( E = \{ z \in \mathcal{C} ; |z| \leq 1 \} \), then we can describe \( h^I(E) \) for \( 0 \leq t < \pi/2 \).

If a weight \( W \) satisfies the condition \((A_p)\) then \( \log W \) belongs to BMO and so there exist two real valued function \( u \) and \( v \) in \( L^\infty_R \) such that \( \log W = u + \tilde{v} \) where \( \tilde{v} \) denotes the harmonic conjugate with \( \tilde{v}(0) = 0 \). For \( W = e^{u+\tilde{v}} \), put
\[
t_W = \|v\|^t = \inf \{ \|v - \tilde{s} - a\|_\infty ; s \in L^\infty_R, a \in R \}.
\]
In the previous paper [7, (1) of Theorem 1], we showed that \( \sigma(T^W_{\phi} \mathcal{C}) \subseteq h^I(R(\phi)) \) for \( t = t_W \). This implies a theorem of Brown and Halmos (cf. [2, Corollary 7.19]) for \( W \equiv 1 \), that is, \( \sigma(T^W_{\phi}) \subseteq h^I(R(\phi)) \). In this paper, we generalize this result for \( T^W_{\phi} \), that is, if \( t = \frac{\pi}{2} \left( 1 - \frac{2}{\max(p, q)} \right) + \frac{2}{p} t_W \) then \( \sigma(T^W_{\phi}) \subseteq h^I(R(\phi)) \) because \( t = t_W \) for \( p = 2 \).

In this paper, we use the following theorems about the invertibility of Toeplitz operators on \( \mathcal{H}^p(W) \) or \( \mathcal{H}^p \). The first one is due to H.Widom, A.Devinatz and R.Rochberg (cf. [1, Theorem 5.3], [6]). The second one is due to N.Krupnik (cf. [1, Theorem 5.22]).

**Theorem WDR.** Suppose \( 1 < p < \infty \) and \( W = |h|^p \) satisfies the condition \((A_p)\), where \( h \) is an outer function in \( \mathcal{H}^p \). Then the following conditions on \( \phi \) and \( W \) are equivalent.

1. \( T^W_{\phi} \) is an invertible operator on \( \mathcal{H}^p(W) \).
2. \( \phi = k(h_0/h_0)(h/h) \), where \( k \) is an invertible function in \( \mathcal{H}^\infty \) and \( h_0 \) is an outer function in \( \mathcal{H}^p \) with \( |h_0|^p \) satisfying the condition \((A_p)\).
3. \( \phi = \gamma \exp(U - iV) \), where \( \gamma \) is constant with \( |\gamma| = 1 \), \( U \) is a bounded real function in \( L^1 \) and \( W \exp \left( \frac{p}{2} V \right) \) satisfies \((A_p)\).

**Theorem K.** Suppose \( 1 < p < \infty \) and \( 1/p + 1/q = 1 \), and \( \phi \) is a function in \( \mathcal{L}^\infty \). The following are equivalent.

1. Both \( T^p_{\phi} \) and \( T^q_{\phi} \) are invertible on \( \mathcal{H}^p \) and \( \mathcal{H}^q \), respectively.
2. \( T^\ell_{\phi} \) is invertible for all \( \ell \) with \( \min\{p, q\} \leq \ell \leq \max\{p, q\} \).
3. \( \phi = ke^{U+iV} \), where \( k \) is an invertible function in \( \mathcal{H}^\infty \), \( U \) and \( V \) are bounded real functions and \( ||V||_\infty < \pi/\max\{p, q\} \).
In this paper, \( W \in (A_p) \) means that \( W \) satisfies the condition \((A_p)\).

§2. Arbitrary symbols

Corollary 1 was proved in the previous paper [7, Theorem 1]. Corollary 2 was proved for \( p \geq 2 \) in [7, Theorem 3]. Corollaries 1 and 2 are just the generalizations of a theorem of Brown and Halmos (cf. [2, Proposition 7.19]). Theorem 2 for \( p = 2 \) was proved in [7, (1) of Theorem 2]. I. Spitkovsky [10] showed that the set of all weights \( W \) for which \( \sigma(T^W_p) = \sigma(T^\phi) \) for any \( \phi \) in \( L^\infty \) does not depend on \( p \). Hence Theorem 2 for \( 1 < p < \infty \) follows. We give another proof.

