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ATTRACTORS OF ASYMPTOTICALLY PERIODIC MULTIVALUED DYNAMICAL SYSTEMS GOVERNED BY TIME-DEPENDENT SUBDIFFERENTIALS

NORIAKI YAMAZAKI

Abstract. Let us consider a nonlinear evolution equation associated with time-dependent subdifferential in a separable Hilbert space. In this paper we treat an asymptotically periodic system which means that time-dependent terms converge to some time-periodic ones as time goes to $+\infty$. Then we consider the large-time behaviour of solutions without uniqueness. In such a situation the corresponding dynamical systems are multivalued. In fact we discuss the stability of multivalued semiflows from the view-point of attractors. Namely, the main object of this paper is to construct a global attractor for the asymptotically periodic multivalued dynamical system, and to discuss the relationship to one for the limiting periodic system.

1 Introduction

In this paper let us consider a non-autonomous system in a real separable Hilbert space $H$ of the form

$$v'(t) + \partial \varphi^t(v(t)) + G(t,v(t)) \ni f(t) \quad \text{in} \quad H, \quad t > s \ (\geq 0),$$

(1.1)

where $v' = \frac{dv}{dt}$, $\partial \varphi^t$ is a subdifferential of time-dependent proper lower semicontinuous (l.s.c.) convex function $\varphi^t$ on $H$, $G(t, \cdot)$ is a multivalued perturbation small relative to $\varphi^t$, and $f$ is a forcing term.

In the case when $G(t, \cdot) \equiv 0$, many mathematicians studied the existence-uniqueness, asymptotic stability, time periodic and almost periodic problem for (1.1) (cf. [7], [8], [13], [14], [15], [16], [18], [23], [24]).

For the multivalued nonmonotone perturbation $G(t, \cdot)$, Ôtani has already shown the existence of solution for (1.1) in [21]. The large-time behavior of solutions for (1.1) was

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discussed by [28] from the view-point of attractors. For the time periodic case, assuming the periodicity conditions with same period $T_0$, $0 < T_0 < +\infty$, i.e.

$$\varphi^t = \varphi^{t+T_0}, \quad G(t, \cdot) = G(t+T_0, \cdot), \quad f(t) = f(t+T_0), \quad \forall t \in R_+ := [0, \infty),$$

the existence of periodic solution for (1.1) was proved in [22]. Moreover, the periodic stability was discussed in [29]. In fact, the author showed the existence and characterization of time-periodic global attractors for (1.1).

In this paper, for a given positive number $T_0 > 0$ let us treat the case when $\varphi^t, G(t, \cdot)$ and $f(t)$ are asymptotically $T_0$-periodic in time. Namely we assume that

$$\varphi^t - \varphi^t_p \longrightarrow 0, \quad G(t, \cdot) - G_p(t, \cdot) \longrightarrow 0, \quad f(t) - f_p(t) \longrightarrow 0 \quad (1.2)$$

in appropriate senses as $t \rightarrow +\infty$, where $\varphi^t_p = \varphi^{t+T_0}_p$, $G_p(t, \cdot) = G_p(t+T_0, \cdot)$ and $f_p(t) = f_p(t+T_0)$ for any $t \in R_+$. By the asymptotically $T_0$-periodic stability (1.2), we have the limiting $T_0$-periodic system for (1.1) of the form:

$$u'(t) + \partial \varphi^t_p(u(t)) + G_p(t, u(t)) \ni f_p(t) \quad \text{in} \ H, \quad t > s \geq 0. \quad (1.3)$$

In the case when $G(t, \cdot)$ and $G_p(t, \cdot)$ are single-valued, the asymptotically $T_0$-periodic problem has already been discussed in [11]. In order to guarantee the uniqueness of solutions for the Cauchy problem of (1.1) and (1.3), they assumed some conditions on $\varphi^t, \varphi^t_p, G(t, \cdot)$ and $G_p(t, \cdot)$. Then, they discussed the asymptotically $T_0$-periodic stability for (1.1) from the view-point of attractors (cf. [11]). The main object of this paper is to develop the result obtained in [11] in order to consider the large-time behaviour of solution for (1.1) without uniqueness. Namely, we would like to construct the attractor for the asymptotically $T_0$-periodic multivalued flows associated with (1.1). Moreover we shall discuss the relationship to the $T_0$-periodic attractor for (1.3) obtained in [29].

In the next Section 2, we recall the known results for the Cauchy problem of (1.1). In Section 3 we consider the limiting $T_0$-periodic problem (1.3) and recall the abstract results obtained in [29]. In Section 4, we introduce the notion of a metric topology on the family \{ $\varphi^t; t \geq 0$ \} which was constructed in [16]. And we present and prove the main results in this paper. In proving main results, we generalize the results obtained in [11] and [30]. In the final section we apply our abstract results to the parabolic variational inequality with asymptotically $T_0$-periodic double obstacles. Then we can discuss the asymptotic stability for the asymptotically $T_0$-periodic double obstacle problem without uniqueness of solutions.

**Notation.** Throughout this paper, let $H$ be a (real) separable Hilbert space with norm $|\cdot|_H$ and inner product $(\cdot, \cdot)_H$. For a proper l.s.c. convex function $\varphi$ on $H$ we use the notation $D(\varphi)$, $\partial \varphi$ and $D(\partial \varphi)$ to indicate the effective domain, subdifferential and its domain of $\varphi$, respectively; for their precise definitions and basic properties see [4].

For two non-empty sets $A$ and $B$ in $H$, we define the so-called Hausdorff semi-distance

$$\text{dist}_H(A, B) := \sup_{x \in A} \inf_{y \in B} |x - y|_H.$$


2 Preliminaries

In this section let us recall the known results for a nonlinear evolution equation in $H$ of the form:

$$u'(t) + \partial \varphi^t(u(t)) + G(t, u(t)) \ni f(t) \quad \text{in } H, \quad t \in J,$$

(2.1)

where $J$ is an interval in $R_+$, $\partial \varphi^t$ is the subdifferential of a time-dependent proper l.s.c. and convex function $\varphi^t$ on $H$, $G(t, \cdot)$ is a multivalued operator from a subset $D(G(t, \cdot)) \subset H$ into $H$ for each $t \in R_+$ and $f$ is a given function in $L^2_{\text{loc}}(J; H)$.

We begin with the definition of solution for (2.1).

**Definition 2.1.** (i) For a compact interval $J := [t_0, t_1] \subset R_+$ and $f \in L^2(J; H)$, a function $u : J \to H$ is called a solution of (2.1) on $J$, if $u \in C(J; H) \cap W^{1,2}_{\text{loc}}((t_0, t_1]; H)$, $\varphi^t(u(\cdot)) \in L^1(J)$, $u(t) \in D(\partial \varphi^t)$ for a.e. $t \in J$, and if there exists a function $g \in L^2_{\text{loc}}(J; H)$ such that $g(t) \in G(t, u(t))$ for a.e. $t \in J$ and

$$f(t) - g(t) - u'(t) \in \partial \varphi^t(u(t)), \quad \text{a.e. } t \in J.$$

(ii) For any interval $J$ in $R_+$ and $f \in L^2_{\text{loc}}(J; H)$, a function $u : J \to H$ is called a solution of (2.1) on $J$, if it is a solution of (2.1) on every compact subinterval of $J$ in the sense of (i).

(iii) Let $J$ be any interval in $R_+$ with initial time $s \in R_+$. For $f \in L^2_{\text{loc}}(J; H)$, a function $u : J \to H$ is called a solution of the Cauchy problem for (2.1) on $J$ with given initial value $u_0 \in H$, if it is a solution of (2.1) on $J$ satisfying $u(s) = u_0$.

Throughout this paper, let $\{a_r\} := \{a_r; r \geq 0\}$ and $\{b_r\} := \{b_r; r \geq 0\}$ be families of real functions in $W^{1,2}_{\text{loc}}(R_+)$ and $W^{1,1}_{\text{loc}}(R_+)$, respectively, such that

$$\sup_{t \in R_+} |a_r'|_{L^2(t,t+1)} + \sup_{t \in R_+} |b_r'|_{L^1(t,t+1)} < +\infty \quad \text{for each } r \geq 0.$$

Now we define the class $\Phi(\{a_r\}, \{b_r\})$ of time-dependent convex function $\varphi^t$.

**Definition 2.2.** $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$ if and only if $\varphi^t$ is a proper l.s.c. convex function on $H$ satisfying the following properties $(\Phi 1)$-$(\Phi 3)$:

$(\Phi 1)$ For each $r > 0$, $s$, $t \in R_+$ and $z \in D(\varphi^s)$ with $|z|_H \leq r$, there exists $\tilde{z} \in D(\varphi^t)$ such that

$$|\tilde{z} - z|_H \leq \left| a_r(t) - a_r(s) \right| (1 + |\varphi^s(z)|^{1/2})$$

and

$$\varphi^t(\tilde{z}) - \varphi^s(z) \leq |b_r(t) - b_r(s)| (1 + |\varphi^s(z)|).$$

$(\Phi 2)$ There exists a positive constant $C_1 > 0$ such that

$$\varphi^t(z) \geq C_1 |z|_H^2, \quad \forall t \in R_+, \forall z \in D(\varphi^t).$$

$(\Phi 3)$ For each $k > 0$ and $t \in R_+$, the level set $\{ z \in H; \varphi^t(z) \leq k \}$ is compact in $H$. 

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Next, we introduce the class $\mathcal{G}(\{\varphi^t\})$ of time-dependent multivalued perturbation $G(t, \cdot)$ associated with $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$.

**Definition 2.3.** $\{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^t\})$ if and only if $G(t, \cdot)$ is a multivalued operator from $D(G(t, \cdot)) \subset H$ into $H$ which fulfills the following conditions (G1)-(G5):

1. **(G1)** $D(\varphi^t) \subset D(G(t, \cdot)) \subset H$ for any $t \in R_+$. And for any interval $J \subset R_+$ and $v \in L^2_{loc}(J; H)$ with $v(t) \in D(\varphi^t)$ for a.e. $t \in J$, there exists a strongly measurable function $g(\cdot)$ on $J$ such that
   \[ g(t) \in G(t, v(t)) \text{ for a.e. } t \in J. \]

2. **(G2)** $G(t, z)$ is a convex subset of $H$ for any $z \in D(\varphi^t)$ and $t \in R_+$.

