BACKWARD SHIFT INVARIANT SUBSPACES IN THE BIDISC III

KEIJI IZUCHI, TAKAHIKO NAKAZI, AND MICHIO SETO

Abstract. In the previous paper, we give a characterization of backward shift invariant subspaces of the Hardy space in the bidisc which satisfy the doubly commuting condition $S_z^* S_w^* = S_z^* S_z$ for the compression operators $S_z$ and $S_w$. In this paper, we give a characterization of backward shift invariant subspaces satisfying $S_z^2 S_w^* = S_w^* S_z^2$.

1. Introduction.

Let $\Gamma^2$ be the 2-dimensional unit torus. We denote by $(z, w) = (e^{i\theta}, e^{i\phi})$ the variables in $\Gamma^2 = \Gamma_z \times \Gamma_w$. Let $L^2 = L^2(\Gamma^2)$ be the usual Lebesgue space on $\Gamma^2$ with the norm $\|f\|_2 = (\int_{\Gamma^2} |f(e^{i\theta}, e^{i\phi})|^2 d\theta d\phi/(2\pi)^2)^{1/2}$. Then $L^2$ is a Hilbert space with the usual inner product. For $f \in L^2$, the Fourier coefficients are given by

$$
\hat{f}(n, m) = \int_{\Gamma^2} f(e^{i\theta}, e^{i\phi}) e^{-in\theta} e^{-im\phi} d\theta d\phi/(2\pi)^2 = \langle f, z^n w^m \rangle.
$$

Let $H^2 = H^2(\Gamma^2)$ be the Hardy space on $\Gamma^2$, that is,

$$
H^2 = \{ f \in L^2; \hat{f}(n, m) = 0 \text{ if } n < 0 \text{ or } m < 0 \}.
$$

For $f \in H^2$, we can write $f$ as

$$
f = \sum_{i,j=0}^{\infty} \oplus a_{i,j} z^i w^j, \quad \text{where } \sum_{i,j=0}^{\infty} |a_{i,j}|^2 < \infty.
$$

Let $P$ be the orthogonal projection from $L^2$ onto $H^2$. For a closed subspace $M$ of $L^2$, we denote by $P_M$ the orthogonal projection from $L^2$ onto $M$. For a function $\psi \in L^\infty$, let $L_{\psi} f = \psi f$ for $f \in L^2$. The Toeplitz operator $T_{\psi}$ is defined by $T_{\psi} f = P L_{\psi} f$ for $f \in H^2$. It is well known that $T_{\psi}^* = T_{\psi}^*$ and $T_{\psi}^n T_{\psi m} = T_{\psi m} T_{\psi n}$ for $n, m \geq 1$. A function $f \in H^2$ is called inner if $|f| = 1$ on $\Gamma^2$ almost everywhere. A closed subspace $M$ of $H^2$ is called invariant if $z M \subset M$ and $w M \subset M$. In one variable

---

1The first author was partially supported by Grant-in-Aid for Scientific Research (No.13440043), Ministry of Education, Science and Culture, Japan.

1991 Mathematics Subject Classification. Primary 47A15; Secondary 32A35.

Key words and phrases. Hardy space, backward shift, invariant subspace.
case, an invariant subspace \( M \) of \( H^2(\Gamma) \) has a form \( M = qH^2(\Gamma) \), where \( q \) is inner. This is the well-known Beurling theorem [B]. In two variable case, the structure of invariant subspaces of \( H^2 \) is very complicated, see [AC], [DY], [Na1], and [R].

Let \( M \) be an invariant subspace of \( H^2 \). Then \( T_z^*(H^2 \ominus M) \subset (H^2 \ominus M) \) and \( T_w^*(H^2 \ominus M) \subset (H^2 \ominus M) \). A closed subspace \( N \) of \( H^2 \) is called \textit{backward shift invariant} if \( H^2 \ominus N \) is invariant. In [IY], the first author and Yang studied backward shift invariant subspaces \( N \) on which \( T_z^* \) is strictly contractive. See [CR] and [S] for studies of backward shift invariant subspaces on the unit circle \( \Gamma \).

Let \( M \) be an invariant subspace of \( H^2 \) and \( \psi \in L^\infty \). Let \( V_\psi \) be the operator on \( M \) defined by \( V_\psi = P_M L_\psi|_M \). Then \( V_z = T_z \) and \( V_z^* = V_z \) on \( M \). In [M], Mandrekar proved that \( V_\psi V_w^* = V_\psi V_z^* \) on \( M \) if and only if \( M \) is Beurling type, that is, \( M = qH^2 \) for some inner function \( q \) in \( H^\infty \), see also [CS] and [Na2].

For \( \psi \in L^\infty \), let \( S_\psi = P_N L_\psi|_N \). Then we have \( S_\psi^* = S_\overline{\psi} \) and \( S_z^* = T_z^* \) on \( N \). In the previous paper [INS], we characterized backward shift invariant subspaces \( N \) which satisfy the condition \( S_z S_w^* = S_w^* S_z \) on \( N \) as follows.

\textbf{Theorem A.} Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( N \neq H^2 \). Then \( S_z S_w^* = S_w^* S_z \) on \( N \) if and only if \( N \) has one of the following forms:

\begin{enumerate}
  \item \( N = H^2 \ominus q_1(z)H^2 \),
  \item \( N = H^2 \ominus q_2(w)H^2 \),
  \item \( N = (H^2 \ominus q_1(z)H^2) \cap (H^2 \ominus q_2(w)H^2) \),
\end{enumerate}

where \( q_1(z) \) and \( q_2(w) \) are one variable inner functions.

This paper is a continuation of previous papers [IN] and [INS]. Here we are interesting in backward shift invariant subspaces \( N \) of \( H^2 \) satisfying that \( S_z^n S_w^m = S_w^m S_z^n \) for given positive integers \( n \) and \( m \). Up to now, we can not give a complete characterization of \( N \) satisfying \( S_z^n S_w^m = S_w^m S_z^n \). If \( S_z^n S_w^m = S_w^m S_z^n \), then trivially \( S_z S_w^* = S_w S_z^* = S_w^m S_z^n \). But the converse is not true. In this paper, we concentrated on the case \( m = 1 \). One reason is the work of this paper deeply concerns with the problem of the normality of the operators \( S_{z^n} \). For many backward shift invariant subspaces \( N \), \( S_{z^n} \) are not normal operators, see [Y]. If \( S_{z^n} \) is normal, since \( S_{z^n} S_w = S_w S_{z^n} \), by the Fuglede-Putnam theorem we have \( S_{z^n} S_w^* = S_w^* S_{z^n} \). So, the results obtained in this paper will be a big help for the above mentioned problem. We will discuss on this subject in the forthcoming paper.
In Section 2, we give characterizations of \( N \) satisfying \( S^z_n S^w_m = 0 \) and \( S^*_w S^z_n = 0 \), respectively. If \( S^z_n S^w_m = 0 \), then \( S^*_w S^z_n = 0 \). The converse is not true. In Section 3, we study \( N \) satisfying \( S^z_n S^*_w = S^*_w S^z_n \), and give a necessary condition for \( S^z_n S^*_w = S^*_w S^z_n \). In Section 4, we study \( N \) satisfying \( S^z_2 S^*_w = S^*_w S^z_2 \). We gave a complete characterization of such \( N \). In [INS], we gave two different type of proofs of Theorem A. One of them is used in Section 3 and the other is used in Section 4.

Let \( H^2(\Gamma_z) \) and \( H^2(\Gamma_w) \) be the Hardy spaces on the unit circle in variables \( z \) and \( w \), respectively. We think that \( H^2(\Gamma_z) \subset H^2(\Gamma_w) \subset H^2 \). For a subset \( E \) of \( H^2 \), we denote by \( \mathcal{E} \) the closed linear span of \( E \). A function \( b(z) = (z - \alpha)/(1 - \bar{\alpha}z), |\alpha| < 1 \), is called a simple Blaschke product.

