<table>
<thead>
<tr>
<th>Title</th>
<th>Backward shift invariant subspaces in the bidisc III</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Izuchi, Keiji; Nakazi, Takahiko; Seto, Michio</td>
</tr>
<tr>
<td>Citation</td>
<td>Hokkaido University Preprint Series in Mathematics, 650, 1-21</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2004</td>
</tr>
<tr>
<td>DOI</td>
<td>10.14943/83803</td>
</tr>
<tr>
<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/69457">http://hdl.handle.net/2115/69457</a></td>
</tr>
<tr>
<td>Type</td>
<td>bulletin (article)</td>
</tr>
<tr>
<td>File Information</td>
<td>pre650.pdf</td>
</tr>
</tbody>
</table>
BACKWARD SHIFT INVARIANT SUBSPACES IN THE BIDISC III

KEIJI IZUCHI, TAKAHIKO NAKAZI, AND MICHIO SETO

Abstract. In the previous paper, we give a characterization of backward shift invariant subspaces of the Hardy space in the bidisc which satisfy the doubly commuting condition $S^*_z S^*_w = S^*_w S^*_z$ for the compression operators $S_z$ and $S_w$. In this paper, we give a characterization of backward shift invariant subspaces satisfying $S^2_z S^*_w = S^*_w S^2_z$.

1. Introduction.

Let $\Gamma^2$ be the 2-dimensional unit torus. We denote by $(z, w) = (e^{i\theta}, e^{i\phi})$ the variables in $\Gamma^2 = \Gamma_z \times \Gamma_w$. Let $L^2 = L^2(\Gamma^2)$ be the usual Lebesgue space on $\Gamma^2$ with the norm $\|f\|_2 = (\int_{\Gamma^2} |f(e^{i\theta}, e^{i\phi})|^2 d\theta d\phi / (2\pi)^2)^{1/2}$. Then $L^2$ is a Hilbert space with the usual inner product. For $f \in L^2$, the Fourier coefficients are given by

$$\hat{f}(n, m) = \int_{\Gamma^2} f(e^{i\theta}, e^{i\phi})e^{-in\theta} e^{-im\phi} d\theta d\phi / (2\pi)^2 = \langle f, z^n w^m \rangle.$$ 

Let $H^2 = H^2(\Gamma^2)$ be the Hardy space on $\Gamma^2$, that is,

$$H^2 = \{ f \in L^2; \hat{f}(n, m) = 0 \text{ if } n < 0 \text{ or } m < 0 \}.$$ 

For $f \in H^2$, we can write $f$ as

$$f = \sum_{i,j=0}^{\infty} \oplus a_{i,j} z^i w^j, \text{ where } \sum_{i,j=0}^{\infty} |a_{i,j}|^2 < \infty.$$ 

Let $P$ be the orthogonal projection from $L^2$ onto $H^2$. For a closed subspace $M$ of $L^2$, we denote by $P_M$ the orthogonal projection from $L^2$ onto $M$. For a function $\psi \in L^\infty$, let $L_\psi f = \psi f$ for $f \in L^2$. The Toeplitz operator $T_\psi$ is defined by $T_\psi f = P L_\psi f$ for $f \in H^2$. It is well known that $T^*_\psi = T^*_{\overline{\psi}}$ and $T^*_n T^*_{wm} = T^*_{wm} T^*_n$ for $n, m \geq 1$. A function $f \in H^2$ is called inner if $|f| = 1$ on $\Gamma^2$ almost everywhere. A closed subspace $M$ of $H^2$ is called invariant if $z M \subset M$ and $w M \subset M$. In one variable

---

1The first author was partially supported by Grant-in-Aid for Scientific Research (No.13440043), Ministry of Education, Science and Culture, Japan.

1991 Mathematics Subject Classification. Primary 47A15; Secondary 32A35.

Key words and phrases. Hardy space, backward shift, invariant subspace.
case, an invariant subspace $M$ of $H^2(\Gamma)$ has a form $M = qH^2(\Gamma)$, where $q$ is inner. This is the well known Beurling theorem [B]. In two variable case, the structure of invariant subspaces of $H^2$ is very complicated, see [AC], [DY], [Na1], and [R].

Let $M$ be an invariant subspace of $H^2$. Then $T_z^*(H^2 \ominus M) \subset (H^2 \ominus M)$ and $T_w^*(H^2 \ominus M) \subset (H^2 \ominus M)$. A closed subspace $N$ of $H^2$ is called backward shift invariant if $H^2 \ominus N$ is invariant. In [IY], the first author and Yang studied backward shift invariant subspaces $N$ on which $T_z^*$ is strictly contractive. See [CR] and [S] for studies of backward shift invariant subspaces on the unit circle $\Gamma$.

Let $M$ be an invariant subspace of $H^2$ and $\psi \in L^\infty$. Let $V_\psi$ be the operator on $M$ defined by $V_\psi = P_M L_\psi|_M$. Then $V_\psi = T_z$ and $V_z^* = V_z$ on $M$. In [M], Mandrekar proved that $V_\psi V_w^* = V_w^* V_\psi$ on $M$ if and only if $M$ is Beurling type, that is, $M = qH^2$ for some inner function $q$ in $H^\infty$, see also [CS] and [Na2].

For $\psi \in L^\infty$, let $S_\psi = P_N L_\psi|_N$. Then we have $S_\psi^* = S_\psi$ and $S_z^* = T_z^*$ on $N$. In the previous paper [INS], we characterized backward shift invariant subspaces $N$ which satisfy the condition $S_z S_w^* = S_w^* S_z$ on $N$ as follows.

**Theorem A.** Let $N$ be a backward shift invariant subspace of $H^2$ and $N \neq H^2$. Then $S_z S_w^* = S_w^* S_z$ on $N$ if and only if $N$ has one of the following forms;

(i) $N = H^2 \ominus q_1(z)H^2$,
(ii) $N = H^2 \ominus q_2(w)H^2$,
(iii) $N = (H^2 \ominus q_1(z)H^2) \cap (H^2 \ominus q_2(w)H^2),$

where $q_1(z)$ and $q_2(w)$ are one variable inner functions.

This paper is a continuation of previous papers [IN] and [INS]. Here we are interesting in backward shift invariant subspaces $N$ of $H^2$ satisfying that $S_z^n S_w^m = S_w^m S_z^n$ for given positive integers $n$ and $m$. Up to now, we can not give a complete characterization of $N$ satisfying $S_z^n S_w^m = S_w^m S_z^n$. If $S_z^n S_w^m = S_w^m S_z^n$, then trivially $S_z^n S_w^m = S_w^m S_z^n$. But the converse is not true. In this paper, we concentrated on the case $m = 1$. One reason is the work of this paper deeply concerns with the problem of the normality of the operators $S_z^n$. For many backward shift invariant subspaces $N$, $S_z^n$ are not normal operators, see [Y]. If $S_z^n$ is normal, since $S_z^n S_w = S_w S_z^n$, by the Fuglede-Putnam theorem we have $S_z^n S_w^* = S_w^* S_z^n$. So, the results obtained in this paper will be a big help for the above mentioned problem. We will discuss on this subject in the forthcoming paper.
In Section 2, we give characterizations of $N$ satisfying $S_z^n S_w^m = 0$ and $S_w^m S_z^n = 0$, respectively. If $S_z^n S_w^m = 0$, then $S_w^m S_z^n = 0$. The converse is not true. In Section 3, we study $N$ satisfying $S_z^n S_w^* = S_w^* S_z^n$, and give a necessary condition for $S_z^n S_w^* = S_w^* S_z^n$. In Section 4, we study $N$ satisfying $S_z^n S_w^* = S_w^* S_z^n$. We gave a complete characterization of such $N$. In [INS], we gave two different types of proofs of Theorem A. One of them is used in Section 3 and the other is used in Section 4.

