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ATTRACTORS OF ASYMPTOTICALLY PERIODIC MULTIVALUED DYNAMICAL SYSTEMS GOVERNED BY TIME-DEPENDENT SUBDIFFERENTIALS

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Abstract. Let us consider a nonlinear evolution equation associated with time-dependent subdifferential in a separable Hilbert space. In this paper we treat an asymptotically periodic system which means that time-dependent terms converge to some time-periodic ones as time goes to $+\infty$. Then we consider the large-time behaviour of solutions without uniqueness. In such a situation the corresponding dynamical systems are multivalued. In fact we discuss the stability of multivalued semiflows from the view-point of attractors. Namely, the main object of this paper is to construct a global attractor for the asymptotically periodic multivalued dynamical system, and to discuss the relationship to one for the limiting periodic system.

1 Introduction

In this paper let us consider a non-autonomous system in a real separable Hilbert space H of the form

$$v'(t) + \partial\varphi^t(v(t)) + G(t, v(t)) \ni f(t) \quad \text{in } H, \quad t > s (\geq 0), \quad (1.1)$$

where $v' = \frac{dv}{dt}$, $\partial\varphi^t$ is a subdifferential of time-dependent proper lower semicontinuous (l.s.c.) convex function φ^t on H , $G(t, \cdot)$ is a multivalued perturbation small relative to φ^t , and f is a forcing term.

In the case when $G(t, \cdot) \equiv 0$, many mathematicians studied the existence-uniqueness, asymptotic stability, time periodic and almost periodic problem for (1.1) (cf. [7], [8], [13], [14], [15], [16], [18], [23], [24]).

For the multivalued nonmonotone perturbation $G(t, \cdot)$, Ôtani has already shown the existence of solution for (1.1) in [21]. The large-time behavior of solutions for (1.1) was

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discussed by [28] from the view-point of attractors. For the time periodic case, assuming the periodicity conditions with same period T_0 , $0 < T_0 < +\infty$, i.e.

$$\varphi^t = \varphi^{t+T_0}, \quad G(t, \cdot) = G(t + T_0, \cdot), \quad f(t) = f(t + T_0), \quad \forall t \in R_+ := [0, \infty),$$

the existence of periodic solution for (1.1) was proved in [22]. Moreover, the periodic stability was discussed in [29]. In fact, the author showed the existence and characterization of time-periodic global attractors for (1.1).

In this paper, for a given positive number $T_0 > 0$ let us treat the case when φ^t , $G(t, \cdot)$ and $f(t)$ are asymptotically T_0 -periodic in time. Namely we assume that

$$\varphi^t - \varphi_p^t \longrightarrow 0, \quad G(t, \cdot) - G_p(t, \cdot) \longrightarrow 0, \quad f(t) - f_p(t) \longrightarrow 0 \quad (1.2)$$

in appropriate senses as $t \rightarrow +\infty$, where $\varphi_p^t = \varphi_p^{t+T_0}$, $G_p(t, \cdot) = G_p(t + T_0, \cdot)$ and $f_p(t) = f_p(t + T_0)$ for any $t \in R_+$. By the asymptotically T_0 -periodic stability (1.2), we have the limiting T_0 -periodic system for (1.1) of the form:

$$u'(t) + \partial\varphi_p^t(u(t)) + G_p(t, u(t)) \ni f_p(t) \quad \text{in } H, \quad t > s (\geq 0). \quad (1.3)$$

In the case when $G(t, \cdot)$ and $G_p(t, \cdot)$ are single-valued, the asymptotically T_0 -periodic problem has already been discussed in [11]. In order to guarantee the uniqueness of solutions for the Cauchy problem of (1.1) and (1.3), they assumed some conditions on φ^t , φ_p^t , $G(t, \cdot)$ and $G_p(t, \cdot)$. Then, they discussed the asymptotically T_0 -periodic stability for (1.1) from the view-point of attractors (cf. [11]). The main object of this paper is to develop the result obtained in [11] in order to consider the large-time behaviour of solution for (1.1) without uniqueness. Namely, we would like to construct the attractor for the asymptotically T_0 -periodic multivalued flows associated with (1.1). Moreover we shall discuss the relationship to the T_0 -periodic attractor for (1.3) obtained in [29].

In the next Section 2, we recall the known results for the Cauchy problem of (1.1). In Section 3 we consider the limiting T_0 -periodic problem (1.3) and recall the abstract results obtained in [29]. In Section 4, we introduce the notion of a metric topology on the family $\{\varphi^t; t \geq 0\}$ which was constructed in [16]. And we present and prove the main results in this paper. In proving main results, we generalize the results obtained in [11] and [30]. In the final section we apply our abstract results to the parabolic variational inequality with asymptotically T_0 -periodic double obstacles. Then we can discuss the asymptotic stability for the asymptotically T_0 -periodic double obstacle problem without uniqueness of solutions.

Notation. Throughout this paper, let H be a (real) separable Hilbert space with norm $|\cdot|_H$ and inner product $(\cdot, \cdot)_H$. For a proper l.s.c. convex function φ on H we use the notation $D(\varphi)$, $\partial\varphi$ and $D(\partial\varphi)$ to indicate the effective domain, subdifferential and its domain of φ , respectively; for their precise definitions and basic properties see [4].

For two non-empty sets A and B in H , we define the so-called Hausdorff semi-distance

$$\text{dist}_H(A, B) := \sup_{x \in A} \inf_{y \in B} |x - y|_H.$$

2 Preliminaries

In this section let us recall the known results for a nonlinear evolution equation in H of the form:

$$u'(t) + \partial\varphi^t(u(t)) + G(t, u(t)) \ni f(t) \quad \text{in } H, \quad t \in J, \quad (2.1)$$

where J is an interval in R_+ , $\partial\varphi^t$ is the subdifferential of a time-dependent proper l.s.c. and convex function φ^t on H , $G(t, \cdot)$ is a multivalued operator from a subset $D(G(t, \cdot)) \subset H$ into H for each $t \in R_+$ and f is a given function in $L^2_{loc}(J; H)$.

We begin with the definition of solution for (2.1).

Definition 2.1. (i) For a compact interval $J := [t_0, t_1] \subset R_+$ and $f \in L^2(J; H)$, a function $u : J \rightarrow H$ is called a solution of (2.1) on J , if $u \in C(J; H) \cap W^{1,2}_{loc}((t_0, t_1]; H)$, $\varphi^{(\cdot)}(u(\cdot)) \in L^1(J)$, $u(t) \in D(\partial\varphi^t)$ for a.e. $t \in J$, and if there exists a function $g \in L^2_{loc}(J; H)$ such that $g(t) \in G(t, u(t))$ for a.e. $t \in J$ and

$$f(t) - g(t) - u'(t) \in \partial\varphi^t(u(t)), \quad \text{a.e. } t \in J.$$

(ii) For any interval J in R_+ and $f \in L^2_{loc}(J; H)$, a function $u : J \rightarrow H$ is called a solution of (2.1) on J , if it is a solution of (2.1) on every compact subinterval of J in the sense of (i).

(iii) Let J be any interval in R_+ with initial time $s \in R_+$. For $f \in L^2_{loc}(J; H)$, a function $u : J \rightarrow H$ is called a solution of the Cauchy problem for (2.1) on J with given initial value $u_0 \in H$, if it is a solution of (2.1) on J satisfying $u(s) = u_0$.

Throughout this paper, let $\{a_r\} := \{a_r; r \geq 0\}$ and $\{b_r\} := \{b_r; r \geq 0\}$ be families of real functions in $W^{1,2}_{loc}(R_+)$ and $W^{1,1}_{loc}(R_+)$, respectively, such that

$$\sup_{t \in R_+} |a'_r|_{L^2(t, t+1)} + \sup_{t \in R_+} |b'_r|_{L^1(t, t+1)} < +\infty \quad \text{for each } r \geq 0.$$

Now we define the class $\Phi(\{a_r\}, \{b_r\})$ of time-dependent convex function φ^t .

Definition 2.2. $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$ if and only if φ^t is a proper l.s.c. convex function on H satisfying the following properties ($\Phi 1$)-($\Phi 3$):

($\Phi 1$) For each $r > 0$, $s, t \in R_+$ and $z \in D(\varphi^s)$ with $|z|_H \leq r$, there exists $\tilde{z} \in D(\varphi^t)$ such that

$$|\tilde{z} - z|_H \leq |a_r(t) - a_r(s)|(1 + |\varphi^s(z)|^{\frac{1}{2}})$$

and

$$\varphi^t(\tilde{z}) - \varphi^s(z) \leq |b_r(t) - b_r(s)|(1 + |\varphi^s(z)|).$$

($\Phi 2$) There exists a positive constant $C_1 > 0$ such that

$$\varphi^t(z) \geq C_1 |z|_H^2, \quad \forall t \in R_+, \quad \forall z \in D(\varphi^t).$$

($\Phi 3$) For each $k > 0$ and $t \in R_+$, the level set $\{z \in H; \varphi^t(z) \leq k\}$ is compact in H .

Next, we introduce the class $\mathcal{G}(\{\varphi^t\})$ of time-dependent multivalued perturbation $G(t, \cdot)$ associated with $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$.

Definition 2.3. $\{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^t\})$ if and only if $G(t, \cdot)$ is a multivalued operator from $D(G(t, \cdot)) \subset H$ into H which fulfills the following conditions (G1)-(G5):

(G1) $D(\varphi^t) \subset D(G(t, \cdot)) \subset H$ for any $t \in R_+$. And for any interval $J \subset R_+$ and $v \in L^2_{loc}(J; H)$ with $v(t) \in D(\varphi^t)$ for a.e. $t \in J$, there exists a strongly measurable function $g(\cdot)$ on J such that

$$g(t) \in G(t, v(t)) \text{ for a.e. } t \in J.$$

(G2) $G(t, z)$ is a convex subset of H for any $z \in D(\varphi^t)$ and $t \in R_+$.

(G3) There are positive constants C_2, C_3 such that

$$|g|_H^2 \leq C_2 \varphi^t(z) + C_3, \quad \forall t \in R_+, \forall z \in D(\varphi^t), \forall g \in G(t, z).$$

(G4) (demi-closedness) If $z_n \in D(\varphi^{t_n}), g_n \in G(t_n, z_n), \{t_n\} \subset R_+, \{\varphi^{t_n}(z_n)\}$ is bounded, $z_n \rightarrow z$ in $H, t_n \rightarrow t$ and $g_n \rightarrow g$ weakly in H as $n \rightarrow +\infty$, then $g \in G(t, z)$.

