SELFSIMILAR EXPANDING SOLUTIONS IN A SECTOR FOR A CRYSTALLINE FLOW*

MI-HO GIGA[1] , YOSHIKAZU GIGA[2] AND HIDEKATA HONTANI[3]

Abstract. For a given sector a selfsimilar expanding solution to a crystalline flow is constructed. The solution is shown to be unique. Because of selfsimilarity the problem is reduced to solve a system of algebraic equations of degree two. The solution is constructed by a method of continuity and obtained by solving associated ordinary differential equations. The selfsimilar expanding solution is useful to construct a crystalline flow from an arbitrary polygon not necessarily admissible.

key words. crystalline flow, selfsimilar expanding solutions, a priori estimate

AMS subject classification. 15A99, 34A12, 74N05, 94A08

1 Introduction.

A curvature flow is important to describe motion of phase boundaries in material sciences [12], [19] and also to modify contours in image analysis [15]. A crystalline flow is considered as a discrete version of an anisotropic curvature flow in the plane. It was introduced by J. Taylor [18] and independently by S. Angenent and M. Gurtin [1] almost a decade ago. Let us give a typical example of anisotropic curvature flow equations in $\mathbf{R}^2$:

$$V = -\mathrm{div}\xi(\vec{n}) \quad \text{on } \Gamma_t \ ; \tag{1.1}$$

here V denotes the normal velocity of an evolving curve $\{\Gamma_t\}$ in the direction of unit normal $\vec{n}$ and $\xi = \nabla\gamma$ with the interfacial energy density $\gamma(p)$ which is positively one-homogeneous and convex in $\mathbf{R}^2$; div denotes the surface divergence. If $\gamma(p) = |p|$, then (1.1) is nothing but the curve shortening equation. If γ is piecewise linear, then (1.1) is so singular that it cannot be interpreted as a conventional partial differential equation. This is a typical situation where a crystalline flow arises. To understand (1.1) one restricts $\{\Gamma_t\}$ in a special class of polygonal curves, which is often called 'admissible' [18], [1]. The boundary of the Wulff shape

$$W_\gamma = \{x \in \mathbf{R}^2;\ x \cdot m \leq \gamma(m) \quad \text{for all} \quad m \in \mathbf{R}^2\}.$$

is a typical example of an admissible polygon (crystal). Its weighted curvature $-\mathrm{div}\xi(\vec{n})$ formally equals -1 if $\vec{n}$ is taken outward. Thus if W_γ is a regular polygon centered at the origin, then it is reasonable to say that $\Gamma_t = t^{1/2}\partial W_\gamma$ is a 'solution' of (1.1). We say that a polygon Γ is an *admissible crystal* if the orientation of each edge (facet) is one of that in ∂W_γ and the orientations of adjacent facets should be adjacent in ∂W_γ. We say that $\{\Gamma_t\}$ is an *admissible evolving crystal* if Γ_t is an admissible crystal and the motion of all vertices of Γ_t is C^1 in time t. Let $S_j(t)$ denote j-th facet of Γ_t. Then (1.1) is formally of the form:

$$V_j(t) = \Lambda_j(t) \quad \text{on } S_j(t) \tag{1.2}$$

*This work was partly supported by Grant-in-Aid for formation of COE 'Mathematics of Nonlinear Structures via Singularities'

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The work of this author was supported partly by the Grant-in-Aid for Scientific Research, No.1420411, No.1563408, the Japan Society for the Promotion of Science (JSPS)

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with *crystalline curvature*

$$\Lambda_j(t) = \chi_j \Delta(\vec{n}_j)/L_j(t)$$

and $V_j(t)$ denotes the normal velocity of $S_j(t)$ in the direction of $\vec{n}_j$; here $L_j(t)$ denotes the length of $S_j(t)$ and χ_j is the transition number (see §2). The quantity $\Delta(\vec{n}_j)$ is the length of the facet of ∂W_γ whose orientation equals $\vec{n}_j$. Together with a transport equation (the first displayed formula in §3) we have a finite system (3.1) of ordinary differential equations (ODEs), which is at least solvable locally-in-time. The resulting flow $\{\Gamma_t\}$ is often called a crystalline flow.

Our goal in this paper is to construct a selfsimilar expanding admissible evolving crystal $\{\Gamma_t\}_{t>0}$ satisfying (1.2) (shortly, selfsimilar expanding solution) such that it tends to the boundary of a sector as time tends to zero. We also prove its uniqueness. (Note that a selfsimilar expanding solution is not necessarily a dilation of the Wulff shape as numerical calculations in [13] show.) There are several reasons such a solution is important. Here is a partial list:

(a) Useful to construct a solution when initial data is a nonadmissible polygon;

(b) Useful to construct a selfsimilar expanding solution to (1.1) for general γ when Γ_t lives in a sector and touches the boundary of the sector with prescribed contact angle.

Before discussing these reasons we mention relation between crystalline flow and curvature flow with smooth strictly convex interfacial energy density. One comes up with two natural questions.

(i) Is a crystalline flow approximated by a curvature flow with smooth convex γ?

(ii) Is a flow with smooth γ approximated by a crystalline flow with a piecewise linear γ? If so, crystalline flow provides a good numerical algorithm to calculate (1.1).

Fortunately, these problems are affirmatively settled by now; [4], [6], [8] for (i) and [4], [11], [10], [6], [8] for (ii). In [4], [6], [8] general notions of a 'solution' to (1.1) based on variational principle or comparison principle are given. Moreover, it is shown that the 'solution' in this sense exists for a general initial simple curve not necessarily admissible. Starting from a general polygon is important for a crystalline algorithm (ii). Practically, for a given curve it is easier to approximate it by a polygon rather than by an admissible crystal. However, it is not clear what is a crystalline flow starting from a general polygon although existence of a 'solution' is known in an abstract level [4], [5] [8]. It is also numerically calculated by [3] when Γ_t is a graph by a variational inequality. Suppose that the initial polygon is non-admissible and that crystalline curvature of a pair of adjacent facets S_A and S_B equals zero. Assume that between orientations of S_A and S_B there are orientations of ∂W_γ. Then one expects that some new facets (with their missing orientations) are created instantaneously. If one has a selfsimilar expanding solution (with respect to a point where S_A intersects S_B), then this solution provides newly created facets. (In fact, it is a unique solution in the level set sense [8]. Even if the crystalline curvature of at least one of S_A and S_B is not zero, a selfsimilar expanding solution represents a leading term of the length of newly created facets. We do not touch these two problems in this paper.) A selfsimilar expanding solution gives a definite way to create new facets (at least when S_A and S_B do not move), so it is useful to implement numerical algorithm starting with a non-admissible polygon. We gave such an applications in [13] for multi-scale analysis of contour shape to extract its structure.

When the Frank diagram of γ (1-level set of γ) is strictly convex and smooth so that W_γ is smooth and strictly convex, we know that there is a (unique) selfsimilar expanding solution to

(1.1) touching the boundary of a given sector with prescribed angle [14], [2]. Our selfsimilar expanding solution to (1.2) together with approximation theory [6], [8] provides a selfsimilar expanding solution (under prescribed contact angle condition) for more general γ whose Frank diagram is not necessarily strictly convex. As in [14], [2] our selfsimilar solutions represents the large time behaviour of general solutions. We shall discuss this topic in a forthcoming paper. Note that our selfsimilar expanding solution is different from a selfsimilar shrinking solution studied, for example, in [16], [17].

Let us briefly mention the method of the proof. The uniqueness is proved by a geometric method (§2) – comparison principle [18], [9]. To construct a selfsimilar expanding solution it suffices to find a solution of ODEs with $L_j(t) = t^{1/2}/a_j$ with $a_j > 0$. The problem is reduced to find a positive a_j's solving a system of algebraic equations of degree two for a_j's. If the number of missing orientations is small, say one or two, then one can solve it directly. However, in general it is terrible. The equation for a_j's is of the form

$$\begin{pmatrix} 1/a_1 \\ \vdots \\ 1/a_n \end{pmatrix} = H \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

and $H = (H_{ij})$ is a tridiagonal matrix; see (3.2). We solve this equation analytically by introducing extra parameter s so that for our algebraic system is decoupled for $s = 0$ to get $a_j = 1/\sqrt{H_{jj}}$ and for $s = 1$ it agrees with (3.2). We differentiate it with respect to s to get ODEs. We solve the ODEs from $s = 0$ to $s = 1$ by establishing a priori estimates. A crucial step is to calculate the determinant of H, in particular to prove its positiveness. An explicit and beautiful formula of $\det H$ is given in Lemma 4.2 up to an explicit constant. It is represented by angles of Wulff shapes. Note that our method provides not only the existence but also a way to calculate a numerical value of a_j's by solving the ODEs numerically.

