BACKWARD SHIFT INVARIANT SUBSPACES IN THE BIDISC III

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Abstract. In the previous paper, we give a characterization of backward shift invariant subspaces of the Hardy space in the bidisc which satisfy the doubly commuting condition $S_z S_w^* = S_w^* S_z$ for the compression operators $S_z$ and $S_w$. In this paper, we give a characterization of backward shift invariant subspaces satisfying $S_z^2 S_w^* = S_w^* S_z^2$.

1. Introduction.

Let $\Gamma^2$ be the 2-dimensional unit torus. We denote by $(z, w) = (e^{i\theta}, e^{i\phi})$ the variables in $\Gamma^2 = \Gamma_z \times \Gamma_w$. Let $L^2 = L^2(\Gamma^2)$ be the usual Lebesgue space on $\Gamma^2$ with the norm $\|f\|_2 = (\int_{\Gamma^2} |f(e^{i\theta}, e^{i\phi})|^2 d\theta d\phi / (2\pi)^2)^{1/2}$. Then $L^2$ is a Hilbert space with the usual inner product. For $f \in L^2$, the Fourier coefficients are given by

$$\hat{f}(n, m) = \int_{\Gamma^2} f(e^{i\theta}, e^{i\phi}) e^{-in\theta} e^{-im\phi} d\theta d\phi / (2\pi)^2 = \langle f, z^n w^m \rangle.$$ 

Let $H^2 = H^2(\Gamma^2)$ be the Hardy space on $\Gamma^2$, that is,

$$H^2 = \{ f \in L^2; \hat{f}(n, m) = 0 \text{ if } n < 0 \text{ or } m < 0 \}.$$ 

For $f \in H^2$, we can write $f$ as

$$f = \sum_{i, j=0}^{\infty} \oplus a_{i, j} z^i w^j,$$ 

where $\sum_{i, j=0}^{\infty} |a_{i, j}|^2 < \infty$.

Let $P$ be the orthogonal projection from $L^2$ onto $H^2$. For a closed subspace $M$ of $L^2$, we denote by $P_M$ the orthogonal projection from $L^2$ onto $M$. For a function $\psi \in L^{\infty}$, let $L_\psi f = \psi f$ for $f \in L^2$. The Toeplitz operator $T_\psi$ is defined by $T_\psi f = P L_\psi f$ for $f \in H^2$. It is well known that $T_\psi^* = T_{\overline{\psi}}$ and $T_n^* T_{wm} = T_{wm} T_{z_n}^*$ for $n, m \geq 1$. A function $f \in H^2$ is called inner if $|f| = 1$ on $\Gamma^2$ almost everywhere. A closed subspace $M$ of $H^2$ is called invariant if $z M \subseteq M$ and $w M \subseteq M$. In one variable

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case, an invariant subspace $M$ of $H^2(\Gamma)$ has a form $M = qH^2(\Gamma)$, where $q$ is inner. This is the well known Beurling theorem [B]. In two variable case, the structure of invariant subspaces of $H^2$ is very complicated, see [AC], [DY], [Na1], and [R].

Let $M$ be an invariant subspace of $H^2$. Then $T_z^*(H^2 \ominus M) \subset (H^2 \ominus M)$ and $T_w^*(H^2 \ominus M) \subset (H^2 \ominus M)$. A closed subspace $N$ of $H^2$ is called backward shift invariant if $H^2 \ominus N$ is invariant. In [IY], the first author and Yang studied backward shift invariant subspaces $N$ on which $T_z^*$ is strictly contractive. See [CR] and [S] for studies of backward shift invariant subspaces on the unit circle $\Gamma$.

Let $M$ be an invariant subspace of $H^2$ and $\psi \in L^\infty$. Let $V_\psi$ be the operator on $M$ defined by $V_\psi = P_M L_\psi | M$. Then $V_z = T_z$ and $V_z^* = V_z$ on $M$. In [M], Mandrekar proved that $V_\psi V_w^* = V_w^* V_\psi$ on $M$ if and only if $M$ is Beurling type, that is, $M = qH^2$ for some inner function $q$ in $H^\infty$, see also [CS] and [Na2].

For $\psi \in L^\infty$, let $S_\psi = P_N L_\psi | N$. Then we have $S_\psi^* = S_\psi$ and $S_z = T_z^*$ on $N$. In the previous paper [INS], we characterized backward shift invariant subspaces $N$ which satisfy the condition $S_z S_w^* = S_w^* S_z$ on $N$ as follows.

**Theorem A.** Let $N$ be a backward shift invariant subspace of $H^2$ and $N \neq H^2$. Then $S_z S_w^* = S_w^* S_z$ on $N$ if and only if $N$ has one of the following forms;

(i) $N = H^2 \ominus q_1(z)H^2$,
(ii) $N = H^2 \ominus q_2(w)H^2$,
(iii) $N = (H^2 \ominus q_1(z)H^2) \cap (H^2 \ominus q_2(w)H^2)$,

where $q_1(z)$ and $q_2(w)$ are one variable inner functions.

This paper is a continuation of previous papers [IN] and [INS]. Here we are interesting in backward shift invariant subspaces $N$ of $H^2$ satisfying that $S_z^n S_w^m = S_w^m S_z^n$ for given positive integers $n$ and $m$. Up to now, we can not give a complete characterization of $N$ satisfying $S_z^n S_w^m = S_w^m S_z^n$. If $S_z S_w^* = S_w^* S_z$, then trivially $S_z^n S_w^m = S_w^m S_z^n$. But the converse is not true. In this paper, we concentrated on the case $m = 1$. One reason is the work of this paper deeply concerns with the problem of the normality of the operators $S_z^n$. For many backward shift invariant subspaces $N$, $S_z^n$ are not normal operators, see [Y]. If $S_z^n$ is normal, since $S_z^n S_w = S_w S_z^n$, by the Fuglede-Putnam theorem we have $S_z^n S_w^* = S_w^* S_z^n$. So, the results obtained in this paper will be a big help for the above mentioned problem. We will discuss on this subject in the forthcoming paper.
In Section 2, we give characterizations of $N$ satisfying $S_z^n S_{w^m} = 0$ and $S_{w^m} S_z^n = 0$, respectively. If $S_z^n S_{w^m} = 0$, then $S_{w^m} S_z^n = 0$. The converse is not true. In Section 3, we study $N$ satisfying $S_z^n S_w^* = S_w^* S_z^n$, and give a necessary condition for $S_z^n S_w^* = S_w^* S_z^n$. In Section 4, we study $N$ satisfying $S_z^n S_{w^m} = 0$. We gave a complete characterization of such $N$. In [INS], we gave two different type of proofs of Theorem A. One of them is used in Section 3 and the other is used in Section 4.

Let $H^2(\Gamma_z)$ and $H^2(\Gamma_w)$ be the Hardy spaces on the unit circle in variables $z$ and $w$, respectively. We think that $H^2(\Gamma_z)$ and $H^2(\Gamma_w)$ are isomorphic. For a subset $E$ of $H^2$, we denote by $\overline{E}$ the closed linear span of $E$. A function $b(z) = (z - \alpha)/(1 - \overline{\alpha}z), |\alpha| < 1$, is called a simple Blaschke product.

2. $S_z^n S_{w^m} = 0$ or $S_{w^m} S_z^n = 0$.

Let $n$ and $m$ be positive integers. In this section, we study backward shift invariant subspaces $N$ of $H^2$ satisfying $S_z^n S_{w^m} = 0$ and $S_{w^m} S_z^n = 0$, respectively.

**Lemma 2.1.** Let $N$ be a backward shift invariant subspace of $H^2$. Then we have the following.

(i) $S_z^n = S_z^n$.

