<table>
<thead>
<tr>
<th>Title</th>
<th>Attractors of asymptotically periodic multivalued dynamical systems governed by time-dependent subdifferentials</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Yamazaki, Noriaki</td>
</tr>
<tr>
<td>Citation</td>
<td>Hokkaido University Preprint Series in Mathematics, 645, 1-27</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2004</td>
</tr>
<tr>
<td>DOI</td>
<td>10.14943/83798</td>
</tr>
<tr>
<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/69452">http://hdl.handle.net/2115/69452</a></td>
</tr>
<tr>
<td>Type</td>
<td>bulletin (article)</td>
</tr>
<tr>
<td>File Information</td>
<td>pre645.pdf</td>
</tr>
</tbody>
</table>
ATTRACTORS OF ASYMPTOTICALLY PERIODIC
MULTIVALUED DYNAMICAL SYSTEMS GOVERNED
BY TIME-DEPENDENT SUBDIFFERENTIALS

NORIAKI YAMAZAKI

Abstract. Let us consider a nonlinear evolution equation associated with time-dependent subdifferential in a separable Hilbert space. In this paper we treat an asymptotically periodic system which means that time-dependent terms converge to some time-periodic ones as time goes to $+\infty$. Then we consider the large-time behaviour of solutions without uniqueness. In such a situation the corresponding dynamical systems are multivalued. In fact we discuss the stability of multivalued semiflows from the view-point of attractors. Namely, the main object of this paper is to construct a global attractor for the asymptotically periodic multivalued dynamical system, and to discuss the relationship to one for the limiting periodic system.

1 Introduction

In this paper let us consider a non-autonomous system in a real separable Hilbert space $H$ of the form

$$v'(t) + \partial\varphi^t(v(t)) + G(t, v(t)) \ni f(t) \quad \text{in } H, \quad t > s \geq 0,$$

where $v' = \frac{dv}{dt}$, $\partial\varphi^t$ is a subdifferential of time-dependent proper lower semicontinuous (l.s.c.) convex function $\varphi^t$ on $H$, $G(t, \cdot)$ is a multivalued perturbation small relative to $\varphi^t$, and $f$ is a forcing term.

In the case when $G(t, \cdot) \equiv 0$, many mathematicians studied the existence-uniqueness, asymptotic stability, time periodic and almost periodic problem for (1.1) (cf. [7], [8], [13], [14], [15], [16], [18], [23], [24]).

For the multivalued nonmonotone perturbation $G(t, \cdot)$, Otani has already shown the existence of solution for (1.1) in [21]. The large-time behavior of solutions for (1.1) was
discussed by [28] from the view-point of attractors. For the time periodic case, assuming
the periodicity conditions with same period $T_0$, $0 < T_0 < +\infty$, i.e.

$$
\varphi^t = \varphi^{t+T_0}, \quad G(t, \cdot) = G(t + T_0, \cdot), \quad f(t) = f(t + T_0), \quad \forall t \in \mathbb{R}_+ := [0, \infty),
$$

the existence of periodic solution for (1.1) was proved in [22]. Moreover, the periodic stability was discussed in [29]. In fact, the author showed the existence and characterization of time-periodic global attractors for (1.1).

In this paper, for a given positive number $T_0 > 0$ let us treat the case when $\varphi^t$, $G(t, \cdot)$ and $f(t)$ are asymptotically $T_0$-periodic in time. Namely we assume that

$$
\varphi^t - \varphi_p^t \longrightarrow 0, \quad G(t, \cdot) - G_p(t, \cdot) \longrightarrow 0, \quad f(t) - f_p(t) \longrightarrow 0 \quad (1.2)
$$

in appropriate senses as $t \to +\infty$, where $\varphi_p^t = \varphi^{t+T_0}$, $G_p(t, \cdot) = G_p(t + T_0, \cdot)$ and $f_p(t) = f_p(t + T_0)$ for any $t \in \mathbb{R}_+$. By the asymptotically $T_0$-periodic stability (1.2), we have the limiting $T_0$-periodic system for (1.1) of the form:

$$
u'(t) + \partial \varphi^t_p(u(t)) + G_p(t, u(t)) \ni f_p(t) \quad \text{in} \ H, \quad t > s \ (\geq 0). \quad (1.3)
$$

In the case when $G(t, \cdot)$ and $G_p(t, \cdot)$ are single-valued, the asymptotically $T_0$-periodic problem has already been discussed in [11]. In order to guarantee the uniqueness of solutions for the Cauchy problem of (1.1) and (1.3), they assumed some conditions on $\varphi^t, \varphi^t_p, G(t, \cdot)$ and $G_p(t, \cdot)$. Then, they discussed the asymptotically $T_0$-periodic stability for (1.1) from the view-point of attractors (cf. [11]). The main object of this paper is to develop the result obtained in [11] in order to consider the large-time behaviour of solution for (1.1) without uniqueness. Namely, we would like to construct the attractor for the asymptotically $T_0$-periodic multivalued flows associated with (1.1). Moreover we shall discuss the relationship to the $T_0$-periodic attractor for (1.3) obtained in [29].

In the next Section 2, we recall the known results for the Cauchy problem of (1.1). In Section 3 we consider the limiting $T_0$-periodic problem (1.3) and recall the abstract results obtained in [29]. In Section 4, we introduce the notion of a metric topology on the family $\{\varphi^t; t \geq 0\}$ which was constructed in [16]. And we present and prove the main results in this paper. In proving main results, we generalize the results obtained in [11] and [30]. In the final section we apply our abstract results to the parabolic variational inequality with asymptotically $T_0$-periodic double obstacles. Then we can discuss the asymptotic stability for the asymptotically $T_0$-periodic double obstacle problem without uniqueness of solutions.

**Notation.** Throughout this paper, let $H$ be a (real) separable Hilbert space with norm $| \cdot |_H$ and inner product $(\cdot, \cdot)_H$. For a proper l.s.c. convex function $\varphi$ on $H$ we use the notation $D(\varphi)$, $\partial \varphi$ and $D(\partial \varphi)$ to indicate the effective domain, subdifferential and its domain of $\varphi$, respectively; for their precise definitions and basic properties see [4].

For two non-empty sets $A$ and $B$ in $H$, we define the so-called Hausdorff semi-distance

$$
\text{dist}_H(A, B) := \sup_{x \in A} \inf_{y \in B} |x - y|_H.
$$
2 Preliminaries

In this section let us recall the known results for a nonlinear evolution equation in $H$ of the form:

$$u'(t) + \partial\varphi^t(u(t)) + G(t, u(t)) \ni f(t) \quad \text{in } H, \quad t \in J, \quad (2.1)$$

where $J$ is an interval in $R_+$, $\partial\varphi^t$ is the subdifferential of a time-dependent proper l.s.c. and convex function $\varphi^t$ on $H$; $G(t, \cdot)$ is a multivalued operator from a subset $D(G(t, \cdot)) \subset H$ into $H$ for each $t \in R_+$ and $f$ is a given function in $L^2_{\text{loc}}(J; H)$.

We begin with the definition of solution for (2.1).

**Definition 2.1.** (i) For a compact interval $J := [t_0, t_1] \subset R_+$ and $f \in L^2(J; H)$, a function $u : J \to H$ is called a solution of (2.1) on $J$, if $u \in C(J; H) \cap W^1_{\text{loc}}((t_0, t_1]; H)$, $\varphi^t(u(\cdot)) \in L^1(J)$, $u(t) \in D(\partial\varphi^t)$ for a.e. $t \in J$, and if there exists a function $g \in L^2_{\text{loc}}(J; H)$ such that $g(t) \in G(t, u(t))$ for a.e. $t \in J$ and

$$f(t) - g(t) - u'(t) \in \partial\varphi^t(u(t)), \quad \text{a.e. } t \in J.$$

(ii) For any interval $J$ in $R_+$ and $f \in L^2_{\text{loc}}(J; H)$, a function $u : J \to H$ is called a solution of (2.1) on $J$, if it is a solution of (2.1) on every compact subinterval of $J$ in the sense of (i).

(iii) Let $J$ be any interval in $R_+$ with initial time $s \in R_+$. For $f \in L^2_{\text{loc}}(J; H)$, a function $u : J \to H$ is called a solution of the Cauchy problem for (2.1) on $J$ with given initial value $u_0 \in H$, if it is a solution of (2.1) on $J$ satisfying $u(s) = u_0$.

Throughout this paper, let $\{a_r\} := \{a_r; r \geq 0\}$ and $\{b_r\} := \{b_r; r \geq 0\}$ be families of real functions in $W^1_{\text{loc}}(R_+)$ and $W^1_{\text{loc}}(R_+)$, respectively, such that

$$\sup_{t \in R_+} |a'_r|_{L^2(t,t+1)} + \sup_{t \in R_+} |b'_r|_{L^1(t,t+1)} < +\infty \quad \text{for each } r \geq 0.$$

Now we define the class $\Phi(\{a_r\}, \{b_r\})$ of time-dependent convex function $\varphi^t$.

**Definition 2.2.** $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$ if and only if $\varphi^t$ is a proper l.s.c. convex function on $H$ satisfying the following properties (Φ1)-(Φ3):

(Φ1) For each $r > 0$, $s$, $t \in R_+$ and $z \in D(\varphi^s)$ with $|z|_H \leq r$, there exists $\tilde{z} \in D(\varphi^t)$ such that

$$|\tilde{z} - z|_H \leq |a_r(t) - a_r(s)|(1 + |\varphi^s(z)|^{1/2})$$

and

$$\varphi^t(\tilde{z}) - \varphi^s(z) \leq |b_r(t) - b_r(s)|(1 + |\varphi^s(z)|).$$

(Φ2) There exists a positive constant $C_1 > 0$ such that

$$\varphi^t(z) \geq C_1|z|_H^2, \quad \forall t \in R_+, \forall z \in D(\varphi^t).$$

(Φ3) For each $k > 0$ and $t \in R_+$, the level set $\{z \in H; \varphi^t(z) \leq k\}$ is compact in $H$. 

3
Next, we introduce the class $\mathcal{G}(\{\varphi_1\})$ of time-dependent multivalued perturbation $G(t, \cdot)$ associated with $\{\varphi_i\} \in \Phi(\{a_r\}, \{b_r\})$.

**Definition 2.3.** $\{G(t, \cdot)\} \in \mathcal{G}(\{\varphi_i\})$ if and only if $G(t, \cdot)$ is a multivalued operator from $D(G(t, \cdot)) \subset H$ into $H$ which fulfills the following conditions (G1)-(G5):

(G1) $D(\varphi_i) \subset D(G(t, \cdot)) \subset H$ for any $t \in R_+$. And for any interval $J \subset R_+$ and $v \in L^2_{loc}(J; H)$ with $v(t) \in D(\varphi_i)$ for a.e. $t \in J$, there exists a strongly measurable function $g(\cdot)$ on $J$ such that

$$g(t) \in G(t, v(t)) \text{ for a.e. } t \in J.$$

(G2) $G(t, z)$ is a convex subset of $H$ for any $z \in D(\varphi_i)$ and $t \in R_+$.

