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# EXPLICIT REPRESENTATION OF FINITE PREDICTOR COEFFICIENTS AND ITS APPLICATIONS

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ABSTRACT. We consider the finite-past predictor coefficients of stationary time series, and establish an explicit representation for them, in terms of the MA and AR coefficients. The proof involves the alternate iteration of projection operators associated with the infinite past and the infinite future. We provide several applications, which include rates of convergence of the finite predictor coefficients, an equality of Baxter-type for long memory processes, and a simple representation of the partial autocorrelation function  $\alpha(\cdot)$ . We use the last result to obtain the precise asymptotic behavior of  $\alpha(\cdot)$  with remainder, for the fractional ARIMA processes.

## 1. INTRODUCTION

Let  $\{X_k\} = \{X_k : k \in \mathbf{Z}\}$  be a real, zero-mean, weakly stationary process, defined on a probability space  $(\Omega, \mathcal{F}, P)$ , which we shall simply call a *stationary process*. We denote by  $H$  the real Hilbert space spanned by  $\{X_k : k \in \mathbf{Z}\}$  in  $L^2(\Omega, \mathcal{F}, P)$ . The norm of  $H$  is given by  $\|Y\| := E[Y^2]^{1/2}$ . For  $n \in \mathbf{N}$ , we denote by  $H_{[-n, -1]}$  and  $H_{(-\infty, -1]}$  the subspaces of  $H$  spanned by  $\{X_{-n}, \dots, X_{-1}\}$  and  $\{X_k : k \leq -1\}$ , respectively. We write  $P_{[-n, -1]}$  and  $P_{(-\infty, -1]}$  for the orthogonal projection operators of  $H$  onto  $H_{[-n, -1]}$  and  $H_{(-\infty, -1]}$ , respectively. The projection  $P_{[-n, -1]}X_0$  (respectively,  $P_{(-\infty, -1]}X_0$ ) stands for the best linear predictor of the future value  $X_0$  based on the finite past  $\{X_{-n}, \dots, X_{-1}\}$  (respectively, the infinite past  $\{X_k : k \leq -1\}$ ), and its mean square prediction error is given by  $\sigma_n^2 := \|X_0 - P_{[-n, -1]}X_0\|^2$  (respectively,  $\sigma^2 := \|X_0 - P_{(-\infty, -1]}X_0\|^2$ ).

For nondeterministic  $\{X_k\}$  (see Section 2.1), the finite predictor coefficients  $\phi_{n,j}$  are the uniquely determined ones in

$$(1.1) \quad P_{[-n, -1]}X_0 = \sum_{j=1}^n \phi_{n,j}X_{-j}.$$

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As is well known, we can calculate the numerical values of  $\phi_{n,1}, \dots, \phi_{n,n}$  as well as the mean square prediction error  $\sigma_n^2$  from the values  $\gamma(0), \dots, \gamma(n)$  of the autocovariance function of  $\{X_k\}$ , using recursive algorithms such as the Durbin–Levinson algorithm (see Brockwell and Davis [5], Sections 3.4 and 5.2). The recursive methods are of great practical importance in time series analysis. However, they are not necessarily effective in problems of theoretical character, in particular, those related to the asymptotic behavior as  $n \rightarrow \infty$ .

A classical problem of this type is the rate of convergence of  $\sigma_n^2 - \sigma^2 \downarrow 0$  as  $n \rightarrow \infty$ . See for instance Ginovian [7], where references to earlier work — by Grenander and Rosenblatt, Grenander and Szegö, Baxter, Ibragimov and many others — are given. The arguments in these references are closely related to the theory of orthogonal polynomials as described in [9].

A new approach to a related problem was introduced by the first author [14]. For the partial autocorrelation function  $\alpha(\cdot)$  of a stationary process  $\{X_k\}$  with short or long memory, the asymptotic behavior of  $|\alpha(n)|$  as  $n \rightarrow \infty$  was obtained using a representation of the mean square prediction error  $\sigma_n^2$  in terms of the MA (moving-average) coefficients  $c_k$  and the AR (autoregressive) coefficients  $a_k$  (see Section 2.2 for the definitions of  $c_k$  and  $a_k$ ). More precisely, after the precise behavior of  $\sigma_n^2 - \sigma^2$  was obtained by arguments involving  $c_k$  and  $a_k$ , that of  $|\alpha(n)|$  was derived from the result by a Tauberian argument, and in so doing the necessary Tauberian condition was verified using the representation of  $\sigma_n^2$ . The partial autocorrelation  $\alpha(n)$ , which is equal to  $\phi_{n,n}$ , is the correlation between  $X_0$  and  $X_n$ , adjusted for the intervening observations  $X_1, \dots, X_{n-1}$  (see (3.22) for definition). The partial autocorrelation function (PACF) is one of the core concepts in time series analysis.

By the same approach but with extra complication, similar results on  $|\alpha(n)|$  were obtained in [15] and [17] for the fractional ARIMA (autoregressive integrated moving-average) processes. The fractional ARIMA model is an important parametric model including a class of long memory processes. It was independently introduced by Granger and Joyeux [8] and Hosking [12] (see Example 2.4).

The advantage of such approach, i.e., that via  $c_k$  and  $a_k$ , has become more apparent in [16] where a representation of the partial autocorrelation function  $\alpha(\cdot)$  itself, in terms of  $c_k$  and  $a_k$ , was derived for the first time. The representation enabled us to study the behavior of  $\alpha(\cdot)$  more directly without using Tauberian arguments.

As a result, proofs were simplified considerably, and results were improved in several ways. In particular, the asymptotic behavior of  $\alpha(n)$  as  $n \rightarrow \infty$ , rather than that of  $|\alpha(n)|$ , was obtained. For example, it was shown that, for the fractional ARIMA( $p, d, q$ ) processes with  $d \in (-1/2, 1/2) \setminus \{0\}$ ,  $\alpha(n) \sim d/n$  as  $n \rightarrow \infty$ .

In this paper, our main interest is in the finite predictor coefficients  $\phi_{n,j}$  which are among the most basic quantities in the prediction theory for  $\{X_k\}$ . After we establish an explicit representation of the type above for  $\phi_{n,j}$ , i.e., that in terms of the MA coefficients  $c_k$  and the AR coefficients  $a_k$ , we give several applications of the representation.

For  $n \in \mathbf{N}$ , we write  $H_{[-n, \infty)}$  for the subspace of  $H$  spanned by  $\{X_k : k \geq -n\}$ , and  $P_{[-n, \infty)}$  for the orthogonal projection operator of  $H$  onto  $H_{[-n, \infty)}$ . Our method for the proof of the representation of  $\phi_{n,j}$  involves the alternate iteration of  $P_{(-\infty, -1]}$  and  $P_{[-n, \infty)}$  as well as Theorem 3.1 in [14] which concerns the key equality  $H_{(-\infty, -1]} \cap H_{[-n, \infty)} = H_{[-n, -1]}$ . By the method, we first establish a general theorem for the finite predictor coefficients (Theorem 2.2). The result yields the representation of  $\phi_{n,j}$  in terms of  $c_k$  and  $a_k$  when Wiener's prediction formula is available (Theorem 2.3). In applications, it is essential that  $\phi_{n,j}$  is expressed in terms of absolutely convergent series made up of  $c_k$  and  $a_k$ . We derive such an expression (Theorem 2.7) under some conditions on  $c_k$  and  $a_k$ , that is, (A1) or (A2) in Section 2.3. The condition (A1) corresponds to short memory processes, while (A2) to long memory processes.

The first application of the representation of  $\phi_{n,j}$  concerns the rate of convergence of  $\phi_{n,j}$  to its limit as  $n \rightarrow \infty$ . If we let  $n \rightarrow \infty$ , then, under suitable conditions,  $\phi_{n,j}$  converges to the infinite prediction coefficient  $\phi_j$  in

$$(1.2) \quad P_{(-\infty, -1]} X_0 = \sum_{j=1}^{\infty} \phi_j X_{-j}.$$

The rate at which  $\phi_{n,j}$  converges to  $\phi_j$  is an interesting question. A textbook treatment of this problem can be found in Pourahmadi [20], Section 7.6; this book, as well as [5], provides excellent background of time series analysis and prediction theory for the present paper. Using the representation of  $\phi_{n,j}$ , we give precise results on the rate of convergence for a class of intermediate memory processes (Theorem 3.7) as well as for a class of long memory processes which includes the fractional ARIMA( $p, d, q$ ) processes with  $d \in (0, 1/2)$  (Theorem 3.3).

The second application of the representation of  $\phi_{n,j}$  is related to the additional error  $\|P_{[-n,-1]}X_0 - \sum_{j=1}^n \phi_j X_{-j}\|$  that arises when we use the infinite predictor coefficients  $\phi_j$  instead of the finite ones  $\phi_{n,j}$  (see Baxter [1], page 138, and Cheng and Pourahmadi [6], page 118). There exists a known inequality that deals with this problem, and is commonly referred to as *Baxter's inequality* (see [1]; see also Berg [3], [6] and Section 6.6.2 in [20]). It takes the form

$$(1.3) \quad \sum_{j=1}^n |\phi_{n,j} - \phi_j| \leq M \sum_{k=n+1}^{\infty} |\phi_k|$$

with some positive constant  $M$ . The original inequality (1.3) of Baxter was an assertion for short memory processes, and his proof in [1] used the theory of orthogonal polynomials. By simple arguments based on the representation of  $\phi_{n,j}$ , we prove (1.3) not only for short memory processes but also for long memory processes including the fractional ARIMA processes (Theorems 4.3 and 4.5). In the short memory case, the proof gives an  $M$  explicitly in terms of  $a_k$  and  $c_k$ .

The third application is a representation of the PACF  $\alpha(\cdot)$  in terms of  $a_k$  and  $c_k$ . The proof involves the representation of  $\phi_{n,j}$  and the equality

$$(1.4) \quad \phi_{n,j} - \phi_{n+1,j} = \phi_{n,n+1-j} \alpha(n+1),$$

which is a part of the Durbin-Levinson algorithm (see, e.g., [5], (5.2.4), page 171). As stated above, we already have such a representation of  $\alpha(\cdot)$  in [16]. The representation of the finite predictor coefficients also gives this type of result since  $\phi_{n,n}$  is nothing but  $\alpha(n)$ . Among these results, the new representation of  $\alpha(\cdot)$  (Theorem 5.4) is of special interest in view of its simplicity. We illustrate the usefulness by applying the result to two classes of processes, that is, a class of short memory processes and the fractional ARIMA model. For the latter, we derive the asymptotic behavior of  $\alpha(n)$  with remainder (Theorem 5.6), which is a refinement of the results in [15, 16].

In Section 2, we prove the representation of the finite predictor coefficients  $\phi_{n,j}$ . In Section 3, we apply it to the rate of convergence of  $\phi_{n,j}$ . In Section 4, we prove inequalities of Baxter-type using the representation of  $\phi_{n,j}$ . Finally in Section 5, we prove a new representation of the PACF  $\alpha(\cdot)$  using that of  $\phi_{n,j}$ , and then apply it to derive the asymptotic behavior of  $\alpha(\cdot)$  for a class of short memory processes and the fractional ARIMA model.

## 2. FINITE PREDICTOR COEFFICIENTS

Let  $\{X_n\} = \{X_n : n \in \mathbf{Z}\}$  be a stationary process; as stated in Section 1, this means that  $\{X_n\}$  is a real, zero-mean, weakly stationary process, defined on a probability space  $(\Omega, \mathcal{F}, P)$ . The *autocovariance function*  $\gamma(\cdot)$  of  $\{X_n\}$  is defined by

$$\gamma(n) := E[X_n X_0], \quad n \in \mathbf{Z}.$$

If there exists an even, nonnegative, and integrable function  $\Delta(\cdot)$  on  $[-\pi, \pi]$  such that

$$\gamma(n) = \int_{-\pi}^{\pi} e^{in\lambda} \Delta(\lambda) d\lambda, \quad n \in \mathbf{Z},$$

then  $\Delta(\cdot)$  is called a *spectral density* of  $\{X_n\}$ . As is well known,  $\{X_n\}$  is *purely nondeterministic* (PND) if and only if it has a positive spectral density such that  $\int_{-\pi}^{\pi} |\log \Delta(\lambda)| d\lambda < \infty$  (see Section 5.7 in [5] and Chapter II in [21]). In this paper, we say that a stationary process  $\{X_n\}$  is *long memory* (respectively, *short memory*) if  $\sum_{k=-\infty}^{\infty} |\gamma(k)| = \infty$  (respectively,  $< \infty$ ). See Beran [2], page 6, and [5], Section 13.2.