**Theorem 1.** Suppose \( W \) satisfies the condition \((A_p) \cap (A_q)\) where \( 1 < p < \infty \) and \( 1/p + 1/q = 1 \), and \( t = \frac{\pi}{2} \left( \frac{1}{2} - \frac{1}{\max(p,q)} \right) + \frac{2}{p} W \). If \( \phi \) is a function in \( L^\infty \), then

\[
\mathcal{R}(\phi) \subseteq \sigma(T^W_p) \subseteq h^t(\mathcal{R}(\phi)).
\]

**Proof.** By Theorem WDR, it is clear that \( \mathcal{R}(\phi) \subseteq \sigma(T^W_p) \). We will show that \( \sigma(T^W_p) \subseteq h^t(\mathcal{R}(\phi)) \). Suppose \( \lambda \notin h^t(\mathcal{R}(\phi)) \). Then by definition \( \lambda \notin \cup \{ (E_{\ell m})^0 : E_{ij}^\alpha \beta \supseteq \mathcal{R}(\phi) \} \) and \( \ell = -i, m = -j, |\theta(\alpha, \beta)| = \pi - 2t \). Then \( (\phi - \lambda)/|\phi - \lambda| = e^{i\xi} \lambda \) where \( 0 \leq s_\lambda \leq \pi - 2t - 2\varepsilon \) a.e. or \( \lambda \leq -2t + 2\varepsilon \leq s_\lambda \leq 0 \) a.e. for some \( \varepsilon > 0 \). Hence \( |s_\lambda + \frac{\pi}{2} - t + \varepsilon| \leq \frac{\pi}{2} - t - \varepsilon \) a.e. or \( |s_\lambda + \frac{\pi}{2} - t - \varepsilon| \leq \frac{\pi}{2} + t - \varepsilon \) a.e. Let \( W = |h|^p \) and \( h^p = \exp(u + \tilde{v} + i(\tilde{u} - v)) \). Then

\[
\frac{\phi - \lambda}{|\phi - \lambda|} \tilde{h} = \exp i(s_\lambda + \frac{2}{p}(v - \tilde{u}))
\]

and

\[
\|s_\lambda + \frac{2}{p}(v - \tilde{u})\|' = \|s_\lambda + \frac{2}{p}v\|' \leq \frac{\pi}{2} - t - \varepsilon + \frac{2}{p}\|v\|' = \frac{\pi}{2} - \frac{2}{2} \left( 1 - \frac{2}{\max(p,q)} \right) - \frac{2}{p} t_W - \varepsilon + \frac{2}{p} t_W = \frac{\pi}{\max(p,q)} - \varepsilon.
\]

By Theorem K, \( T^\phi_{\lambda - \lambda} h^p \) is invertible and so by Theorem WDR \( T^W_{\phi - \lambda} h^p \) is invertible. Thus \( \lambda \notin \sigma(T^W_{\phi - \lambda}) \).

**Corollary 1.** Suppose \( W = e^{u_i + \bar{v}_i} \) is a Helson-Szegő weight and \( t = t_W \). If \( \phi \) is a function in \( L^\infty \), then \( \mathcal{R}(\phi) \subseteq \sigma(T^W_p) \subseteq h^t(\mathcal{R}(\phi)) \).
Corollary 2. Suppose $W \equiv 1$, $1 < p < \infty$ and $1/p + 1/q = 1$ and $t = |p - 2|\pi/2p$.
If $\phi$ is a function in $L^\infty$, then $\mathcal{R}(\phi) \subseteq \sigma(T^p_\phi) \subseteq h^t(\mathcal{R}(\phi))$.

Proof. Since $W \equiv 1$, $t = \pi/2 \left(1 - \frac{2}{\max(p, q)}\right)$. If $p \geq 2$, then $t = \pi/2 \left(1 - \frac{2}{p}\right) = \frac{\pi(p - 2)}{2p}$. If $1 < p < 2$, then $t = \pi/2 \left(1 - \frac{2}{q}\right) = \frac{\pi(2 - p)}{2p}$ because $q = \frac{p}{p - 1}$.