(G3) There are positive constants $C_2, C_3$ such that

$$|g|^2_H \leq 2C_2\varphi_i(z) + C_3, \quad \forall t \in R_+, \forall z \in D(\varphi_i), \forall g \in G(t, z).$$

(G4) (demi-closedness) If $z_n \in D(\varphi_{i+n}), g_n \in G(t_n, z_n), \{t_n\} \subset R_+, \{\varphi_{i+n}(z_n)\}$ is bounded, $z_n \to z$ in $H$, $t_n \to t$ and $g_n \to g$ weakly in $H$ as $n \to +\infty$, then $g \in G(t, z)$.

(G5) For each bounded subset $B$ of $H$, there exist positive constants $C_4(B)$ and $C_5(B)$ such that

$$\varphi_i(z) + (g, z - b)_H \geq C_4(B)|z|^2_H - C_5(B),$$

$$\forall t \in R_+, \forall g \in G(t, z), \forall z \in D(\varphi_i), \forall b \in B.$$

For given $\{\varphi_i\} \in \Phi(\{a_r\}, \{b_r\}), \{G(t, \cdot)\} \in \mathcal{G}(\{\varphi_i\})$ and a forcing term $f \in L^2_{loc}(R_+; H)$, we consider the following evolution equation

\[(E)_s \quad u'(t) + \partial \varphi_i(u(t)) + G(t, u(t)) \ni f(t) \quad \text{in } H, \quad t > s\]

for each $s \in R_+$.

Now let us recall the known results on the existence and global estimates of solutions for the Cauchy problem of $(E)_s$:

(A) [Existence of solution for $(E)_s$] (cf. [21, Theorem II, III])

The Cauchy problem for $(E)_s$ has at least one solution $u$ on $J = [s, +\infty)$ such that $(\cdot - s)^{\frac{1}{2}} u' \in L^2_{loc}(J; H)$, $(\cdot - s)\varphi_i(u(\cdot)) \in L^\infty_{loc}(J)$ and $\varphi_i(u(\cdot))$ is absolutely continuous on any compact subinterval of $(s, +\infty)$, provided that $u_0 \in D(\varphi_i)$. In particular, if $u_0 \in D(\varphi_i)$, then the solution $u$ satisfies that $u' \in L^2_{loc}(J; H)$ and $\varphi_i(u(\cdot))$ is absolutely continuous on any compact interval in $J$.

(B) [Global boundedness of solutions for $(E)_s$] (cf. [25, Theorem 2.2])

Suppose that

$$S_f := \sup_{t \in R_+} |f|_{L^2(t, t+1; H)} < +\infty.$$
Then, the solution \( u \) of the Cauchy problem for \( (E)_s \) on \( [s, +\infty) \) satisfies the following global estimate:

\[
\sup_{t \geq s} |u(t)|_H^2 + \sup_{t \geq s} \int_t^{t+1} \varphi'(u(\tau)) d\tau \leq N_1(1 + S_f^2 + |u_0|_H^2),
\]

where \( N_1 \) is a positive constant independent of \( f, s \in R_+ \) and \( u_0 \in \overline{D(\varphi^s)} \). Moreover, for each \( \delta > 0 \) and each bounded subset \( B \) of \( H \), there is a constant \( N_2(\delta, B) > 0 \), depending only on \( \delta > 0 \) and \( B \), such that

\[
\sup_{t \geq s+\delta} |u''(t)|_{L^2(t,t+1;H)} + \sup_{t \geq s+\delta} \varphi'(u(t)) \leq N_2(\delta, B)
\]

for the solution \( u \) of the Cauchy problem for \( (E)_s \) on \( [s, +\infty) \) with \( s \in R_+ \) and \( u_0 \in \overline{D(\varphi^s)} \cap B \).

Next, let us remember a notion of convergence of convex functions.

**Definition 2.4.** (cf. [20]) Let \( \psi, \psi_n \ (n \in N) \) be proper l.s.c. and convex functions on \( H \). Then we say that \( \psi_n \) converges to \( \psi \) on \( H \) as \( n \to +\infty \) in the sense of Mosco, if the following two conditions (i) and (ii) are satisfied:

(i) for any subsequence \( \{\psi_{n_k}\} \subset \{\psi_n\} \), if \( z_k \to z \) weakly in \( H \) as \( k \to +\infty \), then

\[
\liminf_{k \to +\infty} \psi_{n_k}(z_k) \geq \psi(z).
\]

(ii) for any \( z \in D(\psi) \), there is a sequence \( \{z_n\} \) in \( H \) such that

\[
z_n \to z \text{ in } H \text{ as } n \to +\infty, \quad \lim_{n \to +\infty} \psi_n(z_n) = \psi(z).
\]

Now, we recall a convergence result (cf. [25, Lemma 4.1]) as follows.

(C) Let \( \{\varphi_n^t\} \in \Phi(\{a_r\}, \{b_r\}) \), \( \{G_n(t, \cdot)\} \in \mathcal{G}(\{\varphi_n^t\}) \) with common positive constants \( C_1, C_2, C_3, C_4(B) \) and \( C_5(B) \), \( \{f_n\} \subset L^2(J; H) \), \( J = [s, t_1] \subset R_+ \) and \( u_{0,n} \in \overline{D(\varphi_n^s)} \) for \( n = 1, 2, \cdots \). Assume that

(i) \( \varphi_n^t \) converges to \( \varphi^t \) on \( H \) in the sense of Mosco [20] for each \( t \in J \) (as \( n \to +\infty \)) and \( \bigcup_{n=1}^{+\infty} \{z \in H; \ \varphi_n^t(z) \leq k \} \) is relatively compact in \( H \) for every real \( k > 0 \) and \( t \in J \), where \( \{\varphi_n^t\} \in \Phi(\{a_r\}, \{b_r\}) \) and \( \varphi_n^t = \varphi^t \) if \( n = +\infty \).

(ii) if \( z_n \in D(\varphi_n^s), g_n \in G_n(t_n, z_n), \{t_n\} \subset R_+, \{\varphi_n^s(z_n)\} \) is bounded, \( z_n \to z \) in \( H \), \( t_n \to t \) and \( g_n \to g \) weakly in \( H \) as \( n \to +\infty \), then \( g \in G(t, z) \), where \( \{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^t\}) \).

(iii) \( f_n \rightharpoonup f \) weakly in \( L^2(J; H) \) for some \( f \in L^2(J; H) \) and \( u_{0,n} \to u_0 \) in \( H \) for some \( u_0 \in \overline{D(\varphi^s)} \).

5
Denote by $u$ the solution of the Cauchy problem for $(E)_s$ on $J$ with $u(s) = u_0$ and by $u_n$ the solution of the Cauchy problem for $(E)_s$ with $\varphi^t$, $G$, $f$ replaced by $\varphi'_n$, $G_n$, $f_n$, and with $u_n(s) = u_{0,n}$. Then $u_n$ converges to $u$ on $J$ in the sense that

$$ u_n \to u \text{ in } C(J;H), \quad (\cdot - s)^{1/2}u'_n \to (\cdot - s)^{1/2}u \text{ weakly in } L^2(J;H), $$

$$ \int_J \varphi'_n(u_n(t))dt \to \int_J \varphi'(u(t))dt \quad \text{as } n \to +\infty. $$

### 3 Attractor for periodic multivalued dynamical system

In this section let us recall the known results obtained in [29] for a $T_0$-periodic system in $H$, of the form:

$$(P)_s \quad u'(t) + \partial \varphi^t_p(u(t)) + G_p(t,u(t)) \ni f_p(t) \quad \text{in } H, \quad t > s$$

for each $s \in R_+$, where $\varphi^t_p$, $G_p(t,\cdot)$ and $f_p(t)$ are $T_0$-periodic, namely periodic in time with the same period $T_0$, $0 < T_0 < +\infty$.

#### Definition 3.1.

Let $T_0$ be a positive number. Then

(i) $\Phi_p(\{a_r\}, \{b_r\}; T_0)$ is the set of all $\{\varphi^t_p\} \in \Phi(\{a_r\}, \{b_r\})$ satisfying $T_0$-periodicity condition:

$$ \varphi^{t+T_0}_p(\cdot) = \varphi^t_p(\cdot) \quad \text{on } H, \quad \forall t \in R_+. $$

(ii) $G_p(\{\varphi^t_p\}; T_0)$ is the set of all $\{G_p(t,\cdot)\} \in G(\{\varphi^t_p\})$ satisfying $T_0$-periodicity condition:

$$ G_p(t + T_0, \cdot) = G_p(t, \cdot) \quad \text{in } H, \quad \forall t \in R_+. $$

Throughout this section we assume that $\{\varphi^t_p\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0)$, $\{G_p(t,\cdot)\} \in G_p(\{\varphi^t_p\}; T_0)$ and $f_p \in L^1_{loc}(R_+; H)$ is $T_0$-periodic in time, namely

$$ f_p(t + T_0) = f_p(t) \quad \text{in } H, \quad \forall t \in R_+. $$

Here we note that $(P)_s$ can be considered as $(E)_s$ in Section 2. So, by the result (A) in Section 2, the Cauchy problem for $(P)_s$ has at least one solution $u$ on $[s, +\infty)$. Hence we can define the multivalued dynamical process associated with $(P)_s$ as follows:

#### Definition 3.2.

For every $0 \leq s \leq t < +\infty$ we denote by $U(t,s)$ the mapping from $\overline{D}(\varphi^s_p)$ into $\overline{D}(\varphi^t_p)$ which assigns to each $u_0 \in \overline{D}(\varphi^s_p)$ the set

$$ U(t,s)u_0 := \left\{ z \in H \bigg| \begin{array}{c} \text{There is a solution } u \text{ of } (P)_s \text{ on } [s, +\infty) \\ \text{such that} \\ u(s) = u_0 \text{ and } u(t) = z. \end{array} \right\}. $$

Then we easily see the following properties of $\{U(t,s)\} := \{U(t,s); 0 \leq s \leq t < +\infty\}$:
(U1) \( U(s, s) = I \) on \( \overline{D(\varphi^p)} \) for any \( s \in R_+ \);

(U2) \( U(t_2, s)z = U(t_2, t_1)U(t_1, s)z \) for any \( 0 \leq s \leq t_1 \leq t_2 < +\infty \) and \( z \in \overline{D(\varphi^p)} \);

(U3) \( U(t + T_0, s + T_0)z = U(t, s)z \) for any \( 0 \leq s \leq t < +\infty \) and \( z \in \overline{D(\varphi^p)} \), that is, \( U \) is \( T_0 \)-periodic.

(U4) \( \{U(t, s)\} \) has the following demi-closedness:

- If \( 0 \leq s_n \leq t_n < +\infty \), \( s_n \to s \), \( t_n \to t \), \( z_n \in \overline{D(\varphi^p)} \), \( z \in \overline{D(\varphi^p)} \), \( z_n \to z \) in \( H \)
  and a element \( w_n \in U(t_n, s_n)z_n \) converges to some element \( w \in H \) as \( n \to +\infty \),
  then \( w \in U(t, s)z \).

