Brown-Halmos Type Theorems Of Weighted Toeplitz Operators II

By

Takahiko Nakazi*

* This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education

2000 Mathematics Subject Classification : 47 B 35

Key words and phrases : Toeplitz operator, weighted Hardy space, spectrum
Abstract. The spectra of the Toeplitz operators on the weighted Hardy space $H^p(Wd\theta/2\pi)$ are studied. For example, the theorems of Brown-Halmos type and Hartman-Wintner type are proved. These generalize results in the previous paper which were proved for $p = 2$. 
§1. Introduction

Let \( m = d\theta/2\pi \) be the normalized Lebesgue measure on the unit circle \( T \) and let \( W \) be a non-negative integrable function on \( T \) which does not vanish identically. Suppose \( 1 \leq p \leq \infty \). Let \( L^p(W) = L^p(Wdm) \) and \( L^p(W) = L^p \) when \( W \equiv 1 \). Let \( H^p(W) \) denote the closure in \( L^p(W) \) of the set \( \mathcal{P} \) of all analytic polyomials when \( p \neq \infty \). We will write \( H^p(W) = H^p \) when \( W \equiv 1 \), and then this is a usual Hardy space. \( H^\infty \) denotes the weak * closure of \( \mathcal{P} \) in \( L^\infty \). \( P \) denotes the projection from the set \( \mathcal{C} \) of all trigonometric polynomials to \( \mathcal{P} \). For \( 1 < p < \infty \), \( P \) can be extended to a bounded map of \( L^p(W) \) onto \( H^p(W) \) if and only if \( W \) satisfies the condition \( (A_p) \) (see [3, Theorem 6.2 of Chapter VII]). This is the well known theorem of Hunt, Muckenhoupt and Wheeden, which is a generalization of the theorem of Helson and Szegö (see [3, Theorem 3.2 of Chapter IV]).

Assuming that a weight \( W \) satisfies the condition \( (A_p) \) for \( 1 < p < \infty \), we define a Toeplitz operator \( T_\phi^{W,p} \) on \( H^p(W) \) as follows. For \( \phi \) in \( L^\infty \), suppose that

\[
T_\phi^{W,p} f = P(\phi f) \quad (f \in H^p(W)).
\]

If \( W \equiv 1 \), we will write \( T_\phi^{W,p} = T_\phi^p \).

In this paper, we study the spectrum \( \sigma(T_\phi^{W,p}) \) of a Toeplitz operator \( T_\phi^{W,p} \). For any weights \( W \) in \( (A_p) \) and for any \( \phi \) in \( L^\infty \), the symbol \( \phi \) for invertible \( T_\phi^{W,p} \) was completely described by H.Widom, A.Devinatz and R.Rochberg (see Theorem WDR in this section). This is one of our main tools. In the previous paper [7, (1) of Theorem 1], for \( p = 2 \) we gave a generalization of a theorem of Brown and Halmos [2, Proposition 7.19] to arbitrary weight in \( (A_2) \). In §2 we generalize this theorem for arbitrary \( p \). I.Spitkovsky [10] showed that the set of all weights \( W \) for which \( \sigma(T_\phi^{W,p}) = \sigma(T_\phi^p) \) for all \( \phi \) in \( L^\infty \) does not depend on \( p \). In §2 we give another proof of this result. In fact we describe such a set of weights by using [4, Theorem 2.12]. This also generalizes (1) of Theorem 2 of the previous paper [7].

When \( \phi \) is a continuous function and \( W \equiv 1 \), the spectrum of \( T_\phi^p \) was completely described (cf. [2, Corollary 7.28]). In §3 we prove \( \sigma(T_\phi^{W,p}) = \sigma(T_\phi^p) \) for any continuous function \( \phi \) whenever \( W \) satisfies the condition \( (A_p) \). In the previous paper [7, (2) of Theorem 1], for \( p = 2 \) we gave a generalization of a theorem of Hartman and Wintner (cf. [2, Theorem 7.20]) to arbitrary weight in \( (A_2) \). In §3 we improve this theorem for \( p = 2 \) and we generalize this theorem for arbitrary \( p \) and arbitrary weight in \( (A_p) \). For each inner function \( q \), \( \text{sing} \ q \) denotes the subset of \( \partial D \) on which \( q \) can not be analytically extended. For two inner functions \( q_1 \) and \( q_2 \), M.Lee and D.Sarason [5] showed that \( \sigma(T_\phi) = D \) if \( \phi = \bar{q}_1 q_2 \) and \( \text{sing} \ q_1 \neq \text{sing} \ q_2 \).