2. \( S^z_n S^*_w = 0 \) or \( S^*_w S^z_n = 0 \).

Let \( n \) and \( m \) be positive integers. In this section, we study backward shift invariant subspaces \( N \) of \( H^2 \) satisfying \( S^z_n S^w_m = 0 \) and \( S^*_w S^z_n = 0 \), respectively.

**Lemma 2.1.** Let \( N \) be a backward shift invariant subspace of \( H^2 \). Then we have the following.

(i) \( S^z_n = S^z_n \).

(ii) \( S^w_m S^z_n = S^z_n S^w_m \) and \( S^*_n S^*_w = S^*_w S^*_n \).

(iii) If \( S^z_n S^w_m N \neq \{0\} \), then there exists \( f \in N \) such that \( (S^z_n S^w_m f) \neq 0 \).

**Proof.** All assertions are not difficult to prove.

The following theorem says that the structure of backward shift invariant subspaces satisfying \( S^z_n S^w_m = 0 \) is simple.

**Theorem 2.2.** Let \( N \) be a backward shift invariant subspace of \( H^2 \). Then \( S^z_n S^w_m = 0 \) if and only if \( N \) satisfies one of the following conditions;

(i) \( N \subset \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \).

(ii) \( N \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \).

**Proof.** Suppose that \( S^z_n S^w_m = 0 \). Then

\[
S^*_w N \perp S^*_z N.
\]

Since \( N \) is backward shift invariant, if \( S^*_w N = \{0\} \) then \( N \) satisfies condition (i). If \( S^*_z N = \{0\} \), then \( N \) satisfies (ii).

Next, suppose that

\[
S^*_w N \neq \{0\} \quad \text{and} \quad S^*_z N \neq \{0\}.
\]
We shall lead a contradiction. By (2.1), \( S_{wn}^* S_{zn}^* N \perp S_{wn}^* N \). By Lemma 2.1(ii), \( S_{wn}^* S_{zn}^* N = S_{zn}^* S_{wn}^* N = \{0\} \). Then
\[
(2.3) \quad S_{zn}^* N \subset \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)
\]
and
\[
(2.4) \quad S_{wn}^* N \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w).
\]
By (2.2) and (2.3), there exists a nonnegative integer \( j, 0 \leq j \leq m - 1 \), such that
\[
(2.5) \quad \{0\} \neq S_{wn}^* S_{zn}^* N \subset H^2(\Gamma_z).
\]
By Lemma 2.1(iii), there exists \( g \in N \) such that
\[
(2.6) \quad (S_{wn}^* S_{zn}^*)^\sim(0,0) \neq 0.
\]
Also by (2.2) and (2.4), there exist \( f \in N \) and a nonnegative integer \( i, 0 \leq i \leq n - 1 \), such that
\[
(2.7) \quad S_{zi}^* S_{wn}^* f \in H^2(\Gamma_w)
\]
and
\[
(2.8) \quad (S_{zi}^* S_{wn}^* f)^\sim(0,0) \neq 0.
\]
Then
\[
0 = \langle S_{wn}^* S_{zi}^* f, S_{zn}^* S_{wn}^* g \rangle \quad \text{by (2.1)}
\]
\[
= \langle S_{zi}^* S_{wn}^* f, S_{wn}^* S_{zn}^* g \rangle \quad \text{by Lemma 2.1(ii)}
\]
\[
= (S_{zi}^* S_{wn}^* f)^\sim(0,0) \langle S_{wn}^* S_{zn}^* g \rangle^\sim(0,0) \quad \text{by (2.5) and (2.7)}
\]
\[
\neq 0 \quad \text{by (2.6) and (2.8)}.
\]
This is a desired contradiction.

The converse is trivial. \( \square \)

**Corollary 2.3.** Let \( N \) be a backward shift invariant subspace of \( H^2 \). Then \( S_{zn} S_{wn}^* = 0 \) if and only if either \( S_{zn} = 0 \) or \( S_{wn}^* = 0 \). Hence if \( S_{zn} S_{wn}^* = 0 \), then \( S_{wn}^* S_{zn} = 0 \).

**Lemma 2.4.** Let \( M_1 \) and \( M_2 \) be closed subspaces of \( H^2 \) such that
\[
M_1 \subset \sum_{j=0}^{m} \oplus w^j H^2(\Gamma_z) \quad \text{and} \quad M_2 \subset \sum_{i=0}^{n} \oplus z^i H^2(\Gamma_w).
\]
Then \( M_1 + M_2 \) is closed.
Proof. We denote by \((z^jw^j)_{M_1}\) and \((z^jw^j)_{M_2}\) the orthogonal projections of \(z^jw^j\) to the spaces \(M_1\) and \(M_2\), respectively. Let
\[
M_1' = M_1 \ominus \left(\{(z^jw^j)_{M_1}; 0 \leq i \leq n, 0 \leq j \leq m\}\right)
\]
and
\[
M_2' = M_2 \ominus \left(\{(z^jw^j)_{M_2}; 0 \leq i \leq n, 0 \leq j \leq m\}\right).
\]
Then \(M_1'\) and \(M_2'\) are closed subspaces of \(H^2\),
\[
M_1' \subset z^{n+1}\left(\sum_{j=0}^{m} \odot w^j H^2(\Gamma_z)\right), \quad M_2' \subset w^{m+1}\left(\sum_{i=0}^{n} \odot z^i H^2(\Gamma_w)\right),
\]
and
\[
M_1' + M_2' \perp \{(z^jw^j); 0 \leq i \leq n, 0 \leq j \leq m\}.
\]
Since
\[
z^{n+1}\left(\sum_{j=0}^{m} \odot w^j H^2(\Gamma_z)\right) \perp w^{m+1}\left(\sum_{i=0}^{n} \odot z^i H^2(\Gamma_w)\right),
\]
\(M_1' + M_2' = M_1' \oplus M_2'\) is closed. Hence
\[
M_1 + M_2 = M_1' + M_2' + \{(z^jw^j)_{M_1}, (z^jw^j)_{M_2}; 0 \leq i \leq n, 0 \leq j \leq m\}
\]
is closed.

**Theorem 2.5.** Let \(N\) be a backward shift invariant subspace of \(H^2\). Then \(S_{w^m} S_{z^n} = 0\) if and only if

(i) \(N \ominus \left(\{N \cap \sum_{i=0}^{n-1} \odot z^i H^2(\Gamma_w)\}\right) \subset N \cap \sum_{j=0}^{m-1} \odot w^j H^2(\Gamma_z)\).

Proof. Suppose that \(S_{w^m} S_{z^n} = 0\). Then \(S_{z^n} N \subset \sum_{j=0}^{m-1} \odot w^j H^2(\Gamma_z)\).

Since \(S_{z^n} S_{w^m} = 0\), \(S_{w^m} N \subset \sum_{i=0}^{n-1} \odot z^i H^2(\Gamma_w)\). First, we prove the following

\[
(2.9) \quad N = \left(N \cap \sum_{j=0}^{m-1} \odot w^j H^2(\Gamma_z)\right) + \left(N \cap \sum_{i=0}^{n-1} \odot z^i H^2(\Gamma_w)\right).
\]

Let

\[
(2.10) \quad K = \left(N \cap \sum_{j=0}^{m-1} \odot w^j H^2(\Gamma_z)\right) + \left(N \cap \sum_{i=0}^{n-1} \odot z^i H^2(\Gamma_w)\right).
\]

By Lemma 2.4, \(K\) is closed and \(N = K \oplus (N \ominus K)\). To prove (i), it is sufficient to prove \(N \ominus K = \{0\}\). We have \(N \ominus K \subset N \cap (S_{z^n} N)\perp \cap (S_{w^m} N)\perp\). Let \(f \in N \ominus K\). Then \(f \perp S_{z^n} N\), so that \(S_{z^n} f \perp N\). Since
$S_{z_n} f \in N$, $S_{z_n} f = 0$. Hence $f \in \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$. By (2.10), $f \in K$.