3. **(G3)** There are positive constants $C_2, C_3$ such that
   \[ |g|^2_H \leq C_2 \varphi^t(z) + C_3, \quad \forall t \in R_+, \forall z \in D(\varphi^t), \forall g \in G(t, z). \]

4. **(G4)** (demi-closedness) If $z_n \in D(\varphi^{a_n}), g_n \in G(t_n, z_n), \{t_n\} \subset R_+, \{\varphi^{a_n}(z_n)\}$ is bounded, $z_n \to z$ in $H$, $t_n \to t$ and $g_n \to g$ weakly in $H$ as $n \to +\infty$, then $g \in G(t, z)$.

5. **(G5)** For each bounded subset $B$ of $H$, there exist positive constants $C_4(B)$ and $C_5(B)$ such that
   \[ \varphi^t(z) + (g, z - b)_H \geq C_4(B)\|z\|^2_H - C_5(B), \]
   \[ \forall t \in R_+, \forall g \in G(t, z), \forall z \in D(\varphi^t), \forall b \in B. \]

For given $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\}), \{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^t\})$ and a forcing term $f \in L^2_{loc}(R_+; H)$, we consider the following evolution equation

\[ (E)_s \quad u'(t) + \partial \varphi^t(u(t)) + G(t, u(t)) \ni f(t) \quad \text{in } H, \quad t > s \]

for each $s \in R_+$.

Now let us recall the known results on the existence and global estimates of solutions for the Cauchy problem of $(E)_s$:

1. **(A)** [Existence of solution for $(E)_s$] (cf. [21, Theorem II, III])

   The Cauchy problem for $(E)_s$ has at least one solution $u$ on $J = [s, +\infty)$ such that $(\cdot - s)^{1/2}u' \in L^2_{loc}(J; H), (\cdot - s)^{1/2}\varphi^t(u(\cdot)) \in L^\infty_{loc}(J) \text{ and } \varphi^t(u(\cdot)) \text{ is absolutely continuous on any compact subinterval of } (s, +\infty)$, provided that $u_0 \in D(\varphi^s)$. In particular, if $u_0 \in D(\varphi^s)$, then the solution $u$ satisfies that $u' \in L^2_{loc}(J; H)$ and $\varphi^t(u(\cdot))$ is absolutely continuous on any compact interval in $J$.

2. **(B)** [Global boundedness of solutions for $(E)_s$] (cf. [25, Theorem 2.2])

   Suppose that
   \[ S_f := \sup_{t \in R_+} |f|_{L^2(t, t+1; H)} < +\infty. \]
Then, the solution $u$ of the Cauchy problem for $(E)_s$ on $[s, +\infty)$ satisfies the following global estimate:

$$
\sup_{t \geq s} |u(t)|^2_H + \sup_{t \geq s} \int_t^{t+1} \varphi^\tau(u(\tau))d\tau \leq N_1(1 + S^2_j + |u_0|^2_H),
$$

where $N_1$ is a positive constant independent of $f$, $s \in R_+$ and $u_0 \in \overline{D(\varphi^s)}$. Moreover, for each $\delta > 0$ and each bounded subset $B$ of $H$, there is a constant $N_2(\delta, B) > 0$, depending only on $\delta > 0$ and $B$, such that

$$
\sup_{t \geq s + \delta} |u'|^2_{L^2(t, t+1; H)} + \sup_{t \geq s + \delta} \varphi^\prime(u(t)) \leq N_2(\delta, B)
$$

for the solution $u$ of the Cauchy problem for $(E)_s$ on $[s, +\infty)$ with $s \in R_+$ and $u_0 \in \overline{D(\varphi^s)} \cap B$.

Next, let us remember a notion of convergence of convex functions.

**Definition 2.4.** (cf. [20]) Let $\psi, \psi_n (n \in N)$ be proper l.s.c. and convex functions on $H$. Then we say that $\psi_n$ converges to $\psi$ on $H$ as $n \to +\infty$ in the sense of Mosco, if the following two conditions (i) and (ii) are satisfied:

(i) for any subsequence $\{\psi_{n_k}\} \subset \{\psi_n\}$, if $z_k \to z$ weakly in $H$ as $k \to +\infty$, then

$$
\liminf_{k \to +\infty} \psi_{n_k}(z_k) \geq \psi(z).
$$

(ii) for any $z \in D(\psi)$, there is a sequence $\{z_n\}$ in $H$ such that

$$
z_n \to z \text{ in } H \text{ as } n \to +\infty, \quad \lim_{n \to +\infty} \psi_n(z_n) = \psi(z).
$$

Now, we recall a convergence result (cf. [25, Lemma 4.1]) as follows.

(C) Let $\{\varphi_n^t\} \in \Phi(\{a_r\}, \{b_r\})$, $\{G_n(t, \cdot)\} \in \mathcal{G}(\{\varphi_n^t\})$ with common positive constants $C_1, C_2, C_3, C_4(B)$ and $C_5(B)$, $\{f_n\} \subset L^2(J; H)$, $J = [s, t_1] \subset R_+$ and $u_{0,n} \in \overline{D(\varphi_n^s)}$ for $n = 1, 2, \cdots$. Assume that

(i) $\varphi_n^t$ converges to $\varphi^t$ on $H$ in the sense of Mosco [20] for each $t \in J$ (as $n \to +\infty$) and $\bigcup_{n=1}^{+\infty} \{z \in H; \varphi_n^t(z) \leq k\}$ is relatively compact in $H$ for every real $k > 0$ and $t \in J$, where $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$ and $\varphi_n^t = \varphi^t$ if $n = +\infty$.

(ii) if $z_n \in D(\varphi_n^s), g_n \in G_n(t_n, z_n)$, $\{t_n\} \subset R_+$, $\{\varphi_n^s(z_n)\}$ is bounded, $z_n \to z$ in $H$, $t_n \to t$ and $g_n \to g$ weakly in $H$ as $n \to +\infty$, then $g \in G(t, z)$, where $\{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^t\})$.

(iii) $f_n \to f$ weakly in $L^2(J; H)$ for some $f \in L^2(J; H)$ and $u_{0,n} \to u_0$ in $H$ for some $u_0 \in \overline{D(\varphi^s)}$.  

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Denote by \( u \) the solution of the Cauchy problem for (E) on \( J \) with \( u(s) = u_0 \) and by \( u_n \) the solution of the Cauchy problem for (E) with \( \varphi^t, G, f \) replaced by \( \varphi^t_n, G_n, f_n \), and with \( u_n(s) = u_{0,n} \). Then \( u_n \) converges to \( u \) on \( J \) in the sense that
\[
\int_J \varphi^t_n(u_n(t))dt \to \int_J \varphi^t(u(t))dt \quad \text{as} \quad n \to +\infty.
\]

### 3 Attractor for periodic multivalued dynamical system

In this section let us recall the known results obtained in [29] for a \( T_0 \)-periodic system in \( H \), of the form:
\[
(P)_s \quad u'(t) + \partial \varphi^t_p(u(t)) + G_p(t, u(t)) \ni f_p(t) \quad \text{in} \quad H, \quad t > s
\]
for each \( s \in R_+ \), where \( \varphi^t_p, G_p(t, \cdot) \) and \( f_p(t) \) are \( T_0 \)-periodic, namely periodic in time with the same period \( T_0, 0 < T_0 < +\infty \).

**Definition 3.1.** Let \( T_0 \) be a positive number. Then
(i) \( \Phi_p(\{a_r\}, \{b_r\}; T_0) \) is the set of all \( \{\varphi^t_p\} \in \Phi(\{a_r\}, \{b_r\}) \) satisfying \( T_0 \)-periodicity condition:
\[
\varphi^{t+T_0}_p(\cdot) = \varphi^t_p(\cdot) \quad \text{on} \quad H, \quad \forall t \in R_+.
\]
(ii) \( G_p(\{\varphi^t_p\}; T_0) \) is the set of all \( \{G_p(t, \cdot)\} \in G(\{\varphi^t_p\}) \) satisfying \( T_0 \)-periodicity condition:
\[
G_p(t + T_0, \cdot) = G_p(t, \cdot) \quad \text{in} \quad H, \quad \forall t \in R_+.
\]

Throughout this section we assume that \( \{\varphi^t_p\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0) \), \( \{G_p(t, \cdot)\} \in G_p(\{\varphi^t_p\}; T_0) \) and \( f_p \in L_{loc}(R_+; H) \) is \( T_0 \)-periodic in time, namely
\[
f_p(t + T_0) = f_p(t) \quad \text{in} \quad H, \quad \forall t \in R_+.
\]
Here we note that \( (P)_s \) can be considered as \( (E)_s \) in Section 2. So, by the result (A) in Section 2, the Cauchy problem for \( (P)_s \) has at least one solution \( u \) on \([s, +\infty)\). Hence we can define the multivalued dynamical process associated with \( (P)_s \) as follows:

**Definition 3.2.** For every \( 0 \leq s \leq t < +\infty \) we denote by \( U(t, s) \) the mapping from \( \overline{D(\varphi^s_p)} \) into \( \overline{D(\varphi^t_p)} \) which assigns to each \( u_0 \in \overline{D(\varphi^s_p)} \) the set
\[
U(t, s)u_0 := \left\{ z \in H \left| \begin{array}{l}
\text{There is a solution} \ u \text{ of} \ (P)_s \text{ on} \ [s, +\infty) \\
\text{such that} \\
u(s) = u_0 \text{ and} \ u(t) = z.
\end{array} \right. \right\}. \quad (3.4)
\]

Then we easily see the following properties of \( \{U(t, s)\} := \{U(t, s); 0 \leq s \leq t < +\infty\} \):
(U1) $U(s,s) = I$ on $D(\varphi_p^0)$ for any $s \in R_+$;

(U2) $U(t_2,s)z = U(t_2,t_1)U(t_1,s)z$ for any $0 \leq s \leq t_1 \leq t_2 < +\infty$ and $z \in D(\varphi_p^0)$;

(U3) $U(t + T_0, s + T_0)z = U(t, s)z$ for any $0 \leq s \leq t < +\infty$ and $z \in D(\varphi_p^0)$, that is, $U$ is $T_0$-periodic.

(U4) $\{U(t,s)\}$ has the following demi-closedness:

- If $0 \leq s_n \leq t_n < +\infty$, $s_n \to s$, $t_n \to t$, $z_n \in D(\varphi_p^{s_n})$, $z \in D(\varphi_p^s)$, $z_n \to z$ in $H$
and a element $w_n \in U(t_n, s_n)z_n$ converges to some element $w \in H$ as $n \to +\infty$, then $w \in U(t,s)z$

Next we define the discrete dynamical system in order to construct a global attractor for $(P)_s$.

