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BACKWARD SHIFT INVARIANT SUBSPACES IN THE BIDISC III

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ABSTRACT. In the previous paper, we give a characterization of backward shift invariant subspaces of the Hardy space in the bidisc which satisfy the doubly commuting condition $S_zS_w^* = S_w^*S_z$ for the compression operators $S_z$ and $S_w$. In this paper, we give a characterization of backward shift invariant subspaces satisfying $S_z^2S_w^* = S_w^*S_z^2$.

1. Introduction.

Let $\Gamma^2$ be the 2-dimensional unit torus. We denote by $(z, w) = (e^{i\theta}, e^{i\phi})$ the variables in $\Gamma^2 = \Gamma_z \times \Gamma_w$. Let $L^2 = L^2(\Gamma^2)$ be the usual Lebesgue space on $\Gamma^2$ with the norm $\|f\|_2 = (\int_{\Gamma^2} |f(e^{i\theta}, e^{i\phi})|^2 d\theta d\phi/(2\pi)^2)^{1/2}$. Then $L^2$ is a Hilbert space with the usual inner product. For $f \in L^2$, the Fourier coefficients are given by

$$\hat{f}(n, m) = \int_{\Gamma^2} f(e^{i\theta}, e^{i\phi}) e^{-in\theta} e^{-im\phi} d\theta d\phi/(2\pi)^2 = \langle f, z^nw^m \rangle.$$

Let $H^2 = H^2(\Gamma^2)$ be the Hardy space on $\Gamma^2$, that is,

$$H^2 = \{f \in L^2; \hat{f}(n, m) = 0 \text{ if } n < 0 \text{ or } m < 0 \}.$$

For $f \in H^2$, we can write $f$ as

$$f = \sum_{i,j=0}^{\infty} \oplus a_{i,j} z^i w^j, \text{ where } \sum_{i,j=0}^{\infty} |a_{i,j}|^2 < \infty. $$

Let $P$ be the orthogonal projection from $L^2$ onto $H^2$. For a closed subspace $M$ of $L^2$, we denote by $P_M$ the orthogonal projection from $L^2$ onto $M$. For a function $\psi \in L^\infty$, let $L_\psi f = \psi f$ for $f \in L^2$. The Toeplitz operator $T_\psi$ is defined by $T_\psi f = PL_\psi f$ for $f \in H^2$. It is well known that $T_\psi^* = T_{\overline{\psi}}$ and $T_{z^n}^*T_{w^m} = T_{w^m}T_{z^n}$ for $n, m \geq 1$. A function $f \in H^2$ is called inner if $|f| = 1$ on $\Gamma^2$ almost everywhere. A closed subspace $M$ of $H^2$ is called invariant if $zM \subset M$ and $wM \subset M$. In one variable

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case, an invariant subspace $M$ of $H^2(\Gamma)$ has a form $M = qH^2(\Gamma)$, where $q$ is inner. This is the well known Beurling theorem [B]. In two variable case, the structure of invariant subspaces of $H^2$ is very complicated, see [AC], [DY], [Na1], and [R].

Let $M$ be an invariant subspace of $H^2$. Then $T_z^*(H^2 \ominus M) \subset (H^2 \ominus M)$ and $T_w^*(H^2 \ominus M) \subset (H^2 \ominus M)$. A closed subspace $N$ of $H^2$ is called backward shift invariant if $H^2 \ominus N$ is invariant. In [IY], the first author and Yang studied backward shift invariant subspaces $N$ on which $T_z^*$ is strictly contractive. See [CR] and [S] for studies of backward shift invariant subspaces on the unit circle $\Gamma$.

Let $M$ be an invariant subspace of $H^2$ and $\psi \in L^\infty$. Let $V_\psi$ be the operator on $M$ defined by $V_\psi = P_M L_\psi |_M$. Then $V_z = T_z$ and $V_z^* = V_z$ on $M$. In [M], Mandrekar proved that $V_z V^*_w = V^*_w V_z$ on $M$ if and only if $M$ is Beurling type, that is, $M = qH^2$ for some inner function $q$ in $H^\infty$, see also [CS] and [Na2].

For $\psi \in L^\infty$, let $S_\psi = P_N L_\psi |_N$. Then we have $S_\psi^* = S^*_\psi$ and $S_z = T_z^*$ on $N$. In the previous paper [INS], we characterized backward shift invariant subspaces $N$ which satisfy the condition $S_z S_w^* = S_w^* S_z$ on $N$ as follows.

**Theorem A.** Let $N$ be a backward shift invariant subspace of $H^2$ and $N \neq H^2$. Then $S_z S_w^* = S_w^* S_z$ on $N$ if and only if $N$ has one of the following forms:

(i) $N = H^2 \ominus q_1(z)H^2$,
(ii) $N = H^2 \ominus q_2(w)H^2$,
(iii) $N = (H^2 \ominus q_1(z)H^2) \cap (H^2 \ominus q_2(w)H^2),$

where $q_1(z)$ and $q_2(w)$ are one variable inner functions.

This paper is a continuation of previous papers [IN] and [INS]. Here we are interesting in backward shift invariant subspaces $N$ of $H^2$ satisfying that $S_z^n S_{w^m}^* = S_{w^m}^* S_z^n$ for given positive integers $n$ and $m$. Up to now, we can not give a complete characterization of $N$ satisfying $S_z^n S_{w^m}^* = S_{w^m}^* S_z^n$. If $S_z S_w^* = S_w^* S_z$, then trivially $S_z^n S_{w^m}^* = S_{w^m}^* S_z^n$. But the converse is not true. In this paper, we concentrated on the case $m = 1$. One reason is the work of this paper deeply concerns with the problem of the normality of the operators $S_z^n$. For many backward shift invariant subspaces $N$, $S_z^n$ are not normal operators, see [Y]. If $S_z^n$ is normal, since $S_z^n S_w = S_w S_z^n$, by the Fuglede-Putnam theorem we have $S_z^n S_w^* = S_w^* S_z^n$. So, the results obtained in this paper will be a big help for the above mentioned problem. We will discuss on this subject in the forthcoming paper.
In Section 2, we give characterizations of \( N \) satisfying \( S_z^n S_{w^m} = 0 \) and \( S_{w^m} S_z^n = 0 \), respectively. If \( S_z^n S_{w^m} = 0 \), then \( S_{w^m} S_z^n = 0 \). The converse is not true. In Section 3, we study \( N \) satisfying \( S_z^n S_w^* = S_w^* S_z^n \), and give a necessary condition for \( S_z^n S_w^* = S_w^* S_z^n \). In Section 4, we study \( N \) satisfying \( S_{z^2} S_w^* = S_w^* S_{z^2} \). We gave a complete characterization of such \( N \). In [INS], we gave two different type of proofs of Theorem A. One of them is used in Section 3 and the other is used in Section 4.

Let \( H^2(\Gamma_z) \) and \( H^2(\Gamma_w) \) be the Hardy spaces on the unit circle in variables \( z \) and \( w \), respectively. We think that \( H^2(\Gamma_z) \subset H^2 \) and \( H^2(\Gamma_w) \subset H^2 \). For a subset \( E \) of \( H^2 \), we denote by \([E]\) the closed linear span of \( E \). A function \( b(z) = (z - \alpha)/(1 - \overline{\alpha}z), |\alpha| < 1 \), is called a simple Blaschke product.

2. \( S_z^n S_w^* = 0 \) or \( S_{w^m} S_z^n = 0 \).

Let \( n \) and \( m \) be positive integers. In this section, we study backward shift invariant subspaces \( N \) of \( H^2 \) satisfying \( S_z^n S_{w^m} = 0 \) and \( S_{w^m} S_z^n = 0 \), respectively.

**Lemma 2.1.** Let \( N \) be a backward shift invariant subspace of \( H^2 \). Then we have the following.

(i) \( S_z^n = S_z^n \).

