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ABSTRACT. In the previous paper, we give a characterization of backward shift invariant subspaces of the Hardy space in the bidisc which satisfy the doubly commuting condition $S_z S_w^* = S_w^* S_z$ for the compression operators $S_z$ and $S_w$. In this paper, we give a characterization of backward shift invariant subspaces satisfying $S_z^2 S_w^* = S_w^* S_z^2$.

1. Introduction.

Let $\Gamma^2$ be the 2-dimensional unit torus. We denote by $(z, w) = (e^{i\theta}, e^{i\phi})$ the variables in $\Gamma^2 = \Gamma_z \times \Gamma_w$. Let $L^2 = L^2(\Gamma^2)$ be the usual Lebesgue space on $\Gamma^2$ with the norm $\|f\|_2 = (\int_{\Gamma^2} |f(e^{i\theta}, e^{i\phi})|^2 d\theta d\phi/(2\pi)^2)^{1/2}$. Then $L^2$ is a Hilbert space with the usual inner product. For $f \in L^2$, the Fourier coefficients are given by

$$\hat{f}(n, m) = \int_{\Gamma^2} f(e^{i\theta}, e^{i\phi}) e^{-int} e^{-im\phi} d\theta d\phi/(2\pi)^2 = \langle f, z^n w^m \rangle.$$ 

Let $H^2 = H^2(\Gamma^2)$ be the Hardy space on $\Gamma^2$, that is,

$$H^2 = \{f \in L^2; \hat{f}(n, m) = 0 \text{ if } n < 0 \text{ or } m < 0 \}.$$ 

For $f \in H^2$, we can write $f$ as

$$f = \sum_{i,j=0}^{\infty} \oplus a_{i,j} z^i w^j, \text{ where } \sum_{i,j=0}^{\infty} |a_{i,j}|^2 < \infty.$$ 

Let $P$ be the orthogonal projection from $L^2$ onto $H^2$. For a closed subspace $M$ of $L^2$, we denote by $P_M$ the orthogonal projection from $L^2$ onto $M$. For a function $\psi \in L^\infty$, let $L_{\psi} f = \psi f$ for $f \in L^2$. The Toeplitz operator $T_{\psi}$ is defined by $T_{\psi} f = PL_{\psi} f$ for $f \in H^2$. It is well known that $T_{\psi}^* = T_{\overline{\psi}}$ and $T_{\psi}^* T_{w^m} = T_{w^m} T_{\psi}^*$ for $n, m \geq 1$. A function $f \in H^2$ is called inner if $|f| = 1$ on $\Gamma^2$ almost everywhere. A closed subspace $M$ of $H^2$ is called invariant if $z M \subset M$ and $w M \subset M$. In one variable

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case, an invariant subspace $M$ of $H^2(\Gamma)$ has a form $M = qH^2(\Gamma)$, where $q$ is inner. This is the well known Beurling theorem [B]. In two variable case, the structure of invariant subspaces of $H^2$ is very complicated, see [AC], [DY], [Na1], and [R].

Let $M$ be an invariant subspace of $H^2$. Then $T_z^*(H^2 \ominus M) \subset (H^2 \ominus M)$ and $T_z^*(H^2 \ominus M) \subset (H^2 \ominus M)$. A closed subspace $N$ of $H^2$ is called \textit{backward shift invariant} if $H^2 \ominus N$ is invariant. In [Y], the first author and Yang studied backward shift invariant subspaces $N$ on which $T_z^*$ is strictly contractive. See [CR] and [S] for studies of backward shift invariant subspaces on the unit circle $\Gamma$.

Let $M$ be an invariant subspace of $H^2$ and $\psi \in L^\infty$. Let $V_\psi$ be the operator on $M$ defined by $V_\psi = P_M L_\psi|_M$. Then $V_z = T_z$ and $V_z^* = V_z$ on $M$. In [M], Mandrekar proved that $V_z V_w^* = V_w^* V_z$ on $M$ if and only if $M$ is Beurling type, that is, $M = qH^2$ for some inner function $q$ in $H^\infty$, see also [CS] and [Na2].

For $\psi \in L^\infty$, let $S_\psi = P_N L_\psi|_N$. Then we have $S_\psi^* = S_\psi$ and $S_z^* = T_z^*$ on $N$. In the previous paper [INS], we characterized backward shift invariant subspaces $N$ which satisfy the condition $S_z^* S_w^* = S_w^* S_z$ on $N$ as follows.

\textbf{Theorem A.} Let $N$ be a backward shift invariant subspace of $H^2$ and $N \neq H^2$. Then $S_z^* S_w^* = S_w^* S_z$ on $N$ if and only if $N$ has one of the following forms:

(i) $N = H^2 \ominus q_1(z) H^2$,
(ii) $N = H^2 \ominus q_2(w) H^2$,
(iii) $N = (H^2 \ominus q_1(z) H^2) \cap (H^2 \ominus q_2(w) H^2)$,

where $q_1(z)$ and $q_2(w)$ are one variable inner functions.

This paper is a continuation of previous papers [IN] and [INS]. Here we are interesting in backward shift invariant subspaces $N$ of $H^2$ satisfying that $S_z^n S_w^m = S_w^m S_z^n$ for given positive integers $n$ and $m$. Up to now, we can not give a complete characterization of $N$ satisfying $S_z^n S_w^m = S_w^m S_z^n$. If $S_z S_w^* = S_w^* S_z$, then trivially $S_z^n S_w^m = S_w^m S_z^n$. But the converse is not true. In this paper, we concentrated on the case $m = 1$. One reason is the work of this paper deeply concerns with the problem of the normality of the operators $S_z^n$. For many backward shift invariant subspaces $N$, $S_z^n$ are not normal operators, see [Y]. If $S_z^n$ is normal, since $S_z^n S_w = S_w S_z^n$, by the Fuglede-Putnam theorem we have $S_z^n S_w^* = S_w^* S_z^n$. So, the results obtained in this paper will be a big help for the above mentioned problem. We will discuss on this subject in the forthcoming paper.
In Section 2, we give characterizations of $N$ satisfying $S_{z^n}S_{w^m}^* = 0$ and $S_{w^m}^*S_{z^n} = 0$, respectively. If $S_{z^n}S_{w^m}^* = 0$, then $S_{w^m}^*S_{z^n} = 0$. The converse is not true. In Section 3, we study $N$ satisfying $S_{z^n}S_{w^m}^* = S_{w^m}^*S_{z^n}$, and give a necessary condition for $S_{z^n}S_{w^m}^* = S_{w^m}^*S_{z^n}$. In Section 4, we study $N$ satisfying $S_{z^n}^*S_{w^m} = S_{w^m}S_{z^n}$. We gave a complete characterization of such $N$. In [INS], we gave two different type of proofs of Theorem A. One of them is used in Section 3 and the other is used in Section 4.

Let $H^2(\Gamma_z)$ and $H^2(\Gamma_w)$ be the Hardy spaces on the unit circle in variables $z$ and $w$, respectively. We think that $H^2(\Gamma_z)$ and $H^2(\Gamma_w)$ are the same. For a subset $E$ of $H^2$, we denote by $\overline{E}$ the closed linear span of $E$. A function $b(z) = (z - \alpha)/(1 - \overline{\alpha}z)$, $|\alpha| < 1$, is called a simple Blaschke product.

2. $S_{z^n}S_{w^m}^* = 0$ or $S_{w^m}^*S_{z^n} = 0$.

Let $n$ and $m$ be positive integers. In this section, we study backward shift invariant subspaces $N$ of $H^2$ satisfying $S_{z^n}S_{w^m} = 0$ and $S_{w^m}^*S_{z^n} = 0$, respectively.