Next we define the discrete dynamical system in order to construct a global attractor for \((P)_s\).

**Definition 3.3.** Let \( U(\cdot, \cdot) \) be the solution operator for \((P)_s\) defined by Definition 3.2. Then

(i) For each \( \tau \in R_+ \), we denote by \( U_\tau \) the \( T_0 \)-step mapping from \( \overline{D(\varphi^p)} \) into \( \overline{D(\varphi^p)} \), namely,

\[
U_\tau := U(\tau + T_0, \tau).
\]

(2) For any \( k \in Z_+ := N \cup \{0\} \), we define

\[
U_\tau^k := U_\tau \circ U_\tau \circ \cdots \circ U_\tau.
\]

Clearly we have \( U_\tau^k = U(\tau + kT_0, \tau) \) for any \( \tau \in R_+ \) and \( k \in Z_+ \).

Now, let us recall the known result on the existence of global attractors for discrete multivalued dynamical systems \( U_\tau \) associated with \((P)_s\).

**Theorem 3.1.** (cf. [29, Theorem 3.1]) Assume that \( \{\varphi^p\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0) \), \( \{G_p(t, \cdot)\} \in G_p(\{\varphi^p\}; T_0) \), \( f_p \in L^2_{\text{loc}}(R_+; H) \) satisfies the \( T_0 \)-periodicity condition (3.3). Then, for each \( \tau \in R_+ \), there exists a subset \( \mathcal{A}_\tau \) of \( \overline{D(\varphi^p)} \) such that

(i) \( \mathcal{A}_\tau \) is non-empty and compact in \( H \);

(ii) for each bounded set \( B \) in \( H \) and each number \( \epsilon > 0 \) there exists \( N_{B, \epsilon} \in N \) such that

\[
dist_H(U_\tau^kz, \mathcal{A}_\tau) < \epsilon
\]

for all \( z \in \overline{D(\varphi^p)} \cap B \) and all \( k \geq N_{B, \epsilon} \);

(iii) \( U_\tau^k \mathcal{A}_\tau = \mathcal{A}_\tau \) for any \( k \in N \).

**Remark 3.1.** By [29, Lemma 3.1] we can get the compact absorbing set \( B_{0, \epsilon} \) of \( \overline{D(\varphi^p)} \) for \( U_\tau \) such that for each bounded subset \( B \) of \( H \) there is a positive integer \( n_B \) (independent of \( \tau \in R_+ \)) satisfying

\[
U_\tau^n \left( \overline{D(\varphi^p)} \cap B \right) \subset B_{0, \epsilon} \quad \text{for all } n \geq n_B.
\]
Then we observe that the global attractor $A_\tau$ is given by the $\omega$-limit set of the absorbing set $B_{0,\tau}$ for $U_\tau$, i.e.

$$A_\tau = \bigcap_{n \in \mathbb{Z}^+} \bigcup_{k \geq n} U_\tau^k B_{0,\tau}.$$ 

The next theorem is concerned with a relationship between two global attractors $A_s$ and $A_\tau$. For detail proof, see [29].

**Theorem 3.2.** (cf. [29, Theorem 3.2]) Suppose the same assumptions are made as in Theorem 3.1. Let $A_s$ and $A_\tau$ be the global attractors for $U_s$ and $U_\tau$, with $0 \leq s \leq \tau \leq T_0$, respectively. Then, we have

$$A_\tau = U(\tau, s) A_s,$$

where $U(\tau, s)$ is the $T_0$-periodic process given in Definition 3.2.

**Remark 3.2.** By Theorem 3.1 (iii) and Theorem 3.2, we see that the global attractor $A_\tau$ for $U_\tau$ is $T_0$-periodic in $\tau$. In fact, for each $\tau \in R^+$ choose $m_\tau \in \mathbb{Z}^+$ and $\sigma_\tau \in [0, T_0)$ so that $\tau = \sigma_\tau + m_\tau T_0$. Then, we have $A_\tau = A_{\sigma_\tau}$.

The third known result is the existence of a global attractor for the $T_0$-periodic multivalued dynamical system $(P)_s$.

**Theorem 3.3.** (cf. [29, Theorem 3.3]) Under the same assumptions as Theorem 3.1, put

$$A := \bigcup_{0 \leq \tau \leq T_0} A_\tau,$$

where $A_\tau$ is as obtained in Theorem 3.1. Then, $A$ has the following properties:

(i) $A$ is non-empty and compact in $H$;

(ii) for each bounded set $B$ in $H$ and each number $\epsilon > 0$ there exists a finite time $T_{B, \epsilon} > 0$ such that

$$\text{dist}_H(U(t + \tau, \tau)z, A) < \epsilon$$

for all $\tau \in R^+$, all $z \in D(\varphi^\tau) \cap B$ and all $t \geq T_{B, \epsilon}$.

**Remark 3.3.** In [29, Section 4] the characterization of the $T_0$-periodic global attractor was discussed. The author proved that for each time $\tau \in R^+$ the global attractor $A_\tau$ for the discrete multivalued dynamical system $U_\tau$ coincides with the cross-section of the family of all global bounded complete trajectories for the $T_0$-periodic system $(P)_s$.

### 4 Attractors of asymptotically periodic multivalued dynamical system

Throughout this section, let $M > 0$ be a fixed (sufficiently) large positive number. Now we put

$$\Psi_M := \left\{ \psi; \; \text{\psi is proper, l.s.c. and convex on } H, \; \exists z \in D(\psi) \text{ s.t. } |z|_H \leq M, \; \psi(z) \leq M \right\}.$$
Then let us introduce the notion of a metric topology on \( \Psi_M \) which was introduced in [16].

Given \( \varphi, \psi \in \Psi_M \), we define \( \rho(\varphi, \psi; \cdot) : D(\varphi) \to R \) by putting

\[
\rho(\varphi, \psi; z) = \inf \{\max(|y - z|_H, \psi(y) - \varphi(z)) : y \in D(\psi)\}
\]

for each \( z \in D(\varphi) \), and for each \( r \geq M \)

\[
\rho_r(\varphi, \psi) := \sup_{z \in L_p(r)} \rho(\varphi, \psi; z),
\]

where \( L_p(r) := \{z \in D(\varphi); |z|_H \leq r, \varphi(z) \leq r\} \). Moreover, for each \( r \geq M \), we define the functional \( \pi_r(\cdot, \cdot) \) on \( \Psi_M \times \Psi_M \) by

\[
\pi_r(\varphi, \psi) := \rho_r(\varphi, \psi) + \rho_r(\psi, \varphi) \quad \text{for } \varphi, \psi \in \Psi_M.
\]

Then, according to [16, Proposition 3.1], we can define a complete metric topology on \( \Psi_M \) so that the convergence \( \psi_n \to \psi \) in \( \Psi_M \) (as \( n \to +\infty \)) if and only if

\[
\pi_r(\psi_n, \psi) \to 0 \quad \text{for every } r \geq M.
\]

Now by using the above topology on \( \Psi_M \), we consider an asymptotically \( T_0 \)-periodic system as follows.

**Definition 4.1.** Assume \( \{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\}) \cap \Psi_M, \{G(t, \cdot)\} \in G(\{\varphi^t\}) \) and \( f \in L_{loc}^2(R_+; H) \). Then the system

\[
(\text{AP}) \quad v'(t) + \partial \varphi^t(v(t)) + G(t, v(t)) \ni f(t) \quad \text{in } H, \ t > s (\geq 0)
\]

is asymptotically \( T_0 \)-periodic, if there are \( \{\varphi_p^t\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0) \cap \Psi_M, \{G_p(t, \cdot)\} \in G_p(\{\varphi_p^t\}; T_0) \) and a \( T_0 \)-periodic function \( f_p \in L_{loc}^2(R_+; H) \) such that

**A1** (Convergence of \( \varphi^t - \varphi_p^t \to 0 \) as \( t \to +\infty \)) For each \( r \geq M \),

\[
J_m^{(r)} := \sup_{\sigma \in [0, T_0]} \pi_r(\varphi^m T_0 + \sigma, \varphi_p^\sigma) \to 0 \quad \text{as } m \to +\infty;
\]

**A2** (Convergence of \( G(t, \cdot) - G_p(t, \cdot) \to 0 \) as \( t \to +\infty \)) If \( \{\tau_n\} \subset [0, T_0], \{m_n\} \subset Z_+, m_n \to +\infty, z_n \in D(\varphi^m T_0 + \tau_n), g_n \in G(m_n T_0 + \tau_n, z_n), \{\varphi^m T_0 + \tau_n(z_n)\} \) is bounded, \( z_n \to z \) in \( H, \tau_n \to \tau \) and \( g_n \to g \) weakly in \( H \) (as \( n \to +\infty \)), then

\[
g \in G_p(\tau, z);
\]

**A3** (Convergence of \( f(t) - f_p(t) \to 0 \) as \( t \to +\infty \))

\[
|f(m T_0 + \cdot) - f_p|_{L^2(0, T_0; H)} \to 0 \quad \text{as } m \to +\infty.
\]
By Definition 4.1 we easily see that a limiting system for \((AP)_s\) is a \(T_0\)-periodic one of the form:

\[
(P)_s \quad u'(t) + \partial \varphi^T_p(u(t)) + G_p(t, u(t)) \ni f_p(t) \quad \text{in } H, \ t > s (\geq 0).
\]

Here we note that \((AP)_s\) is also considered as \((E)_s\). So, by the result (A) in Section 2, the Cauchy problem for \((AP)_s\) has at least one solution \(v \) on \([s, +\infty)\). Hence we can define the multivalued dynamical system associated with \((AP)_s\) as follows:

**Definition 4.2.** For every \(0 \leq s \leq t < +\infty\) we denote by \(E(t, s)\) the mapping from \(\overline{D(\varphi^s)}\) into \(\overline{D(\varphi^t)}\) which assigns to each \(v_0 \in \overline{D(\varphi^s)}\) the set

\[
E(t, s)v_0 := \left\{ z \in H \left| \begin{array}{l}
\text{There is a solution } v \text{ of } (AP)_s \text{ on } [s, +\infty) \\
\text{such that } v(s) = v_0 \text{ and } v(t) = z.
\end{array} \right. \right\}.
\]

Then we easily see that \(\{E(t, s)\} := \{E(t, s) ; 0 \leq s \leq t < +\infty\}\) has the following evolution properties:

**E1)** \(E(s, s) = I \) on \(\overline{D(\varphi^s)}\) for any \(s \in R_+;\)

**E2)** \(E(t_2, s)z = E(t_2, t_1)E(t_1, s)z\) for any \(0 \leq s \leq t_1 \leq t_2 < +\infty\) and \(z \in \overline{D(\varphi^s)};\)

