ATTRACTORS OF ASYMPTOTICALLY PERIODIC MULTIVALUED DYNAMICAL SYSTEMS GOVERNED BY TIME-DEPENDENT SUBDIFFERENTIALS

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Abstract. Let us consider a nonlinear evolution equation associated with time-dependent subdifferential in a separable Hilbert space. In this paper we treat an asymptotically periodic system which means that time-dependent terms converge to some time-periodic ones as time goes to $+\infty$. Then we consider the large-time behaviour of solutions without uniqueness. In such a situation the corresponding dynamical systems are multivalued. In fact we discuss the stability of multivalued semiflows from the viewpoint of attractors. Namely, the main object of this paper is to construct a global attractor for the asymptotically periodic multivalued dynamical system, and to discuss the relationship to one for the limiting periodic system.

1 Introduction

In this paper let us consider a non-autonomous system in a real separable Hilbert space $H$ of the form

$$v'(t) + \partial \varphi_t(v(t)) + G(t, v(t)) \ni f(t) \quad \text{in} \ H, \quad t > s \ (\geq 0), \quad (1.1)$$

where $v' = \frac{dv}{dt}$, $\partial \varphi_t$ is a subdifferential of time-dependent proper lower semicontinuous (l.s.c.) convex function $\varphi^t$ on $H$, $G(t, \cdot)$ is a multivalued perturbation small relative to $\varphi^t$, and $f$ is a forcing term.

In the case when $G(t, \cdot) \equiv 0$, many mathematicians studied the existence-uniqueness, asymptotic stability, time periodic and almost periodic problem for (1.1) (cf. [7], [8], [13], [14], [15], [16], [18], [23], [24]).

For the multivalued nonmonotone perturbation $G(t, \cdot)$, Ôtani has already shown the existence of solution for (1.1) in [21]. The large-time behavior of solutions for (1.1) was...
discussed by [28] from the view-point of attractors. For the time periodic case, assuming
the periodicity conditions with same period $T_0$, $0 < T_0 < +\infty$, i.e.
\[ \varphi^t = \varphi^{t+T_0}, \quad G(t, \cdot) = G(t + T_0, \cdot), \quad f(t) = f(t + T_0), \quad \forall t \in \mathbb{R}_+ := [0, \infty), \]
the existence of periodic solution for (1.1) was proved in [22]. Moreover, the periodic sta-
tility was discussed in [29]. In fact, the author showed the existence and characterization
of time-periodic global attractors for (1.1).

In this paper, for a given positive number $T_0 > 0$ let us treat the case when $\varphi^t, G(t, \cdot)$
and $f(t)$ are asymptotically $T_0$-periodic in time. Namely we assume that
\[ \varphi^t - \varphi^t_p \longrightarrow 0, \quad G(t, \cdot) - G_p(t, \cdot) \longrightarrow 0, \quad f(t) - f_p(t) \longrightarrow 0 \quad (1.2) \]
in appropriate senses as $t \to +\infty$, where $\varphi^t_p = \varphi^{t+T_0}_p$, $G_p(t, \cdot) = G_p(t + T_0, \cdot)$ and $f_p(t) = f_p(t + T_0)$ for any $t \in \mathbb{R}_+$. By the asymptotically $T_0$-periodic stability (1.2), we have the
limiting $T_0$-periodic system for (1.1) of the form:
\[ u'(t) + \partial \varphi^t_p(u(t)) + G_p(t, u(t)) \ni f_p(t) \quad \text{in} \ H, \quad t > s \ (\geq 0). \quad (1.3) \]
In the case when $G(t, \cdot)$ and $G_p(t, \cdot)$ are single-valued, the asymptotically $T_0$-periodic
problem has already been discussed in [11]. In order to guarantee the uniqueness of
solutions for the Cauchy problem of (1.1) and (1.3), they assumed some conditions on
$\varphi^t, \varphi^t_p, G(t, \cdot)$ and $G_p(t, \cdot)$. Then, they discussed the asymptotically $T_0$-periodic stability
for (1.1) from the view-point of attractors (cf. [11]). The main object of this paper is
to develop the result obtained in [11] in order to consider the large-time behaviour of
solution for (1.1) without uniqueness. Namely, we would like to construct the attractor
for the asymptotically $T_0$-periodic multivalued flows associated with (1.1). Moreover we
shall discuss the relationship to the $T_0$-periodic attractor for (1.3) obtained in [29].

In the next Section 2, we recall the known results for the Cauchy problem of (1.1). In
Section 3 we consider the limiting $T_0$-periodic problem (1.3) and recall the abstract results
obtained in [29]. In Section 4, we introduce the notion of a metric topology on the family
$\{ \varphi^t; t \geq 0 \}$ which was constructed in [16]. And we present and prove the main results in
this paper. In proving main results, we generalize the results obtained in [11] and [30].
In the final section we apply our abstract results to the parabolic variational inequality
with asymptotically $T_0$-periodic double obstacles. Then we can discuss the asymptotic
stability for the asymptotically $T_0$-periodic double obstacle problem without uniqueness
of solutions.

**Notation.** Throughout this paper, let $H$ be a (real) separable Hilbert space with norm
$| \cdot |_H$ and inner product $(\cdot, \cdot)_H$. For a proper l.s.c. convex function $\varphi$ on $H$ we use the
notation $D(\varphi), \partial \varphi$ and $D(\partial \varphi)$ to indicate the effective domain, subdifferential and its
domain of $\varphi$, respectively; for their precise definitions and basic properties see [4].

For two non-empty sets $A$ and $B$ in $H$, we define the so-called Hausdorff semi-distance
\[ \text{dist}_H(A, B) := \sup_{x \in A} \inf_{y \in B} |x - y|_H. \]
2 Preliminaries

In this section let us recall the known results for a nonlinear evolution equation in $H$ of the form:

$$u'(t) + \partial \varphi^t(u(t)) + G(t, u(t)) \ni f(t) \quad \text{in} \; H, \quad t \in J,$$

where $J$ is an interval in $R_+$, $\partial \varphi^t$ is the subdifferential of a time-dependent proper l.s.c. and convex function $\varphi^t$ on $H$, $G(t, \cdot)$ is a multivalued operator from a subset $D(G(t, \cdot)) \subset H$ into $H$ for each $t \in R_+$ and $f$ is a given function in $L^1_{loc}(J; H)$.

We begin with the definition of solution for (2.1).

**Definition 2.1.** (i) For a compact interval $J := [t_0, t_1] \subset R_+$ and $f \in L^2(J; H)$, a function $u : J \to H$ is called a solution of (2.1) on $J$, if $u \in C(J; H) \cap W^{1,2}_{loc}((t_0, t_1]; H)$, $\varphi^t(u(t)) \in L^1(J)$, $u(t) \in D(\partial \varphi^t)$ for a.e. $t \in J$, and if there exists a function $g \in L^1_{loc}(J; H)$ such that $g(t) \in G(t, u(t))$ for a.e. $t \in J$ and

$$f(t) - g(t) - u'(t) \in \partial \varphi^t(u(t)), \quad \text{a.e.} \; t \in J.$$  

(ii) For any interval $J$ in $R_+$ and $f \in L^2_{loc}(J; H)$, a function $u : J \to H$ is called a solution of (2.1) on $J$, if it is a solution of (2.1) on every compact subinterval of $J$ in the sense of (i).

(iii) Let $J$ be any interval in $R_+$ with initial time $s \in R_+$. For $f \in L^2_{loc}(J; H)$, a function $u : J \to H$ is called a solution of the Cauchy problem for (2.1) on $J$ with given initial value $u_0 \in H$, if it is a solution of (2.1) on $J$ satisfying $u(s) = u_0$.

Throughout this paper, let $\{a_r\} := \{a_r; r \geq 0\}$ and $\{b_r\} := \{b_r; r \geq 0\}$ be families of real functions in $W^{1,2}_{loc}(R_+)$ and $W^{1,1}_{loc}(R_+)$, respectively, such that

$$\sup_{t \in R_+} |a'_r|_{L^2(t,t+1)} + \sup_{t \in R_+} |b'_r|_{L^1(t,t+1)} < +\infty \quad \text{for each} \; r \geq 0.$$  

Now we define the class $\Phi(\{a_r\}, \{b_r\})$ of time-dependent convex function $\varphi^t$.

**Definition 2.2.** $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$ if and only if $\varphi^t$ is a proper l.s.c. convex function on $H$ satisfying the following properties ($\Phi1$)-($\Phi3$):

($\Phi1$) For each $r > 0$, $s, \; t \in R_+$ and $z \in D(\varphi^s)$ with $|z|_H \leq r$, there exists $\tilde{z} \in D(\varphi^t)$ such that

$$|\tilde{z} - z|_H \leq |a_r(t) - a_r(s)|(1 + |\varphi^s(z)|^{\frac{1}{r}})$$

and

$$\varphi^t(\tilde{z}) - \varphi^s(z) \leq |b_r(t) - b_r(s)|(1 + |\varphi^s(z)|).$$

($\Phi2$) There exists a positive constant $C_1 > 0$ such that

$$\varphi^t(z) \geq C_1 |z|_H^2, \quad \forall t \in R_+, \; \forall z \in D(\varphi^t).$$

($\Phi3$) For each $k > 0$ and $t \in R_+$, the level set $\{z \in H; \varphi^t(z) \leq k\}$ is compact in $H$.  

3
Next, we introduce the class $G(\{\varphi^t\})$ of time-dependent multivalued perturbation $G(t, \cdot)$ associated with $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$.

**Definition 2.3.** $G(t, \cdot) \in G(\{\varphi^t\})$ if and only if $G(t, \cdot)$ is a multivalued operator from $D(G(t, \cdot)) \subset H$ into $H$ which fulfills the following conditions (G1)-(G5):

**G1** $D(\varphi^t) \subset D(G(t, \cdot)) \subset H$ for any $t \in R_+$. And for any interval $J \subset R_+$ and $v \in L^2_{\text{loc}}(J; H)$ with $v(t) \in D(\varphi^t)$ for a.e. $t \in J$, there exists a strongly measurable function $g(\cdot)$ on $J$ such that

$$g(t) \in G(t, v(t)) \text{ for a.e. } t \in J.$$ 

**G2** $G(t, z)$ is a convex subset of $H$ for any $z \in D(\varphi^t)$ and $t \in R_+$.

**G3** There are positive constants $C_2, C_3$ such that

$$|g|^2_H \leq C_2 \varphi^t(z) + C_3, \quad \forall t \in R_+, \forall z \in D(\varphi^t), \forall g \in G(t, z).$$

**G4** (demi-closedness) If $z_n \in D(\varphi^{t_n})$, $g_n \in G(t_n, z_n)$, $\{t_n\} \subset R_+$, $\{\varphi^{t_n}(z_n)\}$ is bounded, $z_n \rightarrow z$ in $H$, $t_n \rightarrow t$ and $g_n \rightarrow g$ weakly in $H$ as $n \rightarrow +\infty$, then $g \in G(t, z)$.