**Definition 3.3.** Let $U(\cdot, \cdot)$ be the solution operator for $(P)_s$, defined by Definition 3.2. Then

(i) For each $\tau \in R_+$, we denote by $U_\tau$ the $T_0$-step mapping from $D(\varphi_p^\tau)$ into $D(\varphi_p^{\tau + T_0}) = D(\varphi_p^0)$, namely,

$$U_\tau := U(\tau + T_0, \tau).$$

(2) For any $k \in Z_+ := N \cup \{0\}$, we define

$$U_\tau^k := U_\tau \circ U_\tau \circ \cdots \circ U_\tau \text{ (k-th iteration)}.$$

Clearly we have $U_\tau^k = U(\tau + kT_0, \tau)$ for any $\tau \in R_+$ and $k \in Z_+$.

Now, let us recall the known result on the existence of global attractors for discrete multivalued dynamical systems $U_\tau$ associated with $(P)_s$.

**Theorem 3.1.** (cf. [29, Theorem 3.1]) Assume that $\{\varphi_p^\tau\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0)$, $\{G_p(t, \cdot)\} \in G_p(\{\varphi_p^\tau\}; T_0)$, $f_p \in L^2_{loc}(R_+; H)$ satisfies the $T_0$-periodicity condition (3.3). Then, for each $\tau \in R_+$, there exists a subset $A_\tau$ of $D(\varphi_p^\tau)$ such that

(i) $A_\tau$ is non-empty and compact in $H$;

(ii) for each bounded set $B$ in $H$ and each number $\epsilon > 0$ there exists $N_{B,\epsilon} \in N$ such that

$$\text{dist}_H(U_\tau^k z, A_\tau) < \epsilon$$

for all $z \in D(\varphi_p^\tau) \cap B$ and all $k \geq N_{B,\epsilon}$;

(iii) $U_\tau^k A_\tau = A_\tau$ for any $k \in N$.

**Remark 3.1.** By [29, Lemma 3.1] we can get the compact absorbing set $B_{0,\tau}$ of $D(\varphi_p^\tau)$ for $U_\tau$ such that for each bounded subset $B$ of $H$ there is a positive integer $n_B$ (independent of $\tau \in R_+$) satisfying

$$U_\tau^n (D(\varphi_p^\tau) \cap B) \subset B_{0,\tau} \text{ for all } n \geq n_B.$$

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Then we observe that the global attractor $A_\tau$ is given by the $\omega$-limit set of the absorbing set $B_{0,\tau}$ for $U_\tau$, i.e.

$$A_\tau = \cap_{n \in \mathbb{Z}_+} \bigcup_{k \geq n} U^k_\tau B_{0,\tau}. $$

The next theorem is concerned with a relationship between two global attractors $A_s$ and $A_\tau$. For detail proof, see [29].

**Theorem 3.2.** (cf. [29, Theorem 3.2]) Suppose the same assumptions are made as in Theorem 3.1. Let $A_s$ and $A_\tau$ be the global attractors for $U_s$ and $U_\tau$, with $0 \leq s \leq \tau \leq T_0$, respectively. Then, we have

$$A_\tau = U(\tau, s)A_s,$$

where $U(\tau, s)$ is the $T_0$-periodic process given in Definition 3.2.

**Remark 3.2.** By Theorem 3.1 (iii) and Theorem 3.2, we see that the global attractor $A_\tau$ for $U_\tau$ is $T_0$-periodic in $\tau$. In fact, for each $\tau \in \mathbb{R}_+$ choose $m_\tau \in \mathbb{Z}_+$ and $\sigma_\tau \in [0, T_0)$ so that $\tau = \sigma_\tau + m_\tau T_0$. Then, we have $A_\tau = A_{\sigma_\tau}$.

The third known result is the existence of a global attractor for the $T_0$-periodic multivalued dynamical system $(P)_s$.

**Theorem 3.3.** (cf. [29, Theorem 3.3]) Under the same assumptions as Theorem 3.1, put

$$A := \bigcup_{0 \leq \tau \leq T_0} A_\tau,$$

where $A_\tau$ is as obtained in Theorem 3.1. Then, $A$ has the following properties:

(i) $A$ is non-empty and compact in $H$;

(ii) for each bounded set $B$ in $H$ and each number $\epsilon > 0$ there exists a finite time $T_{B,\epsilon} > 0$ such that

$$\text{dist}_H(U(t + \tau, \tau)z, A) < \epsilon$$

for all $\tau \in \mathbb{R}_+$, all $z \in D(\varphi^\tau_0) \cap B$ and all $t \geq T_{B,\epsilon}$.

**Remark 3.3.** In [29, Section 4] the characterization of the $T_0$-periodic global attractor was discussed. The author proved that for each time $\tau \in \mathbb{R}_+$ the global attractor $A_\tau$ for the discrete multivalued dynamical system $U_\tau$ coincides with the cross-section of the family of all global bounded complete trajectories for the $T_0$-periodic system $(P)_s$.

## 4 Attractors of asymptotically periodic multivalued dynamical system

Throughout this section, let $M > 0$ be a fixed (sufficiently) large positive number. Now we put

$$\Psi_M := \left\{ \psi; \psi \text{ is proper, l.s.c. and convex on } H, \exists z \in D(\psi) \text{ s.t. } |z|_H \leq M, \psi(z) \leq M \right\}. $$

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Then let us introduce the notion of a metric topology on $\Psi_M$ which was introduced in [16].

Given $\varphi, \psi \in \Psi_M$, we define $\rho(\varphi, \psi; \cdot) : D(\varphi) \to R$ by putting

$$\rho(\varphi, \psi; z) = \inf\{\max(|y - z|_H, \psi(y) - \varphi(z)); y \in D(\psi)\}$$

for each $z \in D(\varphi)$, and for each $r \geq M$

$$\rho_r(\varphi, \psi) := \sup_{z \in L_r(r)} \rho(\varphi, \psi; z),$$

where $L_r(r) := \{z \in D(\varphi); |z|_H \leq r, \varphi(z) \leq r\}$. Moreover, for each $r \geq M$, we define the functional $\pi_r(\cdot, \cdot)$ on $\Psi_M \times \Psi_M$ by

$$\pi_r(\varphi, \psi) := \rho_r(\varphi, \psi) + \rho_r(\psi, \varphi) \quad \text{for } \varphi, \psi \in \Psi_M.$$

Then, according to [16, Proposition 3.1], we can define a complete metric topology on $\Psi_M$ so that the convergence $\psi_n \to \psi$ in $\Psi_M$ (as $n \to +\infty$) if and only if

$$\pi_r(\psi_n, \psi) \to 0 \quad \text{for every } r \geq M.$$

Now by using the above topology on $\Psi_M$, we consider an asymptotically $T_0$-periodic system as follows.

**Definition 4.1.** Assume $\{\varphi^l\} \in \Phi(\{a_r\}, \{b_r\}) \cap \Psi_M$, $\{G(t, \cdot)\} \in G(\{\varphi^l\})$ and $f \in L^2_{loc}(R_+; H)$. Then the system

$$(AP)_s \quad v'(t) + \partial \varphi^l(v(t)) + G(t, v(t)) \ni f(t) \quad \text{in } H, \ t > s (\geq 0)$$

is asymptotically $T_0$-periodic, if there are $\{\varphi^l\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0) \cap \Psi_M$, $\{G_p(t, \cdot)\} \in G_p(\{\varphi^l\}; T_0)$ and a $T_0$-periodic function $f_p \in L^2_{loc}(R_+; H)$ such that

(A1) (Convergence of $\varphi^l - \varphi^l_p \to 0$ as $t \to +\infty$) For each $r \geq M$,

$$J_m^{(r)} := \sup_{\sigma \in [0,T_0]} \pi_r(\varphi^mT_0 + \sigma^r; \varphi^r_p \to 0 \quad \text{as } m \to +\infty;$$

(A2) (Convergence of $G(t, \cdot) - G_p(t, \cdot) \to 0$ as $t \to +\infty$) If $\{\tau_n\} \subset [0, T_0]$, $\{m_n\} \subset Z_+$, $m_n \to +\infty$, $z_n \in D(\varphi^{m_nT_0 + \tau_n})$, $g_n \in G(m_nT_0 + \tau_n, z_n)$, $\{\varphi^{m_nT_0 + \tau_n}(z_n)\}$ is bounded, $z_n \to z$ in $H$, $\tau_n \to \tau$ and $g_n \to g$ weakly in $H$ (as $n \to +\infty$), then

$$g \in G_p(\tau, z);$$

(A3) (Convergence of $f(t) - f_p(t) \to 0$ as $t \to +\infty$)

$$|f(mT_0 + \cdot) - f_p|_{L^2(0,T_0; H)} \to 0 \quad \text{as } m \to +\infty.$$
By Definition 4.1 we easily see that a limiting system for \((AP)_s\) is a \(T_0\)-periodic one of the form:

\[
(P)_s \quad u'(t) + \partial \varphi_p^s(u(t)) + G_p(t, u(t)) \equiv f_p(t) \quad \text{in } H, \; t > s \; (\geq 0).
\]

Here we note that \((AP)_s\) is also considered as \((E)_s\). So, by the result (A) in Section 2, the Cauchy problem for \((AP)_s\) has at least one solution \(v\) on \([s, +\infty)\). Hence we can define the multivalued dynamical system associated with \((AP)_s\) as follows:

**Definition 4.2.** For every \(0 \leq s \leq t < +\infty\) we denote by \(E(t, s)\) the mapping from \(\overline{D(\varphi^s)}\) into \(\overline{D(\varphi^t)}\) which assigns to each \(v_0 \in \overline{D(\varphi^s)}\) the set

\[
E(t, s)v_0 := \left\{ z \in H \mid \text{There is a solution } v \text{ of } (AP)_s \text{ on } [s, +\infty) \text{ such that } v(s) = v_0 \text{ and } v(t) = z. \right\}.
\]

Then we easily see that \(\{E(t, s)\} := \{E(t, s); 0 \leq s \leq t < +\infty\}\) has the following evolution properties:

1. **(E1)** \(E(s, s) = I\) on \(\overline{D(\varphi^s)}\) for any \(s \in R_+\);
2. **(E2)** \(E(t_2, s)z = E(t_2, t_1)E(t_1, s)z\) for any \(0 \leq s \leq t_1 \leq t_2 < +\infty\) and \(z \in \overline{D(\varphi^s)}\);
3. **(E3)** \(\{E(t, s)\}\) has the following demi-closedness:
   - If \(0 \leq s_n \leq t_n < +\infty, s_n \to s, t_n \to t, z_n \in \overline{D(\varphi^{s_n})}, z \in \overline{D(\varphi^s)}, z_n \to z\) in \(H\) and a element \(w_n \in E(t_n, s_n)z_n\) converges to some element \(w \in H\) as \(n \to +\infty\), then \(w \in E(t, s)z\).

