ABSTRACT. In the previous paper, we give a characterization of backward shift invariant subspaces of the Hardy space in the bidisc which satisfy the doubly commuting condition $S_zS_w^* = S_w^*S_z$ for the compression operators $S_z$ and $S_w$. In this paper, we give a characterization of backward shift invariant subspaces satisfying $S_z^2S_w^* = S_w^*S_z^2$.

1. INTRODUCTION.

Let $\Gamma^2$ be the 2-dimensional unit torus. We denote by $(z, w) = (e^{i\theta}, e^{i\phi})$ the variables in $\Gamma^2 = \Gamma_z \times \Gamma_w$. Let $L^2 = L^2(\Gamma^2)$ be the usual Lebesgue space on $\Gamma^2$ with the norm $\|f\|_2 = (\int_{\Gamma^2} |f(e^{i\theta}, e^{i\phi})|^2 d\theta d\phi / (2\pi)^2)^{1/2}$. Then $L^2$ is a Hilbert space with the usual inner product. For $f \in L^2$, the Fourier coefficients are given by

$$\hat{f}(n, m) = \int_{\Gamma^2} f(e^{i\theta}, e^{i\phi}) e^{-in\theta} e^{-im\phi} d\theta d\phi / (2\pi)^2 = \langle f, z^n w^m \rangle.$$ 

Let $H^2 = H^2(\Gamma^2)$ be the Hardy space on $\Gamma^2$, that is,

$$H^2 = \{ f \in L^2; \hat{f}(n, m) = 0 \text{ if } n < 0 \text{ or } m < 0 \}.$$

For $f \in H^2$, we can write $f$ as

$$f = \sum_{i,j=0}^{\infty} a_{i,j} z^i w^j, \text{ where } \sum_{i,j=0}^{\infty} |a_{i,j}|^2 < \infty.$$

Let $P$ be the orthogonal projection from $L^2$ onto $H^2$. For a closed subspace $M$ of $L^2$, we denote by $P_M$ the orthogonal projection from $L^2$ onto $M$. For a function $\psi \in L^\infty$, let $L_\psi f = \psi f$ for $f \in L^2$. The Toeplitz operator $T_\psi$ is defined by $T_\psi f = PL_\psi f$ for $f \in H^2$. It is well known that $T_\psi^* = T_{\psi^*}$ and $T_\psi^* T_m = T_{wm} T_{\psi^*}^*$ for $n, m \geq 1$. A function $f \in H^2$ is called inner if $|f| = 1$ on $\Gamma^2$ almost everywhere. A closed subspace $M$ of $H^2$ is called invariant if $z M \subset M$ and $w M \subset M$. In one variable
case, an invariant subspace $M$ of $H^2(\Gamma)$ has a form $M = qH^2(\Gamma)$, where $q$ is inner. This is the well known Beurling theorem [B]. In two variable case, the structure of invariant subspaces of $H^2$ is very complicated, see [AC], [DY], [Na1], and [R].

Let $M$ be an invariant subspace of $H^2$. Then $T^*_z(H^2 \ominus M) \subset (H^2 \ominus M)$ and $T^*_w(H^2 \ominus M) \subset (H^2 \ominus M)$. A closed subspace $N$ of $H^2$ is called backward shift invariant if $H^2 \ominus N$ is invariant. In [IY], the first author and Yang studied backward shift invariant subspaces $N$ on which $T^*_z$ is strictly contractive. See [CR] and [S] for studies of backward shift invariant subspaces on the unit circle $\Gamma$.

Let $M$ be an invariant subspace of $H^2$ and $\psi \in L^\infty$. Let $V_\psi$ be the operator on $M$ defined by $V_\psi = P_M L_\psi | M$. Then $V_z = T_z$ and $V^*_z = V_z$ on $M$. In [M], Mandrekar proved that $V_z V^*_w = V^*_w V_z$ on $M$ if and only if $M$ is Beurling type, that is, $M = qH^2$ for some inner function $q$ in $H^\infty$, see also [CS] and [Na2].

For $\psi \in L^\infty$, let $S_\psi = P_N L_\psi | N$. Then we have $S_\psi^* = S^*_\psi$ and $S^*_z = T^*_z$ on $N$. In the previous paper [INS], we characterized backward shift invariant subspaces $N$ which satisfy the condition $S_z S^*_w = S^*_w S_z$ on $N$ as follows.

**Theorem A.** Let $N$ be a backward shift invariant subspace of $H^2$ and $N \neq H^2$. Then $S_z S^*_w = S^*_w S_z$ on $N$ if and only if $N$ has one of the following forms:

(i) $N = H^2 \ominus q_1(z)H^2$,
(ii) $N = H^2 \ominus q_2(w)H^2$,
(iii) $N = (H^2 \ominus q_1(z)H^2) \cap (H^2 \ominus q_2(w)H^2)$,

where $q_1(z)$ and $q_2(w)$ are one variable inner functions.

This paper is a continuation of previous papers [IN] and [INS]. Here we are interesting in backward shift invariant subspaces $N$ of $H^2$ satisfying $S_z^n S^*_w^m = S^*_w^m S_z^n$ for given positive integers $n$ and $m$. Up to now, we can not give a complete characterization of $N$ satisfying $S_z^n S^*_w^m = S^*_w^m S_z^n$. If $S_z^n S^*_w^m = S^*_w^m S_z^n$, then trivially $S_z^n S^*_w^m = S^*_w^m S_z^n$. But the converse is not true. In this paper, we concentrated on the case $m = 1$. One reason is the work of this paper deeply concerns with the problem of the normality of the operators $S_z^n$. For many backward shift invariant subspaces $N$, $S_z^n$ are not normal operators, see [Y]. If $S_z^n$ is normal, since $S_z^n S_w = S_w S_z^n$, by the Fuglede-Putnam theorem we have $S_z^n S^*_w = S^*_w S_z^n$. So, the results obtained in this paper will be a big help for the above mentioned problem. We will discuss on this subject in the forthcoming paper.
In Section 2, we give characterizations of $N$ satisfying $S^*_{zw} = 0$ and $S^*_{wm} S^*_{zn} = 0$, respectively. If $S^*_{zw} = 0$, then $S^*_{wm} S^*_{zn} = 0$. The converse is not true. In Section 3, we study $N$ satisfying $S^*_{zw} = S^*_{zw} S^*_{zn}$, and give a necessary condition for $S^*_{zw} S^*_{zn} = S^*_{zn} S^*_{zw}$. In Section 4, we study $N$ satisfying $S^*_{zw} S^*_{zn} = S^*_{zn} S^*_{zw}$. We gave a complete characterization of such $N$. In [INS], we gave two different type of proofs of Theorem A. One of them is used in Section 3 and the other is used in Section 4.

Let $H^2_z$ and $H^2_w$ be the Hardy spaces on the unit circle in variables $z$ and $w$, respectively. We think that $H^2_z$ and $H^2_w$. For a subset $E$ of $H^2$, we denote by $[E]$ the closed linear span of $E$. A function $b(z) = (z - \alpha)/(1 - \overline{\alpha} z), |\alpha| < 1$, is called a simple Blaschke product.

Let $S^*_{zw} = 0$ or $S^*_{zn} S^*_{zw} = 0$.

Let $n$ and $m$ be positive integers. In this section, we study backward shift invariant subspaces $N$ of $H^2$ satisfying $S^*_{zw} = 0$ and $S^*_{wm} S^*_{zn} = 0$, respectively.

Lemma 2.1. Let $N$ be a backward shift invariant subspace of $H^2$. Then we have the following.

(i) $S^*_{zn} = S^*_{zn}$.

