BACKWARD SHIFT INVARIANT SUBSPACES IN THE BIDISC III

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ABSTRACT. In the previous paper, we give a characterization of backward shift invariant subspaces of the Hardy space in the bidisc which satisfy the doubly commuting condition $S_z S_w^* = S_w^* S_z$ for the compression operators $S_z$ and $S_w$. In this paper, we give a characterization of backward shift invariant subspaces satisfying $S_z^2 S_w^* = S_w^* S_z^2$.

1. INTRODUCTION.

Let $\Gamma^2$ be the 2-dimensional unit torus. We denote by $(z, w) = (e^{i\theta}, e^{i\phi})$ the variables in $\Gamma^2 = \Gamma_z \times \Gamma_w$. Let $L^2 = L^2(\Gamma^2)$ be the usual Lebesgue space on $\Gamma^2$ with the norm $\|f\|_2 = (\int_{\Gamma^2} |f(e^{i\theta}, e^{i\phi})|^2 d\theta d\phi / (2\pi)^2)^{1/2}$. Then $L^2$ is a Hilbert space with the usual inner product. For $f \in L^2$, the Fourier coefficients are given by

$$\hat{f}(n, m) = \int_{\Gamma^2} f(e^{i\theta}, e^{i\phi}) e^{in\theta} e^{-im\phi} d\theta d\phi / (2\pi)^2 = \langle f, z^n w^m \rangle.$$ 

Let $H^2 = H^2(\Gamma^2)$ be the Hardy space on $\Gamma^2$, that is,

$$H^2 = \{ f \in L^2; \hat{f}(n, m) = 0 \text{ if } n < 0 \text{ or } m < 0 \}.$$ 

For $f \in H^2$, we can write $f$ as

$$f = \sum_{i,j=0}^{\infty} \oplus a_{i,j} z^i w^j, \text{ where } \sum_{i,j=0}^{\infty} |a_{i,j}|^2 < \infty.$$ 

Let $P$ be the orthogonal projection from $L^2$ onto $H^2$. For a closed subspace $M$ of $L^2$, we denote by $P_M$ the orthogonal projection from $L^2$ onto $M$. For a function $\psi \in L^\infty$, let $L_\psi f = \psi f$ for $f \in L^2$. The Toeplitz operator $T_\psi$ is defined by $T_\psi f = PL_\psi f$ for $f \in H^2$. It is well known that $T_\psi^* = T_\psi$ and $T_z^n T_w^m = T_w^m T_z^n$ for $n, m \geq 1$. A function $f \in H^2$ is called inner if $|f| = 1$ on $\Gamma^2$ almost everywhere. A closed subspace $M$ of $H^2$ is called invariant if $z M \subset M$ and $w M \subset M$.

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case, an invariant subspace $M$ of $H^2(\Gamma)$ has a form $M = qH^2(\Gamma)$, where $q$ is inner. This is the well known Beurling theorem [B]. In two variable case, the structure of invariant subspaces of $H^2$ is very complicated, see [AC], [DY], [Na1], and [R].

Let $M$ be an invariant subspace of $H^2$. Then $T^*_z(H^2 \ominus M) \subset (H^2 \ominus M)$ and $T^*_w(H^2 \ominus M) \subset (H^2 \ominus M)$. A closed subspace $N$ of $H^2$ is called \textit{backward shift invariant} if $H^2 \ominus N$ is invariant. In [IY], the first author and Yang studied backward shift invariant subspaces $N$ on which $T^*_z$ is strictly contractive. See [CR] and [S] for studies of backward shift invariant subspaces on the unit circle $\Gamma$.

Let $M$ be an invariant subspace of $H^2$ and $\psi \in L^\infty$. Let $V_\psi$ be the operator on $M$ defined by $V_\psi = P_M L_\psi |_M$. Then $V_\psi = T_\psi$ on $M$. In [M], Mandrekar proved that $V_\psi V^*_\psi = \psi V_\psi$ on $M$ if and only if $M$ is Beurling type, that is, $M = qH^2$ for some inner function $q$ in $H^\infty$, see also [CS] and [Na2].

For $\psi \in L^\infty$, let $S_\psi = P_N L_\psi |_N$. Then we have $S_\psi^* = S_\psi$ and $S_\psi = T^*_\psi$ on $N$. In the previous paper [INS], we characterized backward shift invariant subspaces $N$ which satisfy the condition $S_\psi S^*_w = S^*_w S_\psi$ on $N$ as follows.

\textbf{Theorem A.} Let $N$ be a backward shift invariant subspace of $H^2$ and $N \neq H^2$. Then $S_\psi S^*_w = S^*_w S_\psi$ on $N$ if and only if $N$ has one of the following forms;

\begin{itemize}
  \item[(i)] $N = H^2 \ominus q_1(z)H^2$,
  \item[(ii)] $N = H^2 \ominus q_2(w)H^2$,
  \item[(iii)] $N = (H^2 \ominus q_1(z)H^2) \cap (H^2 \ominus q_2(w)H^2),$
\end{itemize}

where $q_1(z)$ and $q_2(w)$ are one variable inner functions.

This paper is a continuation of previous papers [IN] and [INS]. Here we are interesting in backward shift invariant subspaces $N$ of $H^2$ satisfying that $S_n^* S^*_w = S^*_w S_n$ for given positive integers $n$ and $m$. Up to now, we can not give a complete characterization of $N$ satisfying $S_n^* S^*_w = S^*_w S_n$. If $S_n^* S^*_w = S^*_w S_n$, then trivially $S_n^* S^*_w = S^*_w S_n$. But the converse is not true. In this paper, we concentrated on the case $m = 1$. One reason is the work of this paper deeply concerns with the problem of the normality of the operators $S_n^*$. For many backward shift invariant subspaces $N$, $S_n^*$ are not normal operators, see [Y]. If $S_n^*$ is normal, since $S_n^* S_w = S_w S_n^*$, by the Fuglede-Putnam theorem we have $S_n^* S_w = S_w S_n^*$. So, the results obtained in this paper will be a big help for the above mentioned problem. We will discuss on this subject in the forthcoming paper.
In Section 2, we give characterizations of $N$ satisfying $S_z^n S_w^m = 0$ and $S_w^m S_z^n = 0$, respectively. If $S_z^n S_w^m = 0$, then $S_w^m S_z^n = 0$. The converse is not true. In Section 3, we study $N$ satisfying $S_z^n S_w^* = S_w^* S_z^n$, and give a necessary condition for $S_z^n S_w^* = S_w^* S_z^n$. In Section 4, we study $N$ satisfying $S_z^n S_w^* = S_w^* S_z^n$. We gave a complete characterization of such $N$. In [INS], we gave two different type of proofs of Theorem A. One of them is used in Section 3 and the other is used in Section 4.

Let $H^2(\Gamma_z)$ and $H^2(\Gamma_w)$ be the Hardy spaces on the unit circle in variables $z$ and $w$, respectively. We think that $H^2(\Gamma_z) \subset H^2$ and $H^2(\Gamma_w) \subset H^2$. For a subset $E$ of $H^2$, we denote by $[E]$ the closed linear span of $E$. A function $b(z) = (z - \alpha)/(1 - \overline{\alpha}z), |\alpha| < 1$, is called a simple Blaschke product.