Let $H^2(\Gamma_z)$ and $H^2(\Gamma_w)$ be the Hardy spaces on the unit circle in variables $z$ and $w$, respectively. We think that $H^2(\Gamma_z)$ and $H^2(\Gamma_w)$. For a subset $E$ of $H^2$, we denote by $[E]$ the closed linear span of $E$. A function $b(z) = (z - \alpha)/(1 - \bar{\alpha}z), |\alpha| < 1$, is called a simple Blaschke product.

2. $S_z^n S_w^* = 0$ or $S_w^* S_z^n = 0$.

Let $n$ and $m$ be positive integers. In this section, we study backward shift invariant subspaces $N$ of $H^2$ satisfying $S_z^n S_w^m = 0$ and $S_w^m S_z^n = 0$, respectively.

**Lemma 2.1.** Let $N$ be a backward shift invariant subspace of $H^2$. Then we have the following.

(i) $S_z^n = S_z^n$.

(ii) $S_w^m S_z^n = S_z^n S_w^m$ and $S_z^n S_w^m = S_w^m S_z^n$.

(iii) If $S_z^n S_w^m N \neq \{0\}$, then there exists $f \in N$ such that $(S_z^n S_w^m f) \cap (0, 0) \neq 0$.

**Proof.** All assertions are not difficult to prove.

The following theorem says that the structure of backward shift invariant subspaces satisfying $S_z^n S_w^m = 0$ is simple.

**Theorem 2.2.** Let $N$ be a backward shift invariant subspace of $H^2$. Then $S_z^n S_w^m = 0$ if and only if $N$ satisfies one of the following conditions;

(i) $N \subset \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$.

(ii) $N \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$.

**Proof.** Suppose that $S_z^n S_w^m = 0$. Then

$$S_w^m N \perp S_z^n N.$$  

Since $N$ is backward shift invariant, if $S_w^m N = \{0\}$ then $N$ satisfies condition (i). If $S_z^n N = \{0\}$, then $N$ satisfies (ii).

Next, suppose that

$$S_w^m N \neq \{0\} \text{ and } S_z^n N \neq \{0\}.$$


We shall lead a contradiction. By (2.1), $S_{w^m}^* S_{z^n}^* N \perp S_{z^n}^* S_{w^m}^* N$. By Lemma 2.1(ii), $S_{w^m}^* S_{z^n}^* N = S_{z^n}^* S_{w^m}^* N = \{0\}$. Then

\[(2.3) \quad S_{z^n}^* N \subset \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)\]

and

\[(2.4) \quad S_{w^m}^* N \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w).\]

By (2.2) and (2.3), there exists a nonnegative integer $j$, $0 \leq j \leq m - 1$, such that

\[(2.5) \quad \{0\} \neq S_{w^j}^* S_{z^n}^* N \subset H^2(\Gamma_z).\]

By Lemma 2.1(iii), there exists $g \in N$ such that

\[(2.6) \quad (S_{w^j}^* S_{z^n}^* g) \cap (0, 0) \neq 0.\]

Also by (2.2) and (2.4), there exist $f \in N$ and a nonnegative integer $i$, $0 \leq i \leq n - 1$, such that

\[(2.7) \quad S_{z^i}^* S_{w^m}^* f \in H^2(\Gamma_w)\]

and

\[(2.8) \quad (S_{z^i}^* S_{w^m}^* f) \cap (0, 0) \neq 0.\]

Then

\[
0 = \langle S_{w^m}^* S_{z^i}^* f, S_{z^n}^* S_{w^j}^* g \rangle \quad \text{by (2.1)}
\]

\[
= \langle S_{z^i}^* S_{w^m}^* f, S_{w^j}^* S_{z^n}^* g \rangle \quad \text{by Lemma 2.1(ii)}
\]

\[
= \langle S_{z^i}^* S_{w^m}^* f \cap (0, 0), (S_{w^m}^* S_{z^n}^* g) \cap (0, 0) \rangle \quad \text{by (2.5) and (2.7)}
\]

\[
\neq 0 \quad \text{by (2.6) and (2.8).}
\]

This is a desired contradiction.

The converse is trivial. \qed

**Corollary 2.3.** Let $N$ be a backward shift invariant subspace of $H^2$. Then $S_{z^n} S_{w^m}^* = 0$ if and only if either $S_{z^n}^* = 0$ or $S_{w^m}^* = 0$. Hence if $S_{z^n} S_{w^m}^* = 0$, then $S_{w^m}^* S_{z^n} = 0$.

**Lemma 2.4.** Let $M_1$ and $M_2$ be closed subspaces of $H^2$ such that

\[
M_1 \subset \sum_{j=0}^{m} \oplus w^j H^2(\Gamma_z) \quad \text{and} \quad M_2 \subset \sum_{i=0}^{n} \oplus z^i H^2(\Gamma_w).\]

Then $M_1 + M_2$ is closed.
Proof. We denote by \((z^i w^j)_{M_1}\) and \((z^i w^j)_{M_2}\) the orthogonal projections of \(z^i w^j\) to the spaces \(M_1\) and \(M_2\), respectively. Let 
\[
M'_1 = M_1 \ominus \left( \{ (z^i w^j)_{M_1}, 0 \leq i \leq n, 0 \leq j \leq m \} \right)
\]
and 
\[
M'_2 = M_2 \ominus \left( \{ (z^i w^j)_{M_2}, 0 \leq i \leq n, 0 \leq j \leq m \} \right).
\]
Then \(M'_1\) and \(M'_2\) are closed subspaces of \(H^2\), 
\[
M'_1 \subset z^{n+1} \left( \sum_{j=0}^{m} \oplus w^j H^2(\Gamma_z) \right), \quad M'_2 \subset w^{m+1} \left( \sum_{i=0}^{n} \oplus z^i H^2(\Gamma_w) \right),
\]
and 
\[
M'_1 + M'_2 \perp \{ z^i w^j, 0 \leq i \leq n, 0 \leq j \leq m \}.
\]
Since 
\[
z^{n+1} \left( \sum_{j=0}^{m} \oplus w^j H^2(\Gamma_z) \right) \perp w^{m+1} \left( \sum_{i=0}^{n} \oplus z^i H^2(\Gamma_w) \right),
\]
\(M'_1 + M'_2 = M'_1 \oplus M'_2\) is closed. Hence 
\[
M_1 + M_2 = M'_1 + M'_2 + \{ (z^i w^j)_{M_1}, (z^i w^j)_{M_2}, 0 \leq i \leq n, 0 \leq j \leq m \}
\]
is closed. \(\square\)

**Theorem 2.5.** Let \(N\) be a backward shift invariant subspace of \(H^2\). Then \(S_{w^n} S_{z^n} = 0\) if and only if
\[
(i) \quad N \ominus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \subset N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z).
\]

**Proof.** Suppose that \(S_{w^n} S_{z^n} = 0\). Then \(S_{z^n} N \subset \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)\). Since \(S_{z^n} S_{w^n} = 0\), \(S_{w^n} N \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\). First, we prove the following 
\[
(2.9) \quad N = \left( N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \right) + \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right).
\]
Let 
\[
(2.10) \quad K = \left( N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \right) + \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right),
\]
By Lemma 2.4, \(K\) is closed and \(N = K \oplus (N \ominus K)\). To prove (i), it is sufficient to prove \(N \ominus K = \{0\}\). We have \(N \ominus K \subset N \cap (S_{z^n} N)^\perp \cap (S_{w^n} N)^\perp\). Let \(f \in N \ominus K\). Then \(f \perp S_{z^n} N\), so that \(S_{z^n} f \perp N\). Since
$S_n^* f \in N$, $S_n^* f = 0$. Hence $f \in \bigoplus_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$. By (2.10), $f \in K$. This shows $f = 0$, so that $N \ominus K = \{0\}$. Thus we get (2.9).