This shows $f = 0$, so that $N \cap K = \{0\}$. Thus we get (2.9).

Let

\[(2.11) \quad N_1 = \left\{ f \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z); f \perp N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right\}.
\]

Then $N_1$ is a closed subspace and

\[
N_1 \oplus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \subset N.
\]

If the equality holds in the above, (i) holds. So we assume that

\[
N_1 \oplus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \nsubseteq N.
\]

We shall lead a contradiction. Let

\[(2.12) \quad N_2 = N \ominus \left( N_1 \oplus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right).
\]

Then $N_2 \neq \{0\}$ and $N = N_1 \oplus N_2 \oplus (N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w))$.

Let $g \in N_2$ be such that $g \neq 0$. We shall prove that

\[(2.13) \quad g \notin N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \quad \text{and} \quad g \notin N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z).
\]

The fact $g \notin N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$ is trivial. Suppose that $g \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$. By (2.11), $g \in N_1$. Since $g \in N_2$, by (2.12) we have $g \perp N_1$. Hence $g = 0$. This is a contradiction. Thus we get (2.13).

Next, we shall prove that

\[(2.14) \quad S_{z_n}^* g \nsubseteq N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z).
\]

To prove this, suppose not. Then $S_{z_n}^* g \perp N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$. By (2.10), $g = g_1 + g_2$, where $g_1 \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$ and $g_2 \in N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$. Then $S_{z_n}^* g = S_{z_n}^* g_1 \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$. Therefore $S_{z_n}^* g = 0$, so that $g \in N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$. This contradicts (2.13). Thus we get (2.14).

By (2.14), there exists $h_0$ such that $h_0 \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$ and $\langle g, S_{z_n} h_0 \rangle = \langle S_{z_n}^* g, h_0 \rangle \neq 0$. Since $S_{z_n} h_0 \in N$, by (2.12) we have $S_{z_n} h_0 = h_1 \oplus h_2 \oplus h_3$, where $h_1 \in N_1, h_2 \in N_2$, and $h_3 \in$
Let \( n \) be \( \in \mathbb{N} \), satisfying (assumption. Thus we have (i).

Thus we get
\[
P_N z^n h_0 = S_{z^n} h_0 \perp N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w).
\]

Thus we get \( h_3 = 0 \). By (2.12), \( S_{w^m} N_1 = \{0\} \). Hence \( S_{w^m} S_{z^n} h_0 = S_{w^m} h_2 \). By (2.13) and \( h_2 \in \mathbb{N} \), \( h_2 \notin \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \). This implies that \( S_{w^m} h_2 \neq 0 \). Hence \( S_{w^m} S_{z^n} \neq 0 \). This contradicts our starting assumption. Thus we have (i).

To prove the converse, suppose that condition (i) holds. Then
\[
N = \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \oplus L, \quad \text{where } L \subset N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z).
\]

Let \( F = F_1 + F_2 \in N \), where \( F_1 \in N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \) and \( F_2 \in L \). Since \( z^n F \perp N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \), \( S_{z^n} F \in L \). Hence \( S_{w^m} S_{z^n} F = 0 \). Thus we get \( S_{w^m} S_{z^n} = 0 \).

By Theorem 2.2, the structure of backward shift invariant subspaces \( N \) satisfying \( S_{z^n} S_{w^m} = 0 \) is simple. By Theorem 2.5, the structure of backward shift invariant subspaces \( N \) satisfying \( S_{w^m} S_{z^n} = 0 \) is not so simple. When \( n = m = 1 \), we have the following.

**Theorem 2.6.** Let \( N \) be a backward shift invariant subspace of \( H^2 \).

Then \( S_w S_z = 0 \) if and only if \( N \) has one of the following forms;

(i) \( N = H^2(\Gamma_z) \ominus q(z) H^2(\Gamma_z) \) for some inner function \( q(z) \).

(ii) \( N = H^2(\Gamma_w) \ominus q(w) H^2(\Gamma_w) \) for some inner function \( q(w) \).

(iii) Either \( N = H^2(\Gamma_z) + H^2(\Gamma_w) \), or \( N = H^2(\Gamma_z) + (H^2(\Gamma_w) \ominus q_2(w) H^2(\Gamma_w)) \), or \( N = (H^2(\Gamma_z) \ominus q_1(z) H^2(\Gamma_z)) + H^2(\Gamma_w) \), where \( q_1(z) \) and \( q_2(w) \) are inner functions.

(iv) \( N = (H^2(\Gamma_z) \ominus q_1(z) H^2(\Gamma_z)) + (H^2(\Gamma_w) \ominus q_2(w) H^2(\Gamma_w)) \), where \( q_1(z) \), \( q_2(w) \) are nonconstant inner functions and \( \hat{q}_1(0) \hat{q}_2(0) = 0 \).

In (iii) and (iv), since \( 1 \in N \), we may take \( q_1 \) and \( q_2 \) as \( \hat{q}_1(0) = \hat{q}_2(0) = 0 \).

**Proof.** By Theorem 2.2, \( S_w S_z = 0 \) if and only if either (i) or (ii) holds. By Theorem 2.5, \( S_w S_z = 0 \) if and only if
\[
(2.15) \quad N \ominus (N \cap H^2(\Gamma_w)) \subset N \cap H^2(\Gamma_z).
\]

If either (i) or (ii) holds, by Corollary 2.3 we have \( S_w S_z = 0 \). Suppose that \( N \) satisfies either (iii) or (iv). Then clearly \( 1 \in N \). Since \( N \) has a special form, it is not difficult to see that
\[
N \ominus (N \cap H^2(\Gamma_w)) = \{ f \in N \cap H^2(\Gamma_z); \hat{f}(0) = 0 \}.
\]

Hence (2.15) holds.
Next, suppose that (2.15) holds. Then we have

\begin{equation}
N = (N \cap H^2(\Gamma_z)) + (N \cap H^2(\Gamma_w)).
\end{equation}

If either $N \cap H^2(\Gamma_z) = \{0\}$ or $N \cap H^2(\Gamma_w) = \{0\}$, then $S_zS_w^* = 0$, and by Corollary 2.3, $S_z^*S_w = 0$. Hence either (i) or (ii) holds. Suppose that $N \cap H^2(\Gamma_z) \neq \{0\}$ and $N \cap H^2(\Gamma_w) \neq \{0\}$. We shall prove $1 \in N$. To prove this, suppose that $1 \notin N$. Let $1_w$ be the orthogonal projection of $1$ to $N \cap H^2(\Gamma_w)$. Then $1_w \notin H^2(\Gamma_z)$. Since $N \cap H^2(\Gamma_z) \neq \{0\}$, there exists $f \in N \cap H^2(\Gamma_z)$ such that $\hat{f}(0) \neq 0$. Let $f_1 = f - \hat{f}(0)1_w \in N$. Then $f_1 \notin H^2(\Gamma_z)$. Let $h \in N \cap H^2(\Gamma_w)$. Since $f \in H^2(\Gamma_z)$, $f - \hat{f}(0) \perp h$. Since $1 - 1_w \perp N \cap H^2(\Gamma_w)$,

\[ \langle f_1, h \rangle = \langle \hat{f}(0)(1 - 1_w) + f - \hat{f}(0), h \rangle = 0. \]

Hence $f_1 \in N \ominus (N \cap H^2(\Gamma_w))$. Thus (2.15) does not hold. Therefore $1 \in N$. Since $N \cap H^2(\Gamma_z)$ and $N \cap H^2(\Gamma_w)$ are nonzero backward shift invariant subspaces, by (2.16) $N$ has one of forms in (iii) and (iv). \(\square\)

3. $S_z^nS_w^* = S_w^*S_z^n$.