**Lemma 2.1.** Let $N$ be a backward shift invariant subspace of $H^2$. Then we have the following.

(i) $S_{z^n} = S_{z^n}$.

(ii) $S_{w^m}S_{z^n} = S_{z^n}S_{w^m}$ and $S_{z^n}^*S_{w^m} = S_{w^m}^*S_{z^n}$.

(iii) If $S_{z^n}^*S_{w^m}^*N \neq \{0\}$, then there exists $f \in N$ such that $(S_{z^n}^*S_{w^m}^*f)(0,0) \neq 0$.

**Proof.** All assertions are not difficult to prove. \qed

The following theorem says that the structure of backward shift invariant subspaces satisfying $S_{z^n}S_{w^m} = 0$ is simple.

**Theorem 2.2.** Let $N$ be a backward shift invariant subspace of $H^2$. Then $S_{z^n}S_{w^m} = 0$ if and only if $N$ satisfies one of the following conditions;

(i) $N \subseteq \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$.

(ii) $N \subseteq \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$.

**Proof.** Suppose that $S_{z^n}S_{w^m} = 0$. Then

\begin{equation}
S_{w^m}^*N \perp S_{z^n}^*N.
\end{equation}

Since $N$ is backward shift invariant, if $S_{w^m}^*N = \{0\}$ then $N$ satisfies condition (i). If $S_{z^n}^*N = \{0\}$, then $N$ satisfies (ii).

Next, suppose that

\begin{equation}
S_{w^m}^*N \neq \{0\} \text{ and } S_{z^n}^*N \neq \{0\}.
\end{equation}
We shall lead a contradiction. By (2.1), $S^*_n S^*_m S^*_m N \perp S^*_n S^*_m N$. By Lemma 2.1(ii), $S^*_n S^*_m N = S^*_n S^*_m N = \{0\}$. Then

\[(2.3) \quad S^*_m N \subset \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)\]

and

\[(2.4) \quad S^*_w N \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w).\]

By (2.2) and (2.3), there exists a nonnegative integer $j, 0 \leq j \leq m - 1$, such that

\[(2.5) \quad \{0\} \neq S^*_w S^*_m N \subset H^2(\Gamma_z).\]

By Lemma 2.1(iii), there exists $g \in N$ such that

\[(2.6) \quad (S^*_w S^*_m g)^\gamma(0, 0) \neq 0.\]

Also by (2.2) and (2.4), there exist $f \in N$ and a nonnegative integer $i, 0 \leq i \leq n - 1$, such that

\[(2.7) \quad S^*_z S^*_w f \in H^2(\Gamma_w)\]

and

\[(2.8) \quad (S^*_z S^*_w f)^\gamma(0, 0) \neq 0.\]

Then

\[
0 = \langle S^*_w S^*_m f, S^*_z S^*_w f, S^*_z S^*_w g \rangle \quad \text{by (2.1)} \\
= \langle S^*_z S^*_w f, S^*_z S^*_w g \rangle \quad \text{by Lemma 2.1(ii)} \\
= (S^*_z S^*_w f)^\gamma(0, 0) (S^*_w S^*_z g)^\gamma(0, 0) \quad \text{by (2.5) and (2.7)} \\
\neq 0 \quad \text{by (2.6) and (2.8).}
\]

This is a desired contradiction.

The converse is trivial. \(\square\)

**Corollary 2.3.** Let $N$ be a backward shift invariant subspace of $H^2$. Then $S^*_n S^*_m = 0$ if and only if either $S^*_n = 0$ or $S^*_m = 0$. Hence if $S^*_n S^*_m = 0$, then $S^*_n S^*_m = 0$.

**Lemma 2.4.** Let $M_1$ and $M_2$ be closed subspaces of $H^2$ such that

\[
M_1 \subset \sum_{j=0}^{m} \oplus w^j H^2(\Gamma_z) \quad \text{and} \quad M_2 \subset \sum_{i=0}^{n} \oplus z^i H^2(\Gamma_w).\]

Then $M_1 + M_2$ is closed.
Proof. We denote by \((z^jw^j)_{M_1}\) and \((z^jw^j)_{M_2}\) the orthogonal projections of \(z^jw^j\) to the spaces \(M_1\) and \(M_2\), respectively. Let

\[
M_1' = M_1 \ominus \left( \left\{ (z^jw^j)_{M_1}; 0 \leq i \leq n, 0 \leq j \leq m \right\} \right)
\]

and

\[
M_2' = M_2 \ominus \left( \left\{ (z^jw^j)_{M_2}; 0 \leq i \leq n, 0 \leq j \leq m \right\} \right).
\]

Then \(M_1'\) and \(M_2'\) are closed subspaces of \(H^2\),

\[
M_1' \subset z^{n+1} \left( \sum_{j=0}^{m} \oplus w^j H^2(\Gamma_z) \right), \quad M_2' \subset w^{m+1} \left( \sum_{i=0}^{n} \oplus z^i H^2(\Gamma_w) \right),
\]

and

\[
M_1' + M_2' \perp \left\{ z^jw^j; 0 \leq i \leq n, 0 \leq j \leq m \right\}.
\]

Since

\[
z^{n+1} \left( \sum_{j=0}^{m} \oplus w^j H^2(\Gamma_z) \right) \perp w^{m+1} \left( \sum_{i=0}^{n} \oplus z^i H^2(\Gamma_w) \right),
\]

\(M_1' + M_2' = M_1' \oplus M_2'\) is closed. Hence

\[
M_1 + M_2 = M_1' + M_2' + \left\{ (z^jw^j)_{M_1}, (z^jw^j)_{M_2}; 0 \leq i \leq n, 0 \leq j \leq m \right\}
\]

is closed. \(\square\)

Theorem 2.5. Let \(N\) be a backward shift invariant subspace of \(H^2\). Then \(S_{w^n}S_{z^n} = 0\) if and only if

(i) \(N \ominus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \subset N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)\).

Proof. Suppose that \(S_{w^n}S_{z^n} = 0\). Then \(S_{z^n}N \subset \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)\).

Since \(S_{z^n}S_{w^n} = 0\), \(S_{w^n}N \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\). First, we prove the following

\[
N = \left( N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \right) + \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right).
\]

Let

\[
K = \left( N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \right) + \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right),
\]

By Lemma 2.4, \(K\) is closed and \(N = K \oplus (N \ominus K)\). To prove (i), it is sufficient to prove \(N \ominus K = \{0\}\). We have \(N \ominus K \subset N \cap (S_{z^n}N)^\perp \cap (S_{w^n}N)^\perp\). Let \(f \in N \ominus K\). Then \(f \perp S_{z^n}N\), so that \(S_{z^n}f \perp N\). Since
$S^*_{zn} f \in N$, $S^*_{zn} f = 0$. Hence $f \in \sum_{i=0}^{m-1} \oplus w^j H^2(\Gamma_w)$. By (2.10), $f \in K$. This shows $f = 0$, so that $N \oplus K = \{0\}$. Thus we get (2.9).

Let

$$(2.11) \quad N_1 = \left\{ f \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) ; f \perp N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right\}.$$ 

Then $N_1$ is a closed subspace and

$$N_1 \oplus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \subset N.$$ 

If the equality holds in the above, (i) holds. So we assume that

$$N_1 \oplus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \subsetneq N.$$ 

We shall lead a contradiction. Let

$$(2.12) \quad N_2 = N \ominus \left( N_1 \oplus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right).$$ 

Then $N_2 \neq \{0\}$ and $N = N_1 \oplus N_2 \oplus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right)$.

Let $g \in N_2$ be such that $g \neq 0$. We shall prove that

$$(2.13) \quad g \notin N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \quad \text{and} \quad g \notin N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z).$$ 

The fact $g \notin N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$ is trivial. Suppose that $g \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$. By (2.11), $g \in N_1$. Since $g \in N_2$, by (2.12) we have $g \perp N_1$. Hence $g = 0$. This is a contradiction. Thus we get (2.13).