**E3)** \(\{E(t, s)\}\) has the following demi-closedness:

- If \(0 \leq s_n \leq t_n < +\infty, s_n \to s, t_n \to t, z_n \in \overline{D(\varphi^{s_n})}, z \in \overline{D(\varphi^t)}, z_n \to z\) in \(H\) and a element \(w_n \in E(t_n, s_n)z_n\) converges to some element \(w \in H\) as \(n \to +\infty,\) then \(w \in E(t, s)z\)

We begin with the definition of a discrete \(\omega\)-limit set for \(E(\cdot, \cdot).\)

**Definition 4.3.** (Discrete \(\omega\)-limit set for \(E(\cdot, \cdot)\)) Let \(\tau \in R_+\) be fixed. Let \(B(H)\) be a family of bounded subsets of \(H\). Then for each \(B \in B(H),\) the set

\[
\omega_\tau(B) := \bigcap_{n \in Z_+} \bigcup_{k \geq n, m \geq Z_+} E(kT_0 + mT_0 + \tau, mT_0 + \tau)(\overline{D(\varphi^{mT_0+\tau})} \cap B)
\]

is called the discrete \(\omega\)-limit set of \(B\) under \(E(\cdot, \cdot).\)

**Remark 4.1.** By definition of the discrete \(\omega\)-limit set \(\omega_\tau(B),\) it is easy to see that \(x \in \omega_\tau(B)\) if and only if there exist sequences \(\{k_n\} \subset Z_+ \) with \(k_n \uparrow +\infty, \{m_n\} \subset Z_+,\) \(\{z_n\} \subset B\) with \(z_n \in \overline{D(\varphi^{m_nT_0+\tau})}\) and \(\{x_n\} \subset H\) with \(x_n \in E(k_nT_0 + m_nT_0 + \tau, m_nT_0 + \tau)z_n\) such that

\[
x_n \to x \text{ in } H \text{ as } n \to +\infty.
\]

Now let us mention main theorems in this paper.

**Theorem 4.1.** (Discrete attractors of \((AP)_\tau\)) For each \(\tau \in R_+,\) let \(A_\tau\) be the global attractor of \(T_0\)-periodic dynamical systems \(U_\tau,\) which is obtained in Section 3. For \(\{\varphi^t\} \in\)
\[ \Phi(\{a_r\}, \{b_r\}) \cap \Psi_M, \{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^t\}) \text{ and } f \in L^2_{\text{loc}}(\mathbb{R}_+; H), \text{ we assume that the system } (AP)_s \text{ is asymptotically } T_0\text{-periodic. Here we put } \\
A^*_s := \bigcup_{B \in \mathcal{B}(H)} \omega_*(B). \quad (4.1) \]

Then, we have

(i) \( A^*_s(\subset D(\varphi^*_t)) \) is non-empty and compact in \( H \);

(ii) for each bounded set \( B \in \mathcal{B}(H) \) and each number \( \epsilon > 0 \) there exists \( N_{B,s} \in \mathbb{N} \) such that

\[ \text{dist}_H(E(kT_0 + \tau, \tau)z, A^*_s) < \epsilon \]

for all \( z \in \overline{D(\varphi^*_t)} \cap B \) and all \( k \geq N_{B,s} \);

(iii) \( A^*_s \subset U^l A^*_s \subset A^*_s \) for any \( l \in \mathbb{N} \), where \( U^l \) is the discrete dynamical system for \( (P)_\tau \) given in Definition 3.3.

Remark 4.2. By the definition of the discrete \( \omega \)-limit set \( \omega_*(B) \) and \( A^*_s \), we easily see that

\[ A^*_s = A^*_{\tau+nT_0}, \quad \forall n \in \mathbb{N}. \]

Hence \( A^*_s \) is \( T_0 \)-periodic in time in a sense of the above.

The second main theorem is concerned with a relationship between two attractors \( A^*_s \) and \( A^*_\tau \).

**Theorem 4.2.** Suppose the same assumptions are made as in Theorem 4.1. Let \( A^*_s \) and \( A^*_\tau \) be discrete attractors for \( E(\cdot, s) \) and \( E(\cdot, \tau) \) with \( 0 \leq s \leq \tau < +\infty \), respectively. Then,

\[ A^*_\tau \subset U(\tau, s)A^*_s. \]

where \( U(\tau, s) \) is the \( T_0 \)-periodic process for \( (P)_s \) which is given in Definition 3.2.

By Theorems 4.1-4.2, we can get the attractor for asymptotic \( T_0 \)-periodic system \( (AP)_\tau \).

**Theorem 4.3.** (Global attractor for \( (AP)_\tau \)) Suppose the same assumptions are made as in Theorem 4.1. For any \( \tau \in \mathbb{R}_+ \), let \( A^*_\tau \) be the discrete attractor for \( E(\cdot, \tau) \) obtained in Theorem 4.1. Here we put

\[ A^* := \bigcup_{\tau \in [0, T_0]} A^*_\tau. \quad (4.2) \]

Then, for any bounded set \( B \in \mathcal{B}(H) \),

\[ \bigcap_{s \geq 0} \bigcup_{t \geq s, \tau \in \mathbb{R}_+} E(t + \tau, \tau)(D(\varphi^\tau) \cap B) \subset A^*. \quad (4.3) \]
By Theorem 4.3, the set \( \mathcal{A}^* \) can be called the global attractor of \((\text{AP})_\tau\).

Here we give some key lemmas.

**Lemma 4.1.** If \( \{s_n\} \subset R_+ \), \( \{\tau_n\} \subset R_+ \), \( s \in R_+ \), \( \tau \in R_+ \), \( s_n \to s \), \( \tau_n \to \tau \), \( \{m_n\} \subset Z_+ \) with \( m_n \to +\infty \), \( z_n \in D(\varphi^{m_n T_0 + s_n}) \), \( z \in D(\varphi^\tau_p) \), \( z_n \to z \) in \( H \) and a element \( w_n \in E(m_n T_0 + \tau_n + s_n, m_n T_0 + s_n) z_n \) converges to some element \( w \in H \) as \( n \to +\infty \), then \( w \in U(\tau + s, s) z \)

**Proof.** Since \( \tau_n \to \tau \), without loss of generality we may assume that there exists a finite time \( T > 0 \) such that \( \{\tau_n\} \subset [0, T] \) and \( \tau \in [0, T] \). By \( w_n \in E(m_n T_0 + \tau_n + s_n, m_n T_0 + s_n) z_n \), there is a solution \( v_n \) of \((\text{AP})_{m_n T_0 + s_n} \) on \([m_n T_0 + s_n, +\infty)\) such that

\[
v_n(m_n T_0 + \tau_n + s_n) = w_n \quad \text{and} \quad v_n(m_n T_0 + s_n) = z_n.
\]

Now we put \( u_n(t) := v_n(t + m_n T_0 + s_n) \), then we easily see that \( u_n \) is the solution for

\[
\begin{align*}
u_n'(t) + \partial \varphi^{t + m_n T_0 + s_n}(u_n(t)) + G(t + m_n T_0 + s_n, u_n(t)) &\ni f(t + m_n T_0 + s_n), \quad t > 0, \\
\end{align*}
\]

\[
u_n(0) = z.
\]

Let \( \delta \in (0, 1) \) be fixed. Since \( z_n \to z \) in \( H \) as \( n \to +\infty \), \( \{z_n\} \) is bounded in \( H \). Hence, from global estimates of solutions (cf. (B) in Section 2) it follows that there is a positive constant \( M_\delta > 0 \) (independent of \( n \)) satisfying

\[
\sup_{t \geq \delta} |u_n(t)|_H^2 + \sup_{t \geq \delta} |u_n'|^2_{L^2(t, t + 1; H)} + \sup_{t \geq \delta} \varphi^{t + m_n T_0 + s_n}(u_n(t)) \leq M_\delta. \tag{4.4}
\]

By [16, Lemma 4.1] we note that the convergence assumption (A1) implies

\[
\varphi^{t + m_n T_0 + s_n} \longrightarrow \varphi^t_p \quad \text{in the sense of Mosco} \ [20] \tag{4.5}
\]

for each \( t \geq 0 \) as \( n \to +\infty \). Moreover by the same argument in [10, Lemma 3.1] we can prove that

\[
\bigcup_{n=1}^{+\infty} \{z \in H; \ \varphi^{t + m_n T_0 + s_n}(z) \leq k\} \quad \text{is relatively compact in} \ H \tag{4.6}
\]

for every real \( k > 0 \) and \( t \geq 0 \), where \( \varphi^{t + m_n T_0 + s_n} = \varphi^t_p \) if \( n = +\infty \). Therefore, by (4.4)-(4.6), (A2), (A3) and the convergence result (C) in Section 2, (by taking a subsequence of \( \{n\} \), if necessary) we see that there is a function \( u_\delta \) such that

\[
u_\delta'(t) + \partial \varphi^t_p(u_\delta(t)) + G_p(t + s, u_\delta(t)) \ni f_p(t + s), \quad t > \delta.
\]

By the standard diagonal process and the same argument in [21, Lemma 3.10], we can construct the solution \( u \) on \([0, +\infty)\) satisfying

\[
\begin{align*}
\begin{cases}
 u'(t) + \partial \varphi^t_p(u(t)) + G_p(t + s, u(t)) \ni f_p(t + s), \quad t > 0, \\
u(0) = z
\end{cases}
\end{align*}
\]

and

\[
u \to u \quad \text{in} \ C([0, T]; H) \quad \text{as} \ n \to +\infty. \tag{4.7}
\]
Then, by (4.7) and \( u_n(\tau_n) = w_n \) we have \( u(\tau) = w \), which implies that \( w \in U(\tau + s, s)z \).

By (B) in Section 2, for each \( B \in \mathcal{B}(H) \) we can choose constants \( r_B > 0 \) and \( M_B > 0 \) so that

\[
|v|_H \leq r_B \quad \text{and} \quad \varphi^{t+s}(v) \leq M_B, \tag{4.8}
\]

for any \( s \in R_+, t \geq T_0, z \in \overline{D(\varphi^s)} \cap B \) and \( v \in E(t + s)z \). Hence it follows from condition (A1) that for each \( m \in Z_+, \tau \in [0, T_0], n \in N \) and \( z \in D(\varphi^m) \cap B \) there is \( \tilde{z} := \tilde{z}_{mT_0 + \tau, z, nT_0} \in D(\varphi^\tau) \) such that

\[
|\tilde{z} - v|_H \leq J_{m+n}^{(r_B + M_B + \mu)},
\]

(hence \( |\tilde{z}|_H \leq r_B + J_{m+n}^{(r_B + M_B + \mu)} \))

and

\[
\varphi^\tau(\tilde{z}) - \varphi^{nT_0 + mT_0 + \tau}(v) \leq J_{m+n}^{(r_B + M_B + \mu)},
\]

(hence \( \varphi^\tau(\tilde{z}) \leq M_B + J_{m+n}^{(r_B + M_B + \mu)} \)).

where \( v \in E(nT_0 + mT_0 + \tau, mT_0 + \tau)z \).

