### Instructions for use

<table>
<thead>
<tr>
<th>Title</th>
<th>Attractors of asymptotically periodic multivalued dynamical systems governed by time-dependent subdifferentials</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Yamazaki, Noriaki</td>
</tr>
<tr>
<td>Citation</td>
<td>Hokkaido University Preprint Series in Mathematics, 645, 1-27</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2004</td>
</tr>
<tr>
<td>DOI</td>
<td>10.14943/83798</td>
</tr>
<tr>
<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/69452">http://hdl.handle.net/2115/69452</a></td>
</tr>
<tr>
<td>Type</td>
<td>bulletin (article)</td>
</tr>
<tr>
<td>File Information</td>
<td>pre645.pdf</td>
</tr>
</tbody>
</table>

**Hokkaido University Collection of Scholarly and Academic Papers**
ATTRAJECTORS OF ASYMPTOTICALLY PERIODIC MULTIVALENT DYNAMICAL SYSTEMS GOVERNED BY TIME-DEPENDENT SUBDIFFERENTIALS

NORIAKI YAMAZAKI

Abstract. Let us consider a nonlinear evolution equation associated with time-dependent subdifferential in a separable Hilbert space. In this paper we treat an asymptotically periodic system which means that time-dependent terms converge to some time-periodic ones as time goes to $+\infty$. Then we consider the large-time behaviour of solutions without uniqueness. In such a situation the corresponding dynamical systems are multivalued. In fact we discuss the stability of multivalued semiflows from the view-point of attractors. Namely, the main object of this paper is to construct a global attractor for the asymptotically periodic multivalued dynamical system, and to discuss the relationship to one for the limiting periodic system.

1 Introduction

In this paper let us consider a non-autonomous system in a real separable Hilbert space $H$ of the form

\[ v'(t) + \partial \varphi^t(v(t)) + G(t, v(t)) \ni f(t) \quad \text{in } H, \quad t > s \geq 0, \tag{1.1} \]

where $v' = \frac{dv}{dt}$, $\partial \varphi^t$ is a subdifferential of time-dependent proper lower semicontinuous (l.s.c.) convex function $\varphi^t$ on $H$, $G(t, \cdot)$ is a multivalued perturbation small relative to $\varphi^t$, and $f$ is a forcing term.

In the case when $G(t, \cdot) \equiv 0$, many mathematicians studied the existence-uniqueness, asymptotic stability, time periodic and almost periodic problem for (1.1) (cf. [7], [8], [13], [14], [15], [16], [18], [23], [24]).

For the multivalued nonmonotone perturbation $G(t, \cdot)$, Ôtani has already shown the existence of solution for (1.1) in [21]. The large-time behavior of solutions for (1.1) was...
discussed by [28] from the viewpoint of attractors. For the time periodic case, assuming
the periodicity conditions with same period \( T_0, 0 < T_0 < +\infty \), i.e.
\[
\varphi^t = \varphi^{t+T_0}, \quad G(t, \cdot) = G(t + T_0, \cdot), \quad f(t) = f(t + T_0), \quad \forall t \in \mathbb{R}_+ := [0, \infty),
\]
the existence of periodic solution for (1.1) was proved in [22]. Moreover, the periodic sta-
tility was discussed in [29]. In fact, the author showed the existence and characterization
of time-periodic global attractors for (1.1).

In this paper, for a given positive number \( T_0 > 0 \) let us treat the case when \( \varphi^t, G(t, \cdot) \)
and \( f(t) \) are asymptotically \( T_0 \)-periodic in time. Namely we assume that
\[
\varphi^t - \varphi^t_\ast \longrightarrow 0, \quad G(t, \cdot) - G_p(t, \cdot) \longrightarrow 0, \quad f(t) - f_p(t) \longrightarrow 0 \tag{1.2}
\]
in appropriate senses as \( t \to +\infty \), where \( \varphi^t_\ast = \varphi^{t+T_0}, G_p(t, \cdot) = G_p(t + T_0, \cdot) \) and \( f_p(t) = f_p(t + T_0) \) for any \( t \in \mathbb{R}_+ \). By the asymptotically \( T_0 \)-periodic stability (1.2), we have the
limiting \( T_0 \)-periodic system for (1.1) of the form:
\[
u'(t) + \partial \varphi^t_p(u(t)) + G_p(t, u(t)) \ni f_p(t) \quad \text{in} \ H, \quad t > s \geq 0. \tag{1.3}
\]
In the case when \( G(t, \cdot) \) and \( G_p(t, \cdot) \) are single-valued, the asymptotically \( T_0 \)-periodic
problem has already been discussed in [11]. In order to guarantee the uniqueness of
solutions for the Cauchy problem of (1.1) and (1.3), they assumed some conditions on
\( \varphi^t, \varphi^t_\ast, G(t, \cdot) \) and \( G_p(t, \cdot) \). Then, they discussed the asymptotically \( T_0 \)-periodic stability
for (1.1) from the viewpoint of attractors (cf. [11]). The main object of this paper is
to develop the result obtained in [11] in order to consider the large-time behaviour of
solution for (1.1) without uniqueness. Namely, we would like to construct the attractor
for the asymptotically \( T_0 \)-periodic multivalued flows associated with (1.1). Moreover we
shall discuss the relationship to the \( T_0 \)-periodic attractor for (1.3) obtained in [29].

In the next Section 2, we recall the known results for the Cauchy problem of (1.1). In
Section 3 we consider the limiting \( T_0 \)-periodic problem (1.3) and recall the abstract results
obtained in [29]. In Section 4, we introduce the notion of a metric topology on the family
\( \{\varphi^t; t \geq 0\} \) which was constructed in [16]. And we present and prove the main results in
this paper. In proving main results, we generalize the results obtained in [11] and [30].
In the final section we apply our abstract results to the parabolic variational inequality
with asymptotically \( T_0 \)-periodic double obstacles. Then we can discuss the asymptotic
stability for the asymptotically \( T_0 \)-periodic double obstacle problem without uniqueness
of solutions.

**Notation.** Throughout this paper, let \( H \) be a (real) separable Hilbert space with norm
\( \| \cdot \|_H \) and inner product \( (\cdot, \cdot)_H \). For a proper l.s.c. convex function \( \varphi \) on \( H \) we use the
notation \( D(\varphi), \partial \varphi \) and \( D(\partial \varphi) \) to indicate the effective domain, subdifferential and its
domain of \( \varphi \), respectively; for their precise definitions and basic properties see [4].

For two non-empty sets \( A \) and \( B \) in \( H \), we define the so-called Hausdorff semi-distance
\[
dist_H(A, B) := \sup_{x \in A} \inf_{y \in B} |x - y|_H.
\]
2 Preliminaries

In this section let us recall the known results for a nonlinear evolution equation in $H$ of the form:

$$u'(t) + \partial \varphi^i(u(t)) + G(t, u(t)) \ni f(t) \quad \text{in} \quad H, \quad t \in J,$$

(2.1)

where $J$ is an interval in $R_+$, $\partial \varphi^i$ is the subdifferential of a time-dependent proper l.s.c. and convex function $\varphi^i$ on $H$, $G(t, \cdot)$ is a multivalued operator from a subset $D(G(t, \cdot)) \subset H$ into $H$ for each $t \in R_+$ and $f$ is a given function in $L^2_{\text{loc}}(J; H)$. We begin with the definition of solution for (2.1).

**Definition 2.1.** (i) For a compact interval $J := [t_0, t_1] \subset R_+$ and $f \in L^2(J; H)$, a function $u : J \to H$ is called a solution of (2.1) on $J$, if $u \in C(J; H) \cap W^{1,2}_{\text{loc}}([t_0, t_1]; H)$, $\varphi^i(u(\cdot)) \in L^1(J)$, $u(t) \in D(\partial \varphi^i)$ for a.e. $t \in J$, and if there exists a function $g \in L^2_{\text{loc}}(J; H)$ such that $g(t) \in G(t, u(t))$ for a.e. $t \in J$ and

$$f(t) - g(t) - u'(t) \in \partial \varphi^i(u(t)), \quad \text{a.e.} \quad t \in J.$$

(ii) For any interval $J$ in $R_+$ and $f \in L^2_{\text{loc}}(J; H)$, a function $u : J \to H$ is called a solution of (2.1) on $J$, if it is a solution of (2.1) on every compact subinterval of $J$ in the sense of (i).

(iii) Let $J$ be any interval in $R_+$ with initial time $s \in R_+$. For $f \in L^2_{\text{loc}}(J; H)$, a function $u : J \to H$ is called a solution of the Cauchy problem for (2.1) on $J$ with given initial value $u_0 \in H$, if it is a solution of (2.1) on $J$ satisfying $u(s) = u_0$.

Throughout this paper, let $\{a_r\}_r := \{a_r; r \geq 0\}$ and $\{b_r\}_r := \{b_r; r \geq 0\}$ be families of real functions in $W^{1,2}_{\text{loc}}(R_+)$ and $W^{1,1}_{\text{loc}}(R_+)$, respectively, such that

$$\sup_{t \in R_+} |a'_r|_{L^2(t,t+1)} + \sup_{t \in R_+} |b'_r|_{L^1(t,t+1)} < +\infty \quad \text{for each} \quad r \geq 0.$$

Now we define the class $\Phi(\{a_r\}_r, \{b_r\}_r)$ of time-dependent convex function $\varphi^i$.

**Definition 2.2.** $\{\varphi^i\}_r \in \Phi(\{a_r\}_r, \{b_r\}_r)$ if and only if $\varphi^i$ is a proper l.s.c. convex function on $H$ satisfying the following properties ($\Phi 1$)-($\Phi 3$):

($\Phi 1$) For each $r > 0$, $s$, $t \in R_+$ and $z \in D(\varphi^s)$ with $|z|_H \leq r$, there exists $\bar{z} \in D(\varphi^t)$ such that

$$|\bar{z} - z|_H \leq |a_r(t) - a_r(s)|(1 + |\varphi^s(z)|^{\frac{1}{r}})$$

and

$$\varphi'(\bar{z}) - \varphi^s(z) \leq |b_r(t) - b_r(s)|(1 + |\varphi^s(z)|).$$

($\Phi 2$) There exists a positive constant $C_1 > 0$ such that

$$\varphi^i(z) \geq C_1 |z|_H^2, \quad \forall t \in R_+, \forall z \in D(\varphi^i).$$

($\Phi 3$) For each $k > 0$ and $t \in R_+$, the level set $\{z \in H; \varphi^i(z) \leq k\}$ is compact in $H$. 
Next, we introduce the class $G(\{\varphi^t\})$ of time-dependent multivalued perturbation $G(t, \cdot)$ associated with $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$.