For \( \alpha = \alpha_1 + i\alpha_2 \in \mathcal{C} \) and \( \beta = \beta_1 + i\beta_2 \in \mathcal{C} \), put \( \langle \alpha, \beta \rangle = \alpha_1 \beta_1 + \alpha_2 \beta_2 \) and \( \theta(\alpha, \beta) = \arccos(\langle \alpha, \beta \rangle/||\alpha||||\beta||) \) for \( \alpha \neq 0 \) and \( \beta \neq 0 \). Set

\[
\ell_\alpha^+ = \{ z \in \mathcal{C} \mid \langle z, \alpha \rangle \geq 1 \} \quad \text{and} \quad \ell_\alpha^- = \{ z \in \mathcal{C} \mid \langle z, \alpha \rangle \leq 1 \}
\]

and let \( \mathcal{E}_{\alpha_\beta} \) denote \( \ell_\alpha^+ \cap \ell_\beta^- \) where \( i = + \) or \( - \) and \( j = + \) or \( - \). For each pair \( (\alpha, \beta) \)

\[
\mathcal{C} = \mathcal{E}_{\alpha_\beta}^+ \cup \mathcal{E}_{\alpha_\beta}^- \cup \mathcal{E}_{\alpha_\beta}^{++} \cup \mathcal{E}_{\alpha_\beta}^{--}
\]
and if $\ell = -i$ and $m = -j$, then
$$
(\mathcal{E}_{i\beta}^{t,m})^c = \mathcal{C} \setminus \mathcal{E}_{i\beta}^{t,m} \supset \mathcal{E}_{i\beta}^{t,j}.
$$

For any bounded subset $E$ in $\mathcal{C}$, there exists a pair $(\alpha, \beta)$ such that $\mathcal{E}_{i\beta}^{t,j} \supset E$ for some $(i,j)$. When $0 \leq t < \pi/2$, put
$$
h^t(E) = \bigcap \{ (\mathcal{E}_{i\beta}^{t,m})^c ; \mathcal{E}_{i\beta}^{t,j} \supset E \text{ and } \ell = -i, m = -j, |\theta(\alpha, \beta)| = \pi - 2t \}
$$
for a subset $E$ in $\mathcal{C}$. If $t = 0$, then $h^0(E)$ is the closed convex hull of $E$. If $E$ is a simple set such that $E = [a, b]$ or $E = \{ z \in \mathcal{C} ; |z| \leq 1 \}$, then we can describe $h^t(E)$ for $0 \leq t < \pi/2$.

If a weight $W$ satisfies the condition $(A_p)$ then $\log W$ belongs to BMO and so there exist two real valued functions $u$ and $v$ in $L_R^\infty$ such that $\log W = u + \hat{v}$ where $\hat{v}$ denotes the harmonic conjugate with $\hat{v}(0) = 0$. For $W = e^{u + \hat{v}}$, put
$$
t_W = \| v \|_p' = \inf \{ \| v - \hat{s} - a \|_\infty ; s \in L_R^\infty, a \in R \}.
$$
In the previous paper [7, (1) of Theorem 1], we showed that $\sigma(T_{i\beta}^{W,2}) \subseteq h^t(R(\phi))$ for $t = t_W$. This implies a theorem of Brown and Halmos (cf. [2, Corollary 7.19]) for $W \equiv 1$, that is, $\sigma(T_{i\beta}^{2}) \subseteq h^0(R(\phi))$. In this paper, we generalize this result for $T_{i\beta}^{W,p}$, that is, if $t = \frac{\pi}{2} \left( 1 - \frac{2}{\max(p, q)} \right) + \frac{2}{p} t_W$ then $\sigma(T_{i\beta}^{W,p}) \subseteq h^t(R(\phi))$ because $t = t_W$ for $p = 2$.