This paper is organized as follows. In section 2 we state our main results and give a proof of uniqueness. In section 3 we derive the ODEs to solve the algebraic equation (3.1). We prove the existence of a solution admitting several estimates for matrices established in section 4, which is main technical part of this paper.

2 Main theorems.

To state our main results we formulate our problem. Let ∂C be the boundary of a given oriented cone C in $\mathbf{R}^2$ of the form $\partial C = \ell_A \cup \ell_B$, where ℓ_A and ℓ_B are maximal half lines starting from the origin O and are indexed clockwisely as Figure 2.1. In this paper, we assume that $\vec{n}_j = (\cos\theta_j, \sin\theta_j)$ be the outer unit normal of ℓ_j for $j = A, B$ with $|\theta_A - \theta_B| < \pi$. Let n be a nonnegative integer. Let $\Theta = \{\theta_j;\ j = 1, \cdots, n\}$ with $\theta_A > \theta_1 > \theta_2 > \cdots > \theta_n > \theta_B$ (resp. $\theta_A < \theta_1 < \theta_2 < \cdots < \theta_n < \theta_B$) if $\theta_A > \theta_B$ (resp. $\theta_A < \theta_B$). We call Θ as a *set of admissible angles*. We interpret that Θ is an empty set if $n = 0$.

We call a simple oriented polygonal curve S as an *admissible crystal associated with* C if S is of the form $S = \bigcup_{j=1}^{n} S_j \cup S_A \cup S_B$, where S_j is a maximal, nontrivial and closed segment with the outer unit normal $\vec{n}_j = (\cos\theta_j, \sin\theta_j)$ for $j \in \{1, \cdots, n\} \cup \{A, B\}$ and S_j for $j = A, B$ is a half line contained in ℓ_j. We implicitly assume that segments S_j's are numbered clockwisely. Figure 2.1 shows examples of C. Figure 2.2 shows examples of admissible crystals S associated with C.

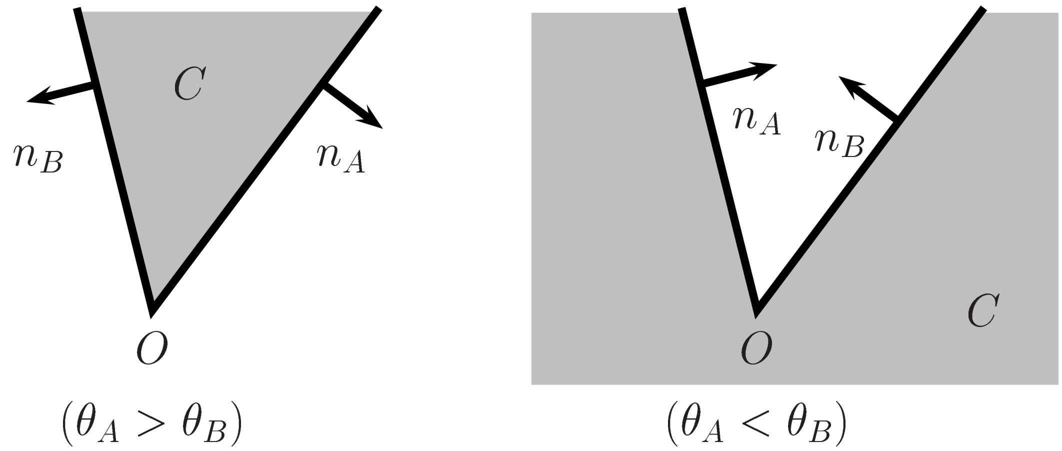

FIG. 2.1: *Oriented cones C*

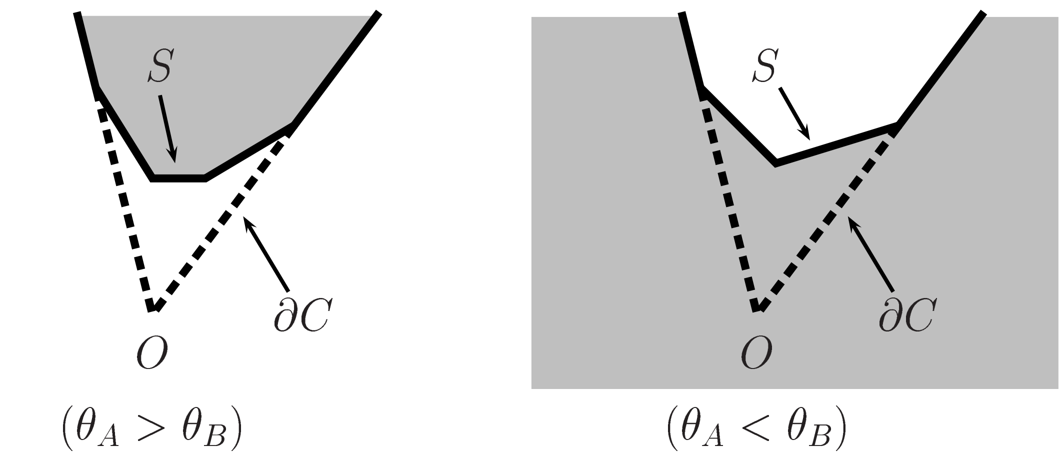

FIG. 2.2: *Admissible crystals S*

We say that a family of polygon $\{S(t)\}_{t\in J}$ belongs to a *set of orientation-preserving evolving curves* $\mathcal{S}$ if $S(t)$ is an admissible crystal for all $t \in J$ and each corner moves continuously differentiably in time, where J is a time interval. These conditions imply that the orientation of each line (facet) is preserved for $t \in J$.

In this paper, we consider an orientation-preserving evolving curve $\{S(t)\}_{t\in J} \in \mathcal{S}$ governed by so-called *crystalline curvature* or *weighted curvature* $\Lambda_j(t) := \dfrac{\chi_j(t)\Delta_j}{L_j(t)}$ for given $\Delta_j > 0$ ($j = 1, \cdots, n$), where $S(t) = \displaystyle\bigcup_{j=1}^{n} S_j(t) \cup S_A(t) \cup S_B(t)$ and $L_j(t)$ is the length of facet $S_j(t)$. The quantity $\chi_j(t)$ is a transition number with

$$\chi_j(t) := \begin{cases} 1, & \text{if } S(t) \text{ is concave around } S_j(t), \\ -1, & \text{if } S(t) \text{ is convex around } S_j(t), \\ 0, & \text{otherwise} \end{cases}$$

(see Figure 2.3). We shall give notion of a selfsimilar expanding solution.

Definition. An orientation-preserving evolving curve $\{S(t)\}_{t>0} \in \mathcal{S}$ is called a *selfsimilar*

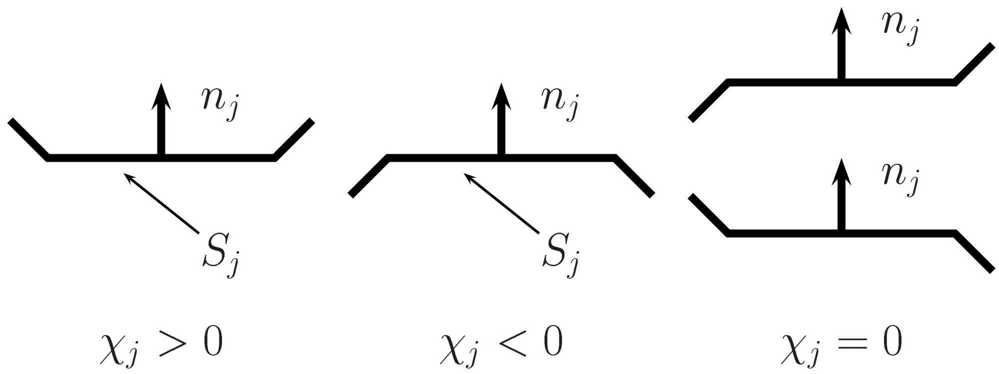

FIG. 2.3: *Transition number* χ_j

expanding solution to (1.2) if there exists an admissible crystal S_* associated with C such that

$$S(t) = t^{1/2} S_* = \{t^{1/2}x; x \in S_*\} \quad \text{for} \quad t > 0\ ; \tag{2.1}$$

$$V_j(t) = \Lambda_j(t) \quad \text{for} \quad t > 0, \quad j = 1, \cdots, n, \tag{2.2}$$

where $S(t)$ is of the form $S(t) = \bigcup_{j=1}^{n} S_j(t) \cup S_A(t) \cup S_B(t)$ and $V_j(t)$ is the normal velocity of facet $S_j(t)$. We note that for a selfsimilar solution $\{S(t)\}_{t>0}$, the transition number is unique independent of j and t, i.e. $\chi_j(t) = -1$ (resp. 1) if $\theta_A > \theta_B$ (resp. $\theta_A < \theta_B$) for all $j = 1, \cdots, n$ and $t > 0$.