**Definition 2.3.** $G(t, \cdot) \in G(\{\varphi^t\})$ if and only if $G(t, \cdot)$ is a multivalued operator from $D(G(t, \cdot)) \subset H$ into $H$ which fulfills the following conditions (G1)-(G5):

(G1) \( D(\varphi^t) \subset D(G(t, \cdot)) \subset H \) for any $t \in R_+$. And for any interval $J \subset R_+$ and $v \in L^2_{loc}(J; H)$ with $v(t) \in D(\varphi^t)$ for a.e. $t \in J$, there exists a strongly measurable function $g(\cdot)$ on $J$ such that
\[
g(t) \in G(t, v(t)) \text{ for a.e. } t \in J.
\]

(G2) \( G(t, z) \) is a convex subset of $H$ for any $z \in D(\varphi^t)$ and $t \in R_+$.

(G3) There are positive constants $C_2, C_3$ such that
\[
|g|^2_H \leq C_2 \varphi^t(z) + C_3, \quad \forall t \in R_+, \forall z \in D(\varphi^t), \forall g \in G(t, z).
\]

(G4) (demi-closedness) If $z_n \in D(\varphi^{t_n})$, $g_n \in G(t_n, z_n)$, $\{t_n\} \subset R_+$, $\{\varphi^{t_n}(z_n)\}$ is bounded, $z_n \to z$ in $H$, $t_n \to t$ and $g_n \to g$ weakly in $H$ as $n \to +\infty$, then $g \in G(t, z)$.

(G5) For each bounded subset $B$ of $H$, there exist positive constants $C_4(B)$ and $C_5(B)$ such that
\[
\varphi^t(z) + (g, z - b)_H \geq C_4(B)|z|^2_H - C_5(B),
\]
\[
\forall t \in R_+, \forall g \in G(t, z), \forall z \in D(\varphi^t), \forall b \in B.
\]

For given $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$, $\{G(t, \cdot)\} \in G(\{\varphi^t\})$ and a forcing term $f \in L^2_{loc}(R_+; H)$, we consider the following evolution equation
\[
(E)_s \quad u'(t) + \partial \varphi^t(u(t)) + G(t, u(t)) \ni f(t) \quad \text{in } H, \quad t > s
\]
for each $s \in R_+$.

Now let us recall the known results on the existence and global estimates of solutions for the Cauchy problem of $(E)_s$:

(A) [Existence of solution for $(E)_s$] (cf. [21, Theorem II, III])
The Cauchy problem for $(E)_s$ has at least one solution $u$ on $J = [s, +\infty)$ such that $(\cdot - s)^{\frac{1}{4}}u' \in L^2_{loc}(J; H)$, $(\cdot - s)^{\frac{1}{4}}\varphi(u(\cdot))) \in L^\infty_{loc}(J)$ and $\varphi(u(\cdot))$ is absolutely continuous on any compact subinterval of $(s, +\infty)$, provided that $u_0 \in D(\varphi^s)$. In particular, if $u_0 \in D(\varphi^s)$, then the solution $u$ satisfies that $u' \in L^2_{loc}(J; H)$ and $\varphi(u(\cdot))$ is absolutely continuous on any compact interval in $J$.

(B) [Global boundedness of solutions for $(E)_s$] (cf. [25, Theorem 2.2])
Suppose that
\[
S_f := \sup_{t \in R_+} |f|_{L^2(t,t+1; H)} < +\infty.
\]
Then, the solution \( u \) of the Cauchy problem for \( (E)_s \) on \([s, +\infty)\) satisfies the following global estimate:

\[
\sup_{t \geq s} |u(t)|^2_H + \sup_{t \geq s} \int_t^{t+1} \varphi^r(u(\tau))d\tau \leq N_1(1 + S^2_f + |u_0|^2_H),
\]

where \( N_1 \) is a positive constant independent of \( f, s \in R_+ \) and \( u_0 \in \overline{D(\varphi^s)} \). Moreover, for each \( \delta > 0 \) and each bounded subset \( B \) of \( H \), there is a constant \( N_2(\delta, B) > 0 \), depending only on \( \delta > 0 \) and \( B \), such that

\[
\sup_{t \geq s+\delta} |u''|^2_{L^2(t, t+1; H)} + \sup_{t \geq s+\delta} \varphi^r(u(t)) \leq N_2(\delta, B)
\]

for the solution \( u \) of the Cauchy problem for \( (E)_s \) on \([s, +\infty)\) with \( s \in R_+ \) and \( u_0 \in \overline{D(\varphi^s)} \cap B \).

Next, let us remember a notion of convergence of convex functions.

**Definition 2.4.** (cf. [20]) Let \( \psi, \psi_n (n \in N) \) be proper l.s.c. and convex functions on \( H \). Then we say that \( \psi_n \) converges to \( \psi \) on \( H \) as \( n \to +\infty \) in the sense of Mosco, if the following two conditions (i) and (ii) are satisfied:

(i) for any subsequence \( \{\psi_{n_k}\} \subset \{\psi_n\} \), if \( z_k \to z \) weakly in \( H \) as \( k \to +\infty \), then

\[
\liminf_{k \to +\infty} \psi_{n_k}(z_k) \geq \psi(z).
\]

(ii) for any \( z \in D(\psi) \), there is a sequence \( \{z_n\} \) in \( H \) such that

\[
z_n \to z \text{ in } H \text{ as } n \to +\infty, \quad \lim_{n \to +\infty} \psi_n(z_n) = \psi(z).
\]

Now, we recall a convergence result (cf. [25, Lemma 4.1]) as follows.

(C) Let \( \{\varphi^t_n\} \in \Phi(\{a_r\}, \{b_r\}) \), \( \{G_n(t, \cdot)\} \in G(\{\varphi^t_n\}) \) with common positive constants \( C_1, C_2, C_3, C_4(B) \) and \( C_5(B) \), \( \{f_n\} \subset L^2(J; H) \), \( J = [s, t] \subset R_+ \) and \( u_{0,n} \in \overline{D(\varphi^s_n)} \) for \( n = 1, 2, \cdots \). Assume that

(i) \( \varphi^t_n \) converges to \( \varphi^t \) on \( H \) in the sense of Mosco [20] for each \( t \in J \) (as \( n \to +\infty \)) and

\[
\bigcup_{n=1}^{+\infty} \{z \in H; \varphi^t_n(z) \leq k\} \text{ is relatively compact in } H \text{ for every real } k > 0 \text{ and } t \in J, \text{ where } \{\varphi^t_n\} \in \Phi(\{a_r\}, \{b_r\}) \text{ and } \varphi^t_n = \varphi^t \text{ if } n = +\infty.
\]

(ii) if \( z_n \in D(\varphi^s_n) \), \( g_n \in G_n(t_n, z_n) \), \( \{t_n\} \subset R_+ \), \( \{\varphi^s_n(z_n)\} \) is bounded, \( z_n \to z \) in \( H \), \( t_n \to t \) and \( g_n \to g \) weakly in \( H \) as \( n \to +\infty \), then \( g \in G(t, z) \), where

\[\{G(t, \cdot)\} \in G(\{\varphi^t\})\].

(iii) \( f_n \to f \) weakly in \( L^2(J; H) \) for some \( f \in L^2(J; H) \) and \( u_{0,n} \to u_0 \) in \( H \) for some \( u_0 \in \overline{D(\varphi^s)} \).
Denote by $u$ the solution of the Cauchy problem for $(E)_s$ on $J$ with $u(s) = u_0$ and by $u_n$ the solution of the Cauchy problem for $(E)_s$ with $\varphi^t, G, f$ replaced by $\varphi^t_n, G_n, f_n$, and with $u_n(s) = u_{0,n}$. Then $u_n$ converges to $u$ on $J$ in the sense that

$$
u_n \to u \text{ in } C(J; H), \ (\cdot - s)^{\frac{1}{2}}u'_n \to (\cdot - s)^{\frac{1}{2}}u' \text{ weakly in } L^2(J; H),$$

$$
\int_J \varphi^t_n(u_n(t))dt \to \int_J \varphi^t(u(t))dt \quad \text{as } n \to +\infty.
$$

3 Attractor for periodic multivalued dynamical system

In this section let us recall the known results obtained in [29] for a $T_0$-periodic system in $H$, of the form:

$$(P)_s \quad u'(t) + \partial \varphi^t_p(u(t)) + G_p(t, u(t)) \ni f_p(t) \quad \text{in } H, \quad t > s$$

for each $s \in R_+$, where $\varphi^t_p, G_p(t, \cdot)$ and $f_p(t)$ are $T_0$-periodic, namely periodic in time with the same period $T_0$, $0 < T_0 < +\infty$.

**Definition 3.1.** Let $T_0$ be a positive number. Then

(i) $\Phi_p(\{a_r\}, \{b_r\}; T_0)$ is the set of all $\{\varphi^t_p\} \in \Phi(\{a_r\}, \{b_r\})$ satisfying $T_0$-periodicity condition:

$$\varphi^{t+T_0}_p(\cdot) = \varphi^t_p(\cdot) \quad \text{on } H, \quad \forall t \in R_+. \tag{3.1}$$

(ii) $G_p(\{\varphi^t_p\}; T_0)$ is the set of all $\{G_p(t, \cdot)\} \in G(\{\varphi^t_p\})$ satisfying $T_0$-periodicity condition:

$$G_p(t + T_0, \cdot) = G_p(t, \cdot) \quad \text{in } H, \quad \forall t \in R_+. \tag{3.2}$$

Throughout this section we assume that $\{\varphi^t_p\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0)$, $\{G_p(t, \cdot)\} \in G_p(\{\varphi^t_p\}; T_0)$ and $f_p \in L^1_{loc}(R_+; H)$ is $T_0$-periodic in time, namely

$$f_p(t + T_0) = f_p(t) \quad \text{in } H, \quad \forall t \in R_+. \tag{3.3}$$

Here we note that $(P)_s$ can be considered as $(E)_s$ in Section 2. So, by the result (A) in Section 2, the Cauchy problem for $(P)_s$ has at least one solution $u$ on $[s, +\infty)$. Hence we can define the multivalued dynamical process associated with $(P)_s$ as follows:

**Definition 3.2.** For every $0 \leq s \leq t < +\infty$ we denote by $U(t, s)$ the mapping from $D(\varphi^s_p)$ into $D(\varphi^t_p)$ which assigns to each $u_0 \in D(\varphi^s_p)$ the set

$$U(t, s)u_0 := \left\{ z \in H \left| \begin{array}{l} \text{There is a solution } u \text{ of } (P)_s \text{ on } [s, +\infty) \\ \text{such that } \\ u(s) = u_0 \text{ and } u(t) = z. \end{array} \right. \right\}. \tag{3.4}$$

Then we easily see the following properties of $\{U(t, s)\} := \{U(t, s); 0 \leq s \leq t < +\infty\}$:
(U1) \( U(s,s) = I \) on \( \overline{D(\varphi_p^s)} \) for any \( s \in R_+ \);

(U2) \( U(t_2,s)z = U(t_2,t_1)U(t_1,s)z \) for any \( 0 \leq s \leq t_1 \leq t_2 < +\infty \) and \( z \in \overline{D(\varphi_p^s)} \);

(U3) \( U(t+T_0,s+T_0)z = U(t,s)z \) for any \( 0 \leq s \leq t < +\infty \) and \( z \in \overline{D(\varphi_p^s)} \), that is, \( U \) is \( T_0 \)-periodic.