In this paper, we use the following theorems about the invertibility of Toeplitz operators on $H^p(W)$ or $H^p$. The first one is due to H.Widom, A.Devinatz and R.Rochberg (cf. [1, Theorem 5.3], [6]). The second one is due to N.Krupnik (cf. [1, Theorem 5.22]).

**Theorem WDR.** Suppose $1 < p < \infty$ and $W = |h|^p$ satisfies the condition $(A_p)$, where $h$ is an outer function in $H^p$. Then the following conditions on $\phi$ and $W$ are equivalent.

1. $T_{i\beta}^{W,p}$ is an invertible operator on $H^p(W)$.
2. $\phi = k(\tilde{h}_0/h_0)(\tilde{h}/h)$, where $k$ is an invertible function in $H^\infty$ and $h_0$ is an outer function in $H^p$ with $|h_0|^p$ satisfying the condition $(A_p)$.
3. $\phi = \gamma \exp(U - iV)$, where $\gamma$ is constant with $|\gamma| = 1$, $U$ is a bounded real function in $L^1$ and $W \exp \left( \frac{p}{2} V \right)$ satisfies $(A_p)$.

**Theorem K.** Suppose $1 < p < \infty$ and $1/p + 1/q = 1$, and $\phi$ is a function in $L^\infty$. The following are equivalent.

1. Both $T_{i\beta}^{0}$ and $T_{i\beta}^{1}$ are invertible on $H^p$ and $H^q$, respectively.
2. $T_{i\beta}^{\ell}$ is invertible for all $\ell$ with $\min(p, q) \leq \ell \leq \max(p, q)$.
3. $\phi = ke^{U+iV}$, where $k$ is an invertible function in $H^\infty$, $U$ and $V$ are bounded real functions and $\|V\|_\infty < \pi/\max(p, q)$. 

4
In this paper, $W \in (A_p)$ means that $W$ satisfies the condition $(A_p)$.

§2. Arbitrary symbols

Corollary 1 was proved in the previous paper [7, Theorem 1]. Corollary 2 was proved for $p \geq 2$ in [7, Theorem 3]. Corollaries 1 and 2 are just the generalizations of a theorem of Brown and Halmos (cf. [2, Proposition 7.19]). Theorem 2 for $p = 2$ was proved in [7, (1) of Theorem 2]. I. Spitkovsky [10] showed that the set of all weights $W$ for which $\sigma(T_W^\phi) = \sigma(T_\phi)$ for any $\phi$ in $L^\infty$ does not depend on $p$. Hence Theorem 2 for $1 < p < \infty$ follows. We give another proof.

**Theorem 1.** Suppose $W$ satisfies the condition $(A_p) \cap (A_q)$ where $1 < p < \infty$ and $1/p + 1/q = 1$, and $t = \frac{\pi}{2} \left( 1 - \frac{2}{\max(p,q)} \right) + \frac{2}{p} t_W$. If $\phi$ is a function in $L^\infty$, then $\mathcal{R}(\phi) \subseteq \sigma(T_W^{\phi,p}) \subseteq h^t(\mathcal{R}(\phi))$.

Proof. By Theorem WDR, it is clear that $\mathcal{R}(\phi) \subseteq \sigma(T_W^{\phi,p})$. We will show that $\sigma(T_W^{\phi,p}) \subseteq h^t(\mathcal{R}(\phi))$. Suppose $\lambda \notin h^t(\mathcal{R}(\phi))$. Then by definition $\lambda \in \mathcal{R}(\phi)$ and $\ell = -i$, $m = -j$, $|\theta(\alpha, \beta)| = \pi - 2t$. Then $(\phi - \lambda)/|\phi - \lambda| = e^{i s_\lambda}$ where $0 \leq s_\lambda \leq \pi - 2t - 2\varepsilon$ a.e. or $-\pi + 2t + 2\varepsilon \leq s_\lambda \leq 0$ a.e. for some $\varepsilon > 0$. Hence $|s_\lambda - \frac{\pi}{2} + t + \varepsilon| \leq \frac{\pi}{2} - t - \varepsilon$ a.e. or $|s_\lambda + \frac{\pi}{2} - t - \varepsilon| \leq \frac{\pi}{2} - t - \varepsilon$ a.e. Let $W = |h|^p$ and $h^p = \exp(u + \tilde{v} + i(\tilde{u} - v))$. Then

