BACKWARD SHIFT INVARIANT SUBSPACES IN THE BIDISC III

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ABSTRACT. In the previous paper, we give a characterization of backward shift invariant subspaces of the Hardy space in the bidisc which satisfy the doubly commuting condition $S_z S_w^* = S_w^* S_z$ for the compression operators S_z and S_w. In this paper, we give a characterization of backward shift invariant subspaces satisfying $S_z^2 S_w^* = S_w^* S_z^2$.

1. INTRODUCTION.

Let Γ^2 be the 2-dimensional unit torus. We denote by $(z, w) = (e^{i\theta}, e^{i\phi})$ the variables in $\Gamma^2 = \Gamma_z \times \Gamma_w$. Let $L^2 = L^2(\Gamma^2)$ be the usual Lebesgue space on Γ^2 with the norm $\|f\|_2 = (\int_{\Gamma^2} |f(e^{i\theta}, e^{i\phi})|^2 d\theta d\phi/(2\pi)^2)^{1/2}$. Then L^2 is a Hilbert space with the usual inner product. For $f \in L^2$, the Fourier coefficients are given by

$$\hat{f}(n,m) = \int_{\Gamma^2} f(e^{i\theta}, e^{i\phi}) e^{-in\theta} e^{-im\phi}\, d\theta d\phi/(2\pi)^2 = \langle f, z^n w^m \rangle.$$

Let $H^2 = H^2(\Gamma^2)$ be the Hardy space on Γ^2, that is,

$$H^2 = \{f \in L^2; \hat{f}(n,m) = 0 \text{ if } n < 0 \text{ or } m < 0\}.$$

For $f \in H^2$, we can write f as

$$f = \sum_{i,j=0}^{\infty} \oplus a_{i,j} z^i w^j, \ \text{ where } \sum_{i,j=0}^{\infty} |a_{i,j}|^2 < \infty.$$

Let P be the orthogonal projection from L^2 onto H^2. For a closed subspace M of L^2, we denote by P_M the orthogonal projection from L^2 onto M. For a function $\psi \in L^\infty$, let $L_\psi f = \psi f$ for $f \in L^2$. The Toeplitz operator T_ψ is defined by $T_\psi f = P L_\psi f$ for $f \in H^2$. It is well known that $T_\psi^* = T_{\overline{\psi}}$ and $T_{z^n}^* T_{w^m} = T_{w^m} T_{z^n}^*$ for $n, m \geq 1$. A function $f \in H^2$ is called inner if $|f| = 1$ on Γ^2 almost everywhere. A closed subspace M of H^2 is called invariant if $zM \subset M$ and $wM \subset M$. In one variable

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case, an invariant subspace M of $H^2(\Gamma)$ has a form $M = qH^2(\Gamma)$, where q is inner. This is the well known Beurling theorem [B]. In two variable case, the structure of invariant subspaces of H^2 is very complicated, see [AC], [DY], [Na1], and [R].

Let M be an invariant subspace of H^2. Then $T_z^*(H^2 \ominus M) \subset (H^2 \ominus M)$ and $T_w^*(H^2 \ominus M) \subset (H^2 \ominus M)$. A closed subspace N of H^2 is called *backward shift invariant* if $H^2 \ominus N$ is invariant. In [IY], the first author and Yang studied backward shift invariant subspaces N on which T_z^* is strictly contractive. See [CR] and [S] for studies of backward shift invariant subspaces on the unit circle Γ.

Let M be an invariant subspace of H^2 and $\psi \in L^\infty$. Let V_ψ be the operator on M defined by $V_\psi = P_M L_\psi|_M$. Then $V_z = T_z$ and $V_z^* = V_{\overline{z}}$ on M. In [M], Mandrekar proved that $V_z V_w^* = V_w^* V_z$ on M if and only if M is Beurling type, that is, $M = qH^2$ for some inner function q in H^∞, see also [CS] and [Na2].

For $\psi \in L^\infty$, let $S_\psi = P_N L_\psi|_N$. Then we have $S_\psi^* = S_{\overline{\psi}}$ and $S_z^* = T_z^*$ on N. In the previous paper [INS], we characterized backward shift invariant subspaces N which satisfy the condition $S_z S_w^* = S_w^* S_z$ on N as follows.

Theorem A. *Let N be a backward shift invariant subspace of H^2 and $N \neq H^2$. Then $S_z S_w^* = S_w^* S_z$ on N if and only if N has one of the following forms;*

(i) $N = H^2 \ominus q_1(z)H^2$,
(ii) $N = H^2 \ominus q_2(w)H^2$,
(iii) $N = (H^2 \ominus q_1(z)H^2) \cap (H^2 \ominus q_2(w)H^2)$,

where $q_1(z)$ and $q_2(w)$ are one variable inner functions.

This paper is a continuation of previous papers [IN] and [INS]. Here we are interesting in backward shift invariant subspaces N of H^2 satisfying that $S_{z^n} S_{w^m}^* = S_{w^m}^* S_{z^n}$ for given positive integers n and m. Up to now, we can not give a complete characterization of N satisfying $S_{z^n} S_{w^m}^* = S_{w^m}^* S_{z^n}$. If $S_z S_w^* = S_w^* S_z$, then trivially $S_{z^n} S_{w^m}^* = S_{w^m}^* S_{z^n}$. But the converse is not true. In this paper, we concentrated on the case $m = 1$. One reason is the work of this paper deeply concerns with the problem of the normality of the operators S_{z^n}. For many backward shift invariant subspaces N, S_{z^n} are not normal operators, see [Y]. If S_{z^n} is normal, since $S_{z^n} S_w = S_w S_{z^n}$, by the Fuglede-Putnam theorem we have $S_{z^n} S_w^* = S_w^* S_{z^n}$. So, the results obtained in this paper will be a big help for the above mentioned problem. We will discuss on this subject in the forthcoming paper.

In Section 2, we give characterizations of N satisfying $S_{z^n}S^*_{w^m} = 0$ and $S^*_{w^m}S_{z^n} = 0$, respectively. If $S_{z^n}S^*_{w^m} = 0$, then $S^*_{w^m}S_{z^n} = 0$. The converse is not true. In Section 3, we study N satisfying $S_{z^n}S^*_w = S^*_wS_{z^n}$, and give a necessary condition for $S_{z^n}S^*_w = S^*_wS_{z^n}$. In Section 4, we study N satisfying $S_{z^2}S^*_w = S^*_wS_{z^2}$. We gave a complete characterization of such N. In [INS], we gave two different type of proofs of Theorem A. One of them is used in Section 3 and the other is used in Section 4.

Let $H^2(\Gamma_z)$ and $H^2(\Gamma_w)$ be the Hardy spaces on the unit circle Γ in variables z and w, respectively. We think that $H^2(\Gamma_z) \subset H^2$ and $H^2(\Gamma_w) \subset H^2$. For a subset E of H^2, we denote by $[E]$ the closed linear span of E. A function $b(z) = (z-\alpha)/(1-\overline{\alpha}z), |\alpha| < 1$, is called a simple Blaschke product.

2. $S_{z^n}S^*_w = 0$ or $S^*_wS_{z^n} = 0$.

Let n and m be positive integers. In this section, we study backward shift invariant subspaces N of H^2 satisfying $S_{z^n}S^*_{w^m} = 0$ and $S^*_{w^m}S_{z^n} = 0$, respectively.

Lemma 2.1. *Let N be a backward shift invariant subspace of H^2. Then we have the following.*

(i) $S^n_z = S_{z^n}$.
(ii) $S_{w^m}S_{z^n} = S_{z^n}S_{w^m}$ *and* $S^*_{z^n}S^*_{w^m} = S^*_{w^m}S^*_{z^n}$.
(iii) *If* $S^*_{z^n}S^*_{w^m}N \neq \{0\}$, *then there exists* $f \in N$ *such that* $(S^*_{z^n}S^*_{w^m}f)\widehat{}\,(0,0) \neq 0$.