Theorem 2. Suppose $W$ satisfies the condition $(A_p)$ for some $p$ with $1 < p < \infty$. Then, $t_W = 0$ if and only if $\sigma(T^w_\phi) = \sigma(T^p_\phi)$ for any $\phi$ in $L^\infty$.

Proof. Suppose that $\sigma(T^w_\phi) = \sigma(T^p_\phi)$ for any $\phi$ in $L^\infty$. If $\phi = \tilde{h}_0/h_0$ and $h_0$ is an outer function with $|h_0|^p \in (A_p)$, then $T^p_\phi$ is invertible and so $T^w_\phi$ is invertible. Put $h_0 = \exp \frac{1}{p}(u_0 + \tilde{v}_0 + i(\tilde{u}_0 - v_0))$ where $u_0 \in L^\infty_R$ and $v_0 \in L^\infty_R$. Then

$$\phi = \frac{\tilde{h}_0}{h_0} = \exp i \frac{2}{p}(v_0 - \tilde{u}_0).$$

Since $T^w_\phi$ is invertible, by Theorem WDR $W|h_0|^p = W \exp(\tilde{v}_0 + u_0)$ belongs to $(A_p)$. Thus $W(A_p) \subseteq (A_p)$ and so by [4, Theorem 2.12] $t_W = 0$.

Conversely if $t_W = 0$ then log $W$ belongs to the closure of $L^\infty$ in BMO. Hence $W(A_p) = (A_p)$ by [4, Theorem 2.12]. Let $W = |h|^p$ and $h$ an outer function in $H^p$. By Theorem WDR in Introduction, $T^w_\phi$ is invertible if and only if $T^w_{\phi/|\phi|}$ is invertible and $\phi$ is invertible in $L^\infty$. If $T^w_{\phi/|\phi|}$ is invertible then by Theorem WDR

$$\frac{\phi}{|\phi|} = \frac{h}{|h|} \frac{\tilde{h}_0}{h_0}$$

for some outer function $h_0$ with $|h_0|^p \in (A_p)$. Since $W(A_p) = (A_p)$, $|h_0|^p|h|^{-p} \in (A_p)$ and $\phi = \frac{\tilde{h}_0 h^{-1}}{h_0 h^{-1}}$. This implies that $T^p_{\phi/|\phi|}$ is invertible. Thus $\sigma(T^w_{\phi/|\phi|}) \supseteq \sigma(T^p_\phi)$ for any $\phi$ in $L^\infty$. If $T^w_{\phi/|\phi|}$ is invertible then $\phi/|\phi| = \tilde{h}_1/h_1$ for some outer function $h_1$ with $|h_1|^p \in (A_p)$. Since $W(A_p) = (A_p)$, $|h|^p|h_1|^p \in (A_p)$ and so

$$\frac{\phi}{|\phi|} = \frac{h}{h_1} \frac{\tilde{h}_1}{h_1 h}.$$ 

Hence $T^w_{\phi/|\phi|}$ is invertible. Thus $\sigma(T^w_{\phi/|\phi|}) \subseteq \sigma(T^p_\phi)$ for any $\phi$ in $L^\infty$. Therefore $\sigma(T^w_{\phi/|\phi|}) = \sigma(T^p_\phi)$ for any $\phi$ in $L^\infty$.

§3. Special symbols

In this section, we study the spectrum of a Toeplitz operator whose symbol is continuous, real-valued or the quotient of two inner functions. Theorem 3 generalizes
Theorem 3. Let $1 < p < \infty$. If $\phi$ is a continuous function on $T$ then

$$\sigma(T_{\phi}^{W,p}) = \mathcal{R}(\phi) \cup \{\lambda \in \mathcal{C} \mid i_\ell(\phi, \lambda) \neq 0\}$$

for any $W$ in $(A_p)$, where $i_\ell(\phi, \lambda)$ is the winding number of the curve determined by $\phi$ with respect to $\lambda$.