(ii) \( S_{w^m} S_z^n = S_z^n S_{w^m} \) and \( S_z^n S_{w^m} = S_{w^m} S_z^n \).

(iii) If \( S_{w^m} S_z^n N \neq \{0\} \), then there exists \( f \in N \) such that \((S_{w^m} S_z^n f)\cap (0,0) \neq 0\).

**Proof.** All assertions are not difficult to prove.

The following theorem says that the structure of backward shift invariant subspaces satisfying \( S_z^n S_{w^m} = 0 \) is simple.

**Theorem 2.2.** Let \( N \) be a backward shift invariant subspace of \( H^2 \). Then \( S_z^n S_{w^m} = 0 \) if and only if \( N \) satisfies one of the following conditions:

(i) \( N \subset \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \).

(ii) \( N \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \).

**Proof.** Suppose that \( S_z^n S_{w^m} = 0 \). Then

\[
S_{w^m} N \perp S_z^n N.
\]

Since \( N \) is backward shift invariant, if \( S_{w^m} N = \{0\} \) then \( N \) satisfies condition (i). If \( S_z^n N = \{0\} \), then \( N \) satisfies (ii).

Next, suppose that

\[
S_{w^m} N \neq \{0\} \quad \text{and} \quad S_z^n N \neq \{0\}.
\]
We shall lead a contradiction. By (2.1), $S_{w^m}^* S_{zn}^* N \perp S_{zn}^* S_{w^m}^* N$. By Lemma 2.1(ii), $S_{w^m}^* S_{zn}^* N = S_{zn}^* S_{w^m}^* N = \{0\}$. Then

\begin{equation}
S_{zn}^* N \subset \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)
\end{equation}

and

\begin{equation}
S_{w^m}^* N \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w).
\end{equation}

By (2.2) and (2.3), there exists a nonnegative integer $j, 0 \leq j \leq m-1$, such that

\begin{equation}
\{0\} \neq S_{wj}^* S_{zn}^* N \subset H^2(\Gamma_z).
\end{equation}

By Lemma 2.1(iii), there exists $g \in N$ such that

\begin{equation}
(S_{wj}^* S_{zn}^* g)^\gamma(0,0) \neq 0.
\end{equation}

Also by (2.2) and (2.4), there exist $f \in N$ and a nonnegative integer $i, 0 \leq i \leq n-1$, such that

\begin{equation}
S_{zi}^* S_{w^m}^* f \in H^2(\Gamma_w)
\end{equation}

and

\begin{equation}
(S_{zi}^* S_{w^m}^* f)^\gamma(0,0) \neq 0.
\end{equation}

Then

\[
0 = \langle S_{w^m}^* S_{zi}^* f, S_{zn}^* S_{w^m}^* g \rangle \quad \text{by (2.1)}
\]

\[
= \langle S_{zi}^* S_{w^m}^* f, S_{wj}^* S_{zn}^* g \rangle \quad \text{by Lemma 2.1(ii)}
\]

\[
= (S_{zi}^* S_{w^m}^* f)^\gamma(0,0) (S_{wj}^* S_{zn}^* g)^\gamma(0,0) \quad \text{by (2.5) and (2.7)}
\]

\[
\neq 0 \quad \text{by (2.6) and (2.8)}.
\]

This is a desired contradiction.

The converse is trivial. \qed

**Corollary 2.3.** Let $N$ be a backward shift invariant subspace of $H^2$. Then $S_{zn}^* S_{w^m}^* = 0$ if and only if either $S_{zn}^* = 0$ or $S_{w^m}^* = 0$. Hence if $S_{zn}^* S_{w^m}^* = 0$, then $S_{zn}^* S_{w^m}^* = 0$.

**Lemma 2.4.** Let $M_1$ and $M_2$ be closed subspaces of $H^2$ such that

\[
M_1 \subset \sum_{j=0}^{m} \oplus w^j H^2(\Gamma_z) \quad \text{and} \quad M_2 \subset \sum_{i=0}^{n} \oplus z^i H^2(\Gamma_w).
\]

Then $M_1 + M_2$ is closed.
Proof. We denote by \((z^i w^j)_{M_1}\) and \((z^i w^j)_{M_2}\) the orthogonal projections of \(z^i w^j\) to the spaces \(M_1\) and \(M_2\), respectively. Let

\[
M'_1 = M_1 \ominus \left( \left\{ (z^i w^j)_{M_1}; 0 \leq i \leq n, 0 \leq j \leq m \right\} \right)
\]

and

\[
M'_2 = M_2 \ominus \left( \left\{ (z^i w^j)_{M_2}; 0 \leq i \leq n, 0 \leq j \leq m \right\} \right).
\]

Then \(M'_1\) and \(M'_2\) are closed subspaces of \(H^2\),

\[
M'_1 \subset z^{n+1} \left( \sum_{j=0}^{m} \oplus w^j H^2(\Gamma_z) \right), \quad M'_2 \subset w^{m+1} \left( \sum_{i=0}^{n} \oplus z^i H^2(\Gamma_w) \right),
\]

and

\[
M'_1 + M'_2 \perp \left\{ (z^i w^j)_{M_1}; 0 \leq i \leq n, 0 \leq j \leq m \right\}.
\]

Since

\[
z^{n+1} \left( \sum_{j=0}^{m} \oplus w^j H^2(\Gamma_z) \right) \perp w^{m+1} \left( \sum_{i=0}^{n} \oplus z^i H^2(\Gamma_w) \right),
\]

\(M'_1 + M'_2 = M'_1 \oplus M'_2\) is closed. Hence

\[
M_1 + M_2 = M'_1 + M'_2 + \left\{ (z^i w^j)_{M_1}; (z^i w^j)_{M_2}; 0 \leq i \leq n, 0 \leq j \leq m \right\}
\]

is closed. \(\square\)

**Theorem 2.5.** Let \(N\) be a backward shift invariant subspace of \(H^2\). Then \(S_{w^m}^* S_{z^n} = 0\) if and only if

\[
N \ominus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \subset N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z).
\]

Proof. Suppose that \(S_{w^m}^* S_{z^n} = 0\). Then \(S_{z^n} N \subset \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)\). Since \(S_{z^n}^* S_{w^m} = 0\), \(S_{w^m} N \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\). First, we prove the following

\[
N = \left( N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \right) + \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right).
\]

Let

\[
K = \left( N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \right) + \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right),
\]

By Lemma 2.4, \(K\) is closed and \(N = K \oplus (N \ominus K)\). To prove (i), it is sufficient to prove \(N \ominus K = \{0\}\). We have \(N \ominus K \subset N \cap (S_{z^n} N) \perp (S_{w^m} N) \perp\). Let \(f \in N \ominus K\). Then \(f \perp S_{z^n} N\), so that \(S_{z^n} f \perp N\). Since
$S^*_zf \in N, S^*_xf = 0$. Hence $f \in \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w)$. By (2.10), $f \in K$. This shows $f = 0$, so that $N \oplus K = \{0\}$. Thus we get (2.9).

Let

$$\tag{2.11} N_1 = \left\{ f \in N \cap \sum_{j=0}^{m-1} \oplus w^jH^2(\Gamma_z); f \perp N \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w) \right\}.$$  

Then $N_1$ is a closed subspace and

$$N_1 \oplus \left( N \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w) \right) \subset N.$$  

If the equality holds in the above, (i) holds. So we assume that

$$N_1 \oplus \left( N \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w) \right) \nsubseteq N.$$  

We shall lead a contradiction. Let

$$\tag{2.12} N_2 = N \ominus \left( N_1 \oplus \left( N \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w) \right) \right).$$  

Then $N_2 \neq \{0\}$ and $N = N_1 \oplus N_2 \oplus (N \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w))$.