Our main results are the existence and its uniqueness of a selfsimilar expanding solution governed by crystalline curvature. It is stated as follows.

THEOREM 2.1 (EXISTENCE). *Let C be a given oriented cone in $\mathbf{R}^2$. Let $\Theta = \{\theta_j;\ j = 1, \cdots, n\ \}$ (with nonnegative integer n) be a set of admissible angles. Let Δ_j be a positive number for $j = 1, \cdots, n$. Then there exists a selfsimilar expanding solution $\{S(t)\}_{t>0}$ such that $S(+0)$ agrees with ∂C.*

THEOREM 2.2 (UNIQUENESS). *Under the same hypotheses of Theorem 2.1 there is at most one selfsimilar expanding solution $\{S(t)\}_{t>0}$ such that $S(+0) = \partial C$.*

We shall prove Theorem 2.1 in §3 based on key a priori estimates shown in §4. In the rest of this section we shall prove Theorem 2.2 by geometric argument.

Proof of Theorem 2.2. Let $\{S(t)\}_{t>0}$ and $\{R(t)\}_{t>0}$ be selfsimilar expanding solutions such that $S(+0)$ and $R(+0)$ agree with the boundary of ∂C. We may assume that $\theta_A > \theta_B$, i.e. the cone C is convex. We may also assume that $n \geq 1$. Then transition numbers of all facets of $S(t)$ and $R(t)$ are -1. Let $S(t)$ (resp. $R(t)$) be of the form $S(t) = \bigcup_{j=1}^{n} S_j(t) \cup S_A(t) \cup S_B(t)$ (resp. $R(t) = \bigcup_{k=1}^{n} R_k(t) \cup R_A(t) \cup R_B(t)$). For convenience we introduce an unbounded region

$D_S(t) \subset \mathbf{R}^2$ enclosed by $S(t)$ for $t > 0$; let $D_S(t)$ denote the closure of the interior region bounded by curve $S(t)$ (see Figure 2.4). Let $\tilde{R}(t)$ be of the form $\tilde{R}(t) = \bigcup_{k=1}^{n} R_k(t)$.

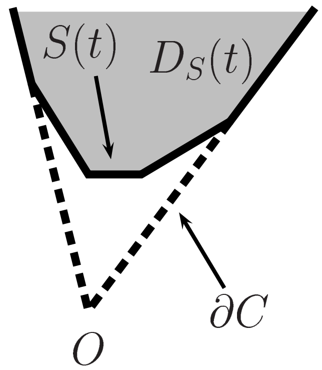

FIG. 2.4: *Region $D_S(t)$ enclosed by $S(t)$*

Suppose that $S \not\equiv R$. We may assume that $\tilde{R}(1) \cap \text{ int } D_S(1) \neq \emptyset$. By this assumption, $t_0 \in (0,1)$ holds for $t_0 := \sup\{t|\tilde{R}(t) \cap \text{ int } D_S(1) = \emptyset\ \}$. Since $\tilde{R}(0)$ is a singleton and $S(1-t_0) \neq \partial C$, there exists $\delta > 0$ such that $\tilde{R}(\delta) \cap \text{ int } D_S(t_1) = \emptyset$ with $t_1 := 1 - t_0$. We fix such δ. Setting $t_2 := \sup\{\tau > 0;\ \tilde{R}(\sigma + \delta) \cap \text{int } D_S(\sigma + t_1) = \emptyset \text{ for } \sigma \in (0,\tau)\}$, we have $0 < t_2 < t_0$. Since $\tilde{R}(\sigma+\delta)$ touches $D_S(\sigma+t_1)$ first time at $\sigma = t_2$, there exists a facet $R_j(t_2+\delta)$ of $\tilde{R}(t_2 + \delta)$ and a facet $S_j(t_2 + t_1)$ of $S(t_2 + t_1)$ such that the normal of $R_j(t_2 + \delta)$ coincides with that of $S_j(t_2 + t_1)$, we conclude that $R_j(t_2 + \delta) \cap S_j(t_2 + t_1) \neq \emptyset$ and that the length of $R_j(t_2 + \delta)$ does not equal the length of $S_j(t_2 + t_1)$. By geometry the length of $R_j(t_2 + \delta)$ is greater than the length of $S_j(t_2 + t_1)$, so that the weighted curvature of$R_j(t_2 + \delta)$ is negative and is greater than that of $S_j(t_2 + t_1)$ (cf. [18], [9]). So the normal velocity of $R_j(t_2 + \delta)$ is negative and is greater than that of $S_j(t_2 + t_1)$, which contradicts $0 < t_2$. □

Remark. (i) The evolution equation (2.2) can be viewed as a crystalline curvature flow equation (1.1) (or (1.2)) with a suitable polygonal Wulff shape. Indeed, if $\theta_A > \theta_B$ for example, then there exists a convex polygon W such that the set $\mathcal{N}$ of the orientations of all facets in W includes all $\vec{n}_j$'s with $j \in \{1, \cdots, n\}$ and the length of facet with $\vec{n}_j$ equals Δ_j and that $\mathcal{N}$ does not include any $\vec{m} = (\cos\theta, \sin\theta)$ for $\theta \neq \theta_j, \theta \in (\theta_B, \theta_A)$. We may assume that W contains the origin as an interior point. The corresponding interfacial energy density γ is given as a support function: $\gamma(x) = \sup\{x \cdot p; p \in W\}$ for $x \in \mathbf{R}^2$. The case $\theta_A < \theta_B$ can be treated in a similar way.

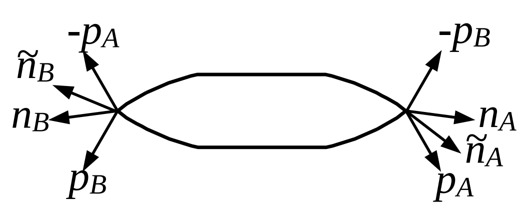

FIG. 2.5: *Wulff shape W*

(ii) If we dilate the Wulff shape so that the length of j-th facet equals $\lambda\Delta_j$, then $\sqrt{\lambda}S(t)$ is the corresponding selfsimilar expanding solution to (1.2) with Δ_j replaced by $\lambda\Delta_j$, where $S(t)$ is defined by (2.1) with (2.2).
(iii) Here is a numerical example of a profile S_* of the selfsimilar expanding solutions for given two different sectors with a fixed Wulff shape having many facets so that it looks a smooth curve. See Figures 2.5, 2.6 and 2.7. We use a Newton type iteration which is closely related to our ODEs (3.4) to find numerical values of a_j's.