(G5) For each bounded subset B of H , there exist positive constants $C_4(B)$ and $C_5(B)$ such that

$$\begin{aligned} \varphi^t(z) + (g, z - b)_H &\geq C_4(B)|z|_H^2 - C_5(B), \\ \forall t \in R_+, \forall g \in G(t, z), \forall z \in D(\varphi^t), \forall b \in B. \end{aligned}$$

For given $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\}), \{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^t\})$ and a forcing term $f \in L^2_{loc}(R_+; H)$, we consider the following evolution equation

$$(E)_s \quad u'(t) + \partial\varphi^t(u(t)) + G(t, u(t)) \ni f(t) \quad \text{in } H, \quad t > s$$

for each $s \in R_+$.

Now let us recall the known results on the existence and global estimates of solutions for the Cauchy problem of $(E)_s$:

(A) [Existence of solution for $(E)_s$] (cf. [21, Theorem II, III])

The Cauchy problem for $(E)_s$ has at least one solution u on $J = [s, +\infty)$ such that $(\cdot - s)^{\frac{1}{2}}u' \in L^2_{loc}(J; H), (\cdot - s)\varphi^{(\cdot)}(u(\cdot)) \in L^\infty_{loc}(J)$ and $\varphi^{(\cdot)}(u(\cdot))$ is absolutely continuous on any compact subinterval of $(s, +\infty)$, provided that $u_0 \in \overline{D(\varphi^s)}$. In particular, if $u_0 \in D(\varphi^s)$, then the solution u satisfies that $u' \in L^2_{loc}(J; H)$ and $\varphi^{(\cdot)}(u(\cdot))$ is absolutely continuous on any compact interval in J .

(B) [Global boundedness of solutions for $(E)_s$] (cf. [25, Theorem 2.2])

Suppose that

$$S_f := \sup_{t \in R_+} |f|_{L^2(t, t+1; H)} < +\infty.$$

Then, the solution u of the Cauchy problem for $(E)_s$ on $[s, +\infty)$ satisfies the following global estimate:

$$\sup_{t \geq s} |u(t)|_H^2 + \sup_{t \geq s} \int_t^{t+1} \varphi^\tau(u(\tau)) d\tau \leq N_1(1 + S_f^2 + |u_0|_H^2),$$

where N_1 is a positive constant independent of f , $s \in R_+$ and $u_0 \in \overline{D(\varphi^s)}$. Moreover, for each $\delta > 0$ and each bounded subset B of H , there is a constant $N_2(\delta, B) > 0$, depending only on $\delta > 0$ and B , such that

$$\sup_{t \geq s+\delta} |u'|_{L^2(t, t+1; H)}^2 + \sup_{t \geq s+\delta} \varphi^t(u(t)) \leq N_2(\delta, B)$$

for the solution u of the Cauchy problem for $(E)_s$ on $[s, +\infty)$ with $s \in R_+$ and $u_0 \in \overline{D(\varphi^s)} \cap B$.

Next, let us remember a notion of convergence of convex functions.

Definition 2.4. (cf. [20]) Let ψ , ψ_n ($n \in N$) be proper l.s.c. and convex functions on H . Then we say that ψ_n converges to ψ on H as $n \rightarrow +\infty$ in the sense of Mosco, if the following two conditions (i) and (ii) are satisfied:

(i) for any subsequence $\{\psi_{n_k}\} \subset \{\psi_n\}$, if $z_k \rightarrow z$ weakly in H as $k \rightarrow +\infty$, then

$$\liminf_{k \rightarrow +\infty} \psi_{n_k}(z_k) \geq \psi(z).$$

(ii) for any $z \in D(\psi)$, there is a sequence $\{z_n\}$ in H such that

$$z_n \rightarrow z \text{ in } H \text{ as } n \rightarrow +\infty, \quad \lim_{n \rightarrow +\infty} \psi_n(z_n) = \psi(z).$$

Now, we recall a convergence result (cf. [25, Lemma 4.1]) as follows.

(C) Let $\{\varphi_n^t\} \in \Phi(\{a_r\}, \{b_r\})$, $\{G_n(t, \cdot)\} \in \mathcal{G}(\{\varphi_n^t\})$ with common positive constants $C_1, C_2, C_3, C_4(B)$ and $C_5(B)$, $\{f_n\} \subset L^2(J; H)$, $J = [s, t_1] \subset R_+$ and $u_{0,n} \in \overline{D(\varphi_n^s)}$ for $n = 1, 2, \dots$. Assume that

(i) φ_n^t converges to φ^t on H in the sense of Mosco [20] for each $t \in J$ (as $n \rightarrow +\infty$) and $\bigcup_{n=1}^{+\infty} \{z \in H; \varphi_n^t(z) \leq k\}$ is relatively compact in H for every real $k > 0$ and $t \in J$, where $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$ and $\varphi_n^t = \varphi^t$ if $n = +\infty$.

(ii) if $z_n \in D(\varphi_n^{t_n})$, $g_n \in G_n(t_n, z_n)$, $\{t_n\} \subset R_+$, $\{\varphi_n^{t_n}(z_n)\}$ is bounded, $z_n \rightarrow z$ in H , $t_n \rightarrow t$ and $g_n \rightarrow g$ weakly in H as $n \rightarrow +\infty$, then $g \in G(t, z)$, where $\{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^t\})$.

(iii) $f_n \rightarrow f$ weakly in $L^2(J; H)$ for some $f \in L^2(J; H)$ and $u_{0,n} \rightarrow u_0$ in H for some $u_0 \in \overline{D(\varphi^s)}$.

Denote by u the solution of the Cauchy problem for $(E)_s$ on J with $u(s) = u_0$ and by u_n the solution of the Cauchy problem for $(E)_s$ with φ^t , G , f replaced by φ_n^t , G_n , f_n , and with $u_n(s) = u_{0,n}$. Then u_n converges to u on J in the sense that

$$u_n \rightarrow u \text{ in } C(J; H), \quad (\cdot - s)^{\frac{1}{2}} u'_n \rightarrow (\cdot - s)^{\frac{1}{2}} u' \text{ weakly in } L^2(J; H),$$

$$\int_J \varphi_n^t(u_n(t)) dt \rightarrow \int_J \varphi^t(u(t)) dt \quad \text{as } n \rightarrow +\infty.$$

3 Attractor for periodic multivalued dynamical system

In this section let us recall the known results obtained in [29] for a T_0 -periodic system in H , of the form:

$$(P)_s \quad u'(t) + \partial\varphi_p^t(u(t)) + G_p(t, u(t)) \ni f_p(t) \quad \text{in } H, \quad t > s$$

for each $s \in R_+$, where φ_p^t , $G_p(t, \cdot)$ and $f_p(t)$ are T_0 -periodic, namely periodic in time with the same period T_0 , $0 < T_0 < +\infty$.

Definition 3.1. Let T_0 be a positive number. Then

(i) $\Phi_p(\{a_r\}, \{b_r\}; T_0)$ is the set of all $\{\varphi_p^t\} \in \Phi(\{a_r\}, \{b_r\})$ satisfying T_0 -periodicity condition:

$$\varphi_p^{t+T_0}(\cdot) = \varphi_p^t(\cdot) \quad \text{on } H, \quad \forall t \in R_+. \quad (3.1)$$

(ii) $\mathcal{G}_p(\{\varphi_p^t\}; T_0)$ is the set of all $\{G_p(t, \cdot)\} \in \mathcal{G}(\{\varphi_p^t\})$ satisfying T_0 -periodicity condition:

$$G_p(t + T_0, \cdot) = G_p(t, \cdot) \quad \text{in } H, \quad \forall t \in R_+. \quad (3.2)$$

Throughout this section we assume that $\{\varphi_p^t\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0)$, $\{G_p(t, \cdot)\} \in \mathcal{G}_p(\{\varphi_p^t\}; T_0)$ and $f_p \in L^2_{loc}(R_+; H)$ is T_0 -periodic in time, namely

$$f_p(t + T_0) = f_p(t) \quad \text{in } H, \quad \forall t \in R_+. \quad (3.3)$$

Here we note that $(P)_s$ can be considered as $(E)_s$ in Section 2. So, by the result (A) in Section 2, the Cauchy problem for $(P)_s$ has at least one solution u on $[s, +\infty)$. Hence we can define the multivalued dynamical process associated with $(P)_s$ as follows:

Definition 3.2. For every $0 \leq s \leq t < +\infty$ we denote by $U(t, s)$ the mapping from $\overline{D(\varphi_p^s)}$ into $\overline{D(\varphi_p^t)}$ which assigns to each $u_0 \in \overline{D(\varphi_p^s)}$ the set

$$U(t, s)u_0 := \left\{ z \in H \left| \begin{array}{l} \text{There is a solution } u \text{ of } (P)_s \text{ on } [s, +\infty) \\ \text{such that} \\ u(s) = u_0 \text{ and } u(t) = z. \end{array} \right. \right\}. \quad (3.4)$$

Then we easily see the following properties of $\{U(t, s)\} := \{U(t, s); 0 \leq s \leq t < +\infty\}$:

- (U1) $U(s, s) = I$ on $\overline{D(\varphi_p^s)}$ for any $s \in R_+$;
- (U2) $U(t_2, s)z = U(t_2, t_1)U(t_1, s)z$ for any $0 \leq s \leq t_1 \leq t_2 < +\infty$ and $z \in \overline{D(\varphi_p^s)}$;
- (U3) $U(t + T_0, s + T_0)z = U(t, s)z$ for any $0 \leq s \leq t < +\infty$ and $z \in \overline{D(\varphi_p^s)}$, that is, U is T_0 -periodic.
- (U4) $\{U(t, s)\}$ has the following demi-closedness:
- If $0 \leq s_n \leq t_n < +\infty$, $s_n \rightarrow s$, $t_n \rightarrow t$, $z_n \in \overline{D(\varphi_p^{s_n})}$, $z \in \overline{D(\varphi_p^s)}$, $z_n \rightarrow z$ in H and a element $w_n \in U(t_n, s_n)z_n$ converges to some element $w \in H$ as $n \rightarrow +\infty$, then $w \in U(t, s)z$

Next we define the discrete dynamical system in order to construct a global attractor for $(P)_s$.

Definition 3.3. Let $U(\cdot, \cdot)$ be the solution operator for $(P)_s$ defined by Definition 3.2. Then

(i) For each $\tau \in R_+$, we denote by U_τ the T_0 -step mapping from $\overline{D(\varphi_p^\tau)}$ into $\overline{D(\varphi_p^{\tau+T_0})} = \overline{D(\varphi_p^\tau)}$, namely,

$$U_\tau := U(\tau + T_0, \tau).$$

(2) For any $k \in Z_+ := N \cup \{0\}$, we define

$$U_\tau^k := \underbrace{U_\tau \circ U_\tau \circ \cdots \circ U_\tau}_{k\text{-th iteration}}.$$

Clearly we have $U_\tau^k = U(\tau + kT_0, \tau)$ for any $\tau \in R_+$ and $k \in Z_+$.