Since \( J_{k}^{(r_B + M_B + \mu)} \rightarrow 0 \) as \( k \rightarrow +\infty \), there is a number \( N_0 \in N \) such that

\[ J_{k}^{(r_B + M_B + \mu)} \leq 1, \quad \forall k > N_0. \]

Now, put \( J_0 := 1 + \sup_{1 \leq k \leq N_0} J_{k}^{(r_B + M_B + \mu)} < +\infty \). Then, we define the bounded set \( \mathcal{B}_\tau \) by

\[ \mathcal{B}_\tau := \{ z \in H; |z|_H \leq r_B + J_0 \} \cap \overline{D(\varphi^\tau)}. \]

Let \( B_{0, \tau} \) be the compact absorbing set for \( U_\tau \) introduced by Remark 3.1. Then, we see that there exists a number \( \bar{N} \in N \) so that

\[ U_l^{\mathcal{B}_\tau} \subset B_{0, \tau}, \quad \forall l \geq \bar{N}. \]  \tag{4.9}

The next lemma is very important to prove Theorem 4.1 (iii).

**Lemma 4.2.** Let \( \tau \in R_+ \) and \( B_{0, \tau} \) be the compact absorbing set for \( U_\tau \). Then we have

\[ \omega_\tau(B) \subset B_{0, \tau}, \quad \forall B \in \mathcal{B}(H). \]

**Proof.** At first we assume \( \tau \in [0, T_0] \).

For each \( B \in \mathcal{B}(H) \), let \( x \) be any element of \( \omega_\tau(B) \). Then, it follows from Remark 4.1 that there exist sequences \( \{k_n\} \subset Z_+ \) with \( k_n \rightarrow +\infty \), \( \{m_n\} \subset Z_+ \), \( \{z_n\} \subset B \) with \( z_n \in \overline{D(\varphi^{m_nT_0 + \tau})} \) and \( \{x_n\} \subset H \) with \( x_n \in E(k_nT_0 + m_nT_0 + \tau, m_nT_0 + \tau)z_n \) such that

\[ x_n \rightarrow x \quad \text{in} \quad H \quad \text{as} \quad n \rightarrow +\infty. \]  \tag{4.10}

Let \( \bar{N} \) be the positive integer obtained in (4.9). Then by (E2) we have

\[ x_n \in E(k_nT_0 + m_nT_0 + \tau, k_nT_0 - \bar{N}T_0 + m_nT_0 + \tau) \]

13
\[ \circ E(k_n T_0 - \tilde{N} T_0 + m_n T_0 + \tau, m_n T_0 + \tau) z_n \]  
(4.11)

for any \( n \) with \( k_n \geq \tilde{N} + 1 \).

Hence, there exists an element \( y_n \in E(k_n T_0 - \tilde{N} T_0 + m_n T_0 + \tau, m_n T_0 + \tau) z_n \) such that

\[ x_n \in E(k_n T_0 + m_n T_0 + \tau, k_n T_0 - \tilde{N} T_0 + m_n T_0 + \tau) y_n. \]  
(4.12)

Since \( \{z_n\} \subset B \), we see that

\[ |y_n|_H \leq r_B \quad \text{and} \quad \varphi^{k_n T_0 - \tilde{N} T_0 + m_n T_0 + \tau}(y_n) \leq M_B \quad \text{for any} \quad n \quad \text{with} \quad k_n \geq \tilde{N} + 1, \]

where \( r_B \) and \( M_B \) are same positive constants in (4.8).

From the convergence condition (A1) it follows that for \( y_n \in E(k_n T_0 - \tilde{N} T_0 + m_n T_0 + \tau, m_n T_0 + \tau) z_n \) there is \( \tilde{z}_n \in D(\varphi_p^\tau) \) such that

\[ |\tilde{z}_n - y_n|_H \leq J^{(r_B + M_B + M)}_{k_n - \tilde{N} + m_n}, \]

(hence \( |\tilde{z}_n|_H \leq r_B + J^{(r_B + M_B + M)}_{k_n - \tilde{N} + m_n} \))

and

\[ \varphi_p^\tau(\tilde{z}_n) \leq M_B + J^{(r_B + M_B + M)}_{k_n - \tilde{N} + m_n}. \]

Since \( \{\tilde{z}_n \subset D(\varphi_p^\tau) ; n \in N \quad \text{with} \quad k_n \geq \tilde{N} + 1\} \subset \tilde{B}_\tau \) is relatively compact in \( H \), we may assume that

\[ \tilde{z}_n \longrightarrow \tilde{z}_\infty \quad \text{in} \quad H \quad \text{as} \quad n \rightarrow +\infty \]

for some \( \tilde{z}_\infty \in H \). Then we easily see that \( \tilde{z}_\infty \in \tilde{B}_\tau \) and

\[ y_n \longrightarrow \tilde{z}_\infty \quad \text{in} \quad H \quad \text{as} \quad n \rightarrow +\infty. \]  
(4.13)

By Lemma 4.1 and (4.10)-(4.13), we observe that

\[ x \in U(\tilde{N} T_0 + \tau, \tau) \tilde{z}_\infty, \]

which implies that

\[ x \in U(\tilde{N} T_0 + \tau, \tau) \tilde{B}_\tau = U_N \tilde{B}_\tau \subset B_{0,\tau}. \]

Hence we have

\[ \omega_\tau(B) \subset B_{0,\tau}. \]

For the general case of \( \tau \in R_+ \), choose positive numbers \( i_\tau \in N \) and \( \tau_0 \in [0, T_0] \) so that \( \tau = \tau_0 + i_\tau T_0 \). Then, we can show \( \omega_\tau(B) \subset B_{0,\tau} \) by the same argument as above. \( \diamond \)

**Proof of Theorem 4.1.** On account of Lemma 4.2 we can get \( A_\tau^* \subset B_{0,\tau} \). Hence, Theorem 4.1 (i) holds. Also, by (4.1) and Remark 4.1 we observe that Theorem 4.1 (ii) holds.

Now, we prove Theorem 4.1 (iii). At first, let us prove that \( A_\tau^* \subset U_\tau^* A_\tau^* \) for any \( l \in N \).
Let \( x \) be any element of \( \mathcal{A}_\tau^* \). By the definition of \( \mathcal{A}_\tau^* \), there are sequences \( \{B_n\} \subset \mathcal{B}(H) \) and \( \{x_n\} \subset H \) with \( x_n \in \omega_\tau(B_n) \) such that

\[
x_n \to x \text{ in } H \quad \text{as } n \to +\infty. \tag{4.14}
\]

Then, for each \( n \) it follows from Remark 4.1 that there exist sequences \( \{k_{n,j}\} \subset \mathbb{Z}_+ \) with \( k_{n,j} \to +\infty \), \( \{m_{n,j}\} \subset \mathbb{Z}_+ \), \( \{z_{n,j}\} \subset B_n \) with \( z_{n,j} \in D(\varphi^{m_{n,j}l_0+\tau}) \) and \( \{v_{n,j}\} \subset H \) with \( v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j} \) such that

\[
v_{n,j} \to x_n \text{ in } H \quad \text{as } j \to +\infty. \tag{4.15}
\]

Let \( l \) be any number in \( \mathbb{N} \), then we see that

\[
v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau) \circ E(k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j}
\]

for \( j \) with \( k_{n,j} \geq l + 1 \). So, there exists an element \( w_{n,j} \in E(k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j} \) such that

\[
v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau)w_{n,j}. \tag{4.16}
\]

By global estimates (B) in Section 2, \( \{w_{n,j} \in H \mid j \in \mathbb{N} \text{ with } k_{n,j} \geq l+1\} \) is relatively compact in \( H \) for each \( n \). Therefore we may assume that the element \( v_{n,j} \) converges to some element \( \tilde{w}_{n,\infty} \in H \) as \( j \to +\infty \). Clearly, \( \tilde{w}_{n,\infty} \in \omega_\tau(B_n) \). Moreover, it follows from Lemma 4.1 and (4.15)-(4.16) that

\[
x_n \in U(lT_0 + \tau, \tau)\tilde{w}_{n,\infty} \subset U(lT_0 + \tau, \tau)\omega_\tau(B_n),
\]

hence, we have

\[
x_n \in \bigcup_{n \geq 1} U_l^j \omega_\tau(B_n), \quad \forall n \geq 1. \tag{4.17}
\]

Here, by the closedness of \( U(\cdot, \cdot) \) we note that for each subset \( X \) of \( B_0, \tau \),

\[
\overline{U_l^jX} \subset U_l^jX, \quad \forall l \in \mathbb{N}. \tag{4.18}
\]

Taking account of Lemma 4.2, (4.14), (4.17) and (4.18), we observe that

\[
x \in \bigcup_{n \geq 1} U_l^j \omega_\tau(B_n) = U_l^j \bigcup_{n \geq 1} \omega_\tau(B_n) \subset U_l^j \bigcup_{n \geq 1} \omega_\tau(B_n) \subset U_l^j \mathcal{A}_\tau^*,
\]

which implies that \( \mathcal{A}_\tau^* \) is semi-invariant under the \( T_0 \)-periodic dynamical systems \( U_{\tau} \), i.e.

\[
\mathcal{A}_\tau^* \subset U_{\tau} \mathcal{A}_\tau^*, \quad \forall l \in \mathbb{N}. \tag{4.19}
\]
Next we shall prove that $U^l \mathcal{A}_r \subset \mathcal{A}_r$ for any $l \in N$. By (4.19), for each $l \in N$

$$U^l \mathcal{A}_r \subset U^l U^n \mathcal{A}_r = U^{l+n} \mathcal{A}_r, \quad \forall n \in N. \quad (4.20)$$

By $\mathcal{A}_r \subset B_{0,\tau}$, (4.20) and the attractive property of $\mathcal{A}_r$, we have

$$U^l \mathcal{A}_r \subset \mathcal{A}_r, \quad \forall l \in N.$$

Therefore we conclude that

$$\mathcal{A}_r \subset U^l \mathcal{A}_r \subset \mathcal{A}_r, \quad \forall l \in N.$$

\[\Diamond\]

**Proof of Theorem 4.2.** Let $x$ be any element of $\mathcal{A}_r$. Then by the definition of $\mathcal{A}_r$, there exist sequences $\{B_n\} \subset \mathcal{B}(H)$ and $\{x_n\} \subset H$ with $x_n \in \omega_\tau(B_n)$ such that

$$x_n \to x \text{ in } H \quad \text{as } n \to +\infty. \quad (4.21)$$

From Remark 4.1 it follows that for each $n$, there are sequences $\{k_{n,j}\} \subset \mathbb{Z}_+$ with $k_{n,j} \to +\infty$, $\{m_{n,j}\} \subset \mathbb{Z}_+$, $\{z_{n,j}\} \subset B_n$ with $z_{n,j} \in D(\varphi^{m_{n,j}T_0+\tau})$ and $\{v_{n,j}\} \subset H$ with $v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j}$ such that

$$v_{n,j} \to x_n \text{ in } H \quad \text{as } j \to +\infty. \quad (4.22)$$

Note that for given $s, \tau \in \mathbb{R}_+$ with $s \leq \tau$ there is a positive number $l_s \in N$ satisfying

$$s \leq \tau \leq l_s T_0 + s.$$ 

By using the property (E2) we see that

$$v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 + m_{n,j}T_0 + s)$$

$\circ E(k_{n,j}T_0 + m_{n,j}T_0 + s, T_0 + m_{n,j}T_0 + l_s T_0 + s)$

$\circ E(T_0 + m_{n,j}T_0 + l_s T_0 + s, m_{n,j}T_0 + \tau)z_{n,j}$

for any $j \in \mathbb{Z}_+$ with $k_{n,j} \geq l_s + 2$. Here we can take elements $w_{n,j} \in H$ and $y_{n,j} \in H$ so that

$$v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 + m_{n,j}T_0 + s)w_{n,j}, \quad (4.23)$$

$$w_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + s, T_0 + m_{n,j}T_0 + l_s T_0 + s)y_{n,j} \quad (4.24)$$

and

$$y_{n,j} \in E(T_0 + m_{n,j}T_0 + l_s T_0 + s, m_{n,j}T_0 + \tau)z_{n,j}. \quad (4.25)$$