Let $g \in N_2$ be such that $g \neq 0$. We shall prove that

$$\tag{2.13} g \notin N \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w) \quad \text{and} \quad g \notin N \cap \sum_{j=0}^{m-1} \oplus w^jH^2(\Gamma_z).$$  

The fact $g \notin N \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w)$ is trivial. Suppose that $g \in N \cap \sum_{j=0}^{m-1} \oplus w^jH^2(\Gamma_z)$. By (2.11), $g \in N_1$. Since $g \in N_2$, by (2.12) we have $g \perp N_1$. Hence $g = 0$. This is a contradiction. Thus we get (2.13).

Next, we shall prove that

$$\tag{2.14} S^*_zg \perp N \cap \sum_{j=0}^{m-1} \oplus w^jH^2(\Gamma_z).$$  

To prove this, suppose not. Then $S^*_zg \perp N \cap \sum_{j=0}^{m-1} \oplus w^jH^2(\Gamma_z)$. By (2.10), $g = g_1 + g_2$, where $g_1 \in N \cap \sum_{j=0}^{m-1} \oplus w^jH^2(\Gamma_z)$ and $g_2 \in N \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w)$. Then $S^*_zg = S^*_zg_1 \in N \cap \sum_{j=0}^{m-1} \oplus w^jH^2(\Gamma_z)$. Therefore $S^*_zg = 0$, so that $g \in N \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w)$. This contradicts (2.13). Thus we get (2.14).

By (2.14), there exists $h_0$ such that $h_0 \in N \cap \sum_{j=0}^{m-1} \oplus w^jH^2(\Gamma_z)$ and $\langle g, S^*_zh_0 \rangle = \langle S^*_zg, h_0 \rangle \neq 0$. Since $S^*_z, h_0 \in N$, by (2.12) we have $S^*_zh_0 = h_1 + h_2 + h_3$, where $h_1 \in N_1, h_2 \in N_2$, and $h_3 \in \cdots$
Then Theorem 2.6. simple. When $n$ satisfying (assumption. Thus we have (i). That we get $h$.

Thus we get $\sum_{i=0}^{n-1} z^i H^2(\Gamma_w)$.

Thus we have $h_2 \neq 0$. Since $z^n h_0 \perp \sum_{i=0}^{n-1} z^i H^2(\Gamma_w)$,

$$P_N z^n h_0 = S_{z^n} h_0 \perp N \cap \sum_{i=0}^{n-1} z^i H^2(\Gamma_w).$$

Thus we get $h_3 = 0$. By (2.12), $S_{w_m} N_1 = \{0\}$. Hence $S_{w_m} S_{z^n} h_0 = S_{w_m} h_2$. By (2.13) and $h_2 \in N_2$, $h_2 \notin \sum_{j=0}^{m-1} w^j H^2(\Gamma_z)$. This implies that $S_{w_m} h_2 \neq 0$. Hence $S_{w_m} S_{z^n} \neq 0$. This contradicts our starting assumption. Thus we have (i).

To prove the converse, suppose that condition (i) holds. Then $N = N \cap \sum_{i=0}^{n-1} z^i H^2(\Gamma_w) \oplus L$, where $L \subset N \cap \sum_{j=0}^{m-1} w^j H^2(\Gamma_z)$. Let $F = F_1 + F_2 \in N$, where $F_1 \in N \cap \sum_{i=0}^{n-1} z^i H^2(\Gamma_w)$ and $F_2 \in L$. Since $z^n F \perp N \cap \sum_{i=0}^{n-1} z^i H^2(\Gamma_w)$, $S_{z^n} F \in L$. Hence $S_{w_m} S_{z^n} F = 0$. Thus we get $S_{w_m} S_{z^n} = 0$. \hfill \Box

By Theorem 2.2, the structure of backward shift invariant subspaces $N$ satisfying $S_{z^n} S_{w_m} = 0$ is simple. By Theorem 2.5, the structure of backward shift invariant subspaces $N$ satisfying $S_{w_m} S_{z^n} = 0$ is not so simple. When $n = m = 1$, we have the following.

**Theorem 2.6.** Let $N$ be a backward shift invariant subspace of $H^2$. Then $S_{w} S_{z} = 0$ if and only if $N$ has one of the following forms;

(i) $N = H^2(\Gamma_z) \ominus q(z) H^2(\Gamma_z)$ for some inner function $q(z)$.

(ii) $N = H^2(\Gamma_u) \ominus q(w) H^2(\Gamma_u)$ for some inner function $q(w)$.

(iii) Either $N = H^2(\Gamma_z) + H^2(\Gamma_w)$, or $N = H^2(\Gamma_z) + H^2(\Gamma_w) \ominus q_2(w) H^2(\Gamma_u)$, or $N = (H^2(\Gamma_z) \ominus q_1(z) H^2(\Gamma_z)) + H^2(\Gamma_w)$, where $q_1(z)$ and $q_2(w)$ are inner functions.

(iv) $N = (H^2(\Gamma_z) \ominus q_1(z) H^2(\Gamma_z)) + (H^2(\Gamma_u) \ominus q_2(w) H^2(\Gamma_u))$, where $q_1(z), q_2(w)$ are nonconstant inner functions and 1 $\in N$, we may take $q_1$ and $q_2$ as $q_1(0) = q_2(0) = 0$.

In (iii) and (iv), since $1 \in N$, we may take $q_1$ and $q_2$ as $q_1(0) = q_2(0) = 0$.

**Proof.** By Theorem 2.2, $S_{w} S_{z} = 0$ if and only if either (i) or (ii) holds. By Theorem 2.5, $S_{w} S_{z} = 0$ if and only if

$$(2.15) \quad N \ominus (N \cap H^2(\Gamma_w)) \subset N \cap H^2(\Gamma_z).$$

If either (i) or (ii) holds, by Corollary 2.3 we have $S_{w} S_{z} = 0$. Suppose that $N$ satisfies either (iii) or (iv). Then clearly $1 \in N$. Since $N$ has a special form, it is not difficult to see that

$$N \ominus (N \cap H^2(\Gamma_w)) = \{ f \in N \cap H^2(\Gamma_z); \hat{f}(0) = 0 \}.$$ 

Hence (2.15) holds.
Next, suppose that (2.15) holds. Then we have

\[ N = (N \cap H^2(\Gamma_z)) + (N \cap H^2(\Gamma_w)). \]

If either \( N \cap H^2(\Gamma_z) = \{0\} \) or \( N \cap H^2(\Gamma_w) = \{0\} \), then \( S_zS_w^* = 0 \), and

by Corollary 2.3, \( S_z^*S_w = 0 \). Hence either (i) or (ii) holds. Suppose that \( N \cap H^2(\Gamma_z) \neq \{0\} \) and \( N \cap H^2(\Gamma_w) \neq \{0\} \). We shall prove \( 1 \in N \). To prove this, suppose that \( 1 \notin N \). Let \( 1_w \) be the orthogonal projection of 1 to \( N \cap H^2(\Gamma_w) \). Then \( 1_w \notin H^2(\Gamma_z) \). Since \( N \cap H^2(\Gamma_z) \neq \{0\} \), there exists \( f \in N \cap H^2(\Gamma_z) \) such that \( \hat{f}(0) \neq 0 \). Let \( f_1 = f - \hat{f}(0)1_w \in N \).

Then \( f_1 \notin H^2(\Gamma_z) \). Let \( h \in N \cap H^2(\Gamma_w) \). Since \( f \in H^2(\Gamma_z) \), \( f - \hat{f}(0) \perp h \). Since \( 1 - 1_w \perp N \cap H^2(\Gamma_w) \),

\[ \langle f_1, h \rangle = \langle \hat{f}(0)(1 - 1_w) + f - \hat{f}(0), h \rangle = 0. \]

Hence \( f_1 \notin N \ominus (N \cap H^2(\Gamma_w)) \). Thus (2.15) does not hold. Therefore \( 1 \in N \). Since \( N \cap H^2(\Gamma_z) \) and \( N \cap H^2(\Gamma_w) \) are nonzero backward shift invariant subspaces, by (2.16) \( N \) has one of forms in (iii) and (iv). \( \square \)

3. \( S_{z^n}S_{w}^* = S_{w}^*S_{z^n} \).