Let

$$N_1 = \left\{ f \in N \cap \bigoplus_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z); f \perp N \cap \bigoplus_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right\}. $$

Then $N_1$ is a closed subspace and

$$N_1 \oplus \left( N \cap \bigoplus_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \subset N. $$

If the equality holds in the above, (i) holds. So we assume that

$$N_1 \oplus \left( N \cap \bigoplus_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \subsetneq N. $$

We shall lead a contradiction. Let

$$N_2 = N \ominus \left( N_1 \oplus \left( N \cap \bigoplus_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right). $$

Then $N_2 \neq \{0\}$ and $N = N_1 \oplus N_2 \oplus (N \cap \bigoplus_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w))$.

Let $g \in N_2$ be such that $g \neq 0$. We shall prove that

$$g \notin N \cap \bigoplus_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \quad \text{and} \quad g \notin N \cap \bigoplus_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z).$$

The fact $g \notin N \cap \bigoplus_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$ is trivial. Suppose that $g \in N \cap \bigoplus_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$. By (2.11), $g \notin N_1$. Since $g \in N_2$, by (2.12) we have $g \perp N_1$. Hence $g = 0$. This is a contradiction. Thus we get (2.13).

Next, we shall prove that

$$S_n^* g \perp N \cap \bigoplus_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z).$$

To prove this, suppose not. Then $S_n^* g \perp N \cap \bigoplus_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$. By (2.10), $g = g_1 + g_2$, where $g_1 \in N \cap \bigoplus_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$ and $g_2 \in N \cap \bigoplus_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$. Then $S_n^* g = S_n^* g_1 \in N \cap \bigoplus_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$.

Therefore $S_n^* g = 0$, so that $g \in N \cap \bigoplus_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$. This contradicts (2.13). Thus we get (2.14).

By (2.14), there exists $h_0$ such that $h_0 \in N \cap \bigoplus_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$ and $\langle g, S_n^* h_0 \rangle = \langle S_n^* g, h_0 \rangle \neq 0$. Since $S_n^* h_0 \in N$, by (2.12) we have $S_n^* h_0 = h_1 \perp h_2 \perp h_3$, where $h_1 \in N_1, h_2 \in N_2, \text{and} \ h_3 \in$
\[ N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w). \] Since \( g \in N_2 \) and \( \langle g, S_w h_0 \rangle \neq 0 \), we have \( h_2 \neq 0 \). Since \( z^n h_0 \perp \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w), \)

\[ P_N z^n h_0 = S_w h_0 \perp N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w). \]

Thus we get \( h_3 = 0 \). By (2.12), \( S_w N_1 = \{0\} \). Hence \( S_w^* S_w h_0 = S_w^* h_0 \). By (2.13) and \( h_2 \in N_2 \), \( h_2 \notin \sum_{j=0}^{m-1} \oplus z^j H^2(\Gamma_z) \). This implies that \( S_w^* h_2 \neq 0 \). Hence \( S_w^* S_w \neq 0 \). This contradicts our starting assumption. Thus we have (i).

To prove the converse, suppose that condition (i) holds. Then \( N = (N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)) \oplus L \), where \( L \subset N \cap \sum_{i=0}^{m-1} \oplus z^i H^2(\Gamma_z) \). Let \( F = F_1 + F_2 \in N \), where \( F_1 \in N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \) and \( F_2 \in L \). Since \( z^n F \perp N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w), S_w F \in L \). Hence \( S_w^* S_w F = 0 \). Thus we get \( S_w^* S_w = 0 \).

By Theorem 2.2, the structure of backward shift invariant subspaces \( N \) satisfying \( S_w S_w^* = 0 \) is simple. By Theorem 2.5, the structure of backward shift invariant subspaces \( N \) satisfying \( S_w^* S_w = 0 \) is not so simple. When \( n = m = 1 \), we have the following.

**Theorem 2.6.** Let \( N \) be a backward shift invariant subspace of \( H^2 \).

Then \( S_w^* S_w = 0 \) if and only if \( N \) has one of the following forms:

(i) \( N = H^2(\Gamma_z) \ominus q(z) H^2(\Gamma_z) \) for some inner function \( q(z) \).

(ii) \( N = H^2(\Gamma_w) \ominus q(w) H^2(\Gamma_w) \) for some inner function \( q(w) \).

(iii) Either \( N = H^2(\Gamma_z) + H^2(\Gamma_w) \), or \( N = H^2(\Gamma_z) + (H^2(\Gamma_w) \ominus q_2(w) H^2(\Gamma_w)) \), or \( N = \bigl( H^2(\Gamma_z) \ominus q_1(z) H^2(\Gamma_z) \bigr) + H^2(\Gamma_w) \), where \( q_1(z) \) and \( q_2(w) \) are inner functions.

(iv) \( N = \bigl( (H^2(\Gamma_z) \ominus q_1(z) H^2(\Gamma_z)) + (H^2(\Gamma_w) \ominus q_2(w) H^2(\Gamma_w)) \bigr) \), where \( q_1(z), q_2(w) \) are nonconstant inner functions and \( \hat{q}_1(0) \hat{q}_2(0) = 0 \).

In (iii) and (iv), since \( 1 \in N \), we may take \( q_1 \) and \( q_2 \) as \( \hat{q}_1(0) = \hat{q}_2(0) = 0 \).

**Proof.** By Theorem 2.2, \( S_w S_w^* = 0 \) if and only if either (i) or (ii) holds. By Theorem 2.5, \( S_w^* S_w = 0 \) if and only if

\[(2.15) \quad N \ominus (N \cap H^2(\Gamma_w)) \subset N \cap H^2(\Gamma_z). \]

If either (i) or (ii) holds, by Corollary 2.3 we have \( S_w^* S_w = 0 \). Suppose that \( N \) satisfies either (iii) or (iv). Then clearly \( 1 \in N \). Since \( N \) has a special form, it is not difficult to see that

\[ N \ominus (N \cap H^2(\Gamma_w)) = \{ f \in N \cap H^2(\Gamma_z); \hat{f}(0) = 0 \}. \]

Hence (2.15) holds.
Next, suppose that (2.15) holds. Then we have

\[ N = (N \cap H^2(\Gamma_z)) + (N \cap H^2(\Gamma_w)). \]