$$\frac{\phi - \lambda}{|\phi - \lambda|} \frac{\bar{h}}{h} = \exp i(s_\lambda + \frac{2}{p}(v - \tilde{u}))$$

and

$$\|s_\lambda + \frac{2}{p}(v - \tilde{u})\|^p = \|s_\lambda + \frac{2}{p}v\|^p \leq \frac{\pi}{2} - t - \varepsilon + \frac{2}{p}\|v\|^p = \frac{\pi}{2} - \frac{2}{p} \left( 1 - \frac{2}{\max(p,q)} \right) - \frac{2}{p} t_W - \varepsilon + \frac{2}{p} t_W = \frac{\pi}{2} \frac{\max(p,q)}{2} - \varepsilon.$$

By Theorem $K$, $T_{\phi - \lambda}^{\phi, \frac{h}{|h|^p}}$ is invertible and so by Theorem WDR $T_{\phi - \lambda}^{W,p}$ is invertible. Thus $\lambda \notin \sigma(T_{\phi}^{W,p}).$

**Corollary 1.** Suppose $W = e^{u+i\tilde{v}}$ is a Helson-Szegő weight and $t = t_W$. If $\phi$ is a function in $L^\infty$, then $\mathcal{R}(\phi) \subseteq \sigma(T_{\phi}^{W,p}) \subseteq h^t(\mathcal{R}(\phi))$.  

5
Corollary 2. Suppose $W \equiv 1$, $1 < p < \infty$ and $1/p + 1/q = 1$ and $t = |p - 2|/2p$. If $\phi$ is a function in $L^\infty$, then $\mathcal{R}(\phi) \subseteq \sigma(T^p_\phi) \subseteq h^1(\mathcal{R}(\phi))$.

Proof. Since $W \equiv 1$, $t = \frac{\pi}{2} \left(1 - \frac{2}{\max(p, q)}\right)$. If $p \geq 2$, then $t = \frac{\pi}{2} \left(1 - \frac{2}{p}\right) = \frac{\pi(p - 2)}{2p}$. If $1 < p < 2$, then $t = \frac{\pi}{2} \left(1 - \frac{2}{q}\right) = \frac{\pi(2 - p)}{2p}$ because $q = \frac{p}{p - 1}$.

Theorem 2. Suppose $W$ satisfies the condition $(A_p)$ for some $p$ with $1 < p < \infty$. Then, $t_W = 0$ if and only if $\sigma(T^W_\phi) = \sigma(T^p_\phi)$ for any $\phi$ in $L^\infty$.

Proof. Suppose that $\sigma(T^W_\phi) = \sigma(T^p_\phi)$ for any $\phi$ in $L^\infty$. If $\phi = \tilde{h}_0/h_0$ and $h_0$ is an outer function with $|h_0|^p \in (A_p)$, then $T^p_\phi$ is invertible and so $T^W_\phi$ is invertible. Put $h_0 = \exp \frac{1}{p}(u_0 + \tilde{u}_0 + i(\tilde{u}_0 - v_0))$ where $u_0 \in L^\infty_R$ and $v_0 \in L^\infty_R$. Then

$$\phi = \tilde{h}_0/h_0 = \exp i\frac{1}{p}(v_0 - \tilde{u}_0).$$

Since $T^W_\phi$ is invertible, by Theorem WDR $W|h_0|^p = W \exp(iu_0 + u_0)$ belongs to $(A_p)$. Thus $W(A_p) \subseteq (A_p)$ and so by [4, Theorem 2.12] $t_W = 0$.