The following is the main theorem in this section.

**Theorem 3.1.** Let \( N \) be a backward shift invariant subspace of \( H^2 \), \( N \neq \{0\} \), and \( N \neq H^2 \). Let \( M = H^2 \ominus N \) and \( n \geq 2 \) be a positive integer. If \( S_{z^n}S_{w}^* = S_{w}^*S_{z^n} \), then one of the following conditions holds;

(i) \( S_zS_w^* = S_w^*S_z \),
(ii) \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \) for an inner function \( q_1(z) \) satisfying \( q_1(z) = \prod_{j=1}^{k} b_j(z) \), \( 1 \leq k \leq n \), where \( b_i \) are simple Blaschke products.

The following lemma is a generalization of [IN], and the idea of the proof is the same.

**Lemma 3.2.** Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \ominus N \). Let \( n \) be a positive integer. Then \( S_{z^n}S_{w}^* = S_{w}^*S_{z^n} \) if and only if

\[ M \ominus (z^nM \oplus (M \cap \bigoplus_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w))) \subset (M \cap H^2(\Gamma_z)) \oplus wM. \]

**Proof.** The operators \( T_{z^n} \) and \( T_{w}^* \) on \( H^2 \) have the matrix forms as

\[ T_{z^n} = \begin{pmatrix} * & P_M T_{z^n} |_{N} \\ 0 & S_{z^n} \end{pmatrix}, T_{w}^* = \begin{pmatrix} * & P_N T_{w}^* |_{M} \\ 0 & S_{w}^* \end{pmatrix} \text{ on } H^2 = \begin{pmatrix} M \oplus N \end{pmatrix}. \]
Set $A = P_MT_{zn}|_N$ and $B = P_NT_{zn}|_M$. Since $T_{zn}T_{wn} = T_{wn}T_{zn}$ on $H^2$, $S_{zn}S_{wn} = S_{wn}S_{zn}$ if and only if $BA = 0$. We have $T_{wn}(M \ominus wM) \subset N$. For $f \in H^2$, $T_{wn}f = 0$ if and only if $f \in H^2(\Gamma_z)$. Hence

$$\ker B = \{ f \in M; T_{wn}f \in M \} = \{ f \in M \ominus wM; T_{wn}f = 0 \} \ominus wM = (M \cap H^2(\Gamma_z)) \ominus wM.$$

We denote by $\text{ran } A$ the closed range of $A$. Let $A_1 = P_MT_{zn}P_N$ on $H^2$. Then $\text{ran } A = \text{ran } A_1$. Since $A_1^* = P_NT_{zn}P_M$, we get

$$\ker A_1^* = \{ f \in M; T_{zn}^*f \in M \} \ominus N = (M \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w)) \ominus z^nM \ominus N.$$

Hence

$$\text{ran } A = \text{ran } A_1 = H^2 \ominus \ker A_1^* = M \ominus \left( z^nM \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w) \right) \right).$$

Since $BA = 0$ if and only if $\text{ran } A \subset \ker B$, we get our assertion. \(\square\)

**Proof of Theorem 3.1.** Suppose that $S_{zn}S_{wn} = S_{wn}S_{zn}$. By Lemma 3.2,

$$M \ominus \left( z^nM \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w) \right) \right) \subset (M \cap H^2(\Gamma_z)) \ominus wM.$$

Let

$$K_0 = M \ominus \left( z^nM \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w) \right) \right).$$

Then

(3.1) \hspace{1cm} $K_0 \subset (M \cap H^2(\Gamma_z)) \ominus wM$

and

(3.2) \hspace{1cm} $M \ominus z^nM = K_0 \ominus \left( M \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w) \right).$

Since $M = \sum_{s=0}^{\infty} \oplus z^{ns}(M \ominus z^nM),$

(3.3) \hspace{1cm} $K_0 \perp \sum_{s=0}^{\infty} \oplus z^{ns}\left( M \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w) \right)$

and

(3.4) \hspace{1cm} $M = \left( \sum_{s=0}^{\infty} \oplus z^{ns}K_0 \right) \oplus \left( \sum_{s=0}^{\infty} \oplus z^{ns}\left( M \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w) \right) \right).$
First, suppose that \( K_0 = \{0\} \). In this case, we shall show that condition (i) holds. By (3.2),
\[
\sum_{i=0}^{n-1} \odot z^i (M \odot z M) = M \odot z^n M = M \cap \sum_{i=0}^{n-1} \odot z^i H^2(\Gamma_w).
\]
Let \( f \in M \odot z M \). Then \( f \in M \odot z^n M \) and \( f = \sum_{i=0}^{n-1} \odot z^i h_i(w) \), where \( h_i(w) \in H^2(\Gamma_w) \). Since \( f \in M \odot z M \),
\[
z^{n-1} f = \sum_{i=0}^{n-1} \odot z^{n-1+i} h_i(w) \in M \odot z^n M.
\]
Hence \( f = h_0(w) \), so that \( M \odot z M = M \cap H^2(\Gamma_w) \). By Lemma 3.2 (for the case of \( n = 1 \)), we get \( S_z S_w^* = S_w^* S_z \).

Next, suppose that \( K_0 \neq \{0\} \). The proof of this case is a little bit long, so we divide into several steps.

**Step 1.** In this step, we shall prove that there is a nonconstant inner function \( q_1(z) \) such that
\[
(3.5) \quad K_0 \subset q_1(z) H^2(\Gamma_z) \odot w M.
\]
First we prove that \( K_0 \nsubseteq w M \). To prove this, suppose that \( K_0 \subset w M \). Let \( F \in K_0 \). Then \( F = w f \) for some \( f \in M \). We shall prove that \( f \in K_0 \).
We have
\[
\langle f, \left( \sum_{s=1}^{\infty} \odot z^{ns} K_0 \right) \odot \left( \sum_{s=0}^{\infty} \odot z^{ns} \left( M \cap \sum_{i=0}^{n-1} \odot z^i H^2(\Gamma_w) \right) \right) \rangle
\]
\[
= \langle w f, w \left( \sum_{s=1}^{\infty} \odot z^{ns} K_0 \odot \left( \sum_{s=0}^{\infty} \odot z^{ns} \left( M \cap \sum_{i=0}^{n-1} \odot z^i H^2(\Gamma_w) \right) \right) \right) \rangle
\]
\[
= \langle F, z^n w \left( \sum_{s=1}^{\infty} \odot z^{n(s-1)} K_0 \right) \rangle \quad \text{by (3.3)}
\]
\[
= 0,
\]
where the last equality follows from \( w \sum_{s=1}^{\infty} \odot z^{n(s-1)} K_0 \subset M \) and (3.2).
Hence by (3.4), we get \( f \in K_0 \). Therefore for every positive integer \( p \), we have \( F = w^p f_p \) for some \( f_p \in K_0 \). This leads \( F = 0 \). This is a contradiction. Thus we get \( K_0 \nsubseteq w M \).

Hence by (3.1), \( M \cap H^2(\Gamma_z) \neq \{0\} \). By the Beurling theorem,
\[
(3.6) \quad M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z),
\]
where \( q_1(z) \) is inner. Then \( q_1(z) \in M \) and
\[
(3.7) \quad q_1(z) H^2(\Gamma_z) \perp w M.
\]
If \( q_1(z) \) is constant, we have \( M = H^2 \), so that \( N = \{0\} \). This contradicts our assumption. Hence \( q_1(z) \) is a nonconstant inner function. By (3.1) and (3.6), we get (3.5).