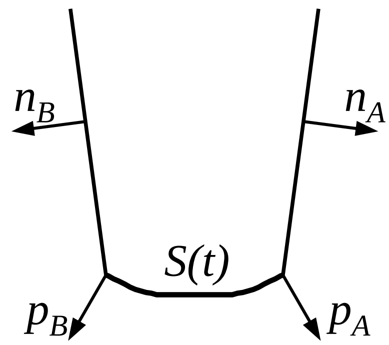

FIG. 2.6: *Selfsimilar expanding solution* $S(t)$

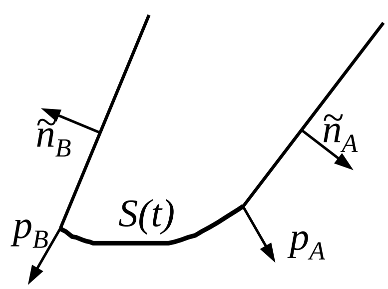

FIG. 2.7: *Selfsimilar expanding solution* $S(t)$

3 Existence of selfsimilar solution of ODE system.

We shall show the existence theorem (Theorem 2.1). When $n = 0$, $\{S(t)\}_{t>0}$ with $S(t) = t^{1/2}(\partial C)$ is the desired selfsimilar expanding solution. In the following we suppose $n \geqq 1$. Let a family of polygon $\{S(t)\}_{t>0}$ belongs to $\mathcal{S}$. Then we have a transport equation of $\{S(t)\}_{t>0}$:

$$\frac{dL_j(t)}{dt} = (\cot\varphi_j + \cot\varphi_{j+1})V_j(t) - \frac{1}{\sin\varphi_j}V_{j-1}(t) - \frac{1}{\sin\varphi_{j+1}}V_{j+1}(t), \quad t > 0$$

for $j = 1, \cdots, n$, where $\varphi_j := \theta_j - \theta_{j-1}$ for $j = 1, \cdots, n+1$. Here we set $\theta_0 := \theta_A, \theta_{n+1} := \theta_B$, $V_0 := 0$ and $V_{n+1} := 0$ for convenience. Plugging the governing law $V_j(t) = \Lambda_j(t)$ for $j = 1, \cdots, n$, we have an ODE system:

$$\frac{dL_j(t)}{dt} = \frac{1}{2}\left\{\frac{p_j}{L_j(t)} + \frac{q_{j-1}}{L_{j-1}(t)} + \frac{r_{j+1}}{L_{j+1}(t)}\right\}, \quad t > 0 \tag{3.1}$$

for $j = 1, \cdots, n$, where

$$p_j = 2\chi_j \Delta_j \frac{\sin(\varphi_j + \varphi_{j+1})}{\sin\varphi_j \sin\varphi_{j+1}} \qquad \text{for } j = 1, \cdots, n,$$

$$q_j = -2\chi_j \Delta_j \frac{1}{\sin\varphi_{j+1}} \qquad \text{for } j = 1, \cdots, n-1,$$

$$r_j = -2\chi_j \Delta_j \frac{1}{\sin\varphi_j} \qquad \text{for } j = 2, \cdots, n,$$

and $q_0 = 0$ and $r_{n+1} = 0$. Here we used $\cot\varphi_j + \cot\varphi_{j+1} = \dfrac{\sin(\varphi_j + \varphi_{j+1})}{\sin\varphi_j \sin\varphi_{j+1}}$.

By the assumption on θ_A and θ_B in §2, we note that $|\sum_{j=1}^{n+1}\varphi_j| = |\theta_A - \theta_B| < \pi$. When C is convex (resp. concave), i.e. $\theta_A > \theta_B$ (resp. $\theta_A < \theta_B$), then $\varphi_j < 0$ (resp. $\varphi_j > 0$) for $j = 1, \cdots, n+1$ and $\chi_j < 0$ (resp. $\chi_j > 0$) for $j = 1, \cdots, n$. So, we have $p_j > 0$ for $j = 1, \cdots, n$, $q_j < 0$ for $j = 1, \cdots, n-1$ and $r_j < 0$ for $j = 2, \cdots, n$.

Definition. A family of function $\{L_j(t)\}_{j=1}^n$ is called a *selfsimilar solution* of the ODE system (3.1) if $L_j(t)$ is of the form $L_j(t) = \alpha_j t^{1/2}$ with positive number α_j satisfying (3.1) for $j = 1, \cdots, n$.

Theorem 2.1 is obtained by showing

THEOREM 3.1. *Let n be a positive integer. There exists a selfsimilar solution $\{L_j(t)\}_{j=1}^n$ of the ODE system (3.1).*

When $n = 1$, equation (3.1) yields an ODE:

$$\frac{dL_1(t)}{dt} = \frac{p_1}{2L_1(t)}, \qquad t > 0.$$

Since $(d/dt)\{L_1(t)^2\} = p_1,\ t > 0$, we obtain $L_1(t) = t^{1/2}{p_1}^{1/2}$ for $t > 0$, so that Theorem 2.1 holds for $n = 1$.

In the following we assume that $n \geqq 2$. Our strategy to prove Theorem 3.1 is as follows. Substituting $L_j(t) = \alpha_j t^{1/2}$ to (3.1), we have

$$\begin{pmatrix} 1/a_1 \\ 1/a_2 \\ \vdots \\ 1/a_n \end{pmatrix} = H \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \tag{3.2}$$

with unknowns $a_j = 1/\alpha_j$, where

$$H = \begin{pmatrix} p_1 & r_2 & & & 0 \\ q_1 & p_2 & r_3 & & \\ & \ddots & \ddots & \ddots & \\ & & q_{n-2} & p_{n-1} & r_n \\ & 0 & & q_{n-1} & p_n \end{pmatrix}.$$

In particular

$$H = \begin{pmatrix} p_1 & r_2 \\ q_1 & p_2 \end{pmatrix}$$

when $n=2$. To show the existence of nonlinear algebraic equations (3.2), we consider the following continuation method sometimes called Davidenko's method. Introducing extra parameter $s \geq 0$ and matrix $K(s)$:

$$K(s) = \begin{pmatrix} p_1 & sr_2 & & & \mathbf{0} \\ sq_1 & p_2 & sr_3 & & \\ & \ddots & \ddots & \ddots & \\ & & sq_{n-2} & p_{n-1} & sr_n \\ \mathbf{0} & & & sq_{n-1} & p_n \end{pmatrix},$$

we consider a system of nonlinear algebraic equations:

$$\begin{pmatrix} 1/b_1(s) \\ 1/b_2(s) \\ \vdots \\ 1/b_n(s) \end{pmatrix} = K(s) \begin{pmatrix} b_1(s) \\ b_2(s) \\ \vdots \\ b_n(s) \end{pmatrix} \tag{3.3}$$

with parameter $s \geq 0$ and $b(s) > 0$. Evidently $b_j(0) = 1/\sqrt{p_j}$, since $K(0)$ is a diagonal matrix. If the solution can be extended up to $s=1$, then $b_j(1)$ is a solution of (3.2) since $K(1) = H$. Differentiating (3.3) formally with respect to parameter s, we have

$$-\begin{pmatrix} \vdots \\ b_j'(s)/b_j(s)^2 \\ \vdots \end{pmatrix} = K(s) \begin{pmatrix} \vdots \\ b_j'(s) \\ \vdots \end{pmatrix} - J \begin{pmatrix} \vdots \\ b_j(s) \\ \vdots \end{pmatrix},$$

which yields

$$Q(s, \vec{b}(s)) \begin{pmatrix} \vdots \\ b_j'(s) \\ \vdots \end{pmatrix} = J \begin{pmatrix} \vdots \\ b_j(s) \\ \vdots \end{pmatrix}, \tag{3.4}$$

i.e.

$$Q(s, \vec{b}(s))\, \vec{b}'(s) = J\vec{b}(s) \quad \text{with} \quad \vec{b}(s) = {}^{\mathrm{t}}(b_1(s), \cdots, b_n(s)),$$

where

$$J = -\begin{pmatrix} 0 & r_2 & & & & \\ q_1 & 0 & r_3 & & 0 & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & q_{n-2} & 0 & r_n \\ & & & & q_{n-1} & 0 \end{pmatrix}$$

and

$$Q(s, \vec{h}) = K(s) + \mathrm{diag}(1/h_1^2, \cdots, 1/h_n^2) \quad \text{with} \quad \vec{h} = {}^{\mathrm{t}}(h_1, \cdots, h_n).$$

Here $'$ denotes d/ds, differentiation with respect to s. If the inverse matrix of Q exists, (3.4) formally yields

$$\vec{b}'(s) = G(s, \vec{b}(s))\, \vec{b}(s). \tag{3.5}$$

Here we set $G(s,\vec{h}) = Q^{-1}(s,\vec{h})\ J$. We consider a system of ODEs (3.5) for $s > 0$ with initial condition

$$\vec{b}(0) = \vec{h}^*, \qquad \vec{h}^* := {}^t(1/\sqrt{p_1}, \cdots, 1/\sqrt{p_n}). \tag{3.6}$$

Local-in-time unique existence of a positive solution $\vec{b}(s)$ of (3.5) and (3.6) is guaranteed, since Q is smooth and $\det\, Q \neq 0$ near $(0,\vec{h}^*)$, so that Q^{-1} is smooth near $(0,\vec{h}^*)$.

As we shall prove in Lemma 3.2, the local solution $\vec{b}(s)$ can be extended uniquely up to $s = 1+\tau$ with some $\tau > 0$ (obtained in Theorem 4.1). Then $a_j := b_j(1)$ $(j = 1, \cdots, n)$ is a solution of (3.2), so that $\{L_j(t)\}_{j=1}^n$ with $L_j(t) = t^{1/2}/b_j(1)$ for $t > 0$, $j = 1, \cdots, n$ is a selfsimilar solution of (3.1), which implies Theorem 3.1.