Next, we shall prove that

$$(2.14) \quad S^*_{zn} g \perp N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z).$$ 

To prove this, suppose not. Then $S^*_{zn} g \perp N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$. By (2.10), $g = g_1 + g_2$, where $g_1 \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$ and $g_2 \in N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$. Then $S^*_{zn} g = S^*_{zn} g_1 \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$. Therefore $S^*_{zn} g = 0$, so that $g \in N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$. This contradicts (2.13). Thus we get (2.14).

By (2.14), there exists $h_0$ such that $h_0 \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$ and $\langle g, S^*_{zn} h_0 \rangle = \langle S^*_{zn} g, h_0 \rangle \neq 0$. Since $S^*_{zn} h_0 \in N$, by (2.12) we have $S^*_{zn} h_0 = h_1 \oplus h_2 \oplus h_3$, where $h_1 \in N_1, h_2 \in N_2$, and $h_3 \in N_3.
Let $h$ satisfy $N \cap \sum_{i=0}^{n-1} z^i H^2(\Gamma_w)$. Since $g \in N_2$ and $\langle g, S_{z^n} h_0 \rangle \neq 0$, we have $h_2 \neq 0$. Since $z^n h_0 \perp \sum_{i=0}^{n-1} z^i H^2(\Gamma_w),
\]
P_N z^n h_0 = S_{z^n} h_0 \perp N \cap \sum_{i=0}^{n-1} z^i H^2(\Gamma_w).

Thus we get $h_3 = 0$. By (2.12), $S_{z_m} N_1 = \{0\}$. Hence $S_{z_m} S_{z^n} h_0 = S_{z_m} h_2$. By (2.13) and $h_2 \in N_2$, $h_2 \notin \sum_{j=0}^{m-1} w^j H^2(\Gamma_z)$. This implies that $S_{z_m} h_2 \neq 0$. Hence $S_{z_m} S_{z^n} \neq 0$. This contradicts our starting assumption. Thus we have (i).

To prove the converse, suppose that condition (i) holds. Then $N = \left( N \cap \sum_{i=0}^{n-1} z^i H^2(\Gamma_w) \right) \oplus L$, where $L \subset N \cap \sum_{i=0}^{m-1} w^i H^2(\Gamma_z)$. Let $F = F_1 + F_2 \in N$, where $F_1 \in N \cap \sum_{i=0}^{n-1} z^i H^2(\Gamma_w)$ and $F_2 \in L$. Since $z^n F \perp N \cap \sum_{i=0}^{n-1} z^i H^2(\Gamma_w), S_{z^n} F \in L$. Hence $S_{z_m} S_{z^n} F = 0$. Thus we get $S_{z_m} S_{z^n} = 0$.

By Theorem 2.2, the structure of backward shift invariant subspaces $N$ satisfying $S_{z_m} S_{z^n} = 0$ is simple. By Theorem 2.5, the structure of backward shift invariant subspaces $N$ satisfying $S_{z_m} S_{z^n} = 0$ is not so simple. When $n = m = 1$, we have the following.

**Theorem 2.6.** Let $N$ be a backward shift invariant subspace of $H^2$. Then $S_{z} S_{z^n} = 0$ if and only if $N$ has one of the following forms;

(i) $N = H^2(\Gamma_z) \oplus q(z) H^2(\Gamma_z)$ for some inner function $q(z)$.

(ii) $N = H^2(\Gamma_w) \oplus q(w) H^2(\Gamma_w)$ for some inner function $q(w)$.

(iii) Either $N = H^2(\Gamma_z) + H^2(\Gamma_w)$, or $N = H^2(\Gamma_z) \oplus q_2(w) H^2(\Gamma_w)$, or $N = (H^2(\Gamma_z) \oplus q_1(z) H^2(\Gamma_z)) + H^2(\Gamma_w)$, where $q_1(z)$ and $q_2(w)$ are inner functions.

(iv) $N = (H^2(\Gamma_z) \oplus q_1(z) H^2(\Gamma_z)) + (H^2(\Gamma_w) \oplus q_2(w) H^2(\Gamma_w))$, where $q_1(z), q_2(w)$ are nonconstant inner functions and $\hat{q}_1(0) \hat{q}_2(0) = 0$.

In (iii) and (iv), since $1 \in N$, we may take $q_1$ and $q_2$ as $\hat{q}_1(0) = \hat{q}_2(0) = 0$.

**Proof.** By Theorem 2.2, $S_{z} S_{z^n} = 0$ if and only if either (i) or (ii) holds. By Theorem 2.5, $S_{z} S_{z^n} = 0$ if and only if

$$N \ominus (N \cap H^2(\Gamma_w)) \subset N \cap H^2(\Gamma_z).$$

If either (i) or (ii) holds, by Corollary 2.3 we have $S_{z} S_{z^n} = 0$. Suppose that $N$ satisfies either (iii) or (iv). Then clearly $1 \in N$. Since $N$ has a special form, it is not difficult to see that

$$N \ominus (N \cap H^2(\Gamma_w)) = \{ f \in N \cap H^2(\Gamma_z); \hat{f}(0) = 0 \}.$$  

Hence (2.15) holds.
Next, suppose that (2.15) holds. Then we have
\begin{equation}
N = (N \cap H^2(\Gamma_z)) + (N \cap H^2(\Gamma_w)).
\end{equation}
If either $N \cap H^2(\Gamma_z) = \{0\}$ or $N \cap H^2(\Gamma_w) = \{0\}$, then $S_z S_w^* = 0$, and by Corollary 2.3, $S_z^* S_w = 0$. Hence either (i) or (ii) holds. Suppose that $N \cap H^2(\Gamma_z) \neq \{0\}$ and $N \cap H^2(\Gamma_w) \neq \{0\}$. We shall prove $1 \in N$. To prove this, suppose that $1 \notin N$. Let $1_w$ be the orthogonal projection of 1 to $N \cap H^2(\Gamma_w)$. Then $1_w \notin H^2(\Gamma_z)$. Since $N \cap H^2(\Gamma_z) \neq \{0\}$, there exists $f \in N \cap H^2(\Gamma_z)$ such that $\hat{f}(0) \neq 0$. Let $f_1 = f - \hat{f}(0)1_w \in N$. Then $f_1 \notin H^2(\Gamma_z)$. Let $h \in N \cap H^2(\Gamma_w)$. Since $f \in H^2(\Gamma_z)$, $f - \hat{f}(0) \perp h$. Since $1 - 1_w \perp N \cap H^2(\Gamma_w)$,
\begin{equation*}
\langle f_1, h \rangle = \langle \hat{f}(0)(1 - 1_w) + f - \hat{f}(0), h \rangle = 0.
\end{equation*}
Hence $f_1 \in N \ominus (N \cap H^2(\Gamma_w))$. Thus (2.15) does not hold. Therefore $1 \in N$. Since $N \cap H^2(\Gamma_z)$ and $N \cap H^2(\Gamma_w)$ are nonzero backward shift invariant subspaces, by (2.16) $N$ has one of forms in (iii) and (iv). \qed

3. $S_z S_w^* = S_w^* S_z^n$.

The following is the main theorem in this section.

**Theorem 3.1.** Let $N$ be a backward shift invariant subspace of $H^2$, $N \neq \{0\}$, and $N \neq H^2$. Let $M = H^2 \ominus N$ and $n \geq 2$ be a positive integer. If $S_z S_w^* = S_w^* S_z^n$, then one of the following conditions holds;

(i) $S_z S_w^* = S_w^* S_z$,

(ii) $M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z)$ for an inner function $q_1(z)$ satisfying $q_1(z) = \prod_{j=1}^{k} b_j(z), 1 \leq k \leq n$, where $b_i$ are simple Blaschke products.