Proof. All assertions are not difficult to prove. □

The following theorem says that the structure of backward shift invariant subspaces satisfying $S_{z^n}S^*_{w^m} = 0$ is simple.

Theorem 2.2. *Let N be a backward shift invariant subspace of H^2. Then $S_{z^n}S^*_{w^m} = 0$ if and only if N satisfies one of the following conditions;*

(i) $N \subset \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$.
(ii) $N \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$.

Proof. Suppose that $S_{z^n}S^*_{w^m} = 0$. Then

$$S^*_{w^m}N \perp S^*_{z^n}N. \tag{2.1}$$

Since N is backward shift invariant, if $S^*_{w^m}N = \{0\}$ then N satisfies condition (i). If $S^*_{z^n}N = \{0\}$, then N satisfies (ii).

Next, suppose that

$$S^*_{w^m}N \neq \{0\} \quad \text{and} \quad S^*_{z^n}N \neq \{0\}. \tag{2.2}$$

We shall lead a contradiction. By (2.1), $S^*_{w^m}S^*_{z^n}N \perp S^*_{z^n}S^*_{w^m}N$. By Lemma 2.1(ii), $S^*_{w^m}S^*_{z^n}N = S^*_{z^n}S^*_{w^m}N = \{0\}$. Then

$$S^*_{z^n}N \subset \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \tag{2.3}$$

and

$$S^*_{w^m}N \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w). \tag{2.4}$$

By (2.2) and (2.3), there exists a nonnegative integer $j, 0 \leq j \leq m-1$, such that

$$\{0\} \neq S^*_{w^j}S^*_{z^n}N \subset H^2(\Gamma_z). \tag{2.5}$$

By Lemma 2.1(iii), there exists $g \in N$ such that

$$(S^*_{w^j}S^*_{z^n}g)\hat{}(0,0) \neq 0. \tag{2.6}$$

Also by (2.2) and (2.4), there exist $f \in N$ and a nonnegative integer $i, 0 \leq i \leq n-1$, such that

$$S^*_{z^i}S^*_{w^m}f \in H^2(\Gamma_w) \tag{2.7}$$

and

$$(S^*_{z^i}S^*_{w^m}f)\hat{}(0,0) \neq 0. \tag{2.8}$$

Then

$$\begin{aligned} 0 &= \langle S^*_{w^m}S^*_{z^i}f, S^*_{z^n}S^*_{w^j}g\rangle \qquad \text{by (2.1)} \\ &= \langle S^*_{z^i}S^*_{w^m}f, S^*_{w^j}S^*_{z^n}g\rangle \qquad \text{by Lemma 2.1(ii)} \\ &= (S^*_{z^i}S^*_{w^m}f)\hat{}(0,0)\,\overline{(S^*_{w^j}S^*_{z^n}g)\hat{}(0,0)} \qquad \text{by (2.5) and (2.7)} \\ &\neq 0 \qquad \text{by (2.6) and (2.8).} \end{aligned}$$

This is a desired contradiction.

The converse is trivial. □

Corollary 2.3. *Let N be a backward shift invariant subspace of H^2. Then $S_{z^n}S^*_{w^m} = 0$ if and only if either $S_{z^n} = 0$ or $S^*_{w^m} = 0$. Hence if $S_{z^n}S^*_{w^m} = 0$, then $S^*_{w^m}S_{z^n} = 0$.*

Lemma 2.4. *Let M_1 and M_2 be closed subspaces of H^2 such that*

$$M_1 \subset \sum_{j=0}^{m} \oplus w^j H^2(\Gamma_z) \ \text{ and } \ M_2 \subset \sum_{i=0}^{n} \oplus z^i H^2(\Gamma_w).$$

Then $M_1 + M_2$ is closed.

Proof. We denote by $(z^i w^j)_{M_1}$ and $(z^i w^j)_{M_2}$ the orthogonal projections of $z^i w^j$ to the spaces M_1 and M_2, respectively. Let

$$M_1' = M_1 \ominus \Big(\Big[\{(z^i w^j)_{M_1}; 0 \le i \le n, 0 \le j \le m\}\Big]\Big)$$

and

$$M_2' = M_2 \ominus \Big(\Big[\{(z^i w^j)_{M_2}; 0 \le i \le n, 0 \le j \le m\}\Big]\Big).$$

Then M_1' and M_2' are closed subspaces of H^2,

$$M_1' \subset z^{n+1}\Big(\sum_{j=0}^{m} \oplus w^j H^2(\Gamma_z)\Big), \quad M_2' \subset w^{m+1}\Big(\sum_{i=0}^{n} \oplus z^i H^2(\Gamma_w)\Big),$$

and

$$M_1' + M_2' \perp \Big[\{z^i w^j; 0 \le i \le n, 0 \le j \le m\}\Big].$$

Since

$$z^{n+1}\Big(\sum_{j=0}^{m} \oplus w^j H^2(\Gamma_z)\Big) \perp w^{m+1}\Big(\sum_{i=0}^{n} \oplus z^i H^2(\Gamma_w)\Big),$$

$M_1' + M_2' = M_1' \oplus M_2'$ is closed. Hence

$$M_1 + M_2 = M_1' + M_2' + \Big[\{(z^i w^j)_{M_1}, (z^i w^j)_{M_2}; 0 \le i \le n, 0 \le j \le m\}\Big]$$

is closed. □

Theorem 2.5. *Let N be a backward shift invariant subspace of H^2. Then $S_{w^m}^* S_{z^n} = 0$ if and only if*

$$\text{(i)} \qquad N \ominus \Big(N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\Big) \subset N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z).$$

Proof. Suppose that $S_{w^m}^* S_{z^n} = 0$. Then $S_{z^n} N \subset \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$. Since $S_{z^n}^* S_{w^m} = 0$, $S_{w^m} N \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$. First, we prove the following

$$N = \Big(N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)\Big) + \Big(N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\Big). \tag{2.9}$$

Let

$$K = \Big(N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)\Big) + \Big(N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\Big). \tag{2.10}$$

By Lemma 2.4, K is closed and $N = K \oplus (N \ominus K)$. To prove (i), it is sufficient to prove $N \ominus K = \{0\}$. We have $N \ominus K \subset N \cap (S_{z^n} N)^\perp \cap (S_{w^m} N)^\perp$. Let $f \in N \ominus K$. Then $f \perp S_{z^n} N$, so that $S_{z^n}^* f \perp N$. Since

$S_{z^n}^* f \in N$, $S_{z^n}^* f = 0$. Hence $f \in \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$. By (2.10), $f \in K$. This shows $f = 0$, so that $N \ominus K = \{0\}$. Thus we get (2.9).

Let

$$N_1 = \Big\{ f \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z); f \perp N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \Big\}. \tag{2.11}$$

Then N_1 is a closed subspace and

$$N_1 \oplus \Big(N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \Big) \subset N.$$

If the equality holds in the above, (i) holds. So we assume that

$$N_1 \oplus \Big(N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \Big) \subsetneqq N.$$

We shall lead a contradiction. Let

$$N_2 = N \ominus \Big(N_1 \oplus \Big(N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \Big) \Big). \tag{2.12}$$

Then $N_2 \neq \{0\}$ and $N = N_1 \oplus N_2 \oplus (N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w))$.

Let $g \in N_2$ be such that $g \neq 0$. We shall prove that

$$g \notin N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \quad \text{and} \quad g \notin N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z). \tag{2.13}$$

The fact $g \notin N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$ is trivial. Suppose that $g \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$. By (2.11), $g \in N_1$. Since $g \in N_2$, by (2.12) we have $g \perp N_1$. Hence $g = 0$. This is a contradiction. Thus we get (2.13).