(ii) $S^*_{zw} S^*_{zn} = S^*_{zn} S^*_{zw}$ and $S^*_{zn} S^*_{zw} = S^*_{zn} S^*_{zn}$.

(iii) If $S^*_{zn} S^*_{zw} N \neq \{0\}$, then there exists $f \in N$ such that $(S^*_{zn} S^*_{zw} f) \cap (0, 0) \neq 0$.

Proof. All assertions are not difficult to prove.

The following theorem says that the structure of backward shift invariant subspaces satisfying $S^*_{zn} S^*_{zw} = 0$ is simple.

Theorem 2.2. Let $N$ be a backward shift invariant subspace of $H^2$. Then $S^*_{zn} S^*_{zw} = 0$ if and only if $N$ satisfies one of the following conditions:

(i) $N \subset \sum_{j=0}^{m-1} \oplus w^j H^2_z$.

(ii) $N \subset \sum_{i=0}^{n-1} \oplus z^i H^2_w$.

Proof. Suppose that $S^*_{zn} S^*_{zw} = 0$. Then

(2.1) $S^*_{zm} N \perp S^*_{zn} N$.

Since $N$ is backward shift invariant, if $S^*_{zm} N = \{0\}$ then $N$ satisfies condition (i). If $S^*_{zn} N = \{0\}$, then $N$ satisfies (ii).

Next, suppose that

(2.2) $S^*_{zm} N \neq \{0\}$ and $S^*_{zn} N \neq \{0\}$. 
We shall lead a contradiction. By (2.1), $S_{w_1}^*S_{z_2}^*, N \perp S_{z_2}^*, S_{w_1}^*, N$. By Lemma 2.1(ii), $S_{w_1}^*, S_{z_2}^*, N = S_{z_2}^*, S_{w_1}^*, N = \{0\}$. Then

$$S_{z_2}^*, N \subseteq \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$$

and

$$S_{w_1}^*, N \subseteq \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w).$$

By (2.2) and (2.3), there exists a nonnegative integer $j, 0 \leq j \leq m - 1$, such that

$$\{0\} \neq S_{w_1}^*, S_{z_2}^*, N \subseteq H^2(\Gamma_z).$$

By Lemma 2.1(iii), there exists $g \in N$ such that

$$S_{w_1}^*, S_{z_2}^*, g \geq (0, 0) \neq 0.$$ 

Also by (2.2) and (2.4), there exist $f \in N$ and a nonnegative integer $i, 0 \leq i \leq n - 1$, such that

$$S_{z_2}^*, S_{w_1}^*, f \in H^2(\Gamma_w)$$

and

$$S_{z_2}^*, S_{w_1}^*, f \geq (0, 0) \neq 0.$$ 

Then

$$0 = \langle S_{w_1}^*, S_{z_2}^*, f, S_{z_2}^*, S_{w_1}^*, g \rangle \quad \text{by (2.1)}$$

$$= \langle S_{z_2}^*, S_{w_1}^*, f, S_{w_1}^*, S_{z_2}^*, g \rangle \quad \text{by Lemma 2.1(ii)}$$

$$= (S_{z_2}^*, S_{w_1}^*, f \geq (0, 0) (S_{w_1}^*, S_{z_2}^*, g \geq (0, 0) \neq 0 \quad \text{by (2.5) and (2.7)}$$

$$0 \neq 0 \quad \text{by (2.6) and (2.8).}$$

This is a desired contradiction.

The converse is trivial. □

**Corollary 2.3.** Let $N$ be a backward shift invariant subspace of $H^2$. Then $S_{z_2}^*, S_{w_1}^* = 0$ if and only if either $S_{z_2}^* = 0$ or $S_{w_1}^* = 0$. Hence if $S_{z_2}^*, S_{w_1}^* = 0$, then $S_{w_1}^*, S_{z_2}^* = 0$.

**Lemma 2.4.** Let $M_1$ and $M_2$ be closed subspaces of $H^2$ such that

$$M_1 \subseteq \sum_{j=0}^{m} \oplus w^j H^2(\Gamma_z) \quad \text{and} \quad M_2 \subseteq \sum_{i=0}^{n} \oplus z^i H^2(\Gamma_w).$$

Then $M_1 + M_2$ is closed.
Proof. We denote by \((z^iw^j)_{M_1}\) and \((z^iw^j)_{M_2}\) the orthogonal projections of \(z^iw^j\) to the spaces \(M_1\) and \(M_2\), respectively. Let

\[
M'_1 = M_1 \ominus \left( \{(z^iw^j)_{M_1}; 0 \leq i \leq n, 0 \leq j \leq m\} \right)
\]

and

\[
M'_2 = M_2 \ominus \left( \{(z^iw^j)_{M_2}; 0 \leq i \leq n, 0 \leq j \leq m\} \right).
\]

Then \(M'_1\) and \(M'_2\) are closed subspaces of \(H^2\),

\[
M'_1 \subset z^{n+1} \left( \sum_{j=0}^{m} \oplus w^j H^2(\Gamma_z) \right), \quad M'_2 \subset w^{m+1} \left( \sum_{i=0}^{n} \oplus z^j H^2(\Gamma_w) \right),
\]

and

\[
M'_1 + M'_2 \perp \{(z^iw^j); 0 \leq i \leq n, 0 \leq j \leq m\}.
\]

Since

\[
z^{n+1} \left( \sum_{j=0}^{m} \oplus w^j H^2(\Gamma_z) \right) \perp w^{m+1} \left( \sum_{i=0}^{n} \oplus z^j H^2(\Gamma_w) \right),
\]

\(M'_1 + M'_2 = M'_1 \oplus M'_2\) is closed. Hence

\[
M_1 + M_2 = M'_1 + M'_2 + \{(z^iw^j)_{M_1}, (z^iw^j)_{M_2}; 0 \leq i \leq n, 0 \leq j \leq m\}
\]

is closed. \(\square\)

Theorem 2.5. Let \(N\) be a backward shift invariant subspace of \(H^2\). Then \(S_{w^m}S_{z^n} = 0\) if and only if

\[(i) \quad N \ominus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \subset N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z).\]

Proof. Suppose that \(S_{w^m}S_{z^n} = 0\). Then \(S_{z^n}N \subset \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)\).

Since \(S_{z^n}S_{w^m} = 0\), \(S_{w^m}N \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\). First, we prove the following

\[
N = \left( N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \right) + \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right).
\]

Let

\[
K = \left( N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \right) + \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right).
\]

By Lemma 2.4, \(K\) is closed and \(N = K \ominus (N \ominus K)\). To prove (i), it is sufficient to prove \(N \ominus K = \{0\}\). We have \(N \ominus K \subset N \cap (S_{z^n}N)^\perp \cap (S_{w^m}N)^\perp\). Let \(f \in N \ominus K\). Then \(f \perp S_{z^n}N\), so that \(S_{z^n}f \perp N\). Since
$S^*_z f \in N$, $S^*_z f = 0$. Hence $f \in \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$. By (2.10), $f \in K$. This shows $f = 0$, so that $N \varsubsetneq K = \{0\}$. Thus we get (2.9).

Let

$$
(2.11) \quad N_1 = \left\{ f \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z); f \perp N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right\}.
$$

Then $N_1$ is a closed subspace and

$$
N_1 \oplus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \subset N.
$$

If the equality holds in the above, (i) holds. So we assume that

$$
N_1 \oplus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \nsubseteq N.
$$

We shall lead a contradiction. Let

$$
(2.12) \quad N_2 = N \ominus \left( N_1 \oplus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right).
$$

Then $N_2 \neq \{0\}$ and $N = N_1 \oplus N_2 \oplus (N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w))$.