The following lemma is a generalization of [IN], and the idea of the proof is the same.

**Lemma 3.2.** Let $N$ be a backward shift invariant subspace of $H^2$ and $M = H^2 \ominus N$. Let $n$ be a positive integer. Then $S_z^n S_w^* = S_w^* S_z^n$ if and only if
\begin{equation*}
M \oplus \left( z^n M \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right) \subset (M \cap H^2(\Gamma_z)) \oplus w M.
\end{equation*}

**Proof.** The operators $T_{z^n}$ and $T_w^*$ on $H^2$ have the matrix forms as
\begin{equation*}
T_{z^n} = \begin{pmatrix} * & P_M T_{z^n} | N \\ 0 & S_{z^n} \\
\end{pmatrix}, \quad T_w^* = \begin{pmatrix} * & P_N T_w^* | M \\ 0 & S_w^* \\
\end{pmatrix} \quad \text{on} \quad H^2 = \begin{pmatrix} M \oplus N \\
\end{pmatrix}.
\end{equation*}
Set $A = P_M T_{z^n}|_N$ and $B = P_N T_{w^n}^*|_M$. Since $T_{z^n} S_{w^n}^* = S_{w^n}^* S_{z^n}$ if and only if $BA = 0$. We have $T_{w^n}^* (M \ominus wM) \subset N$. For $f \in H^2$, $T_{w^n}f = 0$ if and only if $f \in H^2(\Gamma_z)$. Hence

$$\ker B = \{f \in M; T_{w^n}f \in M\} = \{f \in M \ominus wM; T_{w^n}f = 0\} \oplus wM = (M \cap H^2(\Gamma_z)) \oplus wM.$$ 

We denote by $[\text{ran } A]$ the closed range of $A$. Let $A_1 = P_M T_{z^n} P_N$ on $H^2$. Then $[\text{ran } A] = [\text{ran } A_1]$. Since $A_1^* = P_N T_{z^n}^* P_M$, we get

$$\ker A_1^* = \{f \in M; T_{z^n}^* f \in M\} \oplus N = (M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)) \oplus z^n M \oplus N.$$ 

Hence

$$[\text{ran } A] = [\text{ran } A_1] = H^2 \ominus \ker A_1^* = M \ominus \left( z^n M \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right).$$ 

Since $BA = 0$ if and only if $[\text{ran } A] \subset \ker B$, we get our assertion. $\square$

**Proof of Theorem 3.1.** Suppose that $S_{z^n} S_{w^n}^* = S_{w^n}^* S_{z^n}$. By Lemma 3.2,

$$M \ominus \left( z^n M \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right) \subset (M \cap H^2(\Gamma_z)) \oplus wM.$$ 

Let

$$K_0 = M \ominus \left( z^n M \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right).$$ 

Then

(3.1) \quad $K_0 \subset (M \cap H^2(\Gamma_z)) \oplus wM$

and

(3.2) \quad $M \ominus z^n M = K_0 \ominus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right).$

Since $M = \sum_{s=0}^{\infty} \oplus z^{ns}(M \ominus z^n M)$,

(3.3) \quad $K_0 \perp \sum_{s=0}^{\infty} \oplus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right)$

and

(3.4) \quad $M = \left( \sum_{s=0}^{\infty} \oplus z^{ns} K_0 \right) \oplus \left( \sum_{s=0}^{\infty} \oplus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right).$
First, suppose that $K_0 = \{0\}$. In this case, we shall show that condition (i) holds. By (3.2),
\[
\sum_{i=0}^{n-1} \ominus z^i (M \ominus z^M) = M \ominus z^n M = M \cap \sum_{i=0}^{n-1} \ominus z^i H^2(\Gamma_w).
\]
Let $f \in M \ominus zM$. Then $f \in M \ominus z^n M$ and $f = \sum_{i=0}^{n-1} \ominus z^i h_i(w)$, where $h_i(w) \in H^2(\Gamma_w)$. Since $f \in M \ominus zM$,
\[
z^{n-1}f = \sum_{i=0}^{n-1} \ominus z^{n-1+i} h_i(w) \in M \ominus z^n M.
\]
Hence $f = h_0(w)$, so that $M \ominus zM = M \cap H^2(\Gamma_w)$. By Lemma 3.2 (for the case of $n = 1$), we get $S_z S_w = S_w^* S_z$.

Next, suppose that $K_0 \neq \{0\}$. The proof of this case is a little bit long, so we devide into several steps.

**Step 1.** In this step, we shall prove that there is a nonconstant inner function $q_1(z)$ such that
\[
K_0 \subset q_1(z) H^2(\Gamma_z) \oplus wM.
\]
First we prove that $K_0 \not\subset wM$. To prove this, suppose that $K_0 \subset wM$. Let $F \in K_0$. Then $F = wF$ for some $F \in M$. We shall prove that $f \in K_0$. We have
\[
\langle f, \left( \sum_{s=1}^{\infty} \ominus z^{ns} K_0 \right) \ominus \left( \sum_{s=0}^{\infty} \ominus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \ominus z^i H^2(\Gamma_w) \right) \right) \rangle
\]
\[
= \langle wF, w \left( \sum_{s=1}^{\infty} \ominus z^{ns} K_0 \ominus \left( \sum_{s=0}^{\infty} \ominus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \ominus z^i H^2(\Gamma_w) \right) \right) \right) \rangle
\]
\[
= \langle F, z^n w \left( \sum_{s=1}^{\infty} \ominus z^{n(s-1)} K_0 \right) \rangle \quad \text{by (3.3)}
\]
\[
= 0,
\]
where the last equality follows from $w \sum_{s=1}^{\infty} \ominus z^{n(s-1)} K_0 \subset M$ and (3.2). Hence by (3.4), we get $f \in K_0$. Therefore for every positive integer $p$, we have $F = w^p f_p$ for some $f_p \in K_0$. This leads $F = 0$. This is a contradiction. Thus we get $K_0 \not\subset wM$.

Hence by (3.1), $M \cap H^2(\Gamma_z) \neq \{0\}$. By the Beurling theorem,
\[
M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z),
\]
where $q_1(z)$ is inner. Then $q_1(z) \in M$ and
\[
q_1(z) H^2(\Gamma_z) \perp wM.
\]
If \( q_1(z) \) is constant, we have \( M = H^2 \), so that \( N = \{0\} \). This contradicts our assumption. Hence \( q_1(z) \) is a nonconstant inner function. By \((3.1)\) and \((3.6)\), we get \((3.5)\).