Lemma 3.2 (Unique solvability up to $s = 1$). *There exists the unique positive solution $\vec{b}(s)$ (i.e. $\vec{b}(s) > \vec{0}$) of the system of ODEs (3.5) and (3.6) for $s \in [0, 1+\tau]$, where τ is a positive number obtained in Theorem 4.1.*

Here we use notation $\vec{x} < \vec{y}$ (resp. $\vec{x} \leqq \vec{y}$) for $\vec{x} = {}^t(x_1, \cdots, x_n)$, $\vec{y} = {}^t(y_1, \cdots, y_n) \in \mathbf{R}^n$ if $x_j < y_j$ (resp. $x_j \leq y_j$) for $j = 1, \cdots, n$.

Proof admitting a priori estimate (Lemma 3.3). Let S_0 be the maximal existence time of (3.5) and (3.6), and set $S_1 := \min(S_0, 1+\tau)$. We note that $G(s,\vec{b}(s))$ is well-defined for $s \in [0, S_1)$ by Lemma 3.3. Integrating (3.5), we have

$$\vec{b}(s) - \vec{b}(0) = \int_0^s G(u,\vec{b}(u))\vec{b}(u)du \quad \text{for} \quad s \in [0, S_1),$$

which implies

$$|\vec{b}(s)| \ \leqq\ |\vec{b}(0)| + \int_0^s |G(u,\vec{b}(u))|_{\text{op}}\ |\vec{b}(u)|du \quad \text{for} \quad s \in [0, S_1).$$

Here $|\cdot|$ denotes the Euclidean norm and $|\cdot|_{\text{op}}$ denotes the operator norm from $\mathbf{R}^n$ to $\mathbf{R}^n$. Lemma 3.3 implies that there exist a constant C_1 independent of s such that

$$0 < \ |G(s,\vec{b}(s))|_{\text{op}} \ \leq C_1 \quad \text{for} \quad s \in [0, S_1),$$

since each component of J is nonnegative. So we have

$$|\vec{b}(s)| \ \leqq\ |\vec{b}(0)| + C_1 \int_0^s |\vec{b}(u)|du \quad \text{for} \quad s \in [0, S_1).$$

Gronwall's Lemma implies

$$|\vec{b}(s)| \ \leqq\ |\vec{b}(0)| \exp(C_1\ s) \ \leqq\ |\vec{h}^*| \exp(C_1\ S_1) =: C_2 \quad \text{for} \quad s \in [0, S_1).$$

Suppose that $S_0 \leqq 1+\tau$. Then Lemma 3.3 (II) yields

$$1/\sqrt{p_j} \ \leq b_j(s) \ \leq C_2 \quad \text{for} \quad s \in [0, S_0),\ j = 1, \cdots, n,$$

which contradicts the definition of the maximal existence time. Thus we have $S_0 > 1+\tau$. $\square$

Lemma 3.3 (A priori estimate). *Let $S_0 > 0$ denote the maximal existence time of a positive solution of the system of ODEs (3.5) and (3.6). Set $S_1 := \min(S_0, 1+\tau)$, where τ is a positive number obtained in Theorem 4.1. Let $\vec{b}(s)$ be the solution of (3.5) and (3.6).*
(I) The derivative of each component of $\vec{b}(s)$ is positive, i.e., $\vec{b}'(s) > \vec{0}$ for $s \in [0, S_1)$.
(II) In particular, $\vec{b}(s) > \vec{h}^*(> \vec{0})$ for $s \in (0, S_1)$.
(III) There exists a constant $C_3 > 0$ independent of s such that

$$0 \leq \{\text{each component of } Q^{-1}(s, \vec{b}(s))\} \leq C_3 \quad for\ s \in [0, S_1).$$

Proof. The main steps of this lemma are proved in the next section, as summarized in Theorem 4.1. Since each component of matrix J is nonnegative and $\vec{b}(s) > 0$ for $s \in [0, S_1)$, we have $J\vec{b}(s) > \vec{0}$. We now observe that (II) and (III) of Theorem 4.1 implies $\vec{b}'(s) = G(s, \vec{b}(s))\vec{b}(s) > 0$ for $s \in [0, S_1)$. Initial condition (3.6) and (I) yield (II). Theorem 4.1 and (II) implies (III). □

4 A priori estimates for matrices.

Theorem 4.1 (A priori estimates). *(I) There exist positive constants C_4 and τ (independent of s and $\vec{h}$) such that*

$$\det Q(s, \vec{h}) > C_4 \prod_{j=1}^{n} p_j (> 0) \quad \text{for all } s \in [0, 1+\tau] \text{ and all } \vec{h} \in (\mathbf{R}_+)^n.$$

(II) The matrix $Q(s, \vec{h})$ has its inverse for $s \in [0, 1+\tau]$ and $\vec{h} \in (\mathbf{R}_+)^n$. Each component of $Q^{-1}(s, \vec{h})$ is smooth on $[0, 1+\tau] \times (\mathbf{R}_+)^n$.
(III) Let $\vec{h}^\sharp \in (\mathbf{R}_+)^n$ with $\vec{h}^\sharp > 0$. There exists constant $C_5 > 0$ (independent of s and $\vec{h}$) such that $0 \leqq \{$each component of $Q^{-1}(s, \vec{h})\} \leqq C_5$ for all $s \in [0, 1+\tau]$ and all $\vec{h} \in (\mathbf{R}_+)^n$ with $\vec{h} \geqq \vec{h}^\sharp$.

Here we use notation $\mathbf{R}_+ := (0, \infty)$, so that $(\mathbf{R}_+)^n = \overbrace{(0,\infty) \times \cdots \times (0,\infty)}^{n}$. To prove the theorem we shall show the following lemma and propositions. To show positiveness of the determinant of matrix Q we consider matrix $M^{k\ell}(s)$:

$$M^{k\ell}(s) := \begin{pmatrix} 1 & sr_{k+1}/p_{k+1} & & & & \mathbf{0} & \\ sq_k/p_k & 1 & sr_{k+2}/p_{k+2} & & & & \\ & sq_{k+1}/p_{k+1} & 1 & sr_{k+3}/p_{k+3} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & \mathbf{0} & & & sq_{\ell-2}/p_{\ell-2} & 1 & sr_\ell/p_\ell \\ & & & & & sq_{\ell-1}/p_{\ell-1} & 1 \end{pmatrix}$$

for $s \geqq 0$ and $k,\ \ell = 1, \cdots, n$ with $k \leqq \ell$. In particular,

$$M^{kk}(s) = \begin{pmatrix} 1 \end{pmatrix} \qquad \text{for } s \geqq 0,\ k = 1, \cdots, n,$$

$$M^{kk+1}(s) = \begin{pmatrix} 1 & sr_{k+1}/p_{k+1} \\ sq_k/p_k & 1 \end{pmatrix} \qquad \text{for } s \geqq 0,\ k = 1, \cdots, n-1.$$

We note that $M^{1n}(s)$ is obtained by dividing each j-th column of $K(s)$ by p_j's. We set $\tilde{M}^{k\ell} = M^{k\ell}(1)$. Fortunately, $\det \tilde{M}^{k\ell}$ is computable. Note that $\det H = \prod_{j=1}^{n} p_j \cdot \det \tilde{M}^{1n}$.