(ii) $S_{w^m} S_z^n = S_z^n S_{w^m}$ and $S_{w^m} S_{w^m} = S_{w^m} S_{w^m}$.

(iii) If $S_{w^m} S_{w^m} N \neq \{0\}$, then there exists $f \in N$ such that $(S_{w^m} S_{w^m} f) \neq 0$.

**Proof.** All assertions are not difficult to prove. \square

The following theorem says that the structure of backward shift invariant subspaces satisfying $S_z^n S_{w^m} = 0$ is simple.

**Theorem 2.2.** Let $N$ be a backward shift invariant subspace of $H^2$. Then $S_z^n S_{w^m} = 0$ if and only if $N$ satisfies one of the following conditions:

(i) $N \subseteq \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$.

(ii) $N \subseteq \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$.

**Proof.** Suppose that $S_z^n S_{w^m} = 0$. Then

(2.1) $S_{w^m} N \perp S_z^n N$.

Since $N$ is backward shift invariant, if $S_{w^m} N = \{0\}$ then $N$ satisfies condition (i). If $S_z^n N = \{0\}$, then $N$ satisfies (ii).

Next, suppose that

(2.2) $S_{w^m} N \neq \{0\}$ and $S_z^n N \neq \{0\}$. 
We shall lead a contradiction. By (2.1), \( S_{w}^*S_{z}^*N \perp S_{w}^*N \). By Lemma 2.1(ii), \( S_{w}^*S_{z}^*N = S_{z}^*S_{w}^*N = \{0\} \). Then

\[
S_{z}^*N \subset \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)
\]

and

\[
S_{w}^*N \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w).
\]

By (2.2) and (2.3), there exists a nonnegative integer \( j, 0 \leq j \leq m - 1 \), such that

\[
\{0\} \neq S_{w}^*S_{z}^*N \subset H^2(\Gamma_z).
\]

By Lemma 2.1(iii), there exists \( g \in N \) such that

\[
(S_{w}^*S_{z}^*)\hat{g}(0,0) \neq 0.
\]

Also by (2.2) and (2.4), there exist \( f \in N \) and a nonnegative integer \( i, 0 \leq i \leq n - 1 \), such that

\[
S_{z}^*S_{w}^*f \in H^2(\Gamma_w)
\]

and

\[
(S_{z}^*S_{w}^*)\hat{f}(0,0) \neq 0.
\]

Then

\[
0 = \langle S_{w}^*S_{z}^*f, S_{z}^*S_{w}^*g \rangle \quad \text{by (2.1)}
\]
\[
= \langle S_{z}^*S_{w}^*f, S_{w}^*S_{z}^*g \rangle \quad \text{by Lemma 2.1(ii)}
\]
\[
= \langle S_{z}^*S_{w}^*f \rangle(0,0) \langle S_{w}^*S_{z}^*g \rangle(0,0) \quad \text{by (2.5) and (2.7)}
\]
\[
\neq 0 \quad \text{by (2.6) and (2.8)}.
\]

This is a desired contradiction.

The converse is trivial. \( \square \)

**Corollary 2.3.** Let \( N \) be a backward shift invariant subspace of \( H^2 \). Then \( S_{z}S_{w}^* = 0 \) if and only if either \( S_{z} = 0 \) or \( S_{w}^* = 0 \). Hence if \( S_{z}S_{w}^* = 0 \), then \( S_{w}^*S_{z} = 0 \).

**Lemma 2.4.** Let \( M_1 \) and \( M_2 \) be closed subspaces of \( H^2 \) such that

\[
M_1 \subset \sum_{j=0}^{m} \oplus w^j H^2(\Gamma_z) \quad \text{and} \quad M_2 \subset \sum_{i=0}^{n} \oplus z^i H^2(\Gamma_w).
\]

Then \( M_1 + M_2 \) is closed.
Proof. We denote by \((z^i w^j)_{M_1}\) and \((z^i w^j)_{M_2}\) the orthogonal projections of \(z^i w^j\) to the spaces \(M_1\) and \(M_2\), respectively. Let
\[
M'_1 = M_1 \ominus \left( \{ (z^i w^j)_{M_1}; 0 \leq i \leq n, 0 \leq j \leq m \} \right)
\]
and
\[
M'_2 = M_2 \ominus \left( \{ (z^i w^j)_{M_2}; 0 \leq i \leq n, 0 \leq j \leq m \} \right).
\]
Then \(M'_1\) and \(M'_2\) are closed subspaces of \(H^2\),
\[
M'_1 \subset z^{n+1} \left( \sum_{j=0}^{m} \oplus w^j H^2(\Gamma_z) \right), \quad M'_2 \subset w^{m+1} \left( \sum_{i=0}^{n} \oplus z^i H^2(\Gamma_w) \right),
\]
and
\[
M'_1 + M'_2 \perp \left( \{ z^i w^j; 0 \leq i \leq n, 0 \leq j \leq m \} \right).
\]
Since
\[
z^{n+1} \left( \sum_{j=0}^{m} \oplus w^j H^2(\Gamma_z) \right) \perp w^{m+1} \left( \sum_{i=0}^{n} \oplus z^i H^2(\Gamma_w) \right),
\]
\(M'_1 + M'_2 = M'_1 \oplus M'_2\) is closed. Hence
\[
M_1 + M_2 = M'_1 + M'_2 + \left( \{ (z^i w^j)_{M_1}, (z^i w^j)_{M_2}; 0 \leq i \leq n, 0 \leq j \leq m \} \right)
\]
is closed.

**Theorem 2.5.** Let \(N\) be a backward shift invariant subspace of \(H^2\). Then \(S_{w^m} S_{z^n} = 0\) if and only if

(i) \(N \ominus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \subset N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)\).

Proof. Suppose that \(S_{w^m} S_{z^n} = 0\). Then \(S_{z^n} N \subset \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)\). Since \(S_{z^n} S_{w^m} = 0\), \(S_{w^m} N \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\). First, we prove the following

\[
(2.9) \quad N = \left( N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \right) + \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right).
\]

Let

\[
(2.10) \quad K = \left( N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \right) + \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right).
\]

By Lemma 2.4, \(K\) is closed and \(N = K \oplus (N \ominus K)\). To prove (i), it is sufficient to prove \(N \ominus K = \{0\}\). We have \(N \ominus K \subset N \cap (S_{z^n} N)^\perp \cap (S_{w^m} N)^\perp\). Let \(f \in N \ominus K\). Then \(f \perp S_{z^n} N\), so that \(S_{z^n} f \perp N\). Since
$S^*_n f \in N$, $S^*_n f = 0$. Hence $f \in \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$. By (2.10), $f \in K$.

This shows $f = 0$, so that $N \oplus K = \{0\}$. Thus we get (2.9).

Let

$$f \in \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z); f \perp \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right\}.

Then $N_1$ is a closed subspace and

$$N_1 \oplus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \subset N.

If the equality holds in the above, (i) holds. So we assume that

$$N_1 \oplus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \not\subset N.

We shall lead a contradiction. Let

$$N_2 = N \ominus \left( N_1 \ominus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right).

Then $N_2 \not= \{0\}$ and $N = N_1 \oplus N_2 \oplus \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$.

Let $g \in N_2$ be such that $g \neq 0$. We shall prove that

$$g \not\in N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \quad \text{and} \quad g \not\in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z).

The fact $g \not\in N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$ is trivial. Suppose that $g \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$.

By (2.11), $g \in N_1$. Since $g \in N_2$, by (2.12) we have $g \perp N_1$. Hence $g = 0$. This is a contradiction. Thus we get (2.13).