The following is the main theorem in this section.

**Theorem 3.1.** Let $N$ be a backward shift invariant subspace of $H^2$, $N \neq \{0\}$, and $N \neq H^2$. Let $M = H^2 \ominus N$ and $n \geq 2$ be a positive integer. If $S_z^nS_w^* = S_w^*S_z^n$, then one of the following conditions holds;

(i) $S_zS_w^* = S_w^*S_z$,

(ii) $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$ for an inner function $q_1(z)$ satisfying $q_1(z) = \prod_{j=1}^{k} b_j(z), 1 \leq k \leq n$, where $b_i$ are simple Blaschke products.

The following lemma is a generalization of [IN], and the idea of the proof is the same.

**Lemma 3.2.** Let $N$ be a backward shift invariant subspace of $H^2$ and $M = H^2 \ominus N$. Let $n$ be a positive integer. Then $S_z^nS_w^* = S_w^*S_z^n$ if and only if

\[ M \ominus \left( z^nM \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w) \right) \right) \subset (M \cap H^2(\Gamma_z)) \oplus wM. \]

**Proof.** The operators $T_z^n$ and $T_w^*$ on $H^2$ have the matrix forms as

\[ T_z^n = \begin{pmatrix} * & P_MT_z^n |_N \\ 0 & S_z^n \end{pmatrix}, T_w^* = \begin{pmatrix} * & 0 \\ P_NT_w^* |_M & S_w^* \end{pmatrix} \text{ on } H^2 = \begin{pmatrix} M \oplus N \end{pmatrix}. \]
Set $A = P_MT_{z^n}|_N$ and $B = P_NT^*_w|_M$. Since $T_{z^n}T^*_w = T^*_wT_{z^n}$ on $H^2$, $S_{z^n}S^*_w = S^*_wS_{z^n}$ if and only if $BA = 0$. We have $T^*_w(M \ominus wM) \subset N$.

For $f \in H^2$, $T^*_wf = 0$ if and only if $f \in H^2(\Gamma_z)$. Hence

$$\ker B = \{f \in M; T^*_wf \in M\} = \{f \in M \ominus wM; T^*_wf = 0\} \ominus wM = (M \cap H^2(\Gamma_z)) \ominus wM.$$ 

We denote by $[\text{ran } A]$ the closed range of $A$. Let $A_1 = P_MT_{z^n}P_N$ on $H^2$. Then $[\text{ran } A] = [\text{ran } A_1]$. Since $A_1^* = P_NT^*_wP_M$, we get

$$\ker A_1^* = \{f \in M; T^*_{z^n}f \in M\} \oplus N = (M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)) \oplus z^n M \oplus N.$$ 

Hence

$$[\text{ran } A] = [\text{ran } A_1] = H^2 \ominus \ker A_1^* = M \ominus \left( z^n M \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right).$$ 

Since $BA = 0$ if and only if $[\text{ran } A] \subset \ker B$, we get our assertion.

**Proof of Theorem 3.1.** Suppose that $S_{z^n}S^*_w = S^*_wS_{z^n}$. By Lemma 3.2,

$$M \ominus \left( z^n M \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right) \subset (M \cap H^2(\Gamma_z)) \ominus wM.$$ 

Let

$$K_0 = M \ominus \left( z^n M \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right).$$ 

Then

(3.1) $$K_0 \subset (M \cap H^2(\Gamma_z)) \ominus wM$$

and

(3.2) $$M \ominus z^n M = K_0 \ominus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right).$$

Since $M = \sum_{s=0}^{\infty} \oplus z^{ns}(M \ominus z^n M)$,

(3.3) $$K_0 \perp \sum_{s=0}^{\infty} \oplus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right)$$

and

(3.4) $$M = \left( \sum_{s=0}^{\infty} \oplus z^{ns} K_0 \right) \ominus \left( \sum_{s=0}^{\infty} \oplus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right).$$
First, suppose that $K_0 = \{0\}$. In this case, we shall show that condition (i) holds. By (3.2),
\[
\sum_{i=0}^{n-1} \oplus z^i (M \ominus z^n M) = M \ominus z^n M = M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w).
\]
Let $f \in M \ominus zM$. Then $f \in M \ominus z^n M$ and $f = \sum_{i=0}^{n-1} \oplus z^i h_i(w)$, where $h_i(w) \in H^2(\Gamma_w)$. Since $f \in M \ominus zM$,
\[
z^{n-1} f = \sum_{i=0}^{n-1} \oplus z^{n-1+i} h_i(w) \in M \ominus z^n M.
\]
Hence $f = h_0(w)$, so that $M \ominus zM = M \cap H^2(\Gamma_w)$. By Lemma 3.2 (for the case of $n = 1$), we get $S_z S_w^* = S_w^* S_z$.

Next, suppose that $K_0 \neq \{0\}$. The proof of this case is a little bit long, so we divide into several steps.

**Step 1.** In this step, we shall prove that there is a nonconstant inner function $q_1(z)$ such that
\[
(3.5) \quad K_0 \subset q_1(z) H^2(\Gamma_z) \oplus wM.
\]
First we prove that $K_0 \not\subset wM$. To prove this, suppose that $K_0 \subset wM$. Let $F \in K_0$. Then $F = w f$ for some $f \in M$. We shall prove that $f \in K_0$. We have
\[
\left\langle f, \left( \sum_{s=1}^{\infty} \oplus z^n M \ominus z^s K_0 \right) + \left( \sum_{s=0}^{\infty} \oplus z^n \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right) \right\rangle
\]
\[
= \left\langle w f, w \left( \sum_{s=1}^{\infty} \oplus z^n M \ominus z^s K_0 \oplus \left( \sum_{s=0}^{\infty} \oplus z^n \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right) \right) \right\rangle
\]
\[
= \left\langle F, z^n w \left( \sum_{s=1}^{\infty} \oplus z^n (s-1) K_0 \right) \right\rangle \quad \text{by (3.3)}
\]
\[
= 0,
\]
where the last equality follows from $w \sum_{s=1}^{\infty} \oplus z^n (s-1) K_0 \subset M$ and (3.2). Hence by (3.4), we get $f \in K_0$. Therefore for every positive integer $p$, we have $F = w^p f_p$ for some $f_p \in K_0$. This leads $F = 0$. This is a contradiction. Thus we get $K_0 \not\subset wM$.

Hence by (3.1), $M \cap H^2(\Gamma_z) \neq \{0\}$. By the Beurling theorem,
\[
(3.6) \quad M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z),
\]
where $q_1(z)$ is inner. Then $q_1(z) \in M$ and
\[
(3.7) \quad q_1(z) H^2(\Gamma_z) \perp wM.
\]
If $q_1(z)$ is constant, we have $M = H^2$, so that $N = \{0\}$. This contradicts our assumption. Hence $q_1(z)$ is a nonconstant inner function. By (3.1) and (3.6), we get (3.5).

*Step 2.* In this step, we prove

\[(3.8) \quad K_0 \subset q_1(z)\left(\sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\right).\]

Let $G \in K_0$. Then by (3.5), $G = q_1(z)h(z) \oplus wg$, where $h(z) \in H^2(\Gamma_z)$ and $g \in M$. Write

$$h(z) = \left(\sum_{i=0}^{n-1} \oplus a_i z^i\right) \oplus z^n h_0(z).$$

Then

$$G = q_1(z)\left(\sum_{i=0}^{n-1} \oplus a_i z^i\right) \oplus q_1(z)z^n h_0(z) \oplus wg.$$  

By (3.6), $q_1(z)z^n h_0(z) \in z^n M$. Since $G \in K_0 \subset M \oplus z^n M$, we have $h_0(z) = 0$. Hence

\[(3.9) \quad G = q_1(z)\left(\sum_{i=0}^{n-1} \oplus a_i z^i\right) \oplus wg.\]

Here we prove that

\[(3.10) \quad g \in K_0.\]

Since $G = q_1(z)h(z) \oplus wg$, we have

$$\langle g, \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rangle = \langle wg, w \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rangle$$

$$= \langle wg, w \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rangle + \langle q_1 h, w \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rangle \quad \text{by (3.7)}$$

$$= \langle G, w \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rangle$$

$$= \langle G, z^n w \sum_{s=1}^{\infty} \oplus z^{n(s-1)} K_0 \rangle$$

$$= 0 \quad \text{by (3.2).}$$
We also have
\[
\langle g, \sum_{s=0}^{\infty} \oplus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \rangle
= \langle wg, w \left( \sum_{s=0}^{\infty} \oplus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right) \rangle
= \langle G, \sum_{s=0}^{\infty} \oplus z^{ns} w \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \rangle
\text{ by (3.7)}
= 0 \text{ by (3.3)}.
\]
Hence by (3.4), we get (3.10).