Next, we shall prove that

$$S_{z^n}^* g \not\perp N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z). \tag{2.14}$$

To prove this, suppose not. Then $S_{z^n}^* g \perp N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$. By (2.10), $g = g_1 + g_2$, where $g_1 \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$ and $g_2 \in N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$. Then $S_{z^n}^* g = S_{z^n}^* g_1 \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$. Therefore $S_{z^n}^* g = 0$, so that $g \in N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$. This contradicts (2.13). Thus we get (2.14).

By (2.14), there exists h_0 such that $h_0 \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$ and $\langle g, S_{z^n} h_0 \rangle = \langle S_{z^n}^* g, h_0 \rangle \neq 0$. Since $S_{z^n} h_0 \in N$, by (2.12) we have $S_{z^n} h_0 = h_1 \oplus h_2 \oplus h_3$, where $h_1 \in N_1, h_2 \in N_2$, and $h_3 \in$

$N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$. Since $g \in N_2$ and $\langle g, S_{z^n} h_0 \rangle \neq 0$, we have $h_2 \neq 0$. Since $z^n h_0 \perp \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$,

$$P_N z^n h_0 = S_{z^n} h_0 \perp N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w).$$

Thus we get $h_3 = 0$. By (2.12), $S_{w^m}^* N_1 = \{0\}$. Hence $S_{w^m}^* S_{z^n} h_0 = S_{w^m}^* h_2$. By (2.13) and $h_2 \in N_2$, $h_2 \notin \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$. This implies that $S_{w^m}^* h_2 \neq 0$. Hence $S_{w^m}^* S_{z^n} \neq 0$. This contradicts our starting assumption. Thus we have (i).

To prove the converse, suppose that condition (i) holds. Then $N = \left(N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\right) \oplus L$, where $L \subset N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$. Let $F = F_1 + F_2 \in N$, where $F_1 \in N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$ and $F_2 \in L$. Since $z^n F \perp N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$, $S_{z^n} F \in L$. Hence $S_{w^m}^* S_{z^n} F = 0$. Thus we get $S_{w^m}^* S_{z^n} = 0$. □

By Theorem 2.2, the structure of backward shift invariant subspaces N satisfying $S_{z^n} S_{w^m}^* = 0$ is simple. By Theorem 2.5, the structure of backward shift invariant subspaces N satisfying $S_{w^m}^* S_{z^n} = 0$ is not so simple. When $n = m = 1$, we have the following.

Theorem 2.6. *Let N be a backward shift invariant subspace of H^2. Then $S_w^* S_z = 0$ if and only if N has one of the following forms;*

(i) *$N = H^2(\Gamma_z) \ominus q(z) H^2(\Gamma_z)$ for some inner function $q(z)$.*

(ii) *$N = H^2(\Gamma_w) \ominus q(w) H^2(\Gamma_w)$ for some inner function $q(w)$.*

(iii) *Either $N = H^2(\Gamma_z) + H^2(\Gamma_w)$, or $N = H^2(\Gamma_z) + (H^2(\Gamma_w) \ominus q_2(w) H^2(\Gamma_w))$, or $N = (H^2(\Gamma_z) \ominus q_1(z) H^2(\Gamma_z)) + H^2(\Gamma_w)$, where $q_1(z)$ and $q_2(w)$ are inner functions.*

(iv) *$N = (H^2(\Gamma_z) \ominus q_1(z) H^2(\Gamma_z)) + (H^2(\Gamma_w) \ominus q_2(w) H^2(\Gamma_w))$, where $q_1(z), q_2(w)$ are nonconstant inner functions and $\hat{q}_1(0)\hat{q}_2(0) = 0$.*

In (iii) and (iv), since $1 \in N$, we may take q_1 and q_2 as $\hat{q}_1(0) = \hat{q}_2(0) = 0$.

Proof. By Theorem 2.2, $S_z S_w^* = 0$ if and only if either (i) or (ii) holds. By Theorem 2.5, $S_w^* S_z = 0$ if and only if

$$N \ominus (N \cap H^2(\Gamma_w)) \subset N \cap H^2(\Gamma_z). \tag{2.15}$$

If either (i) or (ii) holds, by Corollary 2.3 we have $S_w^* S_z = 0$. Suppose that N satisfies either (iii) or (iv). Then clearly $1 \in N$. Since N has a special form, it is not difficult to see that

$$N \ominus (N \cap H^2(\Gamma_w)) = \{f \in N \cap H^2(\Gamma_z); \hat{f}(0) = 0\}.$$

Hence (2.15) holds.

Next, suppose that (2.15) holds. Then we have

$$N = (N \cap H^2(\Gamma_z)) + (N \cap H^2(\Gamma_w)). \tag{2.16}$$

If either $N \cap H^2(\Gamma_z) = \{0\}$ or $N \cap H^2(\Gamma_w) = \{0\}$, then $S_z S_w^* = 0$, and by Corollary 2.3, $S_z^* S_w = 0$. Hence either (i) or (ii) holds. Suppose that $N \cap H^2(\Gamma_z) \neq \{0\}$ and $N \cap H^2(\Gamma_w) \neq \{0\}$. We shall prove $1 \in N$. To prove this, suppose that $1 \notin N$. Let 1_w be the orthogonal projection of 1 to $N \cap H^2(\Gamma_w)$. Then $1_w \notin H^2(\Gamma_z)$. Since $N \cap H^2(\Gamma_z) \neq \{0\}$, there exists $f \in N \cap H^2(\Gamma_z)$ such that $\hat{f}(0) \neq 0$. Let $f_1 = f - \hat{f}(0)1_w \in N$. Then $f_1 \notin H^2(\Gamma_z)$. Let $h \in N \cap H^2(\Gamma_w)$. Since $f \in H^2(\Gamma_z)$, $f - \hat{f}(0) \perp h$. Since $1 - 1_w \perp N \cap H^2(\Gamma_w)$,

$$\langle f_1, h \rangle = \langle \hat{f}(0)(1 - 1_w) + f - \hat{f}(0), h \rangle = 0.$$

Hence $f_1 \in N \ominus (N \cap H^2(\Gamma_w))$. Thus (2.15) does not hold. Therefore $1 \in N$. Since $N \cap H^2(\Gamma_z)$ and $N \cap H^2(\Gamma_w)$ are nonzero backward shift invariant subspaces, by (2.16) N has one of forms in (iii) and (iv). □

3. $S_{z^n} S_w^* = S_w^* S_{z^n}$.

The following is the main theorem in this section.

Theorem 3.1. *Let N be a backward shift invariant subspace of H^2, $N \neq \{0\}$, and $N \neq H^2$. Let $M = H^2 \ominus N$ and $n \geq 2$ be a positive integer. If $S_{z^n} S_w^* = S_w^* S_{z^n}$, then one of the following conditions holds;*

(i) $S_z S_w^* = S_w^* S_z$.

(ii) *$M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z)$ for an inner function $q_1(z)$ satisfying $q_1(z) = \prod_{j=1}^k b_j(z), 1 \leq k \leq n$, where b_i are simple Blaschke products.*

The following lemma is a generalization of [IN], and the idea of the proof is the same.