2. $S_z^n S_w^* = 0$ or $S_w^m S_z^n = 0$.

Let $n$ and $m$ be positive integers. In this section, we study backward shift invariant subspaces $N$ of $H^2$ satisfying $S_z^n S_w^m = 0$ and $S_w^m S_z^n = 0$, respectively.

**Lemma 2.1.** Let $N$ be a backward shift invariant subspace of $H^2$. Then we have the following.

(i) $S_z^n = S_z^n$.
(ii) $S_w^m S_z^n = S_z^n S_w^m$ and $S_z^n S_w^m = S_w^m S_z^n$.
(iii) If $S_z^n S_w^m N \neq \{0\}$, then there exists $f \in N$ such that $(S_z^n S_w^m f) \cap (0, 0) \neq 0$.

**Proof.** All assertions are not difficult to prove. \qed

The following theorem says that the structure of backward shift invariant subspaces satisfying $S_z^n S_w^m = 0$ is simple.

**Theorem 2.2.** Let $N$ be a backward shift invariant subspace of $H^2$. Then $S_z^n S_w^m = 0$ if and only if $N$ satisfies one of the following conditions:

(i) $N \subset \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)$.
(ii) $N \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$.

**Proof.** Suppose that $S_z^n S_w^m = 0$. Then

(2.1) $S_w^m N \perp S_z^n N$.

Since $N$ is backward shift invariant, if $S_w^m N = \{0\}$ then $N$ satisfies condition (i). If $S_z^n N = \{0\}$, then $N$ satisfies (ii).

Next, suppose that

(2.2) $S_w^m N \neq \{0\}$ and $S_z^n N \neq \{0\}$. 

We shall lead a contradiction. By (2.1), \( S_{w_m}^* S_{z^n}^* N \perp S_{z^n}^* S_{w_m}^* N \). By Lemma 2.1(ii), \( S_{w_m}^* S_{z^n}^* N = S_{z^n}^* S_{w_m}^* N = \{0\} \). Then

\[
(2.3) \quad S_{z^n}^* N \subset \bigoplus_{j=0}^{m-1} w^j H^2(\Gamma_z)
\]

and

\[
(2.4) \quad S_{w_m}^* N \subset \bigoplus_{i=0}^{n-1} z^i H^2(\Gamma_w).
\]

By (2.2) and (2.3), there exists a nonnegative integer \( j, 0 \leq j \leq m - 1 \), such that

\[
(2.5) \quad \{0\} \neq S_{w_j}^* S_{z^n}^* N \subset H^2(\Gamma_z).
\]

By Lemma 2.1(iii), there exists \( g \in N \) such that

\[
(2.6) \quad (S_{w_j}^* S_{z^n}^* g)^\gamma(0,0) \neq 0.
\]

Also by (2.2) and (2.4), there exist \( f \in N \) and a nonnegative integer \( i, 0 \leq i \leq n - 1 \), such that

\[
(2.7) \quad S_{z^i}^* S_{w_m}^* f \in H^2(\Gamma_w)
\]

and

\[
(2.8) \quad (S_{z^i}^* S_{w_m}^* f)^\gamma(0,0) \neq 0.
\]

Then

\[
0 = \langle S_{w_m}^* S_{z^n}^* f, S_{z^n}^* S_{w_j}^* g \rangle \quad \text{by (2.1)}
\]

\[
= \langle S_{z^i}^* S_{w_m}^* f, S_{w_j}^* S_{z^n}^* g \rangle \quad \text{by Lemma 2.1(ii)}
\]

\[
= (S_{z^i}^* S_{w_m}^* f)^\gamma(0,0) (S_{w_j}^* S_{z^n}^* g)^\gamma(0,0) \quad \text{by (2.5) and (2.7)}
\]

\[
\neq 0 \quad \text{by (2.6) and (2.8)}.
\]

This is a desired contradiction.

The converse is trivial. \( \square \)

**Corollary 2.3.** Let \( N \) be a backward shift invariant subspace of \( H^2 \). Then \( S_{z^n}^* S_{w_m}^* = 0 \) if and only if either \( S_{z^n}^* = 0 \) or \( S_{w_m}^* = 0 \). Hence if \( S_{z^n}^* S_{w_m}^* = 0 \), then \( S_{w_m}^* S_{z^n}^* = 0 \).

**Lemma 2.4.** Let \( M_1 \) and \( M_2 \) be closed subspaces of \( H^2 \) such that

\[
M_1 \subset \bigoplus_{j=0}^{m} w^j H^2(\Gamma_z) \quad \text{and} \quad M_2 \subset \bigoplus_{i=0}^{n} z^i H^2(\Gamma_w).
\]

Then \( M_1 + M_2 \) is closed.
Proof. We denote by \((z^{i}w^{j})_{M_{1}}\) and \((z^{i}w^{j})_{M_{2}}\) the orthogonal projections of \(z^{i}w^{j}\) to the spaces \(M_{1}\) and \(M_{2}\), respectively. Let
\[
M'_{1} = M_{1} \ominus \left( \left\{(z^{i}w^{j})_{M_{1}}; 0 \leq i \leq n, 0 \leq j \leq m \right\} \right)
\]
and
\[
M'_{2} = M_{2} \ominus \left( \left\{(z^{i}w^{j})_{M_{2}}; 0 \leq i \leq n, 0 \leq j \leq m \right\} \right).
\]
Then \(M'_{1}\) and \(M'_{2}\) are closed subspaces of \(H^{2}\),
\[
M'_{1} \subset z^{n+1} \left( \sum_{j=0}^{m} \oplus w^{j} H^{2}(\Gamma_{z}) \right), \quad M'_{2} \subset w^{m+1} \left( \sum_{i=0}^{n} \oplus z^{i} H^{2}(\Gamma_{w}) \right),
\]
and
\[
M'_{1} \perp M'_{2} \perp \left\{(z^{i}w^{j}; 0 \leq i \leq n, 0 \leq j \leq m \right\}.
\]
Since
\[
z^{n+1} \left( \sum_{j=0}^{m} \oplus w^{j} H^{2}(\Gamma_{z}) \right) \perp w^{m+1} \left( \sum_{i=0}^{n} \oplus z^{i} H^{2}(\Gamma_{w}) \right),
\]
\(M'_{1} + M'_{2} = M'_{1} \oplus M'_{2}\) is closed. Hence
\[
M_{1} + M_{2} = M'_{1} + M'_{2} + \left\{(z^{i}w^{j})_{M_{1}}, (z^{i}w^{j})_{M_{2}}; 0 \leq i \leq n, 0 \leq j \leq m \right\}
\]
is closed. \(\square\)

**Theorem 2.5.** Let \(N\) be a backward shift invariant subspace of \(H^{2}\). Then \(S_{w}^{\ast}S_{z}^{\ast} = 0\) if and only if
\[
\begin{align*}
& (i) \quad N \ominus \left( N \cap \sum_{i=0}^{n-1} \oplus z^{i} H^{2}(\Gamma_{w}) \right) \subset N \cap \sum_{j=0}^{m-1} \oplus w^{j} H^{2}(\Gamma_{z}).
\end{align*}
\]