**G5** For each bounded subset $B$ of $H$, there exist positive constants $C_4(B)$ and $C_5(B)$ such that

$$\varphi^t(z) + (g, z - b)_H \geq C_4(B)|z|^2_H - C_5(B),$$

$$\forall t \in R_+, \forall g \in G(t, z), \forall z \in D(\varphi^t), \forall b \in B.$$ 

For given $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$, $\{G(t, \cdot)\} \in G(\{\varphi^t\})$ and a forcing term $f \in L^2_{\text{loc}}(R_+; H)$, we consider the following evolution equation

$$u'(t) + \partial \varphi^t(u(t)) + G(t, u(t)) \ni f(t) \quad \text{in } H, \quad t > s$$

for each $s \in R_+$.

Now let us recall the known results on the existence and global estimates of solutions for the Cauchy problem of $(E)_s$:

**A** [Existence of solution for $(E)_s$] (cf. [21, Theorem II, III])

The Cauchy problem for $(E)_s$ has at least one solution $u$ on $J = [s, +\infty)$ such that $(\cdot - s)^{\frac{1}{2}} u' \in L^2_{\text{loc}}(J; H)$, $(\cdot - s)\varphi^{(1)}(u(\cdot)) \in L^\infty_{\text{loc}}(J)$ and $\varphi^{(1)}(u(\cdot))$ is absolutely continuous on any compact subinterval of $(s, +\infty)$, provided that $u_0 \in D(\varphi^s)$. In particular, if $u_0 \in D(\varphi^s)$, then the solution $u$ satisfies that $u' \in L^2_{\text{loc}}(J; H)$ and $\varphi^{(1)}(u(\cdot))$ is absolutely continuous on any compact interval in $J$.

**B** [Global boundedness of solutions for $(E)_s$] (cf. [25, Theorem 2.2])

Suppose that

$$S_f := \sup_{t \in R_+} |f|_{L^2(t, t+1; H)} < +\infty.$$
Then, the solution $u$ of the Cauchy problem for $(E)_s$ on $[s, +\infty)$ satisfies the following global estimate:

$$
\sup_{t \geq s} |u(t)|_H^2 + \sup_{t \geq s} \int_{t}^{t+1} \varphi'(u(\tau)) d\tau \leq N_1 (1 + S_f^2 + |u_0|_H^2),
$$

where $N_1$ is a positive constant independent of $f$, $s \in \mathbb{R}_+$ and $u_0 \in \overline{D(\varphi^*)}$. Moreover, for each $\delta > 0$ and each bounded subset $B$ of $H$, there is a constant $N_2(\delta, B) > 0$, depending only on $\delta > 0$ and $B$, such that

$$
\sup_{t \geq s+\delta} |u'|_{L^2(t, t+1; H)}^2 + \sup_{t \geq s+\delta} \varphi'(u(t)) \leq N_2(\delta, B)
$$

for the solution $u$ of the Cauchy problem for $(E)_s$ on $[s, +\infty)$ with $s \in \mathbb{R}_+$ and $u_0 \in \overline{D(\varphi^*)} \cap B$.

Next, let us remember a notion of convergence of convex functions.

**Definition 2.4.** (cf. [20]) Let $\psi, \psi_n (n \in \mathbb{N})$ be proper l.s.c. and convex functions on $H$. Then we say that $\psi_n$ converges to $\psi$ on $H$ as $n \to +\infty$ in the sense of Mosco, if the following two conditions (i) and (ii) are satisfied:

(i) for any subsequence $\{\psi_{n_k}\} \subset \{\psi_n\}$, if $z_k \to z$ weakly in $H$ as $k \to +\infty$, then

$$
\liminf_{k \to +\infty} \psi_{n_k}(z_k) \geq \psi(z).
$$

(ii) for any $z \in D(\psi)$, there is a sequence $\{z_n\}$ in $H$ such that

$$
z_n \to z \text{ in } H \text{ as } n \to +\infty, \quad \lim_{n \to +\infty} \psi_n(z_n) = \psi(z).
$$

Now, we recall a convergence result (cf. [25, Lemma 4.1]) as follows.

**(C)** Let $\{\varphi^t_n\} \in \Phi(\{a_r\}, \{b_r\})$, $\{G_n(t, \cdot)\} \in \mathcal{G}(\{\varphi^t_n\})$ with common positive constants $C_1, C_2, C_3, C_4(B)$ and $C_5(B)$, $\{f_n\} \subset L^2(J; H)$, $J = [s, t_1] \subset \mathbb{R}_+$ and $u_{0, n} \in \overline{D(\varphi^*_n)}$ for $n = 1, 2, \cdots$. Assume that

(i) $\varphi^t_n$ converges to $\varphi^t$ on $H$ in the sense of Mosco [20] for each $t \in J$ (as $n \to +\infty$) and $\bigcup_{n=1}^{+\infty} \{z \in H; \varphi^t_n(z) \leq k\}$ is relatively compact in $H$ for every real $k > 0$ and $t \in J$, where $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$ and $\varphi^t_n = \varphi^t$ if $n = +\infty$.

(ii) if $z_n \in D(\varphi^*_n)$, $g_n \in G_n(t_n, z_n)$, $\{t_n\} \subset \mathbb{R}_+$, $\{\varphi^*_n(z_n)\}$ is bounded, $z_n \to z$ in $H$, $t_n \to t$ and $g_n \to g$ weakly in $H$ as $n \to +\infty$, then $g \in G(t, z)$, where $\{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^t\})$.

(iii) $f_n \to f$ weakly in $L^2(J; H)$ for some $f \in L^2(J; H)$ and $u_{0, n} \to u_0$ in $H$ for some $u_0 \in \overline{D(\varphi^*)}$.  

5
Denote by $u$ the solution of the Cauchy problem for $(E)_s$ on $J$ with $u(s) = u_0$ and by $u_n$ the solution of the Cauchy problem for $(E)_s$ with $\varphi^t$, $G$, $f$ replaced by $\varphi^t_n$, $G_n$, $f_n$, and with $u_n(s) = u_{0,n}$. Then $u_n$ converges to $u$ on $J$ in the sense that

$$u_n \to u \text{ in } C(J; H), \quad (s - s)^{\frac{1}{2}} u'_n \to (s - s)^{\frac{1}{2}} u' \text{ weakly in } L^2(J; H),$$

$$\int_J \varphi^t_n(u_n(t)) dt \to \int_J \varphi^t(u(t)) dt \quad \text{as } n \to +\infty.$$

### 3 Attractor for periodic multivalued dynamical system

In this section let us recall the known results obtained in [29] for a $T_0$-periodic system in $H$, of the form:

$$(P)_s \quad u'(t) + \partial \varphi^t_p(u(t)) + G_p(t, u(t)) \ni f_p(t) \quad \text{in } H, \quad t > s$$

for each $s \in R_+$, where $\varphi^t_p$, $G_p(t, \cdot)$ and $f_p(t)$ are $T_0$-periodic, namely periodic in time with the same period $T_0$, $0 < T_0 < +\infty$.

**Definition 3.1.** Let $T_0$ be a positive number. Then

(i) $\Phi_p(\{a_r\}, \{b_r\}; T_0)$ is the set of all $\{\varphi^t_p\} \in \Phi(\{a_r\}, \{b_r\})$ satisfying $T_0$-periodicity condition:

$$\varphi^{t+T_0}_p(\cdot) = \varphi^t_p(\cdot) \quad \text{on } H, \quad \forall t \in R_+. \quad (3.1)$$

(ii) $\mathcal{G}_p(\{\varphi^t_p\}; T_0)$ is the set of all $\{G_p(t, \cdot)\} \in \mathcal{G}(\{\varphi^t_p\})$ satisfying $T_0$-periodicity condition:

$$G_p(t + T_0, \cdot) = G_p(t, \cdot) \quad \text{in } H, \quad \forall t \in R_+. \quad (3.2)$$

Throughout this section we assume that $\{\varphi^t_p\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0)$, $\{G_p(t, \cdot)\} \in \mathcal{G}_p(\{\varphi^t_p\}; T_0)$ and $f_p \in L^2_{loc}(R_+; H)$ is $T_0$-periodic in time, namely

$$f_p(t + T_0) = f_p(t) \quad \text{in } H, \quad \forall t \in R_+. \quad (3.3)$$

Here we note that $(P)_s$ can be considered as $(E)_s$ in Section 2. So, by the result (A) in Section 2, the Cauchy problem for $(P)_s$ has at least one solution $u$ on $[s, +\infty)$. Hence we can define the multivalued dynamical process associated with $(P)_s$ as follows:

**Definition 3.2.** For every $0 \leq s \leq t < +\infty$ we denote by $U(t, s)$ the mapping from $\overline{D}(\varphi^t_p)$ into $\overline{D}(\varphi^t_p)$ which assigns to each $u_0 \in \overline{D}(\varphi^t_p)$ the set

$$U(t, s)u_0 := \left\{ z \in H \left| \begin{array}{l}
\text{There is a solution } u \text{ of } (P)_s \text{ on } [s, +\infty) \\
\text{such that } \quad u(s) = u_0 \text{ and } u(t) = z.
\end{array} \right. \right\}. \quad (3.4)$$

Then we easily see the following properties of $\{U(t, s)\} := \{U(t, s); 0 \leq s \leq t < +\infty\}$:
(U1) $U(s, s) = I$ on $D(\varphi_p^\tau)$ for any $s \in R_+$;

(U2) $U(t_2, s)z = U(t_2, t_1)U(t_1, s)z$ for any $0 \leq s \leq t_1 \leq t_2 < +\infty$ and $z \in D(\varphi_p^\tau)$;

(U3) $U(t + T_0, s + T_0)z = U(t, s)z$ for any $0 \leq s \leq t < +\infty$ and $z \in D(\varphi_p^{*\tau})$, that is, $U$ is $T_0$-periodic.

(U4) $\{U(t, s)\}$ has the following demi-closedness:

- If $0 \leq s_n \leq t_n < +\infty$, $s_n \to s$, $t_n \to t$, $z_n \in D(\varphi_p^{\tau n})$, $z \in D(\varphi_p^\tau)$, $z_n \to z$ in $H$
- and a element $w_n \in U(t_n, s_n)z_n$ converges to some element $w \in H$ as $n \to +\infty$.

Next we define the discrete dynamical system in order to construct a global attractor for (P)$_s$.

**Definition 3.3.** Let $U(\cdot, \cdot)$ be the solution operator for (P)$_s$ defined by Definition 3.2. Then

(i) For each $\tau \in R_+$, we denote by $U_\tau$ the $T_0$-step mapping from $D(\varphi_p^\tau)$ into $D(\varphi_p^{*\tau + T_0}) = D(\varphi_p^\tau)$, namely,

\[
U_\tau := U(\tau + T_0, \tau).
\]

(ii) For any $k \in Z_+: = N \cup \{0\}$, we define

\[
U_\tau^k := U_\tau \circ U_\tau \circ \cdots \circ U_\tau.
\]

Clearly we have $U_\tau^k = U(\tau + kT_0, \tau)$ for any $\tau \in R_+$ and $k \in Z_+$.