(U4) \( \{U(t,s)\} \) has the following demi-closedness:

- If \( 0 \leq s_n \leq t_n < +\infty \), \( s_n \to s \), \( t_n \to t \), \( z_n \in \overline{D(\varphi_p^{s_n})} \), \( z \in \overline{D(\varphi_p^{s})} \), \( z_n \to z \) in \( H \) and a element \( w_n \in U(t_n,s_n)z_n \) converges to some element \( w \in H \) as \( n \to +\infty \), then \( w \in U(t,s)z \).

Next we define the discrete dynamical system in order to construct a global attractor for \( (P)_s \).

**Definition 3.3.** Let \( U(\cdot,\cdot) \) be the solution operator for \( (P)_s \) defined by Definition 3.2. Then

(i) For each \( \tau \in R_+ \), we denote by \( U_\tau \) the \( T_0 \)-step mapping from \( \overline{D(\varphi_p^\tau)} \) into \( \overline{D(\varphi_p^{\tau+T_0})} = \overline{D(\varphi_p^0)} \), namely,

\[
U_\tau := U(\tau + T_0, \tau).
\]

(2) For any \( k \in Z_+ := N \cup \{0\} \), we define

\[
U_\tau^k := U_\tau \circ U_\tau \circ \cdots \circ U_\tau.
\]

Clearly we have \( U_\tau^k = U(\tau + kT_0, \tau) \) for any \( \tau \in R_+ \) and \( k \in Z_+ \).

Now, let us recall the known result on the existence of global attractors for discrete multivalued dynamical systems \( U_\tau \) associated with \( (P)_s \).

**Theorem 3.1.** (cf. [29, Theorem 3.1]) Assume that \( \{\varphi_p^\tau\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0) \), \( \{G_p(t,\cdot)\} \in G_p(\{t_0\}; T_0) \), \( f_p \in L^2_{loc}(R_+; H) \) satisfies the \( T_0 \)-periodicity condition (3.3). Then, for each \( \tau \in R_+ \), there exists a subset \( A_\tau \) of \( \overline{D(\varphi_p^\tau)} \) such that

(i) \( A_\tau \) is non-empty and compact in \( H \);

(ii) for each bounded set \( B \) in \( H \) and each number \( \epsilon > 0 \) there exists \( N_{B,\epsilon} \in N \) such that

\[
\text{dist}_H(U_\tau^kz, A_\tau) < \epsilon
\]

for all \( z \in \overline{D(\varphi_p^\tau)} \cap B \) and all \( k \geq N_{B,\epsilon} \);

(iii) \( U_\tau^kA_\tau = A_\tau \) for any \( k \in N \).

**Remark 3.1.** By [29, Lemma 3.1] we can get the compact absorbing set \( B_{0,\tau} \) of \( \overline{D(\varphi_p^\tau)} \) for \( U_\tau \) such that for each bounded subset \( B \) of \( H \) there is a positive integer \( n_B \) (independent of \( \tau \in R_+ \)) satisfying

\[
U_\tau^n \left( \overline{D(\varphi_p^\tau)} \cap B \right) \subset B_{0,\tau} \quad \text{for all } n \geq n_B.
\]
Then we observe that the global attractor $A_\tau$ is given by the $\omega$-limit set of the absorbing set $B_{0,\tau}$ for $U_\tau$, i.e.

$$A_\tau = \bigcap_{n \in \mathbb{Z}_+} \bigcup_{k \geq n} U^k_{\tau} B_{0,\tau}.$$ 

The next theorem is concerned with a relationship between two global attractors $A_s$ and $A_\tau$. For detail proof, see [29].

**Theorem 3.2.** (cf. [29, Theorem 3.2]) Suppose the same assumptions are made as in Theorem 3.1. Let $A_s$ and $A_\tau$ be the global attractors for $U_s$ and $U_\tau$, with $0 \leq s \leq \tau \leq T_0$, respectively. Then, we have

$$A_\tau = U(\tau, s) A_s,$$

where $U(\tau, s)$ is the $T_0$-periodic process given in Definition 3.2.

**Remark 3.2.** By Theorem 3.1 (iii) and Theorem 3.2, we see that the global attractor $A_\tau$ for $U_\tau$ is $T_0$-periodic in $\tau$. In fact, for each $\tau \in \mathbb{R}_+$ choose $m_\tau \in \mathbb{Z}_+$ and $\sigma_\tau \in [0, T_0)$ so that $\tau = \sigma_\tau + m_\tau T_0$. Then, we have $A_\tau = A_{\sigma_\tau}$.

The third known result is the existence of a global attractor for the $T_0$-periodic multivalued dynamical system (P)$_s$.

**Theorem 3.3.** (cf. [29, Theorem 3.3]) Under the same assumptions as Theorem 3.1, put

$$A := \bigcup_{0 \leq \tau \leq T_0} A_\tau,$$

where $A_\tau$ is as obtained in Theorem 3.1. Then, $A$ has the following properties:

(i) $A$ is non-empty and compact in $H$;

(ii) for each bounded set $B$ in $H$ and each number $\epsilon > 0$ there exists a finite time $T_{B, \epsilon} > 0$ such that

$$\text{dist}_H(U(t + \tau, \tau)z, A) < \epsilon$$

for all $\tau \in \mathbb{R}_+$, all $z \in D(\varphi^t_\tau) \cap B$ and all $t \geq T_{B, \epsilon}$.

**Remark 3.3.** In [29, Section 4] the characterization of the $T_0$-periodic global attractor was discussed. The author proved that for each time $\tau \in \mathbb{R}_+$ the global attractor $A_\tau$ for the discrete multivalued dynamical system $U_\tau$ coincides with the cross-section of the family of all global bounded complete trajectories for the $T_0$-periodic system (P)$_s$.

## 4 Attractors of asymptotically periodic multivalued dynamical system

Throughout this section, let $M > 0$ be a fixed (sufficiently) large positive number. Now we put

$$\Psi_M := \left\{ \psi; \psi \text{ is proper, l.s.c. and convex on } H, \exists z \in D(\psi) \text{ s.t. } |z|_H \leq M, \psi(z) \leq M \right\}.$$
Then let us introduce the notion of a metric topology on \( \Psi_M \) which was introduced in [16]. Given \( \varphi, \psi \in \Psi_M \), we define \( \rho(\varphi,\psi;\cdot) : D(\varphi) \to \mathbb{R} \) by putting
\[
\rho(\varphi,\psi;z) = \inf \{ \max(|y-z|_H,\psi(y) - \varphi(z)) ; y \in D(\psi) \}
\]
for each \( z \in D(\varphi) \), and for each \( r \geq M \)
\[
\rho_r(\varphi,\psi) := \sup_{z \in L(\varphi;r)} \rho(\varphi,\psi;z),
\]
where \( L(\varphi;r) := \{ z \in D(\varphi) ; |z|_H \leq r, \varphi(z) \leq r \} \). Moreover, for each \( r \geq M \), we define the functional \( \pi_r(\cdot,\cdot) \) on \( \Psi_M \times \Psi_M \) by
\[
\pi_r(\varphi,\psi) := \rho_r(\varphi,\psi) + \rho_r(\psi,\varphi)
\]
for \( \varphi,\psi \in \Psi_M \).

Then, according to [16, Proposition 3.1], we can define a complete metric topology on \( \Psi_M \) so that the convergence \( \psi_n \to \psi \) in \( \Psi_M \) (as \( n \to +\infty \)) if and only if
\[
\pi_r(\psi_n,\psi) \to 0 \quad \text{for every } r \geq M.
\]

Now by using the above topology on \( \Psi_M \), we consider an asymptotically \( T_0 \)-periodic system as follows.

**Definition 4.1.** Assume \( \{ \varphi^i \} \in \Phi(\{a_r\},\{b_r\}) \cap \Psi_M \), \( \{ G(t,\cdot) \} \in \mathcal{G}(\{\varphi^i\}) \) and \( f \in L^2_{loc}(R_+;H) \). Then the system
\[
(\text{AP})_s \quad v'(t) + \partial \varphi^i(v(t)) + G(t,v(t)) \ni f(t) \quad \text{in } H, \quad t > s \quad (\geq 0)
\]
is asymptotically \( T_0 \)-periodic, if there are \( \{ \varphi^i \} \in \Phi_p(\{a_r\},\{b_r\};T_0) \cap \Psi_M \), \( \{ G_p(t,\cdot) \} \in \mathcal{G}_p(\{\varphi^i\};T_0) \) and a \( T_0 \)-periodic function \( f_p \in L^2_{loc}(R_+;H) \) such that

(A1) **(Convergence of \( \varphi^i - \varphi^i_p \to 0 \) as \( t \to +\infty \))** For each \( r \geq M \),
\[
J_m(r) := \sup_{\sigma \in [0,T_0]} \pi_r(\varphi^m,\varphi^i_p) \to 0 \quad \text{as } m \to +\infty;
\]

(A2) **(Convergence of \( G(t,\cdot) - G_p(t,\cdot) \to 0 \) as \( t \to +\infty \))** If \( \{ \tau_n \} \subset [0,T_0] \), \( \{ m_n \} \subset \mathbb{Z}_+ \), \( m_n \to +\infty \), \( z_n \in D(\varphi^m,T_0 + \tau_n) \), \( g_n \in G(m,T_0 + \tau_n,z_n) \), \( \{ \varphi^m,T_0 + \tau_n \} \) is bounded, \( z_n \to z \) in \( H \), \( \tau_n \to \tau \) and \( g_n \to g \) weakly in \( H \) (as \( n \to +\infty \)), then
\[
g \in G_p(\tau,z);
\]

(A3) **(Convergence of \( f(t) - f_p(t) \to 0 \) as \( t \to +\infty \))**
\[
|f(m,T_0 + \cdot) - f_p|_{L^2(0,T_0;H)} \to 0 \quad \text{as } m \to +\infty.
\]
By Definition 4.1 we easily see that a limiting system for \((AP)_s\) is a \(T_0\)-periodic one of the form:

\[
(P)_s \quad u'(t) + \partial \varphi^s_p(u(t)) + G_p(t, u(t)) \ni f_p(t) \quad \text{in } H, \ t > s \ (s \geq 0).
\]

Here we note that \((AP)_s\) is also considered as \((E)_s\). So, by the result (A) in Section 2, the Cauchy problem for \((AP)_s\) has at least one solution \(v\) on \([s, +\infty)\). Hence we can define the multivalued dynamical system associated with \((AP)_s\) as follows:

**Definition 4.2.** For every \(0 \leq s < t < +\infty\) we denote by \(E(t, s)\) the mapping from \(D(\varphi^s)\) into \(D(\varphi^t)\) which assigns to each \(v_0 \in D(\varphi^s)\) the set

\[
E(t, s)v_0 := \left\{ z \in H \left| \text{There is a solution } v \text{ of } (AP)_s \text{ on } [s, +\infty) \right. \left. \text{such that} \right. \right. \\
\left. \left. v(s) = v_0 \right. \left. \text{and } v(t) = z. \right. \right\}.
\]

Then we easily see that \(\{E(t, s)\} := \{E(t, s); 0 \leq s \leq t < +\infty\}\) has the following evolution properties:

(E1) \(E(s, s) = I\) on \(D(\varphi^s)\) for any \(s \in R_+\);

(E2) \(E(t_2, s)z = E(t_2, t_1)E(t_1, s)z\) for any \(0 \leq s \leq t_1 \leq t_2 < +\infty\) and \(z \in D(\varphi^s)\);

(E3) \(\{E(t, s)\}\) has the following demi-closedness:

- If \(0 \leq s_n \leq t_n < +\infty\), \(s_n \rightarrow s, t_n \rightarrow t, z_n \in D(\varphi^{s_n}), z \in D(\varphi^s), z_n \rightarrow z\) in \(H\) and a element \(w_n \in E(t_n, s_n)z_n\) converges to some element \(w \in H\) as \(n \rightarrow +\infty\), then \(w \in E(t, s)z\).