Let $g \in N_2$ be such that $g \neq 0$. We shall prove that

$$
(2.13) \quad g \notin N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \quad \text{and} \quad g \notin N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z).
$$

The fact $g \notin N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$ is trivial. Suppose that $g \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$. By (2.11), $g \in N_1$. Since $g \in N_2$, by (2.12) we have $g \perp N_1$. Hence $g = 0$. This is a contradiction. Thus we get (2.13).

Next, we shall prove that

$$
(2.14) \quad S^*_z g \perp N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z).
$$

To prove this, suppose not. Then $S^*_z g \perp N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$.

By (2.10), $g = g_1 + g_2$, where $g_1 \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$ and $g_2 \in N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$. Then $S^*_z g = S^*_z g_1 \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$. Therefore $S^*_z g = 0$, so that $g \in N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$. This contradicts (2.13). Thus we get (2.14).

By (2.14), there exists $h_0$ such that $h_0 \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$ and $\langle g, S^*_z h_0 \rangle = \langle S^*_z g, h_0 \rangle \neq 0$. Since $S^*_z h_0 \in N$, by (2.12) we have $S^*_z h_0 = h_1 \oplus h_2 \oplus h_3$, where $h_1 \in N_1, h_2 \in N_2$, and $h_3 \in$
Then let Theorem 2.6.\(N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\). Since \(g \in N_2\) and \(\langle g, S_{z^n} h_0 \rangle \neq 0\), we have \(h_2 \neq 0\). Since \(z^n h_0 \perp \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w),\)

\[ P_N z^n h_0 = S_{z^n} h_0 \perp N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w). \]

Thus we get \(h_3 = 0\). By (2.12), \(S_{w_m} N_1 = \{0\}\). Hence \(S_{w_m} S_{z^n} h_0 = S_{w_m} h_2\). By (2.13) and \(h_2 \in N_2\), \(h_2 \notin \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)\). This implies that \(S_{w_m} h_2 \neq 0\). Hence \(S_{w_m} S_{z^n} \neq 0\). This contradicts our starting assumption. Thus we have (i).

To prove the converse, suppose that condition (i) holds. Then \(N = \left(N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\right) \oplus L\), where \(L \subset N \cap \sum_{i=0}^{m-1} \oplus w^i H^2(\Gamma_z)\). Let \(F = F_1 + F_2 \in N\), where \(F_1 \in N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\) and \(F_2 \in L\). Since \(z^n F \perp N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w), S_{z^n} F \in L\). Hence \(S_{w_m} S_{z^n} F = 0\). Thus we get \(S_{w_m} S_{z^n} = 0\).

By Theorem 2.2, the structure of backward shift invariant subspaces \(N\) satisfying \(S_{z^n} S_{w_m} = 0\) is simple. By Theorem 2.5, the structure of backward shift invariant subspaces \(N\) satisfying \(S_{w_m} S_{z^n} = 0\) is not so simple. When \(n = m = 1\), we have the following.

**Theorem 2.6.** Let \(N\) be a backward shift invariant subspace of \(H^2\). Then \(S^*_w S_z = 0\) if and only if \(N\) has one of the following forms;

(i) \(N = H^2(\Gamma_z) \ominus q(z) H^2(\Gamma_z)\) for some inner function \(q(z)\).

(ii) \(N = H^2(\Gamma_w) \ominus q(w) H^2(\Gamma_w)\) for some inner function \(q(w)\).

(iii) Either \(N = H^2(\Gamma_z) + H^2(\Gamma_w)\), or \(N = H^2(\Gamma_z) + H^2(\Gamma_w) \ominus q_2(w) H^2(\Gamma_w)\), or \(N = (H^2(\Gamma_z) \ominus q_1(z) H^2(\Gamma_z)) + H^2(\Gamma_w)\), where \(q_1(z)\) and \(q_2(w)\) are inner functions.

(iv) \(N = (H^2(\Gamma_z) \ominus q_1(z) H^2(\Gamma_z)) + (H^2(\Gamma_w) \ominus q_2(w) H^2(\Gamma_w))\), where \(q_1(z), q_2(w)\) are nonconstant inner functions and \(\hat{q}_1(0) \hat{q}_2(0) = 0\).

In (iii) and (iv), since \(1 \in N\), we may take \(q_1\) and \(q_2\) as \(\hat{q}_1(0) = \hat{q}_2(0) = 0\).

**Proof.** By Theorem 2.2, \(S_z S^*_w = 0\) if and only if either (i) or (ii) holds. By Theorem 2.5, \(S^*_w S_z = 0\) if and only if

\[(2.15) \quad N \ominus (N \cap H^2(\Gamma_w)) \subset N \cap H^2(\Gamma_z).\]

If either (i) or (ii) holds, by Corollary 2.3 we have \(S^*_w S_z = 0\). Suppose that \(N\) satisfies either (iii) or (iv). Then clearly \(1 \in N\). Since \(N\) has a special form, it is not difficult to see that

\[ N \ominus (N \cap H^2(\Gamma_w)) = \{ f \in N \cap H^2(\Gamma_z); \hat{f}(0) = 0 \}. \]

Hence (2.15) holds.
Then, suppose that (2.15) holds. Then we have

\[(2.16) \quad N = (N \cap H^2(\Gamma_z)) + (N \cap H^2(\Gamma_w)).\]

If either \(N \cap H^2(\Gamma_z) = \{0\}\) or \(N \cap H^2(\Gamma_w) = \{0\}\), then \(S_zS_w^* = 0\), and by Corollary 2.3, \(S_z^*S_w = 0\). Hence either (i) or (ii) holds. Suppose that \(N \cap H^2(\Gamma_z) \neq \{0\}\) and \(N \cap H^2(\Gamma_w) \neq \{0\}\). We shall prove \(1 \in N\). To prove this, suppose that \(1 \notin N\). Let \(1_w\) be the orthogonal projection of 1 to \(N \cap H^2(\Gamma_w)\). Then \(1_w \notin H^2(\Gamma_z)\). Since \(N \cap H^2(\Gamma_z) \neq \{0\}\), there exists \(f \in N \cap H^2(\Gamma_z)\) such that \(\hat{f}(0) \neq 0\). Let \(f_1 = f - \hat{f}(0)1_w \in N\). Then \(f_1 \notin H^2(\Gamma_z)\). Let \(h \in N \cap H^2(\Gamma_w)\). Since \(f \in H^2(\Gamma_z)\), \(f - \hat{f}(0) \perp h\). Since \(1 - 1_w \perp N \cap H^2(\Gamma_w)\),

\[\langle f_1, h \rangle = \langle \hat{f}(0)(1 - 1_w) + f - \hat{f}(0), h \rangle = 0.\]

Hence \(f_1 \in N \cap (N \cap H^2(\Gamma_w))\). Thus (2.15) does not hold. Therefore \(1 \in N\). Since \(N \cap H^2(\Gamma_z)\) and \(N \cap H^2(\Gamma_w)\) are nonzero backward shift invariant subspaces, by (2.16) \(N\) has one of forms in (iii) and (iv).

\[3. \quad S_z^nS_w^* = S_w^*S_z^n.\]

The following is the main theorem in this section.

**Theorem 3.1.** Let \(N\) be a backward shift invariant subspace of \(H^2\), \(N \neq \{0\}\), and \(N \neq H^2\). Let \(M = H^2 \ominus N\) and \(n \geq 2\) be a positive integer. If \(S_z^nS_w^* = S_w^*S_z^n\), then one of the following conditions holds;

(i) \(S_zS_w^* = S_wS_z^n\),

(ii) \(M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)\) for an inner function \(q_1(z)\) satisfying \(q_1(z) = \prod_{j=1}^k b_j(z), 1 \leq k \leq n\), where \(b_i\) are simple Blaschke products.