Now, let us recall the known result on the existence of global attractors for discrete multivalued dynamical systems $U_\tau$ associated with (P)$_s$.

**Theorem 3.1.** (cf. [29, Theorem 3.1]) Assume that $\{\varphi_p^\tau\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0)$, $\{G_p(t, \cdot)\} \in G_p(\{\varphi_p^\tau\}; T_0)$, $f_p \in L^2_{loc}(R_+; H)$ satisfies the $T_0$-periodicity condition (3.3). Then, for each $\tau \in R_+$, there exists a subset $\mathcal{A}_\tau$ of $D(\varphi_p^\tau)$ such that

(i) $\mathcal{A}_\tau$ is non-empty and compact in $H$;

(ii) for each bounded set $B$ in $H$ and each number $\epsilon > 0$ there exists $N_{B, \epsilon} \in N$ such that

\[
\text{dist}_H(U_\tau^k z, \mathcal{A}_\tau) < \epsilon
\]

for all $z \in D(\varphi_p^\tau) \cap B$ and all $k \geq N_{B, \epsilon}$;

(iii) $U_\tau^k \mathcal{A}_\tau = \mathcal{A}_\tau$ for any $k \in N$.

**Remark 3.1.** By [29, Lemma 3.1] we can get the compact absorbing set $B_0, \tau$ of $D(\varphi_p^\tau)$ for $U_\tau$ such that for each bounded subset $B$ of $H$ there is a positive integer $n_B$ (independent of $\tau \in R_+$) satisfying

\[
U_\tau^n (D(\varphi_p^\tau) \cap B) \subset B_0, \tau \quad \text{for all } n \geq n_B.
\]
Then we observe that the global attractor $\mathcal{A}_\tau$ is given by the $\omega$-limit set of the absorbing set $B_{0,\tau}$ for $U_\tau$, i.e.

$$\mathcal{A}_\tau = \bigcap_{n \in \mathbb{Z}_+} \bigcup_{k \geq n} U^k \overline{B_{0,\tau}}.$$ 

The next theorem is concerned with a relationship between two global attractors $\mathcal{A}_s$ and $\mathcal{A}_\tau$. For detail proof, see [29].

**Theorem 3.2.** (cf. [29, Theorem 3.2]) Suppose the same assumptions are made as in Theorem 3.1. Let $\mathcal{A}_s$ and $\mathcal{A}_\tau$ be the global attractors for $U_s$ and $U_\tau$, with $0 \leq s \leq \tau \leq T_0$, respectively. Then, we have

$$\mathcal{A}_\tau = U(\tau, s) \mathcal{A}_s,$$

where $U(\tau, s)$ is the $T_0$-periodic process given in Definition 3.2.

**Remark 3.2.** By Theorem 3.1 (iii) and Theorem 3.2, we see that the global attractor $\mathcal{A}_\tau$ for $U_\tau$ is $T_0$-periodic in $\tau$. In fact, for each $\tau \in \mathbb{R}_+$ choose $m_\tau \in \mathbb{Z}_+$ and $\sigma_\tau \in [0, T_0)$ so that $\tau = \sigma_\tau + m_\tau T_0$. Then, we have $\mathcal{A}_\tau = \mathcal{A}_{\sigma_\tau}$.

The third known result is the existence of a global attractor for the $T_0$-periodic multivalued dynamical system $(P)_s$.

**Theorem 3.3.** (cf. [29, Theorem 3.3]) Under the same assumptions as Theorem 3.1, put

$$\mathcal{A} := \bigcup_{0 \leq \tau \leq T_0} \mathcal{A}_\tau,$$

where $\mathcal{A}_\tau$ is as obtained in Theorem 3.1. Then, $\mathcal{A}$ has the following properties:

(i) $\mathcal{A}$ is non-empty and compact in $H$;

(ii) for each bounded set $B$ in $H$ and each number $\epsilon > 0$ there exists a finite time $T_{B, \epsilon} > 0$ such that

$$\text{dist}_H(U(t + \tau, \tau)z, \mathcal{A}) < \epsilon$$

for all $\tau \in \mathbb{R}_+$, all $z \in D(\varphi_P^\tau) \cap B$ and all $t \geq T_{B, \epsilon}$.

**Remark 3.3.** In [29, Section 4] the characterization of the $T_0$-periodic global attractor was discussed. The author proved that for each time $\tau \in \mathbb{R}_+$ the global attractor $\mathcal{A}_\tau$ for the discrete multivalued dynamical system $U_\tau$ coincides with the cross-section of the family of all global bounded complete trajectories for the $T_0$-periodic system $(P)_s$.

### 4 Attractors of asymptotically periodic multivalued dynamical system

Throughout this section, let $M > 0$ be a fixed (sufficiently) large positive number. Now we put

$$\Psi_M := \left\{ \psi; \psi \text{ is proper, l.s.c. and convex on } H, \exists z \in D(\varphi_P^\tau) \text{ s.t. } |z|_H \leq M, \psi(z) \leq M \right\}.$$
Then let us introduce the notion of a metric topology on $\Psi_M$ which was introduced in [16].

Given $\varphi, \psi \in \Psi_M$, we define $\rho(\varphi, \psi; \cdot) : D(\varphi) \to \mathbb{R}$ by putting

$$
\rho(\varphi, \psi; z) = \inf\{\max(|y - z|_H, \psi(y) - \varphi(z)); y \in D(\psi)\}
$$

for each $z \in D(\varphi)$, and for each $r \geq M$

$$
\rho_r(\varphi, \psi) := \sup_{z \in L_\varphi(r)} \rho(\varphi, \psi; z),
$$

where $L_\varphi(r) := \{z \in D(\varphi); |z|_H \leq r, \varphi(z) \leq r\}$. Moreover, for each $r \geq M$, we define the functional $\pi_r(\cdot, \cdot)$ on $\Psi_M \times \Psi_M$ by

$$
\pi_r(\varphi, \psi) := \rho_r(\varphi, \psi) + \rho_r(\psi, \varphi) \quad \text{for } \varphi, \psi \in \Psi_M.
$$

Then, according to [16, Proposition 3.1], we can define a complete metric topology on $\Psi_M$ so that the convergence $\psi_n \to \psi$ in $\Psi_M$ (as $n \to +\infty$) if and only if

$$
\pi_r(\psi_n, \psi) \to 0 \quad \text{for every } r \geq M.
$$

Now by using the above topology on $\Psi_M$, we consider an asymptotically $T_0$-periodic system as follows.

**Definition 4.1.** Assume $\{\varphi^{'i}\} \in \Phi(\{a_r\}, \{b_r\}) \cap \Psi_M$, $\{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^{'i}\})$ and $f \in L^2_{\text{loc}}(R_+; H)$. Then the system

$$(AP)_s \quad v'(t) + \partial \varphi^{'i}(v(t)) + G(t, v(t)) \ni f(t) \quad \text{in } H, \ t > s \ (\geq 0)$$

is asymptotically $T_0$-periodic, if there are $\{\varphi^{'i}_p\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0) \cap \Psi_M$, $\{G_p(t, \cdot)\} \in \mathcal{G}_p(\{\varphi^{'i}_p\}; T_0)$ and a $T_0$-periodic function $f_p \in L^2_{\text{loc}}(R_+; H)$ such that

**A1** (Convergence of $\varphi^{'i} - \varphi^{'i}_p \to 0$ as $t \to +\infty$) For each $r \geq M$,

$$
J^{(r)}_m := \sup_{\sigma \in [0, T_0]} \pi_r(\varphi^{m T_0 + \sigma}, \varphi^{\sigma}) \to 0 \quad \text{as } m \to +\infty;
$$

**A2** (Convergence of $G(t, \cdot) - G_p(t, \cdot) \to 0$ as $t \to +\infty$) If $\{\tau_n\} \subset [0, T_0]$, $\{m_n\} \subset Z_+$, $m_n \to +\infty$, $z_n \in D(\varphi^{m_n T_0 + \tau_n})$, $g_n \in G(m_n T_0 + \tau_n, z_n)$, $\{\varphi^{m_n T_0 + \tau_n}(z_n)\}$ is bounded, $z_n \to z$ in $H$, $\tau_n \to \tau$ and $g_n \to g$ weakly in $H$ (as $n \to +\infty$), then

$$
g \in G_p(\tau, z);
$$

**A3** (Convergence of $f(t) - f_p(t) \to 0$ as $t \to +\infty$)

$$
|f(m T_0 + \cdot) - f_p|_{L^2(0, T_0; H)} \to 0 \quad \text{as } m \to +\infty.
$$
By Definition 4.1 we easily see that a limiting system for $\text{(AP)}_s$ is a $T_0$-periodic one
of the form:

$$
(P)_s \quad u'(t) + \partial \varphi_p'(u(t)) + G_p(t, u(t)) \ni f_p(t) \quad \text{in } H, \ t > s \ (\geq 0).
$$

Here we note that $\text{(AP)}_s$ is also considered as $\text{(E)}_s$. So, by the result (A) in Section 2, the Cauchy problem for $\text{(AP)}_s$ has at least one solution $v$ on $[s, +\infty)$. Hence we can define the multivalued dynamical system associated with $\text{(AP)}_s$ as follows:

**Definition 4.2.** For every $0 \leq s \leq t < +\infty$ we denote by $E(t, s)$ the mapping from $\overline{D(\varphi^s)}$ into $\overline{D(\varphi^t)}$ which assigns to each $v_0 \in \overline{D(\varphi^s)}$ the set

$$
E(t, s)v_0 := \left\{ \begin{array}{l}
\text{There is a solution } v \text{ of } \text{(AP)}_s \text{ on } [s, +\infty) \\
\text{such that}
\quad v(s) = v_0 \text{ and } v(t) = z.
\end{array} \right\}.
$$

Then we easily see that $\{E(t, s)\} := \{E(t, s); 0 \leq s \leq t < +\infty\}$ has the following evolution properties:

- **(E1)** $E(s, s) = I$ on $\overline{D(\varphi^s)}$ for any $s \in R_+$;
- **(E2)** $E(t_2, s)z = E(t_2, t_1)E(t_1, s)z$ for any $0 \leq s \leq t_1 \leq t_2 < +\infty$ and $z \in \overline{D(\varphi^s)}$;
- **(E3)** $\{E(t, s)\}$ has the following demi-closedness:

  - If $0 \leq s_n \leq t_n < +\infty$, $s_n \rightarrow s$, $t_n \rightarrow t$, $z_n \in \overline{D(\varphi^{s_n})}$, $z \in \overline{D(\varphi^s)}$, $z_n \rightarrow z$ in $H$ and a element $w_n \in E(t_n, s_n)z_n$ converges to some element $w \in H$ as $n \rightarrow +\infty$, then $w \in E(t, s)z$.