Applying (3.9) and (3.10) infinitely many times, we have
\[
G = q_1(z) \left( \sum_{i=0}^{n-1} \oplus a_i z^i \right) \oplus wq_1(z) \left( \sum_{i=0}^{n-1} \oplus b_i z^i \right) \oplus \cdots
\in q_1(z) \left( \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right).
\]
Therefore we get (3.8).

**Step 3.** In this step, we study functions in \( M \ominus zM \) and the inner function \( q_1(z) \). By (3.8), there is a closed subspace \( L \) such that \( K_0 = q_1(z)L \) and \( L \subseteq \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \). Then by (3.2),
\[
M \ominus z^n M = q_1(z)L \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right).
\]
Since \( K_0 \neq \{0\} \), \( L \neq \{0\} \). We have \( M \ominus z^n M = \sum_{i=0}^{n-1} \oplus z^i (M \ominus zM) \).
Hence \( M \ominus zM \neq \{0\} \). Let \( F \in M \ominus zM \) be such that \( F \neq 0 \). Then
\[
F = \left( q_1 \sum_{i=0}^{n-1} \oplus z^i f_i \right) \oplus \sum_{i=0}^{n-1} \oplus z^i g_i,
\]
where \( f_i, g_i \in H^2(\Gamma_w) \),
\[
q_1 \sum_{i=0}^{n-1} \oplus z^i f_i \perp M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w),
\]
and
\[
\sum_{i=0}^{n-1} \oplus z^i g_i \in M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w).
\]
Since $n \geq 2$, $zF \in M \ominus z^n M$, so that we have

$$zF = q_1\left(\sum_{i=0}^{n-1} \oplus z^i G_{1,i}\right) + \sum_{i=0}^{n-1} \oplus z^i H_{1,i},$$

where $G_{1,i}, H_{1,i} \in H^2(\Gamma_w)$. Hence (3.14)

$$q_1\left(z\left(\sum_{i=0}^{n-1} \oplus z^i f_i\right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}\right) = \left(\sum_{i=0}^{n-1} \oplus z^i H_{1,i}\right) - z \sum_{i=0}^{n-1} \oplus z^i g_i.$$

Here we devide into two subcases.

**Subcase 1.**

$$z\left(\sum_{i=0}^{n-1} \oplus z^i f_i\right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w) \neq 0.$$  

Then

$$q_1(z) = \frac{\left(\sum_{i=0}^{n-1} \oplus z^i H_{1,i}(w)\right) - z \sum_{i=0}^{n-1} \oplus z^i g_i(w)}{z\left(\sum_{i=0}^{n-1} \oplus z^i f_i(w)\right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w)}.$$

As proved in Step 1, $q_1(z)$ is a nonconstant inner function. Then by the above, we have

(3.15)  

$$q_1(z) = \prod_{j=1}^{k} b_j(z), \quad 1 \leq k \leq n,$$

where $b_j$ are simple Blaschke products.

**Subcase 2.**

$$z\left(\sum_{i=0}^{n-1} \oplus z^i f_i\right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w) = 0.$$  

Then by (3.14), $f_{n-1} = g_{n-1} = 0$, so that by (3.11)

$$F = \left(q_1 \sum_{i=0}^{n-2} \oplus z^i f_i\right) \oplus \sum_{i=0}^{n-2} \oplus z^i g_i.$$  

Since $F \in M \ominus zM$,

$$zF = \left(q_1 \sum_{i=0}^{n-2} \oplus z^{i+1} f_i\right) \oplus \sum_{i=0}^{n-2} \oplus z^{i+1} g_i \in M \ominus z^2 M.$$  

In the same way as above, either (3.15) holds or $f_{n-2} = g_{n-2} = 0$. Repeat the same argument. Then either (3.15) holds or

$$f_{n-1} = f_{n-2} = \cdots = f_1 = g_{n-1} = g_{n-2} = \cdots = g_1 = 0.$$
Step 4. If (3.15) holds, then (ii) holds. So, suppose that (3.15) does not hold. Then by (3.11) and by the above fact, \( F = q_1f_0 \oplus g_0 \), by (3.12) \( q_1f_0 \perp M \cap H^2(\Gamma_w) \), and by (3.13) \( g_0 \in M \cap H^2(\Gamma_w) \) for every \( F \in M \ominus zM \).

If \( g_0 = 0 \) for every \( F \in M \ominus zM \), since \( q_1(z) \in M \) it follows that \( M \ominus zM \subset q_1(z)H^2(\Gamma_w) \subset M \). Since \( M = \sum_{i=0}^{\infty} z^i(M \ominus zM) \), we have \( M = q_1(z)H^2 \), so that \( N = H^2 \ominus q_1(z)H^2 \). By Theorem A, condition (i) holds.

So we assume that \( g_0 \neq 0 \) for some \( F \in M \ominus zM \). We shall prove that

\[
(3.16) \quad (M \ominus zM) \ominus (M \cap H^2(\Gamma_w)) \subset (M \cap H^2(\Gamma_z)) \ominus wM.
\]

We may assume that \( (M \ominus zM) \ominus (M \cap H^2(\Gamma_w)) \neq \{0\} \). Let \( G \in (M \ominus zM) \ominus (M \cap H^2(\Gamma_w)) \) be such that \( G \neq 0 \). Then \( G = q_1(z)h_1(w) \oplus h_2(w) \), where \( q_1(z)h_1(w) \perp M \cap H^2(\Gamma_w) \) and \( h_2(w) \in M \cap H^2(\Gamma_w) \). Hence \( h_2(w) = G - q_1(z)h_1(w) \perp M \cap H^2(\Gamma_w) \). Therefore \( h_2(w) = 0 \). Since \( q_1(z) \in M \),

\[
G = q_1(z)h_1(w) = h_1(0)q_1(z) + wq_1(z) \frac{h_1(w) - h_1(0)}{w} \\
\in \ M \cap H^2(\Gamma_z) \ominus wM.
\]

Thus we get (3.16). By Lemma 3.2 (for \( n = 1 \)), (i) holds.

4. \( S_zS_w^* = S_w^*S_z \) and \( S_zS_w^* \neq S_w^*S_z \).

Let \( N \) be a backward shift invariant subspace of \( H^2 \) and let \( n \) be a positive integer. Let \( M = H^2 \ominus N \). Then \( M \) is an invariant subspace. If both \( S_zS_w^* = S_w^*S_z \) and \( S_zS_w^* \neq S_w^*S_z \) hold, then by Theorem 3.1, \( M \cap H^2(\Gamma_z) = q(z)H^2(\Gamma_z) \) for some nonconstant inner function \( q(z) \).

In this section, we assume that \( q_1(z)H^2 \subset M \) and \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \) for some nonconstant inner function \( q_1(z) \). Let

\[
\tilde{M} = M \ominus q_1(z)H^2 \subset M.
\]

Then \( H^2 \ominus q_1(z)H^2 = \tilde{M} \ominus N \) and \( \tilde{M} \) is \( w \)-invariant. The following lemma is proved in [INS, Lemma 3.2].