\textit{Step 2.} In this step, we prove

\begin{equation}
K_0 \subset q_1(z) \left( \sum_{i=0}^{n-1} z^i H^2(\Gamma_{w_i}) \right).
\end{equation}

Let \( G \in K_0 \). Then by (3.5), \( G = q_1(z) h(z) \oplus wg \), where \( h(z) \in H^2(\Gamma_z) \) and \( g \in M \). Write

\[ h(z) = \left( \sum_{i=0}^{n-1} a_i z^i \right) \oplus z^n h_0(z). \]

Then

\[ G = q_1(z) \left( \sum_{i=0}^{n-1} a_i z^i \right) \oplus q_1(z) z^n h_0(z) \oplus wg. \]

By (3.6), \( q_1(z) z^n h_0(z) \in z^n M \). Since \( G \in K_0 \subset M \oplus z^n M \), we have \( h_0(z) = 0 \). Hence

\begin{equation}
G = q_1(z) \left( \sum_{i=0}^{n-1} a_i z^i \right) \oplus wg.
\end{equation}

Here we prove that

\begin{equation}
g \in K_0.
\end{equation}

Since \( G = q_1(z) h(z) \oplus wg \), we have

\[ \langle g, \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rangle = \langle wg, w \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rangle \]

\[ = \langle wg, w \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rangle + \langle q_1 h, w \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rangle \quad \text{by (3.7)} \]

\[ = \langle G, w \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rangle \]

\[ = \langle G, z^n w \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rangle \]

\[ = 0 \quad \text{by (3.2)}. \]
We also have
\[ \langle g, \sum_{s=0}^{\infty} \oplus z^s (M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)) \rangle \]
\[ = \langle wg, w \left( \sum_{s=0}^{\infty} \oplus z^s (M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)) \right) \rangle \]
\[ = \langle G, \sum_{s=0}^{\infty} \oplus z^s w \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \rangle \]
\[ = 0 \quad \text{by (3.3)}. \]

Hence by (3.4), we get (3.10).

Applying (3.9) and (3.10) infinitely many times, we have

\[ G = q_1(z) \left( \sum_{i=0}^{n-1} \oplus a_i z^i \right) \oplus \ldots \]
\[ \in q_1(z) \left( \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right). \]

Therefore we get (3.8).

**Step 3.** In this step, we study functions in \( M \ominus z M \) and the inner function \( q_1(z) \). By (3.8), there is a closed subspace \( L \) such that \( K_0 = q_1(z) L \) and \( L \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \). Then by (3.2),

\[ M \ominus z^n M = q_1(z) L \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right). \]

Since \( K_0 \neq \{0\}, L \neq \{0\} \). We have \( M \ominus z^n M = \sum_{i=0}^{n-1} \oplus z^i (M \ominus z M) \). Hence \( M \ominus z M \neq \{0\} \). Let \( F \in M \ominus z M \) be such that \( F \neq 0 \). Then

\[ F = \left( q_1 \sum_{i=0}^{n-1} \oplus z^i f_i \right) \oplus \sum_{i=0}^{n-1} \oplus z^i g_i, \]

where \( f_i, g_i \in H^2(\Gamma_w) \).

\[ q_1 \sum_{i=0}^{n-1} \oplus z^i f_i \perp M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w), \]

and

\[ \sum_{i=0}^{n-1} \oplus z^i g_i \in M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w). \]
Since \( n \geq 2 \), \( zF \in M \oplus z^n M \), so that we have
\[
zF = q_1 \left( \sum_{i=0}^{n-1} \oplus z^i G_{1,i} \right) + \sum_{i=0}^{n-1} \oplus z^i H_{1,i},
\]
where \( G_{1,i}, H_{1,i} \in H^2(\Gamma_w) \). Hence
(3.14)
\[
q_1 \left( z \left( \sum_{i=0}^{n-1} \oplus z^i f_i \right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i} \right) = \left( \sum_{i=0}^{n-1} \oplus z^i H_{1,i} \right) - z \sum_{i=0}^{n-1} \oplus z^i g_i.
\]
Here we divide into two subcases.

**Subcase 1.** \( z \left( \sum_{i=0}^{n-1} \oplus z^i f_i(w) \right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w) \neq 0 \).

Then
\[
q_1(z) = \frac{\left( \sum_{i=0}^{n-1} \oplus z^i H_{1,i}(w) \right) - z \sum_{i=0}^{n-1} \oplus z^i g_i(w)}{z \left( \sum_{i=0}^{n-1} \oplus z^i f_i(w) \right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w)}.
\]
As proved in Step 1, \( q_1(z) \) is a nonconstant inner function. Then by the above, we have
(3.15)
\[
q_1(z) = \prod_{j=1}^k b_j(z), \quad 1 \leq k \leq n,
\]
where \( b_j \) are simple Blaschke products.

**Subcase 2.** \( z \left( \sum_{i=0}^{n-1} \oplus z^i f_i(w) \right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w) = 0 \).

Then by (3.14), \( f_{n-1} = g_{n-1} = 0 \), so that by (3.11)
\[
F = \left( q_1 \sum_{i=0}^{n-2} \oplus z^i f_i \right) \oplus \sum_{i=0}^{n-2} \oplus z^i g_i.
\]
Since \( F \in M \ominus zM \),
\[
zF = \left( q_1 \sum_{i=0}^{n-2} \oplus z^{i+1} f_i \right) \oplus \sum_{i=0}^{n-2} \oplus z^{i+1} g_i \in M \ominus z^2 M.
\]
In the same way as above, either (3.15) holds or \( f_{n-2} = g_{n-2} = 0 \).
Repeat the same argument. Then either (3.15) holds or
\[
f_{n-1} = f_{n-2} = \cdots = f_1 = g_{n-1} = g_{n-2} = \cdots = g_1 = 0.
\]
Step 4. If (3.15) holds, then (ii) holds. So, suppose that (3.15) does not hold. Then by (3.11) and by the above fact, $F = q_1 f_0 \oplus g_0$, by (3.12) $q_1 f_0 \perp M \cap H^2(\Gamma_w)$, and by (3.13) $g_0 \in M \cap H^2(\Gamma_w)$ for every $F \in M \ominus zM$.

If $g_0 = 0$ for every $F \in M \ominus zM$, since $q_1(z) \in M$ it follows that $M \ominus zM \subset q_1(z)H^2(\Gamma_w) \subset M$. Since $M = \sum_{n=0}^{\infty} z^n (M \ominus zM)$, we have $M = q_1(z)H^2$, so that $N = H^2 \ominus q_1(z)H^2$. By Theorem A, condition (i) holds.

So we assume that $g_0 \neq 0$ for some $F \in M \ominus zM$. We shall prove that

$$ (3.16) \quad (M \ominus zM) \ominus (M \cap H^2(\Gamma_w)) \subset (M \cap H^2(\Gamma_z)) \oplus wM. $$

We may assume that $(M \ominus zM) \ominus (M \cap H^2(\Gamma_w)) \neq \{0\}$. Let $G \in (M \ominus zM) \ominus (M \cap H^2(\Gamma_w))$ be such that $G \neq 0$. Then $G = q_1(z)h_1(w) \oplus h_2(w)$, where $q_1(z)h_1(w) \perp M \cap H^2(\Gamma_w)$ and $h_2(w) \in M \cap H^2(\Gamma_w)$. Hence $h_2(w) = G - q_1(z)h_1(w) \perp M \cap H^2(\Gamma_w)$. Therefore $h_2(w) = 0$. Since $q_1(z) \in M$, $h_1(w) = q_1(z)h_1(w) + wq_1(z) \frac{h_1(w) - h_1(0)}{w} \in M \cap H^2(\Gamma_z) \oplus wM$.