We begin with the definition of a discrete $\omega$-limit set for $E(\cdot, \cdot)$.

**Definition 4.3.** (Discrete $\omega$-limit set for $E(\cdot, \cdot)$) Let $\tau \in R_+$ be fixed. Let $\mathcal{B}(H)$ be a family of bounded subsets of $H$. Then for each $B \in \mathcal{B}(H)$, the set

$$
\omega_\tau(B) := \bigcap_{n \in Z^+} \bigcup_{k \geq n, m \in Z^+} E(kT_0 + mT_0 + \tau, mT_0 + \tau)(\overline{D(\varphi^{mT_0 + \tau})} \cap B)
$$

is called the discrete $\omega$-limit set of $B$ under $E(\cdot, \cdot)$.

**Remark 4.1.** By definition of the discrete $\omega$-limit set $\omega_\tau(B)$, it is easy to see that $x \in \omega_\tau(B)$ if and only if there exist sequences $\{k_n\} \subset Z^+$ with $k_n \uparrow +\infty$, $\{m_n\} \subset Z^+$, $\{z_n\} \subset B$ with $z_n \in \overline{D(\varphi^{m_nT_0 + \tau})}$ and $\{x_n\} \subset H$ with $x_n \in E(k_nT_0 + m_nT_0 + \tau, m_nT_0 + \tau)z_n$ such that

$$
x_n \longrightarrow x \text{ in } H \text{ as } n \rightarrow +\infty.
$$

Now let us mention main theorems in this paper.

**Theorem 4.1.** (Discrete attractors of $\text{(AP)}_\tau$) For each $\tau \in R_+$, let $\mathcal{A}_\tau$ be the global attractor of $T_0$-periodic dynamical systems $U_\tau$, which is obtained in Section 3. For $\{\varphi^t\} \in$
\[
\Phi(\{a_r\}, \{b_r\}) \cap \Psi_M, \\{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^t\}) \quad \text{and} \quad f \in L^2_{loc}(R_+; H), \text{ we assume that the system } (AP)_s \text{ is asymptotically } T_0\text{-periodic. Here we put }
\]
\[
A^*_s := \bigcup_{B \in \mathcal{B}(H)} \omega_*(B). \quad (4.1)
\]

Then, we have
(i) \(A^*_s(\subset D(\varphi^s))\) is non-empty and compact in \(H\);
(ii) for each bounded set \(B \in \mathcal{B}(H)\) and each number \(\epsilon > 0\) there exists \(N_{B,s} \in N\) such that
\[
\text{dist}_H(E(kT_0 + \tau, \tau)z, A^*_s) < \epsilon
\]
for all \(z \in \overline{D(\varphi^s)} \cap B\) and all \(k \geq N_{B,s}\);
(iii) \(A^*_s \subset U^l_s A^*_s \subset A^*_s\) for any \(l \in N\), where \(U^l_s\) is the discrete dynamical system for \((P)_s\) given in Definition 3.3.

**Remark 4.2.** By the definition of the discrete \(\omega\)-limit set \(\omega_*(B)\) and \(A^*_s\), we easily see that
\[
A^*_s = A^*_{\tau+nT_0}, \quad \forall n \in N.
\]
Hence \(A^*_s\) is \(T_0\)-periodic in time in a sense of the above.

The second main theorem is concerned with a relationship between two attractors \(A^*_s\) and \(A^*_s\).

**Theorem 4.2.** Suppose the same assumptions are made as in Theorem 4.1. Let \(A^*_s\) and \(A^*_s\) be discrete attractors for \(E(\cdot, s)\) and \(E(\cdot, \tau)\) with \(0 \leq s \leq \tau < +\infty\), respectively. Then,
\[
A^*_s \subset U(\tau, s)A^*_s.
\]
where \(U(\tau, s)\) is the \(T_0\)-periodic process for \((P)_s\) which is given in Definition 3.2.

By Theorems 4.1-4.2, we can get the attractor for asymptotic \(T_0\)-periodic system \((AP)_\tau\).

**Theorem 4.3.** (Global attractor for \((AP)_\tau\)) Suppose the same assumptions are made as in Theorem 4.1. For any \(\tau \in R_+\), let \(A^*_\tau\) be the discrete attractor for \(E(\cdot, \tau)\) obtained in Theorem 4.1. Here we put
\[
A^* := \bigcup_{\tau \in [0, T_0]} A^*_\tau. \quad (4.2)
\]
Then, for any bounded set \(B \in \mathcal{B}(H)\),
\[
\bigcap_{s \geq 0} \bigcup_{t \geq s, \tau \in R_+} E(t + \tau, \tau)(\overline{D(\varphi^s)} \cap B) \subset A^*. \quad (4.3)
\]
By Theorem 4.3, the set \( \mathcal{A}^* \) can be called the global attractor of \( \text{(AP)}_\tau \).

Here we give some key lemmas.

**Lemma 4.1.** If \( \{ s_n \} \subset R_+ \), \( \{\tau_n\} \subset R_+ \), \( s \in R_+ \), \( \tau \in R_+ \), \( s_n \to s \), \( \tau_n \to \tau \), \( \{m_n\} \subset Z_+ \) with \( m_n \to +\infty \), \( z_n \in D(\varphi^{m_n T_0 + s_n}) \), \( z \in D(\varphi^\tau_p) \), \( z_n \to z \) in \( H \) and a element \( w_n \in E(m_n T_0 + \tau_n + s_n, m_n T_0 + s_n)z_n \) converges to some element \( w \in H \) as \( n \to +\infty \), then \( w \in U(\tau + s, s)z \).

**Proof.** Since \( \tau_n \to \tau \), without loss of generality we may assume that there exists a finite time \( T > 0 \) such that \( \{\tau_n\} \subset [0, T] \) and \( \tau \in [0, T] \). By \( w_n \in E(m_n T_0 + \tau_n + s_n, m_n T_0 + s_n)z_n \), there is a solution \( v_n \) of \( \text{(AP)}_{m_n T_0 + s_n} \) on \([m_n T_0 + s_n, +\infty)\) such that

\[
v_n(m_n T_0 + \tau_n + s_n) = w_n \quad \text{and} \quad v_n(m_n T_0 + s_n) = z_n.
\]

Now we put \( u_n(t) := v_n(t + m_n T_0 + s_n) \), then we easily see that \( u_n \) is the solution for

\[
\begin{cases}
    u_n'(t) + \partial \varphi^{t + m_n T_0 + s_n}(u_n(t)) + G(t + m_n T_0 + s_n, u_n(t)) \ni f(t + m_n T_0 + s_n), \quad t > 0, \\
    u_n(0) = z.
\end{cases}
\]

Let \( \delta \in (0, 1) \) be fixed. Since \( z_n \to z \) in \( H \) as \( n \to +\infty \), \( \{z_n\} \) is bounded in \( H \). Hence, from global estimates of solutions (cf. (B) in Section 2) it follows that there is a positive constant \( M_\delta > 0 \) (independent of \( n \)) satisfying

\[
\sup_{t \geq \delta} |u_n(t)|_H^2 + \sup_{t \geq \delta} |u_n'(t)|_{L^2(t,t+1;H)}^2 + \sup_{t \geq \delta} \varphi^{t + m_n T_0 + s_n}(u_n(t)) \leq M_\delta.
\] (4.4)

By [16, Lemma 4.1] we note that the convergence assumption \( \text{(A1)} \) implies

\[
\varphi^{t + m_n T_0 + s_n} \rightharpoonup \varphi^t_p \quad \text{in the sense of Mosco} \ [20]
\] (4.5)

for each \( t \geq 0 \) as \( n \to +\infty \). Moreover by the same argument in [10, Lemma 3.1] we can prove that

\[
\bigcup_{n=1}^{+\infty} \{ z \in H : \varphi^{t + m_n T_0 + s_n}(z) \leq k \}
\] is relatively compact in \( H \) (4.6)

for every real \( k > 0 \) and \( t \geq 0 \), where \( \varphi^{t + m_n T_0 + s_n} = \varphi^t_p \) if \( n = +\infty \). Therefore, by (4.4)-(4.6), (A2), (A3) and the convergence result (C) in Section 2, (by taking a subsequence of \( \{n\} \), if necessary) we see that there is a function \( u_\delta \) such that

\[
u_n'(t) + \partial \varphi^t_p(u_n(t)) + G_p(t + s, u_n(t)) \ni f_p(t + s), \quad t > \delta.
\]

By the standard diagonal process and the same argument in [21, Lemma 3.10], we can construct the solution \( u \) on \([0, +\infty)\) satisfying

\[
\begin{cases}
    u'(t) + \partial \varphi^t_p(u(t)) + G_p(t + s, u(t)) \ni f_p(t + s), \quad t > 0, \\
    u(0) = z
\end{cases}
\]

and

\[
\quad u_n \rightharpoonup u \quad \text{in} \quad C([0, T]; H) \quad \text{as} \quad n \to +\infty.
\] (4.7)
Then, by (4.7) and \( u_n(\tau_n) = w_n \) we have \( u(\tau) = w \), which implies that \( w \in U(\tau + s, s)z. \)

\( \diamond \)

By (B) in Section 2, for each \( B \in \mathcal{B}(H) \) we can choose constants \( r_B > 0 \) and \( M_B > 0 \) so that

\[
|v|_H \leq r_B \quad \text{and} \quad \varphi^{j+s}(v) \leq M_B, \tag{4.8}
\]

for any \( s \in R_+ \), \( t \geq T_0 \), \( z \in \overline{D(\varphi)} \cap B \) and \( v \in E(t + s)z \). Hence it follows from condition (A1) that for each \( m \in \mathbb{Z}_+, \tau \in [0, T_0], n \in \mathbb{N} \) and \( z \in \overline{D(\varphi^{mT_0+\tau})} \cap B \) there is \( \check{z} := \check{z}_{mT_0+\tau} \in D(\varphi_p^\tau) \) such that

\[
|\check{z} - v|_H \leq J_{m+n}^{(r_B+M_B+M)},
\]

(hence \( |\check{z}|_H \leq r_B + J_{m+n}^{(r_B+M_B+M)} \))

and

\[
\varphi_p^\tau(\check{z}) - \varphi^{nT_0+mT_0+\tau}(v) \leq J_{m+n}^{(r_B+M_B+M)},
\]

(hence \( \varphi_p^\tau(\check{z}) \leq M_B + J_{m+n}^{(r_B+M_B+M)} \)).

where \( v \in E(nT_0 + mT_0 + \tau, mT_0 + \tau)z \).