As we also stated in Section 1, we denote by  $H$  the closed real linear hull of  $\{X_k : k \in \mathbf{Z}\}$  with respect to the norm  $\|Y\| := E[Y^2]^{1/2}$ . Then  $H$  is a real Hilbert space with inner product  $(Y, Z) := E[YZ]$ . For  $n, m \in \mathbf{Z}$  with  $n \leq m$ , we write  $H_{(-\infty, n]}$ ,  $H_{[n, \infty)}$ ,  $H_{[n, m]}$  and  $H_{\{n\}}$  for the closed subspaces of  $H$  spanned by  $\{X_k : -\infty < k \leq n\}$ ,  $\{X_k : n \leq k < \infty\}$ ,  $\{X_k : n \leq k \leq m\}$ , and  $X_n$ , respectively. Notice that  $H_{\{n\}} = H_{[n, n]}$ . For an interval  $I$ , we write  $P_I$  for the orthogonal projection operator of  $H$  onto  $H_I$ .

**2.1. General result.** Let  $Y \in H$ . For  $n \in \mathbf{N}$ ,  $P_{[-n, -1]}Y$  is the best linear predictor of  $Y$  based on the observations  $X_{-n}, \dots, X_{-1}$ . If  $\{X_k\}$  is *nondeterministic*, that is,  $X_0 \notin H_{(-\infty, -1]}$ , then  $X_{-n}, \dots, X_{-1}$  are linearly independent, whence we can express the predictor  $P_{[-n, -1]}Y$  uniquely in the form

$$(2.1) \quad P_{[-n, -1]}Y = \sum_{j=1}^n \phi_{n,j}(Y) X_{-j}.$$

In this section, we are concerned with the real coefficients  $\phi_{n,j}(Y)$ .

For  $n, k \in \mathbf{N}$ , we define the orthogonal projection operator  $P_n^k$  by

$$(2.2) \quad P_n^k := \begin{cases} P_{(-\infty, -1]}, & k = 1, 3, 5, \dots, \\ P_{[-n, \infty)}, & k = 2, 4, 6, \dots \end{cases}$$

**Lemma 2.1.** *Assume that  $\{X_n\}$  is nondeterministic. Let  $Y$  be an arbitrary element of  $H$ . Then, for  $n, k \in \mathbf{N}$ , there exist unique real coefficients  $\phi_{n,1}^k(Y), \dots, \phi_{n,n}^k(Y)$  as well as  $Z_n^k \in H_{(-\infty, -n-1]}$  for  $k$  odd and  $Z_n^k \in H_{[0, \infty)}$  for  $k$  even, such that*

$$P_n^k P_n^{k-1} \dots P_n^1 Y = \sum_{j=1}^n \phi_{n,j}^k(Y) X_{-j} + Z_n^k.$$

*Proof.* We assume that  $k$  is odd. From Lemma 6.1 in [20] (Regression Lemma), it follows that

$$(2.3) \quad H_{(-\infty, -1]} = H_{(-\infty, -n-1]} + H_{[-n, -1]} \quad (\text{direct sum})$$

(see the proof of Theorem 6.3 in [20]). Since  $X_{-n}, \dots, X_{-1}$  are linearly independent and  $P_n^k P_n^{k-1} \dots P_n^1 Y \in H_{(-\infty, -1]}$ , the lemma for  $k$  odd follows. The case in which  $k$  is even is proved in a similar fashion.  $\square$

It is natural to ask if  $\phi_{n,j}^k(Y)$  converges to  $\phi_{n,j}(Y)$  as  $k \rightarrow \infty$ . The next theorem answers this question in the affirmative for PND  $\{X_n\}$  with spectral density  $\Delta(\cdot)$  such that

$$(2.4) \quad \int_{-\pi}^{\pi} \Delta(\lambda)^{-1} d\lambda < \infty.$$

**Theorem 2.2.** *Let  $n \in \mathbf{N}$  and  $j = 1, \dots, n$ . We assume that*

$$(2.5) \quad \{X_n\} \text{ is purely nondeterministic and satisfies (2.4).}$$

*Then, for every  $Y \in H$ , we have  $\phi_{n,j}(Y) = \lim_{k \rightarrow \infty} \phi_{n,j}^k(Y)$ .*

*Proof.* By Theorem 3.1 in [14], the assumption (2.5) implies

$$(2.6) \quad H_{(-\infty, -1]} \cap H_{[-n, \infty)} = H_{[-n, -1]} \quad (n = 1, 2, \dots).$$

This and von Neumann's theorem (see, e.g., Theorem 9.20 in [20]) yield

$$(2.7) \quad \text{s-}\lim_{m \rightarrow \infty} P_n^k P_n^{k-1} \dots P_n^1 = P_{[-n, -1]} \quad (n = 1, 2, \dots).$$

We put

$$(2.8) \quad \epsilon_k := X_k - P_{(-\infty, k-1]} X_k \quad (k \in \mathbf{Z}).$$

Then, from Lemma 2.1, we see that

$$(P_n^{2k+1} P_n^{2k} \dots P_n^1 Y, \epsilon_{-1}) = \phi_{n,1}^{2k+1}(Y) \cdot \|\epsilon_{-1}\|^2.$$

By (2.7), the left-hand side tends to  $(P_{[-n, -1]} Y, \epsilon_{-1})$  as  $k \rightarrow \infty$ . Thus  $a_{n,1} := \lim_{k \rightarrow \infty} \phi_{n,1}^{2k+1}(Y)$  exists. In the same way, letting  $k \rightarrow \infty$  in

$$(P_n^{2k+1} P_n^{2k} \dots P_n^1 Y, \epsilon_{-2}) = \phi_{n,2}^{2k+1}(Y) \cdot \|\epsilon_{-2}\|^2 + \phi_{n,1}^{2k+1}(Y) \cdot (X_{-1}, \epsilon_{-2}),$$

we find the existence of  $a_{n,2} := \lim_{k \rightarrow \infty} \phi_{n,2}^{2k+1}(Y)$ . Repeating this argument, we see that  $a_{n,j} := \lim_{k \rightarrow \infty} \phi_{n,j}^{2k+1}(Y)$  exists for all  $j = 1, \dots, n$ . Hence  $Z_n := \lim_{k \rightarrow \infty} Z_n^{2k+1}$  also exists in  $H$ , and we have

$$Z_n = P_{[-n,-1]}Y - \sum_{j=1}^n a_{n,j}X_{-j}.$$

Since the right-hand side is in  $H_{[-n,-1]}$ , so is  $Z_n$ . Moreover,  $Z_n \in H_{(-\infty,-n-1]}$  since, for every  $k \geq 1$ ,  $Z_n^{2k+1}$  belongs to the closed subspace  $H_{(-\infty,-n-1]}$ . Combining,  $Z_n \in H_{[-n,-1]} \cap H_{(-\infty,-n-1]}$ . However, by (2.3), this implies  $Z_n = 0$ . Thus  $P_{[-n,-1]}Y = \sum_{j=1}^n a_{n,j}X_{-j}$ . By uniqueness, we obtain  $\phi_{n,j}(Y) = a_{n,j} = \lim_{k \rightarrow \infty} \phi_{n,j}^{2k+1}(Y)$ . Similarly, we have  $\phi_{n,j}(Y) = \lim_{k \rightarrow \infty} \phi_{n,j}^{2k}(Y)$ . Thus the theorem follows.  $\square$

*Remark 1.* A stationary process  $\{X_n\}$  is said to be *purely minimal* if  $X_0$  does not belong to the closed linear span of  $\{X_k : k \in \mathbf{Z}, k \neq 0\}$  in  $H$  (see Section 8.5 in [20]). The assumption (2.5) of Theorem 2.2 is equivalent to saying that  $\{X_n\}$  is purely minimal (see Makagon and Weron [19] and Salehi [23]; see also Theorem 8.10 in [20]).

*Remark 2.* The proofs above show that the conclusion of Theorem 2.2 holds under the following weaker assumption than (2.5):

$$(2.9) \quad \{X_n\} \text{ is nondeterministic and satisfies (2.6).}$$

In fact, we have assumed (2.5) to ensure (2.9). The condition (2.5) holds in most interesting examples, and, unlike (2.9), we can easily check it.

*Remark 3.* From (2.6), we obtain

$$\begin{aligned} H_{(-\infty,-1]} \cap H_{[0,\infty)} &\subset H_{(-\infty,-1]} \cap H_{[-1,\infty)} = H_{\{-1\}}, \\ H_{(-\infty,-1]} \cap H_{[0,\infty)} &\subset H_{(-\infty,0]} \cap H_{[0,\infty)} = H_{\{0\}}, \end{aligned}$$

while  $X_{-1}$  and  $X_0$  are linearly independent if  $\{X_n\}$  is nondeterministic. Thus, (2.9) implies that  $H_{(-\infty,-1]} \cap H_{[0,\infty)} = \{0\}$ . Combining this and the arguments in Helson and Sarason [10], page 6, we see that, under (2.9),  $\{X_n\}$  must be PND, whence has a spectral density  $\Delta(\cdot)$ . It is an open problem to characterize the condition (2.9) in terms of  $\Delta(\cdot)$ .

**2.2. Representation in terms of MA and AR coefficients.** In this section, we assume that the stationary process  $\{X_n\}$  is purely nondeterministic. For  $n \in \mathbf{N}$



and  $m \in \mathbf{N} \cup \{0\}$ , we can express the  $(m+1)$ -step predictor  $P_{[-n,-1]}X_m$  uniquely in the form

$$(2.10) \quad P_{[-n,-1]}X_m = \sum_{j=1}^n \phi_{n,j}^m X_{-j}.$$

We are concerned with representation of the real coefficients  $\phi_{n,j}^m$ , which we call the  $(m+1)$ -step finite predictor coefficients. In the 1-step case  $m=0$ , we have  $\phi_{n,j}^0 = \phi_{n,j}$  by (1.1).

We consider the following outer function:

$$h(z) := \sqrt{2\pi} \exp \left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\lambda} + z}{e^{i\lambda} - z} \log \Delta(\lambda) d\lambda \right\}, \quad z \in \mathbf{C}, |z| < 1.$$

The function  $h(z)$  is holomorphic and has no zeros in  $|z| < 1$ . We define the MA coefficients  $c_n$  by

$$h(z) = \sum_{n=0}^{\infty} c_n z^n, \quad |z| < 1,$$

and the AR coefficients  $a_n$  by

$$-1/h(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < 1$$

(cf. Section 2 in [14]). Both  $\{c_n\}$  and  $\{a_n\}$  are real sequences, and we have  $c_0 > 0$  and  $\sum_0^{\infty} (c_n)^2 < \infty$ . The coefficients  $c_n$  and  $a_n$  are actually those that appear in the following MA( $\infty$ ) and AR( $\infty$ ) representations, respectively, of  $\{X_n\}$  (under suitable condition such as (2.14) below for the latter):

$$(2.11) \quad X_n = \sum_{j=-\infty}^n c_{n-j} \xi_j, \quad n \in \mathbf{Z},$$

$$(2.12) \quad \sum_{j=-\infty}^n a_{n-j} X_j + \xi_n = 0, \quad n \in \mathbf{Z},$$

where  $\{\xi_k\}$  is the innovation process given by  $\xi_k = \epsilon_k / \|\epsilon_k\|$  with  $\epsilon_k$  in (2.8); see, e.g., Chapter II in [21] for (2.11) and (4.9) in [14] for (2.12). By the assumption that  $\{X_k\}$  is PND,  $\{\xi_k\}$  forms a complete orthonormal system of  $H$  such that, for every  $n \in \mathbf{Z}$ , the closed linear span of  $\{\xi_k : -\infty < k \leq n\}$  in  $H$  is equal to  $H_{(-\infty, n]}$ . Notice that the sums in (2.11) and (2.12) may not converge absolutely in  $H$ .

We put

$$b_j^m := \sum_{k=0}^m c_k a_{j+m-k}, \quad m, j \in \mathbf{N} \cup \{0\}.$$

In particular,  $b_j^0 = c_0 a_j$ . For  $n \in \mathbf{N}$  and  $m, j \in \mathbf{N} \cup \{0\}$ , we define  $b_k^m(n, j)$  recursively by

$$(2.13) \quad \begin{cases} b_1^m(n, j) = b_j^m, \\ b_{k+1}^m(n, j) = \sum_{m_1=0}^{\infty} b_{n+1+m_1}^m b_k^{m_1}(n, j), \quad k = 1, 2, \dots \end{cases}$$

From the proof of Theorem 2.3 below, we see that, under the condition

$$(2.14) \quad \sum_{n=0}^{\infty} |a_n| < \infty$$

which ensures the absolute convergence of the sums in (2.12), the sums in (2.13) also converge absolutely. We put, for  $m \in \mathbf{N} \cup \{0\}$ ,  $n \in \mathbf{N}$ , and  $j = 1, 2, \dots, n$ ,

$$g_k^m(n, j) := \begin{cases} b_k^m(n, j), & k = 1, 3, \dots, \\ b_k^m(n, n+1-j), & k = 2, 4, \dots \end{cases}$$

We write  $\sum^{\infty-}$  for the improper sum:  $\sum^{\infty-} = \lim_{M \rightarrow \infty} \sum^M$ . The following theorem gives an explicit representation of the  $(m+1)$ -step finite predictor coefficients  $\phi_{n,j}^m$  in (2.10), in terms of the MA and AR coefficients, under the absolute convergence of the sums in (2.12).