We begin with the definition of a discrete \(\omega\)-limit set for \(E(\cdot, \cdot)\).

**Definition 4.3.** (Discrete \(\omega\)-limit set for \(E(\cdot, \cdot)\)) Let \(\tau \in R_+\) be fixed. Let \(\mathcal{B}(H)\) be a family of bounded subsets of \(H\). Then for each \(B \in \mathcal{B}(H)\), the set

\[
\omega_\tau(B) := \bigcap_{n \in \mathbb{Z}_+} \bigcup_{k \geq n, m \in \mathbb{Z}_+} E(kT_0 + mT_0 + \tau, mT_0 + \tau)(\overline{D(\varphi^{mT_0 + \tau})} \cap B)
\]

is called the discrete \(\omega\)-limit set of \(B\) under \(E(\cdot, \cdot)\).

**Remark 4.1.** By definition of the discrete \(\omega\)-limit set \(\omega_\tau(B)\), it is easy to see that \(x \in \omega_\tau(B)\) if and only if there exist sequences \(\{k_n\} \subset \mathbb{Z}_+, k_n \uparrow +\infty, \{m_n\} \subset \mathbb{Z}_+, \{z_n\} \subset B\) with \(z_n \in \overline{D(\varphi^{m_nT_0 + \tau})}\) and \(\{x_n\} \subset H\) with \(x_n \in E(k_nT_0 + m_nT_0 + \tau, m_nT_0 + \tau)z_n\), such that

\[
x_n \to x \text{ in } H \text{ as } n \to +\infty.
\]

Now let us mention main theorems in this paper.

**Theorem 4.1.** (Discrete attractors of \((AP)_\tau\)) For each \(\tau \in R_+\), let \(A_\tau\) be the global attractor of \(T_0\)-periodic dynamical systems \(U_\tau\), which is obtained in Section 3. For \(\{\varphi^s\} \in \mathcal{B}(H)\),
\[ \Phi(\{a_r\}, \{b_r\}) \cap \Psi_M, \{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^i\}) \text{ and } f \in L^2_{\text{loc}}(R_+; H), \]
we assume that the system 
\[ (AP)_s \text{ is asymptotically } T_0\text{-periodic}. \]
Here we put
\[ A^*_r := \bigcup_{B \in \mathcal{B}(H)} \omega_\tau(B). \]  
(4.1)

Then, we have

(i) \( A^*_r(\subset D(\varphi^*_r)) \) is non-empty and compact in \( H \);

(ii) for each bounded set \( B \in \mathcal{B}(H) \) and each number \( \epsilon > 0 \) there exists \( N_{B, \epsilon} \in \mathbb{N} \) such that
\[ \text{dist}_H(E(kT_0 + \tau, \tau)z, A^*_r) < \epsilon \]
for all \( z \in \overline{D(\varphi^*_r)} \cap B \) and all \( k \geq N_{B, \epsilon} \);

(iii) \( A^*_r \subset U_l^1 A^*_r \subset A_r \) for any \( l \in \mathbb{N} \), where \( U_r \) is the discrete dynamical system for \( (P)_\tau \) given in Definition 3.3.

Remark 4.2. By the definition of the discrete \( \omega \)-limit set \( \omega_\tau(B) \) and \( A^*_r \), we easily see that
\[ A^*_r = A^*_r + nT_0, \quad \forall n \in \mathbb{N}. \]

Hence \( A^*_r \) is \( T_0 \)-periodic in time in a sense of the above.

The second main theorem is concerned with a relationship between two attractors \( A^*_s \) and \( A^*_r \).

Theorem 4.2. Suppose the same assumptions are made as in Theorem 4.1. Let \( A^*_s \) and \( A^*_r \) be discrete attractors for \( E(\cdot, s) \) and \( E(\cdot, \tau) \) with \( 0 \leq s \leq \tau < +\infty \), respectively. Then,
\[ A^*_r \subset U(\tau, s)A^*_s. \]

where \( U(\tau, s) \) is the \( T_0 \)-periodic process for \( (P)_s \) which is given in Definition 3.2.

By Theorems 4.1-4.2, we can get the attractor for asymptotic \( T_0 \)-periodic system \( (AP)_\tau \).

Theorem 4.3. (Global attractor for \( (AP)_\tau \)) Suppose the same assumptions are made as in Theorem 4.1. For any \( \tau \in R_+ \), let \( A^*_r \) be the discrete attractor for \( E(\cdot, \tau) \) obtained in Theorem 4.1. Here we put
\[ A^* := \bigcup_{\tau \in [0, T_0]} A^*_r. \]
(4.2)

Then, for any bounded set \( B \in \mathcal{B}(H) \),
\[ \bigcap_{s \geq 0} \bigcup_{t \geq s, \tau \in R_+} E(t + \tau, \tau)(D(\varphi^*_r) \cap B) \subset A^*. \]
(4.3)
By Theorem 4.3, the set $\mathcal{A}^*$ can be called the global attractor of $(AP)_\tau$.

Here we give some key lemmas.

**Lemma 4.1.** If $\{s_n\} \subset \mathbb{R}_+, \{\tau_n\} \subset \mathbb{R}_+, s \in \mathbb{R}_+, \tau \in \mathbb{R}_+, s_n \to s, \tau_n \to \tau, \{m_n\} \subset \mathbb{Z}_+$ with $m_n \to +\infty$, $z_n \in D(\varphi^{m_nT_0+s_n})$, $z \in D(\varphi^s)$, $z_n \to z$ in $H$ and a element $w_n \in E(m_nT_0+\tau_n+s_n, m_nT_0+s_n)z_n$ converges to some element $w \in H$ as $n \to +\infty$, then $w \in U(\tau+s,s)z$.

**Proof.** Since $\tau_n \to \tau$, without loss of generality we may assume that there exists a finite time $T > 0$ such that $\{\tau_n\} \subset [0, T]$ and $\tau \in [0, T]$. By $w_n \in E(m_nT_0+\tau_n+s_n, m_nT_0+s_n)z_n$, there is a solution $v_n$ of $(AP)_{m_nT_0+s_n}$ on $[m_nT_0 + s_n, +\infty)$ such that

$$v_n(m_nT_0 + \tau_n + s_n) = w_n$$

and $v_n(m_nT_0 + s_n) = z_n$.

Now we put $u_n(t) := v_n(t + m_nT_0 + s_n)$, then we easily see that $u_n$ is the solution for

$$
\begin{cases}
  u_n'(t) + \partial \varphi^{t+m_nT_0+s_n}(u_n(t)) + G(t + m_nT_0 + s_n, u_n(t)) \ni f(t + m_nT_0 + s_n), & t > 0, \\
  u_n(0) = z.
\end{cases}
$$

Let $\delta \in (0, 1)$ be fixed. Since $z_n \to z$ in $H$ as $n \to +\infty$, $\{z_n\}$ is bounded in $H$. Hence, from global estimates of solutions (cf. (B) in Section 2) it follows that there is a positive constant $M_\delta > 0$ (independent of $n$) satisfying

$$
\sup_{t \geq \delta} |u_n(t)|_H^2 + \sup_{t \geq \delta} |u_n'|_{L^2(t,t+1;H)}^2 + \sup_{t \geq \delta} \varphi^{t+m_nT_0+s_n}(u_n(t)) \leq M_\delta. \tag{4.4}
$$

By [16, Lemma 4.1] we note that the convergence assumption (A1) implies

$$
\varphi^{t+m_nT_0+s_n} \to \varphi^{t+s} \text{ in the sense of Mosco [20] \tag{4.5}}
$$

for each $t \geq 0$ as $n \to +\infty$. Moreover by the same argument in [10, Lemma 3.1] we can prove that

$$
\bigcup_{n=1}^{+\infty} \{z \in H; \varphi^{t+m_nT_0+s_n}(z) \leq k\} \text{ is relatively compact in } H \tag{4.6}
$$

for every real $k > 0$ and $t \geq 0$, where $\varphi^{t+m_nT_0+s_n} = \varphi^{t+s}$ if $n = +\infty$. Therefore, by (4.4)-(4.6), (A2), (A3) and the convergence result (C) in Section 2, (by taking a subsequence of $\{n\}$, if necessary) we see that there is a function $u_\delta$ such that

$$
\varphi^{t+s}(u(t)) + G_p(t+s, u(t)) \ni f_p(t+s), \quad t > \delta.
$$

By the standard diagonal process and the same argument in [21, Lemma 3.10], we can construct the solution $u$ on $[0, +\infty)$ satisfying

$$
\begin{cases}
  u'(t) + \partial \varphi^{t+s}(u(t)) + G_p(t+s, u(t)) \ni f_p(t+s), & t > 0, \\
  u(0) = z
\end{cases}
$$

and

$$
u_n \to u \text{ in } C([0,T]; H) \text{ as } n \to +\infty. \tag{4.7}$$
Then, by (4.7) and \( u_n(\tau_n) = w_n \) we have \( u(\tau) = w \), which implies that \( w \in U(\tau + s, s)z \).

By (B) in Section 2, for each \( B \in \mathcal{B}(H) \) we can choose constants \( r_B > 0 \) and \( M_B > 0 \) so that

\[
|v|_H \leq r_B \quad \text{and} \quad \varphi^{t+s}(v) \leq M_B, \tag{4.8}
\]

for any \( s \in R_+ \), \( t \geq T_0 \), \( z \in \overline{D(\varphi^t)} \cap B \) and \( v \in E(t + s, s)z \). Hence it follows from condition (A1) that for each \( m \in Z_+ \), \( \tau \in [0, T_0] \), \( n \in N \) and \( z \in \overline{D(\varphi^{mT_0+\tau})} \cap B \) there is \( \tilde{z} := \tilde{z}_{mT_0+\tau, z, nT_0} \in D(\varphi^\tau) \) such that

\[
|\tilde{z} - v|_H \leq J^{(r_B+MB+M)}_{m+n},
\]

(hence \( |\tilde{z}|_H \leq r_B + J^{(r_B+MB+M)}_{m+n} \))

and

\[
\varphi^\tau(\tilde{z}) - \varphi^{nT_0+mT_0+\tau}(v) \leq J^{(r_B+MB+M)}_{m+n},
\]

(hence \( \varphi^\tau(\tilde{z}) \leq M_B + J^{(r_B+MB+M)}_{m+n} \)).

where \( v \in E(nT_0 + mT_0 + \tau, mT_0 + \tau)z \).