Since \( J_{k}^{(r_B+M_B+M)} \to 0 \) as \( k \to +\infty \), there is a number \( N_0 \in \mathbb{N} \) such that

\[
J_{k}^{(r_B+M_B+M)} \leq 1, \quad \forall k > N_0.
\]

Now, put \( J_0 := 1 + \sup_{1 \leq k \leq N_0} J_{k}^{(r_B+M_B+M)} < +\infty \). Then, we define the bounded set \( \check{B}_\tau \) by

\[
\check{B}_\tau := \{ z \in H; |z|_H \leq r_B + J_0 \} \cap \overline{D(\varphi_p^\tau)}.
\]

Let \( B_{0,\tau} \) be the compact absorbing set for \( U_\tau \) introduced by Remark 3.1. Then, we see that there exists a number \( \check{N} \in \mathbb{N} \) so that

\[
U_\tau^l \check{B}_\tau \subset B_{0,\tau}, \quad \forall l \geq \check{N}. \tag{4.9}
\]

The next lemma is very important to prove Theorem 4.1 (iii).

**Lemma 4.2.** Let \( \tau \in R_+ \) and \( B_{0,\tau} \) be the compact absorbing set for \( U_\tau \). Then we have

\[
\omega(\tau)(B) \subset B_{0,\tau}, \quad \forall B \in \mathcal{B}(H).
\]

**Proof.** At first we assume \( \tau \in [0, T_0] \).

For each \( B \in \mathcal{B}(H) \), let \( x \) be any element of \( \omega(\tau)(B) \). Then, it follows from Remark 4.1 that there exist sequences \( \{k_n\} \subset \mathbb{Z}_+ \) with \( k_n \to +\infty \), \( \{m_n\} \subset \mathbb{Z}_+ \), \( \{z_n\} \subset B \) with \( z_n \in \overline{D(\varphi^{m_nT_0+\tau})} \) and \( \{x_n\} \subset H \) with \( x_n \in E(k_nT_0 + m_nT_0 + \tau, m_nT_0 + \tau)z_n \) such that

\[
x_n \longrightarrow x \quad \text{in} \quad H \quad \text{as} \quad n \to +\infty. \tag{4.10}
\]

Let \( \check{N} \) be the positive integer obtained in (4.9). Then by (E2) we have

\[
x_n \in E(k_nT_0 + m_nT_0 + \tau, k_nT_0 - \check{N}T_0 + m_nT_0 + \tau)
\]

13
\begin{equation}
\circ E(k_n T_0 - \tilde{N} T_0 + m_n T_0 + \tau, m_n T_0 + \tau) z_n
\end{equation}

for any \( n \) with \( k_n \geq \tilde{N} + 1 \).

Hence, there exists an element \( y_n \in E(k_n T_0 - \tilde{N} T_0 + m_n T_0 + \tau, m_n T_0 + \tau) z_n \) such that

\begin{equation}
x_n \in E(k_n T_0 + m_n T_0 + \tau, k_n T_0 - \tilde{N} T_0 + m_n T_0 + \tau) y_n.
\end{equation}

Since \( \{ z_n \} \subset B \), we see that

\[ |y_n|_H \leq r_B \quad \text{and} \quad \varphi^{k_n T_0 - \tilde{N} T_0 + m_n T_0 + \tau}(y_n) \leq M_B \quad \text{for any} \quad n \quad \text{with} \quad k_n \geq \tilde{N} + 1, \]

where \( r_B \) and \( M_B \) are same positive constants in (4.8).

From the convergence condition (A1) it follows that for \( y_n \in E(k_n T_0 - \tilde{N} T_0 + m_n T_0 + \tau, m_n T_0 + \tau) z_n \) there is \( \tilde{z}_n \in D(\varphi_p^\tau) \) such that

\[ |\tilde{z}_n - y_n|_H \leq J_{k_n - \tilde{N} + m_n}^{(r_B + M_B + t)} \]

(hence \( |\tilde{z}_n|_H \leq r_B + J_{k_n - \tilde{N} + m_n}^{(r_B + M_B + t)} \))

and

\[ \varphi_p^\tau(\tilde{z}_n) \leq M_B + J_{k_n - \tilde{N} + m_n}^{(r_B + M_B + t)}. \]

Since \( \{ \tilde{z}_n \in D(\varphi_p^\tau) ; n \in N \text{ with } k_n \geq \tilde{N} + 1 \} \subset \tilde{B}_\tau \) is relatively compact in \( H \), we may assume that

\[ \tilde{z}_n \longrightarrow \tilde{z}_\infty \text{ in } H \quad \text{as} \quad n \rightarrow +\infty \]

for some \( \tilde{z}_\infty \in H \). Then we easily see that \( \tilde{z}_\infty \in \tilde{B}_\tau \) and

\[ y_n \longrightarrow \tilde{z}_\infty \text{ in } H \quad \text{as} \quad n \rightarrow +\infty. \]  

(4.13)

By Lemma 4.1 and (4.10)-(4.13), we observe that

\[ x \in U(\tilde{N} T_0 + \tau, \tau) \tilde{z}_\infty, \]

which implies that

\[ x \in U(\tilde{N} T_0 + \tau, \tau) \tilde{B}_\tau = U_\tau \tilde{B}_\tau \subset B_{0, \tau}. \]

Hence we have

\[ \omega_\tau(B) \subset B_{0, \tau}. \]

For the general case of \( \tau \in R_+ \), choose positive numbers \( i_\tau \in N \) and \( \tau_0 \in [0, T_0] \) so that \( \tau = \tau_0 + i_\tau T_0 \). Then, we can show \( \omega_\tau(B) \subset B_{0, \tau} \) by the same argument as above. \( \diamond \)

**Proof of Theorem 4.1.** On account of Lemma 4.2 we can get \( A_\tau^* \subset B_{0, \tau} \). Hence, Theorem 4.1 (i) holds. Also, by (4.1) and Remark 4.1 we observe that Theorem 4.1 (ii) holds.

Now, we prove Theorem 4.1 (iii). At first, let us prove that \( A_\tau^* \subset U^l_\tau A_\tau^* \) for any \( l \in N \).
Let \( x \) be any element of \( \mathcal{A}_r^* \). By the definition of \( \mathcal{A}_r^* \), there are sequences \( \{B_n\} \subset \mathcal{B}(H) \) and \( \{x_n\} \subset H \) with \( x_n \in \omega_r(B_n) \) such that

\[
x_n \to x \quad \text{in} \quad H \quad \text{as} \quad n \to +\infty. \tag{4.14}
\]

Then, for each \( n \) it follows from Remark 4.1 that there exist sequences \( \{k_{n,j}\} \subset \mathbb{Z}_+ \) with \( k_{n,j} \to +\infty \), \( \{m_{n,j}\} \subset \mathbb{Z}_+ \), \( \{z_{n,j}\} \subset B_n \) with \( z_{n,j} \in D(\varphi^{m_{n,j}T_0 + \tau}) \) and \( \{v_{n,j}\} \subset H \) with \( v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j} \) such that

\[
\begin{align*}
v_{n,j} &\to x_n \quad \text{in} \quad H \quad \text{as} \quad j \to +\infty. \tag{4.15}
\end{align*}
\]

Let \( l \) be any number in \( \mathbb{N} \), then we see that

\[
v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau)
\]

\[
\circ E(k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j}
\]

for \( j \) with \( k_{n,j} \geq l + 1 \). So, there exists an element \( w_{n,j} \in E(k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j} \) such that

\[
v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau)w_{n,j}. \tag{4.16}
\]

By global estimates \((B)\) in Section 2, the set \( \{w_{n,j} \in H : j \in \mathbb{N} \text{ with } k_{n,j} \geq l + 1\} \) is relatively compact in \( H \) for each \( n \). Therefore we may assume that the element \( w_{n,j} \) converges to some element \( \tilde{w}_{n,\infty} \in H \) as \( j \to +\infty \). Clearly, \( \tilde{w}_{n,\infty} \in \omega_r(B_n) \). Moreover, it follows from Lemma 4.1 and (4.15)-(4.16) that

\[
x_n \in U(lT_0 + \tau, \tau)\tilde{w}_{n,\infty} \subset U(lT_0 + \tau, \tau)\omega_r(B_n),
\]

hence, we have

\[
x_n \in \bigcup_{n \geq 1} U^l_\tau \omega_r(B_n), \quad \forall n \geq 1. \tag{4.17}
\]

Here, by the closedness of \( U(\cdot, \cdot) \) we note that for each subset \( X \) of \( B_{0,\tau} \),

\[
\overline{U^l_\tau X} \subset U^l_\tau \overline{X}, \quad \forall l \in \mathbb{N}. \tag{4.18}
\]

Taking account of Lemma 4.2, (4.14), (4.17) and (4.18), we observe that

\[
x \in \overline{\bigcup_{n \geq 1} U^l_\tau \omega_r(B_n)}
\]

\[
= U^l_\tau \overline{\bigcup_{n \geq 1} \omega_r(B_n)}
\]

\[
\subset U^l_\tau \overline{\bigcup_{n \geq 1} \omega_r(B_n)}
\]

\[
\subset U^l_\tau \mathcal{A}_r^*,
\]

which implies that \( \mathcal{A}_r^* \) is semi-invariant under the \( T_0 \)-periodic dynamical systems \( U_\tau \), i.e.

\[
\mathcal{A}_r^* \subset U^l_\tau \mathcal{A}_r^*, \quad \forall l \in \mathbb{N}. \tag{4.19}
\]
Next we shall prove that $U^l_{\tau}A^*_r \subset A_r$ for any $l \in N$. By (4.19), for each $l \in N$

$$U^l_{\tau}A^*_r \subset U^l_{\tau}U^m_{\tau}A^*_r = U^{l+n}_{\tau}A^*_r, \quad \forall n \in N. \quad (4.20)$$

By $A^*_r \subset B_{0,\tau}$, (4.20) and the attractive property of $A_r$, we have

$$U^l_{\tau}A^*_r \subset A_r, \quad \forall l \in N.$$

Therefore we conclude that

$$A^*_r \subset U^l_{\tau}A^*_r \subset A_r, \quad \forall l \in N. \quad \triangleright$$

**Proof of Theorem 4.2.** Let $x$ be any element of $A^*_r$. Then by the definition of $A^*_r$, there exist sequences $\{B_n\} \subset B(H)$ and $\{x_n\} \subset H$ with $x_n \in \omega_{\tau}(B_n)$ such that

$$x_n \rightarrow x \text{ in } H \quad \text{as } n \rightarrow +\infty. \quad (4.21)$$

From Remark 4.1 it follows that for each $n$, there are sequences $\{k_{n,j}\} \subset Z_+$ with $k_{n,j} \rightarrow +\infty$, $\{m_{n,j}\} \subset Z_+$, $\{z_{n,j}\} \subset B_n$ with $z_{n,j} \in D(e^{m_{n,j}T_0+\tau})$ and $\{v_{n,j}\} \subset H$ with $v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j}$ such that

$$v_{n,j} \rightarrow x_n \text{ in } H \quad \text{as } j \rightarrow +\infty. \quad (4.22)$$