Thus we get (3.16). By Lemma 3.2 (for $n = 1$), (i) holds. \hfill \square

4. $S_z S^*_w = S^*_w S_z$ and $S_z S^*_w \neq S^*_w S_z$.

Let $N$ be a backward shift invariant subspace of $H^2$ and let $n$ be a positive integer. Let $M = H^2 \ominus N$. Then $M$ is an invariant subspace. If both $S_z S^*_w = S^*_w S_z$ and $S_z S^*_w \neq S^*_w S_z$ hold, then by Theorem 3.1, $M \cap H^2(\Gamma_z) = q(z)H^2(\Gamma_z)$ for some nonconstant inner function $q(z)$.

In this section, we assume that $q_1(z)H^2 \subset M$ and $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$ for some nonconstant inner function $q_1(z)$. Let

$$ \tilde{M} = M \ominus q_1(z)H^2 \subset M. $$

Then $H^2 \ominus q_1(z)H^2 = \tilde{M} \ominus N$ and $\tilde{M}$ is $w$-invariant. The following lemma is proved in [INS, Lemma 3.2].

**Lemma 4.1.** Let $f \in \tilde{M}$. Then $T^*_w f \in \tilde{M}$ if and only if $f \in w\tilde{M}$.

We denote by $P_\perp$ the orthogonal projection from $H^2$ onto $H^2 \ominus q_1(z)H^2$. Then we have a Toeplitz type operator $Q_{z^n}$ on $H^2 \ominus q_1(z)H^2$ such that

$$ Q_{z^n} : H^2 \ominus q_1(z)H^2 \ni f \rightarrow P_\perp(T^*_z f) \in H^2 \ominus q_1(z)H^2. $$
Since \( z^n M \subset M \), \( Q_{z^n} \tilde{M} \subset \tilde{M} \) and \( Q_{z^n}^{\ast} = Q_{z^n} \). Then \( Q_{z^n} \) has the following matrix form:

\[
Q_{z^n} = \begin{pmatrix}
* & P_{\tilde{M}} T_{z^n} | N \\
0 & S_{z^n}^* 
\end{pmatrix}
\quad \text{on } H^2 \ominus q_1(z) H^2 = \begin{pmatrix}
\tilde{M} \\
\oplus \\
N
\end{pmatrix}.
\]

Since \( H^2 \ominus q_1(z) H^2 \) is backward shift invariant, \( T_w^* (H^2 \ominus q_1(z) H^2) \subset H^2 \ominus q_1(z) H^2 \). Since \( T_w^* N \subset N \), the operator \( T_w^* \) on \( H^2 \ominus q_1(z) H^2 \) has the following matrix form:

\[
T_w^* = \begin{pmatrix}
* & 0 \\
P_N T_w^* | \tilde{M} & S_w^*
\end{pmatrix}
\quad \text{on } H^2 \ominus q_1(z) H^2 = \begin{pmatrix}
\tilde{M} \\
\oplus \\
N
\end{pmatrix}.
\]

Set

\[(4.1) \quad A = P_{\tilde{M}} T_{z^n} | N \quad \text{and} \quad B = P_N T_w^* | \tilde{M}.\]

By [INS, Lemma 3.3], \( T_w Q_{z^n} = Q_{z^n} T_w^* \) on \( H^2 \ominus q_1(z) H^2 \). Hence we have the following.

**Lemma 4.2.** \( T_w Q_{z^n} = Q_{z^n} T_w^* \) on \( H^2 \ominus q_1(z) H^2 \).

**Lemma 4.3.** \( S_w S_{z^n}^* = S_{z^n} S_w^* \) if and only if \( BA = 0 \).

**Proof.** By Lemma 4.2, \( T_w^* Q_{z^n} = Q_{z^n} T_w^* \) on \( H^2 \ominus q_1(z) H^2 \). Then \( BA + S_w^* S_{z^n} = S_{z^n} S_w^* \). Hence \( S_{z^n} S_w^* = S_{z^n} S_w^* \) if and only if \( BA = 0 \).

The following is a slight generalization of [INS, Theorem 3.5].

**Theorem 4.4.** Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \ominus N \). Suppose that \( M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z) \), where \( q_1(z) \) is a nonconstant inner function. Let \( \tilde{M} = M \ominus q_1(z) H^2 \). Then the following conditions are equivalent.

(i) \( S_{z^n} S_w^* = S_{z^n} S_w^* \) on \( N \).

(ii) \( \tilde{M} \cap \{ f \in \tilde{M} ; T_{z^n}^* f \in \tilde{M} \} \subset w \tilde{M} \).

(iii) \( T_{z^n} \tilde{M} \subset \tilde{M} \).

**Proof.** The idea of the proof is similar to the one of [INS, Theorem 3.5]. For the sake of completeness, we give the proof.

(i) \( \Leftrightarrow \) (ii) By Lemma 4.3, condition (i) is equivalent to \( BA = 0 \). By (4.1) and Lemma 4.1, \( \ker B = \{ f \in \tilde{M} ; T_{z^n}^* f \in \tilde{M} \} = w \tilde{M} \). Put \( A_1 = P_{\tilde{M}} T_{z^n} P_N \) on \( \tilde{M} \ominus N \). Then \( \ker A_1 = \mathbb{ran} A \). Since \( A_1 = P_N T_{z^n} P_M \), \( \ker A_1 = N \oplus \{ f \in \tilde{M} ; T_{z^n}^* f \in \tilde{M} \} \). Hence \( \ker A_1 = \mathbb{ran} A \). Then \( \tilde{M} \ominus \{ f \in \tilde{M} ; T_{z^n}^* f \in \tilde{M} \} \subset w \tilde{M} \). Therefore \( BA = 0 \) if and only if \( \tilde{M} \ominus \{ f \in \tilde{M} ; T_{z^n}^* f \in \tilde{M} \} \subset w \tilde{M} \). Thus we get (i) \( \Leftrightarrow \) (ii).
(ii) \( \Rightarrow \) (iii) Suppose that \( \hat{M} \cap \{ f \in \hat{M}; T_n^* f \in \hat{M} \} \subset w\hat{M} \). Since \( \{ f \in \hat{M}; T_n^* f \in \hat{M} \} \) is closed, \( \hat{M} \cap w\hat{M} \subset \{ f \in \hat{M}; T_n^* f \in \hat{M} \} \). Since \( w\hat{M} \subset \hat{M} \), \( \hat{M} = \sum_{j=0}^{\infty} \oplus w^j(\hat{M} \cap w\hat{M}) \). Since \( T_n^* w^j f = w^j T_n^* f \) for \( f \in H^2 \), we have \( T_n^* \hat{M} \subset \hat{M} \).

(iii) \( \Rightarrow \) (ii) is trivial. \( \Box \)

For \( f \in H^2(\Gamma_z) \), write \( f^*(z) = T_n^* f(z) = \overline{\varphi}(f(z) - \hat{f}(0)) \).

**Lemma 4.5.** Let \( b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}_i z) \), \( |\alpha_i| < 1 \), and \( 1 \leq i \leq n \). Then

(i) \( T_n^* z = 1 \), \( T_n^* b_1 = b_1^*(z) = (1 - |\alpha_1|^2)/(1 - \overline{\alpha}_1 z) \), and \( T_n^* b_i(z) = \overline{\alpha}_i b_i^*(z) \).

(ii) \( T_n^* (b_1(z)b_2^*(z)) = (1 - |\alpha_2|^2)b_1^*(z) + \overline{\alpha}_2 b_1(z)b_2^*(z) \).

(iii) \( H^2(\Gamma_z) \cap (\prod_{j=1}^{k} b_j(z)) H^2(\Gamma_z) = \sum_{j=1}^{k} \oplus [b_1(z) \cdots b_{j-1}(z)b_j^*(z)] \).