The following lemma is a generalization of [IN], and the idea of the proof is the same.

**Lemma 3.2.** Let \(N\) be a backward shift invariant subspace of \(H^2\) and \(M = H^2 \ominus N\). Let \(n\) be a positive integer. Then \(S_z^nS_w^* = S_wS_z^n\) if and only if

\[M \ominus (z^nM \oplus (M \cap \sum_{i=0}^{n-1} \ominus z^iH^2(\Gamma_w))) \subset (M \cap H^2(\Gamma_z)) \oplus wM.\]

**Proof.** The operators \(T_{z^n}\) and \(T_{w^*}\) on \(H^2\) have the matrix forms as

\[T_{z^n} = \begin{pmatrix} * & P_MT_{z^n}|_N \\ 0 & S_z^n \end{pmatrix}, T_{w^*} = \begin{pmatrix} * & 0 \\ P_NT_{w^*}|_M & S_w^* \end{pmatrix}\]

on \(H^2 = \begin{pmatrix} M \\ N \end{pmatrix}\).
Set $A = P_M T_{z^n} \vert_N$ and $B = P_N T_w \vert_M$. Since $T_{z^n} T_w^* = T_w^* T_{z^n}$ on $H^2$, $S_{z^n} S_w^* = S_w^* S_{z^n}$ if and only if $BA = 0$. We have $T_w^* (M \ominus wM) \subset N$. For $f \in H^2$, $T_w^* f = 0$ if and only if $f \in H^2(\Gamma_z)$. Hence
\[
\ker B = \{ f \in M; T_w^* f \in M \} = \{ f \in M \ominus wM; T_w^* f = 0 \} \oplus wM = (M \cap H^2(\Gamma_z)) \oplus wM.
\]

We denote by $[\text{ran } A]$ the closed range of $A$. Let $A_1 = P_M T_{z^n} P_N$ on $H^2$. Then $[\text{ran } A] = [\text{ran } A_1]$. Since $A_1^* = P_N T_{z^n}^* P_M$, we get $\ker A_1^* = \{ f \in M; T_{z^n}^* f \in M \} \oplus N = \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \oplus z^n M \oplus N$.

Hence
\[
[\text{ran } A] = [\text{ran } A_1] = H^2 \ominus \ker A_1^* = M \ominus \left( z^n M \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right).
\]

Since $BA = 0$ if and only if $[\text{ran } A] \subset \ker B$, we get our assertion.

**Proof of Theorem 3.1.** Suppose that $S_{z^n} S_w^* = S_w^* S_{z^n}$. By Lemma 3.2,
\[
M \ominus \left( z^n M \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right) \subset (M \cap H^2(\Gamma_z)) \oplus wM.
\]

Let
\[
K_0 = M \ominus \left( z^n M \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right).
\]

Then
\[
(3.1) \quad K_0 \subset (M \cap H^2(\Gamma_z)) \oplus wM
\]
and
\[
(3.2) \quad M \ominus z^n M = K_0 \ominus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right).
\]

Since $M = \sum_{s=0}^{\infty} \oplus z^{ns}(M \ominus z^n M)$,
\[
(3.3) \quad K_0 \perp \sum_{s=0}^{\infty} \oplus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right)
\]
and
\[
(3.4) \quad M = \left( \sum_{s=0}^{\infty} \oplus z^{ns} K_0 \right) \oplus \left( \sum_{s=0}^{\infty} \oplus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right).
\]
First, suppose that $K_0 = \{0\}$. In this case, we shall show that condition (i) holds. By (3.2),
\[
\sum_{i=0}^{n-1} \oplus z^i (M \ominus z^i M) = M \ominus z^n M = M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w).
\]
Let $f \in M \ominus zM$. Then $f \in M \ominus z^n M$ and $f = \sum_{i=0}^{n-1} \oplus z^i h_i(w)$, where $h_i(w) \in H^2(\Gamma_w)$. Since $f \in M \ominus zM$,
\[
z^{n-1} f = \sum_{i=0}^{n-1} \oplus z^{n-1+i} h_i(w) \in M \ominus z^n M.
\]
Hence $f = h_0(w)$, so that $M \ominus zM = M \cap H^2(\Gamma_w)$. By Lemma 3.2 (for the case of $n = 1$), we get $S_z S_w^* = S_w S_z$.

Next, suppose that $K_0 \neq \{0\}$. The proof of this case is a little bit long, so we devide into several steps.

**Step 1.** In this step, we shall prove that there is a nonconstant inner function $q_1(z)$ such that
\[
K_0 \subset q_1(z) H^2(\Gamma_z) \oplus wM.
\]
First we prove that $K_0 \nsubseteq wM$. To prove this, suppose that $K_0 \subset wM$. Let $F \in K_0$. Then $F = w f$ for some $f \in M$. We shall prove that $f \in K_0$. We have
\[
\langle f, \left( \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \right) \oplus \left( \sum_{s=0}^{\infty} \oplus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right) \rangle
= \langle w f, w \left( \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \oplus \left( \sum_{s=0}^{\infty} \oplus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right) \right) \rangle
= \langle F, z^{n} w \left( \sum_{s=1}^{\infty} \oplus z^{n(s-1) K_0} \right) \rangle \quad \text{by (3.3)}
= 0,
\]
where the last equality follows from $w \sum_{s=1}^{\infty} \oplus z^{n(s-1) K_0} \subset M$ and (3.2). Hence by (3.4), we get $f \in K_0$. Therefore for every positive integer $p$, we have $F = w^p f_p$ for some $f_p \in K_0$. This leads $F = 0$. This is a contradiction. Thus we get $K_0 \nsubseteq wM$.

Hence by (3.1), $M \cap H^2(\Gamma_z) \neq \{0\}$. By the Beurling theorem,
\[
M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z),
\]
where $q_1(z)$ is inner. Then $q_1(z) \in M$ and
\[
q_1(z) H^2(\Gamma_z) \perp wM.
\]
If \( q_1(z) \) is constant, we have \( M = H^2 \), so that \( N = \{0\} \). This contradicts our assumption. Hence \( q_1(z) \) is a nonconstant inner function. By (3.1) and (3.6), we get (3.5).

**Step 2.** In this step, we prove

\[
K_0 \subset q_1(z) \left( \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right).
\]

Let \( G \in K_0 \). Then by (3.5), \( G = q_1(z)h(z) \oplus wg \), where \( h(z) \in H^2(\Gamma_z) \) and \( g \in M \). Write

\[
h(z) = \left( \sum_{i=0}^{n-1} \oplus a_i z^i \right) \oplus z^n h_0(z).
\]

Then

\[
G = q_1(z) \left( \sum_{i=0}^{n-1} \oplus a_i z^i \right) \oplus q_1(z)z^n h_0(z) \oplus wg.
\]

By (3.6), \( q_1(z)z^n h_0(z) \in z^nM \). Since \( G \in K_0 \subset M \oplus z^nM \), we have \( h_0(z) = 0 \). Hence

\[
G = q_1(z) \left( \sum_{i=0}^{n-1} \oplus a_i z^i \right) \oplus wg.
\]

Here we prove that

\[
g \in K_0.
\]