LEMMA 4.2. *For $k,\ \ell = 1, \cdots, n$ with $k \leqq \ell$,*

$$\det \tilde{M}^{k\ell} = \frac{\sin\left(\displaystyle\sum_{j=k}^{\ell+1} \varphi_j\right)}{\left(\displaystyle\prod_{j=k}^{\ell} \nu_j\right)\left(\displaystyle\prod_{j=k}^{\ell+1} \sin\varphi_j\right)}, \tag{4.1}$$

where

$$\nu_j := \cot\varphi_{j+1} + \cot\varphi_j \left(= \frac{\sin(\varphi_j + \varphi_{j+1})}{\sin\varphi_j \sin\varphi_{j+1}}\right). \tag{4.2}$$

Proof. Quantities q_j/p_j and r_j/p_j appearing in matrix $M^{k\ell}(s)$ can be calculated as follows:

$$\frac{q_j}{p_j} = \frac{-1}{\nu_j \sin\varphi_{j+1}}\ , \quad \frac{r_j}{p_j} = \frac{-1}{\nu_j \sin\varphi_j}.$$

We may assume that $k = 1$ without loss of generality. We shall prove (4.1) by induction. Let m^ℓ denote the right hand side of (4.1) with $k = 1$.
(i) Using equality (4.2), we have

$$m^1 = \frac{\sin(\varphi_1 + \varphi_2)}{(\cot\varphi_2 + \cot\varphi_1)\sin\varphi_1 \sin\varphi_2} = 1 = \det \tilde{M}^{11},$$

which implies (4.1) with $k = 1 = \ell$.
(ii) Next we shall show (4.1) with $k = 1,\ \ell = 2$. We have

$$\det \tilde{M}^{12} = 1 - \frac{-1}{\nu_1 \sin\varphi_2} \cdot \frac{-1}{\nu_2 \sin\varphi_2} = \frac{d_2}{\nu_1\nu_2 \sin\varphi_1 \sin^2\varphi_2 \sin\varphi_3} \tag{4.3}$$

with $d_2 = (\nu_1\nu_2\sin^2\varphi_2 - 1)\sin\varphi_1\sin\varphi_3$. Equality (4.2) yields $d_2 = \sin(\varphi_1+\varphi_2)\sin(\varphi_2+\varphi_3) - \sin\varphi_1\sin\varphi_3$. Using the identity

$$\sin\alpha\sin\beta = -\{\cos(\alpha+\beta) - \cos(\alpha-\beta)\}/2, \tag{4.4}$$

we have

$$d_2 = -\{\cos(\varphi_1 + 2\varphi_2 + \varphi_3) - \cos(\varphi_1 + \varphi_3)\}/2.$$

By the identity

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}, \tag{4.5}$$

we have $d_2 = \sin(\varphi_1+\varphi_2+\varphi_3)\sin\varphi_2$. Substituting this to (4.3), we obtain

$$\det \tilde{M}^{12} = \frac{\sin(\varphi_1+\varphi_2+\varphi_3)}{\nu_1\nu_2\sin\varphi_1\sin\varphi_2\sin\varphi_3} = m^2,$$

which is (4.1) for $k = 1, \ell = 2$.
(iii) We assume that $\ell \geqq 3$ and (4.1) holds for $\tilde{M}^{11}, \tilde{M}^{12}, \cdots, \tilde{M}^{1\ \ell-1}$. We have

$$\det \tilde{M}^{1\ell} = \det \tilde{M}^{1\ \ell-1} - \frac{r_\ell}{p_\ell}\frac{q_{\ell-1}}{p_{\ell-1}} \det \tilde{M}^{1\ \ell-2},$$

since

$$\tilde{M}^{1i} = \left(\begin{array}{c|c} \begin{matrix} & & \\ & \tilde{M}^{1\ i-1} & \\ & & \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \\ r_i/p_i \end{matrix} \\ \hline \begin{matrix} 0 & \cdots & 0 & q_{i-1}/p_{i-1} \end{matrix} & 1 \end{array}\right) \quad \text{for } i = 2, \cdots, \ell.$$

The assumption of the induction yields

$$\det \tilde{M}^{1\ell} = \frac{\sin\left(\sum_{j=1}^{\ell} \varphi_j\right)}{\left(\prod_{j=1}^{\ell-1} \nu_j\right)\left(\prod_{j=1}^{\ell} \sin\varphi_j\right)} - \frac{-1}{\nu_\ell \sin\varphi_\ell} \cdot \frac{-1}{\nu_{\ell-1}\sin\varphi_\ell} \cdot \frac{\sin\left(\sum_{j=1}^{\ell-1} \varphi_j\right)}{\left(\prod_{j=1}^{\ell-2} \nu_j\right)\left(\prod_{j=1}^{\ell-1} \sin\varphi_j\right)}.$$

Elementary calculation yields

$$\det \tilde{M}^{1\ell} = \frac{d_\ell}{\left(\prod_{j=1}^{\ell} \nu_j\right)\left(\prod_{j=1}^{\ell+1} \sin\varphi_j\right)\sin\varphi_\ell} \tag{4.6}$$

with

$$d_\ell = \sin\left(\sum_{j=1}^{\ell} \varphi_j\right)\sin(\varphi_\ell + \varphi_{\ell+1}) - \sin\left(\sum_{j=1}^{\ell-1} \varphi_j\right)\sin\varphi_{\ell+1}.$$

The identity (4.4) yields

$$d_\ell = \frac{-1}{2}\left\{\cos\left(\sum_{j=1}^{\ell+1} \varphi_j + \varphi_\ell\right) - \cos\left(\sum_{j=1}^{\ell-1} \varphi_j + \varphi_{\ell+1}\right)\right\},$$

and the identity (4.5) yields

$$d_\ell = \sin\left(\sum_{j=1}^{\ell+1} \varphi_j\right)\sin\varphi_\ell.$$

Substituting this to (4.6), we obtain (4.1) for $\det \tilde{M}^{1\ell}$. By induction the proof is now complete.
□

PROPOSITION 4.3. *The identities*

$$\frac{d}{ds}\left\{\det M^{k\ell}(s)\right\} = -s\left\{\sum_{j=k+1}^{\ell-2}\frac{q_j}{p_j}\frac{r_{j+1}}{p_{j+1}}\det M^{k\ j-1}(s)\cdot\det M^{j+2\ \ell}(s)\right.$$
$$+\sum_{j=k+2}^{\ell-1}\frac{q_{j-1}}{p_{j-1}}\frac{r_j}{p_j}\det M^{k\ j-2}(s)\cdot\det M^{j+1\ \ell}(s) \tag{4.6}$$
$$\left.+2\frac{q_k}{p_k}\frac{r_{k+1}}{p_{k+1}}\det M^{k+2\ \ell}(s)+2\frac{q_{\ell-1}}{p_{\ell-1}}\frac{r_\ell}{p_\ell}\det M^{k\ \ell-2}(s)\right\}$$

for $s \geqq 0$ *and* $k,\ell = 1,\cdots,n$ *with* $k+2<\ell$ *hold. Moreover*

$$\frac{d}{ds}\left\{\det M^{kk}(s)\right\} = 0 \qquad \text{for } s\geqq 0 \text{ and } k=1,\cdots,n;$$
$$\frac{d}{ds}\left\{\det M^{k\ k+1}(s)\right\} = -2s\frac{q_k}{p_k}\frac{r_{k+1}}{p_{k+1}} \qquad \text{for } s\geqq 0 \text{ and } k=1,\cdots,n-1;$$
$$\frac{d}{ds}\left\{\det M^{k\ k+2}(s)\right\} = -2s\left(\frac{q_k}{p_k}\frac{r_{k+1}}{p_{k+1}}+\frac{q_{k+1}}{p_{k+1}}\frac{r_{k+2}}{p_{k+2}}\right) \qquad \text{for } s\geqq 0 \text{ and } k=1,\cdots,n-2.$$

Proof. We observe that for $s \geqq 0$ and $k,\ell = 1,\cdots,n$ with $k+2<\ell$

$$\frac{d}{ds}\{\det M^{k\ell}(s)\}$$
$$=\sum_{j=k+1}^{\ell-1}\det\left(\begin{array}{c|c|c} & 0 & \\ & \vdots & \\ M^{k\ j-1}(s) & 0 & 0 \\ & r_j/p_j & \\ \hline 0\ \cdots\ 0\ \ sq_{j-1}/p_{j-1} & 0 & sr_{j+1}/p_{j+1}\ \ 0\ \cdots\ 0 \\ \hline & q_j/p_j & \\ 0 & 0 & M^{j+1\ \ell}(s) \\ & \vdots & \\ & 0 & \end{array}\right)$$
$$+\det\left(\begin{array}{c|c} 0 & sr_{k+1}/p_{k+1}\ \ 0\ \cdots\ 0 \\ \hline q_k/p_k & \\ 0 & M^{k+1\ \ell}(s) \\ \vdots & \\ 0 & \end{array}\right)+\det\left(\begin{array}{c|c} & 0 \\ & \vdots \\ M^{k\ \ell-1}(s) & 0 \\ & r_\ell/p_\ell \\ \hline 0\ \cdots\ 0\ \ sq_{\ell-1}/p_{\ell-1} & 0 \end{array}\right)$$
$$=-\sum_{j=k+1}^{\ell-2}\frac{q_j}{p_j}\det\left(\begin{array}{c|c|c} & 0 & \\ M^{k\ j-1}(s) & \vdots & 0 \\ & 0 & \\ \hline 0\ \cdots\ 0\ \ sq_{j-1}/p_{j-1} & sr_{j+1}/p_{j+1} & 0\ \cdots\ 0 \\ \hline & sq_{j+1}/p_{j+1} & \\ 0 & 0 & M^{j+2\ \ell}(s) \\ & \vdots & \\ & 0 & \end{array}\right)$$