Conversely if $t_W = 0$ then log $W$ belongs to the closure of $L^\infty$ in BMO. Hence $W(A_p) = (A_p)$ by [4, Theorem 2.12]. Let $W = |h|^p$ and $h$ an outer function in $H^p$. By Theorem WDR in Introduction, $T^W_\phi$ is invertible if and only if $T^W_{\phi/|\phi|}$ is invertible and $\phi$ is invertible in $L^\infty$. If $T^W_{\phi/|\phi|}$ is invertible then by Theorem WDR

$$\frac{\phi}{|\phi|} = \frac{h}{\bar{h}}\frac{\tilde{h}_0}{\tilde{h}}$$

for some outer function $h_0$ with $|h_0|^p \in (A_p)$. Since $W(A_p) = (A_p)$, $|h|^p|\tilde{h}^{-p} \in (A_p)$ and $\phi = \bar{h}_0 h^{-1}/h_0 h^{-1}$. This implies that $T^p_{\phi/|\phi|}$ is invertible. Thus $\sigma(T^W_{\phi/|\phi|}) \supseteq \sigma(T^p_\phi)$ for any $\phi$ in $L^\infty$. If $T^p_{\phi/|\phi|}$ is invertible then $\phi/|\phi| = \tilde{h}_1/h_1$ for some outer function $h_1$ with $|h_1|^p \in (A_p)$. Since $W(A_p) = (A_p)$, $|h|^p|h_1|^p \in (A_p)$ and so

$$\frac{\phi}{|\phi|} = \frac{h}{\bar{h}}\frac{\tilde{h}_1 h}{h_1 h}.$$ 

Hence $T^W_{\phi/|\phi|}$ is invertible. Thus $\sigma(T^W_{\phi/|\phi|}) \subseteq \sigma(T^p_\phi)$ for any $\phi$ in $L^\infty$. Therefore $\sigma(T^W_\phi) = \sigma(T^p_\phi)$ for any $\phi$ in $L^\infty$.

§3. Special symbols

In this section, we study the spectrum of a Toeplitz operator whose symbol is continuous, real-valued or the quotient of two inner functions. Theorem 3 generalizes
(3) of Theorem 3 in the previous paper [7]. Theorem 4 generalizes (2) of Theorem 1 in [7]. Theorem 5 generalizes and improves Corollary 1 in [7]. Corollary 3 improves (3) of Theorem 1 in [7].

**Theorem 3.** Let $1 < p < \infty$. If $\phi$ is a continuous function on $T$ then
\[
\sigma(T_{\phi}^{W,p}) = \mathcal{R}(\phi) \cup \{\lambda \in \mathcal{C} \mid i_t(\phi, \lambda) \neq 0\}
\]
for any $W$ in $(A_p)$, where $i_t(\phi, \lambda)$ is the winding number of the curve determined by $\phi$ with respect to $\lambda$.

Proof. If $\lambda \not\in \mathcal{R}(\phi)$ and $i_t(\phi, \lambda) = 0$ then $(\phi - \lambda)/|\phi - \lambda| = e^{is_\lambda}$ where $s_\lambda \in \mathcal{C}$ and so $W \exp \frac{P}{2}(-s_\lambda)$ belongs to $(A_p)$. By Theorem WDR this implies that $\lambda \not\in \sigma(T_{\phi}^{W,p})$.

Conversely if $\lambda \not\in \sigma(T_{\phi}^{W,p})$ then $\lambda \not\in \mathcal{R}(\phi)$. Hence
\[
\frac{\phi - \lambda}{|\phi - \lambda|} = z^\ell e^{is_\lambda}
\]
where $\ell$ is an integer and $s_\lambda \in \mathcal{C}$. Since $T_{\phi-\lambda}^{W,p}$ is invertible, by Theorem WDR there exists an outer function $h_1$ such that
\[
\frac{\phi - \lambda}{|\phi - \lambda|} = h \frac{h_1}{h_1}
\]
where $|h_1|^p \in (A_p)$ and $W = |h|^p \in (A_q)$ and $h$ is an outer function. Then $|h|^{-q} \in (A_q)$ where $1/p + 1/q = 1$ and so $f = h^{-1}h_1$ belongs to $H^t$ for some $t > 1$. Put $g^2 = \exp(-s_\lambda + is_\lambda)$ then $g \in \bigcap_{1 \leq s < \infty} H^s$ and so $gf$ belongs to $H^1$. Similarly we can show that $(gf)^{-1}$ belongs to $H^1$. Then if $\ell \geq 0$ then $z^{\ell}gf = \overline{gf}$ and $z^{\ell}(gf)^2$ is nonnegative in $H^{1/2}$. Hence $\ell = 0$ because $H^{1/2}$ does not contain any nonconstant nonnegative functions. If $\ell \leq 0$ then $z^{\ell}(gf)^{-2}$ is nonnegative in $H^{1/2}$ and so $\ell = 0$. Thus $(\phi - \lambda)/|\phi - \lambda| = e^{is_\lambda}$ and so $\lambda \not\in \sigma(T_{\phi}^{W,p})$ because $e^{is_\lambda} = g/\overline{g}$ and $|g^{-1}|^p \in (A_p)$.