Now, let us recall the known result on the existence of global attractors for discrete multivalued dynamical systems U_τ associated with $(P)_s$.

Theorem 3.1. (cf. [29, Theorem 3.1]) *Assume that $\{\varphi_p^t\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0)$, $\{G_p(t, \cdot)\} \in \mathcal{G}_p(\{\varphi_p^t\}; T_0)$, $f_p \in L_{loc}^2(R_+; H)$ satisfies the T_0 -periodicity condition (3.3). Then, for each $\tau \in R_+$, there exists a subset \mathcal{A}_τ of $\overline{D(\varphi_p^\tau)}$ such that*

- (i) \mathcal{A}_τ is non-empty and compact in H ;
- (ii) for each bounded set B in H and each number $\epsilon > 0$ there exists $N_{B, \epsilon} \in N$ such that

$$\text{dist}_H(U_\tau^k z, \mathcal{A}_\tau) < \epsilon$$

for all $z \in \overline{D(\varphi_p^\tau)} \cap B$ and all $k \geq N_{B, \epsilon}$;

- (iii) $U_\tau^k \mathcal{A}_\tau = \mathcal{A}_\tau$ for any $k \in N$.

Remark 3.1. By [29, Lemma 3.1] we can get the compact absorbing set $B_{0, \tau}$ of $\overline{D(\varphi_p^\tau)}$ for U_τ such that for each bounded subset B of H there is a positive integer n_B (independent of $\tau \in R_+$) satisfying

$$U_\tau^n (\overline{D(\varphi_p^\tau)} \cap B) \subset B_{0, \tau} \quad \text{for all } n \geq n_B.$$

Then we observe that the global attractor \mathcal{A}_τ is given by the ω -limit set of the absorbing set $B_{0,\tau}$ for U_τ , i.e.

$$\mathcal{A}_\tau = \bigcap_{n \in \mathbb{Z}_+} \overline{\bigcup_{k \geq n} U_\tau^k B_{0,\tau}}.$$

The next theorem is concerned with a relationship between two global attractors \mathcal{A}_s and \mathcal{A}_τ . For detail proof, see [29].

Theorem 3.2. (cf. [29, Theorem 3.2]) *Suppose the same assumptions are made as in Theorem 3.1. Let \mathcal{A}_s and \mathcal{A}_τ be the global attractors for U_s and U_τ , with $0 \leq s \leq \tau \leq T_0$, respectively. Then, we have*

$$\mathcal{A}_\tau = U(\tau, s)\mathcal{A}_s,$$

where $U(\tau, s)$ is the T_0 -periodic process given in Definition 3.2.

Remark 3.2. By Theorem 3.1 (iii) and Theorem 3.2, we see that the global attractor \mathcal{A}_τ for U_τ is T_0 -periodic in τ . In fact, for each $\tau \in R_+$ choose $m_\tau \in \mathbb{Z}_+$ and $\sigma_\tau \in [0, T_0]$ so that $\tau = \sigma_\tau + m_\tau T_0$. Then, we have $\mathcal{A}_\tau = \mathcal{A}_{\sigma_\tau}$.

The third known result is the existence of a global attractor for the T_0 -periodic multivalued dynamical system $(P)_s$.

Theorem 3.3. (cf. [29, Theorem 3.3]) *Under the same assumptions as Theorem 3.1, put*

$$\mathcal{A} := \bigcup_{0 \leq \tau \leq T_0} \mathcal{A}_\tau,$$

where \mathcal{A}_τ is as obtained in Theorem 3.1. Then, \mathcal{A} has the following properties:

- (i) \mathcal{A} is non-empty and compact in H ;
- (ii) for each bounded set B in H and each number $\epsilon > 0$ there exists a finite time $T_{B,\epsilon} > 0$ such that

$$\text{dist}_H(U(t + \tau, \tau)z, \mathcal{A}) < \epsilon$$

for all $\tau \in R_+$, all $z \in \overline{D(\varphi_p^\tau)} \cap B$ and all $t \geq T_{B,\epsilon}$.

Remark 3.3. In [29, Section 4] the characterization of the T_0 -periodic global attractor was discussed. The author proved that for each time $\tau \in R_+$ the global attractor \mathcal{A}_τ for the discrete multivalued dynamical system U_τ coincides with the cross-section of the family of all global bounded complete trajectories for the T_0 -periodic system $(P)_s$.

4 Attractors of asymptotically periodic multivalued dynamical system

Throughout this section, let $M > 0$ be a fixed (sufficiently) large positive number. Now we put

$$\Psi_M := \left\{ \psi; \begin{array}{l} \psi \text{ is proper, l.s.c. and convex on } H, \\ \exists z \in D(\psi) \text{ s.t. } |z|_H \leq M, \psi(z) \leq M \end{array} \right\}.$$

Then let us introduce the notion of a metric topology on Ψ_M which was introduced in [16].

Given $\varphi, \psi \in \Psi_M$, we define $\rho(\varphi, \psi; \cdot) : D(\varphi) \rightarrow R$ by putting

$$\rho(\varphi, \psi; z) = \inf\{\max(|y - z|_H, \psi(y) - \varphi(z)); y \in D(\psi)\}$$

for each $z \in D(\varphi)$, and for each $r \geq M$

$$\rho_r(\varphi, \psi) := \sup_{z \in L_\varphi(r)} \rho(\varphi, \psi; z),$$

where $L_\varphi(r) := \{z \in D(\varphi); |z|_H \leq r, \varphi(z) \leq r\}$. Moreover, for each $r \geq M$, we define the functional $\pi_r(\cdot, \cdot)$ on $\Psi_M \times \Psi_M$ by

$$\pi_r(\varphi, \psi) := \rho_r(\varphi, \psi) + \rho_r(\psi, \varphi) \quad \text{for } \varphi, \psi \in \Psi_M.$$

Then, according to [16, Proposition 3.1], we can define a complete metric topology on Ψ_M so that the convergence $\psi_n \rightarrow \psi$ in Ψ_M (as $n \rightarrow +\infty$) if and only if

$$\pi_r(\psi_n, \psi) \rightarrow 0 \quad \text{for every } r \geq M.$$

Now by using the above topology on Ψ_M , we consider an asymptotically T_0 -periodic system as follows.

Definition 4.1. Assume $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\}) \cap \Psi_M$, $\{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^t\})$ and $f \in L_{loc}^2(R_+; H)$. Then the system

$$(AP)_s \quad v'(t) + \partial\varphi^t(v(t)) + G(t, v(t)) \ni f(t) \quad \text{in } H, t > s (\geq 0)$$

is asymptotically T_0 -periodic, if there are $\{\varphi_p^t\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0) \cap \Psi_M$, $\{G_p(t, \cdot)\} \in \mathcal{G}_p(\{\varphi_p^t\}; T_0)$ and a T_0 -periodic function $f_p \in L_{loc}^2(R_+; H)$ such that

(A1) (Convergence of $\varphi^t - \varphi_p^t \rightarrow 0$ as $t \rightarrow +\infty$) For each $r \geq M$,

$$J_m^{(r)} := \sup_{\sigma \in [0, T_0]} \pi_r(\varphi^{mT_0+\sigma}, \varphi_p^\sigma) \rightarrow 0 \quad \text{as } m \rightarrow +\infty;$$

(A2) (Convergence of $G(t, \cdot) - G_p(t, \cdot) \rightarrow 0$ as $t \rightarrow +\infty$) If $\{\tau_n\} \subset [0, T_0]$, $\{m_n\} \subset Z_+$, $m_n \rightarrow +\infty$, $z_n \in D(\varphi^{m_n T_0 + \tau_n})$, $g_n \in G(m_n T_0 + \tau_n, z_n)$, $\{\varphi^{m_n T_0 + \tau_n}(z_n)\}$ is bounded, $z_n \rightarrow z$ in H , $\tau_n \rightarrow \tau$ and $g_n \rightarrow g$ weakly in H (as $n \rightarrow +\infty$), then

$$g \in G_p(\tau, z);$$

(A3) (Convergence of $f(t) - f_p(t) \rightarrow 0$ as $t \rightarrow +\infty$)

$$|f(mT_0 + \cdot) - f_p|_{L^2(0, T_0; H)} \rightarrow 0 \quad \text{as } m \rightarrow +\infty.$$

By Definition 4.1 we easily see that a limiting system for $(AP)_s$ is a T_0 -periodic one of the form:

$$(P)_s \quad u'(t) + \partial\varphi_p^t(u(t)) + G_p(t, u(t)) \ni f_p(t) \quad \text{in } H, \quad t > s (\geq 0).$$

Here we note that $(AP)_s$ is also considered as $(E)_s$. So, by the result (A) in Section 2, the Cauchy problem for $(AP)_s$ has at least one solution v on $[s, +\infty)$. Hence we can define the multivalued dynamical system associated with $(AP)_s$ as follows:

Definition 4.2. For every $0 \leq s \leq t < +\infty$ we denote by $E(t, s)$ the mapping from $\overline{D(\varphi^s)}$ into $\overline{D(\varphi^t)}$ which assigns to each $v_0 \in \overline{D(\varphi^s)}$ the set

$$E(t, s)v_0 := \left\{ z \in H \left| \begin{array}{l} \text{There is a solution } v \text{ of } (AP)_s \text{ on } [s, +\infty) \\ \text{such that} \\ v(s) = v_0 \text{ and } v(t) = z. \end{array} \right. \right\}.$$

Then we easily see that $\{E(t, s)\} := \{E(t, s); 0 \leq s \leq t < +\infty\}$ has the following evolution properties:

(E1) $E(s, s) = I$ on $\overline{D(\varphi^s)}$ for any $s \in R_+$;

(E2) $E(t_2, s)z = E(t_2, t_1)E(t_1, s)z$ for any $0 \leq s \leq t_1 \leq t_2 < +\infty$ and $z \in \overline{D(\varphi^s)}$;

(E3) $\{E(t, s)\}$ has the following demi-closedness:

- If $0 \leq s_n \leq t_n < +\infty$, $s_n \rightarrow s$, $t_n \rightarrow t$, $z_n \in \overline{D(\varphi^{s_n})}$, $z \in \overline{D(\varphi^s)}$, $z_n \rightarrow z$ in H and a element $w_n \in E(t_n, s_n)z_n$ converges to some element $w \in H$ as $n \rightarrow +\infty$, then $w \in E(t, s)z$

We begin with the definition of a discrete ω -limit set for $E(\cdot, \cdot)$.