Next, we shall prove that

$$S^*_n g \perp N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z).

To prove this, suppose not. Then $S^*_n g \perp N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$.

By (2.10), $g = g_1 + g_2$, where $g_1 \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$ and $g_2 \in N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$. Then $S^*_n g = S^*_n g_1 \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$.

Therefore $S^*_n g = 0$, so that $g \in N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$. This contradicts (2.13). Thus we get (2.14).

By (2.14), there exists $h_0$ such that $h_0 \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$ and $\langle g, S^*_n h_0 \rangle = \langle S^*_n g, h_0 \rangle \neq 0$. Since $S^*_n h_0 \in N$, by (2.12) we have $S^*_n h_0 = h_1 \oplus h_2 \oplus h_3$, where $h_1 \in N_1, h_2 \in N_2$, and $h_3 \in$
$N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$. Since $g \in N_2$ and $\langle g, S_{z^n} h_0 \rangle \neq 0$, we have $h_2 \neq 0$. Since $z^n h_0 \perp \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$,

$$P_N z^n h_0 = S_{z^n} h_0 \perp N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w).$$

Thus we get $h_3 = 0$. By (2.12), $S_{w_m} N_1 = \{ 0 \}$. Hence $S_{w_m} S_{z^n} h_0 = S_{w_m} h_2$. By (2.13) and $h_2 \in N_2$, $h_2 \notin \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$. This implies that $S_{w_m} h_2 \neq 0$. Hence $S_{w_m} S_{z^n} \neq 0$. This contradicts our starting assumption. Thus we have (i).

To prove the converse, suppose that condition (i) holds. Then $N = \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \oplus L$, where $L \subset N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$. Let $F = F_1 + F_2 \in N$, where $F_1 \in N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$ and $F_2 \in L$. Since $z^n F \perp N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$, $S_{z^n} F \in L$. Hence $S_{w_m} S_{z^n} F = 0$. Thus we get $S_{w_m} S_{z^n} = 0$.

By Theorem 2.2, the structure of backward shift invariant subspaces $N$ satisfying $S_{z^n} S_{w_m} = 0$ is simple. By Theorem 2.5, the structure of backward shift invariant subspaces $N$ satisfying $S_{w_m} S_{z^n} = 0$ is not so simple. When $n = m = 1$, we have the following.

**Theorem 2.6.** Let $N$ be a backward shift invariant subspace of $H^2$. Then $S_{w_m} S_{z^n} = 0$ if and only if $N$ has one of the following forms:

(i) $N = H^2(\Gamma_z) \ominus q(z) H^2(\Gamma_z)$ for some inner function $q(z)$.

(ii) $N = H^2(\Gamma_w) \ominus q(w) H^2(\Gamma_w)$ for some inner function $q(w)$.

(iii) Either $N = H^2(\Gamma_z) + H^2(\Gamma_w)$, or $N = H^2(\Gamma_z) \ominus q_2(w) H^2(\Gamma_w)$, or $N = (H^2(\Gamma_z) \ominus q_1(z) H^2(\Gamma_z)) + H^2(\Gamma_w)$, where $q_1(z)$ and $q_2(w)$ are inner functions.

(iv) $N = (H^2(\Gamma_z) \ominus q_1(z) H^2(\Gamma_z)) + (H^2(\Gamma_w) \ominus q_2(w) H^2(\Gamma_w))$, where $q_1(z)$, $q_2(w)$ are nonconstant inner functions and $\hat{q}_1(0) \hat{q}_2(0) = 0$.

In (iii) and (iv), since $1 \in N$, we may take $q_1$ and $q_2$ as $\hat{q}_1(0) = \hat{q}_2(0) = 0$.

**Proof.** By Theorem 2.2, $S_{w_m} S_{z^n} = 0$ if and only if either (i) or (ii) holds. By Theorem 2.5, $S_{w_m} S_{z^n} = 0$ if and only if

(2.15) $N \ominus (N \cap H^2(\Gamma_w)) \subset N \cap H^2(\Gamma_z)$.

If either (i) or (ii) holds, by Corollary 2.3 we have $S_{w_m} S_{z^n} = 0$. Suppose that $N$ satisfies either (iii) or (iv). Then clearly $1 \in N$. Since $N$ has a special form, it is not difficult to see that

$$N \ominus (N \cap H^2(\Gamma_w)) = \{ f \in N \cap H^2(\Gamma_z); \hat{f}(0) = 0 \}.$$

Hence (2.15) holds.
Next, suppose that (2.15) holds. Then we have

\[(2.16) \quad N = (N \cap H^2(\Gamma_z)) + (N \cap H^2(\Gamma_w)).\]

If either \(N \cap H^2(\Gamma_z) = \{0\}\) or \(N \cap H^2(\Gamma_w) = \{0\}\), then \(S_z S_w^* = 0\), and by Corollary 2.3, \(S_z^* S_w = 0\). Hence either (i) or (ii) holds. Suppose that \(N \cap H^2(\Gamma_z) \neq \{0\}\) and \(N \cap H^2(\Gamma_w) \neq \{0\}\). We shall prove \(1 \in N\). To prove this, suppose that \(1 \notin N\). Let \(1_w\) be the orthogonal projection of 1 to \(N \cap H^2(\Gamma_w)\). Then \(1_w \notin H^2(\Gamma_z)\). Since \(N \cap H^2(\Gamma_z) \neq \{0\}\), there exists \(f \in N \cap H^2(\Gamma_z)\) such that \(\hat{f}(0) \neq 0\). Let \(f_1 = f - \hat{f}(0)1_w \in N\). Then \(f_1 \notin H^2(\Gamma_z)\). Let \(h \in N \cap H^2(\Gamma_w)\). Since \(f \in H^2(\Gamma_z)\), \(f - \hat{f}(0) \perp h\). Since \(1 - 1_w \perp N \cap H^2(\Gamma_w)\),

\[\langle f_1, h \rangle = \langle \hat{f}(0)(1 - 1_w) + f - \hat{f}(0), h \rangle = 0.\]

Hence \(f_1 \in N \oplus (N \cap H^2(\Gamma_w))\). Thus (2.15) does not hold. Therefore \(1 \in N\). Since \(N \cap H^2(\Gamma_z)\) and \(N \cap H^2(\Gamma_w)\) are nonzero backward shift invariant subspaces, by (2.16) \(N\) has one of forms in (iii) and (iv). \(\square\)

3. \(S_z S_w^* = S_w^* S_z^n\).

The following is the main theorem in this section.

**Theorem 3.1.** Let \(N\) be a backward shift invariant subspace of \(H^2\), \(N \neq \{0\}\), and \(N \neq H^2\). Let \(M = H^2 \ominus N\) and \(n \geq 2\) be a positive integer. If \(S_z S_w^* = S_w^* S_z^n\), then one of the following conditions holds:

(i) \(S_z S_w^* = S_w^* S_z\),

(ii) \(M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z)\) for an inner function \(q_1(z)\) satisfying \(q_1(z) = \prod_{j=1}^{k} b_j(z), 1 \leq k \leq n\), where \(b_i\) are simple Blaschke products.

The following lemma is a generalization of [IN], and the idea of the proof is the same.