**Theorem 2.3.** *We assume that the AR coefficients  $a_n$  of a purely nondeterministic stationary process  $\{X_n\}$  satisfy (2.14). Then we have  $\phi_{n,j}^m = \sum_{k=1}^{\infty-} g_k^m(n, j)$  for  $n \in \mathbf{N}$ ,  $m \in \mathbf{N} \cup \{0\}$  and  $j = 1, \dots, n$ , that is,*

$$P_{[-n,-1]} X_m = \sum_{j=1}^n \left\{ \sum_{k=1}^{\infty-} g_k^m(n, j) \right\} X_{-j}.$$

*Proof.* For  $m \in \mathbf{N} \cup \{0\}$  and  $n \in \mathbf{N}$ , we have the following Wiener prediction formulas (cf. [14, Theorem 4.4]):

$$(2.15) \quad P_{(-\infty,-1]} X_m = \sum_{j=1}^{\infty} b_j^m X_{-j},$$

$$(2.16) \quad P_{[-n,\infty)} X_{-n-1-m} = \sum_{j=1}^{\infty} b_j^m X_{-n-1+j},$$

the sums converging absolutely in  $H$ . Recall  $P_n^k$  from (2.2). From (2.15), we have

$$P_n^1 X_m = \sum_{j=1}^n g_1^m(n, j) X_{-j} + \sum_{m_1=0}^{\infty} b_{n+1+m_1}^m X_{-n-1-m_1}.$$

From this and (2.16), it follows that

$$\begin{aligned} P_n^2 P_n^1 X_m &= \sum_{j=1}^n g_1^m(n, j) X_{-j} + \sum_{m_1=0}^{\infty} b_{n+1+m_1}^m \sum_{j=1}^{\infty} b_j^{m_1} X_{-n-1+j} \\ &= \sum_{j=1}^n \{g_1^m(n, j) + g_2^m(n, j)\} X_{-j} + \sum_{m_1=0}^{\infty} b_{n+1+m_1}^m \sum_{m_2=0}^{\infty} b_{n+1+m_2}^{m_1} X_{m_2}. \end{aligned}$$

Similarly,

$$\begin{aligned}
P_n^3 P_n^2 P_n^1 X_m &= \sum_{j=1}^n \{g_1^m(n, j) + g_2^m(n, j)\} X_{-j} \\
&\quad + \sum_{m_1=0}^{\infty} b_{n+1+m_1}^m \sum_{m_2=0}^{\infty} b_{n+1+m_2}^{m_1} \sum_{j=1}^{\infty} b_j^{m_2} X_{-j} \\
&= \sum_{j=1}^n \{g_1^m(n, j) + g_2^m(n, j) + g_3^m(n, j)\} X_{-j} \\
&\quad + \sum_{m_1=0}^{\infty} b_{n+1+m_1}^m \sum_{m_2=0}^{\infty} b_{n+1+m_2}^{m_1} \sum_{m_3=0}^{\infty} b_{n+1+m_3}^{m_2} X_{-n-1-m_3}.
\end{aligned}$$

Repeating this argument, we see that  $\phi_{n,j}^k(X_m)$  in Lemma 2.1 with  $Y = X_m$  are given by  $\phi_{n,j}^k(X_m) = \sum_{l=1}^k g_l^m(n, j)$ . The condition (2.14) implies  $\sum_0^\infty (a_n)^2 < \infty$ , whence (2.4) (cf. [14], Proposition 4.2). Thus the theorem follows from Theorem 2.2.  $\square$

**2.3. Representation by absolutely convergent series.** In later applications, it is essential to express the finite predictor coefficients  $\phi_{n,j}$  in (1.1) by an absolutely convergent series made up of  $a_k$  and  $c_k$ . In this section, we first give such an expression for  $b_k^m(n, j)$ . In the 1-step case  $m = 0$ , the result gives the desired representation for  $\phi_{n,j}$ .

We write  $\mathcal{R}_0$  for the class of *slowly varying functions* at infinity: the class of positive, measurable  $\ell(\cdot)$ , defined on some neighborhood  $[A, \infty)$  of infinity, such that  $\lim_{x \rightarrow \infty} \ell(\lambda x)/\ell(x) = 1$  for all  $\lambda > 0$  (see Chapter 1 in [4] for background).

Throughout this section, we assume that the stationary process  $\{X_n\}$  satisfies one of the following conditions (A1) and (A2):

(A1)  $\{X_n\}$  is purely nondeterministic, and  $\{a_n\}$  and  $\{c_n\}$  satisfy, respectively, (2.14) and

$$(2.17) \quad \sum_{n=0}^{\infty} |c_n| < \infty.$$

(A2)  $\{X_n\}$  is purely nondeterministic, and, for  $d \in (0, 1/2)$  and  $\ell(\cdot) \in \mathcal{R}_0$ ,  $\{c_n\}$  and  $\{a_n\}$  satisfy, respectively,

$$(2.18) \quad c_n \sim n^{-(1-d)} \ell(n), \quad n \rightarrow \infty,$$

$$(2.19) \quad a_n \sim n^{-(1+d)} \frac{1}{\ell(n)} \cdot \frac{d \sin(\pi d)}{\pi}, \quad n \rightarrow \infty.$$

It should be noticed that (2.19) implies (2.14). By (2.11), the autocovariance function  $\gamma(\cdot)$  has the expression

$$(2.20) \quad \gamma(n) = \sum_{k=0}^{\infty} c_{|n|+k} c_k, \quad n \in \mathbf{Z}.$$

Hence, (2.17) implies that

$$\sum_{n=0}^{\infty} |\gamma(n)| \leq \left( \sum_{k=0}^{\infty} |c_k| \right)^2 < \infty.$$

Thus  $\{X_n\}$  is short memory under (A1). On the other hand, by (2.20) and Proposition 4.3 in [13], (2.18) implies that

$$(2.21) \quad \gamma(n) \sim n^{-(1-2d)} \ell(n)^2 B(d, 1-2d), \quad n \rightarrow \infty.$$

Since  $0 < 1 - 2d < 1$ , we see that  $\{X_n\}$  is long memory under (A2). We remark that, under suitable conditions, (2.18), (2.19) and (2.21) are equivalent (cf. [14], Theorem 5.1).

**Example 2.4.** For  $d \in (-1/2, 1/2)$  and  $p, q \in \mathbf{N} \cup \{0\}$ , a stationary process  $\{X_n\}$  is said to be a fractional ARIMA( $p, d, q$ ) process if it has a spectral density  $\Delta(\cdot)$  of the form

$$\Delta(\lambda) = \frac{1}{2\pi} \frac{|\theta(e^{i\lambda})|^2}{|\phi(e^{i\lambda})|^2} |1 - e^{i\lambda}|^{-2d}, \quad -\pi \leq \lambda \leq \pi,$$

where  $\phi(z)$  and  $\theta(z)$  are polynomials with real coefficients of degrees  $p, q$ , respectively. We assume that  $\phi(z)$  and  $\theta(z)$  have no common zeros, and that neither  $\phi(z)$  nor  $\theta(z)$  has zeros in the closed unit disk  $\{z \in \mathbf{C} : |z| \leq 1\}$ . We also assume without loss of generality that  $\theta(0)/\phi(0) > 0$ . Then the outer function  $h(\cdot)$  is given by  $h(z) = (1-z)^{-d} \theta(z)/\phi(z)$  (see, e.g., Section 2 in [15]). If  $0 < d < 1/2$ , then  $\{X_n\}$  satisfies (A2) for some constant function  $\ell(\cdot)$  (see Corollary 3.1 in [18]). If  $d = 0$ , then  $\{X_n\}$  is also called an ARMA( $p, q$ ) process (see [5], Chapter 3), and both  $\{c_n\}$  and  $\{a_n\}$  decay exponentially, whence (A1) is satisfied.

We put

$$B_n := \sum_{v=0}^{\infty} |c_v a_{n+v}|, \quad n \in \mathbf{N} \cup \{0\}.$$

For  $n, k, u, v \in \mathbf{N} \cup \{0\}$ , we define  $D_k(n, u, v)$  recursively by

$$\begin{cases} D_0(n, u, v) := \delta_{uv}, \\ D_{k+1}(n, u, v) := \sum_{w=0}^{\infty} B_{n+v+w} D_k(n, u, w). \end{cases}$$

We have, for example,

$$D_3(n, u, v) = \sum_{v_1=0}^{\infty} \sum_{v_2=0}^{\infty} B_{n+v+v_1} B_{n+v_1+v_2} B_{n+v_2+u}.$$

By the Fubini–Tonelli theorem, we have  $D_k(n, u, v) = D_k(n, v, u)$ .

**Lemma 2.5.** *The conditions (A1) and (A2) imply, for  $k, n, v \in \mathbf{N} \cup \{0\}$ ,*

$$\sum_{u=0}^{\infty} D_k(n, u, v) < \infty \quad \text{and} \quad \sum_{u=0}^{\infty} D_k(n, u, v)^2 < \infty,$$

*respectively. In particular, we have  $D_k(n, u, v) < \infty$  for  $k, n, u, v \in \mathbf{N} \cup \{0\}$ .*

*Proof.* First we assume (A1). Then

$$\sum_{m=0}^{\infty} B_m \leq \left\{ \sum_{u=0}^{\infty} |c_u| \right\} \left\{ \sum_{u=0}^{\infty} |a_u| \right\} < \infty.$$

This and the nonnegativity of  $B_m$  imply, for example,

$$\begin{aligned} \sum_{u=0}^{\infty} D_3(n, u, v) &= \sum_{u=0}^{\infty} \sum_{v_1=0}^{\infty} \sum_{v_2=0}^{\infty} B_{n+v+v_1} B_{n+v_1+v_2} B_{n+v_2+u} \\ &\leq \left\{ \sum_{m=0}^{\infty} B_m \right\}^3 < \infty. \end{aligned}$$

The general case can be proved in the same way.

Next we assume (A2). The proof in this case is the same as that of Lemma 2.1 in [16]. By (A2) and Proposition 4.3 in [13], we have  $B_n = O(n^{-1})$  as  $n \rightarrow \infty$ . Therefore, for  $n \in \mathbf{N}$ ,  $f_u \mapsto \sum_{v=0}^{\infty} B_{n+u+v} f_v$  defines a bounded linear operator on  $l^2$  (see Chapter IX in [11]). Since  $D_{k+1}(n, u, v) = \sum_w B_{n+u+w} D_k(n, w, v)$ , we obtain the desired result by induction on  $k$ .  $\square$

We put

$$(2.22) \quad \beta_n := \sum_{v=0}^{\infty} c_v a_{v+n}, \quad n = 0, 1, \dots$$

In view of Lemma 2.5, we may define  $\delta_k(n, u, v)$  recursively by, for  $k, n, u, v \in \mathbf{N} \cup \{0\}$ ,

$$(2.23) \quad \begin{cases} \delta_0(n, u, v) = \delta_{uv}, \\ \delta_{k+1}(n, u, v) = \sum_{w=0}^{\infty} \beta_{n+v+w} \delta_k(n, u, w). \end{cases}$$

By Lemma 2.5 and the Fubini theorem, we have  $\delta_k(n, u, v) = \delta_k(n, v, u)$ .

The following theorem expresses  $b_k^m(n, j)$  by an absolutely convergent series.

**Theorem 2.6.** *We assume either (A1) or (A2). Then, for  $n, k \in \mathbf{N}$  and  $m, j \in \mathbf{N} \cup \{0\}$ ,*

$$(2.24) \quad b_k^m(n, j) = \sum_{v=0}^m c_{m-v} \sum_{u=0}^{\infty} a_{j+u} \delta_{k-1}(n+1, u, v),$$

*the sum converging absolutely.*

*Proof.* By Lemma 2.5 and (3.4), we have

$$(2.25) \quad \begin{aligned} \sum_{u=0}^{\infty} |a_{j+u}| D_{k-1}(n+1, u, v) \\ \leq \{ \sup_u D_{k-1}(n+1, u, v) \} \sum_{u=0}^{\infty} |a_{j+u}| < \infty. \end{aligned}$$

Thus, the right-hand side of (2.24), which we denote by  $B_k^m(n, j)$ , converges absolutely. To prove the proposition, it is enough show that  $B_k^m(n, j)$  satisfies the same recursion as (2.13).