Since \( J^{(r_B+MB+M)}_k \rightarrow 0 \) as \( k \rightarrow +\infty \), there is a number \( N_0 \in N \) such that

\[
J^{(r_B+MB+M)}_k \leq 1, \quad \forall k > N_0.
\]

Now, put \( J_0 := 1 + \sup_{1 \leq k \leq N_0} J^{(r_B+MB+M)}_k < +\infty \). Then, we define the bounded set \( \tilde{B}_\tau \) by

\[
\tilde{B}_\tau := \{ z \in H; |z|_H \leq r_B + J_0 \} \cap \overline{D(\varphi^\tau)}.
\]

Let \( B_{0,\tau} \) be the compact absorbing set for \( U_\tau \) introduced by Remark 3.1. Then, we see that there exists a number \( \tilde{N} \in N \) so that

\[
U^l\tilde{B}_\tau \subset B_{0,\tau}, \quad \forall l \geq \tilde{N}.
\]

(4.9)

The next lemma is very important to prove Theorem 4.1 (iii).

**Lemma 4.2.** Let \( \tau \in R_+ \) and \( B_{0,\tau} \) be the compact absorbing set for \( U_\tau \). Then we have

\[
\omega_\tau(B) \subset B_{0,\tau}, \quad \forall B \in \mathcal{B}(H).
\]

**Proof.** At first we assume \( \tau \in [0, T_0] \).

For each \( B \in \mathcal{B}(H) \), let \( x \) be any element of \( \omega_\tau(B) \). Then, it follows from Remark 4.1 that there exist sequences \( \{k_n\} \subset Z_+ \) with \( k_n \rightarrow +\infty \), \( \{m_n\} \subset Z_+ \), \( \{z_n\} \subset B \) with \( z_n \in \overline{D(\varphi^{m_nT_0+\tau})} \) and \( \{x_n\} \subset H \) with \( x_n \in E(k_nT_0 + m_nT_0 + \tau, m_nT_0 + \tau)z_n \) such that

\[
x_n \rightarrow x \text{ in } H \quad \text{as } n \rightarrow +\infty.
\]

(4.10)

Let \( \tilde{N} \) be the positive integer obtained in (4.9). Then by (E2) we have

\[
x_n \in E(k_nT_0 + m_nT_0 + \tau, k_nT_0 - \tilde{N}T_0 + m_nT_0 + \tau)
\]

\[
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\]
Theorem 4.1 (i) holds. Also, by (4.1) and Remark 4.1 we observe that Theorem 4.1 (ii)
for any \( n \) with \( k_n \geq \bar{N} + 1 \).
Hence, there exists an element \( y_n \in E(k_n T_0 - \bar{N} T_0 + m_n T_0 + \tau, m_n T_0 + \tau)z_n \) such that
\[
x_n \in E(k_n T_0 + m_n T_0 + \tau, k_n T_0 - \bar{N} T_0 + m_n T_0 + \tau)y_n.
\] (4.12)

Since \( \{z_n\} \subset B \), we see that
\[
|y_n|_H \leq r_B \quad \text{and} \quad \varphi^{k_n T_0 - \bar{N} T_0 + m_n T_0 + \tau}(y_n) \leq M_B \quad \text{for any } n \geq \bar{N} + 1,
\]
where \( r_B \) and \( M_B \) are positive constants in (4.8).
From the convergence condition (A1) it follows that for \( y_n \in E(k_n T_0 - \bar{N} T_0 + m_n T_0 + \tau, m_n T_0 + \tau)z_n \) there is \( z_n \in D(\varphi^*_p) \) such that
\[
|z_n - y_n|_H \leq J_{k_n - \bar{N} + m_n}^{(r_B + M_B + M)}.
\] (hence \( |z_n|_H \leq r_B + J_{k_n - \bar{N} + m_n}^{(r_B + M_B + M)} \))
and
\[
\varphi^*_p(z_n) \leq M_B + J_{k_n - \bar{N} + m_n}^{(r_B + M_B + M)}.
\]
Since \( \{z_n \in D(\varphi^*_p) \}; \quad n \in N \) with \( k_n \geq \bar{N} + 1 \}( \subset \bar{B}_* ) \) is relatively compact in \( H \), we may assume that
\[
z_n \longrightarrow \bar{z}_\infty \quad \text{in } H \quad \text{as } n \rightarrow +\infty
\]
for some \( \bar{z}_\infty \in H \). Then we easily see that \( \bar{z}_\infty \in \bar{B}_* \) and
\[
y_n \longrightarrow \bar{z}_\infty \quad \text{in } H \quad \text{as } n \rightarrow +\infty.
\] (4.13)

By Lemma 4.1 and (4.10)-(4.13), we observe that
\[
x \in U(\bar{N} T_0 + \tau, \tau)\bar{z}_\infty,
\]
which implies that
\[
x \in U(\bar{N} T_0 + \tau, \tau)\bar{B}_* = U(\bar{N} T_0 + \tau, \tau)\bar{B}_0 \subset B_{0, \tau}.
\]
Hence we have
\[
\omega_\tau(B) \subset B_{0, \tau}.
\]

For the general case of \( \tau \in \mathbb{R}_+ \), choose positive numbers \( i_\tau \in N \) and \( \tau_0 \in [0, T_0] \) so that \( \tau = \tau_0 + i_\tau T_0 \). Then, we can show \( \omega_\tau(B) \subset B_{0, \tau} \) by the same argument as above. \( \Box \)

**Proof of Theorem 4.1.** On account of Lemma 4.2 we can get \( A^*_\tau \subset B_{0, \tau} \). Hence, Theorem 4.1 (i) holds. Also, by (4.1) and Remark 4.1 we observe that Theorem 4.1 (ii) holds.

Now, we prove Theorem 4.1 (iii). At first, let us prove that \( A^*_\tau \subset U_{\tau l} A^*_\tau \) for any \( l \in N \).
Let \( x \) be any element of \( A^*_\tau \). By the definition of \( A^*_\tau \), there are sequences \( \{B_n\} \subset B(H) \) and \( \{x_n\} \subset H \) with \( x_n \in \omega_\tau(B_n) \) such that

\[
x_n \rightarrow x \text{ in } H \quad \text{as } n \rightarrow +\infty. \tag{4.14}
\]

Then, for each \( n \) it follows from Remark 4.1 that there exist sequences \( \{k_{n,j}\} \subset \mathbb{Z}_+ \) with \( k_{n,j} \rightarrow +\infty \), \( \{m_{n,j}\} \subset \mathbb{Z}_+ \), \( \{z_{n,j}\} \subset B_n \) with \( z_{n,j} \in D(\varphi^{m_{n,j}T_0 + \tau}) \) and \( \{v_{n,j}\} \subset H \) with \( v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j} \) such that

\[
v_{n,j} \rightarrow x_n \text{ in } H \quad \text{as } j \rightarrow +\infty. \tag{4.15}
\]

Let \( l \) be any number in \( N \), then we see that

\[
v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau) \circ E(k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j}
\]

for \( j \) with \( k_{n,j} \geq l + 1 \). So, there exists an element \( w_{n,j} \in E(k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j} \) such that

\[
v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau)w_{n,j}. \tag{4.16}
\]

By global estimates (B) in Section 2, \( \{w_{n,j} \in H : j \in N \text{ with } k_{n,j} \geq l+1\} \) is relatively compact in \( H \) for each \( n \). Therefore we may assume that the element \( w_{n,j} \) converges to some element \( \tilde{w}_{n,\infty} \in H \) as \( j \rightarrow +\infty \). Clearly, \( \tilde{w}_{n,\infty} \in \omega_\tau(B_n) \). Moreover, it follows from Lemma 4.1 and (4.15)-(4.16) that

\[
x_n \in U(lT_0 + \tau, \tau)\tilde{w}_{n,\infty} \subset U(lT_0 + \tau, \tau)\omega_\tau(B_n),
\]

hence, we have

\[
x_n \in \bigcup_{n \geq 1} U_t^l\omega_\tau(B_n), \quad \forall n \geq 1. \tag{4.17}
\]

Here, by the closedness of \( U(\cdot, \cdot) \) we note that for each subset \( X \) of \( B_{0,\tau} \),

\[
\overline{U_t^lX} \subset U_t^l\overline{X}, \quad \forall l \in N. \tag{4.18}
\]

Taking account of Lemma 4.2, (4.14), (4.17) and (4.18), we observe that

\[
x \in \bigcup_{n \geq 1} U_t^l\omega_\tau(B_n) = U_t^l \bigcup_{n \geq 1} \omega_\tau(B_n) \subset U_t^l \bigcup_{n \geq 1} \omega_\tau(B_n) \subset U_t^lA^*_\tau,
\]

which implies that \( A^*_\tau \) is semi-invariant under the \( T_0 \)-periodic dynamical systems \( U_\tau \), i.e.

\[
A^*_\tau \subset U_t^lA^*_\tau, \quad \forall l \in N. \tag{4.19}
\]
Next we shall prove that $U^l_r \mathcal{A}^*_r \subset \mathcal{A}_r$ for any $l \in N$. By (4.19), for each $l \in N$

$$U^l_r \mathcal{A}^*_r \subset U^l_r U^n_r \mathcal{A}^*_r = U^{l+n}_r \mathcal{A}^*_r, \quad \forall n \in N. \quad (4.20)$$

By $\mathcal{A}^*_r \subset B_0$, (4.20) and the attractive property of $\mathcal{A}_r$, we have

$$U^l_r \mathcal{A}^*_r \subset \mathcal{A}_r, \quad \forall l \in N.$$

Therefore we conclude that

$$\mathcal{A}^*_r \subset U^l_r \mathcal{A}^*_r \subset \mathcal{A}_r, \quad \forall l \in N. \quad \Diamond$$

**Proof of Theorem 4.2.** Let $x$ be any element of $\mathcal{A}^*_r$. Then by the definition of $\mathcal{A}^*_r$, there exist sequences $\{B_n\} \subset B(H)$ and $\{x_n\} \subset H$ with $x_n \in \omega_r(B_n)$ such that

$$x_n \longrightarrow x \text{ in } H \quad \text{as } n \rightarrow +\infty. \quad (4.21)$$

From Remark 4.1 it follows that for each $n$, there are sequences $\{k_{n,j}\} \subset Z_+$ with $k_{n,j} \rightarrow +\infty$, $\{m_{n,j}\} \subset Z_+$, $\{z_{n,j}\} \subset B_n$ with $z_{n,j} \in D(\varphi^{m_{n,j}T_0+\tau})$ and $\{v_{n,j}\} \subset H$ with $v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j}$ such that

$$v_{n,j} \longrightarrow x_n \text{ in } H \quad \text{as } j \rightarrow +\infty. \quad (4.22)$$