$$
-\frac{q_{\ell-1}}{p_{\ell-1}} \det \left(\begin{array}{c|c} M^{k\ \ell-2}(s) & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0\ \cdots\ 0\ \ sq_{\ell-2}/p_{\ell-2} & sr_\ell/p_\ell \end{array} \right)
$$

$$
-\sum_{j=k+2}^{\ell-1} \frac{r_j}{p_j} \det \left(\begin{array}{c|c|c} M^{k\ j-2}(s) & \begin{matrix} 0 \\ \vdots \\ 0 \\ sr_{j-1}/p_{j-1} \end{matrix} & 0 \\ \hline 0\ \cdots\ 0 & sq_{j-1}/p_{j-1} & sr_{j+1}/p_{j+1}\ \ 0\ \cdots\ 0 \\ \hline 0 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & M^{j+1\ \ell}(s) \end{array} \right)
$$

$$
-\frac{r_{k+1}}{p_{k+1}} \det \left(\begin{array}{c|c} sq_k/p_k & sr_{k+2}/p_{k+2}\ \ 0\cdots 0 \\ \hline \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & M^{k+2\ \ell}(s) \end{array} \right)
$$

$$
-s\frac{q_k}{p_k}\ \frac{r_{k+1}}{p_{k+1}} \det M^{k+2\ \ell}(s) - s\frac{r_\ell}{p_\ell}\ \frac{q_{\ell-1}}{p_{\ell-1}} \det M^{k\ \ell-2}(s),
$$

which yields (4.6). A direct calculation yields the desired formulas for $k+2 \geq \ell$. □

Proposition 4.4.
(I) For $k, \ell = 1, \cdots, n$ with $k \leqq \ell$, $\det M^{k\ell}(1) = \det \tilde{M}^{k\ell} > 0$.
(II) There exists $\tau > 0$ such that

$$
\frac{d}{ds}\{\det M^{k\ell}(s)\} \leqq 0 \quad \textit{and} \quad \det M^{k\ell}(s) \geqq \det M^{k\ell}(1+\tau) > 0
$$

for $s \in [0, 1+\tau]$ and $k, \ell = 1, \cdots, n$ with $k \leqq \ell$.

Proof. (I) For all $j = 1, \cdots, n+1$ we see that $\varphi_j < 0$ (resp. $\varphi_j > 0$) if the cone C is convex (resp. concave). Since the original assumption in §2 yields $\left|\sum_{j=1}^{n+1} \varphi_j\right| = |\theta_A - \theta_B| < \pi$, identity (4.1) in Lemma 4.2 implies (I).
(II) We shall prove by induction.
(i) For $k = 1, \cdots, n$, $\det M^{kk}(s) = 1$, so that $(d/ds)\{\det M^{kk}(s)\} = 0$ for $s \in \mathbf{R}$.
(ii) For $k = 1, \cdots, n-1$, $(d/ds)\{\det M^{k\ k+1}(s)\} \leqq 0$ for $s \geqq 0$ by Proposition 4.3. By (I) with $\ell = k+1$, there exists $\tau^{k\ k+1} > 0$ such that $\det M^{k\ k+1}(s) > 0$ for $s \in [0, 1+\tau^{k\ k+1}]$.
(iii) Next we consider when $n \geqq 3$. Let m be $2, 3, \cdots, n-1$. For a moment we fix m. Assume that for $k = 1, 2, \cdots, n-(m-1)$, $i = 0, 1, \cdots, m-1$, there exists $\tau^{k\ k+i} > 0$ such that $\det M^{k\ k+i}(s) > 0$ for $s \in [0, 1+\tau^{k\ k+i}]$. For $k = 1, 2, \cdots, n-m$, identity (4.6) in Proposition 4.3 with $\ell = k+m$ yields $(d/ds)\{\det M^{k\ k+m}(s)\} \leqq 0$ for $s \in [0, 1+\hat{\tau}^{k\ k+m}]$ with $\hat{\tau}^{k\ k+m} = \min\{\min\{\tau^{kj}; j = k+1, k+2, \cdots, k+m-2\}, \min\{\tau^{j\ k+m}; j = k+2, k+3, \cdots, k+m-1\}\}$, since $p_j > 0$ $(j = 1, \cdots, n)$, $q_j < 0$ $(j = 1, \cdots, n-1)$ and $r_j < 0$ $(j = 2, \cdots, n)$. By (I) with $\ell = k+m$, there exists $\tau^{k\ k+m} > 0$ such that $\det M^{k\ k+m}(s) > 0$ for $s \in [0, 1+\tau^{k\ k+m}]$. □

Proof of Theorem 4.1. (I) Setting

$$W^{k\ell}\left(s,\vec{h}^{k\ell}\right) := M^{k\ell}(s) + \mathrm{diag}\left(\frac{1}{p_k h_k^2}, \frac{1}{p_{k+1}h_{k+1}^2}, \cdots, \frac{1}{p_\ell h_\ell^2}\right)$$

for $s \geqq 0, \vec{h}^{k\ell} = {}^{\mathrm{t}}(h_k, h_{k+1}, \cdots, h_\ell) \in (\mathbf{R}_+)^{\ell-k+1}$ and $k, \ell = 1, \cdots, n$ with $k \leqq \ell$, we have

$$\det Q^{k\ell}\left(s,\vec{h}^{k\ell}\right) = \det W^{k\ell}\left(s,\vec{h}^{k\ell}\right) \cdot \prod_{j=k}^{\ell} p_j.$$

Here

$$Q^{k\ell}\left(s,\vec{h}^{k\ell}\right) := K^{k\ell}(s) + \mathrm{diag}\left(1/h_k^2, 1/h_{k+1}^2, \cdots, 1/h_\ell^2\right)$$

and

$$K^{k\ell}(s) := \Big(K_{ij}(s)\Big)_{i,j=k,k+1,\cdots,\ell},$$

where $\Big(K_{ij}(s)\Big)_{i,j=1,\cdots,n}$ is the matrix $K(s)$ defined in §3. Proposition 4.4 (II) yields that there exists $\tau > 0$ such that $\det M^{pq}(s) \geqq C_{pq}$ for $s \in [0, 1+\tau]$ and $p, q = 1, \cdots, n$ with $p \leqq q$, where $C_{pq} := \det M^{pq}(1+\tau) > 0$. As we shall show later in Proposition 4.5, we have $\det W^{k\ell}\left(s,\vec{h}^{k\ell}\right) > \det M^{k\ell}(s)$ for $s \in [0, 1+\tau]$ and $\vec{h}^{k\ell} \in (\mathbf{R}_+)^{\ell-k+1}$. Since $p_j > 0$ ($j = 1, \cdots, n$),

$$\det Q^{k\ell}\left(s,\vec{h}^{k\ell}\right) > C_{k\ell}\prod_{j=k}^{\ell} p_j \quad (>0) \tag{4.7}$$

for $s \in [0, 1+\tau]$ and $\vec{h}^{k\ell} \in (\mathbf{R}_+)^{\ell-k+1}$.
(II) Part (II) follows from (I) with $k = 1$ and $\ell = n$ and the definition of Q.
(III) When $n = 1$, $Q(s,\vec{h})$ is a scalar and equals $(p_1 + 1/h_1^2)$, so that $0 < Q^{-1}(s,\vec{h}) = (p_1 + 1/h_1^2)^{-1} < 1/p_1$. We may assume that $n \geqq 2$. Since $Q(s,\vec{h})$ is invertible for $s \in [0,1+\tau]$ and $\vec{h} \in (\mathbf{R}_+)^n$ by (I), the (p,q) component of the inverse matrix of $Q(s,\vec{h})$ equals $\Delta_{qp}(s,\vec{h})/\det Q(s,\vec{h})$ for $p, q = 1, \cdots, n$, where $\Delta_{pq}(s,\vec{h})$ denotes the (p,q) cofactor of $Q(s,\vec{h})$. As we shall observe later in Proposition 4.6, we have