First we have

$$B_1^m(n, j) = \sum_{v=0}^m c_{m-v} \sum_{u=0}^{\infty} a_{j+u} \delta_{uv} = \sum_{v=0}^m c_{m-v} a_{j+v} = b_j^m,$$

as desired. Next, the Fubini–Tonelli theorem and (2.25) yield, for  $k \geq 1$ ,

$$\begin{aligned} \sum_{u=0}^{\infty} a_{j+u} \delta_k(n+1, u, v) \\ &= \sum_{u=0}^{\infty} a_{j+u} \sum_{w=0}^{\infty} \left\{ \sum_{m_1=w}^{\infty} c_{m_1-w} a_{n+1+v+m_1} \right\} \delta_{k-1}(n+1, u, w) \\ &= \sum_{m_1=0}^{\infty} a_{n+1+v+m_1} \sum_{w=0}^{m_1} c_{m_1-w} \sum_{u=0}^{\infty} a_{j+u} \delta_{k-1}(n+1, u, w) \\ &= \sum_{m_1=0}^{\infty} a_{n+1+v+m_1} B_k^{m_1}(n, j), \end{aligned}$$

so that

$$\begin{aligned} B_{k+1}^m(n, j) &= \sum_{v=0}^m c_{m-v} \sum_{m_1=0}^{\infty} a_{n+1+v+m_1} B_k^{m_1}(n, j) \\ &= \sum_{m_1=0}^{\infty} \left\{ \sum_{v=0}^m c_{m-v} a_{n+1+m_1+v} \right\} B_k^{m_1}(n, j) = \sum_{m_1=0}^{\infty} b_{n+1+m_1}^m B_k^{m_1}(n, j). \end{aligned}$$

Thus  $B_k^m(n, j)$  satisfies (2.13).  $\square$

For later applications, we consider the case  $m = 0$  separately. We put

$$d_k(n, j) := \delta_k(n, 0, j), \quad n, k, j \in \mathbf{N} \cup \{0\}.$$

Then, by (2.23),  $d_k(n, j)$  satisfies the following recursion: for  $k, n, j \in \mathbf{N} \cup \{0\}$ ,

$$(2.26) \quad \begin{cases} d_0(n, j) = \delta_{j0}, \\ d_{k+1}(n, j) = \sum_{v=0}^{\infty} \beta_{n+j+v} d_k(n, v). \end{cases}$$

More explicitly,  $d_k(n, j)$  are given by, for  $n, j \in \mathbf{N} \cup \{0\}$ ,

$$d_1(n, j) = \beta_{n+j}, \quad d_2(n, j) = \sum_{v_1=0}^{\infty} \beta_{n+j+v_1} \beta_{n+v_1},$$

and, for  $k = 3, 4, \dots$ ,

$$d_k(n, j) = \sum_{v_1=0}^{\infty} \cdots \sum_{v_{k-1}=0}^{\infty} \beta_{n+j+v_{k-1}} \beta_{n+v_{k-1}+v_{k-2}} \cdots \beta_{n+v_2+v_1} \beta_{n+v_1},$$

the sums converging absolutely.

We put

$$b_k(n, j) := b_k^0(n, j), \quad g_k(n, j) := g_k^0(n, j)$$

for  $(k, n, j)$  for which the right-hand sides are defined. Then, for  $n \in \mathbf{N}$  and  $j = 1, 2, \dots, n$ , we have

$$(2.27) \quad g_k(n, j) = \begin{cases} b_k(n, j), & k = 1, 3, \dots, \\ b_k(n, n+1-j), & k = 2, 4, \dots \end{cases}$$

By Theorem 2.3 and Proposition 2.6, we immediately obtain the following final form of the representation of the 1-step finite predictor coefficients  $\phi_{n,j}$ .

**Theorem 2.7.** *We assume either (A1) or (A2). Then, for  $n \in \mathbf{N}$  and  $j = 1, \dots, n$ , we have  $\phi_{n,j} = \sum_{k=1}^{\infty-} g_k(n, j)$  with (2.27) and*

$$\begin{cases} b_1(n, v) = c_0 a_v, & v \geq 0, \\ b_k(n, v) = c_0 \sum_{u=0}^{\infty} a_{v+u} d_{k-1}(n+1, u), & k \geq 2, v \geq 0, \end{cases}$$

the sum on the right-hand side converging absolutely.

*Remark 4.* Under either (A1) or (A2), the sum  $\sum_{k=1}^{\infty} g_k(n, j)$  converges absolutely for  $n$  large enough and  $j = 1, \dots, n$ . See Propositions 3.4 and 4.4 below.

### 3. RATES OF CONVERGENCE OF FINITE PREDICTOR COEFFICIENTS

If the stationary process  $\{X_n\}$  is PND and satisfies (2.14), then we have the Wiener prediction formula (2.15) with  $m = 0$  or (1.2) with

$$(3.1) \quad \phi_j = c_0 a_j, \quad j \in \mathbf{N}.$$

We call  $\phi_j$  the *infinite predictor coefficients*. It holds that

$$\lim_{n \rightarrow \infty} \phi_{n,j} = \phi_j, \quad j \in \mathbf{N}$$

(cf. Theorem 7.14 in [20]). In this section, we investigate the rate at which  $\phi_{n,j}$  converges to  $\phi_j$  as  $n \rightarrow \infty$ . Notice that, by (2.13), (2.27) and (3.1), we have

$$(3.2) \quad \phi_j = b_1(n, j) = g_1(n, j), \quad n \in \mathbf{N}, \quad j = 1, \dots, n.$$

Thus  $\phi_j$  is the first term of the series  $\sum_{k=1}^{\infty} g_k(n, j)$  in Theorem 2.7 expressing  $\phi_{n,j}$ . This suggests the usefulness of the expression for our purpose.

**3.1. Long memory processes.** In this section, we assume that the stationary process  $\{X_n\}$  satisfies (A2) in Section 2.3. Thus  $\{X_n\}$  is long memory.

For  $u \geq 0$ , we put

$$f_1(u) := \frac{1}{\pi(1+u)}, \quad f_2(u) := \frac{1}{\pi^2} \int_0^\infty \frac{ds_1}{(s_1+1)(s_1+1+u)},$$

and, for  $k = 3, 4, \dots$ ,

$$f_k(u) := \frac{1}{\pi^k} \int_0^\infty ds_{k-1} \cdots \int_0^\infty ds_1 \frac{1}{(s_{k-1}+1)} \times \left\{ \prod_{m=1}^{k-2} \frac{1}{(s_{m+1}+s_m+1)} \right\} \\ \times \frac{1}{(s_1+1+u)}$$

(see [15], Section 3; see also [14], Section 6).

**Lemma 3.1.** (i)  $\sum_{k=1}^{\infty} f_{2k}(0)x^{2k} = (\pi^{-1} \arcsin x)^2$  for  $|x| < 1$ ;

(ii)  $\sum_{k=1}^{\infty} f_{2k-1}(0)x^{2k-1} = \pi^{-1} \arcsin x$  for  $|x| < 1$ .

*Proof.* Let  $j \geq 1$ . We easily see that  $f_{1+j}(u) = \int_0^\infty f_1(s+u)f_j(s)ds$  for  $u \geq 0$ .

Hence, we have, for  $u \geq 0$ ,

$$f_{2+j}(0) = \int_0^\infty f_1(s)f_{j+1}(s)ds = \int_0^\infty ds f_1(s) \int_0^\infty f_1(u+s)f_j(u)du \\ = \int_0^\infty \left\{ \int_0^\infty f_1(s+u)f_1(s)ds \right\} f_j(u)du = \int_0^\infty f_2(u)f_j(u)du.$$

Repeating this argument, we obtain

$$(3.3) \quad \int_0^\infty f_i(u)f_j(u)du = f_{i+j}(0), \quad i, j \in \mathbf{N}.$$

Thus, the assertion (i) follows from Lemma 6.5 in [14], while (ii) from Lemma 3.4 in [16].  $\square$

Recall  $d_k(n, u)$  from Section 2.3.

**Proposition 3.2.** (i) For  $r \in (1, \infty)$ , there exists  $N \in \mathbf{N}$  such that

$$(3.4) \quad 0 < d_k(n, u) \leq \frac{f_k(0)\{r \sin(\pi d)\}^k}{n}, \quad u \in \mathbf{N} \cup \{0\}, \quad k \in \mathbf{N}, \quad n \geq N.$$



(ii) For  $k \in \mathbf{N}$  and  $u \in \mathbf{N} \cup \{0\}$ ,  $d_k(n, u) \sim n^{-1} f_k(0) \sin^k(\pi d)$  as  $n \rightarrow \infty$ .

*Proof.* Let  $r > 1$ . Recall  $\beta_n$  from (2.22). The condition (A2) implies

$$(3.5) \quad \beta_n \sim \frac{\sin(\pi d)}{\pi} n^{-1}, \quad n \rightarrow \infty$$

(see Proposition 4.3 in [13]). Thus, for  $n$  large enough,

$$0 < \beta_{[ns]+n+u} \leq \frac{r^{1/2} \sin(\pi d)}{\pi([ns] + n + u)}, \quad s \geq 0, u \in \mathbf{N} \cup \{0\}.$$

Since we have, for  $n$  large enough,

$$\frac{1}{[ns] + n + u} \leq \frac{r^{1/2}}{n(s+1)}, \quad s \geq 0, u \in \mathbf{N} \cup \{0\},$$

there exists  $N_1 \in \mathbf{N}$  such that

$$(3.6) \quad 0 < \beta_{[ns]+n+u} \leq \frac{r \sin(\pi d)}{\pi(s+1)} n^{-1}, \quad s \geq 0, u \in \mathbf{N} \cup \{0\}, n \geq N_1.$$

In the same way, we can choose  $N_2$  so that

$$(3.7) \quad 0 < \beta_{[ns_2]+[ns_1]+n} \leq \frac{r \sin(\pi d)}{\pi(s_2 + s_1 + 1)} n^{-1}, \quad s_1, s_2 \geq 0, n \geq N_2.$$

Therefore, we have, for  $n \geq N := \max(N_1, N_2)$ ,

$$\begin{aligned} 0 < d_3(n, u) &= \int_0^\infty ds_2 \int_0^\infty ds_1 \cdot \beta_{[s_2]+n} \cdot \beta_{[s_2]+[s_1]+n} \cdot \beta_{[s_1]+n+u} \\ &= n^2 \int_0^\infty ds_2 \int_0^\infty ds_1 \cdot \beta_{[ns_2]+n} \cdot \beta_{[ns_2]+[ns_1]+n} \cdot \beta_{[ns_1]+n+u} \\ &\leq \frac{\{r \sin(\pi d)\}^3}{n} \frac{1}{\pi^3} \int_0^\infty ds_2 \int_0^\infty ds_1 \frac{1}{(s_2+1)(s_2+s_1+1)(s_1+1)} \\ &= \frac{\{r \sin(\pi d)\}^3}{n} f_3(0), \end{aligned}$$

which implies (3.4) with  $k = 3$ . Notice that  $N$  is independent of the choice  $k = 3$ .

We can prove (3.4) for general  $k$  and the same  $N$  in a similar fashion.

We also prove (ii) only for  $k = 3$ ; the general case can be treated in the same way. By (3.5), we have

$$(3.8) \quad \lim_{n \rightarrow \infty} n \beta_{[ns]+n+u} = \frac{\sin(\pi d)}{\pi(s+1)}, \quad s \geq 0, u \in \mathbf{N} \cup \{0\},$$

$$(3.9) \quad \lim_{n \rightarrow \infty} n \beta_{[ns_2]+[ns_1]+n} = \frac{\sin(\pi d)}{\pi(s_2 + s_1 + 1)}, \quad s_1, s_2 \geq 0.$$

By (3.6)–(3.9) and the dominated convergence theorem, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^\infty ds_2 \int_0^\infty ds_1 \cdot n \beta_{[ns_2]+n} \cdot n \beta_{[ns_2]+[ns_1]+n} \cdot n \beta_{[ns_1]+n+u} \\ = \frac{\sin^3(\pi d)}{\pi^3} \int_0^\infty ds_2 \int_0^\infty ds_1 \frac{1}{(s_2+1)(s_2+s_1+1)(s_1+1)}. \end{aligned}$$

This implies  $\lim_n n d_3(n, u) = \sin^3(\pi d) f_3(0)$  or (ii) with  $k = 3$ , as desired.  $\square$

The following theorem gives the rate for long memory processes, at which  $\phi_{n,j}$  converges to  $\phi_j$ . It applies, in particular, to the fractional ARIMA( $p, d, q$ ) processes with  $0 < d < 1/2$ .