Note that for given $s, \tau \in R_+$ with $s \leq \tau$ there is a positive number $l_s \in N$ satisfying

$$s \leq \tau \leq l_sT_0 + s.$$

By using the property (E2) we see that

$$v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 + m_{n,j}T_0 + s)$$

$$\circ E(k_{n,j}T_0 + m_{n,j}T_0 + s, T_0 + m_{n,j}T_0 + l_sT_0 + s)$$

$$\circ E(T_0 + m_{n,j}T_0 + l_sT_0 + s, m_{n,j}T_0 + \tau)z_{n,j}$$

for any $j \in Z_+$ with $k_{n,j} \geq l_s + 2$. Here we can take elements $w_{n,j} \in H$ and $y_{n,j} \in H$ so that

$$v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 + m_{n,j}T_0 + s)w_{n,j}, \quad (4.23)$$

$$w_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + s, T_0 + m_{n,j}T_0 + l_sT_0 + s)y_{n,j} \quad (4.24)$$

and

$$y_{n,j} \in E(T_0 + m_{n,j}T_0 + l_sT_0 + s, m_{n,j}T_0 + \tau)z_{n,j}. \quad (4.25)$$

By $\{z_{n,j}\} \subset B_n$ and the global boundedness result (B) in Section 2, we can get a positive constant $C_n := C_n(B_n) > 0$ satisfying

$$|y_{n,j}|_H \leq C_n, \quad \forall y_{n,j} \in E(T_0 + m_{n,j}T_0 + l_sT_0 + s, m_{n,j}T_0 + \tau)z_{n,j}. \quad (4.26)$$
Here we define the bounded set $B_{C_n}$ by

$$B_{C_n} := \{ b \in H : |b|_H \leq C_n \}.$$ 

From (4.26) and the result (B) in Section 2 it follows that the set

$$\left\{ w_{n,j} \in H; \quad w_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + s, T_0 + m_{n,j}T_0 + l_sT_0 + s)y_{n,j} \right\}$$

for any $j \in \mathbb{Z}_+$ with $k_{n,j} \geq l_s + 2$

is relatively compact in $H$. Hence, we may assume that the element $w_{n,j}$ converges to some element $\tilde{w}_{n,\infty} \in H$ as $j \to +\infty$. Clearly, $\tilde{w}_{n,\infty} \in \omega_s(B_{C_n})$, and it follows from Lemma 4.2 that

$$\omega_s(B_{C_n}) \subset B_{0,s} \subset D(\varphi^s).$$

Moreover, by Lemma 4.1 and (4.22)-(4.23) we have

$$x_n \in U(\tau, s)\tilde{w}_{n,\infty} \subset U(\tau, s)\omega_s(B_{C_n}), \quad \forall n \geq 1,$$

hence, we see that

$$x_n \in \bigcup_{n \geq 1} U(\tau, s)\omega_s(B_{C_n}), \quad \forall n \geq 1.$$ (4.27)

Here, by the closedness of $U(\cdot, \cdot)$, we note that for each subset $X$ of $B_{0,s}$,

$$\overline{U(\tau, s)X} \subset U(\tau, s)X.$$ (4.28)

On account of Lemma 4.2, (4.21), (4.27) and (4.28), we observe that

$$x \in \bigcup_{n \geq 1} U(\tau, s)\omega_s(B_{C_n})$$

$$= U(\tau, s) \bigcup_{n \geq 1} \omega_s(B_{C_n})$$

$$\subset U(\tau, s) \bigcup_{n \geq 1} \omega_s(B_{C_n})$$

$$\subset U(\tau, s)A^*_s,$$

which implies that $A^*_s$ is the subset of $U(\tau, s)A^*_s$, namely

$$A^*_s \subset U(\tau, s)A^*_s.$$

\begin{proof}

Proof of Theorem 4.3. For any $B \in \mathcal{B}(H)$, let $z_0$ be any element of the $\omega$-limit set $\omega_E(B)$ which is define by

$$\omega_E(B) := \bigcap_{s \geq 0, \tau \geq s, \tau \in R_+} E(t + \tau, \tau)(D(\varphi^\tau) \cap B).$$

17
Then we easily see that there exist sequences \( \{t_n\} \subset R_+ \) with \( t_n \uparrow +\infty \), \( \{\tau_n\} \subset R_+ \), \( \{y_n\} \subset B \) with \( y_n \in \overline{D(\varphi \tau_n)} \) and \( \{z_n\} \subset H \) with \( z_n \in E(t_n + \tau_n, \tau_n)y_n \) such that
\[
\begin{align*}
t_n &:= k_n T_0 + t'_n, \quad k_n \in Z_+, \quad k_n \not\to +\infty, \quad t'_n \in [T_0, 2T_0], \quad t'_n \to t_0, \\
\tau_n &:= l_n T_0 + \tau'_n, \quad l_n \in Z_+, \quad \tau'_n \in [0, T_0], \quad \tau_n \to \tau'_0
\end{align*}
\]
and
\[
z_n \to z_0 \quad \text{in } H
\] (4.29)
as \( n \to +\infty \). Without loss of generality, we may assume that
\[
(a) \quad t'_n + \tau'_n \not\to t'_0 + \tau'_0 \quad \text{or} \quad (b) \quad t'_n + \tau'_n \not\to t'_0 + \tau'_0.
\]
Now, assume that (a) holds. Then let us consider the multivalued semiflow
\[
v_n \in E(1 + k_n T_0 + l_n T_0 + t'_0 + \tau'_0, k_n T_0 + l_n T_0 + t'_n + \tau'_n)z_n.
\] (4.30)
Then, there is a solution \( u_n \) on \([k_n T_0 + l_n T_0 + t'_n + \tau'_n, +\infty)\) for
\[
\begin{cases}
  u'_n(t) + \partial \varphi^{t + k_n T_0 + l_n T_0 + t'_0 + \tau'_0}(u_n(t)) + G(t + k_n T_0 + l_n T_0 + t'_n + \tau'_n, u_n(t)) \\
  \quad \ni f(t + k_n T_0 + l_n T_0 + t'_n + \tau'_n), \quad t > 0, \\
  u_n(0) = z_n \quad \text{and} \quad u_n(1 + t'_0 + \tau'_0 - t'_n + \tau'_n) = v_n.
\end{cases}
\]
Since \( z_n \to z_0 \) in \( H \), \( \{z_n\} \) is bounded in \( H \). Therefore by the global estimate (B) in Section 2, we see that
\[
\{v_n \in H; \quad v_n \in E(1 + k_n T_0 + l_n T_0 + t'_0 + \tau'_0, k_n T_0 + l_n T_0 + t'_n + \tau'_n)z_n \}
\]
for any \( n \in N \) is relatively compact in \( H \). Hence we may assume that
\[
v_n \to v \quad \text{in } H \quad \text{for some } v \in H.
\] (4.31)
Now applying Lemma 4.1 with (4.29)-(4.31), we can get
\[
v \in U(1 + t'_0 + \tau'_0, t'_0 + \tau'_0)z_0,
\]
more precisely, (taking the subsequence of \( \{n\} \) if necessary) we observe that
\[
u_n \to u \quad \text{in } C([0, 2]; H) \quad \text{as } n \to +\infty,
\] (4.32)
where \( u \) is the solution \([t'_0 + \tau'_0, +\infty)\) satisfying
\[
\begin{cases}
  u'(t) + \partial \varphi^{t + t'_0 + \tau'_0}(u(t)) + G_p(t + t'_0 + \tau'_0, u(t)) \ni f_p(t + t'_0 + \tau'_0), \quad t > 0, \\
  u(0) = z_0 \quad \text{and} \quad u(1) = v.
\end{cases}
\]
By (4.32) we easily see that
\[
u_n(t'_0 + \tau'_0 - t'_n - \tau'_n) \to z_0 \quad \text{as } n \to +\infty.
\] (4.33)
Note that
\[ u_n(t'_0 + \tau'_0 - t'_n + \tau'_n) \]
\[ \in E(k_n T_0 + l_n T_0 + t'_0 + \tau'_0, \ k_n T_0 + l_n T_0 + t'_n + \tau'_n) z_n \]
\[ = E(k_n T_0 + l_n T_0 + t'_0 + \tau'_0, \ l_n T_0 + \tau'_n) y_n \]
\[ = E(k_n T_0 + l_n T_0 + t'_0 + \tau'_0, \ l_n T_0 + t'_0 + \tau'_0) E(l_n T_0 + t'_0 + \tau'_0, \ l_n T_0 + \tau'_n) y_n. \]

So, we can take an element \( x_n \in E(l_n T_0 + t'_0 + \tau'_0, \ l_n T_0 + \tau'_n) y_n \) such that
\[ u_n(t'_0 + \tau'_0 - t'_n + \tau'_n) \in E(k_n T_0 + l_n T_0 + t'_0 + \tau'_0, \ l_n T_0 + t'_0 + \tau'_0) x_n. \] (4.34)

By \( \{ y_n \} \subset B \) and the global estimate (B) in Section 2, we easily see that \( \{ x_n \} \) is bounded, i.e.
\[ \{ x_n \} \subset \tilde{B} \text{ for some } \tilde{B} \in \mathcal{B}(H). \] (4.35)

Therefore, from Remarks 4.1-4.2 and (4.33)-(4.35) we observe that
\[ z_0 \in \omega_{t'_0 + \tau'_0}(\tilde{B}) \subset \mathcal{A}^*_0 + \tau'_0 \subset \mathcal{A}^*. \]

Thus (4.3) holds.

In the case (b) when \( t'_n + \tau'_n \searrow t'_0 + \tau'_0 \), we can prove (4.3) by the slight modification of the proof as above. \( \diamond \)

Theorem 4.1 implies that the attracting set \( \mathcal{A}^*_s \) for \( (AP)_{\tau} \) is semi-invariant under \( U_{\tau} \) associated with the limiting \( T_0 \)-periodic system \( (P)_s \), in general. Moreover, from Theorem 4.2 we observe that

\[ \mathcal{A}^*_\tau \subset U(\tau, s) \mathcal{A}^*_s \quad \text{for any } 0 \leq s \leq \tau < +\infty. \]

In order to get the invariance of \( \mathcal{A}^*_s \) under \( U_{\tau} \) and \( \mathcal{A}^*_\tau = U(\tau, s) \mathcal{A}^*_s \), let us use a concept of a regular approximation, which was introduced in [17].