We begin with the definition of a discrete \(\omega\)-limit set for \(E(\cdot, \cdot)\).

**Definition 4.3.** (Discrete \(\omega\)-limit set for \(E(\cdot, \cdot)\)) Let \(\tau \in R_+\) be fixed. Let \(B(H)\) be a family of bounded subsets of \(H\). Then for each \(B \in B(H)\), the set

\[
\omega_\tau(B) := \bigcap_{n \in Z_+} \bigcup_{k \geq n, m \in Z_+} E(kT_0 + mT_0 + \tau, mT_0 + \tau)(D(\varphi^{mT_0 + \tau}) \cap B)
\]

is called the discrete \(\omega\)-limit set of \(B\) under \(E(\cdot, \cdot)\).

**Remark 4.1.** By definition of the discrete \(\omega\)-limit set \(\omega_\tau(B)\), it is easy to see that \(x \in \omega_\tau(B)\) if and only if there exist sequences \(\{k_n\} \subset Z_+\) with \(k_n \uparrow +\infty\), \(\{m_n\} \subset Z_+\), \(\{z_n\} \subset B\) with \(z_n \in D(\varphi^{m_nT_0 + \tau})\) and \(\{x_n\} \subset H\) with \(x_n \in E(k_nT_0 + m_nT_0 + \tau, m_nT_0 + \tau)z_n\) such that

\[
x_n \rightarrow x \text{ in } H \text{ as } n \rightarrow +\infty.
\]

Now let us mention main theorems in this paper.

**Theorem 4.1.** (Discrete attractors of \((AP)_\tau\)) For each \(\tau \in R_+\), let \(A_\tau\) be the global attractor of \(T_0\)-periodic dynamical systems \(U_\tau\), which is obtained in Section 3. For \(\{\varphi^i\} \in \)
Then, we have

(i) $A^*_r(\subset D(\varphi^*_r))$ is non-empty and compact in $H$;

(ii) for each bounded set $B \in \mathcal{B}(H)$ and each number $\epsilon > 0$ there exists $N_{B,\epsilon} \in \mathbb{N}$ such that

$$\text{dist}_H(E(kT_0 + \tau, \tau)z, A^*_r) < \epsilon$$

for all $z \in \overline{D(\varphi^*_r)} \cap B$ and all $k \geq N_{B,\epsilon}$;

(iii) $A^*_r \subset U_l^1, A^*_r \subset A_r$ for any $l \in \mathbb{N}$, where $U_r$ is the discrete dynamical system for $(P)_r$ given in Definition 3.3.

**Remark 4.2.** By the definition of the discrete $\omega$-limit set $\omega^*(B)$ and $A^*_r$, we easily see that

$$A^*_r = A^*_r + nT_0, \quad \forall n \in \mathbb{N}.$$ 

Hence $A^*_r$ is $T_0$-periodic in time in a sense of the above.

The second main theorem is concerned with a relationship between two attractors $A^*_s$ and $A^*_r$.

**Theorem 4.2.** Suppose the same assumptions are made as in Theorem 4.1. Let $A^*_s$ and $A^*_r$ be discrete attractors for $E(\cdot, s)$ and $E(\cdot, \tau)$ with $0 \leq s \leq \tau < +\infty$, respectively. Then,

$$A^*_r \subset U(r, s)A^*_s$$

where $U(r, s)$ is the $T_0$-periodic process for $(P)_s$ which is given in Definition 3.2.

By Theorems 4.1-4.2, we can get the attractor for asymptotic $T_0$-periodic system $(AP)_r$.

**Theorem 4.3.** (Global attractor for $(AP)_r$) Suppose the same assumptions are made as in Theorem 4.1. For any $\tau \in R_+$, let $A^*_\tau$ be the discrete attractor for $E(\cdot, \tau)$ obtained in Theorem 4.1. Here we put

$$A^* := \bigcup_{\tau \in [0, T_0]} A^*_\tau. \quad (4.2)$$

Then, for any bounded set $B \in \mathcal{B}(H)$,

$$\bigcap_{s \geq 0} \bigcup_{t \geq s, \tau \in R_+} E(t + \tau, \tau)(\overline{D(\varphi^*_\tau) \cap B}) \subset A^*. \quad (4.3)$$
By Theorem 4.3, the set $\mathcal{A}^*$ can be called the global attractor of (AP)$_\tau$.

Here we give some key lemmas.

**Lemma 4.1.** If $\{s_n\} \subset \mathbb{R}_+, \{\tau_n\} \subset \mathbb{R}_+, s \in \mathbb{R}_+, \tau \in \mathbb{R}_+, s_n \to s$, $\tau_n \to \tau$, $\{m_n\} \subset \mathbb{Z}_+$, with $m_n \to +\infty$, $z_n \in D(\varphi^{m_nT_0+s_n})$, $z \in D(\varphi^s_D)$, $z_n \to z$ in $H$ and a element $w_n \in E(m_nT_0+\tau_n+s_n, m_nT_0+s_n)z_n$ converges to some element $w \in H$ as $n \to +\infty$, then $w \in U(\tau+s,s)z$

**Proof.** Since $\tau_n \to \tau$, without loss of generality we may assume that there exists a finite time $T > 0$ such that $\{\tau_n\} \subset [0, T]$ and $\tau \in [0, T]$. By $w_n \in E(m_nT_0+\tau_n+s_n, m_nT_0+s_n)z_n$, there is a solution $w_n$ of (AP)$_{m_nT_0+s_n}$ on $[m_nT_0+s_n, +\infty)$ such that

$$v_n(m_nT_0+\tau_n+s_n) = w_n \text{ and } v_n(m_nT_0+s_n) = z_n.$$ 

Now we put $u_n(t) := v_n(t + m_nT_0 + s_n)$, then we easily see that $u_n$ is the solution for

$$\begin{cases} u_n'(t) + \partial \varphi^{t+m_nT_0+s_n}(u_n(t)) + G(t + m_nT_0 + s_n, u_n(t)) \ni f(t + m_nT_0 + s_n), \quad t > 0, \\ u_n(0) = z. \end{cases}$$

Let $\delta \in (0, 1)$ be fixed. Since $z_n \to z$ in $H$ as $n \to +\infty$, $\{z_n\}$ is bounded in $H$. Hence, from global estimates of solutions (cf. (B) in Section 2) it follows that there is a positive constant $M_\delta > 0$ (independent of $n$) satisfying

$$\sup_{t \geq \delta} |u_n(t)|^2_H + \sup_{t \geq \delta} |u_n'|^2_{L^2(t, t+1; H)} + \sup_{t \geq \delta} \varphi^{t+m_nT_0+s_n}(u_n(t)) \leq M_\delta.$$  \hspace{1cm} (4.4)

By [16, Lemma 4.1] we note that the convergence assumption (A1) implies

$$\varphi^{t+m_nT_0+s_n} \longrightarrow \varphi^{t+s}$$

in the sense of Mosco [20] \hspace{1cm} (4.5)

for each $t \geq 0$ as $n \to +\infty$. Moreover by the same argument in [10, Lemma 3.1] we can prove that

$$\bigcup_{n=1}^{+\infty} \{z \in H; \varphi^{t+m_nT_0+s_n}(z) \leq k\} \text{ is relatively compact in } H$$ \hspace{1cm} (4.6)

for every real $k > 0$ and $t \geq 0$, where $\varphi^{t+m_nT_0+s_n} = \varphi^{t+s}$ if $n = +\infty$. Therefore, by (4.4)-(4.6), (A2), (A3) and the convergence result (C) in Section 2, (by taking a subsequence of $\{n\}$, if necessary) we see that there is a function $u_\delta$ such that

$$u_\delta'(t) + \partial \varphi^{t+s}_p(u_\delta(t)) + G_p(t+s, u_\delta(t)) \ni f_p(t+s), \quad t > \delta.$$ 

By the standard diagonal process and the same argument in [21, Lemma 3.10], we can construct the solution $u$ on $[0, +\infty)$ satisfying

$$\begin{cases} u'(t) + \partial \varphi^{t+s}_p(u(t)) + G_p(t+s, u(t)) \ni f_p(t+s), \quad t > 0, \\ u(0) = z \end{cases}$$

and

$$u_n \longrightarrow u \text{ in } C([0, T]; H) \text{ as } n \to +\infty.$$ \hspace{1cm} (4.7)
Then, by (4.7) and \( u_n(\tau_n) = w_n \) we have \( u(\tau) = w \), which implies that \( w \in U(\tau + s, s)z \). 

By (B) in Section 2, for each \( B \in \mathcal{B}(H) \) we can choose constants \( r_B > 0 \) and \( M_B > 0 \) so that

\[
|v|_H \leq r_B \quad \text{and} \quad \varphi^{r+s}(v) \leq M_B, \tag{4.8}
\]

for any \( s \in R_+, \ t \geq T_0, \ z \in \overline{D(\varphi^t)} \cap B \) and \( v \in E(t + s, s)z \). Hence it follows from condition (A1) that for each \( m \in Z_+, \ \tau \in [0, T_0], \ n \in N \) and \( z \in \overline{D(\varphi^{mT_0+\tau})} \cap B \) there is \( \tilde{z} := \tilde{z}_{mT_0+\tau,z,nT_0} \in D(\varphi_p^t) \) such that

\[
|\tilde{z} - v|_H \leq J_{m+n}^{(r_B+MB+M)},
\]

(hence \( |\tilde{z}|_H \leq r_B + J_{m+n}^{(r_B+MB+M)} \))

and

\[
\varphi_p^t(\tilde{z}) - \varphi_p^{mT_0+mT_0+\tau}(v) \leq J_{m+n}^{(r_B+MB+M)},
\]

(hence \( \varphi_p^t(\tilde{z}) \leq M_B + J_{m+n}^{(r_B+MB+M)} \)).

where \( v \in E(nT_0 + mT_0 + \tau, mT_0 + \tau)z \).