**Theorem 3.3.** *We assume (A2). Then we have, for  $j \in \mathbf{N}$ ,*

$$\lim_{n \rightarrow \infty} n \{\phi_{n,j} - \phi_j\} = d^2 \sum_{u=j}^{\infty} \phi_u.$$

*Proof.* Let  $r > 1$  be chosen so that  $0 < r \sin(\pi d) < 1$ . By Lemma 3.1,

$$(3.10) \quad \sum_{k=1}^{\infty} f_k(0) \{r \sin(\pi d)\}^k < \infty.$$

Let  $N$  be as in Proposition 3.2 (i). Then, for  $n \geq N$  and  $j = 1, \dots, n$ ,

$$\begin{aligned} n \left| \sum_{u=0}^{\infty} a_{n-j+u} \sum_{k=1}^{\infty} d_{2k-1}(n, u) \right| &\leq \sum_{u=0}^{\infty} |a_{n-j+u}| \sum_{k=1}^{\infty} n d_{2k-1}(n, u) \\ &\leq \left[ \sum_{k=1}^{\infty} f_{2k-1}(0) \{r \sin(\pi d)\}^{2k-1} \right] \sum_{u=n-j}^{\infty} |a_u|, \end{aligned}$$

so that

$$\lim_{n \rightarrow \infty} n \sum_{u=0}^{\infty} a_{n-j+u} \sum_{k=1}^{\infty} d_{2k-1}(n, u) = 0.$$

Proposition 3.2, Lemma 3.1, and the dominated convergence theorem yield

$$\begin{aligned} \lim_{n \rightarrow \infty} n \sum_{u=0}^{\infty} a_{j+u} \sum_{k=1}^{\infty} d_{2k}(n, u) \\ = \left\{ \sum_{k=1}^{\infty} f_{2k}(0) \sin^{2k}(\pi d) \right\} \sum_{u=0}^{\infty} a_{j+u} = d^2 \sum_{u=j}^{\infty} a_u. \end{aligned}$$

Therefore, by Theorem 2.7 and (3.2), we have

$$\begin{aligned} &\lim_{n \rightarrow \infty} n \{\phi_{n-1,j} - \phi_j\} \\ &= \lim_{n \rightarrow \infty} \left\{ n \sum_{k=1}^{\infty} b_{2k+1}(n-1, j) + n \sum_{k=1}^{\infty} b_{2k}(n-1, n-j) \right\} \\ &= \lim_{n \rightarrow \infty} c_0 n \sum_{u=0}^{\infty} a_{j+u} \sum_{k=1}^{\infty} d_{2k}(n, u) + \lim_{n \rightarrow \infty} c_0 n \sum_{u=0}^{\infty} a_{n-j+u} \sum_{k=1}^{\infty} d_{2k-1}(n, u) \\ &= c_0 d^2 \sum_{u=j}^{\infty} a_u = d^2 \sum_{u=j}^{\infty} \phi_u, \end{aligned}$$

as desired. □

From the proof of Theorem 3.3, we also obtain the following proposition.

**Proposition 3.4.** *We assume (A2). Let  $N$  be as in the proof of Theorem 3.3. Then we have  $\sum_{k=1}^{\infty} |g_k(n, j)| < \infty$  for  $n \geq N$  and  $j = 1, \dots, n$ .*

**3.2. Intermediate memory processes.** There are several possible setups for short-memory processes, to which we can apply Theorem 2.7 to obtain the rate of convergence of  $\phi_{n,j}$ . Among them, we consider that of so called “intermediate memory” processes (see [5], page 520). Our class is defined in the following way:

**(A3)** the stationary process  $\{X_n\}$  is purely nondeterministic, and  $\{a_n\}$ ,  $\{c_n\}$  and the autocovariance function  $\gamma(\cdot)$  satisfy the following conditions:

- (i)  $c_n \geq 0$  for all  $n \geq 0$ ;
- (ii)  $\{c_n\}$  is eventually decreasing to zero;
- (iii)  $\{a_n\}$  is eventually decreasing to zero;
- (iv) for  $p \in (1, \infty)$  and  $\ell \in \mathcal{R}_0$ ,  $\gamma(n) \sim n^{-p}\ell(n)$  as  $n \rightarrow \infty$ .

This class of stationary processes was considered in [16]. It is suited for asymptotic analysis. The condition (iv) implies that the stationary processes satisfying (A3) are short memory.

**Example 3.5.** Let  $p > 1$ . If the autocovariance function  $\gamma(\cdot)$  of a stationary process  $\{X_n\}$  is of the form  $\gamma(n) = 1/(1+|n|)^p$ , then it satisfies (A3). See Example in Section 7 of [14].

Throughout this section, we assume that the stationary process  $\{X_n\}$  satisfies (A3). We define a constant  $D$  by

$$D := \sum_{k=-\infty}^{\infty} \gamma(k),$$

which is positive by (A3)(i) and (2.20). By Theorem 5.3 in [14] (see also the proof of Theorem 6.7 in [14]), (A3) implies the following asymptotics:

$$(3.11) \quad c_n \sim n^{-p}\ell(n)D^{-1/2}, \quad n \rightarrow \infty,$$

$$(3.12) \quad a_n \sim n^{-p}\ell(n)D^{-3/2}, \quad n \rightarrow \infty,$$

$$(3.13) \quad \beta_n \sim n^{-p}\ell(n)D^{-1}, \quad n \rightarrow \infty.$$

Thus, in particular,  $\{X_n\}$  satisfies (A1).

We need the following estimates to obtain the rate of convergence for  $\phi_{n,j}$ .

**Proposition 3.6.** (i) For  $r \in (1, \infty)$ , there exists  $N \in \mathbf{N}$  such that

$$(3.14) \quad 0 < \frac{nd_k(n, u)}{(n\beta_n)^k} \leq (r\pi)^k f_k(0), \quad u \in \mathbf{N} \cup \{0\}, k \in \mathbf{N}, n \geq N.$$

(ii) For  $u \in \mathbf{N} \cup \{0\}$ , we have  $d_2(n, u) \sim n\beta_n^2/(2p-1)$  as  $n \rightarrow \infty$ .

*Proof.* Let  $r > 1$ . By (3.13) and the Uniform Convergence Theorem ([4], Theorem 1.5.2), we see that, for  $n$  large enough,

$$0 < \frac{\beta_{[ns]+n+u}}{\beta_n} \leq r^{1/2} \left( \frac{n}{[ns]+n+u} \right)^p, \quad s \geq 0, u \in \mathbf{N} \cup \{0\}.$$

Since we have, for  $n$  large enough,

$$\frac{n}{[ns]+n+u} \leq \frac{r^{1/2}}{(s+1)^p} \leq \frac{r^{1/2}}{s+1}, \quad s \geq 0, u \in \mathbf{N} \cup \{0\},$$

there exists  $N_1 \in \mathbf{N}$  such that

$$(3.15) \quad 0 < \frac{\beta_{[ns]+n+u}}{\beta_n} \leq \frac{r}{1+s}, \quad s \geq 0, u \in \mathbf{N} \cup \{0\}, n \geq N_1.$$

In the same way, we may take  $N_2 \in \mathbf{N}$  so large that

$$(3.16) \quad 0 < \frac{\beta_{[ns_2]+[ns_1]+n}}{\beta_n} \leq \frac{r}{1+s_2+s_1}, \quad s_1, s_2 \geq 0, n \geq N_2.$$

Therefore, we have, for  $n \geq N := \max(N_1, N_2)$ ,

$$\begin{aligned} 0 < \frac{d_3(n, u)}{n^2 \beta_n^3} &= \frac{1}{n^2 \beta_n^3} \int_0^\infty ds_2 \int_0^\infty ds_1 \cdot \beta_{[s_2]+n} \cdot \beta_{[s_2]+[s_1]+n} \cdot \beta_{[s_1]+n+u} \\ &= \int_0^\infty ds_2 \int_0^\infty ds_1 \cdot \frac{\beta_{[ns_2]+n}}{\beta_n} \cdot \frac{\beta_{[ns_2]+[ns_1]+n}}{\beta_n} \cdot \frac{\beta_{[ns_1]+n+u}}{\beta_n} \\ &\leq (r\pi)^3 f_3(0), \end{aligned}$$

which implies (3.14) with  $k = 3$ . Notice that  $N$  is independent of the choice  $k = 3$ .

In the same way, we can prove (3.14) for general  $k$  and the same  $N$ .

By (3.13),

$$(3.17) \quad \lim_{n \rightarrow \infty} \frac{\beta_{[ns]+n+u}}{\beta_n} = \frac{1}{(1+s)^p}, \quad s \geq 0, u \in \mathbf{N} \cup \{0\}.$$

By (3.15), (3.17) and the dominated convergence theorem, we have

$$\frac{d_2(n, u)}{n \beta_n^2} = \int_0^\infty ds \cdot \frac{\beta_{[ns]+n}}{\beta_n} \cdot \frac{\beta_{[ns]+n+u}}{\beta_n} \rightarrow \int_0^\infty \frac{1}{(1+s)^{2p}} ds = \frac{1}{2p-1}$$

as  $n \rightarrow \infty$ . Thus (ii) follows.  $\square$

The following theorem gives the rate for the present class of  $\{X_n\}$ , at which  $\phi_{n,j}$  converges to  $\phi_j$ .

**Theorem 3.7.** *We assume (A3). Then we have, for  $j \in \mathbf{N}$ ,*

$$\begin{aligned} &\phi_{n,j} - \phi_j \\ &\sim n^{-(2p-1)} \ell(n)^2 \frac{1}{2p-1} \left\{ D^{-5/2} c_0 + D^{-2} \sum_{u=j}^\infty \phi_u \right\}, \quad n \rightarrow \infty. \end{aligned}$$

*Proof.* Let  $r > 1$  and  $N$  be as in Proposition 3.6 (i). Let  $t \in (0, 1)$ . Since (3.13) implies  $\lim_{n \rightarrow \infty} n \beta_n = 0$ , there exists an integer  $N'$  such that

$$0 < n \beta_n r \pi < t, \quad n \geq N'.$$

Then we have, for  $l \geq 3$ ,  $n \geq \max(N, N')$  and  $u \geq 0$ ,

$$0 < \frac{d_l(n, u)}{n\beta_n^2} < (n\beta_n)^{l-2}(r\pi)^l f_l(0) < (n\beta_n)(r\pi/t)^3 t^l f_l(0),$$

so that, for  $j = 1, \dots, n$ ,

$$\begin{aligned} & \frac{1}{n\beta_n^2} \left| \sum_{u=0}^{\infty} a_{n-j+u} \sum_{k=2}^{\infty} d_{2k-1}(n, u) \right| \\ & \leq \sum_{u=0}^{\infty} |a_{n-j+u}| \sum_{k=2}^{\infty} \frac{d_{2k-1}(n, u)}{n\beta_n^2} \\ & \leq (n\beta_n) \times (r\pi/t)^3 \left\{ \sum_{k=2}^{\infty} f_{2k-1}(0)t^{2k-1} \right\} \sum_{u=0}^{\infty} |a_u|. \end{aligned}$$

Since (3.13) implies  $n\beta_n^2 \sim n^{-(2p-1)}\ell(n)^2 D^{-2}$ , we obtain

$$(3.18) \quad \lim_{n \rightarrow \infty} \frac{1}{n^{-(2p-1)}\ell(n)^2} \sum_{u=0}^{\infty} a_{n-j+u} \sum_{k=2}^{\infty} d_{2k-1}(n, u) = 0 \quad (j \in \mathbf{N}).$$

In the same way, we have, for  $n$  large enough,

$$\begin{aligned} & \frac{1}{n\beta_n^2} \left| \sum_{u=0}^{\infty} a_{j+u} \sum_{k=2}^{\infty} d_{2k}(n, u) \right| \leq \sum_{u=0}^{\infty} |a_{j+u}| \sum_{k=2}^{\infty} \frac{d_{2k}(n, u)}{n\beta_n^2} \\ & \leq (n\beta_n) \times (r\pi/t)^3 \left\{ \sum_{k=2}^{\infty} f_{2k}(0)t^{2k} \right\} \sum_{u=j}^{\infty} |a_u|, \end{aligned}$$

so that

$$(3.19) \quad \lim_{n \rightarrow \infty} \frac{1}{n^{-(2p-1)}\ell(n)^2} \sum_{u=0}^{\infty} a_{j+u} \sum_{k=2}^{\infty} d_{2k}(n, u) = 0 \quad (j \in \mathbf{N}).$$

Proposition 3.6 and the dominated convergence theorem yield

$$(3.20) \quad \begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n\beta_n^2} \sum_{u=0}^{\infty} a_{j+u} d_2(n, u) &= \lim_{n \rightarrow \infty} \frac{1}{n^{-(2p-1)}\ell(n)^2} \sum_{u=0}^{\infty} a_{j+u} d_2(n, u) \\ &= \frac{D^{-2}}{2p-1} \sum_{u=j}^{\infty} a_u. \end{aligned}$$