Lemma 3.2. *Let N be a backward shift invariant subspace of H^2 and $M = H^2 \ominus N$. Let n be a positive integer. Then $S_{z^n} S_w^* = S_w^* S_{z^n}$ if and only if*

$$M \ominus \left(z^n M \oplus \left(M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right) \subset (M \cap H^2(\Gamma_z)) \oplus wM.$$

Proof. The operators T_{z^n} and T_w^* on H^2 have the matrix forms as

$$T_{z^n} = \begin{pmatrix} * & P_M T_{z^n}|_N \\ 0 & S_{z^n} \end{pmatrix}, T_w^* = \begin{pmatrix} * & 0 \\ P_N T_w^*|_M & S_w^* \end{pmatrix} \quad \text{on } H^2 = \begin{pmatrix} M \\ \oplus \\ N \end{pmatrix}.$$

Set $A = P_M T_{z^n}|_N$ and $B = P_N T_w^*|_M$. Since $T_{z^n} T_w^* = T_w^* T_{z^n}$ on H^2, $S_{z^n} S_w^* = S_w^* S_{z^n}$ if and only if $BA = 0$. We have $T_w^*(M \ominus wM) \subset N$. For $f \in H^2$, $T_w^* f = 0$ if and only if $f \in H^2(\Gamma_z)$. Hence

$$\begin{aligned} \ker B &= \{f \in M; T_w^* f \in M\} \\ &= \{f \in M \ominus wM; T_w^* f = 0\} \oplus wM \\ &= (M \cap H^2(\Gamma_z)) \oplus wM. \end{aligned}$$

We denote by [ran A] the closed range of A. Let $A_1 = P_M T_{z^n} P_N$ on H^2. Then [ran A] = [ran A_1]. Since $A_1^* = P_N T_{z^n}^* P_M$, we get

$$\begin{aligned} \ker A_1^* &= \{f \in M; T_{z^n}^* f \in M\} \oplus N \\ &= \Big(M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\Big) \oplus z^n M \oplus N. \end{aligned}$$

Hence

$$[\operatorname{ran} A] = [\operatorname{ran} A_1] = H^2 \ominus \ker A_1^* = M \ominus \Big(z^n M \oplus \Big(M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\Big)\Big).$$

Since $BA = 0$ if and only if $[\operatorname{ran} A] \subset \ker B$, we get our assertion. □

Proof of Theorem 3.1. Suppose that $S_{z^n} S_w^* = S_w^* S_{z^n}$. By Lemma 3.2,

$$M \ominus \Big(z^n M \oplus \Big(M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\Big)\Big) \subset (M \cap H^2(\Gamma_z)) \oplus wM.$$

Let

$$K_0 = M \ominus \Big(z^n M \oplus \Big(M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\Big)\Big).$$

Then

$$K_0 \subset (M \cap H^2(\Gamma_z)) \oplus wM \tag{3.1}$$

and

$$M \ominus z^n M = K_0 \oplus \Big(M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\Big). \tag{3.2}$$

Since $M = \sum_{s=0}^{\infty} \oplus z^{ns}(M \ominus z^n M)$,

$$K_0 \perp \sum_{s=0}^{\infty} \oplus z^{ns} \Big(M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\Big) \tag{3.3}$$

and

$$M = \Big(\sum_{s=0}^{\infty} \oplus z^{ns} K_0\Big) \oplus \Big(\sum_{s=0}^{\infty} \oplus z^{ns} \Big(M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\Big)\Big). \tag{3.4}$$

First, suppose that $K_0 = \{0\}$. In this case, we shall show that condition (i) holds. By (3.2),

$$\sum_{i=0}^{n-1} \oplus z^i (M \ominus zM) = M \ominus z^n M = M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w).$$

Let $f \in M \ominus zM$. Then $f \in M \ominus z^n M$ and $f = \sum_{i=0}^{n-1} \oplus z^i h_i(w)$, where $h_i(w) \in H^2(\Gamma_w)$. Since $f \in M \ominus zM$,

$$z^{n-1} f = \sum_{i=0}^{n-1} \oplus z^{n-1+i} h_i(w) \in M \ominus z^n M.$$

Hence $f = h_0(w)$, so that $M \ominus zM = M \cap H^2(\Gamma_w)$. By Lemma 3.2 (for the case of $n = 1$), we get $S_z S_w^* = S_w^* S_z$.

Next, suppose that $K_0 \neq \{0\}$. The proof of this case is a little bit long, so we devide into several steps.

Step 1. In this step, we shall prove that there is a nonconstant inner function $q_1(z)$ such that

$$K_0 \subset q_1(z) H^2(\Gamma_z) \oplus wM. \tag{3.5}$$

First we prove that $K_0 \not\subset wM$. To prove this, suppose that $K_0 \subset wM$. Let $F \in K_0$. Then $F = wf$ for some $f \in M$. We shall prove that $f \in K_0$. We have

$$\begin{aligned} &\Big\langle f, \Big(\sum_{s=1}^{\infty} \oplus z^{ns} K_0 \Big) \oplus \Big(\sum_{s=0}^{\infty} \oplus z^{ns} \Big(M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \Big) \Big) \Big\rangle \\ = \ & \Big\langle wf, w\Big(\sum_{s=1}^{\infty} \oplus z^{ns} K_0 \oplus \Big(\sum_{s=0}^{\infty} \oplus z^{ns} \Big(M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \Big) \Big) \Big) \Big\rangle \\ = \ & \Big\langle F, z^n w \Big(\sum_{s=1}^{\infty} \oplus z^{n(s-1)} K_0 \Big) \Big\rangle \qquad \text{by (3.3)} \\ = \ & 0, \end{aligned}$$

where the last equality follows from $w \sum_{s=1}^{\infty} \oplus z^{n(s-1)} K_0 \subset M$ and (3.2). Hence by (3.4), we get $f \in K_0$. Therefore for every positive integer p, we have $F = w^p f_p$ for some $f_p \in K_0$. This leads $F = 0$. This is a contradiction. Thus we get $K_0 \not\subset wM$.

Hence by (3.1), $M \cap H^2(\Gamma_z) \neq \{0\}$. By the Beurling theorem,

$$M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z), \tag{3.6}$$

where $q_1(z)$ is inner. Then $q_1(z) \in M$ and

$$q_1(z) H^2(\Gamma_z) \perp wM. \tag{3.7}$$

If $q_1(z)$ is constant, we have $M = H^2$, so that $N = \{0\}$. This contradicts our assumption. Hence $q_1(z)$ is a nonconstant inner function. By (3.1) and (3.6), we get (3.5).

Step 2. In this step, we prove

$$K_0 \subset q_1(z)\Big(\sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\Big). \tag{3.8}$$

Let $G \in K_0$. Then by (3.5), $G = q_1(z)h(z) \oplus wg$, where $h(z) \in H^2(\Gamma_z)$ and $g \in M$. Write

$$h(z) = \Big(\sum_{i=0}^{n-1} \oplus a_i z^i\Big) \oplus z^n h_0(z).$$

Then

$$G = q_1(z)\Big(\sum_{i=0}^{n-1} \oplus a_i z^i\Big) \oplus q_1(z) z^n h_0(z) \oplus wg.$$

By (3.6), $q_1(z)z^n h_0(z) \in z^n M$. Since $G \in K_0 \subset M \ominus z^n M$, we have $h_0(z) = 0$. Hence

$$G = q_1(z)\Big(\sum_{i=0}^{n-1} \oplus a_i z^i\Big) \oplus wg. \tag{3.9}$$

Here we prove that

$$g \in K_0. \tag{3.10}$$

Since $G = q_1(z)h(z) \oplus wg$, we have

$$\begin{aligned}
\Big\langle g, \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \Big\rangle &= \Big\langle wg, w\sum_{s=1}^{\infty} \oplus z^{ns} K_0 \Big\rangle \\
&= \Big\langle wg, w\sum_{s=1}^{\infty} \oplus z^{ns} K_0 \Big\rangle + \Big\langle q_1 h, w\sum_{s=1}^{\infty} \oplus z^{ns} K_0 \Big\rangle \quad \text{by (3.7)} \\
&= \Big\langle G, w\sum_{s=1}^{\infty} \oplus z^{ns} K_0 \Big\rangle \\
&= \Big\langle G, z^n w\sum_{s=1}^{\infty} \oplus z^{n(s-1)} K_0 \Big\rangle \\
&= 0 \qquad \text{by (3.2).}
\end{aligned}$$

We also have

$$\begin{aligned}
&\Big\langle g, \sum_{s=0}^{\infty} \oplus z^{ns}\Big(M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\Big)\Big\rangle \\
= &\Big\langle wg, w\Big(\sum_{s=0}^{\infty} \oplus z^{ns}\Big(M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\Big)\Big)\Big\rangle \\
= &\Big\langle G, \sum_{s=0}^{\infty} \oplus z^{ns} w\Big(M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\Big)\Big\rangle \quad \text{by (3.7)} \\
= &\ 0 \qquad \text{by (3.3).}
\end{aligned}$$

Hence by (3.4), we get (3.10).