(iv) \( H^2 \cap (\prod_{j=1}^{k} b_j(z)) H^2 = \sum_{j=1}^{k} \oplus [b_1(z) \cdots b_{j-1}(z)b_j^*(z)] H^2(\Gamma_w) \).

**Proof.** It is not difficult to prove (i).

(ii) Since

\[
\varphi b_1(z)b_2^*(z) = \varphi b_1(z) \frac{1 - |\alpha_2|^2}{1 - \overline{\alpha}_2 z} \\
= (1 - |\alpha_2|^2)b_1(z) \left( \frac{1}{\overline{\alpha}_2} + \frac{\alpha_2}{1 - \overline{\alpha}_2 z} \right) \\
= (1 - |\alpha_2|^2)\varphi b_1(z) + \overline{\alpha}_2 b_1(z)b_2^*(z),
\]

we get (ii).

(iii) See [Ni, p.33].

(iv) follows from (iii). \( \Box \)

**Corollary 4.6.** Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \ominus N \). Suppose that \( M \cap H^2(\Gamma_z) = b_1(z)H^2(\Gamma_z) \), where \( b_1(z) \) is a simple Blaschke product. Then \( S_wS_w^* = S_wS_z \).

**Proof.** Let \( b_1(z) = (z - \alpha)/(1 - \overline{\alpha} z) \), \( |\alpha| < 1 \), and \( \hat{M} = M \ominus b_1(z)H^2 \).

Since \( b_1(z) \in M \), \( b_1(z)H^2 \subset M \). By Lemma 4.5(iv), \( \hat{M} \subset b_1(z)H^2(\Gamma_w) \).

By Lemma 4.5(i), \( T_n^* (b_1^*(z)h(w)) = \overline{\alpha}_1 b_1^*(z)h(w) \). Hence \( T_n^* \hat{M} \subset \hat{M} \). By Theorem 4.4, \( S_wS_w^* = S_wS_z \). \( \Box \)

**Corollary 4.7.** Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \ominus N \). Suppose that \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \), where \( q_1(z) \) is an inner function. Let \( n, k \) be positive integers with \( n \geq k + 1 \). Moreover suppose that \( q_1(z) = z^k b(z) \), where \( b \) is a simple Blaschke product, \( b(z) = (z - \alpha)/(1 - \overline{\alpha} z) \), and \( \alpha \neq 0 \). If \( S_wS_w^* = S_w^*S_z^n \), then \( S_z^*S_w^* = S_w^*S_z^k \).
Proof. Let $\tilde{M} = M \ominus q_1(z)H^2$. If $\tilde{M} = \{0\}$, then $M = q_1(z)H^2$. By Theorem A, $S_zS_w^* = S_w^*S_z$. Suppose that $\tilde{M} \neq \{0\}$. By Lemma 4.5(iv),

$$\tilde{M} \subset b^*(z)H^2(\Gamma_w) \oplus \left( \sum_{j=1}^k \oplus z^{j-1}b(z)H^2(\Gamma_w) \right).$$

Let $f \in \tilde{M}$. Then

$$f = b^*(z)h_0(w) + \sum_{j=1}^k \oplus z^{j-1}b(z)h_j(w), \quad h_j(w) \in H^2(\Gamma_w).$$

By Lemma 4.5(i),

$$T_{z^n}^*f = T_{z^n-k}(T_{z^n}^*f)$$

$$= T_{z^n-k}^* \left( \sum_{j=0}^{k} \hat{\alpha}^{(k-j)}b^*(z)h_j(w) \right)$$

$$= \hat{\alpha}^{(n-k)} \sum_{j=0}^{k} \hat{\alpha}^{(k-j)}b^*(z)h_j(w).$$

Since $S_zS_w^* = S_w^*S_z$, by Theorem 4.4 $T_{z^n}^*f \in \tilde{M}$. Since $\alpha \neq 0$,

$$\sum_{j=0}^{k} \hat{\alpha}^{(k-j)}b^*(z)h_j(w) \in \tilde{M}.$$ 

Thus $T_{z^n}^*\tilde{M} \subset \tilde{M}$. By Theorem 4.4, $S_{z^n}S_w^* = S_{w^n}S_z^*$. \hfill $\square$

**Theorem 4.8.** Let $N$ be a backward shift invariant subspace of $H^2$ and $M = H^2 \ominus N$. Suppose that $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$, where $q_1(z)$ is an inner function. Moreover suppose that $q_1(z) = b_1(z)b_2(z)$, where $b_i(z), i = 1,2,$ are simple Blaschke products, $b_i(z) = (z - \alpha_i)/(1 - \bar{\alpha}_iz)$, and $\alpha_1\alpha_2 \neq 0$. Let $n \geq 2$ be a positive integer. Then we have the following.

(i) If $S_zS_{w}^* = S_{w}^*S_z$ and $S_{z^{-1}}S_{w}^* \neq S_{w}^*S_{z^{-1}}$, then $\alpha_1 = \alpha_2$ and $\alpha_1 \neq \alpha_2$.

(ii) If $\alpha_1 = \alpha_2^2$ and $\alpha_1 \neq \alpha_2$, then $S_{z^n}S_{w}^* = S_{w^n}S_z^*$.

**Proof.** Let $\hat{M} = M \ominus q_1(z)H^2$. Suppose that $S_zS_{w}^* = S_{w}^*S_z$ and $S_{z^{-1}}S_{w}^* \neq S_{w}^*S_{z^{-1}}$. By Theorem 4.4, $T_{z^n}^*\hat{M} \subset \tilde{M}$ and $T_{z^{-1}}^*\hat{M} \not\subset \tilde{M}$. By Lemma 4.5(iv),

$$\hat{M} \subset b_1^*(z)H^2(\Gamma_w) \oplus b_1(z)b_2^*(z)H^2(\Gamma_w).$$

Then there exists $f_0 \in \hat{M}$ such that $T_{z^{-1}}^*f_0 \not\in \hat{M}$, and

$$f_0 = b_1^*(z)g_1(w) + b_1(z)b_2^*(z)g_2(w) \in b_1^*(z)H^2(\Gamma_w) \oplus b_1(z)b_2^*(z)H^2(\Gamma_w).$$

(4.2)
By Lemma 4.5,
\[ T_{z_{n-1}}^* f_0 = b_1^* \left( \alpha_1^{(n-1)} g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-2} \alpha_1^{(n-2-j)} \overline{\alpha_2^j} g_2 \right) \right) + \alpha_2^{(n-1)} b_1 b_2^* g_2 \]
and
\[ T_{z_n}^* f_0 = b_1^* \left( \alpha_1^n g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \overline{\alpha_2^j} g_2 \right) \right) + \alpha_2^n b_1 b_2^* g_2. \]

Since \( T_{z_{n-1}}^* f_0 \notin \tilde{M} \) and \( f_0 \in \tilde{M}, T_{z_{n-1}}^* f_0 - \alpha_2^{n-1} f_0 \notin \tilde{M}. \) Then
\[ b_1^* \left( (\alpha_1^{(n-1)} - \alpha_2^{(n-1)}) g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-2} \alpha_1^{(n-2-j)} \overline{\alpha_2^j} g_2 \right) \right) \notin \tilde{M}. \]
Hence
\[ \left( \sum_{j=0}^{n-2} \alpha_1^{(n-2-j)} \overline{\alpha_2^j} \right) b_1^* \left( (\alpha_1 - \alpha_2) g_1 + (1 - |\alpha_2|^2) g_2 \right) \notin \tilde{M}. \]

Since \( 0 \in \tilde{M}, \sum_{j=0}^{n-2} \alpha_1^{(n-2-j)} \overline{\alpha_2^j} \neq 0, \) so that
\[ (4.3) \quad b_1^* \left( (\alpha_1 - \alpha_2) g_1 + (1 - |\alpha_2|^2) g_2 \right) \notin \tilde{M}. \]