Definition 4.3. (Discrete ω -limit set for $E(\cdot, \cdot)$) Let $\tau \in R_+$ be fixed. Let $\mathcal{B}(H)$ be a family of bounded subsets of H . Then for each $B \in \mathcal{B}(H)$, the set

$$\omega_\tau(B) := \bigcap_{n \in Z_+} \overline{\bigcup_{k \geq n, m \in Z_+} E(kT_0 + mT_0 + \tau, mT_0 + \tau)(\overline{D(\varphi^{mT_0 + \tau})} \cap B)}$$

is called the discrete ω -limit set of B under $E(\cdot, \cdot)$.

Remark 4.1. By definition of the discrete ω -limit set $\omega_\tau(B)$, it is easy to see that $x \in \omega_\tau(B)$ if and only if there exist sequences $\{k_n\} \subset Z_+$ with $k_n \uparrow +\infty$, $\{m_n\} \subset Z_+$, $\{z_n\} \subset B$ with $z_n \in \overline{D(\varphi^{m_n T_0 + \tau})}$ and $\{x_n\} \subset H$ with $x_n \in E(k_n T_0 + m_n T_0 + \tau, m_n T_0 + \tau)z_n$ such that

$$x_n \longrightarrow x \text{ in } H \text{ as } n \rightarrow +\infty.$$

Now let us mention main theorems in this paper.

Theorem 4.1. (Discrete attractors of $(AP)_\tau$) For each $\tau \in R_+$, let \mathcal{A}_τ be the global attractor of T_0 -periodic dynamical systems U_τ , which is obtained in Section 3. For $\{\varphi^t\} \in$

$\Phi(\{a_r\}, \{b_r\}) \cap \Psi_M, \{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^t\})$ and $f \in L_{loc}^2(R_+; H)$, we assume that the system $(AP)_s$ is asymptotically T_0 -periodic. Here we put

$$\mathcal{A}_\tau^* := \overline{\bigcup_{B \in \mathcal{B}(H)} \omega_\tau(B)}. \quad (4.1)$$

Then, we have

- (i) $\mathcal{A}_\tau^* \subset D(\varphi_p^\tau)$ is non-empty and compact in H ;
- (ii) for each bounded set $B \in \mathcal{B}(H)$ and each number $\epsilon > 0$ there exists $N_{B, \epsilon} \in N$ such that

$$\text{dist}_H(E(kT_0 + \tau, \tau)z, \mathcal{A}_\tau^*) < \epsilon$$

for all $z \in \overline{D(\varphi^\tau)} \cap B$ and all $k \geq N_{B, \epsilon}$;

- (iii) $\mathcal{A}_\tau^* \subset U_\tau^l \mathcal{A}_\tau^* \subset \mathcal{A}_\tau$ for any $l \in N$, where U_τ is the discrete dynamical system for $(P)_\tau$ given in Definition 3.3.

Remark 4.2. By the definition of the discrete ω -limit set $\omega_\tau(B)$ and \mathcal{A}_τ^* , we easily see that

$$\mathcal{A}_\tau^* = \mathcal{A}_{\tau+nT_0}^*, \quad \forall n \in N.$$

Hence \mathcal{A}_τ^* is T_0 -periodic in time in a sense of the above.

The second main theorem is concerned with a relationship between two attractors \mathcal{A}_s^* and \mathcal{A}_τ^* .

Theorem 4.2. Suppose the same assumptions are made as in Theorem 4.1. Let \mathcal{A}_s^* and \mathcal{A}_τ^* be discrete attractors for $E(\cdot, s)$ and $E(\cdot, \tau)$ with $0 \leq s \leq \tau < +\infty$, respectively. Then,

$$\mathcal{A}_\tau^* \subset U(\tau, s)\mathcal{A}_s^*.$$

where $U(\tau, s)$ is the T_0 -periodic process for $(P)_s$ which is given in Definition 3.2.

By Theorems 4.1-4.2, we can get the attractor for asymptotic T_0 -periodic system $(AP)_\tau$.

Theorem 4.3. (Global attractor for $(AP)_\tau$) Suppose the same assumptions are made as in Theorem 4.1. For any $\tau \in R_+$, let \mathcal{A}_τ^* be the discrete attractor for $E(\cdot, \tau)$ obtained in Theorem 4.1. Here we put

$$\mathcal{A}^* := \bigcup_{\tau \in [0, T_0]} \mathcal{A}_\tau^*. \quad (4.2)$$

Then, for any bounded set $B \in \mathcal{B}(H)$,

$$\bigcap_{s \geq 0} \bigcup_{t \geq s, \tau \in R_+} \overline{E(t + \tau, \tau)(\overline{D(\varphi^\tau)} \cap B)} \subset \mathcal{A}^*. \quad (4.3)$$

By Theorem 4.3, the set \mathcal{A}^* can be called the global attractor of $(\text{AP})_\tau$.

Here we give some key lemmas.

Lemma 4.1. *If $\{s_n\} \subset \mathbb{R}_+$, $\{\tau_n\} \subset \mathbb{R}_+$, $s \in \mathbb{R}_+$, $\tau \in \mathbb{R}_+$, $s_n \rightarrow s$, $\tau_n \rightarrow \tau$, $\{m_n\} \subset \mathbb{Z}_+$ with $m_n \rightarrow +\infty$, $z_n \in \overline{D(\varphi^{m_n T_0 + s_n})}$, $z \in \overline{D(\varphi_p^s)}$, $z_n \rightarrow z$ in H and a element $w_n \in E(m_n T_0 + \tau_n + s_n, m_n T_0 + s_n) z_n$ converges to some element $w \in H$ as $n \rightarrow +\infty$, then $w \in U(\tau + s, s)z$*

Proof. Since $\tau_n \rightarrow \tau$, without loss of generality we may assume that there exists a finite time $T > 0$ such that $\{\tau_n\} \subset [0, T]$ and $\tau \in [0, T]$. By $w_n \in E(m_n T_0 + \tau_n + s_n, m_n T_0 + s_n) z_n$, there is a solution v_n of $(\text{AP})_{m_n T_0 + s_n}$ on $[m_n T_0 + s_n, +\infty)$ such that

$$v_n(m_n T_0 + \tau_n + s_n) = w_n \text{ and } v_n(m_n T_0 + s_n) = z_n.$$

Now we put $u_n(t) := v_n(t + m_n T_0 + s_n)$, then we easily see that u_n is the solution for

$$\begin{cases} u_n'(t) + \partial \varphi^{t+m_n T_0+s_n}(u_n(t)) + G(t + m_n T_0 + s_n, u_n(t)) \ni f(t + m_n T_0 + s_n), & t > 0, \\ u_n(0) = z_n. \end{cases}$$

Let $\delta \in (0, 1)$ be fixed. Since $z_n \rightarrow z$ in H as $n \rightarrow +\infty$, $\{z_n\}$ is bounded in H . Hence, from global estimates of solutions (cf. (B) in Section 2) it follows that there is a positive constant $M_\delta > 0$ (independent of n) satisfying

$$\sup_{t \geq \delta} |u_n(t)|_H^2 + \sup_{t \geq \delta} |u_n'|_{L^2(t, t+1; H)}^2 + \sup_{t \geq \delta} \varphi^{t+m_n T_0+s_n}(u_n(t)) \leq M_\delta. \quad (4.4)$$

By [16, Lemma 4.1] we note that the convergence assumption (A1) implies

$$\varphi^{t+m_n T_0+s_n} \longrightarrow \varphi_p^{t+s} \text{ in the sense of Mosco [20]} \quad (4.5)$$

for each $t \geq 0$ as $n \rightarrow +\infty$. Moreover by the same argument in [10, Lemma 3.1] we can prove that

$$\bigcup_{n=1}^{+\infty} \{z \in H; \varphi^{t+m_n T_0+s_n}(z) \leq k\} \text{ is relatively compact in } H \quad (4.6)$$

for every real $k > 0$ and $t \geq 0$, where $\varphi^{t+m_n T_0+s_n} = \varphi_p^{t+s}$ if $n = +\infty$. Therefore, by (4.4)-(4.6), (A2), (A3) and the convergence result (C) in Section 2, (by taking a subsequence of $\{n\}$, if necessary) we see that there is a function u_δ such that

$$u_\delta'(t) + \partial \varphi_p^{t+s}(u_\delta(t)) + G_p(t + s, u_\delta(t)) \ni f_p(t + s), \quad t > \delta.$$

By the standard diagonal process and the same argument in [21, Lemma 3.10], we can construct the solution u on $[0, +\infty)$ satisfying

$$\begin{cases} u'(t) + \partial \varphi_p^{t+s}(u(t)) + G_p(t + s, u(t)) \ni f_p(t + s), & t > 0, \\ u(0) = z \end{cases}$$

and

$$u_n \longrightarrow u \text{ in } C([0, T]; H) \text{ as } n \rightarrow +\infty. \quad (4.7)$$

Then, by (4.7) and $u_n(\tau_n) = w_n$ we have $u(\tau) = w$, which implies that $w \in U(\tau + s, s)z$. \diamond

By (B) in Section 2, for each $B \in \mathcal{B}(H)$ we can choose constants $r_B > 0$ and $M_B > 0$ so that

$$|v|_H \leq r_B \quad \text{and} \quad \varphi^{t+s}(v) \leq M_B, \quad (4.8)$$

for any $s \in R_+$, $t \geq T_0$, $z \in \overline{D(\varphi^s)} \cap B$ and $v \in E(t + s, s)z$. Hence it follows from condition (A1) that for each $m \in Z_+$, $\tau \in [0, T_0]$, $n \in N$ and $z \in \overline{D(\varphi^{mT_0+\tau})} \cap B$ there is $\tilde{z} := \tilde{z}_{mT_0+\tau, z, nT_0} \in D(\varphi_p^\tau)$ such that

$$|\tilde{z} - v|_H \leq J_{m+n}^{(r_B+M_B+M)},$$

(hence $|\tilde{z}|_H \leq r_B + J_{m+n}^{(r_B+M_B+M)}$)

and

$$\varphi_p^\tau(\tilde{z}) - \varphi^{nT_0+mT_0+\tau}(v) \leq J_{m+n}^{(r_B+M_B+M)},$$

(hence $\varphi_p^\tau(\tilde{z}) \leq M_B + J_{m+n}^{(r_B+M_B+M)}$).

where $v \in E(nT_0 + mT_0 + \tau, mT_0 + \tau)z$.

Since $J_k^{(r_B+M_B+M)} \rightarrow 0$ as $k \rightarrow +\infty$, there is a number $N_0 \in N$ such that

$$J_k^{(r_B+M_B+M)} \leq 1, \quad \forall k > N_0.$$

Now, put $J_0 := 1 + \sup_{1 \leq k \leq N_0} J_k^{(r_B+M_B+M)} < +\infty$. Then, we define the bounded set \tilde{B}_τ by

$$\tilde{B}_\tau := \{z \in H; |z|_H \leq r_B + J_0\} \cap \overline{D(\varphi_p^\tau)}.$$

Let $B_{0,\tau}$ be the compact absorbing set for U_τ introduced by Remark 3.1. Then, we see that there exists a number $\tilde{N} \in N$ so that

$$U_\tau^l \tilde{B}_\tau \subset B_{0,\tau}, \quad \forall l \geq \tilde{N}. \quad (4.9)$$

The next lemma is very important to prove Theorem 4.1 (iii).