Proof. Suppose that \(S_{w}^{\ast}S_{z}^{\ast} = 0\). Then \(S_{z}^{\ast}N \subset \sum_{j=0}^{m-1} \oplus w^{j} H^{2}(\Gamma_{z})\). Since \(S_{z}^{\ast}S_{w}^{\ast} = 0\), \(S_{w}^{\ast}N \subset \sum_{i=0}^{n-1} \oplus z^{i} H^{2}(\Gamma_{w})\). First, we prove the following
\[
(2.9) \quad N = \left( N \cap \sum_{j=0}^{m-1} \oplus w^{j} H^{2}(\Gamma_{z}) \right) + \left( N \cap \sum_{i=0}^{n-1} \oplus z^{i} H^{2}(\Gamma_{w}) \right).
\]

Let
\[
(2.10) \quad K = \left( N \cap \sum_{j=0}^{m-1} \oplus w^{j} H^{2}(\Gamma_{z}) \right) + \left( N \cap \sum_{i=0}^{n-1} \oplus z^{i} H^{2}(\Gamma_{w}) \right).
\]

By Lemma 2.4, \(K\) is closed and \(N = K \oplus (N \ominus K)\). To prove (i), it is sufficient to prove \(N \ominus K = \{0\}\). We have \(N \ominus K \subset N \cap (S_{z}^{\ast}N)^{\perp} \cap (S_{w}^{\ast}N)^{\perp}\). Let \(f \in N \ominus K\). Then \(f \perp S_{z}^{\ast}N\), so that \(S_{z}^{\ast}f \perp N\). Since
\[ S_{n}^* f \in N, \quad S_{n}^* f = 0. \] Hence \( f \in \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_w) \). By (2.10), \( f \in K \).

This shows \( f = 0 \), so that \( N \cap K = \{0\} \). Thus we get (2.9).

Let

\[
(2.11) \quad N_1 = \left\{ f \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_w); \ f \perp N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right\}.
\]

Then \( N_1 \) is a closed subspace and

\[
N_1 \oplus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \subset N.
\]

If the equality holds in the above, (i) holds. So we assume that

\[
N_1 \oplus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \subsetneq N.
\]

We shall lead a contradiction. Let

\[
(2.12) \quad N_2 = N \ominus \left( N_1 \oplus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right).
\]

Then \( N_2 \neq \{0\} \) and \( N = N_1 \oplus N_2 \oplus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \).

Let \( g \in N_2 \) be such that \( g \neq 0 \). We shall prove that

\[
(2.13) \quad g \notin N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \quad \text{and} \quad g \notin N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z).
\]

The fact \( g \notin N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \) is trivial. Suppose that \( g \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \). By (2.11), \( g \in N_1 \). Since \( g \in N_2 \), by (2.12) we have \( g \perp N_1 \). Hence \( g = 0 \). This is a contradiction. Thus we get (2.13).

Next, we shall prove that

\[
(2.14) \quad S_{n}^* g \not\perp N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z).
\]

To prove this, suppose not. Then \( S_{n}^* g \perp N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \).

By (2.10), \( g = g_1 + g_2 \), where \( g_1 \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \) and \( g_2 \in N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \). Then \( S_{n}^* g = S_{n}^* g_1 \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \).

Therefore \( S_{n}^* g = 0 \), so that \( g \in N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \). This contradicts (2.13). Thus we get (2.14).

By (2.14), there exists \( h_0 \) such that \( h_0 \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \) and \( \langle g, S_{n}^* h_0 \rangle = \langle S_{n}^* g, h_0 \rangle \neq 0 \). Since \( S_{n}^* h_0 \in N \), by (2.12) we have \( S_{n}^* h_0 = h_1 \oplus h_2 \oplus h_3 \), where \( h_1 \in N_1, h_2 \in N_2, \) and \( h_3 \in N_3 \).
Then let Theorem 2.6.

N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w). Since g \in N_2 and \langle g, S_n h_0 \rangle \neq 0, we have h_2 \neq 0. Since z^n h_0 \perp \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w),

P_N z^n h_0 = S_n h_0 \perp N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w).

Thus we get h_3 = 0. By (2.12), S_{w,m} N_1 = \{0\}. Hence S_{w,m} S_n h_0 = S_{w,m} h_2. By (2.13) and h_2 \in N_2, h_2 \notin \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z). This implies that S_{w,m} h_2 \neq 0. Hence S_{w,m} S_n \neq 0. This contradicts our starting assumption. Thus we have (i).

To prove the converse, suppose that condition (i) holds. Then N = \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \oplus L, where L \subset N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z). Let F = F_1 + F_2 \in N, where F_1 \in N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) and F_2 \in L. Since z^n F \perp N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w), S_n F \in L. Hence S_{w,m} S_n F = 0. Thus we get S_{w,m} S_n = 0.

By Theorem 2.2, the structure of backward shift invariant subspaces N satisfying S_n S_{w,m} = 0 is simple. By Theorem 2.5, the structure of backward shift invariant subspaces N satisfying S_{w,m} S_n = 0 is not so simple. When n = m = 1, we have the following.

**Theorem 2.6.** Let N be a backward shift invariant subspace of H^2. Then S_w S_z = 0 if and only if N has one of the following forms;

(i) N = H^2(\Gamma_z) \ominus q(z) H^2(\Gamma_z) for some inner function q(z).

(ii) N = H^2(\Gamma_w) \ominus q(w) H^2(\Gamma_w) for some inner function q(w).

(iii) Either N = H^2(\Gamma_z) + H^2(\Gamma_w), or N = H^2(\Gamma_z) \ominus q_2(w) H^2(\Gamma_w), or N = (H^2(\Gamma_z) \ominus q_1(z) H^2(\Gamma_z)) + H^2(\Gamma_w), where q_1(z) and q_2(w) are inner functions.

(iv) N = (H^2(\Gamma_z) \ominus q_1(z) H^2(\Gamma_z)) + (H^2(\Gamma_w) \oplus q_2(w) H^2(\Gamma_w)), where q_1(z), q_2(w) are nonconstant inner functions and \hat{q}_1(0) \hat{q}_2(0) = 0.

In (iii) and (iv), since 1 \in N, we may take q_1 and q_2 as \hat{q}_1(0) = \hat{q}_2(0) = 0.

**Proof.** By Theorem 2.2, S_z S_{w} = 0 if and only if either (i) or (ii) holds. By Theorem 2.5, S_{w,m} S_n = 0 if and only if

(2.15) \quad N \ominus (N \cap H^2(\Gamma_w)) \subset N \cap H^2(\Gamma_z).

If either (i) or (ii) holds, by Corollary 2.3 we have S_{w,m} S_n = 0. Suppose that N satisfies either (iii) or (iv). Then clearly 1 \in N. Since N has a special form, it is not difficult to see that

N \ominus (N \cap H^2(\Gamma_w)) = \{ f \in N \cap H^2(\Gamma_z); \hat{f}(0) = 0 \}.