**Lemma 3.2.** Let \(N\) be a backward shift invariant subspace of \(H^2\) and \(M = H^2 \ominus N\). Let \(n\) be a positive integer. Then \(S_z S_w^* = S_w^* S_z^n\) if and only if

\[M \ominus \left( z^n M \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right) \subset (M \cap H^2(\Gamma_z)) \oplus wM.\]

**Proof.** The operators \(T_{z^n}\) and \(T_{w^*}\) on \(H^2\) have the matrix forms as

\[
T_{z^n} = \begin{pmatrix}
* & P_M T_{z^n}|_N \\
0 & S_{z^n}
\end{pmatrix},
T_{w^*} = \begin{pmatrix}
* & P_M T_{w^*}|_M \\
0 & S_w^*
\end{pmatrix}
\]

on \(H^2 = \begin{pmatrix} M \oplus N \end{pmatrix}\).
Set $A = P_M T_{z^n}|_N$ and $B = P_N T_{w^n}|_M$. Since $T_{z^n} T_{w^n} = T_{w^n} T_{z^n}$ on $H^2$, $S_{z^n} S_{w^n} = S_{w^n} S_{z^n}$ if and only if $BA = 0$. We have $T_{w^n}(M \ominus wM) \subset N$.

For $f \in H^2$, $T_{w^n} f = 0$ if and only if $f \in H^2(\Gamma_z)$. Hence
\[
\ker B = \{ f \in M; T_{w^n} f \in M \} = \{ f \in M \ominus wM; T_{w^n} f = 0 \} \ominus wM = (M \cap H^2(\Gamma_z)) \ominus wM.
\]
We denote by $[\text{ran } A]$ the closed range of $A$. Let $A_1 = P_M T_{z^n} P_N$ on $H^2$. Then $[\text{ran } A] = [\text{ran } A_1]$. Since $A_1^* = P_N T_{w^n} P_M$, we get
\[
\ker A_1^* = \{ f \in M; T_{z^n}^* f \in M \} \ominus N = \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \oplus z^n M \ominus N.
\]
Hence
\[
[\text{ran } A] = [\text{ran } A_1] = H^2 \ominus \ker A_1^* = M \ominus \left( z^n M \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right).
\]
Since $BA = 0$ if and only if $[\text{ran } A] \subset \ker B$, we get our assertion. \quad \Box

**Proof of Theorem 3.1.** Suppose that $S_{z^n} S_{w^n} = S_{w^n} S_{z^n}$. By Lemma 3.2,
\[
M \ominus \left( z^n M \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right) \subset (M \cap H^2(\Gamma_z)) \ominus wM.
\]
Let
\[
K_0 = M \ominus \left( z^n M \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right).
\]
Then
\[
(3.1) \quad K_0 \subset (M \cap H^2(\Gamma_z)) \ominus wM
\]
and
\[
(3.2) \quad M \ominus z^n M = K_0 \ominus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right).
\]
Since $M = \sum_{s=0}^\infty \oplus z^{ns}(M \ominus z^n M)$,
\[
(3.3) \quad K_0 \perp \sum_{s=0}^\infty \oplus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right)
\]
and
\[
(3.4) \quad M = \left( \sum_{s=0}^\infty \oplus z^{ns} K_0 \right) \oplus \left( \sum_{s=0}^\infty \oplus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right).
\]
First, suppose that $K_0 = \{0\}$. In this case, we shall show that condition (i) holds. By (3.2),

$$\sum_{i=0}^{n-1} \oplus z^i(M \ominus z^iM) = M \ominus z^nM = M \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w).$$

Let $f \in M \ominus zM$. Then $f \in M \ominus z^nM$ and $f = \sum_{i=0}^{n-1} \oplus z^i h_i(w)$, where $h_i(w) \in H^2(\Gamma_w)$. Since $f \in M \ominus zM$,

$$z^{n-1}f = \sum_{i=0}^{n-1} \oplus z^{n-1+i} h_i(w) \in M \ominus z^nM.$$

Hence $f = h_0(w)$, so that $M \ominus zM = M \cap H^2(\Gamma_w)$. By Lemma 3.2 (for the case of $n = 1$), we get $S_zS_w = S_w S_z$.

Next, suppose that $K_0 \neq \{0\}$. The proof of this case is a little bit long, so we divide into several steps.

**Step 1.** In this step, we shall prove that there is a nonconstant inner function $q_1(z)$ such that

$$K_0 \subset q_1(z)H^2(\Gamma_z) \oplus wM.$$

First we prove that $K_0 \not\subset wM$. To prove this, suppose that $K_0 \subset wM$. Let $F \in K_0$. Then $F = wf$ for some $f \in M$. We shall prove that $f \in K_0$. We have

$$\langle f, \left(\sum_{s=1}^{\infty} \oplus z^{ns} K_0\right) \oplus \left(\sum_{s=0}^{\infty} \oplus z^{ns} \left(M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\right)\right)\rangle$$

$$= \langle wf, w\left(\sum_{s=1}^{\infty} \oplus z^{ns} K_0 \oplus \left(\sum_{s=0}^{\infty} \oplus z^{ns} \left(M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\right)\right)\right)\rangle$$

$$= \langle F, z^n w \left(\sum_{s=1}^{\infty} \oplus z^{n(s-1)} K_0\right)\rangle$$

by (3.3)

$$= 0,$$

where the last equality follows from $w\sum_{s=1}^{\infty} \oplus z^{n(s-1)} K_0 \subset M$ and (3.2). Hence by (3.4), we get $f \in K_0$. Therefore for every positive integer $p$, we have $F = w^p f_p$ for some $f_p \in K_0$. This leads $F = 0$. This is a contradiction. Thus we get $K_0 \not\subset wM$.

Hence by (3.1), $M \cap H^2(\Gamma_z) \neq \{0\}$. By the Beurling theorem,

$$M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z),$$

where $q_1(z)$ is inner. Then $q_1(z) \in M$ and

$$q_1(z)H^2(\Gamma_z) \perp wM.$$
If \( q_1(z) \) is constant, we have \( M = H^2 \), so that \( N = \{0\} \). This contradicts our assumption. Hence \( q_1(z) \) is a nonconstant inner function. By (3.1) and (3.6), we get (3.5).

**Step 2.** In this step, we prove

(3.8) \[ K_0 \subset q_1(z)\left( \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right). \]

Let \( G \in K_0 \). Then by (3.5), \( G = q_1(z)h(z) \oplus wg \), where \( h(z) \in H^2(\Gamma_z) \) and \( g \in M \). Write

\[ h(z) = \left( \sum_{i=0}^{n-1} \oplus a_i z^i \right) \oplus z^n h_0(z). \]

Then

\[ G = q_1(z)\left( \sum_{i=0}^{n-1} \oplus a_i z^i \right) \oplus q_1(z)z^n h_0(z) \oplus wg. \]

By (3.6), \( q_1(z)z^n h_0(z) \in z^n M \). Since \( G \in K_0 \subset M \oplus z^n M \), we have \( h_0(z) = 0 \). Hence

(3.9) \[ G = q_1(z)\left( \sum_{i=0}^{n-1} \oplus a_i z^i \right) \oplus wg. \]

Here we prove that

(3.10) \[ g \in K_0. \]

Since \( G = q_1(z)h(z) \oplus wg \), we have

\[
\left\langle g, \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \right\rangle = \left\langle wg, w \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \right\rangle \\
= \left\langle wg, w \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \right\rangle + \left\langle q_1 h, w \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \right\rangle \text{ by (3.7)} \\
= \left\langle G, w \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \right\rangle \\
= \left\langle G, z^n w \sum_{s=1}^{\infty} \oplus z^{n(s-1)} K_0 \right\rangle \\
= 0 \text{ by (3.2).}
\]
We also have

\[
\left\langle g, \sum_{s=0}^{\infty} \oplus z^{ns}\left(M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\right)\right\rangle
= \left\langle wg, w\left(\sum_{s=0}^{\infty} \oplus z^{ns}\left(M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\right)\right)\right\rangle
= \left\langle G, \sum_{s=0}^{\infty} \oplus z^{ns}w\left(M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\right)\right\rangle \quad \text{by (3.7)}
= 0 \quad \text{by (3.3)}.
\]

Hence by (3.4), we get (3.10).