**Lemma 4.1.** Let \( f \in \tilde{M} \). Then \( T_w^*f \in \tilde{M} \) if and only if \( f \in w\tilde{M} \).

We denote by \( P_t \) the orthogonal projection from \( H^2 \) onto \( H^2 \ominus q_1(z)H^2 \). Then we have a Toeplitz type operator \( Q_{z^n} \) on \( H^2 \ominus q_1(z)H^2 \) such that

\[
Q_{z^n} : H^2 \ominus q_1(z)H^2 \ni f \to P_{\perp}(T_w^*f) \in H^2 \ominus q_1(z)H^2.
\]
Since \( z^n M \subset M \), \( Q_{z^n} \tilde{M} \subset \tilde{M} \) and \( Q_z^n = Q_{z^n} \). Then \( Q_{z^n} \) has the following matrix form:

\[
Q_{z^n} = \begin{pmatrix}
* & P_M T_{z^n} |_N \\
0 & S_{z^n}
\end{pmatrix}
\quad \text{on} \quad H^2 \ominus q_1(z)H^2 = \begin{pmatrix}
\tilde{M} \\
N
\end{pmatrix}.
\]

Since \( H^2 \ominus q_1(z)H^2 \) is backward shift invariant, \( T_w^* (H^2 \ominus q_1(z)H^2) \subset H^2 \ominus q_1(z)H^2 \). Since \( T_w^* N \subset N \), the operator \( T_w^* \) on \( H^2 \ominus q_1(z)H^2 \) has the following matrix form:

\[
T_w^* = \begin{pmatrix}
* & 0 \\
P_N T_w^* |_{\tilde{M}} & S_w^*
\end{pmatrix}
\quad \text{on} \quad H^2 \ominus q_1(z)H^2 = \begin{pmatrix}
\tilde{M} \\
N
\end{pmatrix}.
\]

Set

\[(4.1) \quad A = P_M T_{z^n} |_N \quad \text{and} \quad B = P_N T_w^* |_{\tilde{M}}.\]

By [INS, Lemma 3.3], \( T_w^* Q_z = Q_z T_w^* \) on \( H^2 \ominus q_1(z)H^2 \). Hence we have the following.

**Lemma 4.2.** \( T_w^* Q_{z^n} = Q_{z^n} T_w^* \) on \( H^2 \ominus q_1(z)H^2 \).

**Lemma 4.3.** \( S_{z^n} S_w^* = S_{z^n} S_w^* \) if and only if \( BA = 0 \).

**Proof.** By Lemma 4.2, \( T_w^* Q_{z^n} = Q_{z^n} T_w^* \) on \( H^2 \ominus q_1(z)H^2 \). Then \( BA + S_w^* S_{z^n} = S_{z^n} S_w^* \). Hence \( S_{z^n} S_w^* = S_{z^n} S_{z^n} \) if and only if \( BA = 0 \).

The following is a slight generalization of [INS, Theorem 3.5].

**Theorem 4.4.** Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \ominus N \). Suppose that \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \), where \( q_1(z) \) is a nonconstant inner function. Let \( \tilde{M} = M \ominus q_1(z)H^2 \). Then the following conditions are equivalent.

(i) \( S_{z^n} S_w^* = S_{z^n} S_w^* \) on \( N \).

(ii) \( \tilde{M} \ominus \{ f \in \tilde{M}; T_{z^n} f \in \tilde{M} \} \subset w\tilde{M} \).

(iii) \( T_{z^n} \tilde{M} \subset \tilde{M} \).

**Proof.** The idea of the proof is similar to the one of [INS, Theorem 3.5]. For the sake of completeness, we give the proof.

(i) \( \Leftrightarrow \) (ii) By Lemma 4.3, condition (i) is equivalent to \( BA = 0 \). By (4.1) and Lemma 4.1, \( \ker B = \{ f \in \tilde{M}; T_{z^n} f \in \tilde{M} \} = w\tilde{M} \). Put \( A_1 = P_M T_{z^n} P_N \) on \( \tilde{M} \ominus N \). Then \( [\text{ran} A] = [\text{ran} A_1] \). Since \( A_{1}^* = P_NT_{z^n} P_M \), \( \ker A_{1}^* = N \ominus \{ f \in \tilde{M}; T_{z^n} f \in \tilde{M} \} \). Hence

\[
[\text{ran} A] = [\text{ran} A_1] = (\tilde{M} \ominus N) \ominus \ker A_{1}^* = \tilde{M} \ominus \{ f \in \tilde{M}; T_{z^n} f \in \tilde{M} \}.
\]

Therefore \( BA = 0 \) if and only if \( \tilde{M} \ominus \{ f \in \tilde{M}; T_{z^n} f \in \tilde{M} \} \subset w\tilde{M} \). Thus we get (i) \( \Leftrightarrow \) (ii).
Since \( b \) and \( M \) are positive integers with \( n < k \), we have \( T_{z,f}^* = T_{z,f} \) for \( f = H^2 \), we have \( T_{z,f}^* \in M \).

(iii) \( \Rightarrow \) (ii) is trivial. \( \square \)

For \( f \in H^2(\Gamma_z) \), write \( f^*(z) = T_{z,f}^*f(z) = z(f(z) - \hat{f}(0)) \).

**Lemma 4.5.** Let \( b_i(z) = (z - \alpha_i)/(1 - \alpha_i z) \), \( |\alpha_i| < 1 \), and \( 1 \leq i \leq n \). Then

(i) \( T_{z}^*z = 1 \), \( T_{z}^*b_1 = b_1^*(z) = (1 - |\alpha_1|^2)/(1 - \alpha_1) \), and \( T_{z}^*b_1(z) = \overline{\alpha_1}b_1^*(z) \).

(ii) \( T_{z}^*(b_1(z)b_2^2(z)) = (1 - |\alpha_2|^2)b_1^*(z) + \overline{\alpha_2}b_1(z)b_2^2(z) \).

(iii) \( H^2(\Gamma_z) \otimes (\prod_{j=1}^k b_j(z)H^2(\Gamma_z) = \prod_{j=1}^k H^2(\Gamma_z) \oplus [b_1(z) \cdots b_{j-1}(z)b_j^*(z)] \).

(iv) \( H^2 \otimes (\prod_{j=1}^k b_j(z)) = \sum_{j=1}^k \oplus [b_1(z) \cdots b_{j-1}(z)b_j^*(z)H^2(\Gamma_w)] \).

**Proof.** It is not difficult to prove (i).

(ii) Since

\[
\overline{\alpha_1}b_1(z)b_2^2(z) = \overline{\alpha_1}b_1(z) \frac{1 - |\alpha_2|^2}{1 - \overline{\alpha_2}z} = (1 - |\alpha_2|^2)b_1(z) \left( \frac{\overline{\alpha_2}}{1 - \overline{\alpha_2}z} \right) = (1 - |\alpha_2|^2)\overline{\alpha_2}b_1(z) + \overline{\alpha_2}b_1(z)b_2^2(z) \text{,}
\]

we get (ii).

(iii) See [Ni, p.33].

(iv) follows from (iii). \( \square \)

**Corollary 4.6.** Let \( N \) be a backward shift invariant subspace of \( H^2 \). Suppose that \( M \cap H^2(\Gamma_z) = b_1(z)H^2(\Gamma_z) \), where \( b_1(z) \) is a simple Blaschke product. Then \( S_zS_w^* = S_w^*S_z \).