Lemma 4.2. *Let $\tau \in R_+$ and $B_{0,\tau}$ be the compact absorbing set for U_τ . Then we have*

$$\omega_\tau(B) \subset B_{0,\tau}, \quad \forall B \in \mathcal{B}(H).$$

Proof. At first we assume $\tau \in [0, T_0]$.

For each $B \in \mathcal{B}(H)$, let x be any element of $\omega_\tau(B)$. Then, it follows from Remark 4.1 that there exist sequences $\{k_n\} \subset Z_+$ with $k_n \rightarrow +\infty$, $\{m_n\} \subset Z_+$, $\{z_n\} \subset B$ with $z_n \in \overline{D(\varphi^{m_n T_0 + \tau})}$ and $\{x_n\} \subset H$ with $x_n \in E(k_n T_0 + m_n T_0 + \tau, m_n T_0 + \tau)z_n$ such that

$$x_n \rightarrow x \text{ in } H \quad \text{as } n \rightarrow +\infty. \quad (4.10)$$

Let \tilde{N} be the positive integer obtained in (4.9). Then by (E2) we have

$$x_n \in E(k_n T_0 + m_n T_0 + \tau, k_n T_0 - \tilde{N} T_0 + m_n T_0 + \tau)$$

$$\circ E(k_n T_0 - \widetilde{N} T_0 + m_n T_0 + \tau, m_n T_0 + \tau) z_n \quad (4.11)$$

for any n with $k_n \geq \widetilde{N} + 1$.

Hence, there exists an element $y_n \in E(k_n T_0 - \widetilde{N} T_0 + m_n T_0 + \tau, m_n T_0 + \tau) z_n$ such that

$$x_n \in E(k_n T_0 + m_n T_0 + \tau, k_n T_0 - \widetilde{N} T_0 + m_n T_0 + \tau) y_n. \quad (4.12)$$

Since $\{z_n\} \subset B$, we see that

$$|y_n|_H \leq r_B \quad \text{and} \quad \varphi^{k_n T_0 - \widetilde{N} T_0 + m_n T_0 + \tau}(y_n) \leq M_B \quad \text{for any } n \text{ with } k_n \geq \widetilde{N} + 1,$$

where r_B and M_B are same positive constants in (4.8).

From the convergence condition (A1) it follows that for $y_n \in E(k_n T_0 - \widetilde{N} T_0 + m_n T_0 + \tau, m_n T_0 + \tau) z_n$ there is $\tilde{z}_n \in D(\varphi_p^\tau)$ such that

$$|\tilde{z}_n - y_n|_H \leq J_{k_n - \widetilde{N} + m_n}^{(r_B + M_B + M)},$$

(hence $|\tilde{z}_n|_H \leq r_B + J_{k_n - \widetilde{N} + m_n}^{(r_B + M_B + M)}$)

and

$$\varphi_p^\tau(\tilde{z}_n) \leq M_B + J_{k_n - \widetilde{N} + m_n}^{(r_B + M_B + M)}.$$

Since $\{\tilde{z}_n \in D(\varphi_p^\tau) ; n \in N \text{ with } k_n \geq \widetilde{N} + 1\} \subset \widetilde{B}_\tau$ is relatively compact in H , we may assume that

$$\tilde{z}_n \longrightarrow \tilde{z}_\infty \text{ in } H \quad \text{as } n \rightarrow +\infty$$

for some $\tilde{z}_\infty \in H$. Then we easily see that $\tilde{z}_\infty \in \widetilde{B}_\tau$ and

$$y_n \longrightarrow \tilde{z}_\infty \text{ in } H \quad \text{as } n \rightarrow +\infty. \quad (4.13)$$

By Lemma 4.1 and (4.10)-(4.13), we observe that

$$x \in U(\widetilde{N} T_0 + \tau, \tau) \tilde{z}_\infty,$$

which implies that

$$x \in U(\widetilde{N} T_0 + \tau, \tau) \widetilde{B}_\tau = U_\tau^{\widetilde{N}} \widetilde{B}_\tau \subset B_{0, \tau}.$$

Hence we have

$$\omega_\tau(B) \subset B_{0, \tau}.$$

For the general case of $\tau \in R_+$, choose positive numbers $i_\tau \in N$ and $\tau_0 \in [0, T_0]$ so that $\tau = \tau_0 + i_\tau T_0$. Then, we can show $\omega_\tau(B) \subset B_{0, \tau}$ by the same argument as above. \diamond

Proof of Theorem 4.1. On account of Lemma 4.2 we can get $\mathcal{A}_\tau^* \subset B_{0, \tau}$. Hence, Theorem 4.1 (i) holds. Also, by (4.1) and Remark 4.1 we observe that Theorem 4.1 (ii) holds.

Now, we prove Theorem 4.1 (iii). At first, let us prove that $\mathcal{A}_\tau^* \subset U_\tau^l \mathcal{A}_\tau^*$ for any $l \in N$.

Let x be any element of \mathcal{A}_τ^* . By the definition of \mathcal{A}_τ^* , there are sequences $\{B_n\} \subset \mathcal{B}(H)$ and $\{x_n\} \subset H$ with $x_n \in \omega_\tau(B_n)$ such that

$$x_n \longrightarrow x \text{ in } H \quad \text{as } n \rightarrow +\infty. \quad (4.14)$$

Then, for each n it follows from Remark 4.1 that there exist sequences $\{k_{n,j}\} \subset Z_+$ with $k_{n,j} \rightarrow +\infty$, $\{m_{n,j}\} \subset Z_+$, $\{z_{n,j}\} \subset B_n$ with $z_{n,j} \in \overline{D(\varphi^{m_{n,j}T_0 + \tau})}$ and $\{v_{n,j}\} \subset H$ with $v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j}$ such that

$$v_{n,j} \longrightarrow x_n \text{ in } H \quad \text{as } j \rightarrow +\infty. \quad (4.15)$$

Let l be any number in N , then we see that

$$\begin{aligned} v_{n,j} \in & E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau) \\ & \circ E(k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j} \end{aligned}$$

for j with $k_{n,j} \geq l + 1$. So, there exists an element $w_{n,j} \in E(k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j}$ such that

$$v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau)w_{n,j}. \quad (4.16)$$

By global estimates (B) in Section 2, $\{w_{n,j} \in H ; j \in N \text{ with } k_{n,j} \geq l + 1\}$ is relatively compact in H for each n . Therefore we may assume that the element $w_{n,j}$ converges to some element $\tilde{w}_{n,\infty} \in H$ as $j \rightarrow +\infty$. Clearly, $\tilde{w}_{n,\infty} \in \omega_\tau(B_n)$. Moreover, it follows from Lemma 4.1 and (4.15)-(4.16) that

$$x_n \in U(lT_0 + \tau, \tau)\tilde{w}_{n,\infty} \subset U(lT_0 + \tau, \tau)\omega_\tau(B_n),$$

hence, we have

$$x_n \in \bigcup_{n \geq 1} U_\tau^l \omega_\tau(B_n), \quad \forall n \geq 1. \quad (4.17)$$

Here, by the closedness of $U(\cdot, \cdot)$ we note that for each subset X of $B_{0,\tau}$,

$$\overline{U_\tau^l X} \subset U_\tau^l \overline{X}, \quad \forall l \in N. \quad (4.18)$$

Taking account of Lemma 4.2, (4.14), (4.17) and (4.18), we observe that

$$\begin{aligned} x & \in \overline{\bigcup_{n \geq 1} U_\tau^l \omega_\tau(B_n)} \\ & = \overline{U_\tau^l \bigcup_{n \geq 1} \omega_\tau(B_n)} \\ & \subset U_\tau^l \overline{\bigcup_{n \geq 1} \omega_\tau(B_n)} \\ & \subset U_\tau^l \mathcal{A}_\tau^*, \end{aligned}$$

which implies that \mathcal{A}_τ^* is semi-invariant under the T_0 -periodic dynamical systems U_τ , i.e.

$$\mathcal{A}_\tau^* \subset U_\tau^l \mathcal{A}_\tau^*, \quad \forall l \in N. \quad (4.19)$$

Next we shall prove that $U_\tau^l \mathcal{A}_\tau^* \subset \mathcal{A}_\tau$ for any $l \in N$. By (4.19), for each $l \in N$

$$U_\tau^l \mathcal{A}_\tau^* \subset U_\tau^l U_\tau^n \mathcal{A}_\tau^* = U_\tau^{l+n} \mathcal{A}_\tau^*, \quad \forall n \in N. \quad (4.20)$$

By $\mathcal{A}_\tau^* \subset B_{0,\tau}$, (4.20) and the attractive property of \mathcal{A}_τ , we have

$$U_\tau^l \mathcal{A}_\tau^* \subset \mathcal{A}_\tau, \quad \forall l \in N.$$

Therefore we conclude that

$$\mathcal{A}_\tau^* \subset U_\tau^l \mathcal{A}_\tau^* \subset \mathcal{A}_\tau, \quad \forall l \in N.$$

◇

Proof of Theorem 4.2. Let x be any element of \mathcal{A}_τ^* . Then by the definition of \mathcal{A}_τ^* , there exist sequences $\{B_n\} \subset \mathcal{B}(H)$ and $\{x_n\} \subset H$ with $x_n \in \omega_\tau(B_n)$ such that

$$x_n \longrightarrow x \text{ in } H \quad \text{as } n \rightarrow +\infty. \quad (4.21)$$

From Remark 4.1 it follows that for each n , there are sequences $\{k_{n,j}\} \subset Z_+$ with $k_{n,j} \rightarrow +\infty$, $\{m_{n,j}\} \subset Z_+$, $\{z_{n,j}\} \subset B_n$ with $z_{n,j} \in \overline{D(\varphi^{m_{n,j}T_0+\tau})}$ and $\{v_{n,j}\} \subset H$ with $v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j}$ such that

$$v_{n,j} \longrightarrow x_n \text{ in } H \quad \text{as } j \rightarrow +\infty. \quad (4.22)$$