Note that for given $s, \tau \in R_+$ with $s \leq \tau$ there is a positive number $l_s \in N$ satisfying

$$s \leq \tau \leq l_s T_0 + s.$$

By using the property (E2) we see that

$$v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 + m_{n,j}T_0 + s)$$

$$\circ E(k_{n,j}T_0 + m_{n,j}T_0 + s, T_0 + m_{n,j}T_0 + l_s T_0 + s)$$

$$\circ E(T_0 + m_{n,j}T_0 + l_s T_0 + s, m_{n,j}T_0 + \tau)z_{n,j}$$

for any $j \in Z_+$ with $k_{n,j} \geq l_s + 2$. Here we can take elements $w_{n,j} \in H$ and $y_{n,j} \in H$ so that

$$v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 + m_{n,j}T_0 + s)w_{n,j}, \quad (4.23)$$

$$w_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + s, T_0 + m_{n,j}T_0 + l_s T_0 + s)y_{n,j} \quad (4.24)$$

and

$$y_{n,j} \in E(T_0 + m_{n,j}T_0 + l_s T_0 + s, m_{n,j}T_0 + \tau)z_{n,j}. \quad (4.25)$$

By $\{z_{n,j}\} \subset B_n$ and the global boundedness result (B) in Section 2, we can get a positive constant $C_n := C_n(B_n) > 0$ satisfying

$$|y_{n,j}| \leq C_n, \quad \forall y_{n,j} \in E(T_0 + m_{n,j}T_0 + l_s T_0 + s, m_{n,j}T_0 + \tau)z_{n,j}. \quad (4.26)$$
Here we define the bounded set $B_{C_n}$ by
\[ B_{C_n} := \{ b \in H : |b|_H \leq C_n \} . \]

From (4.26) and the result (B) in Section 2 it follows that the set
\[ \{ w_{n,j} \in H : w_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + s, T_0 + m_{n,j}T_0 + l_sT_0 + s)y_{n,j} \text{ for any } j \in \mathbb{Z}_+ \text{ with } k_{n,j} \geq l_s + 2 \} \]

is relatively compact in $H$. Hence, we may assume that the element $w_{n,j}$ converges to some element $\tilde{w}_{n,\infty} \in H$ as $j \to +\infty$. Clearly, $\tilde{w}_{n,\infty} \in \omega_s(B_{C_n})$, and it follows from Lemma 4.2 that
\[ \omega_s(B_{C_n}) \subset B_{0,s} \subset D(\varphi^s). \]

Moreover, by Lemma 4.1 and (4.22)-(4.23) we have
\[ x_n \in U(\tau, s)\tilde{w}_{n,\infty} \subset U(\tau, s)\omega_s(B_{C_n}), \quad \forall n \geq 1, \]

hence, we see that
\[ x_n \in \bigcup_{n \geq 1} U(\tau, s)\omega_s(B_{C_n}), \quad \forall n \geq 1. \]  

(4.27)

Here, by the closedness of $U(\cdot, \cdot)$, we note that for each subset $X$ of $B_{0,s}$,
\[ \overline{U(\tau, s)X} \subset U(\tau, s)\overline{X}. \]  

(4.28)

On account of Lemma 4.2, (4.21), (4.27) and (4.28), we observe that
\[ x \in \bigcup_{n \geq 1} U(\tau, s)\omega_s(B_{C_n}) \]
\[ = U(\tau, s) \bigcup_{n \geq 1} \omega_s(B_{C_n}) \]
\[ \subset U(\tau, s) \bigcup_{n \geq 1} \omega_s(B_{C_n}) \]
\[ \subset U(\tau, s)A^*_s, \]

which implies that $A^*_\tau$ is the subset of $U(\tau, s)A^*_s$, namely
\[ A^*_\tau \subset U(\tau, s)A^*_s. \]

\[ \diamond \]

**Proof of Theorem 4.3.** For any $B \in \mathcal{B}(H)$, let $z_0$ be any element of the $\omega$-limit set $\omega_E(B)$ which is define by
\[ \omega_E(B) := \bigcap_{s \geq 0} \bigcup_{t \geq s, \tau \in R_+} E(t + \tau, \tau)(\overline{D(\varphi^s) \cap B}). \]
Then we easily see that there exist sequences \( \{t_n\} \subset R_+ \) with \( t_n \uparrow +\infty \), \( \{\tau_n\} \subset R_+ \), \( \{y_n\} \subset B \) with \( y_n \in \overline{D(\varphi^\tau)} \) and \( \{z_n\} \subset H \) with \( z_n \in E(t_n + \tau_n, \tau_n)y_n \) such that

\[
\begin{align*}
t_n &:= k_n T_0 + t'_n, \quad k_n \in \mathbb{Z}_+, \quad k_n \not\to +\infty, \quad t'_n \in [T_0, 2T_0], \quad t'_n \to t'_0, \\
\tau_n &:= l_n T_0 + \tau'_n, \quad l_n \in \mathbb{Z}_+, \quad \tau'_n \in [0, T_0], \quad \tau_n \to \tau'_0
\end{align*}
\]

and

\[ z_n \to z_0 \quad \text{in } H \quad (4.29) \]

as \( n \to +\infty \). Without loss of generality, we may assume that

\[
\begin{align*}
(a) \quad t'_n + \tau'_n \not\to \tau'_0 & \quad \text{or} \quad (b) \quad t'_n + \tau'_n \searrow t'_0 + \tau'_0.
\end{align*}
\]

Now, assume that (a) holds. Then let us consider the multivalued semiflow

\[
v_n \in E(1 + k_n T_0 + l_n T_0 + t'_0 + \tau'_0, \ k_n T_0 + l_n T_0 + t'_n + \tau'_n)z_n. \quad (4.30)
\]

Then, there is a solution \( u_n \) on \( [k_n T_0 + l_n T_0 + t'_n + \tau'_n, +\infty) \) for

\[
\begin{align*}
u'_n(t) + \partial_{\varphi^t} + k_n T_0 + l_n T_0 + t'_n + \tau'_n (u_n(t)) + G(t + k_n T_0 + l_n T_0 + t'_n + \tau'_n, u_n(t)) \\
\quad \ni f(t + k_n T_0 + l_n T_0 + t'_n + \tau'_n), \quad t > 0,
\end{align*}
\]

\[
u_n(0) = z_n \quad \text{and} \quad u_n(1 + t'_0 + \tau'_0 - t'_n - \tau'_n) = v_n.
\]

Since \( z_n \to z_0 \) in \( H \), \( \{z_n\} \) is bounded in \( H \). Therefore by the global estimate (B) in Section 2, we see that

\[
\begin{align*}
v_n \in E(1 + k_n T_0 + l_n T_0 + t'_0 + \tau'_0, \ k_n T_0 + l_n T_0 + t'_n + \tau'_n)z_n \\
\quad \text{for any } n \in N
\end{align*}
\]

is relatively compact in \( H \). Hence we may assume that

\[ v_n \to v \quad \text{in } H \quad \text{for some } v \in H. \quad (4.31) \]

Now applying Lemma 4.1 with (4.29)-(4.31), we can get

\[ v \in U(1 + t'_0 + \tau'_0, \ t'_0 + \tau'_0)z_0, \]

more precisely, (taking the subsequence of \( \{n\} \) if necessary) we observe that

\[ u_n \to u \quad \text{in } C([0, 2]; H) \quad \text{as } n \to +\infty, \quad (4.32) \]

where \( u \) is the solution \([t'_0 + \tau'_0, +\infty)\) satisfying

\[
\begin{align*}
u'(t) + \partial_{\varphi^t} + t'_0 + \tau'_0 (u(t)) + G_p(t + t'_0 + \tau'_0, u(t)) \ni f_p(t + t'_0 + \tau'_0), \quad t > 0,
\end{align*}
\]

\[ u(0) = z_0 \quad \text{and} \quad u(1) = v. \]

By (4.32) we easily see that

\[ u_n(t'_0 + \tau'_0 - t'_n - \tau'_n) \to z_0 \quad \text{as } n \to +\infty. \quad (4.33) \]
Note that
\[ u_n(t_0' + \tau_n' - t_n' - \tau_n') \in E(k_nT_0 + l_nT_0 + t_0' + \tau_0', k_nT_0 + l_nT_0 + t_n' + \tau_n') \times n \]
\[ = E(k_nT_0 + l_nT_0 + t_0' + \tau_0', l_nT_0 + \tau_n') y_n \]
\[ = E(k_nT_0 + l_nT_0 + t_0' + \tau_0', l_nT_0 + t_0' + \tau_0') E(l_nT_0 + t_0' + \tau_0', l_nT_0 + \tau_n') y_n. \]

So, we can take a element \( x_n \in E(l_nT_0 + t_0' + \tau_0', l_nT_0 + \tau_n') y_n \) such that
\[ u_n(t_0' + \tau_n' - t_n' - \tau_n') \in E(k_nT_0 + l_nT_0 + t_0' + \tau_0', l_nT_0 + t_0' + \tau_0') x_n. \]  
(4.34)

By \( \{y_n\} \subset B \) and the global estimate \((B)\) in Section 2, we easily see that \( \{x_n\} \) is bounded, i.e.
\[ \{x_n\} \subset \tilde{B} \text{ for some } \tilde{B} \in B(H). \]  
(4.35)

Therefore, from Remarks 4.1-4.2 and (4.33)-(4.35) we observe that
\[ z_0 \in \omega_{t_0' + \tau_0'}(\tilde{B}) \subset A^*_{t_0' + \tau_0'} \subset A^*. \]
Thus (4.3) holds.

In the case \((b)\) when \( t_n' + \tau_n' \owns t_0' + \tau_0' \), we can prove (4.3) by the slight modification of the proof as above.

\[ \Diamond \]

Theorem 4.1 implies that the attracting set \( A^* \) for \((AP)\) is semi-invariant under \( U_\tau \) associated with the limiting \( T_0 \)-periodic system \((P)_s\), in general. Moreover, from Theorem 4.2 we observe that
\[ A^*_\tau \subset U(\tau, s)A^*_s \quad \text{for any } 0 \leq s \leq \tau < +\infty. \]

In order to get the invariance of \( A^*_\tau \) under \( U_\tau \) and \( A^*_\tau = U(\tau, s)A^*_s \), let us use a concept of a regular approximation, which was introduced in [17].