**Step 2.** In this step, we prove

\[
(3.8) \quad K_0 \subset q_1(z) \left( \sum_{i=0}^{n-1} \odot z^i H^2(\Gamma_w) \right).
\]

Let \( G \in K_0 \). Then by \((3.5)\), \( G = q_1(z)h(z) \oplus wg \), where \( h(z) \in H^2(\Gamma_z) \) and \( g \in M \). Write

\[
h(z) = \left( \sum_{i=0}^{n-1} \odot a_i z^i \right) \oplus z^n h_0(z).
\]

Then

\[
G = q_1(z) \left( \sum_{i=0}^{n-1} \odot a_i z^i \right) \oplus q_1(z) z^n h_0(z) \oplus wg.
\]

By \((3.6)\), \( q_1(z) z^n h_0(z) \in z^n M \). Since \( G \in K_0 \subset M \ominus z^n M \), we have \( h_0(z) = 0 \). Hence

\[
(3.9) \quad G = q_1(z) \left( \sum_{i=0}^{n-1} \odot a_i z^i \right) \oplus wg.
\]

Here we prove that

\[
(3.10) \quad g \in K_0.
\]

Since \( G = q_1(z)h(z) \oplus wg \), we have

\[
\left< g, \sum_{s=1}^{\infty} \odot z^{ns} K_0 \right> = \left< wg, w \sum_{s=1}^{\infty} \odot z^{ns} K_0 \right>
\]

\[
= \left< wg, w \sum_{s=1}^{\infty} \odot z^{ns} K_0 \right> + \left< q_1 h, w \sum_{s=1}^{\infty} \odot z^{ns} K_0 \right> \quad \text{by \((3.7)\)}
\]

\[
= \left< G, w \sum_{s=1}^{\infty} \odot z^{ns} K_0 \right>
\]

\[
= \left< G, z^n w \sum_{s=1}^{\infty} \odot z^{n(s-1)} K_0 \right>
\]

\[
= 0 \quad \text{by \((3.2)\)}.
\]
We also have
\[ \left< g, \sum_{s=0}^{\infty} \oplus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right> = \left< wg, w \left( \sum_{s=0}^{\infty} \oplus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right) \right> \]
\[ = \left< G, \sum_{s=0}^{\infty} \oplus z^{ns} w \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right> \text{ by } (3.7) \]
\[ = 0 \text{ by } (3.3). \]
Hence by (3.4), we get (3.10).

Applying (3.9) and (3.10) infinitely many times, we have
\[ G = q_1(z) \left( \sum_{i=0}^{n-1} \oplus a_i z^i \right) \oplus wq_1(z) \left( \sum_{i=0}^{n-1} \oplus b_i z^i \right) \oplus \cdots \]
\[ \in q_1(z) \left( \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right). \]
Therefore we get (3.8).

Step 3. In this step, we study functions in \( M \ominus zM \) and the inner function \( q_1(z) \). By (3.8), there is a closed subspace \( L \) such that
\[ K_0 = q_1(z)L \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right). \]
Then by (3.2),
\[ M \ominus z^n M = q_1(z)L \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right). \]
Since \( K_0 \neq \{0\}, L \neq \{0\} \). We have \( M \ominus z^n M = \sum_{i=0}^{n-1} \oplus z^i(M \ominus zM) \). Hence \( M \ominus zM \neq \{0\} \). Let \( F \in M \ominus zM \) be such that \( F \neq 0 \). Then
\[ (3.11) \quad F = \left( q_1 \sum_{i=0}^{n-1} \oplus z^i f_i \right) \oplus \sum_{i=0}^{n-1} \oplus z^i g_i, \]
where \( f_i, g_i \in H^2(\Gamma_w) \),
\[ (3.12) \quad q_1 \sum_{i=0}^{n-1} \oplus z^i f_i \perp M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w), \]
and
\[ (3.13) \quad \sum_{i=0}^{n-1} \oplus z^i g_i \in M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w). \]
Since \( n \geq 2 \), \( zF \in M \ominus z^nM \), so that we have
\[
zF = q_1 \left( \sum_{i=0}^{n-1} \oplus z^i G_{1,i} \right) + \sum_{i=0}^{n-1} \oplus z^i H_{1,i},
\]
where \( G_{1,i}, H_{1,i} \in H^2(\Gamma_w) \).

Hence

(3.14)
\[
q_1 \left( z \left( \sum_{i=0}^{n-1} \oplus z^i f_i \right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i} \right) = \left( \sum_{i=0}^{n-1} \oplus z^i H_{1,i} \right) - z \sum_{i=0}^{n-1} \oplus z^i g_i.
\]

Here we devide into two subcases.

**Subcase 1.**
\[
z \left( \sum_{i=0}^{n-1} \oplus z^i f_i(w) \right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w) \neq 0.
\]

Then
\[
q_1(z) = \frac{\left( \sum_{i=0}^{n-1} \oplus z^i H_{1,i}(w) \right) - z \sum_{i=0}^{n-1} \oplus z^i g_i(w)}{z \left( \sum_{i=0}^{n-1} \oplus z^i f_i(w) \right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w)}.
\]

As proved in Step 1, \( q_1(z) \) is a nonconstant inner function. Then by the above, we have

(3.15)
\[
q_1(z) = \prod_{j=1}^{k} b_j(z), \quad 1 \leq k \leq n,
\]

where \( b_j \) are simple Blaschke products.

**Subcase 2.**
\[
z \left( \sum_{i=0}^{n-1} \oplus z^i f_i(w) \right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w) = 0.
\]

Then by (3.14), \( f_{n-1} = g_{n-1} = 0 \), so that by (3.11)
\[
F = \left( q_1 \sum_{i=0}^{n-2} \oplus z^i f_i \right) \oplus \sum_{i=0}^{n-2} \oplus z^i g_i.
\]

Since \( F \in M \ominus zM \),
\[
zF = \left( q_1 \sum_{i=0}^{n-2} \oplus z^{i+1} f_i \right) \oplus \sum_{i=0}^{n-2} \oplus z^{i+1} g_i \in M \ominus z^2M.
\]

In the same way as above, either (3.15) holds or \( f_{n-2} = g_{n-2} = 0 \). Repeat the same argument. Then either (3.15) holds or
\[
f_{n-1} = f_{n-2} = \cdots = f_1 = g_{n-1} = g_{n-2} = \cdots = g_1 = 0.
\]
Step 4. If (3.15) holds, then (ii) holds. So, suppose that (3.15) does not hold. Then by (3.11) and by the above fact, $F = q_1 f_0 \oplus g_0$, by (3.12) $q_1 f_0 \perp M \cap H^2(\Gamma_w)$, and by (3.13) $g_0 \in M \cap H^2(\Gamma_w)$ for every $F \in M \ominus z M$.

If $g_0 = 0$ for every $F \in M \ominus z M$, since $q_1(z) \in M$ it follows that $M \ominus z M \subset q_1(z)H^2(\Gamma_w) \subset M$. Since $M = \sum_{z \neq 0} \oplus z^i(M \ominus z M)$, we have $M = q_1(z)H^2$, so that $N = H^2 \ominus q_1(z)H^2$. By Theorem A, condition (i) holds.

So we assume that $g_0 \neq 0$ for some $F \in M \ominus z M$. We shall prove that

$$M \ominus z M \ominus (M \cap H^2(\Gamma_w)) \subset (M \cap H^2(\Gamma_z)) \ominus w M.$$  

We may assume that $(M \ominus z M) \ominus (M \cap H^2(\Gamma_w)) \neq \{0\}$. Let $G \in (M \ominus z M) \ominus (M \cap H^2(\Gamma_w))$ be such that $G \neq 0$. Then $G = q_1(z)h_1(w) \oplus h_2(w)$, where $q_1(z)h_1(w) \perp M \cap H^2(\Gamma_w)$ and $h_2(w) \in M \cap H^2(\Gamma_w)$. Hence $h_2(w) = G - q_1(z)h_1(w) \perp M \cap H^2(\Gamma_w)$. Therefore $h_2(w) = 0$. Since $q_1(z) \in M$,

$$G = q_1(z)h_1(w) = h_1(0)q_1(z) \oplus wq_1(z)\frac{h_1(w) - h_1(0)}{w} \in M \cap H^2(\Gamma_z) \ominus w M.$$  

Thus we get (3.16). By Lemma 3.2 (for $n = 1$), (i) holds.

4. $S_z S_w^* S_z S_w^* = S_w^* S_z S_w^* \neq S_w^* S_w^* S_z S_w^*$.