Since \( J_k^{(r_B+MB+M)} \to 0 \) as \( k \to +\infty \), there is a number \( N_0 \in N \) such that

\[
J_k^{(r_B+MB+M)} \leq 1, \quad \forall k > N_0.
\]

Now, put \( J_0 := 1 + \sup_{1 \leq k \leq N_0} J_k^{(r_B+MB+M)} < +\infty \). Then, we define the bounded set \( \overline{B}_\tau \) by

\[
\overline{B}_\tau := \{ z \in H; |z|_H \leq r_B + J_0 \} \cap \overline{D(\varphi^t_p)}.
\]

Let \( B_{0,\tau} \) be the compact absorbing set for \( U_\tau \) introduced by Remark 3.1. Then, we see that there exists a number \( \tilde{N} \in N \) so that

\[
U_{\tau}^l \overline{B}_\tau \subset B_{0,\tau}, \quad \forall l \geq \tilde{N}. \tag{4.9}
\]

The next lemma is very important to prove Theorem 4.1 (iii).

**Lemma 4.2.** Let \( \tau \in R_+ \) and \( B_{0,\tau} \) be the compact absorbing set for \( U_\tau \). Then we have

\[
\omega(\tau)(B) \subset B_{0,\tau}, \quad \forall B \in \mathcal{B}(H).
\]

**Proof.** At first we assume \( \tau \in [0, T_0] \).

For each \( B \in \mathcal{B}(H) \), let \( x \) be any element of \( \omega(\tau)(B) \). Then, it follows from Remark 4.1 that there exist sequences \( \{k_n\} \subset Z_+ \) with \( k_n \to +\infty \), \( \{m_n\} \subset Z_+ \), \( \{z_n\} \subset B \) with \( z_n \in \overline{D(\varphi^{m_nT_0+\tau})} \) and \( \{x_n\} \subset H \) with \( x_n \in E(k_nT_0 + m_nT_0 + \tau, m_nT_0 + \tau)z_n \) such that

\[
x_n \to x \quad \text{in} \quad H \quad \text{as} \quad n \to +\infty. \tag{4.10}
\]

Let \( \tilde{N} \) be the positive integer obtained in (4.9). Then by (E2) we have

\[
x_n \in E(k_nT_0 + m_nT_0 + \tau, k_nT_0 - \tilde{N}T_0 + m_nT_0 + \tau)
\]

...
Consider \( E(k_nT_0 - \tilde{N}T_0 + m_nT_0 + \tau, m_nT_0 + \tau)z_n \) for any \( n \) with \( k_n \geq \tilde{N} + 1 \).

Hence, there exists an element \( y_n \in E(k_nT_0 - \tilde{N}T_0 + m_nT_0 + \tau, m_nT_0 + \tau)z_n \) such that

\[
x_n \in E(k_nT_0 + m_nT_0 + \tau, k_nT_0 - \tilde{N}T_0 + m_nT_0 + \tau)y_n.
\]

Since \( \{z_n\} \subset B \), we see that

\[
|y_n|_H \leq r_B \quad \text{and} \quad \varphi^{k_nT_0 - \tilde{N}T_0 + m_nT_0 + \tau}(y_n) \leq M_B \quad \text{for any} \quad n \geq \tilde{N} + 1,
\]

where \( r_B \) and \( M_B \) are same positive constants in (4.8).

From the convergence condition (A1) it follows that for \( y_n \in E(k_nT_0 - \tilde{N}T_0 + m_nT_0 + \tau, m_nT_0 + \tau)z_n \) there is \( \tilde{z}_n \in D(\varphi_p^\tau) \) such that

\[
|\tilde{z}_n - y_n|_H \leq J_{k_n - \tilde{N} + m_n}^{(r_B + M_B + M)},
\]

(hence \( |\tilde{z}_n|_H \leq r_B + J_{k_n - \tilde{N} + m_n}^{(r_B + M_B + M)} \))

and

\[
\varphi_p^\tau(\tilde{z}_n) \leq M_B + J_{k_n - \tilde{N} + m_n}^{(r_B + M_B + M)}.
\]

Since \( \{\tilde{z}_n \in D(\varphi_p^\tau) ; n \in N \text{ with } k_n \geq \tilde{N} + 1\} \subset \tilde{B}_\tau \) is relatively compact in \( H \), we may assume that

\[
\tilde{z}_n \rightarrow \tilde{z}_\infty \text{ in } H \quad \text{as } n \to +\infty
\]

for some \( \tilde{z}_\infty \in H \). Then we easily see that \( \tilde{z}_\infty \in \tilde{B}_\tau \) and

\[
y_n \rightarrow \tilde{z}_\infty \text{ in } H \quad \text{as } n \to +\infty.
\]

By Lemma 4.1 and (4.10)-(4.13), we observe that

\[
x \in U(\tilde{N}T_0 + \tau, \tau)\tilde{z}_\infty,
\]

which implies that

\[
x \in U(\tilde{N}T_0 + \tau, \tau)\tilde{B}_\tau = U_\tau \tilde{B}_\tau \subset B_{0,\tau}.
\]

Hence we have

\[
\omega_\tau(B) \subset B_{0,\tau}.
\]

For the general case of \( \tau \in R_+ \), choose positive numbers \( i_\tau \in N \) and \( \tau_0 \in [0, T_0] \) so that \( \tau = \tau_0 + i_\tau T_0 \). Then, we can show \( \omega_\tau(B) \subset B_{0,\tau} \) by the same argument as above.

\[ \Box \]

**Proof of Theorem 4.1.** On account of Lemma 4.2 we can get \( A^*_\tau \subset B_{0,\tau} \). Hence, Theorem 4.1 (i) holds. Also, by (4.1) and Remark 4.1 we observe that Theorem 4.1 (ii) holds.

Now, we prove Theorem 4.1 (iii). At first, let us prove that \( A^*_\tau \subset U_\tau^l A^*_\tau \) for any \( l \in N \).
Let $x$ be any element of $\mathcal{A}_\tau^*$. By the definition of $\mathcal{A}_\tau^*$, there are sequences $\{B_n\} \subset \mathcal{B}(H)$ and $\{x_n\} \subset H$ with $x_n \in \omega_\tau(B_n)$ such that

$$x_n \longrightarrow x \text{ in } H \text{ as } n \rightarrow +\infty. \quad (4.14)$$

Then, for each $n$ it follows from Remark 4.1 that there exist sequences $\{k_{n,j}\} \subset \mathbb{Z}_+$ with $k_{n,j} \rightarrow +\infty$, $\{m_{n,j}\} \subset \mathbb{Z}_+$, $\{z_{n,j}\} \subset B_n$ with $z_{n,j} \in \overline{D(\varphi^{m_{n,j}l_0+\tau})}$ and $\{v_{n,j}\} \subset H$ with $v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j}$ such that

$$v_{n,j} \longrightarrow x_n \text{ in } H \text{ as } j \rightarrow +\infty. \quad (4.15)$$

Let $l$ be any number in $\mathbb{N}$, then we see that

$$v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau) \circ E(k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j}$$

for $j$ with $k_{n,j} \geq l + 1$. So, there exists an element $w_{n,j} \in E(k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j}$ such that

$$v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau)w_{n,j}. \quad (4.16)$$

By global estimates (B) in Section 2, $\{w_{n,j} \in H : j \in \mathbb{N} \text{ with } k_{n,j} \geq l + 1\}$ is relatively compact in $H$ for each $n$. Therefore we may assume that the element $w_{n,j}$ converges to some element $\tilde{w}_{n,\infty} \in H$ as $j \rightarrow +\infty$. Clearly, $\tilde{w}_{n,\infty} \in \omega_\tau(B_n)$. Moreover, it follows from Lemma 4.1 and (4.15)-(4.16) that

$$x_n \in U(lT_0 + \tau, \tau)\tilde{w}_{n,\infty} \subset U(lT_0 + \tau, \tau)\omega_\tau(B_n),$$

hence, we have

$$x_n \in \bigcup_{n \geq 1} U_l^l \omega_\tau(B_n), \quad \forall n \geq 1. \quad (4.17)$$

Here, by the closedness of $U(\cdot, \cdot)$ we note that for each subset $X$ of $B_{0,\tau}$,

$$\overline{U_l^lX} \subset U_l^lX, \quad \forall l \in \mathbb{N}. \quad (4.18)$$

Taking account of Lemma 4.2, (4.14), (4.17) and (4.18), we observe that

$$x \in \bigcup_{n \geq 1} U_l^l \omega_\tau(B_n) = U_l^l \bigcup_{n \geq 1} \omega_\tau(B_n) \subset U_l^l \bigcup_{n \geq 1} \omega_\tau(B_n) \subset U_l^l \mathcal{A}_\tau^*,$$

which implies that $\mathcal{A}_\tau^*$ is semi-invariant under the $T_0$-periodic dynamical systems $U_\tau$, i.e.

$$\mathcal{A}_\tau^* \subset U_l^l \mathcal{A}_\tau^*, \quad \forall l \in \mathbb{N}. \quad (4.19)$$
Next we shall prove that \( U^l_* A^* \subset A_r \) for any \( l \in N \). By (4.19), for each \( l \in N \)
\[
U^l_* A^* \subset U^l_* U^n_* A^* = U^{l+n}_* A^*, \quad \forall n \in N.
\] (4.20)

By \( A^*_r \subset B_{0, r} \), (4.20) and the attractive property of \( A_r \), we have
\[
U^l_* A^* \subset A_r, \quad \forall l \in N.
\] Therefore we conclude that
\[
A^*_r \subset U^l_* A^*_r \subset A_r, \quad \forall l \in N.
\]

\[\blacklozenge\]

**Proof of Theorem 4.2.** Let \( x \) be any element of \( A^*_r \). Then by the definition of \( A^*_r \), there exist sequences \( \{B_n\} \subset B(H) \) and \( \{x_n\} \subset H \) with \( x_n \in \omega_r(B_n) \) such that
\[
x_n \longrightarrow x \text{ in } H \quad \text{as } n \rightarrow +\infty.
\] (4.21)

From Remark 4.1 it follows that for each \( n \), there are sequences \( \{k_{n,j}\} \subset Z_+ \) with \( k_{n,j} \rightarrow +\infty \), \( \{m_{n,j}\} \subset Z_+ \), \( \{z_{n,j}\} \subset B_n \) with \( z_{n,j} \in D(\varphi^{m_{n,j}T_0 + \tau}) \) and \( \{v_{n,j}\} \subset H \) with \( v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j} \) such that
\[
v_{n,j} \longrightarrow x_n \text{ in } H \quad \text{as } j \rightarrow +\infty.
\] (4.22)

Note that for given \( s, \tau \in R_+ \) with \( s \leq \tau \) there is a positive number \( l_s \in N \) satisfying
\[
s \leq \tau \leq l_s T_0 + s.
\]