Since \( G = q_1(z)h(z) \oplus wg \), we have

\[
\langle g, \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rangle = \langle wg, w \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rangle
\]

\[
= \langle wg, w \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rangle + \langle q_1 h, w \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rangle \quad \text{by (3.7)}
\]

\[
= \langle G, w \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rangle
\]

\[
= \langle G, z^n w \sum_{s=1}^{\infty} \oplus z^{n(s-1)} K_0 \rangle
\]

\[
= 0 \quad \text{by (3.2)}.
\]
We also have
\[ \langle g, \sum_{s=0}^{\infty} \bigoplus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \bigoplus z^i H^2(\Gamma_w) \right) \rangle = \langle w g, w \left( \sum_{s=0}^{\infty} \bigoplus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \bigoplus z^i H^2(\Gamma_w) \right) \right) \rangle = \langle G, \sum_{s=0}^{\infty} \bigoplus z^{ns} w \left( M \cap \sum_{i=0}^{n-1} \bigoplus z^i H^2(\Gamma_w) \right) \rangle \]
by (3.7),
\[ = 0 \]
by (3.3).
Hence by (3.4), we get (3.10).
Applying (3.9) and (3.10) infinitely many times, we have
\[ G = q_1(z) \left( \sum_{i=0}^{n-1} \bigoplus a_i z^i \right) \oplus \cdots \]
\[ \in q_1(z) \left( \sum_{i=0}^{n-1} \bigoplus z^i H^2(\Gamma_w) \right). \]
Therefore we get (3.8).

**Step 3.** In this step, we study functions in \( M \ominus z M \) and the inner function \( q_1(z) \). By (3.8), there is a closed subspace \( L \) such that \( K_0 = q_1(z)L \) and \( L \subset \bigoplus_{i=0}^{n-1} \bigoplus z^i H^2(\Gamma_w) \). Then by (3.2),
\[ M \ominus z^n M = q_1(z)L \bigoplus \left( M \cap \sum_{i=0}^{n-1} \bigoplus z^i H^2(\Gamma_w) \right). \]
Since \( K_0 \neq \{0\} \), \( L \neq \{0\} \). We have \( M \ominus z^n M = \bigoplus_{i=0}^{n-1} \bigoplus z^i(M \ominus z M) \). Hence \( M \ominus z M \neq \{0\} \). Let \( F \in M \ominus z M \) be such that \( F \neq 0 \). Then
\[ F = \left( q_1 \sum_{i=0}^{n-1} \bigoplus z^i f_i \right) \bigoplus \sum_{i=0}^{n-1} \bigoplus z^i g_i, \]
where \( f_i, g_i \in H^2(\Gamma_w) \),
\[ q_1 \sum_{i=0}^{n-1} \bigoplus z^i f_i \perp M \cap \sum_{i=0}^{n-1} \bigoplus z^i H^2(\Gamma_w), \]
and
\[ \sum_{i=0}^{n-1} \bigoplus z^i g_i \in M \cap \sum_{i=0}^{n-1} \bigoplus z^i H^2(\Gamma_w). \]
Since $n \geq 2$, $zF \in M \ominus z^n M$, so that we have
\[ zF = q_1 \left( \sum_{i=0}^{n-1} \oplus z^i G_{1,i} \right) + \sum_{i=0}^{n-1} \oplus z^i H_{1,i}, \]
where $G_{1,i}, H_{1,i} \in H^2(T_w)$. Hence (3.14)
\[ q_1 \left( z \left( \sum_{i=0}^{n-1} \oplus z^i f_i \right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i} \right) = \left( \sum_{i=0}^{n-1} \oplus z^i H_{1,i} \right) - z \sum_{i=0}^{n-1} \oplus z^i g_i. \]

Here we devide into two subcases.

**Subcase 1.**
\[ z \left( \sum_{i=0}^{n-1} \oplus z^i f_i(w) \right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w) \neq 0. \]

Then
\[ q_1(z) = \frac{\left( \sum_{i=0}^{n-1} \oplus z^i H_{1,i}(w) \right) - z \sum_{i=0}^{n-1} \oplus z^i g_i(w)}{z \left( \sum_{i=0}^{n-1} \oplus z^i f_i(w) \right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w)}. \]

As proved in Step 1, $q_1(z)$ is a nonconstant inner function. Then by the above, we have
\[ (3.15) \quad q_1(z) = \prod_{j=1}^{k} b_j(z), \quad 1 \leq k \leq n, \]
where $b_j$ are simple Blaschke products.

**Subcase 2.**
\[ z \left( \sum_{i=0}^{n-1} \oplus z^i f_i(w) \right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w) \equiv 0. \]

Then by (3.14), $f_{n-1} = g_{n-1} = 0$, so that by (3.11)
\[ F = \left( q_1 \sum_{i=0}^{n-2} \oplus z^i f_i \right) + \sum_{i=0}^{n-2} \oplus z^i g_i. \]

Since $F \in M \ominus zM$,
\[ zF = \left( q_1 \sum_{i=0}^{n-2} \oplus z^{i+1} f_i \right) + \sum_{i=0}^{n-2} \oplus z^{i+1} g_i \in M \ominus z^2 M. \]

In the same way as above, either (3.15) holds or $f_{n-2} = g_{n-2} = 0$. Repeat the same argument. Then either (3.15) holds or
\[ f_{n-1} = f_{n-2} = \cdots = f_1 = g_{n-1} = g_{n-2} = \cdots = g_1 = 0. \]
Step 4. If (3.15) holds, then (ii) holds. So, suppose that (3.15) does not hold. Then by (3.11) and by the above fact, $F = q_1f_0 \oplus g_0$, by (3.12) $q_1f_0 \perp M \cap H^2(\Gamma_w)$, and by (3.13) $g_0 \in M \cap H^2(\Gamma_w)$ for every $F \in M \ominus zM$.

If $g_0 = 0$ for every $F \in M \ominus zM$, since $q_1(z) \in M$ it follows that $M \ominus zM \subset q_1(z)H^2(\Gamma_w) \subset M$. Since $M = \sum_{m=0}^{\infty} z^4(M \ominus zM)$, we have $M = q_1(z)H^2$, so that $N = H^2 \ominus q_1(z)H^2$. By Theorem A, condition (i) holds.

So we assume that $g_0 \neq 0$ for some $F \in M \ominus zM$. We shall prove that

$$\text{(3.16)} \quad (M \ominus zM) \ominus (M \cap H^2(\Gamma_w)) \subset (M \cap H^2(\Gamma_w)) \oplus wM.$$  

We may assume that $(M \ominus zM) \ominus (M \cap H^2(\Gamma_w)) \neq \{0\}$. Let $G \in (M \ominus zM) \ominus (M \cap H^2(\Gamma_w))$ be such that $G \neq 0$. Then $G = q_1(z)h_1(w) \oplus h_2(w)$, where $q_1(z)h_1(w) \perp M \cap H^2(\Gamma_w)$ and $h_2(w) \in M \cap H^2(\Gamma_w)$. Hence $h_2(w) = G - q_1(z)h_1(w) \perp M \cap H^2(\Gamma_w)$. Therefore $h_2(w) = 0$. Since $q_1(z) \in M$,

$$G = q_1(z)h_1(w) = h_1(0)q_1(z) \oplus wq_1(z) \frac{h_1(w) - h_1(0)}{w} \in M \cap H^2(\Gamma_w) \oplus wM.$$ 

Thus we get (3.16). By Lemma 3.2 (for $n = 1$), (i) holds.  

4. $S_2zS_w^* = S_w^*S_2$ and $S_2S_w^* \neq S_w^*S_2$.

Let $N$ be a backward shift invariant subspace of $H^2$ and let $n$ be a positive integer. Let $M = H^2 \ominus N$. Then $M$ is an invariant subspace. If both $S_nS_w^* = S_w^*S_n$ and $S_2S_w^* \neq S_w^*S_2$ hold, then by Theorem 3.1, $M \cap H^2(\Gamma_w) = q_1(z)H^2(\Gamma_w)$ for some nonconstant inner function $q_1(z)$.