As in the proof of Proposition 3.6, we have

$$\begin{aligned} & \frac{1}{na_n\beta_n} \sum_{u=0}^{\infty} a_{n-j+u} d_1(n, u) = \frac{1}{na_n\beta_n} \int_0^{\infty} a_{n-j+[s]} \cdot \beta_{n+[s]} ds \\ & = \int_0^{\infty} \frac{a_{n-j+[ns]}}{a_n} \cdot \frac{\beta_{[ns]+n}}{\beta_n} ds \rightarrow \int_0^{\infty} \frac{1}{(1+s)^{2p}} ds, \quad n \rightarrow \infty \end{aligned}$$

or, by (3.12) and (3.13),

$$(3.21) \quad \lim_{n \rightarrow \infty} \frac{1}{n^{-(2p-1)}\ell(n)^2} \sum_{u=0}^{\infty} a_{n-j+u} d_1(n, u) = \frac{D^{-5/2}}{2p-1}.$$

Combining (3.18)–(3.21) and Theorem 2.7, we obtain

$$\begin{aligned}
& \frac{1}{n^{-(2p-1)\ell(n)^2}} \{\phi_{n-1,j} - \phi_j\} \\
&= \frac{1}{n^{-(2p-1)\ell(n)^2}} \left\{ \sum_{k=1}^{\infty} b_{2k+1}(n-1, j) + \sum_{k=1}^{\infty} b_{2k}(n-1, n-j) \right\} \\
&= \frac{c_0}{n^{-(2p-1)\ell(n)^2}} \left\{ \sum_{u=0}^{\infty} a_{j+u} \sum_{k=1}^{\infty} d_{2k}(n, u) + \sum_{u=0}^{\infty} a_{n-j+u} \sum_{k=1}^{\infty} d_{2k-1}(n, u) \right\} \\
&\rightarrow \frac{c_0}{2p-1} \left\{ D^{-5/2} + D^{-2} \sum_{u=j}^{\infty} a_u \right\} = \frac{1}{2p-1} \left\{ D^{-5/2} c_0 + D^{-2} \sum_{u=j}^{\infty} \phi_u \right\},
\end{aligned}$$

as required.  $\square$

For nondeterministic stationary process  $\{X_n\}$ , the *partial autocorrelation function* (PACF)  $\alpha(\cdot)$  is defined by  $\alpha(1) := \gamma(1)/\gamma(0)$  and

$$(3.22) \quad \alpha(n) := \frac{(X_n - P_{[1,n-1]}X_n, X_0 - P_{[1,n-1]}X_0)}{\|X_n - P_{[1,n-1]}X_n\| \cdot \|X_0 - P_{[1,n-1]}X_0\|}, \quad n = 2, 3, \dots$$

(see Sections 3.4 and 5.2 in [5] for background). By Theorem 1.8 in [16], (A3) implies  $\alpha(n) \sim n^{-p\ell(n)}D^{-1}$  as  $n \rightarrow \infty$ . Since  $p > 1$ , this yields

$$(3.23) \quad \sum_{n=1}^{\infty} |\alpha(n)| < \infty.$$

The implication (A3)  $\Rightarrow$  (3.23) is also verified by Theorem 5.5 below since (3.12) implies  $a_n = O(n^{-(p+1)/2})$  as  $n \rightarrow \infty$ . In view of (3.23) and Theorem 3.7, we have provided a counterexample to Theorem 7.23 in [20] which seems to mistakenly assert that, for each  $j$ ,  $\phi_{n,j}$  converges to  $\phi_j$  exponentially if and only if (3.23) holds. In fact, by Theorem 2.3 in [1] with  $m = 0$  (see also Proposition 4.1 below), (3.23) holds under (A1) which is weaker than (A3).

#### 4. BAXTER'S INEQUALITY

We are concerned with inequalities of Baxter-type which are related to the norm convergence of  $\phi_{n,j}$  to  $\phi_j$ .

**4.1. Baxter's inequality for short memory processes.** In this section, we give a short and easy proof to the original inequality (1.3) of Baxter, i.e., the case  $\lambda = 0$  of Theorem 2.2 in [1]. The proof is based on the representation of  $\phi_{n,j}$  (Theorem 2.7), which is available since the assumption in [1] is the same as (A1) in Section 2.3 by the following proposition.

**Proposition 4.1.** *For a stationary process  $\{X_n\}$ , the following conditions are equivalent:*

- (i)  $\{X_n\}$  satisfies (A1);
- (ii)  $\{X_n\}$  has a positive continuous spectral density and satisfies (2.17);
- (iii)  $\{X_n\}$  has a positive continuous spectral density and satisfies (2.14);
- (iv)  $\{X_n\}$  is short memory, i.e.,  $\sum_{k=-\infty}^{\infty} |\gamma(k)| < \infty$ , and has a positive continuous spectral density.

*Proof.* Notice that if  $\{X_n\}$  has a positive continuous spectral density  $\Delta(\cdot)$  on  $[-\pi, \pi]$ , then it is PND since  $\int_{-\pi}^{\pi} |\log \Delta(\lambda)| d\lambda < \infty$  holds.

Suppose (i). We write  $h(e^{i\lambda})$  for the nontangential limit of  $h(z)$ , i.e.,

$$h(e^{i\lambda}) = \lim_{r \rightarrow 1-0} h(re^{i\lambda}) = \sum_{n=0}^{\infty} c_n e^{in\lambda}, \quad -\pi \leq \lambda \leq \pi.$$

Since (2.17) implies the continuity of  $h(e^{i\lambda})$ , the spectral density  $\Delta(\lambda)$  is also continuous by the equality  $\Delta(\lambda) = 2\pi|h(e^{i\lambda})|^2$ . Letting  $r \rightarrow 1-0$  in

$$\left(\sum_{n=0}^{\infty} c_n r^n e^{in\lambda}\right) \left(\sum_{n=0}^{\infty} a_n r^n e^{in\lambda}\right) = -1, \quad -\pi \leq \lambda \leq \pi,$$

we obtain

$$\left(\sum_{n=0}^{\infty} c_n e^{in\lambda}\right) \left(\sum_{n=0}^{\infty} a_n e^{in\lambda}\right) = -1, \quad -\pi \leq \lambda \leq \pi.$$

This implies that  $h(e^{i\lambda})$ , whence  $\Delta(\lambda)$ , has no zeros on  $[-\pi, \pi]$ . Thus  $\Delta(\cdot)$  is positive, whence (ii) and (iii) follow.

Suppose (iii). In the same way as above, we have

$$\frac{1}{\Delta(\lambda)} = \frac{1}{2\pi} \left| \sum_{n=0}^{\infty} a_n e^{in\lambda} \right|^2, \quad -\pi \leq \lambda \leq \pi.$$

This implies  $\sum_{n=0}^{\infty} a_n e^{in\lambda} \neq 0$  for every  $\lambda \in [-\pi, \pi]$ . By Wiener's theorem for absolutely convergent Fourier series (cf. Lemma 11.6 in [22]), we obtain (2.17) (cf. [3], page 493). Thus (i) follows. The proof of the implication (ii)  $\Rightarrow$  (i) is similar.

By (2.20), (ii) implies (iv). Conversely, we assume (iv). Then we have (2.14), whence (iii), by (3.1) and the arguments in [1], pp. 139–140, which involve the Wiener–Lévy theorem. This completes the proof.  $\square$

We define

$$(4.1) \quad A(j) := \sum_{u=j}^{\infty} |a_u|, \quad F(j) := \left\{ \sum_{v=0}^{\infty} |c_v| \right\} A(j), \quad j = 0, 1, \dots$$

Then  $F(j)$  decreases to zero as  $j \rightarrow \infty$  under (A1). Recall  $d_k(n, j)$  from (2.26).

**Lemma 4.2.** *We assume (A1). Then  $\sum_{u=0}^{\infty} |d_k(n, u)| \leq F(n)^k$  for  $k, n \in \mathbf{N}$ .*

*Proof.* Let  $n \in \mathbf{N}$ . We use induction on  $k$ . Since  $d_1(n, u) = \beta_{n+u}$ , we have

$$\sum_{u=0}^{\infty} |d_1(n, u)| = \sum_{u=0}^{\infty} |\beta_{n+u}| \leq \sum_{v=0}^{\infty} |c_v| \sum_{u=0}^{\infty} |a_{n+u+v}| \leq F(n).$$

We assume  $\sum_{u=0}^{\infty} |d_k(n, u)| \leq F(n)^k$  for  $k \in \mathbf{N}$ . Then

$$\begin{aligned} \sum_{u=0}^{\infty} |d_{k+1}(n, u)| &\leq \sum_{v=0}^{\infty} |d_k(n, v)| \sum_{u=0}^{\infty} |\beta_{n+v+u}| \leq F(n) \sum_{v=0}^{\infty} |d_k(n, v)| \\ &\leq F(n)^{k+1}. \end{aligned}$$

Thus the inequality also holds for  $k + 1$ .  $\square$

Here is Baxter's inequality with  $M$  given explicitly.

**Theorem 4.3.** *We assume one, whence all, of (i)–(iv) in Proposition 4.1. Then, for every  $L \in (1, \infty)$ , there exists  $N \in \mathbf{N}$  such that (1.3) holds for  $n \geq N$  with*

$$M = \left\{ \sum_{v=0}^{\infty} |c_v| \right\} \left\{ \sum_{u=1}^{\infty} |a_u| \right\} L.$$

*Proof.* By Theorem 2.7, Lemma 4.2 and (3.2), we have, for  $n = 1, 2, \dots$ ,

$$\begin{aligned} \sum_{j=1}^n |\phi_{n,j} - \phi_j| &\leq c_0 \sum_{j=1}^n \sum_{u=0}^{\infty} |a_{u+j}| \sum_{k=1}^{\infty} |d_{2k}(n+1, u)| \\ &\quad + c_0 \sum_{j=1}^n \sum_{u=0}^{\infty} |a_{u+n+1-j}| \sum_{k=1}^{\infty} |d_{2k-1}(n+1, u)| \\ &= c_0 \sum_{k=1}^{\infty} \sum_{u=0}^{\infty} |d_k(n+1, u)| \sum_{j=1}^n |a_{u+j}| \\ &\leq c_0 A(1) \sum_{k=1}^{\infty} \sum_{u=0}^{\infty} |d_k(n+1, u)| \leq c_0 A(1) \sum_{k=1}^{\infty} F(n+1)^k. \end{aligned}$$

Choose  $N \in \mathbf{N}$  so that  $0 < 1/\{1 - F(N+1)\} < L$ . If  $n \geq N$ , then  $F(n+1) \leq F(N+1) < 1$ . Therefore,

$$\sum_{k=1}^{\infty} F(n+1)^k = \frac{F(n+1)}{1 - F(n+1)} \leq \frac{F(n+1)}{1 - F(N+1)} \leq LF(n+1).$$

However, by (3.1) and (4.1), we have

$$c_0 F(n+1) = \left\{ \sum_{v=0}^{\infty} |c_v| \right\} \sum_{j=n+1}^{\infty} |\phi_j|.$$

Thus the theorem follows.  $\square$

From the proof of Theorem 4.3, we also obtain the next proposition.

**Proposition 4.4.** *We assume (A1). Choose  $N \in \mathbf{N}$  so that  $F(N+1) < 1$ . Then we have  $\sum_{k=1}^{\infty} |g_k(n, j)| < \infty$  for  $n \geq N$  and  $j = 1, \dots, n$ .*



**4.2. Baxter's inequality for long memory processes.** In this section, we prove Baxter's inequality for long memory stationary processes. The fractional ARIMA( $p, d, q$ ) processes with  $0 < d < 1/2$  are among them.