By $\{z_{n,j}\} \subset B_n$ and the global boundedness result (B) in Section 2, we can get a positive constant $C_n := C_n(B_n) > 0$ satisfying

$$|y_{n,j}|_H \leq C_n, \quad \forall y_{n,j} \in E(T_0 + m_{n,j}T_0 + l_s T_0 + s, m_{n,j}T_0 + \tau)z_{n,j}. \quad (4.26)$$
Here we define the bounded set $B_{C_n}$ by 

$$B_{C_n} := \{ b \in H ; |b|_H \leq C_n \}.$$ 

From (4.26) and the result (B) in Section 2 it follows that the set 

$$\left\{ w_{n,j} \in H ; \begin{array}{l} w_{n,j} \in E(k_{n,j} T_0 + m_{n,j} T_0 + s, T_0 + m_{n,j} T_0 + l_s T_0 + s)y_{n,j} \\ \text{for any } j \in \mathbb{Z}_+ \text{ with } k_{n,j} \geq l_s + 2 \end{array} \right\}$$ 

is relatively compact in $H$. Hence, we may assume that the element $w_{n,j}$ converges to some element $\tilde{w}_{n,\infty} \in H$ as $j \rightarrow +\infty$. Clearly, $\tilde{w}_{n,\infty} \in \omega_s(B_{C_n})$, and it follows from Lemma 4.2 that 

$$\omega_s(B_{C_n}) \subset B_{0,s} \subset \overline{D(\varphi_s^\tau)}.$$ 

Moreover, by Lemma 4.1 and (4.22)-(4.23) we have 

$$\begin{equation}
\begin{aligned}
x_n \in U(\tau,s)\tilde{w}_{n,\infty} \subset U(\tau,s)\omega_s(B_{C_n}), \quad \forall n \geq 1,
\end{aligned}
\end{equation}$$

hence, we see that 

$$x_n \in \bigcup_{n \geq 1} U(\tau,s)\omega_s(B_{C_n}), \quad \forall n \geq 1. \quad (4.27)$$

Here, by the closedness of $U(\cdot,\cdot)$, we note that for each subset $X$ of $B_{0,s}$, 

$$\overline{U(\tau,s)X} \subset U(\tau,s)X. \quad (4.28)$$

On account of Lemma 4.2, (4.21), (4.27) and (4.28), we observe that 

$$x \in \bigcup_{n \geq 1} U(\tau,s)\omega_s(B_{C_n})$$

$$= U(\tau,s) \bigcup_{n \geq 1} \omega_s(B_{C_n})$$

$$\subset U(\tau,s) \bigcup_{n \geq 1} \omega_s(B_{C_n})$$

$$\subset U(\tau,s)\mathcal{A}_s^\tau,$$

which implies that $\mathcal{A}_s^\tau$ is the subset of $U(\tau,s)\mathcal{A}_s^\tau$, namely 

$$\mathcal{A}_s^\tau \subset U(\tau,s)\mathcal{A}_s^\tau.$$

\begin{proof}[Proof of Theorem 4.3] For any $B \in \mathcal{B}(H)$, let $z_0$ be any element of the $\omega$-limit set $\omega_E(B)$ which is define by 

$$\omega_E(B) := \bigcap_{s \geq 0} \bigcup_{t \geq s, \tau \in \mathcal{R}_+} E(t + \tau, \tau)(\overline{D(\varphi^\tau)} \cap B).$$

\end{proof}
Then we easily see that there exist sequences \( \{t_n\} \subset R_+ \) with \( t_n \uparrow +\infty \), \( \{\tau_n\} \subset R_+ \), \( \{y_n\} \subset B \) with \( y_n \in D(\varphi^{\tau_n}) \) and \( \{z_n\} \subset H \) with \( z_n \in E(t_n + \tau_n, \tau_n) y_n \) such that

\[
t_n := k_n T_0 + t'_n, \quad k_n \in Z_+, \quad k_n \not\to \infty, \quad t'_n \in [T_0, 2T_0], \quad t'_n \to t'_0, \\
\tau_n := \tau_n T_0 + \tau'_n, \quad \tau_n \in Z_+, \quad \tau'_n \in [0, T_0], \quad \tau_n \to \tau'_0
\]

and

\[
z_n \to z_0 \quad \text{in } H
\]  

(4.29)
as \( n \to +\infty \). Without loss of generality, we may assume that

\[
(a) \quad t'_n + \tau'_n \not\to t'_0 + \tau'_0 \quad \text{or} \quad (b) \quad t'_n + \tau'_n \not\to t'_0 + \tau'_0.
\]

Now, assume that (a) holds. Then let us consider the multivalued semiflow

\[
v_n \in E(1 + k_n T_0 + l_n T_0 + t'_0 + \tau'_0, \quad k_n T_0 + l_n T_0 + t'_n + \tau'_n) z_n.
\]  

(4.30)
Then, there is a solution \( u_n \) on \( [k_n T_0 + l_n T_0 + t'_n + \tau'_n, +\infty) \) for

\[
\begin{cases}
\begin{align*}
& u'_n(t) + \partial_{t^k + l_n T_0 + t'_n + \tau'_n}(u_n(t)) + G(t + k_n T_0 + l_n T_0 + t'_n + \tau'_n, u_n(t)) \\
& \quad \ni f(t + k_n T_0 + l_n T_0 + t'_n + \tau'_n), \quad t > 0, \\
& u_n(0) = z_n \quad \text{and} \quad u_n(1 + t'_0 + \tau'_0 - t'_n - \tau'_n) = v_n.
\end{align*}
\end{cases}
\]

Since \( z_n \to z_0 \) in \( H \), \( \{z_n\} \) is bounded in \( H \). Therefore by the global estimate (B) in Section 2, we see that

\[
\begin{cases}
\begin{align*}
v_n \in E(1 + k_n T_0 + l_n T_0 + t'_0 + \tau'_0, \quad k_n T_0 + l_n T_0 + t'_n + \tau'_n) z_n \\
\text{for any } n \in N
\end{align*}
\end{cases}
\]
is relatively compact in \( H \). Hence we may assume that

\[
v_n \to v \quad \text{in } H \quad \text{for some } v \in H.
\]  

(4.31)
Now applying Lemma 4.1 with (4.29)-(4.31), we can get

\[
v \in U(1 + t'_0 + \tau'_0, \quad t'_0 + \tau'_0) z_0,
\]
more precisely, (taking the subsequence of \( \{n\} \) if necessary) we observe that

\[
u_n \to u \quad \text{in } C(\lbrack 0, 2 \rbrack; H) \quad \text{as } n \to +\infty,
\]  

(4.32)
where \( u \) is the solution \( [t'_0 + \tau'_0, +\infty) \) satisfying

\[
\begin{cases}
\begin{align*}
& u'(t) + \partial_{t^k + t'_0 + \tau'_0}(u(t)) + G_p(t + t'_0 + \tau'_0, u(t)) \ni f_p(t + t'_0 + \tau'_0), \quad t > 0, \\
u(0) = z_0 \quad \text{and} \quad u(1) = v.
\end{align*}
\end{cases}
\]

By (4.32) we easily see that

\[
u_n(t'_0 + \tau'_0 - t'_n - \tau'_n) \to z_0 \quad \text{as } n \to +\infty.
\]  

(4.33)
Note that
\[
    u_n(t_0' + \tau_0' - t_n' - \tau_n') \\
    \in E(k_nT_0 + l_nT_0 + t_0' + \tau_0', k_nT_0 + l_nT_0 + t_0' + \tau_0')z_n \\
    = E(k_nT_0 + l_nT_0 + t_0' + \tau_0', l_nT_0 + \tau_n')y_n \\
    = E(k_nT_0 + l_nT_0 + t_0' + \tau_0', l_nT_0 + t_0' + \tau_0')E(l_nT_0 + t_0' + \tau_0', l_nT_0 + \tau_n')y_n.
\]

So, we can take a element \( x_n \in E(l_nT_0 + t_0' + \tau_0', l_nT_0 + \tau_n')y_n \) such that
\[
    u_n(t_0' + \tau_0' - t_n' - \tau_n') \in E(k_nT_0 + l_nT_0 + t_0' + \tau_0', l_nT_0 + t_0' + \tau_0')x_n.
\]

By \( \{y_n\} \subset B \) and the global estimate (B) in Section 2, we easily see that \( \{x_n\} \) is bounded, i.e.
\[
    \{x_n\} \subset \tilde{B} \text{ for some } \tilde{B} \in \mathcal{B}(H).
\]

Therefore, from Remarks 4.1-4.2 and (4.33)-(4.35) we observe that
\[
    z_0 \in \omega_{t_0' + \tau_0'}(\tilde{B}) \subset \mathcal{A}^*_\tau \subset A^*.
\]

Thus (4.3) holds.

In the case (b) when \( t_n' + \tau_n' \searrow t_0' + \tau_0' \), we can prove (4.3) by the slight modification of the proof as above. \( \diamond \)

Theorem 4.1 implies that the attracting set \( \mathcal{A}^*_\tau \) for \( \text{(AP)} \) is semi-invariant under \( U_\tau \) associated with the limiting \( T_0 \)-periodic system \( (P)_\tau \), in general. Moreover, from Theorem 4.2 we observe that
\[
    \mathcal{A}^*_\tau \subset U(\tau, s)\mathcal{A}^*_s \quad \text{for any } 0 \leq s \leq \tau < +\infty.
\]

In order to get the invariance of \( \mathcal{A}^*_\tau \) under \( U_\tau \) and \( \mathcal{A}^*_s = U(\tau, s)\mathcal{A}^*_s \), let us use a concept of a regular approximation, which was introduced in [17].