Note that for given $s, \tau \in R_+$ with $s \leq \tau$ there is a positive number $l_s \in N$ satisfying

$$s \leq \tau \leq l_s T_0 + s.$$

By using the property (E2) we see that

$$\begin{aligned} v_{n,j} &\in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 + m_{n,j}T_0 + s) \\ &\quad \circ E(k_{n,j}T_0 + m_{n,j}T_0 + s, T_0 + m_{n,j}T_0 + l_s T_0 + s) \\ &\quad \circ E(T_0 + m_{n,j}T_0 + l_s T_0 + s, m_{n,j}T_0 + \tau)z_{n,j} \end{aligned}$$

for any $j \in Z_+$ with $k_{n,j} \geq l_s + 2$. Here we can take elements $w_{n,j} \in H$ and $y_{n,j} \in H$ so that

$$v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 + m_{n,j}T_0 + s)w_{n,j}, \quad (4.23)$$

$$w_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + s, T_0 + m_{n,j}T_0 + l_s T_0 + s)y_{n,j} \quad (4.24)$$

and

$$y_{n,j} \in E(T_0 + m_{n,j}T_0 + l_s T_0 + s, m_{n,j}T_0 + \tau)z_{n,j}. \quad (4.25)$$

By $\{z_{n,j}\} \subset B_n$ and the global boundedness result (B) in Section 2, we can get a positive constant $C_n := C_n(B_n) > 0$ satisfying

$$|y_{n,j}|_H \leq C_n, \quad \forall y_{n,j} \in E(T_0 + m_{n,j}T_0 + l_s T_0 + s, m_{n,j}T_0 + \tau)z_{n,j}. \quad (4.26)$$

Here we define the bounded set B_{C_n} by

$$B_{C_n} := \{b \in H ; |b|_H \leq C_n\}.$$

From (4.26) and the result (B) in Section 2 it follows that the set

$$\left\{ \begin{array}{l} w_{n,j} \in H ; \\ w_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + s, T_0 + m_{n,j}T_0 + l_s T_0 + s)y_{n,j} \end{array} \right\} \\ \text{for any } j \in Z_+ \text{ with } k_{n,j} \geq l_s + 2$$

is relatively compact in H . Hence, we may assume that the element $w_{n,j}$ converges to some element $\tilde{w}_{n,\infty} \in H$ as $j \rightarrow +\infty$. Clearly, $\tilde{w}_{n,\infty} \in \omega_s(B_{C_n})$, and it follows from Lemma 4.2 that

$$\omega_s(B_{C_n}) \subset B_{0,s} \subset \overline{D(\varphi_p^s)}.$$

Moreover, by Lemma 4.1 and (4.22)-(4.23) we have

$$x_n \in U(\tau, s)\tilde{w}_{n,\infty} \subset U(\tau, s)\omega_s(B_{C_n}), \quad \forall n \geq 1,$$

hence, we see that

$$x_n \in \bigcup_{n \geq 1} U(\tau, s)\omega_s(B_{C_n}), \quad \forall n \geq 1. \quad (4.27)$$

Here, by the closedness of $U(\cdot, \cdot)$, we note that for each subset X of $B_{0,s}$,

$$\overline{U(\tau, s)X} \subset U(\tau, s)\overline{X}. \quad (4.28)$$

On account of Lemma 4.2, (4.21), (4.27) and (4.28), we observe that

$$\begin{aligned} x &\in \overline{\bigcup_{n \geq 1} U(\tau, s)\omega_s(B_{C_n})} \\ &= \overline{U(\tau, s) \bigcup_{n \geq 1} \omega_s(B_{C_n})} \\ &\subset U(\tau, s) \overline{\bigcup_{n \geq 1} \omega_s(B_{C_n})} \\ &\subset U(\tau, s)\mathcal{A}_s^*, \end{aligned}$$

which implies that \mathcal{A}_τ^* is the subset of $U(\tau, s)\mathcal{A}_s^*$, namely

$$\mathcal{A}_\tau^* \subset U(\tau, s)\mathcal{A}_s^*.$$

◇

Proof of Theorem 4.3. For any $B \in \mathcal{B}(H)$, let z_0 be any element of the ω -limit set $\omega_E(B)$ which is define by

$$\omega_E(B) := \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s, \tau \in R_+} E(t + \tau, \tau) \overline{D(\varphi^\tau)} \cap B}.$$

Then we easily see that there exist sequences $\{t_n\} \subset R_+$ with $t_n \uparrow +\infty$, $\{\tau_n\} \subset R_+$, $\{y_n\} \subset B$ with $y_n \in \overline{D(\varphi^{\tau_n})}$ and $\{z_n\} \subset H$ with $z_n \in E(t_n + \tau_n, \tau_n)y_n$ such that

$$\begin{aligned} t_n &:= k_n T_0 + t'_n, \quad k_n \in Z_+, \quad k_n \nearrow +\infty, \quad t'_n \in [T_0, 2T_0], \quad t'_n \rightarrow t'_0, \\ \tau_n &:= l_n T_0 + \tau'_n, \quad l_n \in Z_+, \quad \tau'_n \in [0, T_0], \quad \tau'_n \rightarrow \tau'_0 \end{aligned}$$

and

$$z_n \longrightarrow z_0 \quad \text{in } H \quad (4.29)$$

as $n \rightarrow +\infty$. Without loss of generality, we may assume that

$$(a) \quad t'_n + \tau'_n \nearrow t'_0 + \tau'_0 \quad \text{or} \quad (b) \quad t'_n + \tau'_n \searrow t'_0 + \tau'_0.$$

Now, assume that (a) holds. Then let us consider the multivalued semiflow

$$v_n \in E(1 + k_n T_0 + l_n T_0 + t'_0 + \tau'_0, k_n T_0 + l_n T_0 + t'_n + \tau'_n)z_n. \quad (4.30)$$

Then, there is a solution u_n on $[k_n T_0 + l_n T_0 + t'_n + \tau'_n, +\infty)$ for

$$\begin{cases} u'_n(t) + \partial\varphi^{t+k_n T_0+l_n T_0+t'_n+\tau'_n}(u_n(t)) + G(t + k_n T_0 + l_n T_0 + t'_n + \tau'_n, u_n(t)) \\ \quad \ni f(t + k_n T_0 + l_n T_0 + t'_n + \tau'_n), \quad t > 0, \\ u_n(0) = z_n \quad \text{and} \quad u_n(1 + t'_0 + \tau'_0 - t'_n - \tau'_n) = v_n. \end{cases}$$

Since $z_n \rightarrow z_0$ in H , $\{z_n\}$ is bounded in H . Therefore by the global estimate (B) in Section 2, we see that

$$\left\{ \begin{array}{l} v_n \in E(1 + k_n T_0 + l_n T_0 + t'_0 + \tau'_0, k_n T_0 + l_n T_0 + t'_n + \tau'_n)z_n \\ v_n \in H; \end{array} \right\} \quad \text{for any } n \in N$$

is relatively compact in H . Hence we may assume that

$$v_n \longrightarrow v \quad \text{in } H \quad \text{for some } v \in H. \quad (4.31)$$

Now applying Lemma 4.1 with (4.29)-(4.31), we can get

$$v \in U(1 + t'_0 + \tau'_0, t'_0 + \tau'_0)z_0,$$

more precisely, (taking the subsequence of $\{n\}$ if necessary) we observe that

$$u_n \longrightarrow u \quad \text{in } C([0, 2]; H) \quad \text{as } n \rightarrow +\infty, \quad (4.32)$$

where u is the solution $[t'_0 + \tau'_0, +\infty)$ satisfying

$$\begin{cases} u'(t) + \partial\varphi_p^{t+t'_0+\tau'_0}(u(t)) + G_p(t + t'_0 + \tau'_0, u(t)) \ni f_p(t + t'_0 + \tau'_0), \quad t > 0, \\ u(0) = z_0 \quad \text{and} \quad u(1) = v. \end{cases}$$

By (4.32) we easily see that

$$u_n(t'_0 + \tau'_0 - t'_n - \tau'_n) \longrightarrow z_0 \quad \text{as } n \rightarrow +\infty. \quad (4.33)$$

Note that

$$\begin{aligned}
& u_n(t'_0 + \tau'_0 - t'_n - \tau'_n) \\
& \in E(k_n T_0 + l_n T_0 + t'_0 + \tau'_0, k_n T_0 + l_n T_0 + t'_n + \tau'_n) z_n \\
& = E(k_n T_0 + l_n T_0 + t'_0 + \tau'_0, l_n T_0 + \tau'_n) y_n \\
& = E(k_n T_0 + l_n T_0 + t'_0 + \tau'_0, l_n T_0 + t'_0 + \tau'_0) E(l_n T_0 + t'_0 + \tau'_0, l_n T_0 + \tau'_n) y_n.
\end{aligned}$$

So, we can take a element $x_n \in E(l_n T_0 + t'_0 + \tau'_0, l_n T_0 + \tau'_n) y_n$ such that

$$u_n(t'_0 + \tau'_0 - t'_n - \tau'_n) \in E(k_n T_0 + l_n T_0 + t'_0 + \tau'_0, l_n T_0 + t'_0 + \tau'_0) x_n. \quad (4.34)$$

By $\{y_n\} \subset B$ and the global estimate (B) in Section 2, we easily see that $\{x_n\}$ is bounded, i.e.

$$\{x_n\} \subset \tilde{B} \text{ for some } \tilde{B} \in \mathcal{B}(H). \quad (4.35)$$

Therefore, from Remarks 4.1-4.2 and (4.33)-(4.35) we observe that

$$z_0 \in \omega_{t'_0 + \tau'_0}(\tilde{B}) \subset \mathcal{A}_{t'_0 + \tau'_0}^* \subset \mathcal{A}^*.$$

Thus (4.3) holds.

In the case (b) when $t'_n + \tau'_n \searrow t'_0 + \tau'_0$, we can prove (4.3) by the slight modification of the proof as above. \diamond

Theorem 4.1 implies that the attracting set \mathcal{A}_τ^* for $(\text{AP})_\tau$ is semi-invariant under U_τ associated with the limiting T_0 -periodic system $(\text{P})_s$, in general. Moreover, from Theorem 4.2 we observe that

$$\mathcal{A}_\tau^* \subset U(\tau, s) \mathcal{A}_s^* \quad \text{for any } 0 \leq s \leq \tau < +\infty.$$

In order to get the invariance of \mathcal{A}_τ^* under U_τ and $\mathcal{A}_\tau^* = U(\tau, s) \mathcal{A}_s^*$, let us use a concept of a regular approximation, which was introduced in [17].