If either \( N \cap H^2(\Gamma_z) = \{0\} \) or \( N \cap H^2(\Gamma_w) = \{0\} \), then \( S_zS_w^* = 0 \), and by Corollary 2.3, \( S_z^*S_w = 0 \). Hence either (i) or (ii) holds. Suppose that \( N \cap H^2(\Gamma_z) \neq \{0\} \) and \( N \cap H^2(\Gamma_w) \neq \{0\} \). We shall prove \( 1 \in N \). To prove this, suppose that \( 1 \notin N \). Let \( 1_w \) be the orthogonal projection of 1 to \( N \cap H^2(\Gamma_w) \). Then \( 1_w \notin H^2(\Gamma_z) \). Since \( N \cap H^2(\Gamma_z) \neq \{0\} \), there exists \( f \in N \cap H^2(\Gamma_z) \) such that \( \hat{f}(0) \neq 0 \). Let \( f_1 = f - \hat{f}(0)1_w \in N \). Then \( f_1 \notin H^2(\Gamma_z) \). Let \( h \in N \cap H^2(\Gamma_w) \). Since \( f \in H^2(\Gamma_z) \), \( f - \hat{f}(0) \perp h \). Since \( 1 - 1_w \perp N \cap H^2(\Gamma_w) \),

\[ \langle f_1, h \rangle = \langle \hat{f}(0)(1 - 1_w) + f - \hat{f}(0), h \rangle = 0. \]

Hence \( f_1 \in N \cap (N \cap H^2(\Gamma_w)) \). Thus (2.15) does not hold. Therefore \( 1 \in N \). Since \( N \cap H^2(\Gamma_z) \) and \( N \cap H^2(\Gamma_w) \) are nonzero backward shift invariant subspaces, by (2.16) \( N \) has one of forms in (iii) and (iv).

3. \( S_z^{n}S_w^* = S_w^*S_z^{n} \).

The following is the main theorem in this section.

**Theorem 3.1.** Let \( N \) be a backward shift invariant subspace of \( H^2 \), \( N \neq \{0\} \), and \( N \neq H^2 \). Let \( M = H^2 \ominus N \) and \( n \geq 2 \) be a positive integer. If \( S_z^{n}S_w^* = S_w^*S_z^{n} \), then one of the following conditions holds;

(i) \( S_zS_w^* = S_w^*S_z \),

(ii) \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \) for an inner function \( q_1(z) \) satisfying

\[ q_1(z) = \prod_{j=1}^{k} b_j(z), 1 \leq k \leq n, \text{ where } b_i \text{ are simple Blaschke products}. \]

The following lemma is a generalization of [IN], and the idea of the proof is the same.

**Lemma 3.2.** Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \ominus N \). Let \( n \) be a positive integer. Then \( S_z^{n}S_w^* = S_w^*S_z^{n} \) if and only if

\[ M \ominus \left( z^nM \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w) \right) \right) \subset (M \cap H^2(\Gamma_z)) \oplus wM. \]

**Proof.** The operators \( T_z^n \) and \( T_w^* \) on \( H^2 \) have the matrix forms as

\[
T_z^n = \begin{pmatrix}
* & P_M T_z^n |_N \\
0 & S_z^n
\end{pmatrix},
T_w^* = \begin{pmatrix}
* & P_N T_w^* |_M \\
0 & S_w^*
\end{pmatrix}
\]
on \( H^2 = \begin{pmatrix} M & \oplus \\
N & \end{pmatrix} \).
Set $A = P_MT_{z^n}|_N$ and $B = P_NT_{w^n}|_M$. Since $T_{z^n}T_{w^n}^* = T_{w^n}^*T_{z^n}$ on $H^2$, $S_{z^n}S_{w^n}^* = S_{w^n}S_{z^n}$ if and only if $BA = 0$. We have $T_{w^n}^*(M \ominus wM) \subset N$. For $f \in H^2$, $T_{w^n}^*f = 0$ if and only if $f \in H^2(\Gamma_z)$. Hence
\[
\ker B = \{ f \in M; T_{w^n}^*f \in M \} \\
= \{ f \in M \ominus wM; T_{w^n}^*f = 0 \} \oplus wM \\
= (M \cap H^2(\Gamma_z)) \oplus wM.
\]
We denote by $[\text{ran } A]$ the closed range of $A$. Let $A_1 = P_MT_{z^n}P_N$ on $H^2$. Then $[\text{ran } A] = [\text{ran } A_1]$. Since $A_1^* = P_NT_{z^n}^*P_M$, we get
\[
\ker A_1^* = \{ f \in M; T_{z^n}^*f \in M \} \oplus N \\
= \left( M \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w) \right) \oplus z^nM \oplus N.
\]
Hence
\[
[\text{ran } A] = [\text{ran } A_1] = H^2 \ominus \ker A_1^* = M \ominus \left( z^nM \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w) \right) \right).
\]
Since $BA = 0$ if and only if $[\text{ran } A] \subset \ker B$, we get our assertion. □

**Proof of Theorem 3.1.** Suppose that $S_{z^n}S_{w^n}^* = S_{w^n}S_{z^n}$. By Lemma 3.2,
\[
M \ominus \left( z^nM \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w) \right) \right) \subset (M \cap H^2(\Gamma_z)) \oplus wM.
\]
Let
\[
K_0 = M \ominus \left( z^nM \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w) \right) \right).
\]
Then
\[
(3.1) \quad K_0 \subset (M \cap H^2(\Gamma_z)) \oplus wM
\]
and
\[
(3.2) \quad M \ominus z^nM = K_0 \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w) \right).
\]
Since $M = \sum_{s=0}^{\infty} \oplus z^{ns}(M \ominus z^nM)$,
\[
(3.3) \quad K_0 \perp \sum_{s=0}^{\infty} \oplus z^{ns}\left( M \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w) \right)
\]
and
\[
(3.4) \quad M = \left( \sum_{s=0}^{\infty} \oplus z^{ns}K_0 \right) \oplus \left( \sum_{s=0}^{\infty} \oplus z^{ns}\left( M \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w) \right) \right).
\]
First, suppose that $K_0 = \{0\}$. In this case, we shall show that condition (i) holds. By (3.2),
\[
\sum_{i=0}^{n-1} \oplus z^i(M \ominus z M) = M \ominus z^n M = M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w).
\]
Let $f \in M \ominus z M$. Then $f \in M \ominus z^n M$ and $f = \sum_{i=0}^{n-1} \oplus z^i h_i(w)$, where $h_i(w) \in H^2(\Gamma_w)$. Since $f \in M \ominus z M$,
\[
z^{n-1} f = \sum_{i=0}^{n-1} \oplus z^{n-1+i} h_i(w) \in M \ominus z^n M.
\]
Hence $f = h_0(w)$, so that $M \ominus z M = M \cap H^2(\Gamma_w)$. By Lemma 3.2 (for the case of $n = 1$), we get $S_z S_w = S_{w}^* S_z$.

Next, suppose that $K_0 \neq \{0\}$. The proof of this case is a little bit long, so we devide into several steps.

**Step 1.** In this step, we shall prove that there is a nonconstant inner function $q_1(z)$ such that
\[
(3.5) \quad K_0 \subset q_1(z) H^2(\Gamma_z) \oplus w M.
\]
First we prove that $K_0 \not\subset wM$. To prove this, suppose that $K_0 \subset wM$. Let $F \in K_0$. Then $F = w f$ for some $f \in M$. We shall prove that $f \in K_0$. We have
\[
\langle f, \left( \sum_{s=1}^{\infty} \oplus z^{n s} K_0 \right) \ominus \left( \sum_{s=0}^{\infty} \oplus z^{n s} \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right) \rangle = 0,
\]
where the last equality follows from $w \sum_{s=1}^{\infty} \oplus z^{n(s-1)} K_0 \subset M$ and (3.2). Hence by (3.4), we get $f \in K_0$. Therefore for every positive integer $p$, we have $F = w^p f_p$ for some $f_p \in K_0$. This leads $F = 0$. This is a contradiction. Thus we get $K_0 \not\subset w M$.