Hence (2.15) holds.
Next, suppose that (2.15) holds. Then we have
\[ N = (N \cap H^2(\Gamma_z)) + (N \cap H^2(\Gamma_w)). \]
If either \( N \cap H^2(\Gamma_z) = \{0\} \) or \( N \cap H^2(\Gamma_w) = \{0\} \), then \( S_z S_w^* = 0 \), and by Corollary 2.3, \( S_z^* S_w = 0 \). Hence either (i) or (ii) holds. Suppose that \( N \cap H^2(\Gamma_z) \neq \{0\} \) and \( N \cap H^2(\Gamma_w) \neq \{0\} \). We shall prove \( 1 \in N \). To prove this, suppose that \( 1 \notin N \). Let \( 1_w \) be the orthogonal projection of 1 to \( N \cap H^2(\Gamma_w) \). Then \( 1_w \notin H^2(\Gamma_z) \). Since \( N \cap H^2(\Gamma_z) \neq \{0\} \), there exists \( f \in N \cap H^2(\Gamma_z) \) such that \( \hat{f}(0) \neq 0 \). Let \( f_1 = f - \hat{f}(0)1_w \in N \).

Then \( f_1 \notin H^2(\Gamma_z) \). Let \( h \in N \cap H^2(\Gamma_w) \). Since \( f \in H^2(\Gamma_z) \), \( f - \hat{f}(0) \perp h \). Since \( 1 - 1_w \perp N \cap H^2(\Gamma_w) \),
\[ \langle f_1, h \rangle = \langle \hat{f}(0)(1 - 1_w) + f - \hat{f}(0), h \rangle = 0. \]
Hence \( f_1 \in N \cap H^2(\Gamma_w) \). Thus (2.15) does not hold. Therefore \( 1 \in N \). Since \( N \cap H^2(\Gamma_z) \) and \( N \cap H^2(\Gamma_w) \) are nonzero backward shift invariant subspaces, by (2.16) \( N \) has one of forms in (iii) and (iv). \( \square \)

3. \( S_z S_w^* = S_w^* S_z^* \).

The following is the main theorem in this section.

**Theorem 3.1.** Let \( N \) be a backward shift invariant subspace of \( H^2 \), \( N \neq \{0\} \), and \( N \neq H^2 \). Let \( M = H^2 \ominus N \) and \( n \geq 2 \) be a positive integer. If \( S_z S_w^* = S_w^* S_z^* \), then one of the following conditions holds;

(i) \( S_z S_w^* = S_w^* S_z^* \),

(ii) \( M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z) \) for an inner function \( q_1(z) \) satisfying \( q_1(z) = \prod_{j=1}^{k} b_j(z), 1 \leq k \leq n, \) where \( b_i \) are simple Blaschke products.

The following lemma is a generalization of [IN], and the idea of the proof is the same.

**Lemma 3.2.** Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \ominus N \). Let \( n \) be a positive integer. Then \( S_z S_w^* = S_w^* S_z^* \) if and only if
\[ M \ominus \left( z^n M \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right) \subset (M \cap H^2(\Gamma_z)) \oplus w M. \]

**Proof.** The operators \( T_z^n \) and \( T_w^* \) on \( H^2 \) have the matrix forms as
\[
T_z^n = \begin{pmatrix}
* & P_M T_z^n |_N \\
0 & S_z^n
\end{pmatrix},
T_w^* = \begin{pmatrix}
* & P_N T_w^* |_M \\
0 & S_w^*
\end{pmatrix}
\]
on \( H^2 = \begin{pmatrix}
M \\
N
\end{pmatrix}. \]
Set $A = P_M T_{z^n}|_N$ and $B = P_N T_w^*|_M$. Since $T_{z^n} T_w^* = T_w^* T_{z^n}$ on $H^2$, $S_{z^n} S_w^* = S_w^* S_{z^n}$ if and only if $BA = 0$. We have $T_w^*(M \ominus wM) \subset N$. For $f \in H^2$, $T_w^* f = 0$ if and only if $f \in H^2(\Gamma_z)$. Hence
\[
\ker B = \{ f \in M; T_w^* f \in M \} = \{ f \in M \ominus wM; T_w^* f = 0 \} \oplus wM = (M \cap H^2(\Gamma_z)) \oplus wM.
\]
We denote by $[\text{ran } A]$ the closed range of $A$. Let $A_1 = P_M T_{z^n} P_N$ on $H^2$. Then $[\text{ran } A] = [\text{ran } A_1]$. Since $A_1^* = P_N T_w^* P_M$, we get
\[
\ker A_1^* = \{ f \in M; T_w^* f \in M \} \oplus N = \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \oplus z^n M \oplus N.
\]
Hence
\[
[\text{ran } A] = [\text{ran } A_1] = H^2 \ominus \ker A_1^* = M \ominus \left( z^n M \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right).
\]
Since $BA = 0$ if and only if $[\text{ran } A] \subset \ker B$, we get our assertion. □

**Proof of Theorem 3.1.** Suppose that $S_{z^n} S_w^* = S_w^* S_{z^n}$. By Lemma 3.2,
\[
M \ominus \left( z^n M \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right) \subset (M \cap H^2(\Gamma_z)) \oplus wM.
\]
Let
\[
K_0 = M \ominus \left( z^n M \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right).
\]
Then
\[
(3.1) \quad K_0 \subset (M \cap H^2(\Gamma_z)) \oplus wM
\]
and
\[
(3.2) \quad M \ominus z^n M = K_0 \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right).
\]
Since $M = \sum_{s=0}^{\infty} \oplus z^{ns}(M \ominus z^n M)$,
\[
(3.3) \quad K_0 \perp \sum_{s=0}^{\infty} \oplus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right)
\]
and
\[
(3.4) \quad M = \left( \sum_{s=0}^{\infty} \oplus z^{ns} K_0 \right) \oplus \left( \sum_{s=0}^{\infty} \oplus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right).
\]
First, suppose that $K_0 = \{0\}$. In this case, we shall show that condition (i) holds. By (3.2),
\[
\sum_{i=0}^{n-1} \oplus z^i (M \ominus z^i M) = M \ominus z^n M = M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w).
\]

Let $f \in M \ominus zM$. Then $f \in M \ominus z^n M$ and $f = \sum_{i=0}^{n-1} \oplus z^i h_i(w)$, where $h_i(w) \in H^2(\Gamma_w)$. Since $f \in M \ominus zM$,
\[
z^{n-1} f = \sum_{i=0}^{n-1} \oplus z^{n-1+i} h_i(w) \in M \ominus z^n M.
\]

Hence $f = h_0(w)$, so that $M \ominus zM = M \cap H^2(\Gamma_w)$. By Lemma 3.2 (for the case of $n = 1$), we get $S_z S^*_w = S^*_w S_z$.

Next, suppose that $K_0 \neq \{0\}$. The proof of this case is a little bit long, so we divide into several steps.

**Step 1.** In this step, we shall prove that there is a nonconstant inner function $q_1(z)$ such that
\[
K_0 \subset q_1(z) H^2(\Gamma_z) \oplus wM.
\]

First we prove that $K_0 \not\subset wM$. To prove this, suppose that $K_0 \subset wM$. Let $F \in K_0$. Then $F = wF$ for some $F \in M$. We shall prove that $f \in K_0$. We have
\[
\langle f, \left( \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \right) \oplus \left( \sum_{s=0}^{\infty} \oplus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right) \rangle
\]
\[
= \langle wF, w \left( \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \oplus \left( \sum_{s=0}^{\infty} \oplus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right) \right) \rangle
\]
\[
= \langle F, z^n w \left( \sum_{s=1}^{\infty} \oplus z^{n(s-1)} K_0 \right) \rangle \quad \text{by (3.3)}
\]
\[
= 0,
\]
where the last equality follows from $w \sum_{s=1}^{\infty} \oplus z^{n(s-1)} K_0 \subset M$ and (3.2).