Applying (3.9) and (3.10) infinitely many times, we have

$$\begin{aligned}
G &= q_1(z)\Big(\sum_{i=0}^{n-1} \oplus a_i z^i\Big) \oplus w q_1(z)\Big(\sum_{i=0}^{n-1} \oplus b_i z^i\Big) \oplus \cdots \\
&\in q_1(z)\Big(\sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\Big).
\end{aligned}$$

Therefore we get (3.8).

Step 3. In this step, we study functions in $M \ominus zM$ and the inner function $q_1(z)$. By (3.8), there is a closed subspace L such that $K_0 = q_1(z)L$ and $L \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$. Then by (3.2),

$$M \ominus z^n M = q_1(z)L \oplus \Big(M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\Big).$$

Since $K_0 \neq \{0\}$, $L \neq \{0\}$. We have $M \ominus z^n M = \sum_{i=0}^{n-1} \oplus z^i (M \ominus zM)$. Hence $M \ominus zM \neq \{0\}$. Let $F \in M \ominus zM$ be such that $F \neq 0$. Then

$$F = \Big(q_1 \sum_{i=0}^{n-1} \oplus z^i f_i\Big) \oplus \sum_{i=0}^{n-1} \oplus z^i g_i, \tag{3.11}$$

where $f_i, g_i \in H^2(\Gamma_w)$,

$$q_1 \sum_{i=0}^{n-1} \oplus z^i f_i \perp M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w), \tag{3.12}$$

and

$$\sum_{i=0}^{n-1} \oplus z^i g_i \in M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w). \tag{3.13}$$

Since $n \geq 2$, $zF \in M \ominus z^n M$, so that we have

$$zF = q_1\Big(\sum_{i=0}^{n-1} \oplus z^i G_{1,i}\Big) + \sum_{i=0}^{n-1} \oplus z^i H_{1,i},$$

where $G_{1,i}, H_{1,i} \in H^2(\Gamma_w)$. Hence

(3.14)

$$q_1\Big(z\Big(\sum_{i=0}^{n-1} \oplus z^i f_i\Big) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}\Big) = \Big(\sum_{i=0}^{n-1} \oplus z^i H_{1,i}\Big) - z\sum_{i=0}^{n-1} \oplus z^i g_i.$$

Here we devide into two subcases.

Subcase 1. $$z\Big(\sum_{i=0}^{n-1} \oplus z^i f_i(w)\Big) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w) \not\equiv 0.$$

Then

$$q_1(z) = \frac{\Big(\sum_{i=0}^{n-1} \oplus z^i H_{1,i}(w)\Big) - z\sum_{i=0}^{n-1} \oplus z^i g_i(w)}{z\Big(\sum_{i=0}^{n-1} \oplus z^i f_i(w)\Big) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w)}.$$

As proved in Step 1, $q_1(z)$ is a nonconstant inner function. Then by the above, we have

$$q_1(z) = \prod_{j=1}^{k} b_j(z), \quad 1 \leq k \leq n, \tag{3.15}$$

where b_j are simple Blaschke products.

Subcase 2. $$z\Big(\sum_{i=0}^{n-1} \oplus z^i f_i(w)\Big) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w) \equiv 0.$$

Then by (3.14), $f_{n-1} = g_{n-1} = 0$, so that by (3.11)

$$F = \Big(q_1 \sum_{i=0}^{n-2} \oplus z^i f_i\Big) \oplus \sum_{i=0}^{n-2} \oplus z^i g_i.$$

Since $F \in M \ominus zM$,

$$zF = \Big(q_1 \sum_{i=0}^{n-2} \oplus z^{i+1} f_i\Big) \oplus \sum_{i=0}^{n-2} \oplus z^{i+1} g_i \in M \ominus z^2 M.$$

In the same way as above, either (3.15) holds or $f_{n-2} = g_{n-2} = 0$. Repeat the same argument. Then either (3.15) holds or

$$f_{n-1} = f_{n-2} = \cdots = f_1 = g_{n-1} = g_{n-2} = \cdots = g_1 = 0.$$

Step 4. If (3.15) holds, then (ii) holds. So, suppose that (3.15) does not holds. Then by (3.11) and by the above fact, $F = q_1 f_0 \oplus g_0$, by (3.12) $q_1 f_0 \perp M \cap H^2(\Gamma_w)$, and by (3.13) $g_0 \in M \cap H^2(\Gamma_w)$ for every $F \in M \ominus zM$.

If $g_0 = 0$ for every $F \in M \ominus zM$, since $q_1(z) \in M$ it follows that $M \ominus zM \subset q_1(z)H^2(\Gamma_w) \subset M$. Since $M = \sum_{i=0}^{\infty} \oplus z^i(M \ominus zM)$, we have $M = q_1(z)H^2$, so that $N = H^2 \ominus q_1(z)H^2$. By Theorem A, condition (i) holds.

So we assume that $g_0 \neq 0$ for some $F \in M \ominus zM$. We shall prove that

$$(M \ominus zM) \ominus (M \cap H^2(\Gamma_w)) \subset (M \cap H^2(\Gamma_z)) \oplus wM. \tag{3.16}$$

We may assume that $(M \ominus zM) \ominus (M \cap H^2(\Gamma_w)) \neq \{0\}$. Let $G \in (M \ominus zM) \ominus (M \cap H^2(\Gamma_w))$ be such that $G \neq 0$. Then $G = q_1(z)h_1(w) \oplus h_2(w)$, where $q_1(z)h_1(w) \perp M \cap H^2(\Gamma_w)$ and $h_2(w) \in M \cap H^2(\Gamma_w)$. Hence $h_2(w) = G - q_1(z)h_1(w) \perp M \cap H^2(\Gamma_w)$. Therefore $h_2(w) = 0$. Since $q_1(z) \in M$,

$$\begin{aligned} G = q_1(z)h_1(w) &= h_1(0)q_1(z) \oplus wq_1(z)\frac{h_1(w) - h_1(0)}{w} \\ &\in M \cap H^2(\Gamma_z) \oplus wM. \end{aligned}$$

Thus we get (3.16). By Lemma 3.2 (for $n = 1$), (i) holds. □

4. $S_{z^2}S_w^* = S_w^*S_{z^2}$ AND $S_zS_w^* \neq S_w^*S_z$.

Let N be a backward shift invariant subspace of H^2 and let n be a positive integer. Let $M = H^2 \ominus N$. Then M is an invariant subspace. If both $S_{z^n}S_w^* = S_w^*S_{z^n}$ and $S_zS_w^* \neq S_w^*S_z$ hold, then by Theorem 3.1, $M \cap H^2(\Gamma_z) = q(z)H^2(\Gamma_z)$ for some nonconstant inner function $q(z)$.

In this section, we assume that $q_1(z)H^2 \subset M$ and $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$ for some nonconstant inner function $q_1(z)$. Let

$$\tilde{M} = M \ominus q_1(z)H^2 \subset M.$$

Then $H^2 \ominus q_1(z)H^2 = \tilde{M} \oplus N$ and $\tilde{M}$ is w-invariant. The following lemma is proved in [INS, Lemma 3.2].