Definition 4.4. (Regular approximation) Let $s \in R_+$ be fixed. Let $z \in D(\varphi_p^s)$. Then, we say that $U(t+s, s)z$ is regularly approximated by $E(t+kT_0+s, kT_0+s)$ as $k \rightarrow +\infty$, if for each finite $T > 0$ there are sequences $\{k_n\} \subset Z_+$ with $k_n \rightarrow +\infty$ and $\{z_n\} \subset H$ with $z_n \in D(\varphi^{k_n T_0 + s})$ and $z_n \rightarrow z$ in H satisfying the following property: for any function $u \in W^{1,2}(0, T; H)$ satisfying $u(t) \in U(t+s, s)z$ for all $t \in [0, T]$ there is a sequence $\{u_n\} \subset W^{1,2}(0, T; H)$ such that $u_n(t) \in E(t+k_n T_0+s, k_n T_0+s)z_n$ for all $t \in [0, T]$ and $u_n \rightarrow u$ in $C([0, T]; H)$ as $n \rightarrow +\infty$.

Using the above concept, we can show that the invariance of \mathcal{A}_τ^* under U_τ . Moreover we can get

$$\mathcal{A}_\tau^* = U(\tau, s) \mathcal{A}_s^*.$$

Theorem 4.4 Suppose all assumptions in Theorem 4.1. Let \mathcal{A}_s^* and \mathcal{A}_τ^* be discrete attractors for $E(\cdot, s)$ and $E(\cdot, \tau)$, with $0 \leq s \leq \tau < +\infty$, respectively. Assume that for

any point z of \mathcal{A}_s^* , $U(t+s, s)z$ is regularly approximated by $E(t+kT_0+s, kT_0+s)$ as $k \rightarrow +\infty$. Then we have

$$\mathcal{A}_\tau^* = U(\tau, s)\mathcal{A}_s^*.$$

Proof. By Theorem 4.2, we have only to show that

$$U(\tau, s)\mathcal{A}_s^* \subset \mathcal{A}_\tau^*.$$

To do so, let x be any element of $U(\tau, s)\mathcal{A}_s^*$.

At first, taking account of Definitions 3.2-3.3 and Theorem 4.1 (iii), we see that for each $n \in \mathbb{N}$

$$\begin{aligned} & U_\tau^n U(\tau, s)\mathcal{A}_s^* \\ &= U(nT_0 + \tau, \tau)U(\tau, s)\mathcal{A}_s^* \\ &= U(nT_0 + \tau, nT_0 + s)U(nT_0 + s, s)\mathcal{A}_s^* \\ &= U(\tau, s)U_s^n \mathcal{A}_s^* \\ &\supset U(\tau, s)\mathcal{A}_s^*. \end{aligned} \tag{4.36}$$

Hence, there exists a element $y_n \in \mathcal{A}_s^*$ such that

$$x \in U_\tau^n U(\tau, s)y_n = U(nT_0 + \tau - s + s, s)y_n.$$

By using our assumption as $t = nT_0 + \tau - s$, we observe that for each n , there are sequences $\{k_{n,j}\} \subset \mathbb{Z}_+$, $\{x_{n,j}\} \subset H$ and $\{y_{n,j}\} \subset H$ such that

$$k_{n,j} \rightarrow +\infty, \quad y_{n,j} \in D(\varphi^{k_{n,j}T_0+s}), \quad y_{n,j} \rightarrow y_n \text{ in } H$$

and

$$x_{n,j} \in E(nT_0 + \tau - s + k_{n,j}T_0 + s, k_{n,j}T_0 + s)y_{n,j}, \quad x_{n,j} \rightarrow x \text{ in } H \tag{4.37}$$

as $j \rightarrow +\infty$. Therefore, by the usual diagonal argument, we can find a subsequence $\{j_n\}$ of $\{j\}$ such that $\tilde{x}_n := x_{n,j_n}$, $\tilde{y}_n := y_{n,j_n}$ and $\tilde{k}_n := k_{n,j_n}$ satisfy

$$|\tilde{x}_n - x|_H < \frac{1}{n}, \quad \tilde{x}_n \in E(nT_0 + \tau - s + \tilde{k}_n T_0 + s, \tilde{k}_n T_0 + s)\tilde{y}_n, \quad |\tilde{y}_n - y_n|_H < \frac{1}{n} \tag{4.38}$$

for every $n = 1, 2, \dots$. Since $\{\tilde{y}_n\}$ is bounded in H , there is a bounded set $B \in \mathcal{B}(H)$ so that $\{\tilde{y}_n\} \subset B$.

By (E2), we see that

$$\begin{aligned} \tilde{x}_n &\in E(nT_0 + \tau - s + \tilde{k}_n T_0 + s, \tilde{k}_n T_0 + s)\tilde{y}_n \\ &= E(nT_0 + \tilde{k}_n T_0 + \tau, T_0 + \tilde{k}_n T_0 + \tau)E(T_0 + \tilde{k}_n T_0 + \tau, \tilde{k}_n T_0 + s)\tilde{y}_n, \end{aligned}$$

hence there is an element $\tilde{z}_n \in E(T_0 + \tilde{k}_n T_0 + \tau, \tilde{k}_n T_0 + s)\tilde{y}_n$ such that

$$\tilde{x}_n \in E(nT_0 + \tilde{k}_n T_0 + \tau, T_0 + \tilde{k}_n T_0 + \tau)\tilde{z}_n. \tag{4.39}$$

Since $\{\tilde{y}_n\} \subset B$ and the global estimate (B) in Section 2, we see that $\{\tilde{z}_n\}$ is also bounded in H . Hence, there is a bounded set $\tilde{B} \in \mathcal{B}(H)$ so that $\{\tilde{z}_n\} \subset \tilde{B}$. The above fact (4.37)-(4.39) implies (cf. Remark 4.1) that $x \in \omega_\tau(\tilde{B}) \subset \mathcal{A}_\tau^*$. Thus we have $U(\tau, s)\mathcal{A}_s^* \subset \mathcal{A}_\tau^*$. \diamond

By the same argument in Theorem 4.4, we can get the following corollary:

Corollary. (i) Suppose the same assumptions of Theorem 4.4. Then, by Remark 4.2 we observe that \mathcal{A}_s^* is invariant under the T_0 -periodic dynamical system $U_s(=U(T_0+s, s))$. Namely,

$$\mathcal{A}_s^* = U_s^l \mathcal{A}_s^* \quad \text{for any } l \in \mathbb{N}.$$

(ii) Assume that for any point z of \mathcal{A}_τ , $U(t+\tau, \tau)z$ is regularly approximated by $E(t+kT_0+\tau, kT_0+\tau)$ as $k \rightarrow +\infty$. Then, we have $\mathcal{A}_\tau^* \supset \mathcal{A}_\tau (=U_\tau \mathcal{A}_\tau)$. Hence by Theorem 4.1 (iii) we conclude that

$$\mathcal{A}_\tau^* = \mathcal{A}_\tau.$$

Remark 4.3. If the solution operator $U(t, s)$ is singlevalued, namely the solution for the Cauchy problem of $(P)_s$ is unique, the assumptions of Theorem 4.4 always hold. Thus, Theorems 4.1-4.4 contain the abstract results obtained in [11], which was concerned with the asymptotic T_0 -periodic stability for the singlevalued dynamical system associated with time-dependent subdifferentials.

5 Application to obstacle problems for PDE's

Let Ω be a bounded domain in R^N ($1 \leq N < +\infty$) with smooth boundary $\Gamma = \partial\Omega$, q be a fixed number with $2 \leq q < +\infty$ and T_0 be a fixed positive number. We use the notation

$$a_q(v, z) := \int_{\Omega} |\nabla v|^{q-2} \nabla v \cdot \nabla z dx, \quad \forall v, z \in W^{1,q}(\Omega)$$

and denote by (\cdot, \cdot) the usual inner product in $L^2(\Omega)$.

For prescribed obstacle functions $\sigma_0 \leq \sigma_1$ and each $t \in R_+$ we define the set

$$K(t) := \left\{ z \in W^{1,q}(\Omega); \sigma_0(t, \cdot) \leq z \leq \sigma_1(t, \cdot) \text{ a.e. on } \Omega \right\}.$$

Let f be a function in $L_{loc}^2(R_+; L^2(\Omega))$ and h be a non-negative function on $R_+ \times R$.

Then for given $\mathbf{b} \in L^\infty(\Omega)^N$ we consider an interior asymptotically T_0 -periodic double obstacle problem $(OP)_s^{AP}$ ($s \in R_+$):

- Find functions $v \in C([s, +\infty); L^2(\Omega))$ and $\theta \in L_{loc}^2((s, +\infty); L^2(\Omega))$ such that

$$(OP)_s^{AP} \left\{ \begin{array}{l} v \in L_{loc}^q((s, +\infty); W^{1,q}(\Omega)) \cap W_{loc}^{1,2}((s, +\infty); L^2(\Omega)); \\ v(t) \in K(t) \text{ for a.e. } t \geq s; \\ 0 \leq \theta(t, x) \leq h(t, v(t, x)) \quad \text{a.e. on } (s, +\infty) \times \Omega; \\ (v'(t) + \theta(t) + \mathbf{b} \cdot \nabla v(t) - f(t), v(t) - z) + a_q(v(t), v(t) - z) \leq 0 \\ \text{for any } z \in K(t) \text{ and a.e. } t \geq s. \end{array} \right.$$

The main object of this section is to consider the large-time behaviour of solution for $(\text{OP})_s^{AP}$ assuming asymptotically T_0 -periodicity conditions

$$\sigma_i(t) - \sigma_{i,p}(t) \longrightarrow 0 \quad (i = 0, 1), \quad h(t, \cdot) - h_p(t, \cdot) \longrightarrow 0, \quad f(t) - f_p(t) \longrightarrow 0$$

as $t \rightarrow \infty$ in the sense specified below, where $\sigma_{i,p}(t)$, $h_p(t, \cdot)$, $f_p(t)$ are periodic in time with the same period T_0 . By the above assumptions, the limiting system of $(\text{OP})_s^{AP}$ is a T_0 -periodic one $(\text{OP})_s^P$ as follows:

- Find functions $u \in C([s, +\infty); L^2(\Omega))$ and $\theta \in L_{loc}^2((s, +\infty); L^2(\Omega))$ such that
$$(\text{OP})_s^P \left\{ \begin{array}{l} u \in L_{loc}^q((s, +\infty); W^{1,q}(\Omega)) \cap W_{loc}^{1,2}((s, +\infty); L^2(\Omega)); \\ u(t) \in K_p(t) \text{ for a.e. } t \geq s; \\ 0 \leq \theta(t, x) \leq h_p(t, u(t, x)) \quad \text{a.e. on } (s, +\infty) \times \Omega; \\ (u'(t) + \theta(t) + \mathbf{b} \cdot \nabla u(t) - f_p(t), u(t) - z) + a_q(u(t), u(t) - z) \leq 0 \\ \text{for any } z \in K_p(t) \text{ and a.e. } t \geq s, \end{array} \right.$$

where $K_p(t) := \{z \in W^{1,q}(\Omega); \sigma_{0,p}(t, \cdot) \leq z \leq \sigma_{1,p}(t, \cdot) \text{ a.e. on } \Omega \}$.