Hence by (3.1), $M \cap H^2(\Gamma_z) \neq \{0\}$. By the Beurling theorem,
\[
(3.6) \quad M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z),
\]
where $q_1(z)$ is inner. Then $q_1(z) \in M$ and
\[
(3.7) \quad q_1(z) H^2(\Gamma_z) \perp w M.
\]
If $q_1(z)$ is constant, we have $M = H^2$, so that $N = \{0\}$. This contradicts our assumption. Hence $q_1(z)$ is a nonconstant inner function. By (3.1) and (3.6), we get (3.5).

**Step 2.** In this step, we prove

(3.8) \[ K_0 \subset q_1(z)\left(\sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\right). \]

Let $G \in K_0$. Then by (3.5), $G = q_1(z)h(z) \oplus wg$, where $h(z) \in H^2(\Gamma_z)$ and $g \in M$. Write

\[ h(z) = \left(\sum_{i=0}^{n-1} \oplus a_i z^i\right) \oplus z^n h_0(z). \]

Then

\[ G = q_1(z)\left(\sum_{i=0}^{n-1} \oplus a_i z^i\right) \oplus q_1(z)z^n h_0(z) \oplus wg. \]

By (3.6), $q_1(z)z^n h_0(z) \in z^n M$. Since $G \in K_0 \subset M \oplus z^n M$, we have $h_0(z) = 0$. Hence

(3.9) \[ G = q_1(z)\left(\sum_{i=0}^{n-1} \oplus a_i z^i\right) \oplus wg. \]

Here we prove that

(3.10) \[ g \in K_0. \]

Since $G = q_1(z)h(z) \oplus wg$, we have

\[ \left\langle g, \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rightangle = \left\langle wg, w \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rightangle \]

\[ = \left\langle wg, w \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rightangle + \left\langle q_1 h, w \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rightangle \text{ by (3.7)} \]

\[ = \left\langle G, w \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rightangle \]

\[ = \left\langle G, z^n w \sum_{s=1}^{\infty} \oplus z^{n(s-1)} K_0 \rightangle \]

\[ = 0 \text{ by (3.2)}. \]
We also have
\[ \langle g, \sum_{s=0}^{\infty} \oplus z^{n_s} \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \rangle \]
\[ = \langle wg, w \left( \sum_{s=0}^{\infty} \oplus z^{n_s} \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right) \rangle \]
\[ = \langle G, \sum_{s=0}^{\infty} \oplus z^{n_s} w \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \rangle \] by (3.7)
\[ = 0 \] by (3.3).

Hence by (3.4), we get (3.10).

Applying (3.9) and (3.10) infinitely many times, we have
\[ G = q_1(z) \left( \sum_{i=0}^{n-1} \oplus a_i z^i \right) \oplus \cdots \]
\[ \in q_1(z) \left( \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right). \]

Therefore we get (3.8).

Step 3. In this step, we study functions in \( M \ominus z M \) and the inner function \( q_1(z) \). By (3.8), there is a closed subspace \( L \) such that \( K_0 = q_1(z)L \) and \( L \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \). Then by (3.2),
\[ M \ominus z^n M = q_1(z)L \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right). \]

Since \( K_0 \neq \{0\} \), \( L \neq \{0\} \). We have \( M \ominus z^n M = \sum_{i=0}^{n-1} \oplus z^i (M \ominus z M) \). Hence \( M \ominus z M \neq \{0\} \). Let \( F \in M \ominus z M \) be such that \( F \neq 0 \). Then
\[ F = \left( q_1 \sum_{i=0}^{n-1} \oplus z^i f_i \right) \oplus \sum_{i=0}^{n-1} \oplus z^i g_i, \]
where \( f_i, g_i \in H^2(\Gamma_w) \),
\[ q_1 \sum_{i=0}^{n-1} \oplus z^i f_i \perp M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w), \]
and
\[ \sum_{i=0}^{n-1} \oplus z^i g_i \in M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w). \]
Since $n \geq 2$, $zF \in M \ominus z^n M$, so that we have

$$zF = q_1\left(\sum_{i=0}^{n-1} \oplus z^i G_{1,i}\right) + \sum_{i=0}^{n-1} \oplus z^i H_{1,i},$$

where $G_{1,i}, H_{1,i} \in H^2(\mathbb{T}_w)$. Hence (3.14)

$$q_1\left(z \left(\sum_{i=0}^{n-1} \oplus z^i f_i\right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}\right) = \left(\sum_{i=0}^{n-1} \oplus z^i H_{1,i}\right) - z \sum_{i=0}^{n-1} \oplus z^i g_i.$$

Here we devide into two subcases.

**Subcase 1.**

$$z \left(\sum_{i=0}^{n-1} \oplus z^i f_i(w)\right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w) \neq 0.$$ Then

$$q_1(z) = \frac{\left(\sum_{i=0}^{n-1} \oplus z^i H_{1,i}(w)\right) - z \sum_{i=0}^{n-1} \oplus z^i g_i(w)}{z \left(\sum_{i=0}^{n-1} \oplus z^i f_i(w)\right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w)}.$$ As proved in Step 1, $q_1(z)$ is a nonconstant inner function. Then by the above, we have

(3.15) $$q_1(z) = \prod_{j=1}^{k} b_j(z), \quad 1 \leq k \leq n,$$

where $b_j$ are simple Blaschke products.

**Subcase 2.**

$$z \left(\sum_{i=0}^{n-1} \oplus z^i f_i(w)\right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w) = 0.$$ Then by (3.14), $f_{n-1} = g_{n-1} = 0$, so that by (3.11)

$$F = \left(q_1 \sum_{i=0}^{n-2} \oplus z^i f_i\right) \oplus \sum_{i=0}^{n-2} \oplus z^i g_i.$$ Since $F \in M \ominus z M$,

$$zF = \left(q_1 \sum_{i=0}^{n-2} \oplus z^{i+1} f_i\right) \oplus \sum_{i=0}^{n-2} \oplus z^{i+1} g_i \in M \ominus z^2 M.$$ In the same way as above, either (3.15) holds or $f_{n-2} = g_{n-2} = 0$. Repeat the same argument. Then either (3.15) holds or

$$f_{n-1} = f_{n-2} = \cdots = f_1 = g_{n-1} = g_{n-2} = \cdots = g_1 = 0.$$
Thus we get (3.16). By Lemma 3.2 (for $n$ such that $q$ holds), Lemma is proved in [INS, Lemma 3.2].

Let $M = q(z)H^2$, so that $N = H^2 \ominus q(z)H^2$. By Theorem A, condition (i) holds.

So we assume that $g_0 \neq 0$ for some $F \in M \ominus zM$. We shall prove that

$$ (3.16) \quad (M \ominus zM) \ominus (M \cap H^2(\Gamma_w)) \subset (M \cap H^2(\Gamma_z)) \ominus wM. $$

We may assume that $(M \ominus zM) \ominus (M \cap H^2(\Gamma_w)) \neq \{0\}$. Let $G \in (M \ominus zM) \ominus (M \cap H^2(\Gamma_w))$ be such that $G \neq 0$. Then $G = q(z)h_1(w) \ominus h_2(w)$, where $q(z)h_1(w) \perp M \cap H^2(\Gamma_w)$ and $h_2(w) \in M \cap H^2(\Gamma_w)$. Hence $h_2(w) = G - q(z)h_1(w) \perp M \cap H^2(\Gamma_w)$. Therefore $h_2(w) = 0$. Since $q(z) \in M$,

$$ G = q(z)h_1(w) = h_1(0)q(z) + wq(z)\frac{h_1(w) - h_1(0)}{w}. $$

Thus we get (3.16). By Lemma 3.2 (for $n = 1$), (i) holds.