Lemma 4.1. *Let $f \in \tilde{M}$. Then $T_w^* f \in \tilde{M}$ if and only if $f \in w\tilde{M}$.*

We denote by $P_\perp$ the orthogonal projection from H^2 onto $H^2 \ominus q_1(z)H^2$. Then we have a Toeplitz type operator Q_{z^n} on $H^2 \ominus q_1(z)H^2$ such that

$$Q_{z^n} : H^2 \ominus q_1(z)H^2 \ni f \to P_\perp(T_{z^n} f) \in H^2 \ominus q_1(z)H^2.$$

Since $z^n M \subset M$, $Q_{z^n}\tilde{M} \subset \tilde{M}$ and $Q_z^n = Q_{z^n}$. Then Q_{z^n} has the following matrix form;

$$Q_{z^n} = \begin{pmatrix} * & P_{\tilde{M}} T_{z^n}|_N \\ 0 & S_{z^n} \end{pmatrix} \quad \text{on} \quad H^2 \ominus q_1(z)H^2 = \begin{pmatrix} \tilde{M} \\ \oplus \\ N \end{pmatrix}.$$

Since $H^2 \ominus q_1(z)H^2$ is backward shift invariant, $T_w^*(H^2 \ominus q_1(z)H^2) \subset H^2 \ominus q_1(z)H^2$. Since $T_w^* N \subset N$, the operator T_w^* on $H^2 \ominus q_1(z)H^2$ has the following matrix form;

$$T_w^* = \begin{pmatrix} * & 0 \\ P_N T_w^*|_{\tilde{M}} & S_w^* \end{pmatrix} \quad \text{on} \quad H^2 \ominus q_1(z)H^2 = \begin{pmatrix} \tilde{M} \\ \oplus \\ N \end{pmatrix}.$$

Set

$$A = P_{\tilde{M}} T_{z^n}|_N \quad \text{and} \quad B = P_N T_w^*|_{\tilde{M}}. \tag{4.1}$$

By [INS, Lemma 3.3], $T_w^* Q_z = Q_z T_w^*$ on $H^2 \ominus q_1(z)H^2$. Hence we have the following.

Lemma 4.2. *$T_w^* Q_{z^n} = Q_{z^n} T_w^*$ on $H^2 \ominus q_1(z)H^2$.*

Lemma 4.3. *$S_{z^n} S_w^* = S_w^* S_{z^n}$ if and only if $BA = 0$.*

Proof. By Lemma 4.2, $T_w^* Q_{z^n} = Q_{z^n} T_w^*$ on $H^2 \ominus q_1(z)H^2$. Then $BA + S_w^* S_{z^n} = S_{z^n} S_w^*$. Hence $S_{z^n} S_w^* = S_w^* S_{z^n}$ if and only if $BA = 0$. □

The following is a slight generalization of [INS, Theorem 3.5].

Theorem 4.4. *Let N be a backward shift invariant subspace of H^2 and $M = H^2 \ominus N$. Suppose that $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$, where $q_1(z)$ is a nonconstant inner function. Let $\tilde{M} = M \ominus q_1(z)H^2$. Then the following conditions are equivalent.*

(i) *$S_{z^n} S_w^* = S_w^* S_{z^n}$ on N.*
(ii) *$\tilde{M} \ominus \{f \in \tilde{M}; T_{z^n}^* f \in \tilde{M}\} \subset w\tilde{M}$.*
(iii) *$T_{z^n}^* \tilde{M} \subset \tilde{M}$.*

Proof. The idea of the proof is similar to the one of [INS, Theorem 3.5]. For the sake of completeness, we give the proof.

(i) $\Leftrightarrow$ (ii) By Lemma 4.3, condition (i) is equivalent to $BA = 0$. By (4.1) and Lemma 4.1, $\ker B = \{f \in \tilde{M}; T_w^* f \in \tilde{M}\} = w\tilde{M}$. Put $A_1 = P_{\tilde{M}} T_{z^n} P_N$ on $\tilde{M} \oplus N$. Then $[\text{ran } A] = [\text{ran } A_1]$. Since $A_1^* = P_N T_{z^n}^* P_{\tilde{M}}$, $\ker A_1^* = N \oplus \{f \in \tilde{M}; T_{z^n}^* f \in \tilde{M}\}$. Hence

$$[\text{ran } A] = [\text{ran } A_1] = (\tilde{M} \oplus N) \ominus \ker A_1^* = \tilde{M} \ominus \{f \in \tilde{M}; T_{z^n}^* f \in \tilde{M}\}.$$

Therefore $BA = 0$ if and only if $\tilde{M} \ominus \{f \in \tilde{M}; T_{z^n}^* f \in \tilde{M}\} \subset w\tilde{M}$. Thus we get (i) $\Leftrightarrow$ (ii).

(ii) $\Rightarrow$ (iii) Suppose that $\tilde{M} \ominus \{f \in \tilde{M}; T_{z^n}^* f \in \tilde{M}\} \subset w\tilde{M}$. Since $\{f \in \tilde{M}; T_{z^n}^* f \in \tilde{M}\}$ is closed, $\tilde{M} \ominus w\tilde{M} \subset \{f \in \tilde{M}; T_{z^n}^* f \in \tilde{M}\}$. Since $w\tilde{M} \subset \tilde{M}$, $\tilde{M} = \sum_{j=0}^{\infty} \oplus w^j(\tilde{M} \ominus w\tilde{M})$. Since $T_{z^n}^* w^j f = w^j T_{z^n}^* f$ for $f \in H^2$, we have $T_{z^n}^* \tilde{M} \subset \tilde{M}$.

(iii) $\Rightarrow$ (ii) is trivial. □

For $f \in H^2(\Gamma_z)$, write $f^*(z) = T_z^* f(z) == \overline{z}(f(z) - \hat{f}(0))$.

Lemma 4.5. *Let $b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}_i z), |\alpha_i| < 1,$ and $1 \le i \le n$. Then*

(i) $T_z^* z = 1$, $T_z^* b_1 = b_1^*(z) = (1 - |\alpha_1|^2)/(1 - \overline{\alpha}_1 z)$, *and* $T_z^* b_1^*(z) = \overline{\alpha}_1 b_1^*(z)$.

(ii) $T_z^*(b_1(z) b_2^*(z)) = (1 - |\alpha_2|^2) b_1^*(z) + \overline{\alpha}_2 b_1(z) b_2^*(z)$.

(iii) $H^2(\Gamma_z) \ominus (\prod_{j=1}^k b_j(z)) H^2(\Gamma_z) = \sum_{j=1}^k \oplus [b_1(z) \cdots b_{j-1}(z) b_j^*(z)]$.

(iv) $H^2 \ominus (\prod_{j=1}^k b_j(z)) H^2 = \sum_{j=1}^k \oplus [b_1(z) \cdots b_{j-1}(z) b_j^*(z) H^2(\Gamma_w)]$.

Proof. It is not difficult to prove (i).

(ii) Since

$$\begin{aligned} \overline{z} b_1(z) b_2^*(z) &= \overline{z} b_1(z) \frac{1 - |\alpha_2|^2}{1 - \overline{\alpha}_2 z} \\ &= (1 - |\alpha_2|^2) b_1(z) \Big(\overline{z} + \frac{\overline{\alpha}_2}{1 - \overline{\alpha}_2 z} \Big) \\ &= (1 - |\alpha_2|^2) \overline{z} b_1(z) + \overline{\alpha}_2 b_1(z) b_2^*(z), \end{aligned}$$

we get (ii).

(iii) See [Ni, p.33].