In this section, we assume that $q_1(z)H^2 \subset M$ and $M \cap H^2(\Gamma_w) = q_1(z)H^2(\Gamma_w)$ for some nonconstant inner function $q_1(z)$. Let

$$\tilde{M} = M \ominus q_1(z)H^2 \subset M.$$ 

Then $H^2 \ominus q_1(z)H^2 = \tilde{M} \ominus N$ and $\tilde{M}$ is $w$-invariant. The following lemma is proved in [INS, Lemma 3.2].

**Lemma 4.1.** Let $f \in \tilde{M}$. Then $T_w^*f \in \tilde{M}$ if and only if $f \in w\tilde{M}$.

We denote by $P_\perp$ the orthogonal projection from $H^2$ onto $H^2 \ominus q_1(z)H^2$. Then we have a Toeplitz type operator $Q_{z^n}$ on $H^2 \ominus q_1(z)H^2$ such that

$$Q_{z^n} : H^2 \ominus q_1(z)H^2 \ni f \to P_\perp(T_{z^n}f) \in H^2 \ominus q_1(z)H^2.$$
Since $z^nM \subset M$, $Q_z \tilde{M} \subset \tilde{M}$ and $Q_z^n = Q_z$. Then $Q_z$ has the following matrix form:

$$Q_z = \begin{pmatrix} * & P_M T_z^n |_N \\ 0 & S_z^n \\ \end{pmatrix}$$
on H^2 \oplus q_1(z)H^2 = \begin{pmatrix} \tilde{M} \\ \oplus \\ N \\ \end{pmatrix}.

Since $H^2 \oplus q_1(z)H^2$ is backward shift invariant, $T_w^*(H^2 \oplus q_1(z)H^2) \subset H^2 \oplus q_1(z)H^2$. Since $T_w^*N \subset N$, the operator $T_w^*$ on $H^2 \oplus q_1(z)H^2$ has the following matrix form:

$$T_w^* = \begin{pmatrix} * & 0 \\ P_N T_w^* |_{\tilde{M}} & S_w^* \\ \end{pmatrix}$$
on H^2 \oplus q_1(z)H^2 = \begin{pmatrix} \tilde{M} \\ \oplus \\ N \\ \end{pmatrix}.

Set

(4.1) $A = P_M T_z^n |_N$ and $B = P_N T_w^* |_{\tilde{M}}$.

By [INS, Lemma 3.3], $T_w^*Q_z = Q_z T_w^*$ on $H^2 \oplus q_1(z)H^2$. Hence we have the following.

**Lemma 4.2.** $T_w^*Q_z = Q_z T_w^*$ on $H^2 \oplus q_1(z)H^2$.

**Lemma 4.3.** $S_w^n S_w^* = S_w^* S_w^n$ if and only if $BA = 0$.

**Proof.** By Lemma 4.2, $T_w^* Q_z = Q_z T_w^*$ on $H^2 \oplus q_1(z)H^2$. Then $BA + S_w^n S_w^n = S_w^* S_w^n$. Hence $S_w^n S_w^* = S_w^* S_w^n$ if and only if $BA = 0$. \hfill \Box

The following is a slight generalization of [INS, Theorem 3.5].

**Theorem 4.4.** Let $N$ be a backward shift invariant subspace of $H^2$ and $M = H^2 \ominus N$. Suppose that $M \cap H^2(T_z) = q_1(z)H^2(T_z)$, where $q_1(z)$ is a nonconstant inner function. Let $\tilde{M} = M \ominus q_1(z)H^2$. Then the following conditions are equivalent.

(i) $S_z^n S_w^* = S_w^* S_z^n$ on $N$.

(ii) $\tilde{M} \ominus \{ f \in \tilde{M}; T_z^n f \in \tilde{M} \} \subset w \tilde{M}$.

(iii) $T_z^n \tilde{M} \subset M$.

**Proof.** The idea of the proof is similar to the one of [INS, Theorem 3.5]. For the sake of completeness, we give the proof.

(i) $\Leftrightarrow$ (ii) By Lemma 4.3, condition (i) is equivalent to $BA = 0$. By (4.1) and Lemma 4.1, $\ker B = \{ f \in \tilde{M}; T_w^* f \in \tilde{M} \} = w \tilde{M}$. Put $A_1 = P_M T_z^n P_N$ on $\tilde{M} \ominus N$. Then $[\text{ran } A] = [\text{ran } A_1]$. Since $A_1^* = P_N T_w^* P_M$, $\ker A_1^* = N \ominus \{ f \in \tilde{M}; T_z^n f \in \tilde{M} \}$. Hence

$[\text{ran } A] = [\text{ran } A_1] = (\tilde{M} \ominus N) \ominus \ker A_1^* = \tilde{M} \ominus \{ f \in \tilde{M}; T_z^n f \in \tilde{M} \}$.

Therefore $BA = 0$ if and only if $\tilde{M} \ominus \{ f \in \tilde{M}; T_z^n f \in \tilde{M} \} \subset w \tilde{M}$. Thus we get (i) $\Leftrightarrow$ (ii).
(ii) \(\Rightarrow\) (iii) Suppose that \(\tilde{M} \oplus \{f \in \tilde{M} : T_{z}^*f \in \tilde{M}\} \subset w\tilde{M}\). Since \(\{f \in \tilde{M} : T_{z}^*f \in \tilde{M}\}\) is closed, \(\tilde{M} \subseteq w\tilde{M} \subset \{f \in \tilde{M} : T_{z}^*f \in \tilde{M}\}\). Since \(w\tilde{M} \subset M\), \(M = \bigoplus_{j=0}^{\infty}w^j(\tilde{M} \oplus w\tilde{M})\). Since \(T_{z}^*w^j\tilde{f} = w^jT_{z}^*f\) for \(f \in H^2\), we have \(T_{z}^*\tilde{M} \subset \tilde{M}\).

(iii) \(\Rightarrow\) (ii) is trivial. \(\square\)

For \(f \in H^2(\Gamma_z)\), write \(f^*(z) = T_{z}^*f(z) = C(f(z) - \hat{f}(0))\).

Lemma 4.5. Let \(b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}_iz), |\alpha_i| < 1, \) and \(1 \leq i \leq n\). Then

(i) \(T_{z}^*z = 1, T_{z}^*b_1 = b_1^*(z) = (1 - |\alpha_1|^2)/(1 - \overline{\alpha}_1z), \) and \(T_{z}^*b_i(z) = \overline{\alpha}_ib_i^*(z)\).

(ii) \(T_{z}^*(b_1(z)b_2(z)) = (1 - |\alpha_2|^2)b_2^*(z) + \overline{\alpha}_2b_1(z)b_2^*(z)\).

(iii) \(H^2(\Gamma_z) \oplus (\bigoplus_{j=1}^{k}b_j(z))H^2(\Gamma_z) = \bigoplus_{j=1}^{k}[b_1(z) \cdots b_{j-1}(z)b_j^*(z)H^2(\Gamma_z)]\).

(iv) \(H^2 \oplus (\bigoplus_{j=1}^{k}b_j(z))H^2 = \bigoplus_{j=1}^{k}[b_1(z) \cdots b_{j-1}(z)b_j^*(z)H^2(\Gamma_w)]\).

Proof. It is not difficult to prove (i).