4. $S_zS_w^* = S_w^*S_z$ and $S_zS_w^* \neq S_w^*S_z$.

Let $N$ be a backward shift invariant subspace of $H^2$ and let $n$ be a positive integer. Let $M = H^2 \ominus N$. Then $M$ is an invariant subspace. If both $S_zS_w^* = S_w^*S_z$ and $S_zS_w^* \neq S_w^*S_z$ hold, then by Theorem 3.1, $M \cap H^2(\Gamma_z) = q(z)H^2(\Gamma_z)$ for some nonconstant inner function $q(z)$.

In this section, we assume that $q(z)H^2 \subset M$ and $M \cap H^2(\Gamma_z) = q(z)H^2(\Gamma_z)$ for some nonconstant inner function $q(z)$. Let

$$ \tilde{M} = M \ominus q(z)H^2 \subset M. $$

Then $H^2 \ominus q(z)H^2 = \tilde{M} \ominus N$ and $\tilde{M}$ is $w$-invariant. The following lemma is proved in [INS, Lemma 3.2].

**Lemma 4.1.** Let $f \in \tilde{M}$. Then $T^*_w f \in \tilde{M}$ if and only if $f \in w\tilde{M}$.

We denote by $P_\perp$ the orthogonal projection from $H^2$ onto $H^2 \ominus q(z)H^2$. Then we have a Toeplitz type operator $Q_{z^n}$ on $H^2 \ominus q(z)H^2$ such that

$$ Q_{z^n} : H^2 \ominus q(z)H^2 \ni f \mapsto P_\perp(T_{z^n}f) \in H^2 \ominus q(z)H^2. $$
Since \( z^n M \subset M \), \( Q_z M \subset \tilde{M} \) and \( Q_z^n = Q_z \). Then \( Q_z \) has the following matrix form:

\[
Q_z = \begin{pmatrix} \ast & P_M T_z | N \\ 0 & S_z \\ \end{pmatrix} \quad \text{on} \quad H^2 \oplus q_1(z)H^2 = \begin{pmatrix} \tilde{M} \\ N \end{pmatrix}.
\]

Since \( H^2 \oplus q_1(z)H^2 \) is backward shift invariant, \( T_w^* (H^2 \oplus q_1(z)H^2) \subset H^2 \oplus q_1(z)H^2 \). Since \( T_w^* N \subset N \), the operator \( T_w^* \) on \( H^2 \oplus q_1(z)H^2 \) has the following matrix form:

\[
T_w^* = \begin{pmatrix} \ast & 0 \\ P_N T_w^*_M & S_w^* \\ \end{pmatrix} \quad \text{on} \quad H^2 \oplus q_1(z)H^2 = \begin{pmatrix} \tilde{M} \\ N \end{pmatrix}.
\]

Set

\[
(4.1) \quad A = P_M T_z | N \quad \text{and} \quad B = P_N T_w^*_M.
\]

By [INS, Lemma 3.3], \( T_w^* Q_z = Q_z T_w^* \) on \( H^2 \oplus q_1(z)H^2 \). Hence we have the following.

**Lemma 4.2.** \( T_w^* Q_z = Q_z T_w^* \) on \( H^2 \oplus q_1(z)H^2 \).

**Lemma 4.3.** \( S_z^n S_w^n = S_w^n S_z^n \) if and only if \( BA = 0 \).

**Proof.** By Lemma 4.2, \( T_w^* Q_z = Q_z T_w^* \) on \( H^2 \oplus q_1(z)H^2 \). Then \( BA + S_w^n S_z^n = S_z^n S_w^n \). Hence \( S_z^n S_w^n = S_w^n S_z^n \) if and only if \( BA = 0 \).

The following is a slight generalization of [INS, Theorem 3.5].

**Theorem 4.4.** Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \oplus N \). Suppose that \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \), where \( q_1(z) \) is a nonconstant inner function. Let \( \tilde{M} = M \ominus q_1(z)H^2 \). Then the following conditions are equivalent:

(i) \( S_z^n S_w^n = S_w^n S_z^n \) on \( N \).

(ii) \( \tilde{M} \ominus \{ f \in \tilde{M}; T_z^n f \in \tilde{M} \} \subset w\tilde{M} \).

(iii) \( T_z^n \tilde{M} \subset \tilde{M} \).

**Proof.** The idea of the proof is similar to the one of [INS, Theorem 3.5]. For the sake of completeness, we give the proof.

(i) \( \iff \) (ii) By Lemma 4.3, condition (i) is equivalent to \( BA = 0 \). By (4.1) and Lemma 4.1, \( \ker B = \{ f \in \tilde{M}; T_z^n f \in \tilde{M} \} = w\tilde{M} \). Put \( A_1 = P_M T_z^n P_N \) on \( \tilde{M} \ominus N \). Then \( \text{ran} A = [\text{ran} A_1] \). Since \( A_1^* = P_N T_z^n P_M \), \( \ker A_1^* = N \ominus \{ f \in \tilde{M}; T_z^n f \in \tilde{M} \} \). Hence

\[
\text{ran} A = [\text{ran} A_1] = (\tilde{M} \ominus N) \ominus \ker A_1^* = (\tilde{M} \ominus N \ominus \{ f \in \tilde{M}; T_z^n f \in \tilde{M} \}).
\]

Therefore \( BA = 0 \) if and only if \( \tilde{M} \ominus \{ f \in \tilde{M}; T_z^n f \in \tilde{M} \} \subset w\tilde{M} \).

Thus we get (i) \( \iff \) (ii).
\( \text{Corollary 4.6.} \) Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \ominus N \). Suppose that \( M \cap H^2(\Gamma_z) = b_1(z)H^2(\Gamma_z) \), where \( b_1(z) \) is a simple Blaschke product. Then \( S_zS_w^* = S_w^*S_z \).

\textbf{Proof.} Let \( b_1(z) = (z - \alpha)/(1 - \overline{\alpha}z) \), \( |\alpha| < 1 \), and \( M = M \ominus b_1(z)H^2 \). Since \( b_1(z) \in M \), \( b_1(z)H^2 \subset M \). By Lemma 4.5(iv), \( M \subset b_1(z)H^2(\Gamma_w) \). By Lemma 4.5(i), \( T^*_w(b_1(z)h(w)) = \overline{b_1(z)}h(w) \). Hence \( T^*_wM \subset M \). By Theorem 4.4, \( S_zS_w^* = S_w^*S_z \).