(iv) follows from (iii). □

Corollary 4.6. *Let N be a backward shift invariant subspace of H^2 and $M = H^2 \ominus N$. Suppose that $M \cap H^2(\Gamma_z) = b_1(z) H^2(\Gamma_z)$, where $b_1(z)$ is a simple Blaschke product. Then $S_z S_w^* = S_w^* S_z$.*

Proof. Let $b_1(z) = (z - \alpha)/(1 - \overline{\alpha} z), |\alpha| < 1$, and $\tilde{M} = M \ominus b_1(z) H^2$. Since $b_1(z) \in M$, $b_1(z) H^2 \subset M$. By Lemma 4.5(iv), $\tilde{M} \subset b_1^*(z) H^2(\Gamma_w)$. By Lemma 4.5(i), $T_z^*(b_1^*(z) h(w)) = \overline{\alpha}_1 b_1^*(z) h(w)$. Hence $T_z^* \tilde{M} \subset \tilde{M}$. By Theorem 4.4, $S_z S_w^* = S_w^* S_z$. □

Corollary 4.7. *Let N be a backward shift invariant subspace of H^2 and $M = H^2 \ominus N$. Suppose that $M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z)$, where $q_1(z)$ is an inner function. Let n, k be positive integers with $n \ge k + 1$. Moreover suppose that $q_1(z) = z^k b(z)$, where b is a simple Blaschke product, $b(z) = (z - \alpha)/(1 - \overline{\alpha} z)$, and $\alpha \ne 0$. If $S_{z^n} S_w^* = S_w^* S_{z^n}$, then $S_{z^k} S_w^* = S_w^* S_{z^k}$.*

Proof. Let $\tilde{M} = M \ominus q_1(z)H^2$. If $\tilde{M} = \{0\}$, then $M = q_1(z)H^2$. By Theorem A, $S_z S_w^* = S_w^* S_z$. Suppose that $\tilde{M} \neq \{0\}$. By Lemma 4.5(iv),

$$\tilde{M} \subset b^*(z)H^2(\Gamma_w) \oplus \Big(\sum_{j=1}^{k} \oplus z^{j-1} b(z) H^2(\Gamma_w) \Big).$$

Let $f \in \tilde{M}$. Then

$$f = b^*(z)h_0(w) + \sum_{j=1}^{k} \oplus z^{j-1} b(z) h_j(w), \quad h_j(w) \in H^2(\Gamma_w).$$

By Lemma 4.5(i),

$$\begin{aligned} T_{z^n}^* f &= T_{z^{n-k}}^* (T_{z^k}^* f) \\ &= T_{z^{n-k}}^* \Big(\sum_{j=0}^{k} \overline{\alpha}^{(k-j)} b^*(z) h_j(w) \Big) \\ &= \overline{\alpha}^{(n-k)} \sum_{j=0}^{k} \overline{\alpha}^{(k-j)} b^*(z) h_j(w). \end{aligned}$$

Since $S_{z^n} S_w^* = S_w^* S_{z^n}$, by Theorem 4.4 $T_{z^n}^* f \in \tilde{M}$. Since $\alpha \neq 0$,

$$\sum_{j=0}^{k} \overline{\alpha}^{(k-j)} b^*(z) h_j(w) \in \tilde{M}.$$

Thus $T_{z^k}^* \tilde{M} \subset \tilde{M}$. By Theorem 4.4, $S_{z^k} S_w^* = S_w^* S_{z^k}$. □

Theorem 4.8. *Let N be a backward shift invariant subspace of H^2 and $M = H^2 \ominus N$. Suppose that $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$, where $q_1(z)$ is an inner function. Moreover suppose that $q_1(z) = b_1(z)b_2(z)$, where $b_i(z), i = 1, 2$, are simple Blaschke products, $b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}_i z)$, and $\alpha_1 \alpha_2 \neq 0$. Let $n \geq 2$ be a positive integer. Then we have the following.*

(i) *If $S_{z^n} S_w^* = S_w^* S_{z^n}$ and $S_{z^{n-1}} S_w^* \neq S_w^* S_{z^{n-1}}$, then $\alpha_1^n = \alpha_2^n$ and $\alpha_1 \neq \alpha_2$.*
(ii) *If $\alpha_1^n = \alpha_2^n$ and $\alpha_1 \neq \alpha_2$, then $S_{z^n} S_w^* = S_w^* S_{z^n}$.*

Proof. Let $\tilde{M} = M \ominus q_1(z)H^2$. Suppose that $S_{z^n} S_w^* = S_w^* S_{z^n}$ and $S_{z^{n-1}} S_w^* \neq S_w^* S_{z^{n-1}}$. By Theorem 4.4, $T_{z^n}^* \tilde{M} \subset \tilde{M}$ and $T_{z^{n-1}}^* \tilde{M} \not\subset \tilde{M}$. By Lemma 4.5(iv),

$$\tilde{M} \subset b_1^*(z)H^2(\Gamma_w) \oplus b_1(z) b_2^*(z) H^2(\Gamma_w). \tag{4.2}$$

Then there exists $f_0 \in \tilde{M}$ such that $T_{z^{n-1}}^* f_0 \notin \tilde{M}$, and

$$f_0 = b_1^*(z) g_1(w) + b_1(z) b_2^*(z) g_2(w) \in b_1^*(z)H^2(\Gamma_w) \oplus b_1(z) b_2^*(z) H^2(\Gamma_w).$$

By Lemma 4.5,

$$T^*_{z^{n-1}}f_0 = b_1^*\Big(\overline{\alpha}_1^{(n-1)}g_1 + (1-|\alpha_2|^2)\Big(\sum_{j=0}^{n-2}\overline{\alpha}_1^{(n-2-j)}\overline{\alpha}_2^j\Big)g_2\Big) + \overline{\alpha}_2^{(n-1)}b_1b_2^*g_2$$

and

$$T^*_{z^n}f_0 = b_1^*\Big(\overline{\alpha}_1^n g_1 + (1-|\alpha_2|^2)\Big(\sum_{j=0}^{n-1}\overline{\alpha}_1^{(n-1-j)}\overline{\alpha}_2^j\Big)g_2\Big) + \overline{\alpha}_2^n b_1b_2^*g_2.$$

Since $T^*_{z^{n-1}}f_0 \notin \tilde{M}$ and $f_0 \in \tilde{M}$, $T^*_{z^{(n-1)}}f_0 - \overline{\alpha}_2^{n-1}f_0 \notin \tilde{M}$. Then

$$b_1^*\Big((\overline{\alpha}_1^{(n-1)} - \overline{\alpha}_2^{(n-1)})g_1 + (1-|\alpha_2|^2)\Big(\sum_{j=0}^{n-2}\overline{\alpha}_1^{(n-2-j)}\overline{\alpha}_2^j\Big)g_2\Big) \notin \tilde{M}.$$

Hence

$$\Big(\sum_{j=0}^{n-2}\overline{\alpha}_1^{(n-2-j)}\overline{\alpha}_2^j\Big)b_1^*\Big((\overline{\alpha}_1 - \overline{\alpha}_2)g_1 + (1-|\alpha_2|^2)g_2\Big) \notin \tilde{M}.$$

Since $0 \in \tilde{M}$, $\sum_{j=0}^{n-2}\overline{\alpha}_1^{(n-2-j)}\overline{\alpha}_2^j \neq 0$, so that

$$b_1^*\Big((\overline{\alpha}_1 - \overline{\alpha}_2)g_1 + (1-|\alpha_2|^2)g_2\Big) \notin \tilde{M}. \tag{4.3}$$

Since $T^*_{z^n}f_0 \in \tilde{M}$, $T^*_{z^n}f_0 - \overline{\alpha}_2^n f_0 \in \tilde{M}$. Then

$$b_1^*\Big((\overline{\alpha}_1^n - \overline{\alpha}_2^n)g_1 + (1-|\alpha_2|^2)\Big(\sum_{j=0}^{n-1}\overline{\alpha}_1^{(n-1-j)}\overline{\alpha}_2^j\Big)g_2\Big) \in \tilde{M}.$$

Hence

$$\Big(\sum_{j=0}^{n-1}\overline{\alpha}_1^{(n-1-j)}\overline{\alpha}_2^j\Big)b_1^*\Big((\overline{\alpha}_1 - \overline{\alpha}_2)g_1 + (1-|\alpha_2|^2)g_2\Big) \in \tilde{M}. \tag{4.4}$$

Now we prove (i). Suppose that $\alpha_1 = \alpha_2$. Then $\alpha_1 = \alpha_2 \neq 0$, so that $\sum_{j=0}^{n-1}\overline{\alpha}_1^{(n-1-j)}\overline{\alpha}_2^j \neq 0$. By (4.3) and (4.4), $b_1^*g_2 \notin \tilde{M}$ and $b_1^*g_2 \in \tilde{M}$. This is a contradiction. Hence $\alpha_1 \neq \alpha_2$.