By using the property (E2) we see that
\[
v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 + m_{n,j}T_0 + s)
\]
\[
\circ E(k_{n,j}T_0 + m_{n,j}T_0 + s, T_0 + m_{n,j}T_0 + l_s T_0 + s)
\]
\[
\circ E(T_0 + m_{n,j}T_0 + l_s T_0 + s, m_{n,j}T_0 + \tau)z_{n,j}
\]
for any \( j \in Z_+ \) with \( k_{n,j} \geq l_s + 2 \). Here we can take elements \( w_{n,j} \in H \) and \( y_{n,j} \in H \) so that
\[
v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 + m_{n,j}T_0 + s)w_{n,j}, \quad (4.23)
\]
\[
w_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + s, T_0 + m_{n,j}T_0 + l_s T_0 + s)y_{n,j}
\] (4.24)

and
\[
y_{n,j} \in E(T_0 + m_{n,j}T_0 + l_s T_0 + s, m_{n,j}T_0 + \tau)z_{n,j}. \quad (4.25)
\]

By \( \{z_{n,j}\} \subset B_n \) and the global boundedness result (B) in Section 2, we can get a positive constant \( C_n := C_n(B_n) > 0 \) satisfying
\[
|y_{n,j}|_H \leq C_n, \quad \forall y_{n,j} \in E(T_0 + m_{n,j}T_0 + l_s T_0 + s, m_{n,j}T_0 + \tau)z_{n,j}. \quad (4.26)
\]
Here we define the bounded set $B_{C_n}$ by

$$B_{C_n} := \{ b \in H : |b|_H \leq C_n \}.$$ 

From (4.26) and the result (B) in Section 2 it follows that the set

$$\left\{ w_{n,j} \in H ; \quad w_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + s, T_0 + m_{n,j}T_0 + l_sT_0 + s)y_{n,j} \right\}$$

for any $j \in \mathbb{Z}_+$ with $k_{n,j} \geq l_s + 2$

is relatively compact in $H$. Hence, we may assume that the element $w_{n,j}$ converges to some element $\tilde{w}_{n,\infty} \in H$ as $j \to +\infty$. Clearly, $\tilde{w}_{n,\infty} \in \omega_s(B_{C_n})$, and it follows from Lemma 4.2 that

$$\omega_s(B_{C_n}) \subset B_{0,s} \subset D(\varphi^s).$$

Moreover, by Lemma 4.1 and (4.22)-(4.23) we have

$$x_n \in U(\tau, s)\tilde{w}_{n,\infty} \subset U(\tau, s)\omega_s(B_{C_n}), \quad \forall n \geq 1,$$

hence, we see that

$$x_n \in \bigcup_{n \geq 1} U(\tau, s)\omega_s(B_{C_n}), \quad \forall n \geq 1. \quad (4.27)$$

Here, by the closedness of $U(\cdot, \cdot)$, we note that for each subset $X$ of $B_{0,s}$,

$$\overline{U(\tau, s)X} \subset U(\tau, s)X. \quad (4.28)$$

On account of Lemma 4.2, (4.21), (4.27) and (4.28), we observe that

$$x \in \bigcup_{n \geq 1} U(\tau, s)\omega_s(B_{C_n})$$

$$= U(\tau, s) \bigcup_{n \geq 1} \omega_s(B_{C_n})$$

$$\subset U(\tau, s) \bigcup_{n \geq 1} \omega_s(B_{C_n})$$

$$\subset U(\tau, s)\mathcal{A}^*_s,$$

which implies that $\mathcal{A}^*_s$ is the subset of $U(\tau, s)\mathcal{A}^*_s$, namely

$$\mathcal{A}^*_s \subset U(\tau, s)\mathcal{A}^*_s.$$

\diamond

**Proof of Theorem 4.3.** For any $B \in \mathcal{B}(H)$, let $z_0$ be any element of the $\omega$-limit set $\omega_E(B)$ which is define by

$$\omega_E(B) := \bigcap_{s \geq 0} \bigcup_{t \geq s, \tau \in \mathbb{R}_+} E(t + \tau, \tau)(D(\varphi^s) \cap B).$$
Then we easily see that there exist sequences \( \{t_n\} \subset R_+ \) with \( t_n \uparrow +\infty \), \( \{\tau_n\} \subset R_+ \), \( \{y_n\} \subset B \) with \( y_n \in \overline{D(\varphi^\tau)} \) and \( \{z_n\} \subset H \) with \( z_n \in E(t_n + \tau_n, \tau_n)y_n \) such that
\[
\begin{align*}
t_n &:= k_nT_0 + t_n', \ k_n \in Z_+, \ k_n \not\to +\infty, \ t_n' \in [T_0, 2T_0], \ t_n' \to t_0', \\
\tau_n &:= l_nT_0 + \tau_n', \ l_n \in Z_+, \ \tau_n' \in [0, T_0], \ \tau_n \to \tau_0'
\end{align*}
\]
and
\[
z_n \to z_0 \quad \text{in } H \quad (4.29)
\]
as \( n \to +\infty \). Without loss of generality, we may assume that
\[
(a) \quad t_n' + \tau_n' \not\to t_0' + \tau_0' \quad \text{or} \quad (b) \quad t_n' + \tau_n' \to t_0' + \tau_0'.
\]

Now, assume that (a) holds. Then let us consider the multivalued semiflow
\[
v_n \in E(1 + k_nT_0 + l_nT_0 + t_0' + \tau_0', \ k_nT_0 + l_nT_0 + t_n' + \tau_n')z_n. \quad (4.30)
\]
Then, there is a solution \( u_n \) on \([k_nT_0 + l_nT_0 + t_n' + \tau_n', +\infty)\) for
\[
\begin{align*}
\begin{cases}
u_n'(t) + \partial_{\varphi^t+k_nT_0+l_nT_0+t_n'+\tau_n'}(u_n(t)) + G(t + k_nT_0 + l_nT_0 + t_n' + \tau_n', u_n(t)) \\
u_n(0) = z_n \quad \text{and} \quad u_n(1 + t_0' + \tau_0' - t_n' - \tau_n') = v_n.
\end{cases} &\quad (4.31)
\end{align*}
\]
Since \( z_n \to z_0 \) in \( H \), \( \{z_n\} \) is bounded in \( H \). Therefore by the global estimate (B) in Section 2, we see that
\[
\begin{align*}
\begin{cases}
v_n \in H; \\
v_n \in E(1 + k_nT_0 + l_nT_0 + t_0' + \tau_0', \ k_nT_0 + l_nT_0 + t_n' + \tau_n')z_n
\end{cases} &\quad \text{for any } n \in N
\end{align*}
\]
is relatively compact in \( H \). Hence we may assume that
\[
v_n \to v \text{ in } H \text{ for some } v \in H. \quad (4.32)
\]

Now applying Lemma 4.1 with (4.29)-(4.31), we can get
\[
v \in U(1 + t_0' + \tau_0', t_0' + \tau_0')z_0,
\]
more precisely, (taking the subsequence of \( \{n\} \) if necessary) we observe that
\[
u_n \to u \quad \text{in } C([0,2]; H) \quad \text{as } n \to +\infty,
\]
where \( u \) is the solution \([t_0' + \tau_0', +\infty)\) satisfying
\[
\begin{align*}
\begin{cases}
u'(t) + \partial_{\varphi^t+t_0'+\tau_0'}(u(t)) + G_p(t + t_0' + \tau_0', u(t)) \ni f_p(t + t_0' + \tau_0'), \\
u(0) = z_0 \quad \text{and} \quad u(1) = v.
\end{cases}
\end{align*}
\]
By (4.32) we easily see that
\[
u_n(t_0' + \tau_0' - t_n' - \tau_n') \to z_0 \quad \text{as } n \to +\infty. \quad (4.33)
\]
Note that
\[ u_n(t'_0 + \tau'_0 - t'_n - \tau'_n) \]
\[ \in E(k_nT_0 + l_nT_0 + t'_0 + \tau'_0, \ k_nT_0 + l_nT_0 + t'_n + \tau'_n)z_n \]
\[ = E(k_nT_0 + l_nT_0 + t'_0 + \tau'_0, \ l_nT_0 + \tau'_n)y_n \]
\[ = E(k_nT_0 + l_nT_0 + t'_0 + \tau'_0, \ l_nT_0 + t'_0 + \tau'_0)E(l_nT_0 + t'_0 + \tau'_0, \ l_nT_0 + \tau'_n)y_n. \]

So, we can take a element \( x_n \in E(l_nT_0 + t'_0 + \tau'_0, \ l_nT_0 + \tau'_n)y_n \) such that
\[ u_n(t'_0 + \tau'_0 - t'_n - \tau'_n) \in E(k_nT_0 + l_nT_0 + t'_0 + \tau'_0, \ l_nT_0 + t'_0 + \tau'_0) \]
(4.34)

By \( \{y_n\} \subset B \) and the global estimate (B) in Section 2, we easily see that \( \{x_n\} \) is bounded, i.e.
\[ \{x_n\} \subset \tilde{B} \text{ for some } \tilde{B} \in B(H). \] (4.35)

Therefore, from Remarks 4.1-4.2 and (4.33)-(4.35) we observe that
\[ z_0 \in \omega_{t'_0 + \tau'_0}(\tilde{B}) \subset \mathcal{A}^*_s \subset \mathcal{A}^*. \]

Thus (4.3) holds.

In the case (b) when \( t'_n + \tau'_n \searrow t'_0 + \tau'_0 \), we can prove (4.3) by the slight modification of the proof as above. \( \diamond \)

Theorem 4.1 implies that the attracting set \( \mathcal{A}^*_\tau \) for \( (AP)_\tau \) is semi-invariant under \( U_{\tau} \) associated with the limiting \( T_0 \)-periodic system \( (P)_s \), in general. Moreover, from Theorem 4.2 we observe that
\[ \mathcal{A}^*_\tau \subset U(\tau, s)\mathcal{A}^*_s \text{ for any } 0 \leq s \leq \tau < +\infty. \]

In order to get the invariance of \( \mathcal{A}^*_\tau \) under \( U_{\tau} \) and \( \mathcal{A}^*_s = U(\tau, s)\mathcal{A}^*_s \), let us use a concept of a regular approximation, which was introduced in [17].

**Definition 4.4.** (Regular approximation) Let \( s \in \mathbb{R}_+ \) be fixed. Let \( z \in D(\varphi^s_p) \). Then, we say that \( U(t + s, s)z \) is regularly approximated by \( E(t + kT_0 + s, \ kT_0 + s) \) as \( k \to +\infty \), if for each finite \( T > 0 \) there are sequences \( \{k_n\} \subset \mathbb{Z}_+ \) with \( k_n \to +\infty \) and \( \{z_n\} \subset H \) with \( z_n \in D(\varphi^{k_nT_0+s}) \) and \( z_n \to z \) in \( H \) satisfying the following property: for any function \( u \in W^{1,2}(0, T; H) \) satisfying \( u(t) \in U(t + s, s)z \) for all \( t \in [0, T] \) there is a sequence \( \{u_n\} \subset W^{1,2}(0, T; H) \) such that \( u_n(t) \in E(t + k_nT_0 + s, \ k_nT_0 + s)z_n \) for all \( t \in [0, T] \) and \( u_n \to u \) in \( C([0, T]; H) \) as \( n \to +\infty \).