**Definition 4.4.** (Regular approximation) Let \( s \in R_+ \) be fixed. Let \( z \in D(\varphi_p^s) \). Then, we say that \( U(t+s,s)z \) is regularly approximated by \( E(t+kT_0+s, kT_0+s) \) as \( k \to +\infty \), if for each finite \( T > 0 \) there are sequences \( \{k_n\} \subset Z_+ \) with \( k_n \to +\infty \) and \( \{z_n\} \subset H \) with \( z_n \in D(\varphi_{k_nT_0+s}) \) and \( z_n \to z \) in \( H \) satisfying the following property: for any function \( u \in W^{1,2}(0,T; H) \) satisfying \( u(t) \in U(t+s,s)z \) for all \( t \in [0,T] \) there is a sequence \( \{u_n\} \subset W^{1,2}(0,T; H) \) such that \( u_n(t) \in E(t+k_nT_0+s, k_nT_0+s)z_n \) for all \( t \in [0,T] \) and \( u_n \to u \) in \( C([0,T]; H) \) as \( n \to +\infty \).  

Using the above concept, we can show that the invariance of \( A^*_\tau \) under \( U_\tau \). Moreover we can get
\[ A^*_\tau = U(\tau, s)A^*_s. \]

**Theorem 4.4** Suppose all assumptions in Theorem 4.1. Let \( A^*_\tau \) and \( A^*_\tau \) be discrete attractors for \( E(\cdot,s) \) and \( E(\cdot,\tau) \), with \( 0 \leq s \leq \tau < +\infty \), respectively. Assume that for
any point \( z \) of \( \mathcal{A}_s^* \), \( U(t + s, s)z \) is regularly approximated by \( E(t + kT + s, kT + s) \) as \( k \to +\infty \). Then we have

\[
\mathcal{A}^*_s = U(\tau, s)\mathcal{A}_s^*.
\]

**Proof.** By Theorem 4.2, we have only to show that

\[
U(\tau, s)\mathcal{A}_s^* \subset \mathcal{A}_s^*.
\]

To do so, let \( x \) be any element of \( U(\tau, s)\mathcal{A}_s^* \).

At first, taking account of Definitions 3.2-3.3 and Theorem 4.1 (iii), we see that for each \( n \in N \)

\[
U^nU(\tau, s)\mathcal{A}_s^* = U(nT + \tau, nT + s)U(\tau, s)\mathcal{A}_s^* = U(\tau, s)U^n\mathcal{A}_s^* \subset U(\tau, s)\mathcal{A}_s^*.
\]

Hence, there exists an element \( y_n \in \mathcal{A}_s^* \) such that

\[
x \in U^nU(\tau, s)y_n = U(nT + \tau - s + s, s)y_n.
\]

By using our assumption as \( t = nT + \tau - s \), we observe that for each \( n \), there are sequences \( \{k_{n,j}\} \subset \mathbb{Z}_+ \), \( \{x_{n,j}\} \subset H \) and \( \{y_{n,j}\} \subset H \) such that

\[
k_{n,j} \to +\infty, \quad y_{n,j} \in D(\varphi_{k_{n,j}T_0 + s}), \quad y_{n,j} \to y_n \text{ in } H
\]

and

\[
x_{n,j} \in E(nT + \tau - s + k_{n,j}T_0 + s, k_{n,j}T_0 + s)y_{n,j}, \quad x_{n,j} \to x \text{ in } H
\]

as \( j \to +\infty \). Therefore, by the usual diagonal argument, we can find a subsequence \( \{j_n\} \) of \( \{j\} \) such that \( \bar{x}_n := x_{n,j_n}, \bar{y}_n := y_{n,j_n} \) and \( \bar{k}_n := k_{n,j_n} \) satisfy

\[
|\bar{x}_n - x|_H < \frac{1}{n}, \quad \bar{x}_n \in E(nT + \tau - s + \bar{k}_nT_0 + s, \bar{k}_nT_0 + s)\bar{y}_n, \quad |\bar{y}_n - y_n|_H < \frac{1}{n} \tag{4.38}
\]

for every \( n = 1, 2, \ldots \). Since \( \{\bar{y}_n\} \) is bounded in \( H \), there is a bounded set \( B \in \mathcal{B}(H) \) so that \( \{\bar{y}_n\} \subset B \).

By (E2), we see that

\[
\bar{x}_n \in E(nT + \tau - s + \bar{k}_nT_0 + s, \bar{k}_nT_0 + s)\bar{y}_n
\]

\[
= E(nT + \bar{k}_nT_0 + \tau, T_0 + \bar{k}_nT_0 + \tau)E(T_0 + \bar{k}_nT_0 + \tau, \bar{k}_nT_0 + \tau)\bar{y}_n,
\]

hence there is an element \( \bar{z}_n \in E(T_0 + \bar{k}_nT_0 + \tau, \bar{k}_nT_0 + s)\bar{y}_n \) such that

\[
\bar{x}_n \in E(nT + \bar{k}_nT_0 + \tau, T_0 + \bar{k}_nT_0 + \tau)\bar{z}_n. \tag{4.39}
\]

Since \( \{\bar{y}_n\} \subset B \) and the global estimate (B) in Section 2, we see that \( \{\bar{z}_n\} \) is also bounded in \( H \). Hence, there is a bounded set \( \bar{B} \in \mathcal{B}(H) \) so that \( \{\bar{z}_n\} \subset \bar{B} \). The above fact (4.37)-(4.39) implies (cf. Remark 4.1) that \( x \in \omega_r(\bar{B}) \subset \mathcal{A}_r^* \). Thus we have \( U(\tau, s)\mathcal{A}_s^* \subset \mathcal{A}_r^* \). \( \diamond \)
By the same argument in Theorem 4.4, we can get the following corollary:

**Corollary.** (i) Suppose the same assumptions of Theorem 4.4. Then, by Remark 4.2 we observe that $A^*$ is invariant under the $T_0$-periodic dynamical system $U_s(:= U(T_0 + s, s))$. Namely,

$$A^*_s = U^*_s A^*_s$$

for any $l \in N$.

(ii) Assume that for any point $z$ of $A_\tau$, $U(t + \tau, \tau)z$ is regularly approximated by $E(t + kT_0 + \tau, kT_0 + \tau)$ as $k \to +\infty$. Then, we have $A^*_\tau \supset A_\tau (= U_\tau A_\tau)$. Hence by Theorem 4.1 (iii) we conclude that $A^*_\tau = A_\tau$.

**Remark 4.3.** If the solution operator $U(t, s)$ is singlevalued, namely the solution for the Cauchy problem of (P) is unique, the assumptions of Theorem 4.4 always hold. Thus, Theorems 4.1-4.4 contain the abstract results obtained in [11], which was concerned with the asymptotic $T_0$-periodic stability for the singlevalued dynamical system associated with time-dependent subdifferentials.

### 5 Application to obstacle problems for PDE’s

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ ($1 \leq N < +\infty$) with smooth boundary $\Gamma = \partial \Omega$, $q$ be a fixed number with $2 \leq q < +\infty$ and $T_0$ be a fixed positive number. We use the notation

$$a_q(v, z) := \int_\Omega |\nabla v|^q - 2 \nabla v \cdot \nabla z dx, \quad \forall v, z \in W^{1,q}(\Omega)$$

and denote by $(\cdot, \cdot)$ the usual inner product in $L^2(\Omega)$.

For prescribed obstacle functions $\sigma_0 \leq \sigma_1$ and each $t \in \mathbb{R}_+$ we define the set

$$K(t) := \left\{ z \in W^{1,q}(\Omega); \sigma_0(t, \cdot) \leq z \leq \sigma_1(t, \cdot) \text{ a.e. on } \Omega \right\}.$$

Let $f$ be a function in $L^2_{\text{loc}}(\mathbb{R}_+; L^2(\Omega))$ and $h$ be a non-negative function on $\mathbb{R}_+ \times \mathbb{R}$. Then for given $b \in L^\infty(\Omega)^N$ we consider an interior asymptotically $T_0$-periodic double obstacle problem $(\text{OP})^{AP}_s (s \in \mathbb{R}_+):$

- Find functions $v \in C((s, +\infty); L^2(\Omega))$ and $\theta \in L^2_{\text{loc}}((s, +\infty); L^2(\Omega))$ such that

$$\begin{align*}
(\text{OP})^{AP}_s \left\{ \begin{array}{l}
\forall v \in L^q_{\text{loc}}((s, +\infty); W^{1,q}(\Omega)) \cap W^{1,2}_{\text{loc}}((s, +\infty); L^2(\Omega)) ; \\
v(t) \in K(t) \text{ for a.e. } t \geq s ; \\
0 \leq \theta(t, x) \leq h(t, v(t, x)) \quad \text{a.e. on } (s, +\infty) \times \Omega ; \\
(v'(t) + \theta(t) + b \cdot \nabla v(t) - f(t), v(t) - z) + a_q(v(t), v(t) - z) \leq 0 \\
\text{for any } z \in K(t) \text{ and a.e. } t \geq s .
\end{array} \right. 
\end{align*}$$
The main object of this section is to consider the large-time behaviour of solution for \((OP)^{AP}_s\) assuming asymptotically \(T_0\)-periodicity conditions

\[
\sigma_i(t) - \sigma_{i,p}(t) \to 0 \quad (i = 0, 1), \quad h(t, \cdot) - h_p(t, \cdot) \to 0, \quad f(t) - f_p(t) \to 0
\]
as \(t \to \infty\) in the sense specified below, where \(\sigma_{i,p}(t), h_p(t, \cdot), f_p(t)\) are periodic in time with the same period \(T_0\). By the above assumptions, the limiting system of \((OP)^{AP}_s\) is a \(T_0\)-periodic one \((OP)^P_s\) as follows:

- Find functions \(u \in C((s, +\infty); L^2(\Omega))\) and \(\theta \in L^2_{\text{loc}}((s, +\infty); L^2(\Omega))\) such that

\[
(\text{OP})^P_s \quad \begin{cases}
  u \in L^2_{\text{loc}}((s, +\infty); W^{1,2}(\Omega)) \cap W^{1,2}_{\text{loc}}((s, +\infty); L^2(\Omega)); \\
  u(t) \in K_p(t) \text{ for a.e. } t \geq s; \\
  0 \leq \theta(t, x) \leq h_p(t, u(t, x)) \quad \text{a.e. on } (s, +\infty) \times \Omega; \\
  (u'(t) + \theta(t) + b \cdot \nabla u(t) - f_p(t), u(t) - z) + a_q(u(t), u(t) - z) \leq 0 \\
  \text{for any } z \in K_p(t) \text{ and a.e. } t \geq s,
\end{cases}
\]

where \(K_p(t) := \{ z \in W^{1,2}(\Omega); \sigma_{0,p}(t, \cdot) \leq z \leq \sigma_{1,p}(t, \cdot) \text{ a.e. on } \Omega \}\).