**Theorem 4.** Suppose $W$ satisfies the condition $(A_p) \cap (A_q)$ where $1 < p < \infty$ and $1/p + 1/q = 1$, and $t = \frac{\pi}{2} \left(1 - \frac{2}{\max(p, q)}\right) + \frac{2}{p} t_W$. If $\phi$ is real valued, $a = \text{ess inf} \phi$ and $b = \text{ess sup} \phi$, then
\[
\mathcal{R}(\phi) \subseteq \sigma(T_{\phi}^{W,p}) \subseteq \triangle(c, r) \cap \triangle(\bar{c}, r)
\]
where $c = \frac{a + b}{2} - \frac{i(a - b)}{2} \cos 2t$ and $r = -\frac{a - b}{2} \sin 2t$. If $t_W = 0$ then $[a, b] \subseteq \sigma(T_{\phi}^{W,p})$.

Proof. By Theorem 1, $\sigma(T_{\phi}^{W,p}) \subseteq h^t(\mathcal{R}(\phi)) \subseteq h^t([a, b])$ for $t = \frac{\pi}{2} \left(1 - \frac{2}{\max(p, q)}\right) + \frac{2}{p} t_W$. It is elementary to see that $h^t([a, b]) \subseteq \triangle(c, r) \cap \triangle(\bar{c}, r)$. Suppose $t_W = 0$. 

7
Then \( t = \frac{\pi}{2} \left( 1 - \frac{2}{\max(p,q)} \right) \) and by Theorem 2 \( \sigma(T^{W,p}_\phi) = \sigma(T^p_\phi) \). We will show that \([a,b] \subseteq \sigma(T^p_\phi)\). Suppose \( \lambda \in [a,b] \) and \( \lambda \notin \mathcal{R}(\phi) \), then \( \psi = (\phi - \lambda)/|\phi - \lambda| = 2\chi_E - 1 \) for some measurable set \( E \) in \( \partial D \). If \( \lambda \notin \sigma(T^p_\phi) \), then by Theorem WDR there exists an outer function \( h_1 \) in \( H^p \) with \( h_1^{-1} \) in \( H^q \) such that \( \psi = \bar{h}_1/h_1 \). Since \( T^p_\psi = T^3_\psi \) is also invertible, there exists an outer function \( h_2 \) in \( H^q \) with \( h_2^{-1} \) in \( H^p \) such that \( \psi = \bar{h}_2/h_2 \). Hence

\[
\frac{\bar{h}_1}{h_1} = \frac{h_1}{\bar{h}_1} = \frac{h_2}{\bar{h}_2} = \frac{h_2}{\bar{h}_2}
\]

because \( \psi \) is a real valued function. Hence \( h_1^2 = \bar{h}_1^2 \in H^{p/2} \) and \( h_2^2 = \bar{h}_2^2 \in H^{q/2} \). Therefore \( h_1 \) or \( h_2 \) is constant because \( \max(p/2,q/2) \geq 1 \) and the only real function in \( H^1 \) is constant. Thus \( \psi \) is constant and this contradicts that \( \phi \) is not constant. Thus \([a,b] \subseteq \sigma(T^p_\phi)\).