Using the above concept, we can show that the invariance of \( \mathcal{A}^*_\tau \) under \( U_{\tau} \). Moreover we can get
\[ \mathcal{A}^*_\tau = U(\tau, s)\mathcal{A}^*_s. \]

**Theorem 4.4** Suppose all assumptions in Theorem 4.1. Let \( \mathcal{A}^*_\tau \) and \( \mathcal{A}^*_s \) be discrete attractors for \( E(\cdot, s) \) and \( E(\cdot, \tau) \), with \( 0 \leq s \leq \tau < +\infty \), respectively. Assume that for
any point z of $A_s^*$, $U(t + s, s)z$ is regularly approximated by $E(t + kT_0 + s, kT_0 + s)$ as $k \to +\infty$. Then we have

$$A_s^* = U(\tau, s)A_s^*.$$  

**Proof.** By Theorem 4.2, we have only to show that

$$U(\tau, s)A_s^* \subset A_s^*.$$  

To do so, let $x$ be any element of $U(\tau, s)A_s^*$.

At first, taking account of Definitions 3.2-3.3 and Theorem 4.1 (iii), we see that for each $n \in N$

$$U^n U(\tau, s)A_s^* = U(nT_0 + \tau, \tau)U(\tau, s)A_s^* = U(nT_0 + \tau, nT_0 + s)U(nT_0 + s, s)A_s^*$$  

$$= U(\tau, s)U^n A_s^* = U(\tau, s)A_s^*.$$  

Hence, there exists an element $y_n \in A_s^*$ such that

$$x \in U^n U(\tau, s)y_n = U(nT_0 + \tau - s + s, s)y_n.$$  

By using our assumption as $t = nT_0 + \tau - s$, we observe that for each $n$, there are sequences ${k_{n,j}} \subset Z_+$, ${x_{n,j}} \subset H$ and ${y_{n,j}} \subset H$ such that

$$k_{n,j} \to +\infty, \quad y_{n,j} \in D(\varphi_{k_{n,j}T_0 + s}), \quad y_{n,j} \to y_n \text{ in } H$$

and

$$x_{n,j} \in E(nT_0 + \tau - s + k_{n,j}T_0 + s, k_{n,j}T_0 + s)y_{n,j}, \quad x_{n,j} \to x \text{ in } H$$

as $j \to +\infty$. Therefore, by the usual diagonal argument, we can find a subsequence ${j_n}$ of $\{j\}$ such that $\bar{x}_n := x_{n,j_n}$, $\bar{y}_n := y_{n,j_n}$ and $\bar{k}_n := k_{n,j_n}$ satisfy

$$|\bar{x}_n - x|_H < \frac{1}{n}, \quad \bar{x}_n \in E(nT_0 + \tau - s + \bar{k}_nT_0 + s, \bar{k}_nT_0 + s)\bar{y}_n, \quad |\bar{y}_n - y_n|_H < \frac{1}{n}$$

for every $n = 1, 2, \ldots$. Since $\{\bar{y}_n\}$ is bounded in $H$, there is a bounded set $B \in \mathcal{B}(H)$ so that $\{\bar{y}_n\} \subset B$.

By (E2), we see that

$$\bar{x}_n \in E(nT_0 + \tau - s + \bar{k}_nT_0 + s, \bar{k}_nT_0 + s)\bar{y}_n = E(nT_0 + \bar{k}_nT_0 + \tau, T_0 + \bar{k}_nT_0 + \tau)E(T_0 + \bar{k}_nT_0 + \tau, \bar{k}_nT_0 + s)\bar{y}_n,$$

hence there is an element $\tilde{z}_n \in E(T_0 + \bar{k}_nT_0 + \tau, \bar{k}_nT_0 + s)\bar{y}_n$ such that

$$\bar{x}_n \in E(nT_0 + \bar{k}_nT_0 + \tau, T_0 + \bar{k}_nT_0 + \tau)\tilde{z}_n.$$  

(4.39)

Since $\{\bar{y}_n\} \subset B$ and the global estimate (B) in Section 2, we see that $\{\tilde{z}_n\}$ is also bounded in $H$. Hence, there is a bounded set $\bar{B} \in \mathcal{B}(H)$ so that $\{\tilde{z}_n\} \subset \bar{B}$. The above fact (4.37)-(4.39) implies (cf. Remark 4.1) that $x \in \omega_\tau(\bar{B}) \subset A_s^*$. Thus we have $U(\tau, s)A_s^* \subset A_s^*$. ◊
By the same argument in Theorem 4.4, we can get the following corollary:

**Corollary.** (i) Suppose the same assumptions of Theorem 4.4. Then, by Remark 4.2 we observe that \( \mathcal{A}^*_s \) is invariant under the \( T_0 \)-periodic dynamical system \( U_s(= U(T_0 + s, s)) \). Namely,

\[
\mathcal{A}^*_s = U_s^l \mathcal{A}^*_s \quad \text{for any } l \in \mathbb{N}.
\]

(ii) Assume that for any point \( z \) of \( \mathcal{A}_\tau \), \( U(\tau + \tau, \tau)z \) is regularly approximated by \( E(\tau + kT_0 + \tau, kT_0 + \tau) \) as \( k \to +\infty \). Then, we have \( \mathcal{A}^*_\tau \supset \mathcal{A}_\tau (= U_\tau \mathcal{A}_\tau) \). Hence by Theorem 4.1 (iii) we conclude that

\[
\mathcal{A}^*_\tau = \mathcal{A}_\tau.
\]

**Remark 4.3.** If the solution operator \( U(t, s) \) is singlevalued, namely the solution for the Cauchy problem of (P) is unique, the assumptions of Theorem 4.4 always hold. Thus, Theorems 4.1-4.4 contain the abstract results obtained in [11], which was concerned with the asymptotic \( T_0 \)-periodic stability for the singlevalued dynamical system associated with time-dependent subdifferentials.

## 5 Application to obstacle problems for PDE’s

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) (\( 1 \leq N < +\infty \)) with smooth boundary \( \Gamma = \partial \Omega \), \( q \) be a fixed number with \( 2 \leq q < +\infty \) and \( T_0 \) be a fixed positive number. We use the notation

\[
a_q(v, z) := \int_{\Omega} |\nabla v|^q - 2 \nabla v \cdot \nabla z dx, \quad \forall v, z \in W^{1,q}(\Omega)
\]

and denote by \((\cdot, \cdot)\) the usual inner product in \( L^2(\Omega) \).

For prescribed obstacle functions \( \sigma_0 \leq \sigma_1 \) and each \( t \in \mathbb{R}_+ \) we define the set

\[
K(t) := \{ z \in W^{1,q}(\Omega); \sigma_0(t, \cdot) \leq z \leq \sigma_1(t, \cdot) \text{ a.e. on } \Omega \}.
\]

Let \( f \) be a function in \( L^2_{\text{loc}}(\mathbb{R}_+; L^2(\Omega)) \) and \( h \) be a non-negative function on \( \mathbb{R}_+ \times \mathbb{R} \). Then for given \( b \in L^\infty(\Omega)^N \) we consider an interior asymptotically \( T_0 \)-periodic double obstacle problem \((\text{OP})_{s}^{AP} \) \((s \in \mathbb{R}_+) :\)

- Find functions \( v \in C([s, +\infty); L^2(\Omega)) \) and \( \theta \in L^2_{\text{loc}}((s, +\infty); L^2(\Omega)) \) such that

\[
\begin{cases}
  v \in L^q_{\text{loc}}((s, +\infty); W^{1,q}(\Omega)) \cap W^{1,2}_{\text{loc}}((s, +\infty); L^2(\Omega)); \\
  v(t) \in K(t) \text{ for a.e. } t \geq s; \\
  0 \leq \theta(t, x) \leq h(t, v(t, x)) \text{ a.e. on } (s, +\infty) \times \Omega; \\
  (v'(t) + \theta(t) + b \cdot \nabla v(t) - f(t), v(t) - z) + a_q(v(t), v(t) - z) \leq 0 \\
  \text{for any } z \in K(t) \text{ and a.e. } t \geq s.
\end{cases}
\]
The main object of this section is to consider the large-time behaviour of solution for \((OP)^{AP}_s\) assuming asymptotically \(T_0\)-periodicity conditions

\[
\sigma_i(t) - \sigma_{i,p}(t) \to 0 \quad (i = 0, 1), \quad h(t, \cdot) - h_p(t, \cdot) \to 0, \quad f(t) - f_p(t) \to 0
\]

as \(t \to \infty\) in the sense specified below, where \(\sigma_i(t), h_p(t, \cdot), f_p(t)\) are periodic in time with the same period \(T_0\). By the above assumptions, the limiting system of \((OP)^{AP}_s\) is a \(T_0\)-periodic one \((OP)^P_s\) as follows:

- Find functions \(u \in C([s, +\infty); L^2(\Omega))\) and \(\theta \in L^2_{loc}([s, +\infty); L^2(\Omega))\) such that

\[
\begin{align*}
(\text{OP})^P_s & \\
& \begin{cases}
  u \in L^2_{loc}((s, +\infty); W^{1,q}(\Omega)) \cap W^{1,2}_{loc}((s, +\infty); L^2(\Omega)); \\
  u(t) \in K_p(t) \text{ for a.e. } t \geq s; \\
  0 \leq \theta(t, x) \leq h_p(t, u(t, x)) \text{ a.e. on } (s, +\infty) \times \Omega; \\
  (u'(t) + \theta(t) + b \cdot \nabla u(t) - f_p(t), u(t) - z) + a_q(u(t), u(t) - z) \leq 0
\end{cases}
\end{align*}
\]

for any \(z \in K_p(t)\) and a.e. \(t \geq s\),

where \(K_p(t) := \{ z \in W^{1,q}(\Omega); \sigma_0,p(t, \cdot) \leq z \leq \sigma_1,p(t, \cdot) \text{ a.e. on } \Omega \} \).