\textbf{Corollary 4.7.} Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \ominus N \). Suppose that \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \), where \( q_1(z) \) is an inner function. Let \( n, k \) be positive integers with \( n \geq k + 1 \). Moreover suppose that \( q_1(z) = z^kb(z) \), where \( b \) is a simple Blaschke product, \( b(z) = (z - \alpha)/(1 - \overline{\alpha}z) \), and \( \alpha \neq 0 \). If \( S_zS_w^* = S_w^*S_z^n \), then \( S_zS_w^* = S_w^*S_{z^k} \).
Proof. Let $\tilde{M} = M \ominus q_1(z)H^2$. If $\tilde{M} = \{0\}$, then $M = q_1(z)H^2$. By Theorem A, $S_zS_w^* = S_w^*S_z$. Suppose that $\tilde{M} \neq \{0\}$. By Lemma 4.5(iv),
\[ \tilde{M} \subset b^*(z)H^2(\Gamma_w) \oplus \left( \sum_{j=1}^{k} \oplus z^{j-1}b(z)H^2(\Gamma_w) \right). \]
Let $f \in \tilde{M}$. Then
\[ f = b^*(z)h_0(w) + \sum_{j=1}^{k} \oplus z^{j-1}b(z)h_j(w), \quad h_j(w) \in H^2(\Gamma_w). \]
By Lemma 4.5(i),
\[ T_{z^n}^*f = T_{z^{n-k}}^*(T_{z^k}^*f) = T_{z^{n-k}}^* \left( \sum_{j=0}^{k} \alpha^{(k-j)}b^*(z)h_j(w) \right) = \alpha^{(n-k)} \sum_{j=0}^{k} \alpha^{(k-j)}b^*(z)h_j(w). \]
Since $S_{z^n}S_w^* = S_w^*S_{z^n}$, by Theorem 4.4 $T_{z^n}^*f \in \tilde{M}$. Since $\alpha \neq 0$,
\[ \sum_{j=0}^{k} \alpha^{(k-j)}b^*(z)h_j(w) \in \tilde{M}. \]
Thus $\tilde{T}_{z^n}^*\tilde{M} \subset \tilde{M}$. By Theorem 4.4, $S_{z^n}S_w^* = S_w^*S_{z^n}$.

Theorem 4.8. Let $N$ be a backward shift invariant subspace of $H^2$ and $M = H^2 \ominus N$. Suppose that $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$, where $q_1(z)$ is an inner function. Moreover suppose that $q_1(z) = b_1(z)b_2(z)$, where $b_i(z), i = 1, 2$, are simple Blaschke products, $b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}_iz)$, and $\alpha_1\alpha_2 \neq 0$. Let $n \geq 2$ be a positive integer. Then we have the following.

(i) If $S_{z^n}S_w^* = S_w^*S_{z^n}$ and $S_{z^{n-1}}S_w^* \neq S_w^*S_{z^{n-1}}$, then $\alpha_1 = \alpha_2$ and $\alpha_1 \neq \alpha_2$.

(ii) If $\alpha_1 = \alpha_2$ and $\alpha_1 \neq \alpha_2$, then $S_{z^n}S_w^* = S_w^*S_{z^n}$.

Proof. Let $\tilde{M} = M \ominus q_1(z)H^2$. Suppose that $S_{z^n}S_w^* = S_w^*S_{z^n}$ and $S_{z^{n-1}}S_w^* \neq S_w^*S_{z^{n-1}}$. By Theorem 4.4, $T_{z^n}^*\tilde{M} \subset \tilde{M}$ and $T_{z^{n-1}}^*\tilde{M} \nsubseteq \tilde{M}$. By Lemma 4.5(iv),
\[ \tilde{M} \subset b_1^*(z)H^2(\Gamma_w) \oplus b_1(z)b_2^*(z)H^2(\Gamma_w). \]
Then there exists $f_0 \in \tilde{M}$ such that $T_{z^{n-1}}^*f_0 \notin \tilde{M}$, and
\[ f_0 = b_1^*(z)g_1(w) + b_1(z)b_2^*(z)g_2(w) \in b_1^*(z)H^2(\Gamma_w) \oplus b_1(z)b_2^*(z)H^2(\Gamma_w). \]
By Lemma 4.5,
\[ T_{z_n-1}^* f_0 = b_1^* \left( \alpha_1^{(n-1)} g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-2} \alpha_1^{(n-2-j)} \alpha_2^j \right) g_2 \right) + \alpha_2^{(n-1)} b_1 b_2^* g_2 \]
and
\[ T_{z_n}^* f_0 = b_1^* \left( \alpha_1^n g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \alpha_2^j \right) g_2 \right) + \alpha_2^n b_1 b_2^* g_2. \]
Since \( T_{z_n-1}^* f_0 \not\in \hat{M} \) and \( f_0 \in \hat{M}, T_{z_n-1}^* f_0 - \alpha_2^{n-1} f_0 \not\in \hat{M}. \) Then
\[ b_1^* \left( (\bar{\alpha}_1^{(n-1)} - \alpha_2^{(n-1)}) g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-2} \alpha_1^{(n-2-j)} \alpha_2^j \right) g_2 \right) \not\in \hat{M}. \]
Hence
\[ \left( \sum_{j=0}^{n-2} \alpha_1^{(n-2-j)} \alpha_2^j \right) b_1^* \left( (\bar{\alpha}_1 - \alpha_2) g_1 + (1 - |\alpha_2|^2) g_2 \right) \not\in \hat{M}. \]
Since \( 0 \in \hat{M}, \sum_{j=0}^{n-2} \alpha_1^{(n-2-j)} \alpha_2^j \neq 0 \), so that
\[ (4.3) \quad b_1^* \left( (\bar{\alpha}_1 - \alpha_2) g_1 + (1 - |\alpha_2|^2) g_2 \right) \not\in \hat{M}. \]
Since \( T_{z_n}^* f_0 \in \hat{M}, T_{z_n}^* f_0 - \alpha_2^n f_0 \in \hat{M}. \) Then
\[ b_1^* \left( (\bar{\alpha}_1^n - \alpha_2^n) g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \alpha_2^j \right) g_2 \right) \in \hat{M}. \]
Hence
\[ (4.4) \quad \left( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \alpha_2^j \right) b_1^* \left( (\bar{\alpha}_1 - \alpha_2) g_1 + (1 - |\alpha_2|^2) g_2 \right) \in \hat{M}. \]
Now we prove (i). Suppose that \( \alpha_1 = \alpha_2. \) Then \( \alpha_1 = \alpha_2 \neq 0, \) so that
\[ \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \alpha_2^j \neq 0. \] By (4.3) and (4.4), \( b_1^* g_2 \not\in \hat{M} \) and \( b_1^* g_2 \in \hat{M}. \)
This is a contradiction. Hence \( \alpha_1 \neq \alpha_2. \)
Suppose that \( \alpha_1^n = \alpha_2^n. \) Then \( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \alpha_2^j \neq 0. \) By (4.4),
\[ b_1^* \left( (\bar{\alpha}_1 - \alpha_2) g_1 + (1 - |\alpha_2|^2) g_2 \right) \in \hat{M}. \]
This contradicts (4.3), so that \( \alpha_1^n = \alpha_2^n. \) Thus we get (i).
(ii) Suppose that \( \alpha_1^n = \alpha_2^n \) and \( \alpha_1 \neq \alpha_2. \) Then \( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \alpha_2^j = 0. \)
Let \( f \in \hat{M}. \) Then by (4.2),
\[ f = b_1^* (z) h_1(w) + b_1(z) b_2^* (z) h_2(w). \] Similarly,
we have
\[ T_{n}^{*}f - \bar{\alpha}_{j}f = \left( \sum_{j=0}^{n-1} \bar{\alpha}_{1}^{j}/\bar{\alpha}_{2}^{j} \right) \bar{\alpha}_{1}^{j}[(\bar{\alpha}_{1} - \bar{\alpha}_{2})h_{1} + (1 - |\alpha_{2}|^{2})h_{2}]. \]

Hence \( T_{n}^{*}f = \bar{\alpha}_{j}f \in \tilde{M} \), so that we get \( T_{n}^{*}\tilde{M} \subset \tilde{M} \). By Theorem 4.4, \( S_{\infty}^{*}S_{\infty}^{*} = S_{\infty}^{*}S_{\infty}^{*} \).