**Definition 4.4.** (Regular approximation) Let \( s \in R_+ \) be fixed. Let \( z \in D(\varphi^s) \). Then, we say that \( U(t + s, s)z \) is regularly approximated by \( E(t + kT_0 + s, kT_0 + s) \) as \( k \to +\infty \), if for each finite \( T > 0 \) there are sequences \( \{k_n\} \subset Z_+ \) with \( k_n \to +\infty \) and \( \{z_n\} \subset H \) with \( z_n \in D(\varphi^{k_nT_0+s}) \) and \( z_n \to z \) in \( H \) satisfying the following property: for any function \( u \in W^{1,2}(0, T; H) \) satisfying \( u(t) \in U(t + s, s)z \) for all \( t \in [0, T] \) there is a sequence \( \{u_n\} \subset W^{1,2}(0, T; H) \) such that \( u_n(t) \to E(t + k_nT_0 + s, k_nT_0 + s)z_n \) for all \( t \in [0, T] \) and \( u_n \to u \) in \( C([0, T]; H) \) as \( n \to +\infty \).

Using the above concept, we can show that the invariance of \( \mathcal{A}^*_\tau \) under \( U_\tau \). Moreover we can get
\[
    \mathcal{A}^*_\tau = U(\tau, s)\mathcal{A}^*_s.
\]

**Theorem 4.4** Suppose all assumptions in Theorem 4.1. Let \( \mathcal{A}^*_s \) and \( \mathcal{A}^*_\tau \) be discrete attractors for \( E(\cdot, s) \) and \( E(\cdot, \tau) \), with \( 0 \leq s \leq \tau < +\infty \), respectively. Assume that for
any point $z$ of $\mathcal{A}_{s}^{*}$, $U(t + s, s)z$ is regularly approximated by $E(t + kT_{0} + s, kT_{0} + s)$ as $k \to +\infty$. Then we have

$$\mathcal{A}_{s}^{*} = U(\tau, s)\mathcal{A}_{s}^{*}.$$  

**Proof.** By Theorem 4.2, we have only to show that

$$U(\tau, s)\mathcal{A}_{s}^{*} \subset \mathcal{A}_{s}^{*}.$$  

To do so, let $x$ be any element of $U(\tau, s)\mathcal{A}_{s}^{*}$.

At first, taking account of Definitions 3.2-3.3 and Theorem 4.1 (iii), we see that for each $n \in N$

$$U_{\tau}^{n}U(\tau, s)\mathcal{A}_{s}^{*} = U(nT_{0} + \tau, \tau)U(\tau, s)\mathcal{A}_{s}^{*} = U(nT_{0} + \tau, nT_{0} + s)U(nT_{0} + s, \mathcal{A}_{s}^{*}) = U(\tau, s)U_{s}^{n}\mathcal{A}_{s}^{*} \supset U(\tau, s)\mathcal{A}_{s}^{*}.  \tag{4.36}$$

Hence, there exists a element $y_{n} \in \mathcal{A}_{s}^{*}$ such that

$$x \in U_{\tau}^{n}U(\tau, s)y_{n} = U(nT_{0} + \tau - s + s, s)y_{n}.$$  

By using our assumption as $t = nT_{0} + \tau - s$, we observe that for each $n$, there are sequences $\{k_{n,j}\} \subset \mathbb{Z}_{+}$, $\{x_{n,j}\} \subset H$ and $\{y_{n,j}\} \subset H$ such that

$$k_{n,j} \to +\infty, \quad y_{n,j} \in D(\varphi^{k_{n,j}T_{0} + s}), \quad y_{n,j} \to y_{n} \text{ in } H$$

and

$$x_{n,j} \in E(nT_{0} + \tau - s + k_{n,j}T_{0} + s, k_{n,j}T_{0} + s)y_{n,j}, \quad x_{n,j} \to x \text{ in } H \tag{4.37}$$

as $j \to +\infty$. Therefore, by the usual diagonal argument, we can find a subsequence $\{j_{n}\}$ of $\{j\}$ such that $\bar{x}_{n} := x_{n,j_{n}}$, $\bar{y}_{n} := y_{n,j_{n}}$ and $\bar{k}_{n} := k_{n,j_{n}}$ satisfy

$$|\bar{x}_{n} - x|_{H} < \frac{1}{n}, \quad \bar{x}_{n} \in E(nT_{0} + \tau - s + \bar{k}_{n}T_{0} + s, \bar{k}_{n}T_{0} + s)\bar{y}_{n}, \quad |\bar{y}_{n} - y_{n}|_{H} < \frac{1}{n} \tag{4.38}$$

for every $n = 1, 2, \ldots$. Since $\{\bar{y}_{n}\}$ is bounded in $H$, there is a bounded set $B \in \mathcal{B}(H)$ so that $\{\bar{y}_{n}\} \subset B$.

By (E2), we see that

$$\bar{x}_{n} \in E(nT_{0} + \tau - s + \bar{k}_{n}T_{0} + s, \bar{k}_{n}T_{0} + s)\bar{y}_{n}$$

$$= E(nT_{0} + \bar{k}_{n}T_{0} + \tau, \bar{k}_{n}T_{0} + s, \bar{y}_{n})E(T_{0} + \bar{k}_{n}T_{0} + \tau, \bar{k}_{n}T_{0} + s)\bar{y}_{n},$$

hence there is an element $\tilde{z}_{n} \in E(T_{0} + \bar{k}_{n}T_{0} + \tau, \bar{k}_{n}T_{0} + s)\bar{y}_{n}$ such that

$$\bar{x}_{n} \in E(nT_{0} + \bar{k}_{n}T_{0} + \tau, \bar{k}_{n}T_{0} + \tau)\tilde{z}_{n}. \tag{4.39}$$

Since $\{\bar{y}_{n}\} \subset B$ and the global estimate (B) in Section 2, we see that $\{\tilde{z}_{n}\}$ is also bounded in $H$. Hence, there is a bounded set $\tilde{B} \in \mathcal{B}(H)$ so that $\{\tilde{z}_{n}\} \subset \tilde{B}$. The above fact (4.37)-(4.39) implies (cf. Remark 4.1) that $x \in \omega_{r}(\tilde{B}) \subset \mathcal{A}_{s}^{*}$. Thus we have $U(\tau, s)\mathcal{A}_{s}^{*} \subset \mathcal{A}_{s}^{*}$.  \hfill \diamond
By the same argument in Theorem 4.4, we can get the following corollary:

**Corollary.** (i) Suppose the same assumptions of Theorem 4.4. Then, by Remark 4.2 we observe that \( A^*_s \) is invariant under the \( T_0 \)-periodic dynamical system \( U_s := U(T_0 + s) \). Namely,

\[
A^*_s = U^l_s A^*_s \quad \text{for any } l \in \mathbb{N}.
\]

(ii) Assume that for any point \( z \) of \( A^*_t \), \( U(t + \tau, \tau)z \) is regularly approximated by \( E(t + kT_0 + \tau, kT_0 + \tau) \) as \( k \to +\infty \). Then, we have \( A^*_s \supset A^*_t \). Hence by Theorem 4.1 (iii) we conclude that

\[
A^*_s = A^*_t.
\]

**Remark 4.3.** If the solution operator \( U(t, s) \) is singlevalued, namely the solution for the Cauchy problem of (P) is unique, the assumptions of Theorem 4.4 always hold. Thus, Theorems 4.1-4.4 contain the abstract results obtained in [11], which was concerned with the asymptotic \( T_0 \)-periodic stability for the singlevalued dynamical system associated with time-dependent subdifferentials.

## 5 Application to obstacle problems for PDE’s

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) \((1 \leq N < +\infty)\) with smooth boundary \( \Gamma = \partial \Omega \), \( q \) be a fixed number with \( 2 \leq q < +\infty \) and \( T_0 \) be a fixed positive number. We use the notation

\[
a_q(v, z) := \int_\Omega |\nabla v|^{q-2} \nabla v \cdot \nabla z \, dx, \quad \forall v, z \in W^{1,q} \Omega
\]

and denote by \((\cdot, \cdot)\) the usual inner product in \( L^2(\Omega) \).

For prescribed obstacle functions \( \sigma_0 \leq \sigma_1 \) and each \( t \in \mathbb{R}_+ \) we define the set

\[
K(t) := \{ z \in W^{1,q}(\Omega); \sigma_0(t, \cdot) \leq z \leq \sigma_1(t, \cdot) \text{ a.e. on } \Omega \}.
\]

Let \( f \) be a function in \( L^2_{\text{loc}}(\mathbb{R}_+; L^2(\Omega)) \) and \( h \) be a non-negative function on \( \mathbb{R}_+ \times \mathbb{R} \). Then for given \( b \in L^\infty(\Omega)^N \) we consider an interior asymptotically \( T_0 \)-periodic double obstacle problem \((\text{OP})_{sAP}^* \) \((s \in \mathbb{R}_+)\) :

\[ \begin{cases} 
& v \in L^q_{\text{loc}}((s, +\infty); W^{1,q}(\Omega)) \cap W^{1,2}_{\text{loc}}((s, +\infty); L^2(\Omega)); \\
& v(t) \in K(t) \text{ for a.e. } t \geq s; \\
& 0 \leq \theta(t, x) \leq h(t, v(t, x)) \text{ a.e. on } (s, +\infty) \times \Omega; \\
& (v(t) + \theta(t) + b \cdot \nabla v(t) - f(t), v(t) - z) + a_q(v(t), v(t) - z) \leq 0 \\
& \text{for any } z \in K(t) \text{ and a.e. } t \geq s.
\end{cases} \]

\[ (\text{OP})_{sAP}^* \]
The main object of this section is to consider the large-time behaviour of solution for 
\((\text{OP})_{s}^{AP}\) assuming asymptotically \(T_{0}\)-periodicity conditions
\[
\sigma_{i}(t) - \sigma_{i,p}(t) \to 0 \quad (i = 0, 1), \quad h(t, \cdot) - h_{p}(t, \cdot) \to 0, \quad f(t) - f_{p}(t) \to 0
\]
as \(t \to \infty\) in the sense specified below, where \(\sigma_{i,p}(t), h_{p}(t, \cdot), f_{p}(t)\) are periodic in time with the same period \(T_{0}\). By the above assumptions, the limiting system of \((\text{OP})_{s}^{AP}\) is a \(T_{0}\)-periodic one \((\text{OP})_{s}^{P}\) as follows:

- Find functions \(u \in C((s, +\infty); L^{2}(\Omega))\) and \(\theta \in L^{2}_{\text{loc}}((s, +\infty); L^{2}(\Omega))\) such that

\[
(\text{OP})_{s}^{P}
\begin{cases}
    u \in L^{2}_{\text{loc}}((s, +\infty); W^{1,q}(\Omega)) \cap W^{1,2}_{\text{loc}}((s, +\infty); L^{2}(\Omega)); \\
    u(t) \in K_{p}(t) \text{ for a.e. } t \geq s; \\
    0 \leq \theta(t, x) \leq h_{p}(t, u(t, x)) \quad \text{ a.e. on } (s, +\infty) \times \Omega; \\
    (u'(t) + \theta(t) + b \cdot \nabla u(t) - f_{p}(t), u(t) - z) + a_{q}(u(t), u(t) - z) \leq 0 \\
    \text{ for any } z \in K_{p}(t) \text{ and a.e. } t \geq s,
\end{cases}
\]

where \(K_{p}(t) := \{z \in W^{1,q}(\Omega); \sigma_{0,p}(t, \cdot) \leq z \leq \sigma_{1,p}(t, \cdot) \text{ a.e. on } \Omega \}\)