Proof. If $\lambda \notin \mathcal{R}(\phi)$ and $i_\ell(\phi, \lambda) = 0$ then $(\phi - \lambda)/|\phi - \lambda| = e^{is_\lambda}$ where $s_\lambda \in \mathcal{C}$ and so $W \exp P_2(-\bar{s}_\lambda)$ belongs to $(A_p)$. By Theorem WDR this implies that $\lambda \notin \sigma(T_{\phi}^{W,p})$. Conversely if $\lambda \notin \sigma(T_{\phi}^{W,p})$ then $\lambda \notin \mathcal{R}(\phi)$. Hence

$$\frac{\phi - \lambda}{|\phi - \lambda|} = z^\ell e^{is_\lambda}$$

where $\ell$ is an integer and $s_\lambda \in \mathcal{C}$. Since $T_{\phi-\lambda}^{W,p}$ is invertible, by Theorem WDR there exists an outer function $h_1$ such that

$$\frac{\phi - \lambda}{|\phi - \lambda|} = h_1 h$$

where $|h_1|^p \in (A_p)$ and $W = |h|^p \in (A_p)$ and $h$ is an outer function. Then $|h|^{-q} \in (A_q)$ where $1/p + 1/q = 1$ and so $f = h^{-1}h_1$ belongs to $H^t$ for some $t > 1$. Put $g^2 = \exp(-s_\lambda + is_\lambda)$ then $g \in \bigcap_{1 \leq s < \infty} H^s$ and so $gf$ belongs to $H^1$. Similary we can show that $(gf)^{-1}$ belongs to $H^1$. Then if $\ell \geq 0$ then $z^\ell gf = g\bar{f}$ and $z^\ell(gf)^2$ is nonnegative in $H^{1/2}$. Hence $\ell = 0$ because $H^{1/2}$ does not contain any nonconstant nonnegative functions. If $\ell \leq 0$ then $z^\ell(gf)^{-2}$ is nonnegative in $H^{1/2}$ and so $\ell = 0$. Thus $(\phi - \lambda)/|\phi - \lambda| = e^{is_\lambda}$ and so $\lambda \notin \sigma(T_{\phi}^{W,p})$ because $e^{is_\lambda} = g/\overline{g}$ and $|g^{-1}| \in (A_p)$.

Theorem 4. Suppose $W$ satisfies the condition $(A_p) \cap (A_q)$ where $1 < p < \infty$ and $1/p + 1/q = 1$, and $t = \frac{\pi}{2} \left(1 - \frac{2}{\max(p,q)}\right) + \frac{2}{p}t_W$. If $\phi$ is real valued, $a = \text{ess inf} \phi$ and $b = \text{ess sup} \phi$, then

$$\mathcal{R}(\phi) \subseteq \sigma(T_{\phi}^{W,p}) \subseteq \triangle(c, r) \cap \triangle(\bar{c}, r)$$

where $c = \frac{a + b}{2} - i \frac{a - b}{2} \cos 2t$ and $r = -\frac{a - b}{2} \sin 2t$. If $t_W = 0$ then $[a, b] \subseteq \sigma(T_{\phi}^{W,p})$.

Proof. By Theorem 1, $\sigma(T_{\phi}^{W,p}) \subseteq h^t(\mathcal{R}(\phi)) \subseteq h^t([a, b])$ for $t = \frac{\pi}{2} \left(1 - \frac{2}{\max(p,q)}\right) + \frac{2}{p}t_W$. It is elementary to see that $h^t([a, b]) \subseteq \triangle(c, r) \cap \triangle(\bar{c}, r)$. Suppose $t_W = 0$. 

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Then $t = \frac{\pi}{2} \left( 1 - \frac{2}{\max(p, q)} \right)$ and by Theorem 2 $\sigma(T^W_\phi) = \sigma(T^p_\phi)$. We will show that $[a, b] \subseteq \sigma(T^p_\phi)$. Suppose $\lambda \in [a, b]$ and $\lambda \notin \mathcal{R}(\phi)$, then $\psi = (\phi - \lambda)/|\phi - \lambda| = 2\chi_E - 1$ for some measurable set $E$ in $\partial D$. If $\lambda \notin \sigma(T^p_\phi)$, then by Theorem WDR there exists an outer function $h_1$ in $H^p$ with $h_1^{-1}$ in $H^q$ such that $h_1 = \bar{h}_1/h_1$. Since $T^p_\phi = T^q_\psi$ is also invertible, there exists an outer function $h_2$ in $H^q$ with $h_2^{-1}$ in $H^p$ such that $h = h_2/h_2$. Hence

$$\frac{\bar{h}_1}{h_1} = \frac{h_1}{h_1} \frac{h_2}{h_2} = h_2$$

because $\psi$ is a real valued function. Hence $h_1^2 = \bar{h}_1^2 \in H^{p/2}$ and $h_2^2 = \bar{h}_2^2 \in H^{q/2}$. Therefore $h_1$ or $h_2$ is constant because $\max(p/2, q/2) \geq 1$ and the only real function in $H^1$ is constant. Thus $\psi$ is constant and this contradicts that $\phi$ is not constant. Thus $[a, b] \subseteq \sigma(T^p_\phi)$.