(ii) Since

\[
\overline{\alpha}_1b_1(z)b_2^*(z) = \overline{\alpha}_1b_1(z) \frac{1 - |\alpha_2|^2}{1 - \overline{\alpha}_2z}
\]

\[
= (1 - |\alpha_2|^2)b_2^*(z)\left(\overline{\alpha}_1 + \frac{\overline{\alpha}_2}{1 - \overline{\alpha}_2z}\right)
\]

\[
= (1 - |\alpha_2|^2)b_2^*(z) + \overline{\alpha}_2b_1(z)b_2^*(z),
\]

we get (ii).

(iii) See [Ni, p.33].

(iv) follows from (iii). \(\square\)

Corollary 4.6. Let \(N\) be a backward shift invariant subspace of \(H^2\) and \(M = H^2 \ominus N\). Suppose that \(M \cap H^2(\Gamma_z) = b_1(z)H^2(\Gamma_z)\), where \(b_1(z)\) is a simple Blaschke product. Then \(S_zS_w^* = S_w^*S_z\).

Proof. Let \(b_1(z) = (z - \alpha)/(1 - \overline{\alpha}z), |\alpha| < 1, \) and \(\tilde{M} = M \oplus b_1(z)H^2\). Since \(b_1(z) \in M, b_1(z)H^2 \subset M\). By Lemma 4.5(iv), \(\tilde{M} \subset b_1^*(z)H^2(\Gamma_w)\). By Lemma 4.5(i), \(T_{z}^*(b_1^*(z)h(w)) = \overline{\alpha}_1b_1^*(z)h(w)\). Hence \(T_{z}^*\tilde{M} \subset \tilde{M}\). By Theorem 4.4, \(S_zS_w^* = S_w^*S_z\).

Corollary 4.7. Let \(N\) be a backward shift invariant subspace of \(H^2\) and \(M = H^2 \ominus N\). Suppose that \(M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)\), where \(q_1(z)\) is an inner function. Let \(n, k\) be positive integers with \(n \geq k + 1\). Moreover suppose that \(q_1(z) = z^nb(z), \) where \(b\) is a simple Blaschke product, \(b(z) = (z - \alpha)/(1 - \overline{\alpha}z), \) and \(\alpha \neq 0\). If \(S_zS_w^* = S_w^*S_z^n\), then \(S_zS_w^* = S_w^*S_z^k\).
Proof. Let \( \tilde{M} = M \ominus q_1(z)H^2 \). If \( \tilde{M} = \{0\} \), then \( M = q_1(z)H^2 \). By Theorem A, \( S_zS_w^* = S_w^*S_z \). Suppose that \( \tilde{M} \neq \{0\} \). By Lemma 4.5(iv),
\[
\tilde{M} \subset b^*(z)H^2(\Gamma_w) \oplus \left( \sum_{j=1}^k \oplus z^{j-1}b(z)H^2(\Gamma_w) \right).
\]
Let \( f \in \tilde{M} \). Then
\[
f = b^*(z)h_0(w) + \sum_{j=1}^k \oplus z^{j-1}b(z)h_j(w), \quad h_j(w) \in H^2(\Gamma_w).
\]
By Lemma 4.5(i),
\[
T_{z^n}^*f = T_{z^{n-k}}^*(T_{z^k}^*f)
\]
\[
= T_{z^{n-k}}^* \left( \sum_{j=0}^k \alpha^{(k-j)}b^*(z)h_j(w) \right)
\]
\[
= \alpha^{(n-k)} \sum_{j=0}^k \alpha^{(k-j)}b^*(z)h_j(w).
\]
Since \( S_zS_w^* = S_w^*S_z \), by Theorem 4.4 \( T_{z^n}^*f \in \tilde{M} \). Since \( \alpha \neq 0 \),
\[
\sum_{j=0}^k \alpha^{(k-j)}b^*(z)h_j(w) \in \tilde{M}.
\]
Thus \( T_{z^n}^*\tilde{M} \subset \tilde{M} \). By Theorem 4.4, \( S_zS_w^* = S_w^*S_z \). \( \square \)

**Theorem 4.8.** Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \ominus N \). Suppose that \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \), where \( q_1(z) \) is an inner function. Moreover suppose that \( q_1(z) = b_1(z)b_2(z) \), where \( b_i(z), i = 1, 2 \), are simple Blaschke products, \( b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha_i}z) \), and \( \alpha_1\alpha_2 \neq 0 \). Let \( n \geq 2 \) be a positive integer. Then we have the following.

(i) If \( S_zS_w^* = S_w^*S_z \) and \( S_zS_w^* = S_w^*S_z \), then \( \alpha_1 = \alpha_2 \) and \( \alpha_1 \neq \alpha_2 \).

(ii) If \( \alpha_1 = \alpha_2 \) and \( \alpha_1 \neq \alpha_2 \), then \( S_zS_w^* = S_w^*S_z \).

**Proof.** Let \( \hat{M} = M \ominus q_1(z)H^2 \). Suppose that \( S_zS_w^* = S_w^*S_z \) and \( S_{z^{-1}}S_w^* \neq S_w^*S_{z^{-1}} \). By Theorem 4.4, \( T_{z^n}^*\hat{M} \subset \hat{M} \) and \( T_{z^n}^*\hat{M} \notin \hat{M} \). By Lemma 4.5(iv),
\[
\hat{M} \subset b_1^*(z)H^2(\Gamma_w) \oplus b_1(z)b_2^*(z)H^2(\Gamma_w).
\]
Then there exists \( f_0 \in \hat{M} \) such that \( T_{z^n}^*f_0 \notin \hat{M} \), and
\[
f_0 = b_1^*(z)g_1(w) + b_1(z)b_2^*(z)g_2(w) \in b_1^*(z)H^2(\Gamma_w) \oplus b_1(z)b_2^*(z)H^2(\Gamma_w).
\]
By Lemma 4.5,

\[ T_{z^{n-1}} f_0 = b_1^* \left( \alpha_1^{(n-1)} g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-2} \alpha_1^{(n-2-j)} \alpha_2^{j} \right) g_2 \right) + \alpha_2^{(n-1)} b_1^* b_2^* g_2 \]

and

\[ T_{z^n} f_0 = b_1^* \left( \alpha_1^n g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \alpha_2^{j} \right) g_2 \right) + \alpha_2^n b_1^* b_2^* g_2. \]

Since \( T_{z^{n-1}} f_0 \notin \tilde{M} \) and \( f_0 \in \tilde{M} \), \( T_{z^{n-1}} f_0 - \alpha_2^{n-1} f_0 \notin \tilde{M} \). Then

\[ b_1^* \left( (\alpha_1^{(n-1)} - \alpha_2^{(n-1)}) g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-2} \alpha_1^{(n-2-j)} \alpha_2^{j} \right) g_2 \right) \notin \tilde{M}. \]

Hence

\[ \left( \sum_{j=0}^{n-2} \alpha_1^{(n-2-j)} \alpha_2^{j} \right) b_1^* \left( (\alpha_1 - \alpha_2) g_1 + (1 - |\alpha_2|^2) g_2 \right) \notin \tilde{M}. \]

Since \( 0 \in \tilde{M} \), \( \sum_{j=0}^{n-2} \alpha_1^{(n-2-j)} \alpha_2^{j} \neq 0 \), so that

\[ b_1^* \left( (\alpha_1 - \alpha_2) g_1 + (1 - |\alpha_2|^2) g_2 \right) \notin \tilde{M}. \]

Since \( T_{z^n} f_0 \in \tilde{M} \), \( T_{z^n} f_0 - \alpha_2^n f_0 \in \tilde{M} \). Then

\[ b_1^* \left( (\alpha_1^n - \alpha_2^n) g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \alpha_2^{j} \right) g_2 \right) \in \tilde{M}. \]

Hence

\[ \left( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \alpha_2^{j} \right) b_1^* \left( (\alpha_1 - \alpha_2) g_1 + (1 - |\alpha_2|^2) g_2 \right) \in \tilde{M}. \]

Now we prove (i). Suppose that \( \alpha_1 = \alpha_2 \). Then \( \alpha_1 = \alpha_2 \neq 0 \), so that \( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \alpha_2^j \neq 0 \). By (4.3) and (4.4), \( b_1^* g_2 \notin \tilde{M} \) and \( b_1^* g_2 \in \tilde{M} \). This is a contradiction. Hence \( \alpha_1 \neq \alpha_2 \).