Now we suppose the following conditions:

- \(\sigma_i\) and \(\sigma_{i,p}\) are functions on \(R_+ \times \Omega\) such that

\[
\sup_{t \in R_+} \left| \frac{d\sigma_i}{dt} \right|_{L^2(t,t+1;W^{1,q}(\Omega))} + \sup_{t \in R_+} \left| \frac{d\sigma_{i,p}}{dt} \right|_{L^2(t,t+1;L^\infty(\Omega))} < +\infty,
\]

and \(\sigma_{i,p}\) is a \(T_0\)-periodic obstacle function, i.e.

\[
\sigma_{i,p}(t + T_0, x) = \sigma_{i,p}(t, x) \quad \text{for a.e. } x \in \Omega \text{ and any } t \in R_+
\]

for \(i = 0, 1\). Moreover, there are positive constants \(k_1 > 0\) and \(k_2 > 0\) such that

\[
\sigma_1 - \sigma_0 \geq k_1 \quad \text{and} \quad \sigma_{1,p} - \sigma_{0,p} \geq k_1 \quad \text{a.e. on } R_+ \times \Omega
\]

and

\[
|\sigma_i|_{L^\infty(R_+;W^{1,q}(\Omega))} + |\sigma_i|_{L^\infty(R_+ \times \Omega)} + |\sigma_{i,p}|_{L^\infty(R_+;W^{1,q}(\Omega))} + |\sigma_{i,p}|_{L^\infty(R_+ \times \Omega)} \leq k_2
\]

for \(i = 0, 1\).

- \(h\) and \(h_p\) are non-negative continuous functions on \(R_+ \times R\). There is a positive constant \(L\) such that

\[
|h(t, z_1) - h(t, z_2)| \leq L|z_1 - z_2| \\
|h_p(t, z_1) - h_p(t, z_2)| \leq L|z_1 - z_2|
\]

for all \(t \in R_+, z_i \in R\) and \(i = 1, 2\). Moreover, \(h_p\) is a \(T_0\)-periodic function, i.e. for any \(z \in R\), \(h_p(t + T_0, z) = h_p(t, z)\) for any \(t \in R_+\).
\[ f_p(t + T_0) = f_p(t) \quad \text{in} \ L^2(\Omega), \quad \forall t \in R_+. \]

Moreover, we suppose the following convergence conditions:

- (Convergence of \( \sigma_i(t) - \sigma_{i,p}(t) \to 0 \) as \( t \to +\infty \)) Put
  \[ I_m := \sup_{t \in [0,T_0]} |\sigma_0(mT_0 + t) - \sigma_{0,p}(t)|_{W^{1,\infty}(\Omega)} + \sup_{t \in [0,T_0]} |\sigma_1(mT_0 + t) - \sigma_{1,p}(t)|_{W^{1,\infty}(\Omega)} \]
  \[ + \sup_{t \in [0,T_0]} |\sigma_0(mT_0 + t) - \sigma_{0,p}(t)|_{L^\infty(\Omega)} + \sup_{t \in [0,T_0]} |\sigma_1(mT_0 + t) - \sigma_{1,p}(t)|_{L^\infty(\Omega)} \]

Then,
  \[ I_m \to 0 \quad \text{as} \ m \to +\infty; \]

- (Convergence of \( h(t, \cdot) - h_{p}(t, \cdot) \to 0 \) as \( t \to +\infty \)) For any \( z \in R \),
  \[ \sup_{t \in [0,T_0]} |h(mT_0 + t, z) - h_{p}(t, z)| \to 0 \quad \text{as} \ m \to +\infty; \quad (5.1) \]

- (Convergence of \( f(t) - f_{p}(t) \to 0 \) as \( t \to +\infty \))
  \[ |f(mT_0 + \cdot) - f_{p}|_{L^2(0,T_0;L^2(\Omega))} \to 0 \quad \text{as} \ m \to +\infty. \quad (5.2) \]

Under the above assumptions, let us consider problems \( (OP)^{sP}_{sP} \) and \( (OP)^{sP}_{sP} \).

In order to apply the abstract results in Sections 2-4, we choose \( L^2(\Omega) \) as a real separable Hilbert space \( H \). And we define a family \( \{ \varphi^t \} \) of proper l.s.c. convex functions \( \varphi^t \) on \( L^2(\Omega) \) by

\[
\varphi^t(z) = \begin{cases} 
\frac{1}{q} \int_\Omega |\nabla z|^q \, dx & \text{if } z \in K(t), \\
+\infty & \text{if } z \in L^2(\Omega) \setminus K(t),
\end{cases}
\]

and define \( \varphi^t_{p} \) by replacing \( K(t) \) by \( K_{p}(t) \) in (5.3).

Also, we define a multivalued operator \( G(\cdot, \cdot) \) from \( R_+ \times H^1(\Omega) \) into \( L^2(\Omega) \) by

\[
G(t, z) := \left\{ g \in L^2(\Omega); \quad g = l + b \cdot \nabla z \quad \text{in} \ L^2(\Omega) \right\}
\]

for all \( t \in R_+ \) and \( z \in H^1(\Omega) \). And we define \( G_{p}(\cdot, \cdot) \) by replacing \( h(t, \cdot) \) by \( h_{p}(t, \cdot) \) in (5.4).

By the same argument in [27, Lemma 5.1], we can get the following lemmas.

**Lemma 5.1.** (cf. [27, Lemma 5.1]) Put for any \( r > 0 \) and \( t \in R_+ \)

\[
a_r(t) = b_r(t) := k_3 \int_0^t \left\{ |\sigma^t_{0,p}|_{L^\infty(\Omega)} + |\sigma^t_{0,p}|_{W^{1,\infty}(\Omega)} + |\sigma^t_{1,p}|_{L^\infty(\Omega)} + |\sigma^t_{1,p}|_{W^{1,\infty}(\Omega)} \right\} \, dt
\]

\[ + k_3 \int_0^t \left\{ |\sigma^t_{0,p}|_{L^\infty(\Omega)} + |\sigma^t_{0,p}|_{W^{1,\infty}(\Omega)} + |\sigma^t_{1,p}|_{L^\infty(\Omega)} + |\sigma^t_{1,p}|_{W^{1,\infty}(\Omega)} \right\} \, dt
\]

\[ + k_3 \int_0^t \left\{ |\sigma^t_{0,p}|_{L^\infty(\Omega)} + |\sigma^t_{0,p}|_{W^{1,\infty}(\Omega)} + |\sigma^t_{1,p}|_{L^\infty(\Omega)} + |\sigma^t_{1,p}|_{W^{1,\infty}(\Omega)} \right\} \, dt
\]

\[ + k_3 \int_0^t \left\{ |\sigma^t_{0,p}|_{L^\infty(\Omega)} + |\sigma^t_{0,p}|_{W^{1,\infty}(\Omega)} + |\sigma^t_{1,p}|_{L^\infty(\Omega)} + |\sigma^t_{1,p}|_{W^{1,\infty}(\Omega)} \right\} \, dt
\]

\[ + k_3 \int_0^t \left\{ |\sigma^t_{0,p}|_{L^\infty(\Omega)} + |\sigma^t_{0,p}|_{W^{1,\infty}(\Omega)} + |\sigma^t_{1,p}|_{L^\infty(\Omega)} + |\sigma^t_{1,p}|_{W^{1,\infty}(\Omega)} \right\} \, dt
\]
where $k_3$ is a (sufficiently large) positive constant. Then, $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$ and $\{\varphi^t_p\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0)$.

Moreover we have $\{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^t\})$ and $\{G_p(t, \cdot)\} \in \mathcal{G}_p(\{\varphi^t_p\}; T_0)$.

Lemma 5.2. The convergence assumptions (A1)-(A3) hold.

Proof. We easily see that (A2) and (A3) hold by assumptions (5.1) and (5.2).

Now let us show (A1). For each $t \in R_+$ there are $m \in Z_+$ and $\tau \in [0, T_0]$ so that $t = mT_0 + \tau$.

For each $z_p \in D(\varphi^t_p) = K_p(t)$, we put

$$z := (z_p - \sigma_{0,p}(t)) \frac{\sigma_1(t) - \sigma_0(t)}{\sigma_{1,p}(t) - \sigma_{0,p}(t)} + \sigma_0(t).$$

Then we easily see that $z \in D(\varphi^t) = K(t)$. Moreover, by the same argument in [27, Lemma 5.1], we see that

$$|z - z_p|_{L^2(\Omega)} \leq k_4 I_m \quad \text{and} \quad |\nabla z - \nabla z_p|_{L^q(\Omega)} \leq k_4 I_m (1 + |\nabla z_p|_{L^q(\Omega)}) \quad (5.5)$$

for some constant $k_4 > 0$. Hence we have

$$\varphi^t(z) - \varphi^t_p(z_p) \leq k_5 I_m (1 + \varphi^t_p(z_p)), \quad (5.6)$$

for a sufficiently large $k_5 > 0$.

Conversely, let $z \in D(\varphi^t) = K(t)$ and we put

$$z_p := (z - \sigma_0(t)) \frac{\sigma_{1,p}(t) - \sigma_{0,p}(t)}{\sigma_1(t) - \sigma_0(t)} + \sigma_{0,p}(t).$$

Then, we observe that $z_p \in D(\varphi^t_p) = K_p(t)$ and

$$|z_p - z|_{L^2(\Omega)} \leq k_4 I_m \quad \text{and} \quad \varphi^t_p(z_p) - \varphi^t(z) \leq k_5 I_m (1 + \varphi^t(z)). \quad (5.7)$$

Therefore by (5.5)-(5.7) we see that the convergence assumption (A1) holds. \hfill \diamond

Clearly, the obstacle problem (OP)_{sAP} can be reformulated as an evolution equation (AP), involving the subdifferential of $\varphi^t$ given by (5.3) and the multivalued operator $G(t, \cdot)$ defined by (5.4). Also, the limiting $T_0$-periodic problem (OP)_{sP} can be reformulated as an evolution equation (P). Therefore, by Lemmas 5.1-5.2 we can apply abstract results in Section 2-4. Namely, we can obtain an attractor $\mathcal{A}_s^*$ for (OP)_{sAP}, a $T_0$-periodic attractor $\mathcal{A}_s$ for (OP)_{sP} and the relationships between (OP)_{sAP} and (OP)_{sP}.

Additionally, we assume that $f(t) \equiv f_p(t)$ for any $t \in R_+$ and

$$\sigma_0(t, z) \equiv \sigma_{0,p}(t, z), \quad \sigma_{1,p}(t, z) \equiv \sigma_1(t, z), \quad h_p(t, z) \leq h(t, z)$$

24
for any $0 \leq t < +\infty$ and $z \in R$. Then we easily see that the assumptions of Theorem 4.4 and its Corollary hold. Hence we can get $A^*_s = A_s$ by the same argument in [30, Theorem 5.4].

Unfortunately we do not give assumptions for $\sigma_i(t, \cdot)$, $h(t, \cdot)$ and $f(t)$ in order to get

$$U(\tau, s)A^*_s = A^*_r \subset A_r$$

for any $0 \leq s \leq \tau < +\infty$. (5.8)

It seems difficult to show (5.8), so it is the open problem.

References


26


NORIAKI YAMAZAKI

DEPARTMENT OF MATHEMATICAL SCIENCE, COMMON SUBJECT DIVISION, MURORAN INSTITUTE OF TECHNOLOGY, 27-1 MIZUMOTO-CHÔ, MURORAN, 050-8585, JAPAN

E-mail address: noriaki@mms.muroran-it.ac.jp