Now we suppose the following conditions:

- \(\sigma_i\) and \(\sigma_{i,p}\) are functions on \(R_+ \times \Omega\) such that

\[
\sup_{t \in R_+} \left| \frac{d\sigma_i}{dt} \right|_{L^2(t,t+1; W^{1,2}(\Omega))} + \sup_{t \in R_+} \left| \frac{d\sigma_{i,p}}{dt} \right|_{L^2(t,t+1; L^\infty(\Omega))} < +\infty,
\]

\[
\sup_{t \in R_+} \left| \frac{d\sigma_{i,p}}{dt} \right|_{L^2(t,t+1; W^{1,2}(\Omega))} + \sup_{t \in R_+} \left| \frac{d\sigma_{i,p}}{dt} \right|_{L^2(t,t+1; L^\infty(\Omega))} < +\infty
\]

and \(\sigma_{i,p}\) is a \(T_0\)-periodic obstacle function, i.e.

\[
\sigma_{i,p}(t + T_0, x) = \sigma_{i,p}(t, x) \quad \text{for a.e. } x \in \Omega \text{ and any } t \in R_+
\]

for \(i = 0, 1\). Moreover, there are positive constants \(k_1 > 0\) and \(k_2 > 0\) such that

\[
\sigma_1 - \sigma_0 \geq k_1 \quad \text{and} \quad \sigma_{1,p} - \sigma_{0,p} \geq k_1 \quad \text{a.e. on } R_+ \times \Omega
\]

and

\[
|\sigma_i|_{L^\infty(R_+; W^{1,2}(\Omega))} + |\sigma_i|_{L^\infty(R_+ \times \Omega)} + |\sigma_{i,p}|_{L^\infty(R_+; W^{1,2}(\Omega))} + |\sigma_{i,p}|_{L^\infty(R_+ \times \Omega)} \leq k_2
\]

for \(i = 0, 1, 2\).

- \(h\) and \(h_p\) are non-negative continuous functions on \(R_+ \times R\). There is a positive constant \(L\) such that

\[
|h(t, z_1) - h(t, z_2)| \leq L|z_1 - z_2| \quad |h_p(t, z_1) - h_p(t, z_2)| \leq L|z_1 - z_2|
\]

for all \(t \in R_+, z_i \in R\) and \(i = 1, 2\). Moreover, \(h_p\) is a \(T_0\)-periodic function, i.e. for any \(z \in R\), \(h_p(t + T_0, z) = h_p(t, z)\) for any \(t \in R_+\).
• $f, f_p \in L^2_{loc}(R_+; L^2(\Omega))$, and $f_p$ is a $T_0$-periodic function, i.e.

$$f_p(t + T_0) = f_p(t) \quad \text{in} \ L^2(\Omega), \quad \forall t \in R_+.$$  

Moreover, we suppose the following convergence conditions:

• (Convergence of $\sigma_i(t) - \sigma_{i,p}(t) \to 0$ as $t \to +\infty$) Put

$$I_m := \sup_{t \in [0,T_0]} |\sigma_0(mT_0 + t) - \sigma_{0,p}(t)|_{W^{1,q}(\Omega)} + \sup_{t \in [0,T_0]} |\sigma_1(mT_0 + t) - \sigma_{1,p}(t)|_{W^{1,q}(\Omega)}$$

$$+ \sup_{t \in [0,T_0]} |\sigma_0(mT_0 + t) - \sigma_{0,p}(t)|_{L^\infty(\Omega)} + \sup_{t \in [0,T_0]} |\sigma_1(mT_0 + t) - \sigma_{1,p}(t)|_{L^\infty(\Omega)}$$

Then,

$$I_m \to 0 \quad \text{as} \ m \to +\infty.$$  

• (Convergence of $h(t, \cdot) - h_p(t, \cdot) \to 0$ as $t \to +\infty$) For any $z \in R$,

$$\sup_{t \in [0,T_0]} |h(mT_0 + t, z) - h_p(t, z)| \to 0 \quad \text{as} \ m \to +\infty; \quad (5.1)$$

• (Convergence of $f(t) - f_p(t) \to 0$ as $t \to +\infty$)

$$|f(mT_0 + \cdot) - f_p|_{L^2(0,T_0;L^2(\Omega))} \to 0 \quad \text{as} \ m \to +\infty. \quad (5.2)$$

Under the above assumptions, let us consider problems $(OP)^s_{AP}$ and $(OP)^s_P$.

In order to apply the abstract results in Sections 2-4, we choose $L^2(\Omega)$ as a real separable Hilbert space $H$. And we define a family $\{\varphi^t\}$ of proper l.s.c. convex functions $\varphi^t$ on $L^2(\Omega)$ by

$$\varphi^t(z) = \begin{cases} 
\frac{1}{q} \int_{\Omega} |\nabla z|^q dx & \text{if } z \in K(t), \\
\infty & \text{if } z \in L^2(\Omega) \setminus K(t),
\end{cases} \quad (5.3)$$

and define $\varphi^t_p$ by replacing $K(t)$ by $K_p(t)$ in (5.3).

Also, we define a multivalued operator $G(\cdot, \cdot)$ from $R_+ \times H^1(\Omega)$ into $L^2(\Omega)$ by

$$G(t, z) := \begin{cases} 
g \in L^2(\Omega); \\
g = l + b \cdot \nabla z & \text{in} \ L^2(\Omega) \\
0 \leq l(x) \leq h(t, z(x)) & \text{a.e. on} \ \Omega
\end{cases} \quad (5.4)$$

for all $t \in R_+$ and $z \in H^1(\Omega)$. And we define $G_p(\cdot, \cdot)$ by replacing $h(t, \cdot)$ by $h_p(t, \cdot)$ in (5.4).

By the same argument in [27, Lemma 5.1], we can get the following lemmas.

**Lemma 5.1.** (cf. [27, Lemma 5.1]) Put for any $r > 0$ and $t \in R_+$

$$a_r(t) = b_r(t) := k_3 \int_0^t \left\{|\sigma_{0,p}'|_{L^\infty(\Omega)} + |\sigma_{0,p}'|_{W^{1,\gamma}(\Omega)} + |\sigma_{1,p}'|_{L^\infty(\Omega)} + |\sigma_{1,p}'|_{W^{1,\gamma}(\Omega)} \right\} d\tau$$
where $k_3$ is a (sufficiently large) positive constant. Then, $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$ and $\{\varphi^t_p\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0)$.

Moreover we have $\{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^t\})$ and $\{G_p(t, \cdot)\} \in \mathcal{G}_p(\{\varphi^t_p\}; T_0)$.

**Lemma 5.2.** The convergence assumptions (A1)-(A3) hold.

**Proof.** We easily see that (A2) and (A3) hold by assumptions (5.1) and (5.2).

Now let us show (A1). For each $t \in R_+$ there are $m \in Z_+$ and $\tau \in [0, T_0]$ so that $t = mT_0 + \tau$.

For each $z_p \in D(\varphi^t_p) = K_p(t)$, we put

$$z := (z_p - \sigma_{0,p}(t)) \frac{\sigma_1(t) - \sigma_0(t)}{\sigma_{1,p}(t) - \sigma_{0,p}(t)} + \sigma_0(t).$$

Then we easily see that $z \in D(\varphi^t) = K(t)$. Moreover, by the same argument in [27, Lemma 5.1], we see that

$$|z - z_p|_{L^2(\Omega)} \leq k_4 I_m \quad \text{and} \quad |\nabla z - \nabla z_p|_{L^q(\Omega)} \leq k_4 I_m (1 + |\nabla z_p|_{L^q(\Omega)}) \quad (5.5)$$

for some constant $k_4 > 0$. Hence we have

$$\varphi^t(z) - \varphi^t_p(z_p) \leq k_5 I_m (1 + \varphi^t_p(z_p)), \quad (5.6)$$

for a sufficiently large $k_5 > 0$.

Conversely, let $z \in D(\varphi^t) = K(t)$ and we put

$$z_p := (z - \sigma_0(t)) \frac{\sigma_{1,p}(t) - \sigma_{0,p}(t)}{\sigma_1(t) - \sigma_0(t)} + \sigma_{0,p}(t).$$

Then, we observe that $z_p \in D(\varphi^t_p) = K_p(t)$ and

$$|z_p - z|_{L^2(\Omega)} \leq k_4 I_m \quad \text{and} \quad \varphi^t_p(z_p) - \varphi^t(z) \leq k_5 I_m (1 + \varphi^t(z)). \quad (5.7)$$

Therefore by (5.5)-(5.7) we see that the convergence assumption (A1) holds. \qed

Clearly, the obstacle problem $(\text{OP})^p_s$ can be reformulated as an evolution equation $(\text{AP})_s$ involving the subdifferential of $\varphi^t$ given by (5.3) and the multivalued operator $G(t, \cdot)$ defined by (5.4). Also, the limiting $T_0$-periodic problem $(\text{OP})^p_0$ can be reformulated as an evolution equation (P)$_0$. Therefore, by Lemmas 5.1-5.2 we can apply abstract results in Section 2-4. Namely, we can obtain an attractor $\mathcal{A}^*_s$ for $(\text{OP})^p_0$, a $T_0$-periodic attractor $\mathcal{A}_s$ for $(\text{OP})^p_s$ and the relationships between $(\text{OP})^p_s$ and $(\text{OP})^p_0$.

Additionally, we assume that $f(t) \equiv f_p(t)$ for any $t \in R_+$ and

$$\sigma_0(t, z) \equiv \sigma_{0,p}(t, z), \quad \sigma_{1,p}(t, z) \equiv \sigma_1(t, z), \quad h_p(t, z) \leq h(t, z)$$
for any $0 \leq t < +\infty$ and $z \in R$. Then we easily see that the assumptions of Theorem 4.4 and its Corollary hold. Hence we can get $\mathcal{A}_s^t = \mathcal{A}_s$ by the same argument in [30, Theorem 5.4].

Unfortunately we do not give assumptions for $\sigma_i(t, \cdot), h(t, \cdot)$ and $f(t)$ in order to get

$$U(\tau, s)\mathcal{A}_s^t = \mathcal{A}_t^s \subset \mathcal{A}_t$$ for any $0 \leq s \leq \tau < +\infty.$ \quad (5.8)

It seems difficult to show (5.8), so it is the open problem.

References


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