Hence by (3.4), we get $f \in K_0$. Therefore for every positive integer $p$, we have $F = w^p f_p$ for some $f_p \in K_0$. This leads $F = 0$. This is a contradiction. Thus we get $K_0 \not\subset wM$.

Hence by (3.1), $M \cap H^2(\Gamma_z) \neq \{0\}$. By the Beurling theorem,\n\[
M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z),
\]
where $q_1(z)$ is inner. Then $q_1(z) \in M$ and
\[
q_1(z) H^2(\Gamma_z) \perp wM.
\]
If \(q_1(z)\) is constant, we have \(M = H^2\), so that \(N = \{0\}\). This contradicts our assumption. Hence \(q_1(z)\) is a nonconstant inner function. By (3.1) and (3.6), we get (3.5).

**Step 2.** In this step, we prove

\[
(3.8) \quad K_0 \subset q_1(z) \left( \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right).
\]

Let \(G \in K_0\). Then by (3.5), \(G = q_1(z)h(z) \oplus wg\), where \(h(z) \in H^2(\Gamma_z)\) and \(g \in M\). Write

\[
h(z) = \left( \sum_{i=0}^{n-1} \oplus a_iz^i \right) \ominus z^n h_0(z).
\]

Then

\[
G = q_1(z) \left( \sum_{i=0}^{n-1} \oplus a_iz^i \right) \ominus q_1(z)z^n h_0(z) \oplus wg.
\]

By (3.6), \(q_1(z)z^n h_0(z) \in z^n M\). Since \(G \in K_0 \subset M \ominus z^n M\), we have \(h_0(z) = 0\). Hence

\[
(3.9) \quad G = q_1(z) \left( \sum_{i=0}^{n-1} \oplus a_iz^i \right) \ominus wg.
\]

Here we prove that

\[
(3.10) \quad g \in K_0.
\]

Since \(G = q_1(z)h(z) \oplus wg\), we have

\[
\langle g, \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rangle = \langle wg, w \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rangle = \langle wg, w \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rangle + \langle q_1h, w \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rangle \text{ by (3.7)}
\]

\[
= \langle G, w \sum_{s=1}^{\infty} \oplus z^{ns} K_0 \rangle = \langle G, z^n w \sum_{s=1}^{\infty} \oplus z^{n(s-1)} K_0 \rangle = 0 \text{ by (3.2)}.
\]
We also have
\[
\left\langle g, \sum_{s=0}^{\infty} \oplus z^{ns}\left(M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\right)\right\rangle
= \left\langle wg, w\left(\sum_{s=0}^{\infty} \oplus z^{ns}\left(M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\right)\right)\right\rangle
= \left\langle G, \sum_{s=0}^{\infty} \oplus z^{ns}w\left(M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\right)\right\rangle \quad \text{by (3.7)}
= 0 \quad \text{by (3.3)}.
\]
Hence by (3.4), we get (3.10).

Applying (3.9) and (3.10) infinitely many times, we have
\[
G \quad = \quad q_1(z)\left(\sum_{i=0}^{n-1} \oplus a_i z^i\right) + wq_1(z)\left(\sum_{i=0}^{n-1} \oplus b_i z^i\right) + \cdots
\quad \in \quad q_1(z)\left(\sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\right).
\]
Therefore we get (3.8).

Step 3. In this step, we study functions in \(M \ominus z^nM\) and the inner function \(q_1(z)\). By (3.8), there is a closed subspace \(L\) such that \(K_0 = q_1(z)\) with \(L \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\). Then by (3.2),
\[
M \ominus z^nM = q_1(z)\left(\sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\right).
\]
Since \(K_0 \neq \{0\}\), \(L \neq \{0\}\). We have \(M \ominus z^nM = \sum_{i=0}^{n-1} \oplus z^i(M \ominus zM)\). Hence \(M \ominus zM \neq \{0\}\). Let \(F \in M \ominus zM\) be such that \(F \neq 0\). Then
\[
F = \left(q_1 \sum_{i=0}^{n-1} \oplus z^i f_i\right) \oplus \sum_{i=0}^{n-1} \oplus z^i g_i,
\]
where \(f_i, g_i \in H^2(\Gamma_w)\).
\[
q_1 \sum_{i=0}^{n-1} \oplus z^i f_i \perp M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w),
\]
and
\[
\sum_{i=0}^{n-1} \oplus z^i g_i \in M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w).
\]
Since \( n \geq 2 \), \( zF \in M \ominus z^n M \), so that we have
\[
zF = q_1 \left( \sum_{i=0}^{n-1} \oplus z^i G_{1,i} \right) + \sum_{i=0}^{n-1} \oplus z^i H_{1,i},
\]
where \( G_{1,i}, H_{1,i} \in H^2(\Gamma_w) \). Hence (3.14)
\[
q_1 \left( z \left( \sum_{i=0}^{n-1} \oplus z^i f_i \right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i} \right) = \left( \sum_{i=0}^{n-1} \oplus z^i H_{1,i} \right) - z \sum_{i=0}^{n-1} \oplus z^i g_i.
\]

Here we devide into two subcases.

**Subcase 1.**
\[
z \left( \sum_{i=0}^{n-1} \oplus z^i f_i(w) \right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w) \neq 0.
\]
Then
\[
q_1(z) = \left( \sum_{i=0}^{n-1} \oplus z^i H_{1,i}(w) \right) - z \sum_{i=0}^{n-1} \oplus z^i g_i(w)
\]
\[
\frac{z \left( \sum_{i=0}^{n-1} \oplus z^i f_i(w) \right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w)}{z \left( \sum_{i=0}^{n-1} \oplus z^i f_i(w) \right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w)}.
\]
As proved in Step 1, \( q_1(z) \) is a nonconstant inner function. Then by the above, we have
\[
q_1(z) = \prod_{j=1}^{k} b_j(z), \quad 1 \leq k \leq n,
\]
where \( b_j \) are simple Blaschke products.

**Subcase 2.**
\[
z \left( \sum_{i=0}^{n-1} \oplus z^i f_i(w) \right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w) = 0.
\]
Then by (3.14), \( f_{n-1} = g_1 = 0 \), so that by (3.11)
\[
F = \left( q_1 \sum_{i=0}^{n-2} \oplus z^i f_i \right) \oplus \sum_{i=0}^{n-2} \oplus z^i g_i.
\]
Since \( F \in M \ominus z M \),
\[
zF = \left( q_1 \sum_{i=0}^{n-2} \oplus z^{i+1} f_i \right) \oplus \sum_{i=0}^{n-2} \oplus z^{i+1} g_i \in M \ominus z^2 M.
\]

In the same way as above, either (3.15) holds or \( f_{n-2} = g_{n-2} = 0 \). Repeat the same argument. Then either (3.15) holds or
\[
f_{n-1} = f_{n-2} = \cdots = f_1 = g_{n-1} = g_{n-2} = \cdots = g_1 = 0.
\]
Step 4. If (3.15) holds, then (ii) holds. So, suppose that (3.15) does not hold. Then by (3.11) and by the above fact, \( F = q_1f_0 \perp g_0 \), by (3.12) \( q_1f_0 \perp M \cap H^2(\Gamma_w) \), and by (3.13) \( g_0 \in M \cap H^2(\Gamma_w) \) for every \( F \in M \ominus zM \).