Applying (3.9) and (3.10) infinitely many times, we have

\[
G = q_1(z)\left(\sum_{i=0}^{n-1} \oplus a_i z^i\right) \oplus wq_1(z)\left(\sum_{i=0}^{n-1} \oplus b_i z^i\right) \oplus \cdots
\in q_1(z)\left(\sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\right).
\]

Therefore we get (3.8).

**Step 3.** In this step, we study functions in \(M \ominus z M\) and the inner function \(q_1(z)\). By (3.8), there is a closed subspace \(L\) such that \(K_0 = q_1(z)L\) and \(L \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\). Then by (3.2),

\[
M \ominus z^n M = q_1(z)L \oplus \left(M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\right).
\]

Since \(K_0 \neq \{0\}\), \(L \neq \{0\}\). We have \(M \ominus z^n M = \sum_{i=0}^{n-1} \oplus z^i(M \ominus z M)\). Hence \(M \ominus z M \neq \{0\}\). Let \(F \in M \ominus z M\) be such that \(F \neq 0\). Then

\[
F = \left(q_1 \sum_{i=0}^{n-1} \oplus z^i f_i\right) \oplus \sum_{i=0}^{n-1} \oplus z^i g_i,
\]

where \(f_i, g_i \in H^2(\Gamma_w)\),

\[
q_1 \sum_{i=0}^{n-1} \oplus z^i f_i \perp M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w),
\]

and

\[
\sum_{i=0}^{n-1} \oplus z^i g_i \in M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w).
\]
Since \( n \geq 2 \), \( zF \in M \ominus z^n M \), so that we have

\[
zF = q_1 \left( \sum_{i=0}^{n-1} \oplus z^i G_{1,i} \right) + \sum_{i=0}^{n-1} \oplus z^i H_{1,i},
\]

where \( G_{1,i}, H_{1,i} \in H^2(T_w) \). Hence

(3.14)

\[
q_1 \left( z \left( \sum_{i=0}^{n-1} \oplus z^i f_i \right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i} \right) = \left( \sum_{i=0}^{n-1} \oplus z^i H_{1,i} \right) - z \sum_{i=0}^{n-1} \oplus z^i g_i.
\]

Here we devide into two subcases.

**Subcase 1.**

\[
z \left( \sum_{i=0}^{n-1} \oplus z^i f_i(w) \right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w) \neq 0.
\]

Then

\[
q_1(z) = \frac{\left( \sum_{i=0}^{n-1} \oplus z^i H_{1,i}(w) \right) - z \sum_{i=0}^{n-1} \oplus z^i g_i(w)}{z \left( \sum_{i=0}^{n-1} \oplus z^i f_i(w) \right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w)}.
\]

As proved in Step 1, \( q_1(z) \) is a nonconstant inner function. Then by the above, we have

(3.15)

\[
q_1(z) = \prod_{j=1}^{k} b_j(z), \quad 1 \leq k \leq n,
\]

where \( b_j \) are simple Blaschke products.

**Subcase 2.**

\[
z \left( \sum_{i=0}^{n-1} \oplus z^i f_i(w) \right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w) = 0.
\]

Then by (3.14), \( f_{n-1} = g_{n-1} = 0 \), so that by (3.11)

\[
F = \left( q_1 \sum_{i=0}^{n-2} \oplus z^i f_i \right) + \sum_{i=0}^{n-2} \oplus z^i g_i.
\]

Since \( F \in M \ominus zM \),

\[
zF = \left( q_1 \sum_{i=0}^{n-2} \oplus z^{i+1} f_i \right) + \sum_{i=0}^{n-2} \oplus z^{i+1} g_i \in M \ominus z^2 M.
\]

In the same way as above, either (3.15) holds or \( f_{n-2} = g_{n-2} = 0 \). Repeat the same argument. Then either (3.15) holds or

\[
f_{n-1} = f_{n-2} = \cdots = f_1 = g_{n-1} = g_{n-2} = \cdots = g_1 = 0.
\]
Step 4. If (3.15) holds, then (ii) holds. So, suppose that (3.15) does not hold. Then by (3.11) and by the above fact, \( F = q_1f_0 \oplus g_0 \), by (3.12) \( q_1f_0 \perp M \cap H^2(\Gamma_w) \), and by (3.13) \( g_0 \in M \cap H^2(\Gamma_w) \) for every \( F \in M \ominus zM \).

If \( g_0 = 0 \) for every \( F \in M \ominus zM \), since \( q_1(z) \in M \) it follows that \( M \ominus zM \subset q_1(z)H^2(\Gamma_w) \subset M \). Since \( M = \sum_{i=0}^{\infty} z^i(M \ominus zM) \), we have \( M = q_1(z)H^2 \), so that \( N = H^2 \ominus q_1(z)H^2 \). By Theorem A, condition (i) holds.

So we assume that \( g_0 \neq 0 \) for some \( F \in M \ominus zM \). We shall prove that

\[
(3.16) \quad (M \ominus zM) \ominus (M \cap H^2(\Gamma_w)) \subset (M \cap H^2(\Gamma_z)) \oplus wM.
\]

We may assume that \((M \ominus zM) \ominus (M \cap H^2(\Gamma_w)) \neq \{0\}\). Let \( G \in (M \ominus zM) \ominus (M \cap H^2(\Gamma_w)) \) be such that \( G \neq 0 \). Then \( G = q_1(z)h_1(w) \ominus h_2(w) \), where \( q_1(z)h_1(w) \perp M \cap H^2(\Gamma_w) \) and \( h_2(w) \in M \cap H^2(\Gamma_w) \). Hence \( h_2(w) = G - q_1(z)h_1(w) \perp M \cap H^2(\Gamma_w) \). Therefore \( h_2(w) = 0 \). Since \( q_1(z) \in M \),

\[
G = q_1(z)h_1(w) = h_1(0)q_1(z) + h_1(z)w - h_1(0) \frac{q_1(z)w}{w} \in M \cap H^2(\Gamma_z) \oplus wM.
\]

Thus we get (3.16). By Lemma 3.2 (for \( n = 1 \)), (i) holds.

\[\square\]

4. \( S_zS_w^* = S_w^*S_z \) and \( S_zS_w^* \neq S_w^*S_z \).

Let \( N \) be a backward shift invariant subspace of \( H^2 \) and let \( n \) be a positive integer. Let \( M = H^2 \ominus N \). Then \( M \) is an invariant subspace. If both \( S_nS_w^* = S_w^*S_n \) and \( S_zS_w^* \neq S_w^*S_z \) hold, then by Theorem 3.1, \( M \cap H^2(\Gamma_z) = q(z)H^2(\Gamma_z) \) for some nonconstant inner function \( q(z) \).

In this section, we assume that \( q_1(z)H^2 \subset M \) and \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \) for some nonconstant inner function \( q_1(z) \). Let

\[
\tilde{M} = M \ominus q_1(z)H^2 \subset M.
\]

Then \( H^2 \ominus q_1(z)H^2 = \tilde{M} \ominus N \) and \( \tilde{M} \) is \( w \)-invariant. The following lemma is proved in [INS, Lemma 3.2].