Let $N$ be a backward shift invariant subspace of $H^2$ and let $n$ be a positive integer. Let $M = H^2 \ominus N$. Then $M$ is an invariant subspace. If both $S_z S_w^* = S_w^* S_z$ and $S_z S_w^* \neq S_w^* S_z$ hold, then by Theorem 3.1, $M \cap H^2(\Gamma_z) = q(z)H^2(\Gamma_z)$ for some nonconstant inner function $q(z)$.

In this section, we assume that $q_1(z)H^2 \subset M$ and $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$ for some nonconstant inner function $q_1(z)$. Let

$$\tilde{M} = M \ominus q_1(z)H^2 \subset M.$$  

Then $H^2 \ominus q_1(z)H^2 = \tilde{M} \ominus N$ and $\tilde{M}$ is $w$-invariant. The following lemma is proved in [INS, Lemma 3.2].

Lemma 4.1. Let $f \in \tilde{M}$. Then $T_w f \notin \tilde{M}$ if and only if $f \in w \tilde{M}$.

We denote by $P_\perp$ the orthogonal projection from $H^2$ onto $H^2 \ominus q_1(z)H^2$. Then we have a Toeplitz type operator $Q_{z^n}$ on $H^2 \ominus q_1(z)H^2$ such that

$$Q_{z^n} : H^2 \ominus q_1(z)H^2 \ni f \to P_\perp(T_n f) \in H^2 \ominus q_1(z)H^2.$$  

Since $z^n M \subset M$, $Q_z \tilde{M} \subset \tilde{M}$ and $Q_z^n = Q_z^n$. Then $Q_z^n$ has the following matrix form:

$$Q_z^n = \begin{pmatrix} * & P_{\tilde{M}}T_{z^n}|_N \\ 0 & S_{z^n} \end{pmatrix} \quad \text{on } H^2 \ominus q_1(z) H^2 = \begin{pmatrix} \tilde{M} \\ N \end{pmatrix}.$$

Since $H^2 \ominus q_1(z) H^2$ is backward shift invariant, $T_w^*(H^2 \ominus q_1(z) H^2) \subset H^2 \ominus q_1(z) H^2$. Since $T_w^* N \subset N$, the operator $T_w^*$ on $H^2 \ominus q_1(z) H^2$ has the following matrix form:

$$T_w^* = \begin{pmatrix} * & 0 \\ P_N T_w^*|_{\tilde{M}} & S_w^* \end{pmatrix} \quad \text{on } H^2 \ominus q_1(z) H^2 = \begin{pmatrix} \tilde{M} \\ N \end{pmatrix}.$$

Set

(4.1) $A = P_{\tilde{M}}T_{z^n}|_N$ and $B = P_N T_w^*|_{\tilde{M}}$.

By [INS, Lemma 3.3], $T_w Q_z = Q_z T_w^*$ on $H^2 \ominus q_1(z) H^2$. Hence we have the following.

Lemma 4.2. $T_w Q_z^n = Q_z^n T_w^*$ on $H^2 \ominus q_1(z) H^2$.

Lemma 4.3. $S_w S_w^* = S_w^* S_w$ if and only if $BA = 0$.

Proof. By Lemma 4.2, $T_w^* Q_z^n = Q_z^n T_w^*$ on $H^2 \ominus q_1(z) H^2$. Then $BA + S_w^* S_w = S_w S_w^*$. Hence $S_w S_w^* = S_w^* S_w$ if and only if $BA = 0$. \hfill \Box

The following is a slight generalization of [INS, Theorem 3.5].

Theorem 4.4. Let $N$ be a backward shift invariant subspace of $H^2$ and $M = H^2 \ominus N$. Suppose that $M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z)$, where $q_1(z)$ is a nonconstant inner function. Let $\tilde{M} = M \ominus q_1(z) H^2$. Then the following conditions are equivalent.

(i) $S_{z^n} S_{z^n}^* = S_{z^n}^* S_{z^n}$ on $N$.
(ii) $\tilde{M} \ominus \{ f \in \tilde{M}; T_{z^n}^* f \in \tilde{M} \} \subset w \tilde{M}$.
(iii) $T_{z^n}^* \tilde{M} \subset \tilde{M}$.

Proof. The idea of the proof is similar to the one of [INS, Theorem 3.5]. For the sake of completeness, we give the proof.

(i) $\Leftrightarrow$ (ii) By Lemma 4.3, condition (i) is equivalent to $BA = 0$. By (4.1) and Lemma 4.1, $\ker B = \{ f \in \tilde{M}; T_{z^n}^* f \in \tilde{M} \} = w \tilde{M}$. Put $A_1 = P_{\tilde{M}} T_{z^n} P_N$ on $\tilde{M} \ominus N$. Then $[\text{ran } A] = [\text{ran } A_1]$. Since $A_1^* = P_N T_{z^n}^* P_{\tilde{M}}$, $\ker A_1^* = N \ominus \{ f \in \tilde{M}; T_{z^n}^* f \in \tilde{M} \}$. Hence $[\text{ran } A] = [\text{ran } A_1] = (\tilde{M} \ominus N) \ominus \ker A_1^* = \tilde{M} \ominus \{ f \in \tilde{M}; T_{z^n}^* f \in \tilde{M} \}$. Therefore $BA = 0$ if and only if $\tilde{M} \ominus \{ f \in \tilde{M}; T_{z^n}^* f \in \tilde{M} \} \subset w \tilde{M}$. Thus we get (i) $\Leftrightarrow$ (ii).
(ii) \(\Rightarrow\) (iii) Suppose that \(\hat{M} \ominus \{f \in \hat{M}; T_n^*f \in \hat{M}\} \subset w\hat{M}\). Since \(\{f \in \hat{M}; T_n^*f \in \hat{M}\}\) is closed, \(\hat{M} \ominus w\hat{M} \subset \{f \in \hat{M}; T_n^*f \in \hat{M}\}\). Since \(w\hat{M} \subset \hat{M}, \hat{M} = \sum_{j=0}^\infty w^j(\hat{M} \ominus w\hat{M})\). Since \(T_n^*w^j f = w^jT_n^*f\) for \(f \in H^2, \) we have \(T_n^*\hat{M} \subset \hat{M}\).

(iii) \(\Rightarrow\) (ii) is trivial. \(\square\)

For \(f \in H^2(\Gamma_z), \) write \(f^*(z) = T_n^*f(z) = \overline{\mathfrak{z}}(f(z) - \hat{f}(0)).\)

**Lemma 4.5.** Let \(b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}_iz), |\alpha_i| < 1, \) and \(1 \leq i \leq n.\) Then

(i) \(T_n^*z = 1, T_n^*b_1 = b_1^*(z) = (1 - |\alpha_1|^2)/(1 - \overline{\alpha}_1z), \) and \(T_n^*b_j(z) = \overline{\alpha}_j b_j^*(z).\)

(ii) \(T_n^*(b_1(z)b_2^*(z)) = (1 - |\alpha_2|^2)b_2^*(z) + \overline{\alpha}_2 b_1(z)b_2^*(z).\)

(iii) \(H^2(\Gamma_z) \ominus (\bigoplus_{j=1}^k b_j(z)H^2(\Gamma_z)) = \bigoplus_{j=1}^k [b_1(z) \cdots b_{j-1}(z)b_j^*(z)].\)

(iv) \(H^2(\Gamma_z) \ominus [b_1(z) \cdots b_{j-1}(z)b_j^*(z)H^2(\Gamma_w)].\)

**Proof.** It is not difficult to prove (i).  

(ii) Since

\[
\overline{\alpha} b_2(z) = \overline{\alpha}_1 b_1(z) \left(1 - \frac{|\alpha_2|^2}{1 - \overline{\alpha}_2z}\right)
\]

we get (ii).

(iii) See [Ni, p.33].