If \( g_0 = 0 \) for every \( F \in M \ominus zM \), since \( q_1(z) \in M \) it follows that \( M \ominus zM \subset q_1(z)H^2(\Gamma_w) \subset M \). Since \( M = \sum_{z=0}^{\infty} \ominus z^j(M \ominus zM) \), we have \( M = q_1(z)H^2 \), so that \( N = H^2 \ominus q_1(z)H^2 \). By Theorem A, condition (i) holds.

So we assume that \( g_0 \neq 0 \) for some \( F \in M \ominus zM \). We shall prove that
\[
(3.16) \quad (M \ominus zM) \ominus (M \cap H^2(\Gamma_w)) \subset (M \cap H^2(\Gamma_z)) \ominus wM.
\]

We may assume that \( (M \ominus zM) \ominus (M \cap H^2(\Gamma_w)) \neq \{0\} \). Let \( G \in (M \ominus zM) \ominus (M \cap H^2(\Gamma_w)) \) be such that \( G \neq 0 \). Then \( G = q_1(z)h_1(w) \ominus h_2(w) \), where \( q_1(z)h_1(w) \perp M \cap H^2(\Gamma_w) \) and \( h_2(w) \in M \cap H^2(\Gamma_w) \). Hence \( h_2(w) = G - q_1(z)h_1(w) \perp M \cap H^2(\Gamma_w) \). Therefore \( h_2(w) = 0 \). Since \( q_1(z) \in M \),
\[
G = q_1(z)h_1(w) = h_1(0)q_1(z) + wq_1(z)\frac{h_1(w) - h_1(0)}{w} \in M \cap H^2(\Gamma_z) \ominus wM.
\]

Thus we get (3.16). By Lemma 3.2 (for \( n = 1 \)), (i) holds.

4. \( S_zS_n^* = S_n^*S_z \) and \( S_zS_n^* \neq S_n^*S_z \).

Let \( N \) be a backward shift invariant subspace of \( H^2 \) and let \( n \) be a positive integer. Let \( M = H^2 \ominus N \). Then \( M \) is an invariant subspace. If both \( S_nS_w^* = S_w^*S_n \) and \( S_zS_n^* \neq S_n^*S_z \) hold, then by Theorem 3.1, \( M \cap H^2(\Gamma_z) = q(z)H^2(\Gamma_z) \) for some nonconstant inner function \( q(z) \).

In this section, we assume that \( q_1(z)H^2 \subset M \) and \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \) for some nonconstant inner function \( q_1(z) \). Let
\[
\tilde{M} = M \ominus q_1(z)H^2 \subset M.
\]
Then \( H^2 \ominus q_1(z)H^2 = \tilde{M} \ominus N \) and \( \tilde{M} \) is \( w \)-invariant. The following lemma is proved in [INS, Lemma 3.2].

**Lemma 4.1.** Let \( f \in \tilde{M} \). Then \( T_n^*f \in \tilde{M} \) if and only if \( f \in w\tilde{M} \).

We denote by \( P_\perp \) the orthogonal projection from \( H^2 \) onto \( H^2 \ominus q_1(z)H^2 \). Then we have a Toeplitz type operator \( Q_{z^n} \) on \( H^2 \ominus q_1(z)H^2 \) such that
\[
Q_{z^n} : H^2 \ominus q_1(z)H^2 \ni f \to P_\perp(T_n^*f) \in H^2 \ominus q_1(z)H^2.
\]
Since \( z^n M \subseteq M \), \( Q_z M \subseteq \tilde{M} \) and \( Q_z^n = Q_z \). Then \( Q_z \) has the following matrix form:

\[
Q_z = \begin{pmatrix}
* & P_M T_z |_N \\
0 & S_z^n
\end{pmatrix}
\quad \text{on} \quad H^2 \ominus q_1(z)H^2 = \begin{pmatrix} \tilde{M} \\ N \end{pmatrix}.
\]

Since \( H^2 \ominus q_1(z)H^2 \) is backward shift invariant, \( T_z^* (H^2 \ominus q_1(z)H^2) \subseteq H^2 \ominus q_1(z)H^2 \). Since \( T_z^* N \subseteq N \), the operator \( T_z^* \) on \( H^2 \ominus q_1(z)H^2 \) has the following matrix form:

\[
T_z^* = \begin{pmatrix}
* & 0 \\
P_N T_z^* |_{\tilde{M}} & S_z^*
\end{pmatrix}
\quad \text{on} \quad H^2 \ominus q_1(z)H^2 = \begin{pmatrix} \tilde{M} \\ N \end{pmatrix}.
\]

Set

(4.1) \( A = P_N T_z |_{\tilde{M}} \) and \( B = P_N T_z^* |_{\tilde{M}} \).

By [INS, Lemma 3.3], \( T_z^* Q_z = Q_z T_z^* \) on \( H^2 \ominus q_1(z)H^2 \). Hence we have the following.

**Lemma 4.2.** \( T_z^* Q_z = Q_z T_z^* \) on \( H^2 \ominus q_1(z)H^2 \).

**Lemma 4.3.** \( S_z S_z^* = S_z^* S_z \) if and only if \( BA = 0 \).

**Proof.** By Lemma 4.2, \( T_z^* Q_z = Q_z T_z^* \) on \( H^2 \ominus q_1(z)H^2 \). Then \( BA + S_z S_z^* = S_z^* S_z \). Hence \( S_z S_z^* = S_z^* S_z \) if and only if \( BA = 0 \).

The following is a slight generalization of [INS, Theorem 3.5].

**Theorem 4.4.** Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \ominus N \). Suppose that \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \), where \( q_1(z) \) is a nonconstant inner function. Let \( \hat{M} = M \ominus q_1(z)H^2 \). Then the following conditions are equivalent.

(i) \( S_z S_z^* = S_z^* S_z \) on \( N \).

(ii) \( \hat{M} \ominus \{ f \in \hat{M} \mid T_z f \in \hat{M} \} \subseteq w\hat{M} \).

(iii) \( T_z^* \hat{M} \subseteq \hat{M} \).

**Proof.** The idea of the proof is similar to the one of [INS, Theorem 3.5]. For the sake of completeness, we give the proof.