**Definition 4.4.** (Regular approximation) Let \( s \in \mathbb{R}_+ \) be fixed. Let \( z \in D(\varphi_{p}^s) \). Then, we say that \( U(t+s, s) z \) is regularly approximated by \( E(t + k T_0 + s, k T_0 + s) \) as \( k \to +\infty \), if for each finite \( T > 0 \) there are sequences \( \{ k_n \} \subset \mathbb{Z}_+ \) with \( k_n \to +\infty \) and \( \{ z_n \} \subset H \) with \( z_n \in D(\varphi^{k_n T_0 + s}) \) and \( z_n \to z \) in \( H \) satisfying the following property: for any function \( u \in W^{1,2}(0, T; H) \) satisfying \( u(t) \in U(t+s, s) z \) for all \( t \in [0, T] \) there is a sequence \( \{ u_n \} \subset W^{1,2}(0, T; H) \) such that \( u_n(t) \in E(t + k_n T_0 + s, k_n T_0 + s) z_n \) for all \( t \in [0, T] \) and \( u_n \to u \) in \( C([0, T]; H) \) as \( n \to +\infty \).

Using the above concept, we can show that the invariance of \( \mathcal{A}^*_s \) under \( U_{\tau} \). Moreover we can get
\[ \mathcal{A}^*_\tau = U(\tau, s) \mathcal{A}^*_s. \]

**Theorem 4.4** Suppose all assumptions in Theorem 4.1. Let \( \mathcal{A}^*_s \) and \( \mathcal{A}^*_\tau \) be discrete attractors for \( E(\cdot, s) \) and \( E(\cdot, \tau) \), with \( 0 \leq s \leq \tau < +\infty \), respectively. Assume that for
any point \( z \) of \( \mathcal{A}_s^* \), \( U(t + s, s)z \) is regularly approximated by \( E(t + kT_0 + s, kT_0 + s) \) as \( k \to +\infty \). Then we have
\[
\mathcal{A}_s^* = U(\tau, s)\mathcal{A}_s^*.
\]

**Proof.** By Theorem 4.2, we have only to show that
\[
U(\tau, s)\mathcal{A}_s^* \subset \mathcal{A}_s^*.
\]
To do so, let \( x \) be any element of \( U(\tau, s)\mathcal{A}_s^* \).

At first, taking account of Definitions 3.2-3.3 and Theorem 4.1 (iii), we see that for each \( n \in \mathbb{N} \)
\[
U^n U(\tau, s)\mathcal{A}_s^* = U(nT_0 + \tau, \tau)U(\tau, s)\mathcal{A}_s^* = U(nT_0 + \tau, nT_0 + s)U(nT_0 + s, \tau)\mathcal{A}_s^* = U(\tau, s)U^n \mathcal{A}_s^* \subset U(\tau, s)\mathcal{A}_s^*.
\]

Hence, there exists a element \( y_n \in \mathcal{A}_s^* \) such that
\[
x \in U^n U(\tau, s)y_n = U(nT_0 + \tau - s, s)\mathcal{A}_s^*.
\]

By using our assumption as \( t = nT_0 + \tau - s \), we observe that for each \( n \), there are sequences \( \{k_{n,j}\} \subset \mathbb{Z}_+, \{x_{n,j}\} \subset H \) and \( \{y_{n,j}\} \subset H \) such that
\[
k_{n,j} \to +\infty, \quad y_{n,j} \in D(\varphi^{k_{n,j}T_0 + s}), \quad y_{n,j} \to y_n \text{ in } H
\]
and
\[
x_{n,j} \in E(nT_0 + \tau - s + k_{n,j}T_0 + s, k_{n,j}T_0 + s)y_{n,j}, \quad x_{n,j} \to x \text{ in } H
\]
as \( j \to +\infty \). Therefore, by the usual diagonal argument, we can find a subsequence \( \{j_n\} \) of \( \{j\} \) such that \( \tilde{x}_n := x_{n,j_n}, \tilde{y}_n := y_{n,j_n} \) and \( \tilde{k}_n := k_{n,j_n} \) satisfy
\[
|\tilde{x}_n - x|_H < \frac{1}{n}, \quad \tilde{x}_n \in E(nT_0 + \tau - s + \tilde{k}_nT_0 + s, \tilde{k}_nT_0 + s)\tilde{y}_n, \quad |\tilde{y}_n - y_n|_H < \frac{1}{n}
\]
for every \( n = 1, 2, \ldots \). Since \( \{\tilde{y}_n\} \) is bounded in \( H \), there is a bounded set \( B \in \mathcal{B}(H) \) so that \( \{\tilde{y}_n\} \subset B \).

By (E2), we see that
\[
\tilde{x}_n \in E(nT_0 + \tau - s + \tilde{k}_nT_0 + s, \tilde{k}_nT_0 + s)\tilde{y}_n = E(nT_0 + \tilde{k}_nT_0 + \tau, \tilde{k}_nT_0 + \tau)E(T_0 + \tilde{k}_nT_0 + \tau, \tilde{k}_nT_0 + s)\tilde{y}_n,
\]
hence there is an element \( \tilde{z}_n \in E(T_0 + \tilde{k}_nT_0 + \tau, \tilde{k}_nT_0 + s)\tilde{y}_n \) such that
\[
\tilde{x}_n \in E(nT_0 + \tilde{k}_nT_0 + \tau, \tilde{k}_nT_0 + s)\tilde{y}_n \subset E(T_0 + \tilde{k}_nT_0 + \tau, \tilde{k}_nT_0 + s)\tilde{y}_n.
\]

Since \( \{\tilde{y}_n\} \subset B \) and the global estimate (B) in Section 2, we see that \( \{\tilde{z}_n\} \) is also bounded in \( H \). Hence, there is a bounded set \( \tilde{B} \in \mathcal{B}(H) \) so that \( \{\tilde{z}_n\} \subset \tilde{B} \). The above fact (4.37)-(4.39) implies (cf. Remark 4.1) that \( x \in \omega_{\tau}(\tilde{B}) \subset \mathcal{A}_s^* \). Thus we have \( U(\tau, s)\mathcal{A}_s^* \subset \mathcal{A}_s^* \). \( \diamond \)
By the same argument in Theorem 4.4, we can get the following corollary:

**Corollary.** (i) Suppose the same assumptions of Theorem 4.4. Then, by Remark 4.2 we observe that \( A^*_s \) is invariant under the \( T_0 \)-periodic dynamical system \( U_s := U(T_0 + s, s) \). Namely,

\[
A^*_s = U^l_s A^*_s \quad \text{for any } l \in \mathbb{N}.
\]

(ii) Assume that for any point \( z \) of \( A_\tau \), \( U(t + \tau, \tau)z \) is regularly approximated by \( E(t + kT_0 + \tau, kT_0 + \tau) \) as \( k \to +\infty \). Then, we have \( A^*_\tau \supset A_\tau (= U_\tau A_\tau) \). Hence by Theorem 4.1 (iii) we conclude that

\[
A^*_\tau = A_\tau.
\]

**Remark 4.3.** If the solution operator \( U(t, s) \) is singlevalued, namely the solution for the Cauchy problem of (P) is unique, the assumptions of Theorem 4.4 always hold. Thus, Theorems 4.1-4.4 contain the abstract results obtained in [11], which was concerned with the asymptotic \( T_0 \)-periodic stability for the singlevalued dynamical system associated with time-dependent subdifferentials.

5 Application to obstacle problems for PDE’s

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) \((1 \leq N < +\infty)\) with smooth boundary \( \Gamma = \partial \Omega \), \( q \) be a fixed number with \( 2 \leq q < +\infty \) and \( T_0 \) be a fixed positive number. We use the notation

\[
a_q(v, z) := \int_\Omega |\nabla v|^{q-2} \nabla v \cdot \nabla z \, dx, \quad \forall v, z \in W^{1,q}(\Omega)
\]

and denote by \( (\cdot, \cdot) \) the usual inner product in \( L^2(\Omega) \).

For prescribed obstacle functions \( \sigma_0 \leq \sigma_1 \) and each \( t \in \mathbb{R}_+ \) we define the set

\[
K(t) := \left\{ z \in W^{1,q}(\Omega); \sigma_0(t, \cdot) \leq z \leq \sigma_1(t, \cdot) \quad \text{a.e. on } \Omega \right\}.
\]

Let \( f \) be a function in \( L^2_{\text{loc}}(\mathbb{R}_+; L^2(\Omega)) \) and \( h \) be a non-negative function on \( \mathbb{R}_+ \times \mathbb{R} \). Then for given \( b \in L^\infty(\Omega)^N \) we consider an interior asymptotically \( T_0 \)-periodic double obstacle problem \((\text{OP})^\text{AP}_s \) \((s \in \mathbb{R}_+)\):

- Find functions \( v \in C([s, +\infty); L^2(\Omega)) \) and \( \theta \in L^2_{\text{loc}}((s, +\infty); L^2(\Omega)) \) such that

\[
\begin{cases}
\text{\((\text{OP})^\text{AP}_s\)} \\
0 \leq \theta(t, x) \leq h(t, v(t, x)) \quad \text{a.e. on } (s, +\infty) \times \Omega;
\end{cases}
\]

\[
\begin{cases}
0 \leq \theta(t, x) \leq h(t, v(t, x)) \quad \text{a.e. on } (s, +\infty) \times \Omega;
\end{cases}
\]

\[
\begin{cases}
(v' + \theta(t) + b \cdot \nabla v(t) - f(t), v(t) - z) + a_q(v(t), v(t) - z) \leq 0
\end{cases}
\]

for any \( z \in K(t) \) and a.e. \( t \geq s \).
The main object of this section is to consider the large-time behaviour of solution for \((OP)^AP\) assuming asymptotically \(T_0\)-periodicity conditions
\[
\sigma_i(t) - \sigma_{i,p}(t) \to 0 \quad (i = 0, 1), \quad h(t, \cdot) - h_p(t, \cdot) \to 0, \quad f(t) - f_p(t) \to 0
\]
as \(t \to \infty\) in the sense specified below, where \(\sigma_{i,p}(t)\), \(h_p(t, \cdot)\), \(f_p(t)\) are periodic in time with the same period \(T_0\). By the above assumptions, the limiting system of \((OP)^AP\) is a \(T_0\)-periodic one \((OP)^P\) as follows:

- Find functions \(u \in C((s, +\infty); L^2(\Omega))\) and \(\theta \in L^2_{loc}((s, +\infty); L^2(\Omega))\) such that

\[
(\text{OP})^P_s \quad \begin{cases}
  u \in L^2_{loc}((s, +\infty); W^{1,q}(\Omega)) \cap W^{1,2}_{loc}((s, +\infty); L^2(\Omega)); \\
  u(t) \in K_p(t) \text{ for a.e. } t \geq s; \\
  0 \leq \theta(t, x) \leq h_p(t, u(t, x)) \text{ a.e. on } (s, +\infty) \times \Omega; \\
  (u'(t) + \theta(t) + b \cdot \nabla u(t) - f_p(t), u(t) - z) + a_q(u(t), u(t) - z) \leq 0 \\
  \text{for any } z \in K_p(t) \text{ and a.e. } t \geq s,
\end{cases}
\]

where \(K_p(t) := \{z \in W^{1,q}(\Omega); \sigma_{0,p}(t, \cdot) \leq z \leq \sigma_{1,p}(t, \cdot) \text{ a.e. on } \Omega\}\).