Now we suppose the following conditions:

- \(\sigma_{i}\) and \(\sigma_{i,p}\) are functions on \(R_{+} \times \Omega\) such that

\[
\sup_{t \in R_{+}} \left| \frac{d\sigma_{i}}{dt} \right|_{L^{2}(t, t+1; W^{1,q}(\Omega))} + \sup_{t \in R_{+}} \left| \frac{d\sigma_{i}}{dt} \right|_{L^{2}(t, t+1; L^{\infty}(\Omega))} < +\infty,
\]

\[
\sup_{t \in R_{+}} \left| \frac{d\sigma_{i,p}}{dt} \right|_{L^{2}(t, t+1; W^{1,q}(\Omega))} + \sup_{t \in R_{+}} \left| \frac{d\sigma_{i,p}}{dt} \right|_{L^{2}(t, t+1; L^{\infty}(\Omega))} < +\infty
\]

and \(\sigma_{i,p}\) is a \(T_{0}\)-periodic obstacle function, i.e.

\[
\sigma_{i,p}(t + T_{0}, x) = \sigma_{i,p}(t, x) \quad \text{ for a.e. } x \in \Omega \text{ and any } t \in R_{+}
\]

for \(i = 0, 1\). Moreover, there are positive constants \(k_{1} > 0\) and \(k_{2} > 0\) such that

\[
\sigma_{1} - \sigma_{0} \geq k_{1} \quad \text{ and } \quad \sigma_{1,p} - \sigma_{0,p} \geq k_{1} \quad \text{ a.e. on } R_{+} \times \Omega
\]

and

\[
|\sigma_{i}|_{L^{\infty}(R_{+}; W^{1,q}(\Omega))} + |\sigma_{i}|_{L^{\infty}(R_{+} \times \Omega)} + |\sigma_{i,p}|_{L^{\infty}(R_{+}; W^{1,q}(\Omega))} + |\sigma_{i,p}|_{L^{\infty}(R_{+} \times \Omega)} \leq k_{2}
\]

for \(i = 0, 1\).

- \(h\) and \(h_{p}\) are non-negative continuous functions on \(R_{+} \times R\). There is a positive constant \(L\) such that

\[
|h(t, z_{1}) - h(t, z_{2})| \leq L|z_{1} - z_{2}|
\]

\[
|h_{p}(t, z_{1}) - h_{p}(t, z_{2})| \leq L|z_{1} - z_{2}|
\]

for all \(t \in R_{+}, z_{i} \in R\) and \(i = 1, 2\). Moreover, \(h_{p}\) is a \(T_{0}\)-periodic function, i.e. for any \(z \in R\), \(h_{p}(t + T_{0}, z) = h_{p}(t, z)\) for any \(t \in R_{+}\).
• \( f, f_p \in L^2_{\text{loc}}(R_+; L^2(\Omega)) \), and \( f_p \) is a \( T_0 \)-periodic function, i.e.
\[
f_p(t + T_0) = f_p(t) \quad \text{in } L^2(\Omega), \quad \forall t \in R_+.
\]

Moreover, we suppose the following convergence conditions:

- (Convergence of \( \sigma_i(t) - \sigma_{i,p}(t) \to 0 \) as \( t \to +\infty \)) Put
\[
I_m := \sup_{t \in [0,T_0]} |\sigma_0(t) + t) - \sigma_{0,p}(t)|_{W^{1,\sigma}(\Omega)} + \sup_{t \in [0,T_0]} |\sigma_1(t) + t) - \sigma_{1,p}(t)|_{W^{1,\sigma}(\Omega)}
\]

\[
+ \sup_{t \in [0,T_0]} |\sigma_0(t) + t) - \sigma_{0,p}(t)|_{L^{\infty}(\Omega)} + \sup_{t \in [0,T_0]} |\sigma_1(t) + t) - \sigma_{1,p}(t)|_{L^{\infty}(\Omega)}
\]

Then,
\[
I_m \to 0 \quad \text{as } m \to +\infty;
\]

- (Convergence of \( h(t, \cdot) - h_p(t, \cdot) \to 0 \) as \( t \to +\infty \)) For any \( z \in R \),
\[
\sup_{t \in [0,T_0]} |h(t, t) - h_p(t, t)| \to 0 \quad \text{as } m \to +\infty; \quad (5.1)
\]

- (Convergence of \( f(t) - f_p(t) \to 0 \) as \( t \to +\infty \))
\[
|f(t) - f_p|_{L^2(0,T_0; L^2(\Omega))} \to 0 \quad \text{as } m \to +\infty. \quad (5.2)
\]

Under the above assumptions, let us consider problems (OP)\(^{AP}_s\) and (OP)\(^P\).

In order to apply the abstract results in Sections 2-4, we choose \( L^2(\Omega) \) as a real separable Hilbert space \( H \). And we define a family \( \{\varphi^i\} \) of proper l.s.c. convex functions \( \varphi^i \) on \( L^2(\Omega) \) by

\[
\varphi^i(z) = \left\{ \begin{array}{ll}
\frac{1}{q} \int_{\Omega} |\nabla z|^q dx & \text{if } z \in K(t), \\
\infty & \text{if } z \in L^2(\Omega) \setminus K(t),
\end{array} \right. \quad (5.3)
\]

and define \( \varphi^i_p \) by replacing \( K(t) \) by \( K_p(t) \) in (5.3).

Also, we define a multivalued operator \( G(\cdot, \cdot) \) from \( R_+ \times H^1(\Omega) \) into \( L^2(\Omega) \) by

\[
G(t, z) := \left\{ g \in L^2(\Omega); \quad g = l + b \cdot \nabla z \quad \text{in } L^2(\Omega) \right\}
\]

\[
0 \leq l(x) \leq h(t, z(x)) \quad \text{a.e. on } \Omega \right\}
\]

for all \( t \in R_+ \) and \( z \in H^1(\Omega) \). And we define \( G_p(\cdot, \cdot) \) by replacing \( h(t, \cdot) \) by \( h_p(t, \cdot) \) in \( (5.4) \).

By the same argument in [27, Lemma 5.1], we can get the following lemmas.

**Lemma 5.1.** (cf. [27, Lemma 5.1]) Put for any \( r > 0 \) and \( t \in R_+ \)
\[
a_r(t) = b_r(t) := k_3 \int_0^t \left\{ |\sigma^r_{0,p}|_{L^\infty(\Omega)} + |\sigma^r_{0,p}|_{W^{1,\sigma}(\Omega)} + |\sigma^r_{1,p}|_{L^\infty(\Omega)} + |\sigma^r_{1,p}|_{W^{1,\sigma}(\Omega)} \right\} d\tau
\]

23
where $k_3$ is a (sufficiently large) positive constant. Then, $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$ and $\{\varphi^t_p\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0).

Moreover we have $\{G(t, \cdot)\} \in G(\{\varphi^t\})$ and $\{G_p(t, \cdot)\} \in G_p(\{\varphi^t_p\}; T_0)$.

**Lemma 5.2.** The convergence assumptions (A1)-(A3) hold.

**Proof.** We easily see that (A2) and (A3) hold by assumptions (5.1) and (5.2).

Now let us show (A1). For each $t \in R_+$ there are $m \in Z_+$ and $\tau \in [0, T_0]$ so that $t = mT_0 + \tau$.

For each $z_p \in D(\varphi^t) = K_p(t)$, we put

$$z := (z_p - \sigma_{0,p}(t)) \frac{\sigma_1(t) - \sigma_0(t)}{\sigma_{1,p}(t) - \sigma_{0,p}(t)} + \sigma_0(t).$$

Then we easily see that $z \in D(\varphi^t) = K(t)$. Moreover, by the same argument in [27, Lemma 5.1], we see that

$$|z - z_p|_{L^2(\Omega)} \leq k_4 I_m \quad \text{and} \quad |\nabla z - \nabla z_p|_{L^1(\Omega)} \leq k_4 I_m (1 + |\nabla z_p|_{L^1(\Omega)})$$

(5.5)

for some constant $k_4 > 0$. Hence we have

$$\varphi^t(z) - \varphi^t_p(z_p) \leq k_5 I_m (1 + \varphi^t_p(z_p)),$$

(5.6)

for a sufficiently large $k_5 > 0$.

Conversely, let $z \in D(\varphi^t) = K(t)$ and we put

$$z_p := (z - \sigma_0(t)) \frac{\sigma_{1,p}(t) - \sigma_{0,p}(t)}{\sigma_1(t) - \sigma_0(t)} + \sigma_{0,p}(t).$$

Then, we observe that $z_p \in D(\varphi^t_p) = K_p(t)$ and

$$|z_p - z|_{L^2(\Omega)} \leq k_4 I_m$$

and

$$\varphi^t_p(z_p) - \varphi^t(z) \leq k_5 I_m (1 + \varphi^t(z)).$$

(5.7)

Therefore by (5.5)-(5.7) we see that the convergence assumption (A1) holds. \diamond

Clearly, the obstacle problem $(OP)_{sAP}$ can be reformulated as an evolution equation $(AP)_s$ involving the subdifferential of $\varphi^t$ given by (5.3) and the multivalued operator $G(t, \cdot)$ defined by (5.4). Also, the limiting $T_0$-periodic problem $(OP)_s$ can be reformulated as an evolution equation $(P)_s$. Therefore, by Lemmas 5.1-5.2 we can apply abstract results in Section 2-4. Namely, we can obtain an attractor $A^*_s$ for $(OP)_{sAP}$, a $T_0$-periodic attractor $A_s$ for $(OP)_s$ and the relationships between $(OP)_{sAP}$ and $(OP)_s$.

Additionally, we assume that $f(t) \equiv f_p(t)$ for any $t \in R_+$ and

$$\sigma_0(t, z) \equiv \sigma_{0,p}(t, z), \quad \sigma_{1,p}(t, z) \equiv \sigma_1(t, z), \quad h_p(t, z) \leq h(t, z)$$

24
for any \(0 \leq t < +\infty\) and \(z \in R\). Then we easily see that the assumptions of Theorem 4.4 and its Corollary hold. Hence we can get \(A_s^* = A_s\) by the same argument in [30, Theorem 5.4].

Unfortunately we do not give assumptions for \(\sigma_i(t, \cdot), h(t, \cdot)\) and \(f(t)\) in order to get

\[
U(\tau, s)A_s^* = A_s^* \subset A_s \quad \text{for any } 0 \leq s \leq \tau < +\infty.
\]

(5.8)

It seems difficult to show (5.8), so it is the open problem.

References


25


Noriaki Yamazaki
Department of Mathematical Science, Common Subject Division, Muroran Institute of Technology, 27-1 Mizumoto-chō, Muroran, 050-8585, Japan
E-mail address: noriaki@mmm.muroran-it.ac.jp