For a weight \( W \) in \((A_p)\) and a measurable set \( E \), put

\[
\gamma_+(E,W,p) = \sup\{t > 0 \; ; \; W \exp(t\chi_E) \text{ satisfies } (A_p)\}
\]

and

\[
\gamma_-(E,W,p) = \inf\{t < 0 \; ; \; W \exp(t\chi_E) \text{ satisfies } (A_p)\}.
\]

**Theorem 5.** Let \( W \) satisfy the condition \((A_p)\) and \( 1 < p < \infty \). Suppose \( \phi = a\chi_E + b\chi_{E^c} \) where \( a, b \) are real numbers and \( E \) is measurable set in \( \partial D \) with \( 0 < d\theta(E) < 2\pi \). Then

\[
\sigma(T^{W,p}_\phi) = \{\lambda \in \mathcal{C} \; ; \; \pi \geq \text{Arg} \frac{a - \lambda}{b - \lambda} \geq \frac{2}{p}\gamma_+(E,W,p) \}
\]

or

\[
-\pi \leq \text{Arg} \frac{a - \lambda}{b - \lambda} \leq \frac{2}{p}\gamma_-(E,W,p) \}
\]

where \(-\pi \leq \text{Arg} \leq \pi\). In particular, \( \sigma(T^{W,p}_\phi) \supseteq [a,b] \).

**Proof.** If \( \lambda \neq a, b \), and \( \lambda \) is a real number then

\[
\frac{\phi - \lambda}{|\phi - \lambda|} = \frac{a - \lambda}{|a - \lambda|}\chi_E + \frac{b - \lambda}{|b - \lambda|}\chi_{E^c}.
\]

There exist \( a(\lambda) \) and \( b(\lambda) \) such that \(-\pi \leq a(\lambda), b(\lambda) \leq \pi \) and

\[
\frac{a - \lambda}{|a - \lambda|} = e^{ia(\lambda)}, \quad \frac{b - \lambda}{|b - \lambda|} = e^{ib(\lambda)}.
\]

Hence

\[
\frac{\phi - \lambda}{|\phi - \lambda|} = \exp i\{(a(\lambda) - b(\lambda))\chi_E + b(\lambda)\}
\]
where $0 \leq a(\lambda) - b(\lambda) \leq \pi$ or $-\pi \leq a(\lambda) - b(\lambda) \leq 0$. If $\lambda \notin \sigma(T_{\phi}^{W,p})$, then by Theorem WDR $W \exp \left\{ \frac{p}{2} (a(\lambda) - b(\lambda)) \hat{\chi}_E \right\}$ belongs to $(A_p)$. Hence

$$\sigma(T_{\phi}^{W,p}) \subseteq \left\{ \lambda \in \mathcal{C} ; \pi \geq a(\lambda) - b(\lambda) \geq \frac{2}{p} \gamma_+(E, W, p) \right\}$$

$$\bigcup \left\{ \lambda \in \mathcal{C} ; \frac{2}{p} \gamma_-(E, W, p) \geq a(\lambda) - b(\lambda) \geq -\pi \right\}.$$  

If $\pi \geq a(\lambda) - b(\lambda) > \frac{2}{p} \gamma_+(E, W, p)$ or $-\pi \leq a(\lambda) - b(\lambda) < \frac{2}{p} \gamma_-(E, W, p)$, then

$$W \exp \left\{ \frac{p}{2} (a(\lambda) - b(\lambda)) \hat{\chi}_E \right\}$$

does not belong to $(A_p)$ and so $\lambda \in \sigma(T_{\phi}^{W,p})$. Since $\sigma(T_{\phi}^{W,p})$ is closed,

$$\sigma(T_{\phi}^{W,p}) = \left\{ \lambda \in \mathcal{C} ; \pi \geq a(\lambda) - b(\lambda) \geq \frac{2}{p} \gamma_+(E, W, p) \right\}$$

$$\bigcap \left\{ \lambda \in \mathcal{C} ; \frac{2}{p} \gamma_-(E, W, p) \geq a(\lambda) - b(\lambda) \geq -\pi \right\}.$$  

**Lemma 1.** For a measurable set $E$ in $T$ with $0 < m(E) < 1$, $\|\pi \chi_E - v\|' \geq \pi/2$ for any $v$ in $L^\infty_R$ with $\|v\|_\infty < \pi/2$.