Since \( T_{z_n}^* f_0 \in \tilde{M}, T_{z_n}^* f_0 - \alpha_2^n f_0 \in \tilde{M}. \) Then
\[ b_1^* \left( (\alpha_1^n - \alpha_2^n) g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \overline{\alpha_2^j} g_2 \right) \right) \in \tilde{M}. \]
Hence
\[ (4.4) \quad \left( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \overline{\alpha_2^j} \right) b_1^* \left( (\alpha_1 - \alpha_2) g_1 + (1 - |\alpha_2|^2) g_2 \right) \in \tilde{M}. \]

Now we prove (i). Suppose that \( \alpha_1 = \alpha_2. \) Then \( \alpha_1 = \alpha_2 \neq 0, \) so that
\[ \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \overline{\alpha_2^j} \neq 0. \] By (4.3) and (4.4), \( b_1^* g_2 \notin \tilde{M} \) and \( b_1^* g_2 \in \tilde{M}. \) This is a contradiction. Hence \( \alpha_1 \neq \alpha_2. \)

Suppose that \( \alpha_1^n \neq \alpha_2^n. \) Then \( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \overline{\alpha_2^j} \neq 0. \) By (4.4),
\[ b_1^* \left( (\alpha_1^n - \alpha_2^n) g_1 + (1 - |\alpha_2|^2) g_2 \right) \in \tilde{M}. \]
This contradicts (4.3), so that \( \alpha_1^n = \alpha_2^n. \) Thus we get (i).

(ii) Suppose that \( \alpha_1^n = \alpha_2^n \) and \( \alpha_1 \neq \alpha_2. \) Then \( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \overline{\alpha_2^j} = 0. \) Let \( f \in \tilde{M}. \) Then by (4.2),
\[ f = b_1^* (z) h_1(w) + b_2(z) b_2^* h_2(w). \] Similarly,
we have
\[ T_n^* f - \alpha_2^n f = \left( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \alpha_2^j \right) b_1^* \left( (\alpha_1 - \alpha_2) h_1 + (1 - |\alpha_2|^2) h_2 \right). \]

Hence \( T_n^* f = \overline{\alpha_2^n} f \in \tilde{M} \), so that we get \( T_n^* M \subset \tilde{M} \). By Theorem 4.4, \( S_n^* S_n^* = S_w^* S_n^* \).

\[ \square \]

**Corollary 4.9.** Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \ominus N \). Suppose that \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \), where \( q_1(z) \) is an inner function. Moreover suppose that \( q_1(z) = b_1(z)b_2(z) \), where \( b_i(z), i = 1, 2 \), are simple Blaschke products, \( b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha_i}z) \), and \( \alpha_1\alpha_2 \neq 0 \). Then we have the following.

(i) If \( S_n^* S_n^* = S_w^* S_n^* \) and \( S_n^* S_n^* \neq S_n^* S_n^* \), then \( \alpha_1 + \alpha_2 = 0 \).

(ii) If \( \alpha_1 + \alpha_2 = 0 \), then \( S_n^* S_n^* = S_w^* S_n^* \).

The following is the main theorem in this section.

**Theorem 4.10.** Let \( N \) be a backward shift invariant subspace of \( H^2 \), \( N \neq \{0\} \), and \( N \neq H^2 \). Then \( S_n^* S_n^* = S_w^* S_n^* \) if and only if one of the following conditions holds.

(i) \( S_n^* S_n^* = S_n^* S_n^* \).

(ii) \( S_n^* S_n^* = 0 \).

(iii) There exist two simple Blaschke products \( b_1(z) \) and \( b_2(z) \), \( b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha_i}z) \), such that \( N \subset H^2 \ominus b_1(z)b_2(z)H^2 \) and \( \alpha_1 + \alpha_2 = 0 \).

A backward shift invariant subspace \( N \) satisfying condition (i) is given by Theorem A. Also \( N \) satisfying condition (ii) is given by Theorem 2.2.

**Proof of Theorem 4.10.** Suppose that \( S_n^* S_n^* = S_n^* S_n^* \). Moreover suppose that \( S_n^* S_n^* \neq S_n^* S_n^* \). By Theorem 3.1, \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \) for an inner function \( q_1(z) \) such that either \( q_1(z) = b_1(z) \) or \( q_1(z) = b_1(z)b_2(z) \), where \( b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha_i}z) \), \( |\alpha_i| < 1 \). If \( q_1(z) = b_1(z) \), by Corollary 4.6, (i) holds.

Suppose that \( q_1(z) = b_1(z)b_2(z) \). Moreover suppose that \( \alpha_1 = \alpha_2 = 0 \). Then \( q_1(z) = z^2 \). Hence \( z^2 H^2 \subset M \), so that \( N \subset H^2(\Gamma_w) \oplus zH^2(\Gamma_w) \). Then by Theorem 2.2, \( S_n^* S_n^* = 0 \). Thus (ii) holds.

Suppose that \( \alpha_1 \neq 0 \) and \( \alpha_2 = 0 \). Then \( q_1(z) = zb_1(z) \). By Corollary 4.7, (i) holds.

Suppose that \( \alpha_1 \neq 0 \) and \( \alpha_2 \neq 0 \). Then by Corollary 4.9(i), we get \( \alpha_1 + \alpha_2 = 0 \). Hence (iii) holds.
Next, we prove the converse assertion. Suppose that one of conditions (i), (ii), and (iii) holds. If (i) holds, then trivially $S_z S_w^* = S_w^* S_z$.

If (ii) holds, by Corollary 2.3 we have $S_z S_w^* = S_w^* S_z$.

Suppose that (iii) holds. Then $b_1(z) b_2(z) H^2(\Gamma_z) \subset M$. Hence $M \cap H^2(\Gamma_z)$ equals to either $b_1(z) H^2(\Gamma_z)$, or $b_2(z) H^2(\Gamma_z)$, or $b_1(z) b_2(z) H^2(\Gamma_z)$.

By Corollary 4.6, $S_z S_w^* = S_w^* S_z$ for the first two cases. Hence $S_z S_w^* = S_w^* S_z$ for the last case, by Corollary 4.9(ii), $S_z S_w^* = S_w^* S_z$.

\( \square \)

**Example 4.11.** We give an example of a backward shift invariant subspace $N$ of $H^2$ satisfying $S_z S_w^* = S_w^* S_z$, $S_z S_w^* \neq S_w^* S_z$, and $S_z S_w^* \neq 0$. Let $q_1(z) = b_1(z) b_2(z)$, where $b_i(z)$, $i = 1, 2$, are $b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha_i} z)$, $\alpha_1 + \alpha_2 = 0$, and $\alpha_1 \neq 0$. Let $q_2(w)$ be a non-constant inner function. Let $M = q_1(z) H^2 \oplus b_1(z) b_2(z) q_2(w) H^2(\Gamma_w)$.

Then $M$ is an invariant subspace and $M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z)$. Let $N = H^2 \oplus M$. By Theorem 4.10, $S_z S_w^* = S_w^* S_z$. We have $\widehat{M} = M \oplus q_1(z) H^2 = b_1(z) b_2(z) q_2(w) H^2(\Gamma_w)$. Since

$$T_z b_1(z) b_2(z) q_2(w) = (1 - |\alpha_1|^2) b_1(z) q_2(w) + \overline{\alpha_1} b_1(z) b_2(z) q_2(w),$$

$T_z b_1(z) b_2(z) q_2(w) \notin \widehat{M}$. By Theorem 4.4, $S_z S_w^* \neq S_w^* S_z$. By Theorem 2.2, $S_z S_w^* \neq 0$.

We leave the following problem for the reader.

**Problem 4.12.** Characterize backward invariant subspaces $N$ of $H^2$ satisfying $S_z S_w^* = S_w^* S_z$ for $n \geq 3$.

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