(i) \( \iff \) (ii) By Lemma 4.3, condition (i) is equivalent to \( BA = 0 \). By (4.1) and Lemma 4.1, \( \ker B = \{ f \in \hat{M} \mid T_z f \in \hat{M} \} = w\hat{M} \). Put \( A_1 = P_M T_z P_N \) on \( \hat{M} \ominus N \). Then \( \{ \text{ran} A \} = \{ \text{ran} A_1 \} \). Since \( A_1^* = P_N T_z^* P_M \), \( \ker A_1^* = N \oplus \{ f \in \hat{M} \mid T_z f \in \hat{M} \} \). Hence

\[
\{ \text{ran} A \} = \{ \text{ran} A_1 \} = (\hat{M} \ominus N) \ominus \ker A_1^* = \hat{M} \ominus \{ f \in \hat{M} \mid T_z f \in \hat{M} \}.
\]

Therefore \( BA = 0 \) if and only if \( \hat{M} \ominus \{ f \in \hat{M} \mid T_z f \in \hat{M} \} \subseteq w\hat{M} \). Thus we get (i) \( \iff \) (ii).
(ii) \( \Rightarrow \) (iii) Suppose that \( \tilde{M} \oplus \{ f \in \tilde{M}; T^n_z f \in \tilde{M} \} \subset w\tilde{M} \). Since \( \{ f \in \tilde{M}; T^n_z f \in \tilde{M} \} \) is closed, \( \tilde{M} \oplus w\tilde{M} \subset \{ f \in \tilde{M}; T^n_z f \in \tilde{M} \} \). Since \( w\tilde{M} \subset M \), \( M = \bigoplus_{j=0}^{\infty} w^j (\tilde{M} \oplus w\tilde{M}) \). Since \( T^n_z w^j f = w^j T^n_z f \) for \( f \in H^2 \), we have \( T^n_z \tilde{M} \subset \tilde{M} \).

(iii) \( \Rightarrow \) (ii) is trivial. \( \square \)

For \( f \in H^2(\Gamma_z) \), write \( f^*(z) = T^n_z f(z) = \overline{\varphi} (f(z) - \hat{f}(0)) \).

**Lemma 4.5.** Let \( b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha_i} z), |\alpha_i| < 1, \) and \( 1 \leq i \leq n \). Then

(i) \( T^n_z z = 1, T^n_z b_1 = b_1^*(z) = (1 - |\alpha_1|^2)/(1 - \overline{\alpha_1} z), \) and \( T^n_z b_1^*(z) = \overline{\alpha_1} b_1^*(z) \).

(ii) \( T^n_z (b_1(z) b_2^*(z)) = (1 - |\alpha_2|^2) b_1^*(z) + \overline{\alpha_2} b_1(z) b_2^*(z) \).

(iii) \( H^2(\Gamma_z) \ominus (\bigoplus_{j=1}^k b_j(z)) H^2(\Gamma_z) = \bigoplus_{j=1}^k [b_1(z) \cdot \cdots \cdot b_{j-1}(z) b_j^*(z)] \).

(iv) \( H^2 \ominus (\bigoplus_{j=1}^k b_j(z)) H^2 = \bigoplus_{j=1}^k [b_1(z) \cdot \cdots \cdot b_{j-1}(z) b_j^*(z)] H^2(\Gamma_w) \).

**Proof.** It is not difficult to prove (i).

(ii) Since

\[
\overline{\alpha_1} b_1(z) b_2^*(z) = \overline{\alpha_1} b_1(z) \frac{1 - |\alpha_2|^2}{1 - \overline{\alpha_2} z}
= (1 - |\alpha_2|^2) b_1(z) \left( \overline{\alpha_2} \frac{1}{1 - \overline{\alpha_2} z} \right)
= (1 - |\alpha_2|^2) \overline{\alpha_2} b_1(z) b_2^*(z),
\]

we get (ii).

(iii) See [Ni, p.33].

(iv) follows from (iii). \( \square \)

**Corollary 4.6.** Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \ominus N \). Suppose that \( M \cap H^2(\Gamma_z) = b_1(z) H^2(\Gamma_z) \), where \( b_1(z) \) is a simple Blaschke product. Then \( S_z S^*_w = S^*_w S_z \).

**Proof.** Let \( b_1(z) = (z - \alpha)/(1 - \overline{\alpha} z), |\alpha| < 1, \) and \( \tilde{M} = M \ominus b_1(z) H^2 \).

Since \( b_1(z) \in M \), \( b_1(z) H^2 \subset M \). By Lemma 4.5(iv), \( \tilde{M} \subset b_1(z) H^2(\Gamma_w) \).

By Lemma 4.5(i), \( T^n_z (b_1^*(z) h(w)) = \overline{\alpha_1} b_1^*(z) h(w) \). Hence \( T^n_z \tilde{M} \subset \tilde{M} \). By Theorem 4.4, \( S_z S^*_w = S^*_w S_z \).

**Corollary 4.7.** Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \ominus N \). Suppose that \( M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z) \), where \( q_1(z) \) is an inner function. Let \( n, k \) be positive integers with \( n \geq k + 1 \). Moreover suppose that \( q_1(z) = z^n b(z) \), where \( b(z) \) is a simple Blaschke product, \( b(z) = (z - \alpha)/(1 - \overline{\alpha} z), \) and \( \alpha \neq 0 \). If \( S_z S^*_w = S^*_w S_z^n \), then \( S_z S^*_w = S^*_w S_z^k \).
Proof. Let \( \tilde{M} = M \ominus q_1(z)H^2 \). If \( \tilde{M} = \{0\} \), then \( M = q_1(z)H^2 \). By Theorem A, \( S_zS_w^* = S_w^*S_z \). Suppose that \( \tilde{M} \neq \{0\} \). By Lemma 4.5(iv),

\[
\tilde{M} \subset b^*(z)H^2(\Gamma_w) \oplus \left( \sum_{j=1}^{k} \oplus z^{j-1}b(z)H^2(\Gamma_w) \right).
\]

Let \( f \in \tilde{M} \). Then

\[
f = b^*(z)h_0(w) + \sum_{j=1}^{k} \oplus z^{j-1}b(z)h_j(w), \quad h_j(w) \in H^2(\Gamma_w).
\]

By Lemma 4.5(i),

\[
T_{z^n}^*f = T_{z^n}^*(-k)(T_{z^n}^*f) = T_{z^n}^* \left( \sum_{j=0}^{k} \alpha^{(k-j)}b^*(z)h_j(w) \right) = \alpha^{(n-k)} \sum_{j=0}^{k} \alpha^{(k-j)}b^*(z)h_j(w).
\]

Since \( S_zS_w^* = S_w^*S_z \), by Theorem 4.4 \( T_{z^n}^*f \in \tilde{M} \). Since \( \alpha \neq 0 \),

\[
\sum_{j=0}^{k} \alpha^{(k-j)}b^*(z)h_j(w) \in \tilde{M}.
\]

Thus \( T_{z^n}^*\tilde{M} \subset \tilde{M} \). By Theorem 4.4, \( S_zS_w^* = S_w^*S_z \).

**Theorem 4.8.** Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \ominus N \). Suppose that \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \), where \( q_1(z) \) is an inner function. Moreover suppose that \( q_1(z) = b_1(z)b_2(z) \), where \( b_i(z) \), \( i = 1, 2 \), are simple Blaschke products, \( b_1(z) = (z - \alpha_1)/(1 - \overline{\alpha}_1z) \), and \( \alpha_1\alpha_2 \neq 0 \). Let \( n \geq 2 \) be a positive integer. Then we have the following.

(i) If \( S_zS_w^* = S_w^*S_z \) and \( S_{z^n-1}S_w^* \neq S_w^*S_{z^n-1} \), then \( \alpha_1 = \alpha_2^* \) and \( \alpha_1 \neq \alpha_2 \).