Proof. Suppose $\phi = a \chi_E + b \chi_E^c$ where $a$ and $b$ are real numbers, $a \neq b$ and $0 < m(E) < 1$. For $W = e^{u+\tilde{v}}$ where $u, v \in L^\infty_R$ and $\|v\|_\infty < \pi/2$, $\sigma(T_{\phi}^{W,2}) \supseteq [a, b]$ if and only if $\|\pi \chi_E - v\|' \geq \pi/2$. This is proved in [7, Corollary 1]. Now Theorem 5 shows Lemma 1.

**Corollary 3.** Suppose $W$ satisfies the condition $(A_2)$. If $\phi$ is real valued, $a = \text{ess inf } \phi$ and $b = \text{ess sup } \phi$ then $[a, b] \subseteq \sigma(T_{\phi}^{W,2})$.

Proof. Since $W \in (A_2)$, $W = e^{u+\tilde{v}}$ where $u, v \in L^\infty_R$ and $\|v\|_\infty < \pi/2$. For $\lambda \in [a, b] \cap \mathcal{R}(\phi)^c$, $\frac{\phi - \lambda}{|\phi - \lambda|} = e^{i\ell}$ and $\ell = \pi(1 - \chi_E)$ for some measurable set $E$ in $T$ with $0 < m(E) < 1$. Then, in [7, (3) of Theorem 1], it is proved that $\lambda \in \sigma(T_{\phi}^{W,2})$ if and only if $\|\pi \chi_E - v\|' \geq \pi/2$. Now Lemma 1 implies this corollary.

**Lemma 2.** If $q_1$ and $q_2$ are inner functions and $\tilde{q}_1 q_2 = f/|f| = |g|/g$ where both $f$ and $g$ are in $\cap_{p>1/2} H^p$, then $\text{singq}_1 \neq \text{singq}_2$.

Proof. See the proof of [8, Corollary 5].

**Theorem 6.** Suppose $W$ satisfies the condition $(A_p)$ where $1 < p < \infty$. If $\phi = \tilde{q}_1 q_2$ where $q_1$ and $q_2$ are inner functions with $\text{singq}_1 \neq \text{singq}_2$ then $\sigma(T_{\phi}^{W,p}) = D$.

Proof. Suppose $W = |h|^p$ for some outer function in $H^p$. If $\lambda \in D$ then

$$\tilde{q}_1 q_2 - \lambda = \tilde{q}_1 (q_2 - \lambda q_1) = \tilde{q}_1 q_3 k$$
where \( q_3 \) is inner and \( k \) is invertible in \( H^\infty \). By the proof of [8, Theorem 2(2)] \( \bar{q}_2 q_3 = \frac{f}{|f|} = \frac{|g|}{g} \) where both \( f \) and \( g \) are in \( H^1 \). By Lemma 2 sing\( q_2 = \) sing\( q_3 \) and so \( \text{sing} q_1 \neq \text{sing} q_3 \). If \( \lambda \notin \sigma(T_{\bar{q}_1q_2}^{W,p}) \) then \( 0 \notin \sigma(T_{\bar{q}_1q_3}^{W,p}) \) because \( k \) is invertible in \( H^\infty \). By Theorem WDR

\[
\bar{q}_1q_3 = \frac{h\bar{h}_0}{\bar{h}h_0}
\]

where \( h_0 \) is an outer function in \( H^p \) with \( |h_0|^p \in (A_p) \). Hence \( \bar{q}_1 q_3 = f/|f| = |g|/g \) where \( f = (h/h_0)^2 \) and \( g = (h_0/h)^2 \). Since \( |h|^p \) and \( |h_0|^p \) are in \( (A_p) \), both \( f \) and \( g \) belong to \( H^{1/2} \). This contradicts Lemma 2. Hence \( \lambda \in \sigma(T_{\bar{q}_1q_2}^{W,p}) \) and so \( \sigma(T_{\bar{q}_1q_2}^{W,p}) = \bar{D} \).

Acknowledgement
I should like to thank Professor Kazuya Tachizawa for helpful comments on the paper of R.Johnson and C.J.Neugebauer.
References

1. A. Böttcher and B. Silbermann, Analysis Of Toeplitz Operators, Springer-Verlag


Takahiko Nakazi
Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan
nakazi@math.sci.hokudai.ac.jp