**Corollary 4.9.** Let \( N \) be a backward shift invariant subspace of \( H^{2} \) and \( M = H^{2} \oplus N \). Suppose that \( M \cap H^{2}(\Gamma_{z}) = q_{1}(z)H^{2}(\Gamma_{z}), \) where \( q_{1}(z) \) is an inner function. Moreover suppose that \( q_{1}(z) = b_{1}(z)b_{2}(z), \) where \( b_{i}(z), i = 1, 2, \) are simple Blaschke products, \( b_{i}(z) = (z - \alpha_{i})/(1 - \bar{\alpha}_{i}z), \) and \( \alpha_{1}\alpha_{2} \neq 0 \). Then we have the following.

(i) If \( S_{\infty}^{*}S_{\infty}^{*} = S_{\infty}^{*}S_{\infty}^{*} \) and \( S_{\infty}^{*}S_{\infty}^{*} \neq S_{\infty}^{*}S_{\infty}^{*} \), then \( \alpha_{1} + \alpha_{2} = 0. \)

(ii) If \( \alpha_{1} + \alpha_{2} = 0, \) then \( S_{\infty}^{*}S_{\infty}^{*} = S_{\infty}^{*}S_{\infty}^{*} \).

The following is the main theorem in this section.

**Theorem 4.10.** Let \( N \) be a backward shift invariant subspace of \( H^{2}, \) \( N \neq \{0\}, \) and \( N \neq H^{2}. \) Then \( S_{\infty}^{*}S_{\infty}^{*} = S_{\infty}^{*}S_{\infty}^{*} \) if and only if one of the following conditions holds.

(i) \( S_{\infty}^{*}S_{\infty}^{*} = S_{\infty}^{*}S_{\infty}^{*} \).

(ii) \( S_{\infty}^{*}S_{\infty}^{*} = 0. \)

(iii) There exist two simple Blaschke products \( b_{1}(z) \) and \( b_{2}(z), \) \( b_{i}(z) = (z - \alpha_{i})/(1 - \bar{\alpha}_{i}z), 0 < |\alpha_{i}| < 1, \) such that \( N \subset H^{2} \oplus b_{1}(z)b_{2}(z)H^{2} \) and \( \alpha_{1} + \alpha_{2} = 0. \)

A backward shift invariant subspace \( N \) satisfying condition (i) is given by Theorem A. Also \( N \) satisfying condition (ii) is given by Theorem 2.2.

**Proof of Theorem 4.10.** Suppose that \( S_{\infty}^{*}S_{\infty}^{*} = S_{\infty}^{*}S_{\infty}^{*} \). Moreover suppose that \( S_{\infty}^{*}S_{\infty}^{*} \neq S_{\infty}^{*}S_{\infty}^{*} \). By Theorem 3.1, \( M \cap H^{2}(\Gamma_{z}) = q_{1}(z)H^{2}(\Gamma_{z}) \) for an inner function \( q_{1}(z) \) such that either \( q_{1}(z) = b_{1}(z) \) or \( q_{1}(z) = b_{1}(z)b_{2}(z), \) where \( b_{i}(z) = (z - \alpha_{i})/(1 - \bar{\alpha}_{i}z), |\alpha_{i}| < 1. \) If \( q_{1}(z) = b_{1}(z), \) by Corollary 4.6, (i) holds.

Suppose that \( q_{1}(z) = b_{1}(z)b_{2}(z). \) Moreover suppose that \( \alpha_{1} = \alpha_{2} = 0. \) Then \( q_{1}(z) = z^{2}. \) Hence \( z^{2}H^{2} \subset M, \) so that \( N \subset H^{2}(\Gamma_{w}) \oplus zH^{2}(\Gamma_{w}). \) Then by Theorem 2.2, \( S_{\infty}^{*}S_{\infty}^{*} = 0. \) Thus (ii) holds.

Suppose that \( \alpha_{1} \neq 0 \) and \( \alpha_{2} = 0. \) Then \( q_{1}(z) = zb_{1}(z). \) By Corollary 4.7, (i) holds.

Suppose that \( \alpha_{1} \neq 0 \) and \( \alpha_{2} \neq 0. \) Then by Corollary 4.9(i), we get \( \alpha_{1} + \alpha_{2} = 0. \) Hence (iii) holds.
Next, we prove the converse assertion. Suppose that one of conditions (i), (ii), and (iii) holds. If (i) holds, then trivially $S_z S_w^* = S^*_w S_z$.

If (ii) holds, by Corollary 2.3 we have $S_z S_w^* = S^*_w S_z$.

Suppose that (iii) holds. Then $b_1(z) b_2(z) H^2(\Gamma_z) \subseteq M$. Hence $M \cap H^2(\Gamma_z)$ equals to either $b_1(z) H^2(\Gamma_z)$, or $b_2(z) H^2(\Gamma_z)$, or $b_1(z) b_2(z) H^2(\Gamma_z)$. By Corollary 4.6, $S_z S_w^* = S^*_w S_z$ for the first two cases. Hence $S_z S_w^* = S^*_w S_z$ for the last case, by Corollary 4.9(ii), $S_z S_w^* = S^*_w S_z$.

\[ \square \]

**Example 4.11.** We give an example of a backward shift invariant subspace $N$ of $H^2$ satisfying $S_z S_w^* = S^*_w S_z$, $S_z S_w^* \neq S^*_w S_z$, and $S_z S_w^* \neq 0$. Let $q_1(z) = b_1(z) b_2(z)$, where $b_i(z)$, $i = 1, 2$, are $b_1(z) = (z - \alpha_1)/(1 - \overline{\alpha}_1 z)$, $\alpha_1 + \alpha_2 = 0$, and $\alpha_1 \neq 0$. Let $q_2(w)$ be a non-constant inner function. Let $M = q_1(z) H^2 \oplus b_1(z) b_2(z) q_2(w) H^2(\Gamma_w)$. Then $M$ is an invariant subspace and $M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z)$. Let $N = H^2 \oplus M$. By Theorem 4.10, $S_z S_w^* = S^*_w S_z$. We have $\tilde{M} = M \cap q_1(z) H^2 = b_1(z) b_2(z) q_2(w) H^2(\Gamma_w)$. Since

$$T^* b_1(z) b_2(z) q_2(w) = (1 - |\alpha|^2) b_1(z) q_2(w) + \overline{\alpha}_1 b_1(z) b_2(z) q_2(w),$$

$T^* b_1(z) b_2(z) q_2(w) \notin \tilde{M}$. By Theorem 4.4, $S_z S_w^* \neq S^*_w S_z$. By Theorem 2.2, $S_z S_w^* \neq S^*_w S_z$.

We leave the following problem for the reader.

**Problem 4.12.** Characterize backward invariant subspaces $N$ of $H^2$ satisfying $S_z S_w^* = S^*_w S_z$ for $n \geq 3$.

**Acknowledgement.** The authors would like to thank the referee for many comments improving the original manuscript.

**References**


Department of Mathematics, Niigata University, Niigata, 950-2181, Japan

E-mail address: izuchi@math.sc.niigata-u.ac.jp

Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan

E-mail address: nakazi@math.sci.hokudai.ac.jp

Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan

E-mail address: seto@math.sci.hokudai.ac.jp