**Lemma 4.1.** Let \( f \in \tilde{M} \). Then \( T_w^*f \in \tilde{M} \) if and only if \( f \in w\tilde{M} \).

We denote by \( P_z \) the orthogonal projection from \( H^2 \) onto \( H^2 \ominus q_1(z)H^2 \). Then we have a Toeplitz type operator \( Q_{z^n} \) on \( H^2 \ominus q_1(z)H^2 \) such that

\[
Q_{z^n} : H^2 \ominus q_1(z)H^2 \ni f \rightarrow P_z(T_{z^n}f) \in H^2 \ominus q_1(z)H^2.
\]
Since $z^n M \subset M$, $Q_{z^n} M \subset \tilde{M}$ and $Q_{z^n}^* = Q_{z^n}$. Then $Q_{z^n}$ has the following matrix form:

$$Q_{z^n} = \begin{pmatrix} * & P_{\tilde{M}} T_{z^n}|_N \\ 0 & S_{z^n}^* \end{pmatrix} \quad \text{on} \quad H^2 \ominus q_1(z)H^2 = \begin{pmatrix} \tilde{M} \\ N \end{pmatrix}.$$

Since $H^2 \ominus q_1(z)H^2$ is backward shift invariant, $T_{w}^* (H^2 \ominus q_1(z)H^2) \subset H^2 \ominus q_1(z)H^2$. Since $T_{w}^* N \subset N$, the operator $T_{w}^*$ on $H^2 \ominus q_1(z)H^2$ has the following matrix form:

$$T_w^* = \begin{pmatrix} * & P_N T_w^*|_{\tilde{M}} \\ 0 & S_w^* \end{pmatrix} \quad \text{on} \quad H^2 \ominus q_1(z)H^2 = \begin{pmatrix} \tilde{M} \\ N \end{pmatrix}.$$

Set

(4.1) \quad A = P_{\tilde{M}} T_{z^n}|_N \quad \text{and} \quad B = P_N T_w^*|_{\tilde{M}}.

By [INS, Lemma 3.3], $T_w Q_{z^n} = Q_{z^n} T_w^*$ on $H^2 \ominus q_1(z)H^2$. Hence we have the following.

**Lemma 4.2.** $T_w Q_{z^n} = Q_{z^n} T_w^*$ on $H^2 \ominus q_1(z)H^2$.

**Lemma 4.3.** $S_{w}^* S_{w} = S_{w}^* S_{z^n}$ if and only if $BA = 0$.

**Proof.** By Lemma 4.2, $T_w Q_{z^n} = Q_{z^n} T_w^*$ on $H^2 \ominus q_1(z)H^2$. Then $BA + S_{w}^* S_{z^n} = S_{w}^* S_{z^n}$. Hence $S_{z^n} S_{w}^* = S_{w}^* S_{z^n}$ if and only if $BA = 0$. \hfill \Box

The following is a slight generalization of [INS, Theorem 3.5].

**Theorem 4.4.** Let $N$ be a backward shift invariant subspace of $H^2$ and $M = H^2 \ominus N$. Suppose that $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$, where $q_1(z)$ is a nonconstant inner function. Let $\tilde{M} = M \ominus q_1(z)H^2$. Then the following conditions are equivalent.

(i) $S_{z^n} S_{w}^* = S_{w}^* S_{z^n}$ on $N$.

(ii) $\tilde{M} \ominus \{ f \in \tilde{M} ; T_{z^n} f \in \tilde{M} \} \subset w\tilde{M}$.

(iii) $T_{z^n} \tilde{M} \subset \tilde{M}$.

**Proof.** The idea of the proof is similar to the one of [INS, Theorem 3.5]. For the sake of completeness, we give the proof.

(i) $\Leftrightarrow$ (ii) By Lemma 4.3, condition (i) is equivalent to $BA = 0$. By (4.1) and Lemma 4.1, ker $B = \{ f \in \tilde{M} ; T_{z^n} f \in \tilde{M} \} = w\tilde{M}$. Put $A_1 = P_M T_{z^n} P_N$ on $\tilde{M} \ominus N$. Then [ran $A_1$] = [ran $A_1$]. Since $A_1^* = P_N T_{z^n} P_M$, ker $A_1^* = N \ominus \{ f \in \tilde{M} ; T_{z^n} f \in \tilde{M} \}$. Hence

[ran $A$] = [ran $A_1$] = $(\tilde{M} \ominus N) \ominus \ker A_1^* = \tilde{M} \ominus \{ f \in \tilde{M} ; T_{z^n} f \in \tilde{M} \}.$

Therefore $BA = 0$ if and only if $\tilde{M} \ominus \{ f \in \tilde{M} ; T_{z^n} f \in \tilde{M} \} \subset w\tilde{M}$. Thus we get (i) $\Leftrightarrow$ (ii).
Since \( M \oplus \{ f \in \tilde{M}; T^*_{z_n} f \in \tilde{M} \} \subset w\tilde{M} \). Since \( \{ f \in \tilde{M}; T^*_{z_n} f \in \tilde{M} \} \) is closed, \( \tilde{M} \oplus w\tilde{M} \subset \{ f \in \tilde{M}; T^*_{z_n} f \in \tilde{M} \} \). Since \( w\tilde{M} \subset \tilde{M}, \tilde{M} = \sum_{j=0}^{\infty} \oplus w^j (\tilde{M} \oplus w\tilde{M}) \). Since \( T^*_{z_n} w^j f = w^j T^*_{z_n} f \) for \( f \in H^2 \), we have \( T^*_{z_n} \tilde{M} \subset \tilde{M} \).

(iii) \( \Rightarrow \) (ii) is trivial.

For \( f \in H^2(\Gamma_z) \), write \( f^*(z) = T^*_{z_n} f(z) = \bar{z}(f(z) - \hat{f}(0)) \).

**Lemma 4.5.** Let \( b_i(z) = (z - \alpha_i)/(1 - \bar{\alpha}_i z), |\alpha_i| < 1, \) and \( 1 \leq i \leq n \). Then

(i) \( T^*_{z_n} z = 1, T^*_{z_n} b_1 = b_1^*(z) = (1 - |\alpha_1|^2)/(1 - \bar{\alpha}_1 z), \) and \( T^*_{z_n} b_i(z) = \bar{\alpha}_1 b_i(z) \).

(ii) \( T^*_{z_n}(b_1(z)b_2^*(z)) = \bar{b}_2(z) + \bar{\alpha}_2 b_1(z) b_2^*(z) \).

(iii) \( H^2(\Gamma_z) \ominus (\prod_{j=1}^{k} b_j(z)) H^2(\Gamma_z) = \sum_{j=1}^{k} \oplus [b_1(z) \cdots b_{j-1}(z) b_j^*(z)] \).

(iv) \( H^2 \ominus (\prod_{j=1}^{k} b_j(z)) H^2 = \sum_{j=1}^{k} \oplus [b_1(z) \cdots b_{j-1}(z) b_j^*(z) H^2(\Gamma_w)] \).

**Proof.** It is not difficult to prove (i).

(ii) Since

\[
\bar{\alpha}_1 b_1(z) b_2^*(z) = \bar{\alpha}_1 b_1(z) \left( 1 - |\alpha_2|^2 \right) \frac{1}{1 - \bar{\alpha}_2 z} = (1 - |\alpha_2|^2) b_1(z) \left( \bar{z} + \frac{\bar{\alpha}_2}{1 - \bar{\alpha}_2 z} \right) = (1 - |\alpha_2|^2) \bar{\alpha}_1 b_1(z) + \bar{\alpha}_2 b_1(z) b_2^*(z),
\]

we get (ii).

(iii) See [Ni, p.33].

(iv) follows from (iii).