(iv) follows from (iii). \(\square\)

**Corollary 4.6.** Let \(N\) be a backward shift invariant subspace of \(H^2\) and \(M = H^2 \ominus N.\) Suppose that \(M \cap H^2(\Gamma_z) = b_1(z)H^2(\Gamma_z),\) where \(b_1(z)\) is a simple Blaschke product. Then \(S_w S_w^* = S_w^*S_w\).

**Proof.** Let \(b_1(z) = (z - \alpha)/(1 - \overline{\alpha}z), |\alpha| < 1, \) and \(\hat{M} = M \ominus b_1(z)H^2.\) Since \(b_1(z) \in M, b_1(z)H^2 \subset M.\) By Lemma 4.5(iv), \(\hat{M} \subset b_1(z)H^2(\Gamma_w).\)

By Lemma 4.5(i), \(T_n^*(b_1(z)h(w)) = \overline{\alpha}_1 b_1^*(z)h(w).\) Hence \(T_n^*\hat{M} \subset M.\) By Theorem 4.4, \(S_w S_w^* = S_w^*S_w.\) \(\square\)

**Corollary 4.7.** Let \(N\) be a backward shift invariant subspace of \(H^2\) and \(M = H^2 \ominus N.\) Suppose that \(M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z),\) where \(q_1(z)\) is an inner function. Let \(n, k\) be positive integers with \(n \geq k + 1.\) Moreover suppose that \(q_1(z) = z^nb(z),\) where \(b\) is a simple Blaschke product, \(b(z) = (z - \alpha)/(1 - \overline{\alpha}z), \) and \(\alpha \neq 0.\) If \(S_n S_w^* = S_w^*S_n,\) then \(S_n S_w^* = S_w^*S_n.\)
Proof. Let $\tilde{M} = M \ominus q_1(z)H^2$. If $\tilde{M} = \{0\}$, then $M = q_1(z)H^2$. By Theorem A, $S_2S_w^* = S_w^*S_2$. Suppose that $\tilde{M} \neq \{0\}$. By Lemma 4.5(iv),

$$\tilde{M} \subset b^*(z)H^2(\Gamma_w) \oplus \left( \sum_{j=1}^{k} \oplus z^{j-1}b(z)H^2(\Gamma_w) \right).$$

Let $f \in \tilde{M}$. Then

$$f = b^*(z)h_0(w) + \sum_{j=1}^{k} \oplus z^{j-1}b(z)h_j(w), \quad h_j(w) \in H^2(\Gamma_w).$$

By Lemma 4.5(i),

$$T_{z^n}^*f = T_{z^{n-k}}^*(T_{z^k}^*f)$$

$$= T_{z^{n-k}}^* \left( \sum_{j=0}^{k} \alpha^{(k-j)}b^*(z)h_j(w) \right)$$

$$= \alpha^{(n-k)} \sum_{j=0}^{k} \alpha^{(k-j)}b^*(z)h_j(w).$$

Since $S_2S_w^* = S_w^*S_2$, by Theorem 4.4 $T_{z^n}^*f \in \tilde{M}$. Since $\alpha \neq 0$,

$$\sum_{j=0}^{k} \alpha^{(k-j)}b^*(z)h_j(w) \in \tilde{M}.$$  

Thus $T_{z^n}^*\tilde{M} \subset \tilde{M}$. By Theorem 4.4, $S_{z^n}S_w^* = S_w^*S_{z^n}$.

By Theorem 4.8. Let $N$ be a backward shift invariant subspace of $H^2$ and $M = H^2 \ominus N$. Suppose that $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$, where $q_1(z)$ is an inner function. Moreover suppose that $q_1(z) = b_1(z)b_2(z)$, where $b_i(z), i = 1, 2$, are simple Blaschke products, $b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}iz)$, and $\alpha_1\alpha_2 \neq 0$. Let $n \geq 2$ be a positive integer. Then we have the following.

(i) If $S_2S_w^* = S_w^*S_2$ and $S_{2n-1}S_w^* \neq S_w^*S_{2n-1}$, then $\alpha_1^n = \alpha_2^n$ and $\alpha_1 \neq \alpha_2$.

(ii) If $\alpha_1^n = \alpha_2^n$ and $\alpha_1 \neq \alpha_2$, then $S_2S_w^* = S_w^*S_2$.

Proof. Let $\tilde{M} = M \ominus q_1(z)H^2$. Suppose that $S_2S_w^* = S_w^*S_2$ and $S_{2n-1}S_w^* \neq S_w^*S_{2n-1}$. By Theorem 4.4, $T_{z^n}^*\tilde{M} \subset \tilde{M}$ and $T_{z^{n-1}}^*\tilde{M} \not\subset \tilde{M}$. By Lemma 4.5(iv),

$$\tilde{M} \subset b_1^*(z)H^2(\Gamma_w) \oplus b_1(z)b_2^*(z)H^2(\Gamma_w).$$

Then there exists $f_0 \in \tilde{M}$ such that $T_{z^{n-1}}^*f_0 \not\in \tilde{M}$, and $f_0 = b_1^*(z)g_1(w) + b_1(z)b_2^*(z)g_2(w) \in b_1^*(z)H^2(\Gamma_w) \oplus b_1(z)b_2^*(z)H^2(\Gamma_w)$.  

$\square$
By Lemma 4.5,
\[
T_{z,n-1}^* f_0 = b_1^* (\alpha_1^{(n-1)} g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-2} \alpha_1^{(n-1-j)} \overline{\alpha_2}^j g_2 \right) + \alpha_2^{(n-1)} b_1 b_2^* g_2
\]
and
\[
T_{z,n}^* f_0 = b_1^* (\overline{\alpha_1} g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-1} \overline{\alpha_1}^{(n-1-j)} \overline{\alpha_2}^j g_2 \right) + \alpha_2^2 b_1 b_2^* g_2.
\]
Since \( T_{z,n-1}^* f_0 \not\in \tilde{M} \) and \( f_0 \in \tilde{M}, T_{z,n}^* f_0 - \overline{\alpha_2}^{n-1} f_0 \not\in \tilde{M} \). Then
\[
b_1^* \left( (\overline{\alpha_1} - \overline{\alpha_2}) g_1 + (1 - |\alpha_2|^2) g_2 \right) \not\in \tilde{M}.
\]
Hence
\[
\left( \sum_{j=0}^{n-2} \overline{\alpha_1}^{(n-2-j)} \overline{\alpha_2}^j \right) b_1^* \left( (\overline{\alpha_1} - \overline{\alpha_2}) g_1 + (1 - |\alpha_2|^2) g_2 \right) \not\in \tilde{M}.
\]
Since \( 0 \in \tilde{M}, \sum_{j=0}^{n-2} \overline{\alpha_1}^{(n-2-j)} \overline{\alpha_2}^j \neq 0 \), so that
\[
(4.3) \quad b_1^* \left( (\overline{\alpha_1} - \overline{\alpha_2}) g_1 + (1 - |\alpha_2|^2) g_2 \right) \not\in \tilde{M}.
\]
Since \( T_{z,n}^* f_0 \in \tilde{M}, T_{z,n}^* f_0 - \overline{\alpha_2}^0 f_0 \in \tilde{M} \). Then
\[
b_1^* \left( (\overline{\alpha_1} - \overline{\alpha_2}) g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-1} \overline{\alpha_1}^{(n-1-j)} \overline{\alpha_2}^j \right) g_2 \right) \in \tilde{M}.
\]
Hence
\[
(4.4) \quad \left( \sum_{j=0}^{n-1} \overline{\alpha_1}^{(n-1-j)} \overline{\alpha_2}^j \right) b_1^* \left( (\overline{\alpha_1} - \overline{\alpha_2}) g_1 + (1 - |\alpha_2|^2) g_2 \right) \in \tilde{M}.
\]

Now we prove (i). Suppose that \( \alpha_1 = \alpha_2 \). Then \( \alpha_1 = \alpha_2 \neq 0 \), so that
\[
\sum_{j=0}^{n-1} \overline{\alpha_1}^{(n-1-j)} \overline{\alpha_2}^j \neq 0.
\]
By (4.3) and (4.4), \( b_1^* g_2 \not\in \tilde{M} \) and \( b_1^* g_2 \in \tilde{M} \). This is a contradiction. Hence \( \alpha_1 \neq \alpha_2 \).