Now we suppose the following conditions:

- σ_i and $\sigma_{i,p}$ are functions on $R_+ \times \Omega$ such that

$$\begin{aligned} \sup_{t \in R_+} \left| \frac{d\sigma_i}{dt} \right|_{L^2(t, t+1; W^{1,q}(\Omega))} + \sup_{t \in R_+} \left| \frac{d\sigma_i}{dt} \right|_{L^2(t, t+1; L^\infty(\Omega))} &< +\infty, \\ \sup_{t \in R_+} \left| \frac{d\sigma_{i,p}}{dt} \right|_{L^2(t, t+1; W^{1,q}(\Omega))} + \sup_{t \in R_+} \left| \frac{d\sigma_{i,p}}{dt} \right|_{L^2(t, t+1; L^\infty(\Omega))} &< +\infty \end{aligned}$$

and $\sigma_{i,p}$ is a T_0 -periodic obstacle function, i.e.

$$\sigma_{i,p}(t + T_0, x) = \sigma_{i,p}(t, x) \quad \text{for a.e. } x \in \Omega \text{ and any } t \in R_+$$

for $i = 0, 1$. Moreover, there are positive constants $k_1 > 0$ and $k_2 > 0$ such that

$$\sigma_1 - \sigma_0 \geq k_1 \quad \text{and} \quad \sigma_{1,p} - \sigma_{0,p} \geq k_1 \quad \text{a.e. on } R_+ \times \Omega$$

and

$$|\sigma_i|_{L^\infty(R_+; W^{1,q}(\Omega))} + |\sigma_i|_{L^\infty(R_+ \times \Omega)} + |\sigma_{i,p}|_{L^\infty(R_+; W^{1,q}(\Omega))} + |\sigma_{i,p}|_{L^\infty(R_+ \times \Omega)} \leq k_2$$

for $i = 0, 1$.

- h and h_p are non-negative continuous functions on $R_+ \times R$. There is a positive constant L such that

$$\begin{aligned} |h(t, z_1) - h(t, z_2)| &\leq L|z_1 - z_2| \\ |h_p(t, z_1) - h_p(t, z_2)| &\leq L|z_1 - z_2| \end{aligned}$$

for all $t \in R_+$, $z_i \in R$ and $i = 1, 2$. Moreover, h_p is a T_0 -periodic function, i.e. for any $z \in R$, $h_p(t + T_0, z) = h_p(t, z)$ for any $t \in R_+$.

- $f, f_p \in L^2_{loc}(R_+; L^2(\Omega))$, and f_p is a T_0 -periodic function, i.e.

$$f_p(t + T_0) = f_p(t) \quad \text{in } L^2(\Omega), \quad \forall t \in R_+.$$

Moreover, we suppose the following convergence conditions:

- (Convergence of $\sigma_i(t) - \sigma_{i,p}(t) \rightarrow 0$ as $t \rightarrow +\infty$) Put

$$\begin{aligned} I_m := & \sup_{t \in [0, T_0]} |\sigma_0(mT_0 + t) - \sigma_{0,p}(t)|_{W^{1,q}(\Omega)} + \sup_{t \in [0, T_0]} |\sigma_1(mT_0 + t) - \sigma_{1,p}(t)|_{W^{1,q}(\Omega)} \\ & + \sup_{t \in [0, T_0]} |\sigma_0(mT_0 + t) - \sigma_{0,p}(t)|_{L^\infty(\Omega)} + \sup_{t \in [0, T_0]} |\sigma_1(mT_0 + t) - \sigma_{1,p}(t)|_{L^\infty(\Omega)} \end{aligned}$$

Then,

$$I_m \rightarrow 0 \quad \text{as } m \rightarrow +\infty;$$

- (Convergence of $h(t, \cdot) - h_p(t, \cdot) \rightarrow 0$ as $t \rightarrow +\infty$) For any $z \in R$,

$$\sup_{t \in [0, T_0]} |h(mT_0 + t, z) - h_p(t, z)| \rightarrow 0 \quad \text{as } m \rightarrow +\infty; \quad (5.1)$$

- (Convergence of $f(t) - f_p(t) \rightarrow 0$ as $t \rightarrow +\infty$)

$$|f(mT_0 + \cdot) - f_p|_{L^2(0, T_0; L^2(\Omega))} \rightarrow 0 \quad \text{as } m \rightarrow +\infty. \quad (5.2)$$

Under the above assumptions, let us consider problems $(OP)_s^{AP}$ and $(OP)_s^P$.

In order to apply the abstract results in Sections 2-4, we choose $L^2(\Omega)$ as a real separable Hilbert space H . And we define a family $\{\varphi^t\}$ of proper l.s.c. convex functions φ^t on $L^2(\Omega)$ by

$$\varphi^t(z) = \begin{cases} \frac{1}{q} \int_{\Omega} |\nabla z|^q dx & \text{if } z \in K(t), \\ +\infty & \text{if } z \in L^2(\Omega) \setminus K(t), \end{cases} \quad (5.3)$$

and define φ_p^t by replacing $K(t)$ by $K_p(t)$ in (5.3).

Also, we define a multivalued operator $G(\cdot, \cdot)$ from $R_+ \times H^1(\Omega)$ into $L^2(\Omega)$ by

$$G(t, z) := \left\{ g \in L^2(\Omega); \begin{array}{l} g = l + \mathbf{b} \cdot \nabla z \quad \text{in } L^2(\Omega) \\ 0 \leq l(x) \leq h(t, z(x)) \quad \text{a.e. on } \Omega \end{array} \right\} \quad (5.4)$$

for all $t \in R_+$ and $z \in H^1(\Omega)$. And we define $G_p(\cdot, \cdot)$ by replacing $h(t, \cdot)$ by $h_p(t, \cdot)$ in (5.4).

By the same argument in [27, Lemma 5.1], we can get the following lemmas.

Lemma 5.1. (cf. [27, Lemma 5.1]) *Put for any $r > 0$ and $t \in R_+$*

$$a_r(t) = b_r(t) := k_3 \int_0^t \left\{ |\sigma'_{0,p}|_{L^\infty(\Omega)} + |\sigma'_{0,p}|_{W^{1,q}(\Omega)} + |\sigma'_{1,p}|_{L^\infty(\Omega)} + |\sigma'_{1,p}|_{W^{1,q}(\Omega)} \right\} d\tau$$

$$+k_3 \int_0^t \left\{ |\sigma'_0|_{L^\infty(\Omega)} + |\sigma'_0|_{W^{1,q}(\Omega)} + |\sigma'_1|_{L^\infty(\Omega)} + |\sigma'_1|_{W^{1,q}(\Omega)} \right\} d\tau,$$

where k_3 is a (sufficiently large) positive constant. Then, $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$ and $\{\varphi_p^t\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0)$.

Moreover we have $\{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^t\})$ and $\{G_p(t, \cdot)\} \in \mathcal{G}_p(\{\varphi_p^t\}; T_0)$.

Lemma 5.2. *The convergence assumptions (A1)-(A3) hold.*

Proof. We easily see that (A2) and (A3) hold by assumptions (5.1) and (5.2).

Now let us show (A1). For each $t \in R_+$ there are $m \in Z_+$ and $\tau \in [0, T_0]$ so that $t = mT_0 + \tau$.

For each $z_p \in D(\varphi_p^t) = K_p(t)$, we put

$$z := (z_p - \sigma_{0,p}(t)) \frac{\sigma_1(t) - \sigma_0(t)}{\sigma_{1,p}(t) - \sigma_{0,p}(t)} + \sigma_0(t).$$

Then we easily see that $z \in D(\varphi^t) = K(t)$. Moreover, by the same argument in [27, Lemma 5.1], we see that

$$|z - z_p|_{L^2(\Omega)} \leq k_4 I_m \quad \text{and} \quad |\nabla z - \nabla z_p|_{L^q(\Omega)} \leq k_4 I_m (1 + |\nabla z_p|_{L^q(\Omega)}) \quad (5.5)$$

for some constant $k_4 > 0$. Hence we have

$$\varphi^t(z) - \varphi_p^t(z_p) \leq k_5 I_m (1 + \varphi_p^t(z_p)), \quad (5.6)$$

for a sufficiently large $k_5 > 0$.

Conversely, let $z \in D(\varphi^t) = K(t)$ and we put

$$z_p := (z - \sigma_0(t)) \frac{\sigma_{1,p}(t) - \sigma_{0,p}(t)}{\sigma_1(t) - \sigma_0(t)} + \sigma_{0,p}(t).$$

Then, we observe that $z_p \in D(\varphi_p^t) = K_p(t)$ and

$$|z_p - z|_{L^2(\Omega)} \leq k_4 I_m \quad \text{and} \quad \varphi_p^t(z_p) - \varphi^t(z) \leq k_5 I_m (1 + \varphi^t(z)). \quad (5.7)$$

Therefore by (5.5)-(5.7) we see that the convergence assumption (A1) holds. \diamond

Clearly, the obstacle problem $(\text{OP})_s^{AP}$ can be reformulated as an evolution equation $(\text{AP})_s$ involving the subdifferential of φ^t given by (5.3) and the multivalued operator $G(t, \cdot)$ defined by (5.4). Also, the limiting T_0 -periodic problem $(\text{OP})_s^P$ can be reformulated as an evolution equation $(\text{P})_s$. Therefore, by Lemmas 5.1-5.2 we can apply abstract results in Section 2-4. Namely, we can obtain an attractor \mathcal{A}_s^* for $(\text{OP})_s^{AP}$, a T_0 -periodic attractor \mathcal{A}_s for $(\text{OP})_s^P$ and the relationships between $(\text{OP})_s^{AP}$ and $(\text{OP})_s^P$.

Additionally, we assume that $f(t) \equiv f_p(t)$ for any $t \in R_+$ and

$$\sigma_0(t, z) \equiv \sigma_{0,p}(t, z), \quad \sigma_{1,p}(t, z) \equiv \sigma_1(t, z), \quad h_p(t, z) \leq h(t, z)$$

for any $0 \leq t < +\infty$ and $z \in R$. Then we easily see that the assumptions of Theorem 4.4 and its Corollary hold. Hence we can get $\mathcal{A}_s^* = \mathcal{A}_s$ by the same argument in [30, Theorem 5.4].

Unfortunately we do not give assumptions for $\sigma_i(t, \cdot)$, $h(t, \cdot)$ and $f(t)$ in order to get

$$U(\tau, s)\mathcal{A}_s^* = \mathcal{A}_\tau^* \subset \mathcal{A}_\tau \text{ for any } 0 \leq s \leq \tau < +\infty. \quad (5.8)$$

It seems difficult to show (5.8), so it is the open problem.

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