Suppose that $\alpha_1^n \neq \alpha_2^n$. Then $\sum_{j=0}^{n-1}\overline{\alpha}_1^{(n-1-j)}\overline{\alpha}_2^j \neq 0$. By (4.4),

$$b_1^*\Big((\overline{\alpha}_1 - \overline{\alpha}_2)g_1 + (1-|\alpha_2|^2)g_2\Big) \in \tilde{M}.$$

This contradicts (4.3), so that $\alpha_1^n = \alpha_2^n$. Thus we get (i).

(ii) Suppose that $\alpha_1^n = \alpha_2^n$ and $\alpha_1 \neq \alpha_2$. Then $\sum_{j=0}^{n-1}\overline{\alpha}_1^{(n-1-j)}\overline{\alpha}_2^j = 0$. Let $f \in \tilde{M}$. Then by (4.2), $f = b_1^*(z)h_1(w) + b_1(z)b_2^*(z)h_2(w)$. Similarly, we have

$$T_{z^n}^* f - \overline{\alpha}_2^n f = \Big(\sum_{j=0}^{n-1} \overline{\alpha}_1^{(n-1-j)} \overline{\alpha}_2^j\Big) b_1^* \Big((\overline{\alpha}_1 - \overline{\alpha}_2) h_1 + (1 - |\alpha_2|^2) h_2\Big).$$

Hence $T_{z^n}^* f = \overline{\alpha}_2^n f \in \tilde{M}$, so that we get $T_{z^n}^* \tilde{M} \subset \tilde{M}$. By Theorem 4.4, $S_{z^n} S_w^* = S_w^* S_{z^n}$. □

Corollary 4.9. *Let N be a backward shift invariant subspace of H^2 and $M = H^2 \ominus N$. Suppose that $M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z)$, where $q_1(z)$ is an inner function. Moreover suppose that $q_1(z) = b_1(z) b_2(z)$, where $b_i(z), i = 1, 2$, are simple Blaschke products, $b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}_i z)$, and $\alpha_1 \alpha_2 \neq 0$. Then we have the following.*

(i) *If $S_{z^2} S_w^* = S_w^* S_{z^2}$ and $S_z S_w^* \neq S_w^* S_z$, then $\alpha_1 + \alpha_2 = 0$.*
(ii) *If $\alpha_1 + \alpha_2 = 0$, then $S_{z^2} S_w^* = S_w^* S_{z^2}$.*

The following is the main theorem in this section.

Theorem 4.10. *Let N be a backward shift invariant subspace of H^2, $N \neq \{0\}$, and $N \neq H^2$. Then $S_{z^2} S_w^* = S_w^* S_{z^2}$ if and only if one of the following conditions holds.*

(i) *$S_z S_w^* = S_w^* S_z$.*
(ii) *$S_{z^2} S_w^* = 0$.*
(iii) *There exist two simple Blaschke products $b_1(z)$ and $b_2(z)$, $b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}_i z), 0 < |\alpha_i| < 1$, such that $N \subset H^2 \ominus b_1(z) b_2(z) H^2$ and $\alpha_1 + \alpha_2 = 0$.*

A backward shift invariant subspace N satisfying condition (i) is given by Theorem A. Also N satisfying conditon (ii) is given by Theorem 2.2.

Proof of Theorem 4.10. Suppose that $S_{z^2} S_w^* = S_w^* S_{z^2}$. Moreover suppose that $S_z S_w^* \neq S_w^* S_z$. By Theorem 3.1, $M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z)$ for an inner function $q_1(z)$ such that either $q_1(z) = b_1(z)$ or $q_1(z) = b_1(z) b_2(z)$, where $b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}_i z), |\alpha_i| < 1$. If $q_1(z) = b_1(z)$, by Corollary 4.6, (i) holds.

Suppose that $q_1(z) = b_1(z) b_2(z)$. Moreover suppose that $\alpha_1 = \alpha_2 = 0$. Then $q_1(z) = z^2$. Hence $z^2 H^2 \subset M$, so that $N \subset H^2(\Gamma_w) \oplus z H^2(\Gamma_w)$. Then by Theorem 2.2, $S_{z^2} S_w^* = 0$. Thus (ii) holds.

Suppose that $\alpha_1 \neq 0$ and $\alpha_2 = 0$. Then $q_1(z) = z b_1(z)$. By Corollary 4.7, (i) holds.

Suppose that $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. Then by Corollary 4.9(i), we get $\alpha_1 + \alpha_2 = 0$. Hence (iii) holds.

Next, we prove the converse assertion. Suppose that one of conditions (i), (ii), and (iii) holds. If (i) holds, then trivially $S_{z^2}S_w^* = S_w^*S_{z^2}$. If (ii) holds, by Corollary 2.3 we have $S_{z^2}S_w^* = S_w^*S_{z^2}$.

Suppose that (iii) holds. Then $b_1(z)b_2(z)H^2(\Gamma_z) \subset M$. Hence $M \cap H^2(\Gamma_z)$ equals to either $b_1(z)H^2(\Gamma_z)$, or $b_2(z)H^2(\Gamma_z)$, or $b_1(z)b_2(z)H^2(\Gamma_z)$. By Corollary 4.6, $S_zS_w^* = S_w^*S_z$ for the frist two cases. Hence $S_{z^2}S_w^* = S_w^*S_{z^2}$. For the last case, by Corollary 4.9(ii), $S_{z^2}S_w^* = S_w^*S_{z^2}$. $\square$

Example 4.11. We give an example of a backward shift invariant subspace N of H^2 satisfying $S_{z^2}S_w^* = S_w^*S_{z^2}$, $S_zS_w^* \neq S_w^*S_z$, and $S_{z^2}S_w^* \neq 0$. Let $q_1(z) = b_1(z)b_2(z)$, where $b_i(z), i = 1, 2$, are $b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}_i z)$, $\alpha_1 + \alpha_2 = 0$, and $\alpha_1 \neq 0$. Let $q_2(w)$ be a nonconstant inner function. Let $M = q_1(z)H^2 \oplus b_1(z)b_2^*(z)q_2(w)H^2(\Gamma_w)$. Then M is an invariant subspace and $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$. Let $N = H^2 \ominus M$. By Theorem 4.10, $S_{z^2}S_w^* = S_w^*S_{z^2}$. We have $\tilde{M} = M \ominus q_1(z)H^2 = b_1(z)b_2^*(z)q_2(w)H^2(\Gamma_w)$. Since

$$T_z^* b_1(z)b_2^*(z)q_2(w) = (1 - |\alpha_1|^2)b_1^*(z)q_2(w) + \overline{\alpha}_1 b_1(z)b_2^*(z)q_2(w),$$

$T_z^* b_1(z)b_2^*(z)q_2(w) \notin \tilde{M}$. By Theorem 4.4, $S_zS_w^* \neq S_w^*S_z$. By Theorem 2.2, $S_{z^2}S_w^* \neq 0$.

We leave the following problem for the reader.

Problem 4.12. Characterize backward invariant subspaces N of H^2 satisfying $S_{z^n}S_w^* = S_w^*S_{z^n}$ for $n \geq 3$.

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