Suppose that \( \alpha_1^n \neq \alpha_2^n \). Then \( \sum_{j=0}^{n-1} \overline{\alpha_1}^{(n-1-j)} \overline{\alpha_2}^j \neq 0 \). By (4.4),
\[
b_1^* \left( (\overline{\alpha_1} - \overline{\alpha_2}) g_1 + (1 - |\alpha_2|^2) g_2 \right) \in \tilde{M}.
\]
This contradicts (4.3), so that \( \alpha_1^n = \alpha_2^n \). Thus we get (i).

(ii) Suppose that \( \alpha_1^n = \alpha_2^n \) and \( \alpha_1 \neq \alpha_2 \). Then \( \sum_{j=0}^{n-1} \overline{\alpha_1}^{(n-1-j)} \overline{\alpha_2}^j = 0 \).

Let \( f \in \tilde{M} \). Then by (4.2), \( f = b_1^* (z) h_1 (w) + b_1 (z) b_2^* (z) h_2 (w) \). Similarly,
we have
\[ T^n_z \alpha_2 f - \overline{\alpha}_2 f = \left( \sum_{j=0}^{n-1} \overline{\alpha}_1^{(n-1-j)} \overline{\alpha}_2^j \right) b_1^* \left( (\overline{\alpha}_1 - \overline{\alpha}_2) h_1 + (1 - |\alpha_2|^2) h_2 \right). \]

Hence \( T^n_z f = \overline{\alpha}_2^j f \in \tilde{M} \), so that we get \( T^n_z \tilde{M} \subset \tilde{M} \). By Theorem 4.4, \( S_z S_w^* = S_w^* S_z^* \).

**Corollary 4.9.** Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \ominus N \). Suppose that \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \), where \( q_1(z) \) is an inner function. Moreover suppose that \( q_1(z) = b_1(z)b_2(z) \), where \( b_i(z), i = 1, 2, \) are simple Blaschke products, \( b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}_i z) \), and \( \alpha_1 \alpha_2 \neq 0 \). Then we have the following.

(i) If \( S_z S_w^* = S_w^* S_z \) and \( S_z S_w^* \neq S_w^* S_z \), then \( \alpha_1 + \alpha_2 = 0 \).

(ii) If \( \alpha_1 + \alpha_2 = 0 \), then \( S_z S_w^* = S_w^* S_z \).

The following is the main theorem in this section.

**Theorem 4.10.** Let \( N \) be a backward shift invariant subspace of \( H^2 \), \( N \neq \{0\} \), and \( N \neq H^2 \). Then \( S_z S_w^* = S_w^* S_z \) if and only if one of the following conditions holds.

(i) \( S_z S_w^* = S_w^* S_z \).

(ii) \( S_z S_w^* = 0 \).

(iii) There exist two simple Blaschke products \( b_1(z) \) and \( b_2(z) \), \( b_1(z) = (z - \alpha_i)/(1 - \overline{\alpha}_i z) \), \( 0 < |\alpha_i| < 1 \), such that \( N \subset H^2 \ominus b_1(z)b_2(z)H^2 \) and \( \alpha_1 + \alpha_2 = 0 \).

A backward shift invariant subspace \( N \) satisfying condition (i) is given by Theorem A. Also \( N \) satisfying condition (ii) is given by Theorem 2.2.

**Proof of Theorem 4.10.** Suppose that \( S_z S_w^* = S_w^* S_z \). Moreover suppose that \( S_z S_w^* \neq S_w^* S_z \). By Theorem 3.1, \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \) for an inner function \( q_1(z) \) such that either \( q_1(z) = b_1(z) \) or \( q_1(z) = b_1(z)b_2(z) \), where \( b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}_i z) \), \( |\alpha_i| < 1 \). If \( q_1(z) = b_1(z) \), by Corollary 4.6, (i) holds.

Suppose that \( q_1(z) = b_1(z)b_2(z) \). Moreover suppose that \( \alpha_1 = \alpha_2 = 0 \). Then \( q_1(z) = z^2 \). Hence \( z^2 H^2 \subset M \), so that \( N \subset H^2(\Gamma_w) \oplus z H^2(\Gamma_w) \). Then by Theorem 2.2, \( S_z S_w^* = 0 \). Thus (ii) holds.

Suppose that \( \alpha_1 \neq 0 \) and \( \alpha_2 = 0 \). Then \( q_1(z) = zb_1(z) \). By Corollary 4.7, (i) holds.

Suppose that \( \alpha_1 \neq 0 \) and \( \alpha_2 \neq 0 \). Then by Corollary 4.9(i), we get \( \alpha_1 + \alpha_2 = 0 \). Hence (iii) holds.
Next, we prove the converse assertion. Suppose that one of conditions (i), (ii), and (iii) holds. If (i) holds, then trivially $S_zS_z^* = S_zS_z^*$. If (ii) holds, by Corollary 2.3 we have $S_zS_z^* = S_zS_z^*$. If (iii) holds, then trivially $S_zS_z^* = S_zS_z^*$. Suppose that (iii) holds. Then $b_1(z)b_2(z)H^2(\Gamma_z) \subset M$. Hence $M \cap H^2(\Gamma_z)$ equals to either $b_1(z)H^2(\Gamma_z)$, or $b_2(z)H^2(\Gamma_z)$, or $b_1(z)b_2(z)H^2(\Gamma_z)$. By Corollary 4.6, $S_zS_z^* = S_zS_z^*$ for the first two cases. Hence $S_zS_z^* = S_zS_z^*$. For the last case, by Corollary 4.9(ii), $S_zS_z^* = S_zS_z^*$.

Example 4.11. We give an example of a backward shift invariant subspace $N$ of $H^2$ satisfying $S_zS_z^* = S_zS_z^*$, $S_zS_z^* \neq S_zS_z^*$, and $S_zS_z^* \neq 0$. Let $q_1(z) = b_1(z)b_2(z)$, where $b_i(z), i = 1, 2$, are $b_i(z) = (z - \alpha_i)/(1 - \alpha_i z)$, $\alpha_1 + \alpha_2 = 0$, and $\alpha_1 \neq 0$. Let $q_2(w)$ be a non-constant inner function. Let $M = q_1(z)H^2 \oplus b_1(z)b_2(z)q_2(w)H^2(\Gamma_w)$. Then $M$ is an invariant subspace and $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$. Let $N = H^2 \oplus M$. By Theorem 4.10, $S_zS_z^* = S_zS_z^*$. We have $\tilde{M} = M \ominus q_1(z)H^2 = b_1(z)b_2(z)q_2(w)H^2(\Gamma_w)$. Since

\[ T_z^*b_1(z)b_2(z)q_2(w) = (1 - |\alpha_1|^2)b_1(z)q_2(w) + \overline{\alpha_1}b_1(z)b_2(z)q_2(w), \]

$T_z^*b_1(z)b_2(z)q_2(w) \notin \tilde{M}$. By Theorem 4.4, $S_zS_z^* \neq S_zS_z^*$. By Theorem 2.2, $S_zS_z^* \neq 0$.

We leave the following problem for the reader.

Problem 4.12. Characterize backward invariant subspaces $N$ of $H^2$ satisfying $S_zS_z^* = S_zS_z^*$ for $n \geq 3$.

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