$$\Delta_{pq}(s,\vec{h}) = (-1)^{p+q}\left(\prod_{j=p}^{q-1} q_j\right) s^{q-p} \det Q^{1\ p-1}(s,\vec{h}) \det Q^{q+1\ n}(s,\vec{h}) \quad \text{for} \quad p < q;$$

$$\Delta_{pq}(s,\vec{h}) = (-1)^{p+q}\left(\prod_{j=q+1}^{p} r_j\right) s^{p-q} \det Q^{1\ q-1}(s,\vec{h}) \det Q^{p+1\ n}(s,\vec{h}) \quad \text{for} \quad p > q;$$

$$\Delta_{pp}(s,\vec{h}) = \det Q^{1\ p-1}(s,\vec{h}) \det Q^{p+1\ n}(s,\vec{h})$$

for $s \geqq 0$ and $\vec{h} \in (\mathbf{R}_+)^n$. Here we use convention $\det Q^{10}(s,\vec{h}) = 1 = \det Q^{n+1\ n}(s,\vec{h})$. Since $q_j < 0$ ($j = 1, \cdots, n-1$) and $r_j < 0$ ($j = 2, \cdots, n$), inequality (4.7) yields $\Delta_{pq}(s,\vec{h}) \geqq 0$ for $s \in [0, 1+\tau]$, $\vec{h} \in (\mathbf{R}_+)^n$ and $p, q = 1, \cdots, n$, which implies $\Big\{(p,q) \text{ component of } Q^{-1}(s,\vec{h})\Big\} = \Delta_{qp}(s,\vec{h})/\det Q(s,\vec{h}) \geqq 0$ by (I). On the other hand $\Delta_{pq}(s,\vec{h})$ is bounded from the above for

$s \in [0, 1+\tau]$ and $\vec{h} = (h_1, \cdots, h_n) \in (\mathbf{R}_+)^n$ with $\vec{h} \geq \vec{h}^\sharp$, since $\Delta_{pq}(s, \vec{h})$ is a polynomial of $1/h_1^2, \cdots, 1/h_n^2$ and s. Thus (I) yields that there exists $C_5 > 0$ such that $\Delta_{pq}(s, \vec{h}) / \det Q(s, \vec{h}) \leqq C_5$ for $s \in [0, 1+\tau], \vec{h} \in (\mathbf{R}_+)^n$ with $\vec{h} \geq \vec{h}^\sharp$ and $p, q = 1, \cdots, n$. □

Next we shall show an inequality, which is used in the proof of Theorem 4.1. For $\mu_j > 0$ $(j = 1, \cdots, n)$, $\xi_j \in \mathbf{R}$ $(j = 1, \cdots, n-1)$ and $\eta_j \in \mathbf{R}$ $(j = 2, \cdots, n)$, we set

$$A^{k\ell} := B^{k\ell} + \mathrm{diag}(\mu_k, \mu_{k+1}, \cdots, \mu_\ell),$$

$$B^{k\ell} := \begin{pmatrix} 1 & \eta_{k+1} & & & & & \mathbf{0} \\ \xi_k & 1 & \eta_{k+2} & & & & \\ & \xi_{k+1} & 1 & \eta_{k+3} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & \xi_{\ell-3} & 1 & \eta_{\ell-1} & \\ & \mathbf{0} & & & \xi_{\ell-2} & 1 & \eta_\ell \\ & & & & & \xi_{\ell-1} & 1 \end{pmatrix}$$

for $k, \ell = 1. \cdots, n$ with $k \leqq \ell$. In particular,

$$B^{kk} := \begin{pmatrix} 1 \end{pmatrix} \qquad \text{for } k = 1, \cdots, n;$$

$$B^{k\ k+1} := \begin{pmatrix} 1 & \eta_{k+1} \\ \xi_k & 1 \end{pmatrix} \qquad \text{for } k = 1, \cdots, n-1 \text{ when } n \geqq 2.$$

Proposition 4.5. *Let k and ℓ be $1, \cdots, n$ with $k \leqq \ell$. If $\det B^{pq} > 0$ for $p, q = k, k+1, \cdots, \ell$ with $p \leqq q$ then $\det A^{k\ell} > \det B^{k\ell}$.*

Proof. For $k,\ \ell = 1, \cdots, n$ with $k \leq \ell$ we set

$$C^{k\ell}(r) := B^{k\ell} + r\ \mathrm{diag}(\mu_k, \cdots, \mu_\ell) \qquad \text{for } r > 0,$$

which yields $C^{k\ell}(0) = B^{k\ell}$ and $C^{k\ell}(1) = A^{k\ell}$. We shall prove by induction on $\ell - k = 0, 1, 2, \cdots, n-1$. If $0 \leq \ell - k \leq 1$, direct calculation yields $(d/dr)\{\det C^{k\ell}(r)\} > 0$ for $r > 0$, which implies $\det C^{k\ell}(r) > \det B^{k\ell} > 0$ for $r > 0$. We assume that $k, \ell = 1, \cdots, n$ with $k + 2 \leq \ell$. Suppose that $(d/dr)\{\det C^{pq}(r)\} > 0$ for $r > 0$ and for $p, q = k, k+1, \cdots, \ell$ with $p \leq q \leq p + \ell - k - 1$. An elementary calculation yields

$$\begin{aligned} \frac{d}{dr} \det C^{k\ell}(r) &= \mu_k \cdot \det C^{k+1\ \ell}(r) + \det C^{k\ \ell-1}(r) \cdot \mu_\ell \\ &\quad + \sum_{j=k+1}^{\ell-1} \det C^{k\ j-1}(r) \cdot \mu_j \cdot \det C^{j+1\ \ell}(r) \\ &> 0 \qquad\qquad \text{for } r > 0, \end{aligned}$$

since $\det C^{pq}(r) > \det C^{pq}(0) = \det B^{pq} > 0$ for $r > 0$ and for $p, q = k, k+1, \cdots, \ell$ with $p \leq q \leq p + \ell - k - 1$ by the assumption of the induction. Thus we obtain

$$\det A^{k\ell} = \det C^{k\ell}(1) > \det C^{k\ell}(0) = \det B^{k\ell}$$

for $k \leq \ell$. □

Finally, we shall show identities on a cofactor of tridiagonal matrix, which are used in the proof of Theorem 4.1. For $\lambda_j \in \mathbf{R}$ $(j = 1, \cdots, n)$, $\xi_j \in \mathbf{R}$ $(j = 2, \cdots, n)$ and $\eta_j \in \mathbf{R}$ $(j = 1, \cdots, n-1)$, we set matrices

$$E^{k\ell} := \begin{pmatrix} \lambda_k & \xi_{k+1} & & & & & & \\ \eta_k & \lambda_{k+1} & \xi_{k+2} & & & \mathbf{0} & & \\ & \eta_{k+1} & \lambda_{k+2} & \xi_{k+3} & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & \eta_{\ell-3} & \lambda_{\ell-2} & \xi_{\ell-1} & \\ & \mathbf{0} & & & & \eta_{\ell-2} & \lambda_{\ell-1} & \xi_\ell \\ & & & & & & \eta_{\ell-1} & \lambda_\ell \end{pmatrix}$$

for $k, \ell = 1, \cdots, n$ with $k \leqq \ell$. In particular,

$$E^{kk} := \begin{pmatrix} \lambda_k \end{pmatrix} \qquad \text{for } k = 1, \cdots, n;$$

$$E^{kk+1} := \begin{pmatrix} \lambda_k & \eta_{k+1} \\ \eta_k & \lambda_{k+1} \end{pmatrix} \qquad \text{for } k = 1, \cdots, n-1 \text{ when } n \geqq 2.$$

Let D_{pq} be the (p, q) cofactor of matrix E^{1n} for $p, q = 1, \cdots, n$. The next proposition is obtained by an elementary calculation.

Proposition 4.6 *For $p, q = 1, \cdots, n$ the following identities hold.*
(I) When $p < q$,

$$D_{pq} = (-1)^{p+q} \prod_{j=p}^{q-1} \eta_j \det E^{1\ p-1} \det E^{q+1\ n}.$$

(II) When $p > q$,

$$D_{pq} = (-1)^{p+q} \prod_{j=q+1}^{p} \xi_j \det E^{1\ q-1} \det E^{p+1\ n}.$$

(III) $\qquad D_{pp} = \det E^{1\ p-1} \det E^{p+1\ n}.$
Here we use convention $\det E^{10} = 1 = \det E^{n+1\ n}$.

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