For a weight $W$ in $(A_p)$ and a measurable set $E$, put

$$\gamma_+(E, W, p) = \sup \{ t > 0 ; W \exp(t\chi_E) \text{ satisfies } (A_p) \}$$

and

$$\gamma_-(E, W, p) = \inf \{ t < 0 ; W \exp(t\chi_E) \text{ satisfies } (A_p) \}. $$

**Theorem 5.** Let $W$ satisfy the condition $(A_p)$ and $1 < p < \infty$. Suppose $\phi = a\chi_E + b\chi_{E^c}$ where $a, b$ are real numbers and $E$ is measurable set in $\partial D$ with $0 < d\theta(E) < 2\pi$. Then

$$\sigma(T^W_\phi) = \{ \lambda \in \mathcal{C} ; \pi \geq \text{Arg} \frac{a - \lambda}{b - \lambda} \geq \frac{2}{p} \gamma_+(E, W, p) \}$$

or $-\pi \leq \text{Arg} \frac{a - \lambda}{b - \lambda} \leq \frac{2}{p} \gamma_-(E, W, p) \}$

where $-\pi \leq \text{Arg}z \leq \pi$. In particular, $\sigma(T^W_\phi) \supseteq [a, b]$.

**Proof.** If $\lambda \neq a, b$, and $\lambda$ is a real number then

$$\frac{\phi - \lambda}{|\phi - \lambda|} = \frac{a - \lambda}{|a - \lambda|} \chi_E + \frac{b - \lambda}{|b - \lambda|} \chi_{E^c}. $$

There exist $a(\lambda)$ and $b(\lambda)$ such that $-\pi \leq a(\lambda), b(\lambda) \leq \pi$ and

$$\frac{a - \lambda}{|a - \lambda|} = e^{ia(\lambda)}, \quad \frac{b - \lambda}{|b - \lambda|} = e^{ib(\lambda)}. $$

Hence

$$\frac{\phi - \lambda}{|\phi - \lambda|} = \exp i \{ (a(\lambda) - b(\lambda)) \chi_E + b(\lambda) \}$$
where $0 \leq a(\lambda) - b(\lambda) \leq \pi$ or $-\pi \leq a(\lambda) - b(\lambda) \leq 0$. If $\lambda \not\in \sigma(T^{W,p}_\phi)$, then by Theorem WDR \[ W \exp \left\{ \frac{P}{2} (a(\lambda) - b(\lambda))\hat{\chi}_E \right\} \] belongs to $(A_p)$. Hence

\[ \sigma(T^{W,p}_\phi) \subseteq \left\{ \lambda \in \mathcal{C} \mid \pi \geq a(\lambda) - b(\lambda) \geq \frac{2}{p} \gamma_+(E, W, p) \right\} \]

\[ \cup \left\{ \lambda \in \mathcal{C} \mid \frac{2}{p} \gamma_-(E, W, p) \geq a(\lambda) - b(\lambda) \geq -\pi \right\}. \]

If $\pi \geq a(\lambda) - b(\lambda) > \frac{2}{p} \gamma_+(E, W, p)$ or $-\pi \leq a(\lambda) - b(\lambda) < \frac{2}{p} \gamma_-(E, W, p)$, then

\[ W \exp \left\{ \frac{P}{2} (a(\lambda) - b(\lambda))\hat{\chi}_E \right\} \] does not belong to $(A_p)$ and so $\lambda \in \sigma(T^{W,p}_\phi)$. Since $\sigma(T^{W,p}_\phi)$ is closed,

\[ \sigma(T^{W,p}_\phi) = \left\{ \lambda \in \mathcal{C} \mid \pi \geq a(\lambda) - b(\lambda) \geq \frac{2}{p} \gamma_+(E, W, p) \right\} \]

\[ \cap \left\{ \lambda \in \mathcal{C} \mid \frac{2}{p} \gamma_-(E, W, p) \geq a(\lambda) - b(\lambda) \geq -\pi \right\}. \]

**Lemma 1.** For a measurable set $E$ in $T$ with $0 < m(E) < 1$, $\|\pi\chi_E - v\|' \geq \pi/2$ for any $v$ in $L^n_R$ with $\|v\|_\infty < \pi/2$.