Now we suppose the following conditions:

- \(\sigma_i\) and \(\sigma_{i,p}\) are functions on \(R_+ \times \Omega\) such that

\[
\sup_{t \in R_+} \left| \frac{d\sigma_i}{dt} \right|_{L^2(t,t+1;W^{1,q}(\Omega))} + \sup_{t \in R_+} \left| \frac{d\sigma_{i,p}}{dt} \right|_{L^2(t,t+1;L^\infty(\Omega))} < +\infty,
\]

\[
\sup_{t \in R_+} \left| \frac{d\sigma_{i,p}}{dt} \right|_{L^2(t,t+1;W^{1,q}(\Omega))} + \sup_{t \in R_+} \left| \frac{d\sigma_{i,p}}{dt} \right|_{L^2(t,t+1;L^\infty(\Omega))} < +\infty
\]

and \(\sigma_{i,p}\) is a \(T_0\)-periodic obstacle function, i.e.
\[
\sigma_{i,p}(t + T_0, x) = \sigma_{i,p}(t, x) \quad \text{for a.e. } x \in \Omega \text{ and any } t \in R_+
\]

for \(i = 0, 1\). Moreover, there are positive constants \(k_1 > 0\) and \(k_2 > 0\) such that

\[
\sigma_1 - \sigma_0 \geq k_1 \quad \text{and} \quad \sigma_{1,p} - \sigma_{0,p} \geq k_1 \quad \text{a.e. on } R_+ \times \Omega
\]

and

\[
|\sigma_i|_{L^\infty(R_+;W^{1,q}(\Omega))} + |\sigma_i|_{L^\infty(R_+ \times \Omega)} + |\sigma_{i,p}|_{L^\infty(R_+;W^{1,q}(\Omega))} + |\sigma_{i,p}|_{L^\infty(R_+ \times \Omega)} \leq k_2
\]

for \(i = 0, 1\).

- \(h\) and \(h_p\) are non-negative continuous functions on \(R_+ \times R\). There is a positive constant \(L\) such that

\[
|h(t, z_1) - h(t, z_2)| \leq L|z_1 - z_2|
\]

\[
|h_p(t, z_1) - h_p(t, z_2)| \leq L|z_1 - z_2|
\]

for all \(t \in R_+, \ z_i \in R\) and \(i = 1, 2\). Moreover, \(h_p\) is a \(T_0\)-periodic function, i.e. for any \(z \in R\), \(h_p(t + T_0, z) = h_p(t, z)\) for any \(t \in R_+\).
• \( f, f_p \in L^2_{\text{loc}}(R_+; L^2(\Omega)) \), and \( f_p \) is a \( T_0 \)-periodic function, i.e.

\[
f_p(t + T_0) = f_p(t) \quad \text{in } L^2(\Omega), \quad \forall t \in R_+.
\]

Moreover, we suppose the following convergence conditions:

• \text{(Convergence of } \sigma_i(t) - \sigma_{i,p}(t) \rightarrow 0 \text{ as } t \rightarrow +\infty)\text{ \text{Put }}

\[
I_m := \sup_{t \in [0,T_0]} |\sigma_0(mT_0 + t) - \sigma_{0,p}(t)|_{W^{1,1}(\Omega)} + \sup_{t \in [0,T_0]} |\sigma_1(mT_0 + t) - \sigma_{1,p}(t)|_{W^{1,1}(\Omega)}
\]

\[
+ \sup_{t \in [0,T_0]} |\sigma_0(mT_0 + t) - \sigma_{0,p}(t)|_{L^\infty(\Omega)} + \sup_{t \in [0,T_0]} |\sigma_1(mT_0 + t) - \sigma_{1,p}(t)|_{L^\infty(\Omega)}
\]

Then,

\[
I_m \rightarrow 0 \quad \text{as } m \rightarrow +\infty;
\]

• \text{(Convergence of } h(t, \cdot) - h_p(t, \cdot) \rightarrow 0 \text{ as } t \rightarrow +\infty)\text{ \text{For any } } z \in R,\text{ \text{Put }}

\[
\sup_{t \in [0,T_0]} |h(mT_0 + t, z) - h_p(t, z)| \rightarrow 0 \quad \text{as } m \rightarrow +\infty;
\]

(5.1)

• \text{(Convergence of } f(t) - f_p(t) \rightarrow 0 \text{ as } t \rightarrow +\infty)\text{ \text{Put }}

\[
|f(mT_0 + \cdot) - f_p|_{L^2(0,T_0;L^2(\Omega))} \rightarrow 0 \quad \text{as } m \rightarrow +\infty.
\]

(5.2)

Under the above assumptions, let us consider problems (OP)\( ^{\text{AP}}_s \text{ and (OP)}\( ^{P}_s \text{.}

In order to apply the abstract results in Sections 2-4, we choose \( L^2(\Omega) \) as a real separable Hilbert space \( H \). And we define a family \( \{\varphi_t\} \) of proper l.s.c. convex functions \( \varphi_t \) on \( L^2(\Omega) \) by

\[
\varphi_t(z) = \begin{cases} 
\frac{1}{q} \int_{\Omega} |\nabla z|^q dx & \text{if } z \in K(t), \\
+\infty & \text{if } z \in L^2(\Omega) \setminus K(t),
\end{cases}
\]

(5.3)

and define \( \varphi_{p,t} \) by replacing \( K(t) \) by \( K_p(t) \) in (5.3).

Also, we define a multivalued operator \( G(\cdot, \cdot) \) from \( R_+ \times H^1(\Omega) \) into \( L^2(\Omega) \) by

\[
G(t, z) := \left\{ g \in L^2(\Omega); \begin{array}{l}
g = l + b \cdot \nabla z \quad \text{in } L^2(\Omega) \\
0 \leq l(x) \leq h(t, z(x)) \quad \text{a.e. on } \Omega
\end{array} \right\}
\]

(5.4)

for all \( t \in R_+ \) and \( z \in H^1(\Omega) \). And we define \( G_p(\cdot, \cdot) \) by replacing \( h(t, \cdot) \) by \( h_p(t, \cdot) \) in (5.4).

By the same argument in [27, Lemma 5.1], we can get the following lemmas.

**Lemma 5.1.** (cf. [27, Lemma 5.1]) \text{Put for any } r > 0 \text{ and } t \in R_+ 

\[
a_r(t) = b_r(t) := k_3 \int_0^t \left\{ |\sigma_{0,p}'|_{L^\infty(\Omega)} + |\sigma_{0,p}'|_{W^{1,1}(\Omega)} + |\sigma_{1,p}'|_{L^\infty(\Omega)} + |\sigma_{1,p}'|_{W^{1,1}(\Omega)} \right\} dt
\]

(5.5)
where $k_3$ is a (sufficiently large) positive constant. Then, \( \{ \varphi^t \} \in \Phi(\{a_r\}, \{b_r\}) \) and \( \{ \varphi^t_p \} \in \Phi_p(\{a_r\}, \{b_r\}; T_0) \).

Moreover we have \( \{ G(t, \cdot) \} \in \mathcal{G}(\{ \varphi^t \}) \) and \( \{ G_p(t, \cdot) \} \in \mathcal{G}_p(\{ \varphi^t_p \}; T_0) \).

**Lemma 5.2.** The convergence assumptions (A1)-(A3) hold.

**Proof.** We easily see that (A2) and (A3) hold by assumptions (5.1) and (5.2).

Now let us show (A1). For each \( t \in R_+ \) there are \( m \in Z_+ \) and \( \tau \in [0, T_0] \) so that \( t = mT_0 + \tau \).

For each \( z_p \in D(\varphi^t_p) = K_p(t) \), we put

\[
  z := (z_p - \sigma_{0,p}(t)) \frac{\sigma_1(t) - \sigma_0(t)}{\sigma_{1,p}(t) - \sigma_{0,p}(t)} + \sigma_0(t).
\]

Then we easily see that \( z \in D(\varphi^t) = K(t) \). Moreover, by the same argument in [27, Lemma 5.1], we see that

\[
|z - z_p|_{L^2(\Omega)} \leq k_4 I_m \quad \text{and} \quad |\nabla z - \nabla z_p|_{L^2(\Omega)} \leq k_4 I_m (1 + |\nabla z_p|_{L^2(\Omega)}) \tag{5.5}
\]

for some constant \( k_4 > 0 \). Hence we have

\[
\varphi^t(z) - \varphi^t_p(z_p) \leq k_5 I_m (1 + \varphi^t_p(z_p)), \tag{5.6}
\]

for a sufficiently large \( k_5 > 0 \).

Conversely, let \( z \in D(\varphi^t) = K(t) \) and we put

\[
  z_p := (z - \sigma_0(t)) \frac{\sigma_{1,p}(t) - \sigma_{0,p}(t)}{\sigma_1(t) - \sigma_0(t)} + \sigma_{0,p}(t).
\]

Then, we observe that \( z_p \in D(\varphi^t_p) = K_p(t) \) and

\[
|z_p - z|_{L^2(\Omega)} \leq k_4 I_m \quad \text{and} \quad \varphi^t_p(z_p) - \varphi^t(z) \leq k_5 I_m (1 + \varphi^t(z)). \tag{5.7}
\]

Therefore by (5.5)-(5.7) we see that the convergence assumption (A1) holds. \( \diamond \)

Clearly, the obstacle problem (OP)$_s^{AP}$ can be reformulated as an evolution equation (AP)$_s$ involving the subdifferential of $\varphi^t$ given by (5.3) and the multivalued operator $G(t, \cdot)$ defined by (5.4). Also, the limiting $T_0$-periodic problem (OP)$_s^{AP}$ can be reformulated as an evolution equation (P)$_s$. Therefore, by Lemmas 5.1-5.2 we can apply abstract results in Section 2-4. Namely, we can obtain an attractor $\mathcal{A}_s$ for (OP)$_s^{AP}$, a $T_0$-periodic attractor $\mathcal{A}_s$ for (OP)$_s^{AP}$ and the relationships between (OP)$_s^{AP}$ and (OP)$_s^{P}$.

Additionally, we assume that $f(t) \equiv f_p(t)$ for any $t \in R_+$ and

\[
\sigma_0(t, z) \equiv \sigma_{0,p}(t, z), \quad \sigma_{1,p}(t, z) \equiv \sigma_1(t, z), \quad h_p(t, z) \leq h(t, z)
\]

24
for any $0 \leq t < +\infty$ and $z \in R$. Then we easily see that the assumptions of Theorem 4.4 and its Corollary hold. Hence we can get $A^*_s = A_s$ by the same argument in [30, Theorem 5.4].

Unfortunately we do not give assumptions for $\sigma_i(t, \cdot), h(t, \cdot)$ and $f(t)$ in order to get

$$U(\tau, s)A^*_s = A^*_s \subset A_\tau$$

for any $0 \leq s \leq \tau < +\infty$. (5.8)

It seems difficult to show (5.8), so it is the open problem.

References


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