Suppose that \( \alpha_1^n \neq \alpha_2^n \). Then \( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \alpha_2^j \neq 0 \). By (4.4),

\[ b_1^* \left( (\alpha_1 - \alpha_2) g_1 + (1 - |\alpha_2|^2) g_2 \right) \in \tilde{M}. \]

This contradicts (4.3), so that \( \alpha_1^n = \alpha_2^n \). Thus we get (i).

(ii) Suppose that \( \alpha_1^n = \alpha_2^n \) and \( \alpha_1 \neq \alpha_2 \). Then \( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \alpha_2^j = 0 \).

Let \( f \in \tilde{M} \). Then by (4.2), \( f = b_1^*(z) h_1(w) + b_1(z) b_2^*(z) h_2(w) \). Similarly,
we have
\[ T_n^*f - \overline{\alpha}_2^nf = \left( \sum_{j=0}^{n-1} \overline{\alpha}_1^{(n-1-j)}\overline{\alpha}_2^j \right) b_1^* \left( (\overline{\alpha}_1 - \overline{\alpha}_2)h_1 + (1 - |\alpha_2|^2)h_2 \right). \]

Hence \( T_n^*f = \overline{\alpha}_2^nf \in \tilde{M} \), so that we get \( T_n^*\tilde{M} \subset \tilde{M} \). By Theorem 4.4, \( S_n^w \tilde{M} = S_n^w \).

**Corollary 4.9.** Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \ominus N \). Suppose that \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \), where \( q_1(z) \) is an inner function. Moreover suppose that \( q_1(z) = b_1(z)b_2(z) \), where \( b_i(z), i = 1, 2 \), are simple Blaschke products, \( b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}_iz) \), and \( \alpha_1\alpha_2 \neq 0 \). Then we have the following.

(i) If \( S_2^wS_2^w = S_w^wS_2 \) and \( S_z^wS_z^* \neq S_w^wS_z \), then \( \alpha_1 + \alpha_2 = 0 \).

(ii) If \( \alpha_1 + \alpha_2 = 0 \), then \( S_2^wS_2^* = S_w^wS_2 \).

The following is the main theorem in this section.

**Theorem 4.10.** Let \( N \) be a backward shift invariant subspace of \( H^2 \), \( N \neq \{0\} \), and \( N \neq H^2 \). Then \( S_2^wS_2^* = S_w^wS_2 \) if and only if one of the following conditions holds.

(i) \( S_2^wS_2^* = S_w^wS_2 \).

(ii) \( S_2^wS_2^* = 0 \).

(iii) There exist two simple Blaschke products \( b_1(z) \) and \( b_2(z) \), \( b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}_iz), 0 < |\alpha_i| < 1, such that \( N \subset H^2 \ominus b_1(z)b_2(z)H^2 \) and \( \alpha_1 + \alpha_2 = 0 \).

A backward shift invariant subspace \( N \) satisfying condition (i) is given by Theorem A. Also \( N \) satisfying condition (ii) is given by Theorem 2.2.

**Proof of Theorem 4.10.** Suppose that \( S_2^wS_2^* = S_w^wS_2 \). Moreover suppose that \( S_z^w \neq S_w^wS_2 \). By Theorem 3.1, \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \) for an inner function \( q_1(z) \) such that either \( q_1(z) = b_1(z) \) or \( q_1(z) = b_1(z)b_2(z) \), where \( b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}_iz), 0 < |\alpha_i| < 1 \). If \( q_1(z) = b_1(z) \), by Corollary 4.6, (i) holds.

Suppose that \( q_1(z) = b_1(z)b_2(z) \). Moreover suppose that \( \alpha_1 = \alpha_2 = 0 \). Then \( q_1(z) = z^2 \). Hence \( z^2H^2 \subset M \), so that \( N \subset H^2(\Gamma_w) \oplus zH^2(\Gamma_w) \). Then by Theorem 2.2, \( S_2^wS_2^* = 0 \). Thus (ii) holds.

Suppose that \( \alpha_1 \neq 0 \) and \( \alpha_2 = 0 \). Then \( q_1(z) = zb_1(z) \). By Corollary 4.7, (i) holds.

Suppose that \( \alpha_1 \neq 0 \) and \( \alpha_2 \neq 0 \). Then by Corollary 4.9(i), we get \( \alpha_1 + \alpha_2 = 0 \). Hence (iii) holds.
Next, we prove the converse assertion. Suppose that one of conditions (i), (ii), and (iii) holds. If (i) holds, then trivially $S_z S^*_w = S^*_w S_z$.

If (ii) holds, by Corollary 2.3 we have $S_z S^*_w = S^*_w S_z$.

Suppose that (iii) holds. Then $b_1(z) b_2(z) H^2(\Gamma_z) \subset M$. Hence $M \cap H^2(\Gamma_z)$ equals to either $b_1(z) H^2(\Gamma_z)$, or $b_2(z) H^2(\Gamma_z)$, or $b_1(z) b_2(z) H^2(\Gamma_z)$.

By Corollary 4.6, $S_z S^*_w = S^*_w S_z$ for the first two cases. Hence $S_z S^*_w = S^*_w S_z$. For the last case, by Corollary 4.9(ii), $S_z S^*_w = S^*_w S_z$.

\[ \square \]

**Example 4.11.** We give an example of a backward shift invariant subspace $N$ of $H^2$ satisfying $S_z S^*_w = S^*_w S_z$, $S_z S^*_w \neq S^*_w S_z$, and $S_z S^*_w \neq 0$. Let $q_1(z) = b_1(z) b_2(z)$, where $b_i(z)$, $i = 1, 2$, are $b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha_i} z)$, $\alpha_1 + \alpha_2 = 0$, and $\alpha_1 \neq 0$. Let $q_2(w)$ be a non-constant inner function. Let $M = q_1(z) H^2 \oplus b_1(z) b_2^*(z) q_2(w) H^2(\Gamma_w)$. Then $M$ is an invariant subspace and $M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z)$. Let $N = H^2 \ominus M$. By Theorem 4.10, $S_z S^*_w = S^*_w S_z$. We have $\tilde{M} = M \ominus q_1(z) H^2 = b_1(z) b_2^*(z) q_2(w) H^2(\Gamma_w)$. Since

$$T_z^* b_1(z) b_2^*(z) q_2(w) = (1 - |\alpha_1|^2) b_1^*(z) q_2(w) + \overline{\alpha_1} b_1(z) b_2^*(z) q_2(w),$$

$T_z^* b_1(z) b_2^*(z) q_2(w) \notin \tilde{M}$. By Theorem 4.4, $S_z S^*_w \neq S^*_w S_z$. By Theorem 2.2, $S_z S^*_w \neq 0$.

We leave the following problem for the reader.

**Problem 4.12.** Characterize backward invariant subspaces $N$ of $H^2$ satisfying $S_z S^*_w = S^*_w S_z$ for $n \geq 3$.

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