Proof. Suppose $\phi = a\chi_E + b\chi_E^c$ where $a$ and $b$ are real numbers, $a \neq b$ and $0 < m(E) < 1$. For $W = e^{u+\tilde{v}}$ where $u, v \in L^n_R$ and $\|v\|_\infty < \pi/2$, $\sigma(T^{W,2}_\phi) \supseteq [a, b]$ if and only if $\|\pi\chi_E - v\|' \geq \pi/2$. This is proved in [7, Corollary 1]. Now Theorem 5 shows Lemma 1.

**Corollary 3.** Suppose $W$ satisfies the condition $(A_2)$. If $\phi$ is real valued, $a = \text{ess inf} \phi$ and $b = \text{ess sup} \phi$ then $[a, b] \subseteq \sigma(T^{W,2}_\phi)$.

Proof. Since $W \in (A_2)$, $W = e^{u+\tilde{v}}$ where $u, v \in L^n_R$ and $\|v\|_\infty < \pi/2$. For $\lambda \in [a, b] \cap \mathcal{R}(\phi)^e$, \[ \frac{\phi - \lambda}{|\phi - \lambda|} = e^{\ell} \] and $\ell = \pi(1 - \chi_E)$ for some measurable set $E$ in $T$ with $0 < m(E) < 1$. Then, in [7, (3) of Theorem 1], it is proved that $\lambda \in \sigma(T^{W,2}_\phi)$ if and only if $\|\pi\chi_E - v\|' \geq \pi/2$. Now Lemma 1 implies this corollary.

**Lemma 2.** If $q_1$ and $q_2$ are inner functions and $\tilde{q}_1q_2 = f/|f| = |g|/g$ where both $f$ and $g$ are in $\cap_{p>1/2} H^p$, then $\text{sing} q_1 \neq \text{sing} q_2$.

Proof. See the proof of [8, Corollary 5].

**Theorem 6.** Suppose $W$ satisfies the condition $(A_p)$ where $1 < p < \infty$. If $\phi = \tilde{q}_1q_2$ where $q_1$ and $q_2$ are inner functions with $\text{sing} q_1 \neq \text{sing} q_2$ then $\sigma(T^{W,p}_\phi) = D$.

Proof. Suppose $W = |h|^p$ for some outer function in $H^p$. If $\lambda \in D$ then \[ \tilde{q}_1q_2 - \lambda = \tilde{q}_1(q_2 - \lambda q_1) = \tilde{q}_1q_3k \]
where $q_3$ is inner and $k$ is invertible in $H^\infty$. By the proof of [8, Theorem 2(2)] $\bar{q}_2 q_3 = \frac{f}{|f|} = \frac{|g|}{g}$ where both $f$ and $g$ are in $H^1$. By Lemma 2 $\text{sing} q_2 = \text{sing} q_3$ and so $\text{sing} q_1 \neq \text{sing} q_3$. If $\lambda \notin \sigma(T_{\bar{q}_1 q_2}^{W,p})$ then $0 \notin \sigma(T_{\bar{q}_3 q_3}^{W,p})$ because $k$ is invertible in $H^\infty$. By Theorem WDR

$$\bar{q}_1 q_3 = \frac{h}{h_0} \frac{\bar{h} h_0}{\bar{h} h_0}$$

where $h_0$ is an outer function in $H^p$ with $|h_0|^p \in (A_p)$. Hence $\bar{q}_1 q_3 = f/|f| = |g|/g$ where $f = (h/h_0)^2$ and $g = (h_0/h)^2$. Since $|h|^p$ and $|h_0|^p$ are in $(A_p)$, both $f$ and $g$ belong to $H^{1/2}$. This contradicts Lemma 2. Hence $\lambda \in \sigma(T_{\bar{q}_1 q_2}^{W,p})$ and so $\sigma(T_{\bar{q}_3 q_3}^{W,p}) = \bar{D}$.

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