**Proof.** Let \( b_1(z) = (z - \alpha)/(1 - \alpha z) \), \( |\alpha| < 1 \), and \( \hat{M} \cap b_1(z)H^2 \). Since \( b_1(z) \in M \), \( b_1(z)H^2 \subset M \). By Lemma 4.5(iv), \( \hat{M} \subset b_1^*(z)H^2(\Gamma_w) \). By Lemma 4.5(i), \( T_{z,f}^*(b_1^*(z)h(w)) = \overline{\alpha_1}b_1^*(z)h(w) \). Hence \( T_{z,f}^* \hat{M} \subset \hat{M} \). By Theorem 4.4, \( S_zS_w^* = S_w^*S_z \). \( \square \)

**Corollary 4.7.** Let \( N \) be a backward shift invariant subspace of \( H^2 \). Suppose that \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \), where \( q_1(z) \) is an inner function. Let \( n, k \) be positive integers with \( n \geq k + 1 \). Moreover suppose that \( q_1(z) = z^kb(z) \), where \( b(z) = (z - \alpha)/(1 - \alpha z) \), and \( \alpha \neq 0 \). If \( S_{z^n}S_w^* = S_w^*S_{z^n} \), then \( S_{z^n}S_w^* = S_w^*S_{z^n} \).
Then there exists \( f = b^*(z)H^2 \). If \( \tilde{M} = \{0\} \), then \( \tilde{M} = q_1(z)H^2 \). By Theorem A, \( S_zS_w = S_wS_z \). Suppose that \( \tilde{M} \neq \{0\} \). By Lemma 4.5(iv),

\[
\tilde{M} \subset b^*(z)H^2(\Gamma_w) \oplus \left( \sum_{j=1}^{k} \oplus z^{j-1}b(z)H^2(\Gamma_w) \right).
\]

Let \( f \in \tilde{M} \). Then

\[
f = b^*(z)h_0(w) + \sum_{j=1}^{k} \oplus z^{j-1}b(z)h_j(w), \quad h_j(w) \in H^2(\Gamma_w).
\]

By Lemma 4.5(i),

\[
T_{z^n}^*f = T_{z^{n-k}}^*(T_{z^k}^*f) = T_{z^{n-k}}^* \left( \sum_{j=0}^{k} \alpha^{(k-j)}b^*(z)h_j(w) \right)
\]

\[
= \alpha^{(n-k)} \sum_{j=0}^{k} \alpha^{(k-j)}b^*(z)h_j(w).
\]

Since \( S_zS_w = S_wS_z \), by Theorem 4.4 \( T_{z^n}^*f \in \tilde{M} \). Since \( \alpha \neq 0 \),

\[
\sum_{j=0}^{k} \alpha^{(k-j)}b^*(z)h_j(w) \in \tilde{M}.
\]

Thus \( T_{z^n}^*\tilde{M} \subset \tilde{M} \). By Theorem 4.4, \( S_zS_w = S_wS_z \). \( \square \)

**Theorem 4.8.** Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \oplus N \). Suppose that \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \), where \( q_1(z) \) is an inner function. Moreover suppose that \( q_1(z) = b_1(z)b_2(z) \), where \( b_i(z) \), \( i = 1, 2 \), are simple Blaschke products, \( b_i(z) = (z - \alpha_i)/(1 - \bar{\alpha}_iz) \), and \( \alpha_1\alpha_2 \neq 0 \). Let \( n \geq 2 \) be a positive integer. Then we have the following.

(i) If \( S_zS_w = S_wS_z \) and \( S_{z^n-1}S_w \neq S_wS_{z^n-1} \), then \( \alpha^n = \alpha^2 \) and \( \alpha_1 \neq \alpha_2 \).

(ii) If \( \alpha^n = \alpha^2 \) and \( \alpha_1 \neq \alpha_2 \), then \( S_zS_w = S_wS_z \).

**Proof.** Let \( \tilde{M} = M \oplus q_1(z)H^2 \). Suppose that \( S_zS_w = S_wS_z \) and \( S_{z^n-1}S_w \neq S_wS_{z^n-1} \). By Theorem 4.4, \( T_{z^n}^*\tilde{M} \subset \tilde{M} \) and \( T_{z^n-1}^*\tilde{M} \not\subset \tilde{M} \). By Lemma 4.5(iv),

\[
\tilde{M} \subset b_1^*(z)H^2(\Gamma_w) \oplus b_1(z)b_2(z)H^2(\Gamma_w).
\]

Then there exists \( f_0 \in \tilde{M} \) such that \( T_{z^n-1}^*f_0 \not\in \tilde{M} \), and

\[
f_0 = b_1^*(z)g_1(w) + b_1(z)b_2(z)g_2(w) \in b_1^*(z)H^2(\Gamma_w) \oplus b_1(z)b_2(z)H^2(\Gamma_w).
\]
By Lemma 4.5,
\[ T_{z_{n-1}}^* f_0 = b_1^*(\alpha_1^{(n-1)} g_1 + (1 - |\alpha_2|^2)\left(\sum_{j=0}^{n-2} \alpha_1^{(n-2-j)} \alpha_2^j \right) g_2) + \alpha_2^{(n-1)} b_1 g_2 \]
and
\[ T_{z_n}^* f_0 = b_1^*(\alpha_1^n g_1 + (1 - |\alpha_2|^2)\left(\sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \alpha_2^j \right) g_2) + \alpha_2^n b_1 g_2. \]
Since \( T_{z_{n-1}}^* f_0 \notin \tilde{M} \) and \( f_0 \in \tilde{M} \), \( T_{z_{n-1}}^* f_0 - \alpha_2^{n-1} f_0 \notin \tilde{M} \). Then
\[ b_1^*(\alpha_1^{(n-1)} - \alpha_2^{(n-1)}) g_1 + (1 - |\alpha_2|^2)\left(\sum_{j=0}^{n-2} \alpha_1^{(n-2-j)} \alpha_2^j \right) g_2 \notin \tilde{M}. \]
Hence
\[ \left(\sum_{j=0}^{n-2} \alpha_1^{(n-2-j)} \alpha_2^j \right) b_1^*(\alpha_1 - \alpha_2) g_1 + (1 - |\alpha_2|^2) g_2 \notin \tilde{M}. \]
Since \( 0 \in \tilde{M} \), \( \sum_{j=0}^{n-2} \alpha_1^{(n-2-j)} \alpha_2^j \neq 0 \), so that
\[ (4.3) \quad b_1^*(\alpha_1 - \alpha_2) g_1 + (1 - |\alpha_2|^2) g_2 \notin \tilde{M}. \]
Since \( T_{z_n}^* f_0 \in \tilde{M} \), \( T_{z_n}^* f_0 - \alpha_2^n f_0 \in \tilde{M} \). Then
\[ b_1^*(\alpha_1^n - \alpha_2^n) g_1 + (1 - |\alpha_2|^2)\left(\sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \alpha_2^j \right) g_2 \in \tilde{M}. \]
Hence
\[ (4.4) \quad \left(\sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \alpha_2^j \right) b_1^*(\alpha_1 - \alpha_2) g_1 + (1 - |\alpha_2|^2) g_2 \in \tilde{M}. \]
Now we prove (i). Suppose that \( \alpha_1 = \alpha_2 \). Then \( \alpha_1 = \alpha_2 \neq 0 \), so that
\[ \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \alpha_2^j \neq 0. \]
By (4.3) and (4.4), \( b_1^* g_2 \notin \tilde{M} \) and \( b_1^* g_2 \in \tilde{M} \). This is a contradiction. Hence \( \alpha_1 \neq \alpha_2 \).

Suppose that \( \alpha_1^n \neq \alpha_2^n \). Then \( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \alpha_2^j \neq 0 \). By (4.4),
\[ b_1^*(\alpha_1 - \alpha_2) g_1 + (1 - |\alpha_2|^2) g_2 \in \tilde{M}. \]
This contradicts (4.3), so that \( \alpha_1^n = \alpha_2^n \). Thus we get (i).

(ii) Suppose that \( \alpha_1^n = \alpha_2^n \) and \( \alpha_1 \neq \alpha_2 \). Then \( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \alpha_2^j = 0 \).