**Theorem 4.5.** *We assume (A2). Then there exists a positive constant  $M$  such that (1.3) holds for all  $n \in \mathbf{N}$ .*

*Proof.* Let  $r > 1$  be chosen so that  $0 < r \sin(\pi d) < 1$ . Then we have (3.10). By Proposition 3.2 and (2.19), we may take a positive integer  $N$  such that both (3.4) and  $a_n > 0$  hold for  $n \geq N$ . Pick  $\delta \in (0, d)$ . By (2.19) and Theorem 1.5.6 (iii) in [4] (Potter-type bounds), we may assume that

$$(4.2) \quad a_m/a_n \leq 2 \max\{(n/m)^{1+d-\delta}, (m/n)^{1+d+\delta}\}, \quad m, n \geq N.$$

As in the proof of Theorem 4.3, we have, for  $n \geq N + 3$ ,

$$\sum_{j=1}^{n-1} |\phi_{n-1,j} - \phi_j| \leq c_0 \sum_{k=1}^{\infty} \sum_{u=0}^{\infty} d_k(n, u) \sum_{j=1}^{n-1} |a_{u+j}| = c_0 \{G_1(n) + G_2(n)\},$$

where

$$G_1(n) := \sum_{k=1}^{\infty} \sum_{u=0}^{\infty} d_k(n, u) \sum_{j=1}^{N+1} |a_{u+j}|,$$

$$G_2(n) := \sum_{k=1}^{\infty} \sum_{u=0}^{\infty} d_k(n, u) \sum_{j=N+2}^{n-1} a_{u+j}.$$

For  $n \geq N + 3$ , we have

$$G_1(n) \leq n^{-1} \left[ \sum_{k=1}^{\infty} f_k(0) \{r \sin(\pi d)\}^k \right] \sum_{j=1}^{N+1} \sum_{u=0}^{\infty} |a_{u+j}|,$$

and

$$G_2(n) \leq \sum_{k=1}^{\infty} \int_0^{\infty} du \cdot d_k(n, [u]) \int_N^n a_{[u]+[s]+2} ds$$

$$= na_n \sum_{k=1}^{\infty} \int_0^{\infty} du \cdot nd_k(n, [nu]) \int_{N/n}^1 \frac{a_{[nu]+[ns]+2}}{a_n} ds.$$

By (4.2), we have, for  $u > 0$ ,  $n \geq N + 3$ , and  $N/n \leq s \leq 1$ ,

$$\frac{a_{[nu]+[ns]+2}}{a_n} \leq 2 \max \left\{ \left( \frac{n}{[nu] + [ns] + 2} \right)^{1+\delta+d}, \left( \frac{n}{[nu] + [ns] + 2} \right)^{1-\delta+d} \right\}$$

$$\leq 2 \max\{(u+s)^{-(1+d+\delta)}, (u+s)^{-(1+d-\delta)}\}.$$

Hence, by (3.4),

$$\begin{aligned} G_2(n) &\leq 2 \int_0^1 ds \int_0^\infty \left\{ (u+s)^{-(1+d+\delta)} + (u+s)^{-(1+d-\delta)} \right\} du \\ &\quad \times na_n \left[ \sum_{k=1}^\infty f_k(0) \{r \sin(\pi d)\}^k \right] \\ &\leq 2 \left\{ \frac{1}{(\delta+d)(1-d-\delta)} + \frac{1}{(d-\delta)(1-d+\delta)} \right\} \\ &\quad \times na_n \left[ \sum_{k=1}^\infty f_k(0) \{r \sin(\pi d)\}^k \right]. \end{aligned}$$

Combining these estimates, we obtain

$$\limsup_{n \rightarrow \infty} \left\{ n^d \ell(n) \sum_{j=1}^{n-1} |\phi_{n-1,j} - \phi_j| \right\} < \infty.$$

Since  $\sum_{k=n}^\infty \phi_k = c_0 \sum_{k=n}^\infty a_k \sim c_0 \sin(\pi d) / \{\pi n^d \ell(n)\}$  as  $n \rightarrow \infty$ , the theorem follows.  $\square$

## 5. PARTIAL AUTOCORRELATION FUNCTIONS

**5.1. Representation in terms of MA and AR coefficients.** In this section, we assume that the stationary process  $\{X_n\}$  satisfies either (A1) or (A2) in Section 2.3. The aim of this section is to prove an explicit representation of the PACF  $\alpha(\cdot)$  in terms of the MA and AR coefficients. The formula is much simpler than the earlier one in [16]. The proof is based on Theorem 2.7.

Recall  $\beta_n$  from (2.22). For  $n \in \mathbf{N} \cup \{0\}$ , we define  $\alpha_1(n) := \beta_n$  and, for  $k = 3, 5, \dots$ ,

$$\alpha_k(n) := \sum_{v_1=0}^\infty \cdots \sum_{v_{k-1}=0}^\infty \beta_{n+v_1} \beta_{n+1+v_1+v_2} \cdots \beta_{n+1+v_{k-2}+v_{k-1}} \beta_{n+1+v_{k-1}}.$$

As in the case of  $d_k(n, j)$  in Section 2.3, the sums converge absolutely. We have

$$(5.1) \quad \alpha_{2k+1}(n) = \sum_{v=0}^\infty \beta_{n+v} d_{2k}(n+1, v), \quad n, k \in \mathbf{N} \cup \{0\}.$$

The following proposition is the key to the proof of the main result in this section (i.e., Theorem 5.4 below).

**Proposition 5.1.** *For  $n, j \in \mathbf{N} \cup \{0\}$  and  $k \in \mathbf{N}$ , we have*

$$(5.2) \quad d_{2k}(n, j) - d_{2k}(n+1, j) = \sum_{l=1}^k \alpha_{2k-2l+1}(n) d_{2l-1}(n, j).$$

*Proof.* Let  $n, j \in \mathbf{N} \cup \{0\}$ . We use mathematical induction on  $k$ .

Since  $\alpha_1(n) = \beta_n$  and  $d_1(n, j) = \beta_{n+j}$ , we have

$$d_2(n, j) = \sum_{v=n}^\infty \beta_v \beta_{v+j} = \sum_{v=n}^\infty \alpha_1(v) d_1(v, j),$$

which implies (5.2) with  $k = 1$ .

We assume that (5.2) holds for  $k \in \mathbf{N}$ . By (2.26) and the Fubini–Tonelli theorem, we have

$$d_{2k+2}(n, j) = \sum_{v_2=0}^{\infty} \left[ \sum_{v_1=n}^{\infty} \beta_{v_2+v_1} \beta_{j+v_1} \right] d_{2k}(n, v_2),$$

whence  $d_{2k+2}(n, j) - d_{2k+2}(n+1, j) = \text{I} + \text{II}$ , where

$$\begin{aligned} \text{I} &= \sum_{v_2=0}^{\infty} \left[ \sum_{v_1=0}^{\infty} \beta_{n+v_2+v_1} \beta_{n+j+v_1} \right] [d_{2k}(n, v_2) - d_{2k}(n+1, v_2)], \\ \text{II} &= \sum_{v_2=0}^{\infty} \beta_{n+v_2} \beta_{n+j} d_{2k}(n+1, v_2). \end{aligned}$$

Then, by (2.26), (5.2), and the Fubini–Tonelli theorem,

$$\begin{aligned} \text{I} &= \sum_{v_2=0}^{\infty} \left[ \sum_{v_1=0}^{\infty} \beta_{n+v_2+v_1} \beta_{n+j+v_1} \right] \sum_{l=1}^k \alpha_{2k-2l+1}(n) d_{2l-1}(n, v_2) \\ &= \sum_{l=1}^k \alpha_{2k-2l+1}(n) \sum_{v_1=0}^{\infty} \beta_{n+j+v_1} \sum_{v_2=0}^{\infty} \beta_{n+v_1+v_2} d_{2l-1}(n, v_2) \\ &= \sum_{l=1}^k \alpha_{2k-2l+1}(n) d_{2l+1}(n, j) = \sum_{l=2}^{k+1} \alpha_{2(k+1)-2l+1}(n) d_{2l-1}(n, j), \end{aligned}$$

while, by (5.1), we have

$$\text{II} = \left[ \sum_{v_2=0}^{\infty} \beta_{n+v_2} d_{2k}(n+1, v_2) \right] \beta_{n+j} = \alpha_{2k+1}(n) d_1(n, j).$$

Thus we obtain (5.2) with  $k$  replaced by  $k+1$ , as desired.  $\square$

We derive two kinds of difference equations for  $b_k(n, j)$  from Proposition 5.1.

**Proposition 5.2.** *For  $n, k \in \mathbf{N}$  and  $j \in \mathbf{N} \cup \{0\}$ , we have*

$$(5.3) \quad b_{2k+1}(n, j) - b_{2k+1}(n+1, j) = \sum_{l=1}^k \alpha_{2k-2l+1}(n+1) b_{2l}(n, j).$$

*Proof.* From Theorem 2.7 and Proposition 5.1, we see that

$$\begin{aligned} &b_{2k+1}(n, j) - b_{2k+1}(n+1, j) \\ &= c_0 \sum_{u=0}^{\infty} a_{j+u} \{d_{2k}(n+1, u) - d_{2k}(n+2, u)\} \\ &= c_0 \sum_{u=0}^{\infty} a_{j+u} \sum_{l=1}^k \alpha_{2k-2l+1}(n+1) d_{2l-1}(n+1, j) \\ &= \sum_{l=1}^k \alpha_{2k-2l+1}(n+1) c_0 \sum_{u=0}^{\infty} a_{j+u} d_{2l-1}(n+1, j) \\ &= \sum_{l=1}^k \alpha_{2k-2l+1}(n+1) b_{2l}(n, j). \end{aligned}$$

Thus the proposition follows.  $\square$

**Proposition 5.3.** *For  $n, k \in \mathbf{N}$ , and  $j = 1, \dots, n$ , we have*

$$(5.4) \quad \begin{aligned} &b_{2k}(n, n+1-j) - b_{2k}(n+1, n+2-j) \\ &= \sum_{l=1}^k \alpha_{2k-2l+1}(n+1) b_{2l-1}(n, n+1-j). \end{aligned}$$

*Proof.* Since  $d_1(n+1, u) = \alpha_1(n+1+u)$  and  $b_1(n, j) = c_0 a_j$ , we see from Theorem 2.7 that

$$b_2(n, n+1-j) = c_0 \sum_{u=0}^{\infty} a_{n+1-j+u} d_1(n+1, u) = \sum_{u=n+1}^{\infty} \alpha_1(u) b_1(n, u-j).$$

Similarly, we have

$$b_2(n+1, n+2-j) = c_0 \sum_{u=0}^{\infty} a_{n+2-j+u} d_1(n+2, u) = \sum_{u=n+2}^{\infty} \alpha_1(u) b_1(n, u-j).$$

Thus (5.4) holds for  $k = 1$ .

Suppose that  $k \geq 2$ . Then, from (2.26) and Theorem 2.7,

$$\begin{aligned} b_{2k}(n, n+1-j) &= c_0 \sum_{v=0}^{\infty} a_{n+1-j+v} \sum_{u=0}^{\infty} \beta_{n+1+v+u} d_{2(k-1)}(n+1, u), \\ b_{2k}(n+1, n+2-j) &= c_0 \sum_{v=1}^{\infty} a_{n+1-j+v} \sum_{u=0}^{\infty} \beta_{n+1+v+u} d_{2(k-1)}(n+2, u). \end{aligned}$$

Hence  $b_{2k}(n, n+1-j) - b_{2k}(n+1, n+2-j) = \text{I} + \text{II}$  with

$$\begin{aligned} \text{I} &= c_0 \sum_{v=0}^{\infty} a_{n+1-j+v} \sum_{u=0}^{\infty} \beta_{n+1+v+u} \{d_{2(k-1)}(n+1, u) - d_{2(k-1)}(n+2, u)\}, \\ \text{II} &= c_0 a_{n+1-j} \sum_{u=0}^{\infty} \beta_{n+1+u} d_{2(k-1)}(n+2, u). \end{aligned}$$

By (2.26) and Theorem 2.7, I is equal to

$$\begin{aligned} &c_0 \sum_{v=0}^{\infty} a_{n+1-j+v} \sum_{u=0}^{\infty} \beta_{n+1+v+u} \sum_{l=1}^{k-1} \alpha_{2(k-1)-2l+1}(n+1) d_{2l-1}(n+1, v) \\ &= \sum_{l=1}^{k-1} \alpha_{2(k-1)-2l+1}(n+1) c_0 \sum_{v=0}^{\infty} a_{n+1-j+v} \sum_{u=0}^{\infty} \beta_{n+1+v+u} d_{2l-1}(n+1, u) \\ &= \sum_{l=1}^{k-1} \alpha_{2(k-1)-2l+1}(n+1) b_{2l+1}(n, n+1-j) \\ &= \sum_{l=2}^k \alpha_{2k-2l+1}(n) b_{2l-1}(n, n+1-j), \end{aligned}$$

while, by (5.1) and the equality  $b_1(n, n+1-j) = c a_{n+1-j}$ , we have

$$\text{II} = \alpha_{2k-1}(n+1) b_1(n, n+1-j).$$

Thus we obtain (5.4), as desired.  $\square$

Here is the representation of the PACF  $\alpha(\cdot)$  in terms of the MA and AR coefficients.

**Theorem 5.4.** *We assume either (A1) or (A2). Then  $\alpha(n) = \sum_{k=1}^{\infty-} \alpha_{2k-1}(n)$  for  $n = 2, 3, \dots$*

*Proof.* We have  $b_1(n, j) - b_1(n+1, j) = c_0 a_j - c_0 a_j = 0$  for  $n \in \mathbf{N}$  and  $j = 1, \dots, n$ . This, together with Theorem 2.7 and Propositions 5.2 and 5.3, yields

$$\begin{aligned}
\phi_{n,j} - \phi_{n+1,j} &= \sum_{k=1}^{\infty-} \{g_k(n, j) - g_k(n+1, j)\} \\
&= \sum_{k=1}^{\infty-} \{b_{2k+1}(n, j) - b_{2k+1}(n+1, j) \\
&\quad + b_{2k}(n, n+1-j) - b_{2k}(n+1, n+2-j)\} \\
&= \sum_{k=1}^{\infty-} \sum_{l=1}^k \alpha_{2k-2l+1}(n+1) \{b_{2l}(n, j) + b_{2l-1}(n, n+1-j)\} \\
&= \sum_{l=1}^{\infty-} \{b_{2l}(n, j) + b_{2l-1}(n, n+1-j)\} \sum_{k=l}^{\infty-} \alpha_{2k-2l+1}(n+1) \\
&= \left\{ \sum_{k=1}^{\infty-} \alpha_{2k-1}(n+1) \right\} \phi_{n, n+1-j}.
\end{aligned}$$

Combining this and (1.4), we obtain the theorem.  $\square$

**5.2. Asymptotics for short memory processes.** In this section, we apply Theorem 5.4 to the partial autocorrelation functions of a class of short memory processes.