**Corollary 4.6.** Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \ominus N \). Suppose that \( M \cap H^2(\Gamma_z) = b_1(z) H^2(\Gamma_z) \), where \( b_1(z) \) is a simple Blaschke product. Then \( S_z S^*_w = S^*_w S_z \).

**Proof.** Let \( b_1(z) = (z - \alpha)/(1 - \bar{\alpha} z), |\alpha| < 1, \) and \( \tilde{M} = \tilde{M} \ominus b_1(z) H^2 \). Since \( b_1(z) \in M, b_1(z) H^2 \subset M \). By Lemma 4.5(iv), \( \tilde{M} \subset b_1(z) H^2(\Gamma_w) \). By Lemma 4.5(i), \( T^*_{z_n}(b_1(z) h(w)) = \bar{\alpha}_1 b_1(z) h(w) \). Hence \( T^*_{z_n} \tilde{M} \subset \tilde{M} \). By Theorem 4.4, \( S_z S^*_w = S^*_w S_z \).

**Corollary 4.7.** Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \ominus N \). Suppose that \( M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z) \), where \( q_1(z) \) is an inner function. Let \( n, k \) be positive integers with \( n \geq k + 1 \). Moreover suppose that \( q_1(z) = z^k b(z) \), where \( b \) is a simple Blaschke product, \( b(z) = (z - \alpha)/(1 - \bar{\alpha} z), \) and \( \alpha \neq 0 \). If \( S_z S^*_w = S^*_w S_z^n \), then \( S_z S^*_w = S^*_w S_z^k \).
Proof. Let \( \tilde{M} = M \ominus q_1(z)H^2 \). If \( \tilde{M} = \{0\} \), then \( M = q_1(z)H^2 \). By Theorem A, \( S_zS_w^* = S_w^*S_z \). Suppose that \( M \neq \{0\} \). By Lemma 4.5(iv),
\[
\tilde{M} \subset b^*(z)H^2(\Gamma_w) \oplus \left( \sum_{j=1}^{k} \oplus zj^{-1}b(z)H^2(\Gamma_w) \right).
\]
Let \( f \in \tilde{M} \). Then
\[
f = b^*(z)h_0(w) + \sum_{j=1}^{k} \oplus zj^{-1}b(z)h_j(w), \quad h_j(w) \in H^2(\Gamma_w).
\]
By Lemma 4.5(i),
\[
T_{\alpha}^*f = T_{\alpha}^*(T_{\alpha}^*-k)(T_{\alpha}^*f).
\]
\[
= T_{\alpha}^*(\sum_{j=0}^{k} \bar{\alpha}^{(k-j)}b^*(z)h_j(w))
\]
\[
= \bar{\alpha}^{(n-k)}(\sum_{j=0}^{k} \bar{\alpha}^{(k-j)}b^*(z)h_j(w)).
\]
Since \( S_zS_w^* = S_w^*S_z \), by Theorem 4.4 \( T_{\alpha}^*f \in \tilde{M} \). Since \( \alpha \neq 0 \),
\[
\sum_{j=0}^{k} \bar{\alpha}^{(k-j)}b^*(z)h_j(w) \in \tilde{M}.
\]
Thus \( T_{\alpha}^*\tilde{M} \subset \tilde{M} \). By Theorem 4.4, \( S_z^*S_w^* = S_w^*S_z^* \).

\[ \square \]

**Theorem 4.8.** Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \ominus N \). Suppose that \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \), where \( q_1(z) \) is an inner function. Moreover suppose that \( q_1(z) = b_1(z)b_2(z) \), where \( b_1(z) \), \( b_2(z) \) are simple Blaschke products, \( b_1(z) = (z-\alpha_1)/(1-\bar{\alpha}_1z) \), and \( \alpha_1\alpha_2 \neq 0 \). Let \( n \geq 2 \) be a positive integer. Then we have the following.

(i) If \( S_zS_w^* = S_w^*S_z \) and \( S_{z-1}S_w^* \neq S_w^*S_{z-1} \), then \( \alpha_1 = \alpha_2^0 \) and \( \alpha_2 \neq \alpha_2 \).

(ii) If \( \alpha_1^0 = \alpha_2^0 \) and \( \alpha_1 \neq \alpha_2 \), then \( S_zS_w^* = S_w^*S_z \).

**Proof.** Let \( \tilde{M} = M \ominus q_1(z)H^2 \). Suppose that \( S_zS_w^* = S_w^*S_z \) and \( S_{z-1}S_w^* \neq S_w^*S_{z-1} \). By Theorem 4.4, \( T_{\alpha}^*\tilde{M} \subset \tilde{M} \) and \( T_{\alpha}^*\tilde{M} \notin \tilde{M} \). By Lemma 4.5(iv),
\[
\tilde{M} \subset b_1(z)H^2(\Gamma_w) \oplus b_1(z)b_2(z)H^2(\Gamma_w).
\]
Then there exists \( f_0 \in \tilde{M} \) such that \( T_{\alpha}^*f_0 \notin \tilde{M} \), and
\[
f_0 = b_1(z)g_1(w) + b_1(z)b_2(z)g_2(w) \in b_1(z)H^2(\Gamma_w) \oplus b_1(z)b_2(z)H^2(\Gamma_w).
\]
By Lemma 4.5,
\[ T_{z_{n-1}}^* f_0 = b_1^* \left( \alpha_1^{(n-1)} g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-2} \alpha_1^{(n-2-j)} \frac{1}{\alpha_2^j} g_2 \right) \right) + \alpha_2^{(n-1)} b_1^* b_2^* g_2 \]
and
\[ T_{z_n}^* f_0 = b_1^* \left( \alpha_1^n g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \frac{1}{\alpha_2^j} g_2 \right) \right) + \alpha_2^n b_1^* b_2^* g_2. \]
Since \( T_{z_{n-1}}^* f_0 \notin \tilde{M} \) and \( f_0 \in \tilde{M} \), \( T_{z_{n-1}}^* f_0 - \alpha_2^{n-1} f_0 \notin \tilde{M} \). Then
\[ b_1^* \left( (\alpha_1^{(n-1)} - \alpha_2^{(n-1)}) g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-2} \alpha_1^{(n-2-j)} \frac{1}{\alpha_2^j} g_2 \right) \right) \notin \tilde{M}. \]
Hence
\[ \left( \sum_{j=0}^{n-2} \alpha_1^{(n-2-j)} \frac{1}{\alpha_2^j} \right) b_1^* \left( (\alpha_1 - \alpha_2) g_1 + (1 - |\alpha_2|^2) g_2 \right) \notin \tilde{M}. \]
Since \( 0 \in \tilde{M} \), \( \sum_{j=0}^{n-2} \alpha_1^{(n-2-j)} \frac{1}{\alpha_2^j} \neq 0 \), so that
\[ (4.3) \quad b_1^* \left( (\alpha_1 - \alpha_2) g_1 + (1 - |\alpha_2|^2) g_2 \right) \notin \tilde{M}. \]
Since \( T_{z_n}^* f_0 \in \tilde{M} \), \( T_{z_n}^* f_0 - \alpha_2^n f_0 \in \tilde{M} \). Then
\[ b_1^* \left( (\alpha_1^n - \alpha_2^n) g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \frac{1}{\alpha_2^j} g_2 \right) \right) \in \tilde{M}. \]
Hence
\[ (4.4) \quad \left( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \frac{1}{\alpha_2^j} \right) b_1^* \left( (\alpha_1 - \alpha_2) g_1 + (1 - |\alpha_2|^2) g_2 \right) \in \tilde{M}. \]
Now we prove (i). Suppose that \( \alpha_1 = \alpha_2 \). Then \( \alpha_1 = \alpha_2 \neq 0 \), so that
\[ \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \frac{1}{\alpha_2^j} \neq 0. \]
By (4.3) and (4.4), \( b_1^* g_2 \notin \tilde{M} \) and \( b_1^* g_2 \in \tilde{M} \). This is a contradiction. Hence \( \alpha_1 \neq \alpha_2 \).