Let \( f \in \tilde{M} \). Then by (4.2), \( f = b_1^*(z) h_1(w) + b_1(z) h_2(w) \). Similarly,
we have
\[ T^n_\ast f - \alpha^n f = \left( \sum_{j=0}^{n-1} \overline{\alpha}_1^{(n-1-j)} \overline{\alpha}_2^j \right) b_1^\ast \left( (\overline{\alpha}_1 - \overline{\alpha}_2) h_1 + (1 - |\alpha_2|^2) h_2 \right). \]

Hence \( T^n_\ast f = \overline{\alpha}_2^n f \in \tilde{M} \), so that we get \( T^n_\ast \tilde{M} \subset \tilde{M} \). By Theorem 4.4, \( S_w^\ast S_w^\ast = S_w^\ast S_w^\ast \).

\textbf{Corollary 4.9.} Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \ominus N \). Suppose that \( M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z) \), where \( q_1(z) \) is an inner function. Moreover suppose that \( q_1(z) = b_1(z) b_2(z) \), where \( b_i(z) \), \( i = 1, 2 \), are simple Blaschke products, \( b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}_i z) \), and \( \alpha_1 \alpha_2 \neq 0 \). Then we have the following.

(i) If \( S_{z_2} S_w^\ast = S_w^\ast S_{z_2} \) and \( S_{z_2} S_w^\ast \neq S_w^\ast S_{z_2} \), then \( \alpha_1 + \alpha_2 = 0 \).

(ii) If \( \alpha_1 + \alpha_2 = 0 \), then \( S_{z_2} S_w^\ast = S_w^\ast S_{z_2} \).

The following is the main theorem in this section.

\textbf{Theorem 4.10.} Let \( N \) be a backward shift invariant subspace of \( H^2 \), \( N \neq \{0\} \), and \( N \neq H^2 \). Then \( S_{z_2} S_w^\ast = S_w^\ast S_{z_2} \) if and only if one of the following conditions holds.

(i) \( S_{z_2} S_w^\ast = S_w^\ast S_{z_2} \).

(ii) \( S_{z_2} S_w = 0 \).

(iii) There exist two simple Blaschke products \( b_1(z) \) and \( b_2(z) \), \( b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}_i z) \), \( 0 < |\alpha_i| < 1 \), such that \( N \subset H^2 \ominus b_1(z) b_2(z) H^2 \) and \( \alpha_1 + \alpha_2 = 0 \).

A backward shift invariant subspace \( N \) satisfying condition (i) is given by Theorem A. Also \( N \) satisfying condition (ii) is given by Theorem 2.2.

\textbf{Proof of Theorem 4.10.} Suppose that \( S_{z_2} S_w^\ast = S_w^\ast S_{z_2} \). Moreover suppose that \( S_{z_2} S_w^\ast \neq S_w^\ast S_{z_2} \). By Theorem 3.1, \( M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z) \) for an inner function \( q_1(z) \) such that either \( q_1(z) = b_1(z) \) or \( q_1(z) = b_1(z) b_2(z) \), where \( b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}_i z) \), \( |\alpha_i| < 1 \). If \( q_1(z) = b_1(z) \), by Corollary 4.6, (i) holds.

Suppose that \( q_1(z) = b_1(z) b_2(z) \). Moreover suppose that \( \alpha_1 + \alpha_2 = 0 \). Then \( q_1(z) = z^2 \). Hence \( z^2 H^2 \subset M \), so that \( N \subset H^2(\Gamma_w) \oplus z H^2(\Gamma_w) \). Then by Theorem 2.2, \( S_{z_2} S_w = 0 \). Thus (ii) holds.

Suppose that \( \alpha_1 \neq 0 \) and \( \alpha_2 = 0 \). Then \( q_1(z) = z b_1(z) \). By Corollary 4.7, (i) holds.

Suppose that \( \alpha_1 \neq 0 \) and \( \alpha_2 \neq 0 \). Then by Corollary 4.9(i), we get \( \alpha_1 + \alpha_2 = 0 \). Hence (iii) holds.
Next, we prove the converse assertion. Suppose that one of conditions (i), (ii), and (iii) holds. If (i) holds, then trivially $S_zS_w^* = S_w^*S_z$.

If (ii) holds, by Corollary 2.3 we have $S_zS_w^* = S_w^*S_z$.

Suppose that (iii) holds. Then $b_1(z)b_2(z)H^2(\Gamma_z) \subset M$. Hence $M \cap H^2(\Gamma_z)$ equals to either $b_1(z)H^2(\Gamma_z)$, or $b_2(z)H^2(\Gamma_z)$, or $b_1(z)b_2(z)H^2(\Gamma_z)$.

By Corollary 4.6, $S_zS_w^* = S_w^*S_z$ for the first two cases. Hence $S_zS_w^* = S_w^*S_z$. For the last case, by Corollary 4.9(ii), $S_zS_w^* = S_w^*S_z$.

\begin{example}
Example 4.11. We give an example of a backward shift invariant subspaces (i), (ii), and (iii) holds. If (i) holds, then trivially $S_zS_w^* = S_w^*S_z$.

Next, we prove the converse assertion. Suppose that one of conditions (i), (ii), and (iii) holds. If (i) holds, then trivially $S_zS_w^* = S_w^*S_z$.

Suppose that (iii) holds. Then $b_1(z)b_2(z)H^2(\Gamma_z) \subset M$. Hence $M \cap H^2(\Gamma_z)$ equals to either $b_1(z)H^2(\Gamma_z)$, or $b_2(z)H^2(\Gamma_z)$, or $b_1(z)b_2(z)H^2(\Gamma_z)$.

By Corollary 4.6, $S_zS_w^* = S_w^*S_z$ for the first two cases. Hence $S_zS_w^* = S_w^*S_z$. For the last case, by Corollary 4.9(ii), $S_zS_w^* = S_w^*S_z$.

\end{example}

\begin{problem}
Problem 4.12. Characterize backward invariant subspaces $N$ of $H^2$ satisfying $S_zS_w^* = S_w^*S_z$, $S_zS_w^* \neq S_w^*S_z$, and $S_zS_w^* \neq 0$. Let $q_1(z) = b_1(z)b_2(z)$, where $b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}_iz)$, $\alpha_1 + \alpha_2 = 0$, and $\alpha_1 \neq 0$. Let $q_2(w)$ be a non-constant inner function. Let $M = q_1(z)H^2 \oplus b_1(z)b_2(z)q_2(w)H^2(\Gamma_w)$. Then $M$ is an invariant subspace and $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$.

Let $N = H^2 \oplus M$. By Theorem 4.10, $S_zS_w^* = S_w^*S_z$. We have $\tilde{M} = M \oplus q_1(z)H^2 = b_1(z)b_2(z)q_2(w)H^2(\Gamma_w)$. Since

$$T_z^*b_1(z)b_2(z)q_2(w) = (1 - |\alpha_1|^2)b_1^*(z)q_2(w) + \overline{\alpha}_1b_1(z)b_2(z)q_2(w),$$

$T_z^*b_1(z)b_2(z)q_2(w) \notin \tilde{M}$. By Theorem 2.2, $S_zS_w^* \neq S_w^*S_z$. By Theorem 2.2, $S_zS_w^* = 0$.

We leave the following problem for the reader.

\begin{question}
Problem 4.12. Characterize backward invariant subspaces $N$ of $H^2$ satisfying $S_zS_w^* = S_w^*S_z$, for $n \geq 3$.

\end{question}

Acknowledgement. The authors would like to thank the referee for many comments improving the original manuscript.

\begin{thebibliography}{9}
\addcontentsline{toc}{section}{References}


\end{thebibliography}


**Department of Mathematics, Niigata University, Niigata, 950-2181, Japan**

E-mail address: izuchi@math.sc.niigata-u.ac.jp

**Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan**

E-mail address: nakazi@math.sci.hokudai.ac.jp

**Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan**

E-mail address: seto@math.sci.hokudai.ac.jp