**Theorem 5.5.** *Let  $p > 1$  and let  $\{X_n\}$  be a purely nondeterministic stationary process. We assume (2.17) and*

$$(5.5) \quad a_n = O(n^{-p}) \quad (n \rightarrow \infty).$$

*Then the partial autocorrelation function  $\alpha(\cdot)$  satisfies*

$$(5.6) \quad \alpha(n) = O(n^{-p}) \quad (n \rightarrow \infty).$$

Clearly, (5.5) implies (2.14), so that  $\{X_n\}$  in Theorem 5.5 satisfies (A1).

*Proof.* By (5.5), there exists  $L_1 \in (0, \infty)$  such that

$$|a_n| \leq \frac{L_1}{n^p} \quad (n = 1, 2, \dots).$$

Recall  $\beta_n$  from (2.22) and  $\alpha_{2k+1}(\cdot)$  from the previous section. It holds that

$$|\alpha_1(n)| = |\beta_n| \leq \sum_{v=0}^{\infty} |c_v| \frac{L_1}{(n+v)^p} \leq \frac{L_2}{n^p} \quad (n = 1, 2, \dots)$$

with  $L_2 := L_1 \sum_{k=0}^{\infty} |c_k|$ . Let  $A(\cdot)$  and  $F(\cdot)$  be as in (4.1). By (5.1) and Lemma 4.2, we have, for  $n, k \in \mathbf{N}$ ,

$$|\alpha_{2k+1}(n)| \leq \sum_{v=0}^{\infty} |\beta_{n+v} d_{2k}(n+1, v)| \leq \frac{L_2}{n^p} F(n+1)^{2k}.$$

Choose  $N \in \mathbf{N}$  so that  $F(N+1) < 1$ . Then, combining the estimates above with Theorem 5.4, we see that, for  $n \geq N$ ,

$$|\alpha(n)| \leq \frac{L_2}{n^p} \sum_{k=0}^{\infty} F(n+1)^{2k} = \frac{L_2}{1 - F(n+1)^2} n^{-p} \leq \frac{L_2}{1 - F(N+1)^2} n^{-p}.$$

Thus (5.6) follows.  $\square$

**5.3. Partial autocorrelation functions of FARIMA.** We assume that  $\{X_n\}$  is a fractional ARIMA( $p, d, q$ ) process with

$$p, q \in \mathbf{N} \cup \{0\}, \quad 0 < d < 1/2$$

(see Example 2.4 for the definition). Let  $\alpha(\cdot)$  be the partial autocorrelation function of  $\{X_n\}$ . The aim of this section is to prove the following asymptotic behavior with remainder for  $\alpha(\cdot)$ .

**Theorem 5.6.** *We have  $n\alpha(n) = d + O(n^{-d})$  as  $n \rightarrow \infty$ .*

The rest of this section is devoted to the proof of this theorem.

As before, we denote by  $\{c_n\}$  and  $\{a_n\}$  the MA and AR coefficients, respectively, of  $\{X_n\}$ . We also consider a fractional ARIMA(0,  $d$ , 0) process  $\{X'_n\}$  satisfying  $E[(X'_n)^2] = \Gamma(1-2d)/\Gamma^2(1-d)$ . The AR coefficients  $\{a'_n\}$  and MA coefficients  $\{c'_n\}$  of  $\{X'_n\}$  are given by

$$a'_n = \frac{\Gamma(n-d)d}{\Gamma(n+1)\Gamma(1-d)}, \quad c'_n = \frac{\Gamma(n+d)}{\Gamma(n+1)\Gamma(d)}, \quad n \geq 0$$

(see, e.g., [5], Section 13.2). Notice that  $c'_n > 0$  for  $n \geq 0$  and  $a'_n > 0$  for  $n \geq 1$ . Put

$$\beta'_n := \sum_{v=0}^{\infty} c'_v a'_{n+v}, \quad n \geq 0.$$

**Lemma 5.7.** *We have  $\beta'_n = \sin(\pi d)/\{\pi(n-d)\}$  for  $n \geq 0$ .*

*Proof.* Using the hypergeometric function, we have, for  $n \geq 0$ ,

$$\begin{aligned} \beta'_n &= \frac{\Gamma(n-d)d}{\Gamma(n+1)\Gamma(1-d)} \sum_{v=0}^{\infty} \frac{\Gamma(d+v)}{\Gamma(d)} \cdot \frac{\Gamma(n-d+v)}{\Gamma(n-d)} \cdot \frac{\Gamma(n+1)}{\Gamma(n+1+v)} \cdot \frac{1}{v!} \\ &= \frac{\Gamma(n-d)d}{\Gamma(n+1)\Gamma(1-d)} F(d, n-d; n+1; 1) \\ &= \frac{\Gamma(n-d)d}{\Gamma(n+1)\Gamma(1-d)} \cdot \frac{\Gamma(n+1)}{\Gamma(n+1-d)\Gamma(1+d)} = \frac{\sin(\pi d)}{\pi} \cdot \frac{1}{n-d}, \end{aligned}$$

as desired.  $\square$

**Proposition 5.8.** *There exist a real sequence  $\{\delta_n\}_{n=1}^{\infty}$  and a positive constant  $M$  such that  $\beta_n = \beta'_n\{1 + \delta_n\}$  and  $|\delta_n| \leq Mn^{-d}$  for  $n \geq 1$ .*

*Proof.* By Lemma 2.2 in [15], we have, as  $n \rightarrow \infty$ ,

$$\begin{aligned} \frac{c_n}{n^{d-1}} &= \frac{K_1}{\Gamma(d)} + O(n^{-1}), & \frac{a_n}{n^{-d-1}} &= -\frac{1}{K_1\Gamma(-d)} + O(n^{-1}), \\ \frac{c'_n}{n^{d-1}} &= \frac{1}{\Gamma(d)} + O(n^{-1}), & \frac{a'_n}{n^{d-1}} &= -\frac{1}{\Gamma(-d)} + O(n^{-1}), \end{aligned}$$

where  $K_1 := \theta(1)/\phi(1) > 0$ . Hence we may write

$$c_n = \{K_1 + s_n\}c'_n, \quad n \geq 0, \quad a_n = \{(1/K_1) + t_n\}a'_n, \quad n \geq 1,$$

where  $\{s_n\}$  and  $\{t_n\}$  are sequences satisfying, for some  $L \in (0, \infty)$ ,

$$|s_n| \leq L/(n+1), \quad n \geq 0, \quad |t_n| \leq L/n, \quad n \geq 1.$$

We have, for  $n = 1, 2, \dots$ ,

$$|\beta_n - \beta'_n| \leq \sum_{v=0}^{\infty} |s_v|c'_v a'_{n+v} + \sum_{v=0}^{\infty} |t_{n+v}|c'_v a'_{n+v} + \sum_{v=0}^{\infty} |s_v t_{n+v}|c'_v a'_{n+v}.$$

From  $c'_n/(n+1) \sim 1/\{n^{2-d}\Gamma(d)\}$  as  $n \rightarrow \infty$ , we see that

$$\sum_{v=0}^{\infty} |s_v|c'_v a'_{n+v} \leq L \sum_{v=0}^{\infty} \frac{c'_v}{v+1} a'_{n+v} \sim a'_n L \sum_{v=0}^{\infty} \frac{c'_v}{v+1}, \quad n \rightarrow \infty.$$

Hence, using  $a'_n \sim \text{constant} \cdot n^{-(1+d)}$  as  $n \rightarrow \infty$ , we get

$$\sum_{v=0}^{\infty} |s_v|c'_v a'_{n+v} = O(n^{-(1+d)}), \quad n \rightarrow \infty.$$

Similarly, as  $n \rightarrow \infty$ ,

$$\sum_{v=0}^{\infty} |t_{n+v}|c'_v a'_{n+v} = O(n^{-(2+d)}), \quad \sum_{v=0}^{\infty} |s_v t_{n+v}|c'_v a'_{n+v} = O(n^{-(2+d)}).$$

Combining these and  $\beta'_n \sim \pi^{-1} \sin(\pi d) n^{-1}$ , we obtain the proposition.  $\square$

*Proof of Theorem 5.6.* By Theorem 5.4, the partial autocorrelation functions  $\alpha(\cdot)$  and  $\alpha'(\cdot)$  of  $\{X_n\}$  and  $\{X'_n\}$ , respectively, admit the representations

$$\alpha(n) = \sum_{k=1}^{\infty-} \alpha_{2k-1}(n), \quad \alpha'(n) = \sum_{k=1}^{\infty-} \alpha'_{2k-1}(n),$$

where  $\alpha_k(\cdot)$  are the sequences defined in the previous section for  $\{X_n\}$ , while  $\alpha'_k(\cdot)$  are their counterparts defined for  $\{X'_n\}$ , that is,  $\alpha'_1(n) = \beta'_n$  and, for  $k \geq 3$ ,

$$\alpha'_k(n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_{k-1}=0}^{\infty} \beta'_{n+1+m_1} \beta'_{n+1+m_1+m_2} \cdots \beta'_{n+1+m_{k-2}+m_{k-1}} \beta'_{n+m_{k-1}}.$$

Let  $r > 1$  be chosen so that  $r^2 \sin(\pi d) < 1$  and let  $M$  be as in Proposition 5.8. As in Proposition 3.4 (i), there exists an integer  $N$  independent of  $k$  such that, for  $n \geq N$  and  $k \geq 1$ ,

$$1 + (M/n^d) \leq r, \quad \alpha'_k(n) \leq \frac{1}{n} \{r \sin(\pi d)\}^k f_k(0).$$

By Proposition 5.8, we have, for  $n \geq 1$  and  $v \geq 0$ ,

$$|\beta_{n+v}| \leq (1 + Mn^{-d}) \beta'_{n+v}, \quad |\beta_{n+v} - \beta'_{n+v}| \leq Mn^{-d} \beta'_{n+v}.$$

We also have  $(1+x)^k - 1 \leq kx(1+x)^k$  for  $x \geq 0$ . Hence, for  $n \geq N$ ,

$$\begin{aligned} & |\alpha_3(n) - \alpha'_3(n)| \\ & \leq \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} |\beta_{n+1+m_1} - \beta'_{n+1+m_1}| \cdot |\beta_{n+1+m_1+m_2}| \cdot |\beta_{n+m_2}| \\ & \quad + \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \beta'_{n+1+m_1} \cdot |\beta_{n+1+m_1+m_2} - \beta'_{n+1+m_1+m_2}| \cdot |\beta_{n+m_2}| \\ & \quad + \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \beta'_{n+1+m_1} \cdot \beta'_{n+1+m_1+m_2} \cdot |\beta_{n+m_2} - \beta'_{n+m_2}| \\ & \leq Mn^{-d} \{(1 + Mn^{-d})^2 + (1 + Mn^{-d}) + 1\} \alpha'_3(n) \\ & = \{(1 + Mn^{-d})^3 - 1\} \alpha'_3(n) \leq 3Mn^{-d}(1 + Mn^{-d})^3 \alpha'_3(n) \\ & \leq 3Mn^{-(d+1)} \{r^2 \sin(\pi d)\}^3 f_3(0). \end{aligned}$$

In the same way,

$$|\alpha_k(n) - \alpha'_k(n)| \leq Mkn^{-(d+1)} \{r^2 \sin(\pi d)\}^k f_k(0), \quad k \in \mathbf{N}, \quad n \geq N.$$

Since  $\alpha'(n) = d/(n-d)$  (see Theorem 1 in [12]), it follows that

$$\begin{aligned} \left| \alpha(n) - \frac{d}{n-d} \right| & \leq \sum_{k=1}^{\infty} |\alpha_{2k-1}(n) - \alpha'_{2k-1}(n)| \\ & \leq n^{-(d+1)} M \sum_{k=1}^{\infty} (2k-1) f_{2k-1}(0) \{r^2 \sin(\pi d)\}^{2k-1}. \end{aligned}$$

By Lemma 3.1, we have  $\sum_{k=1}^{\infty} (2k-1) f_{2k-1}(0) \{r^2 \sin(\pi d)\}^{2k-1} < \infty$ , so that

$$\alpha(n) = \frac{d}{n-d} + O\left(n^{-(d+1)}\right) = \frac{d}{n} + O\left(n^{-(d+1)}\right), \quad n \rightarrow \infty.$$

Thus the theorem follows.  $\square$

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