Suppose that \( \alpha_1^n \neq \alpha_2^n \). Then \( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \frac{1}{\alpha_2^j} \neq 0 \). By (4.4),
\[ b_1^* \left( (\alpha_1^n - \alpha_2^n) g_1 + (1 - |\alpha_2|^2) g_2 \right) \in \tilde{M}. \]
This contradicts (4.3), so that \( \alpha_1^n = \alpha_2^n \). Thus we get (i).

(ii) Suppose that \( \alpha_1^n = \alpha_2^n \) and \( \alpha_1 \neq \alpha_2 \). Then \( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \frac{1}{\alpha_2^j} = 0 \).
Let \( f \in \tilde{M} \). Then by (4.2),
\[ f = b_1^*(z) h_1(w) + b_1(z) b_2^*(z) h_2(w). \]
Similarly,
we have

\[ T^*_n f - \bar{\alpha} f = \left( \sum_{j=0}^{n-1} \bar{\alpha}^{(n-1-j)} \bar{\alpha}^j \right) b_1 \left( (\bar{\alpha}_1 - \bar{\alpha}_2) h_1 + (1 - |\alpha_2|^2) h_2 \right). \]

Hence \( T^*_n f = \bar{\alpha} f \in \tilde{M} \), so that we get \( T^*_n \tilde{M} \subset \tilde{M} \). By Theorem 4.4, \( S_z^* S_w^* = S_w^* S_z^* \).

**Corollary 4.9.** Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \ominus N \). Suppose that \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \), where \( q_1(z) \) is an inner function. Moreover suppose that \( q_1(z) = b_1(z)b_2(z) \), where \( b_i(z), i = 1, 2, \) are simple Blaschke products, \( b_i(z) = (z - \alpha_i)/(1 - \bar{\alpha}_i z) \), and \( \alpha_1 \alpha_2 \neq 0 \). Then we have the following.

(i) If \( S_{z_2} S_w^* = S_w^* S_{z_2} \) and \( S_{z_2} S_w^* \neq S_w^* S_z \), then \( \alpha_1 + \alpha_2 = 0 \).

(ii) If \( \alpha_1 + \alpha_2 = 0 \), then \( S_{z_2} S_w^* = S_w^* S_{z_2} \).

The following is the main theorem in this section.

**Theorem 4.10.** Let \( N \) be a backward shift invariant subspace of \( H^2 \), \( N \neq \{0\} \), and \( N \neq H^2 \). Then \( S_{z_2} S_w^* = S_w^* S_{z_2} \) if and only if one of the following conditions holds.

(i) \( S_{z_2} S_w^* = S_w^* S_z \).

(ii) \( S_{z_2} S_w^* = 0 \).

(iii) There exist two simple Blaschke products \( b_1(z) \) and \( b_2(z) \), \( b_i(z) = (z - \alpha_i)/(1 - \bar{\alpha}_i z) \), such that \( N \subset H^2 \ominus b_1(z)b_2(z)H^2 \) and \( \alpha_1 + \alpha_2 = 0 \).

A backward shift invariant subspace \( N \) satisfying condition (i) is given by Theorem A. Also \( N \) satisfying condition (ii) is given by Theorem 2.2.

**Proof of Theorem 4.10.** Suppose that \( S_{z_2} S_w^* = S_w^* S_{z_2} \). Moreover suppose that \( S_{z_2} S_w^* \neq S_w^* S_z \). By Theorem 3.1, \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \) for an inner function \( q_1(z) \) such that either \( q_1(z) = b_1(z) \) or \( q_1(z) = b_1(z)b_2(z) \), where \( b_i(z) = (z - \alpha_i)/(1 - \bar{\alpha}_i z), |\alpha_i| < 1 \). If \( q_1(z) = b_1(z) \), by Corollary 4.6, (i) holds.

Suppose that \( q_1(z) = b_1(z)b_2(z) \). Moreover suppose that \( \alpha_1 = \alpha_2 = 0 \). Then \( q_1(z) = z^2 \). Hence \( z^2 H^2 \subset M \), so that \( N \subset H^2(\Gamma_w) \oplus z H^2(\Gamma_w) \). Then by Theorem 2.2, \( S_{z_2} S_w^* = 0 \). Thus (ii) holds.

Suppose that \( \alpha_1 \neq 0 \) and \( \alpha_2 = 0 \). Then \( q_1(z) = z b_1(z) \). By Corollary 4.7, (i) holds.

Suppose that \( \alpha_1 \neq 0 \) and \( \alpha_2 \neq 0 \). Then by Corollary 4.9(i), we get \( \alpha_1 + \alpha_2 = 0 \). Hence (iii) holds.
Next, we prove the converse assertion. Suppose that one of conditions (i), (ii), and (iii) holds. If (i) holds, then trivially $S_{zz}S_w^* = S_w^*S_{zz}$.

If (ii) holds, by Corollary 2.3 we have $S_{zz}S_w^* = S_w^*S_{zz}$. Suppose that (iii) holds. Then $b_1(z)b_2(z)H^2(\Gamma_z) \subseteq M$. Hence $M \cap H^2(\Gamma_z)$ equals to either $b_1(z)H^2(\Gamma_z)$, or $b_2(z)H^2(\Gamma_z)$, or $b_1(z)b_2(z)H^2(\Gamma_z)$. By Corollary 4.6, $S_{zz}S_w^* = S_w^*S_{zz}$ for the first two cases. Hence $S_{zz}S_w^* = S_w^*S_{zz}$. For the last case, by Corollary 4.9(ii), $S_{zz}S_w^* = S_w^*S_{zz}$.

Example 4.11. We give an example of a backward shift invariant subspace $N$ of $H^2$ satisfying $S_{zz}S_w^* = S_w^*S_{zz}$, $S_{zz}S_w^* \neq S_w^*S_{zz}$, and $S_{zz}S_w^* \neq 0$. Let $q_1(z) = b_1(z)b_2(z)$, where $b_i(z), i = 1, 2$, are $b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}z)$, $\alpha_1 + \alpha_2 = 0$, and $\alpha_1 \neq 0$. Let $q_2(w)$ be a non-constant inner function. Let $M = q_1(z)H^2 \oplus b_1(z)b_2(z)q_2(w)H^2(\Gamma_w)$. Then $M$ is an invariant subspace and $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$. Let $N = H^2 \oplus M$. By Theorem 4.10, $S_{zz}S_w^* = S_w^*S_{zz}$. We have $\tilde{M} = M \oplus q_1(z)H^2 = b_1(z)b_2(z)q_2(w)H^2(\Gamma_w)$. Since

$$T_z^*b_1(z)b_2^*(z)q_2(w) = (1 - |\alpha_1|^2)b_1^*(z)q_2(w) + \overline{\alpha}_1 b_1(z)b_2^*(z)q_2(w),$$

$T_z^*b_1(z)b_2^*(z)q_2(w) \notin \tilde{M}$. By Theorem 4.4, $S_{zz}S_w^* \neq S_w^*S_{zz}$. By Theorem 2.2, $S_{zz}S_w^* \neq 0$.

We leave the following problem for the reader.

Problem 4.12. Characterize backward invariant subspaces $N$ of $H^2$ satisfying $S_{zz}S_w^* = S_w^*S_{zz}$ for $n \geq 3$.

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