(ii) If \( \alpha_1^* = \alpha_2^* \) and \( \alpha_1 \neq \alpha_2 \), then \( S_zS_w^* = S_w^*S_z \).

**Proof.** Let \( \tilde{M} = M \ominus q_1(z)H^2 \). Suppose that \( S_zS_w^* = S_w^*S_z \) and \( S_{z^n-1}S_w^* \neq S_w^*S_{z^n-1} \). By Theorem 4.4, \( T_{z^n}^*\tilde{M} \subset \tilde{M} \) and \( T_{z^n}^*\tilde{M} \not\subset \tilde{M} \).

By Lemma 4.5(iv),

\[
\tilde{M} \subset b^*_1(z)H^2(\Gamma_w) \oplus b_1(z)b_2^*(z)H^2(\Gamma_w).
\]

Then there exists \( f_0 \in \tilde{M} \) such that \( T_{z^n}^*f_0 \notin \tilde{M} \), and

\[
f_0 = b^*_1(z)g_1(w) + b_1(z)b_2^*(z)g_2(w) \in b^*_1(z)H^2(\Gamma_w) \oplus b_1(z)b_2^*(z)H^2(\Gamma_w).
\]
By Lemma 4.5,

\[ T_{z_{n-1}}^* f_0 = b_1^*(\alpha_1^{(n-1)} g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-2} \alpha_1^{(n-2-j)} \bar{\alpha}_2^j \right) g_2) + \bar{\alpha}_2^{(n-1)} b_1^* b_2^* g_2 \]

and

\[ T_{z_n}^* f_0 = b_1^*(\bar{\alpha}_1^n g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-1} \bar{\alpha}_1^{(n-1-j)} \bar{\alpha}_2^j \right) g_2) + \bar{\alpha}_2^n b_1^* b_2^* g_2. \]

Since \( T_{z_{n-1}}^* f_0 \notin \tilde{M} \) and \( f_0 \in \tilde{M}, T_{z_{n-1}}^* f_0 - \bar{\alpha}_2^{n-1} f_0 \notin \tilde{M} \). Then

\[ b_1^* \left( (\alpha_1^{(n-1)} - \bar{\alpha}_2^{(n-1)}) g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-2} \alpha_1^{(n-2-j)} \bar{\alpha}_2^j \right) g_2 \right) \notin \tilde{M}. \]

Hence

\[ \left( \sum_{j=0}^{n-2} \alpha_1^{(n-2-j)} \bar{\alpha}_2^j \right) b_1^* \left( (\alpha_1 - \bar{\alpha}_2) g_1 + (1 - |\alpha_2|^2) g_2 \right) \notin \tilde{M}. \]

Since \( 0 \in \tilde{M}, \sum_{j=0}^{n-2} \alpha_1^{(n-2-j)} \bar{\alpha}_2^j \neq 0 \), so that

\[ (4.3) \quad b_1^* \left( (\alpha_1 - \bar{\alpha}_2) g_1 + (1 - |\alpha_2|^2) g_2 \right) \notin \tilde{M}. \]

Since \( T_{z_n}^* f_0 \in \tilde{M}, T_{z_n}^* f_0 - \bar{\alpha}_2^n f_0 \in \tilde{M} \). Then

\[ b_1^* \left( (\alpha_1^n - \bar{\alpha}_2^n) g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \bar{\alpha}_2^j \right) g_2 \right) \in \tilde{M}. \]

Hence

\[ (4.4) \quad \left( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \bar{\alpha}_2^j \right) b_1^* \left( (\alpha_1 - \bar{\alpha}_2) g_1 + (1 - |\alpha_2|^2) g_2 \right) \in \tilde{M}. \]

Now we prove (i). Suppose that \( \alpha_1 = \alpha_2 \). Then \( \alpha_1 = \alpha_2 \neq 0 \), so that \( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \bar{\alpha}_2^j \neq 0 \). By (4.3) and (4.4), \( b_1^* g_2 \notin \tilde{M} \) and \( b_1^* g_2 \in \tilde{M} \). This is a contradiction. Hence \( \alpha_1 \neq \alpha_2 \).

Suppose that \( \alpha_1^n \neq \alpha_2^n \). Then \( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \bar{\alpha}_2^j \neq 0 \). By (4.4),

\[ b_1^* \left( (\alpha_1 - \bar{\alpha}_2) g_1 + (1 - |\alpha_2|^2) g_2 \right) \in \tilde{M}. \]

This contradicts (4.3), so that \( \alpha_1^n = \alpha_2^n \). Thus we get (i).

(ii) Suppose that \( \alpha_1^n = \alpha_2^n \) and \( \alpha_1 \neq \alpha_2 \). Then \( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \bar{\alpha}_2^j = 0 \).

Let \( f \in \tilde{M} \). Then by (4.2), \( f = b_1^* (z) h_1(w) + b_1(z) b_2^* (z) h_2(w) \). Similarly,
we have
\[ T_n^* f - \alpha_n f = \left( \sum_{j=0}^{n-1} \overline{\alpha}_1^{(n-1-j)} \overline{\alpha}_2^j \right) b_1^* \left( (\overline{\alpha}_1 - \overline{\alpha}_2) h_1 + (1 - |\alpha_2|^2) h_2 \right). \]

Hence \( T_n^* f = \overline{\alpha}_2 f \in \tilde{M} \), so that we get \( \tilde{T}_n \tilde{M} \subset \tilde{M} \). By Theorem 4.4, \( S_w^* S_{\tilde{w}}^* = S_w^* S_{\tilde{w}}^* \).

\begin{corollary}
Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \oplus N \). Suppose that \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \), where \( q_1(z) \) is an inner function. Moreover suppose that \( q_1(z) = b_1(z)b_2(z), \) where \( b_i(z), i = 1, 2, \) are simple Blaschke products, \( b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}_i z), \) and \( \alpha_1 \alpha_2 \neq 0 \). Then we have the following.

(i) If \( S_{\tilde{z}} S_{\tilde{w}}^* = S_{\tilde{z}}^* S_{\tilde{w}} \) and \( S_{\tilde{z}} S_{\tilde{w}}^* \neq S_{\tilde{w}}^* S_{\tilde{w}} \), then \( \alpha_1 + \alpha_2 = 0 \).
(ii) If \( \alpha_1 + \alpha_2 = 0 \), then \( S_{\tilde{z}} S_{\tilde{w}}^* = S_{\tilde{w}}^* S_{\tilde{w}} \).
\end{corollary}

The following is the main theorem in this section.

\begin{theorem}
Let \( N \) be a backward shift invariant subspace of \( H^2 \), \( N \neq \{0\}, \) and \( N \neq H^2 \). Then \( S_{\tilde{z}} S_{\tilde{w}}^* = S_{\tilde{w}}^* S_{\tilde{z}} \) if and only if one of the following conditions holds.

(i) \( S_{\tilde{z}} S_{\tilde{w}}^* = S_{\tilde{w}}^* S_{\tilde{z}} \).
(ii) \( S_{\tilde{z}} S_{\tilde{w}}^* = 0 \).
(iii) There exist two simple Blaschke products \( b_1(z) \) and \( b_2(z), b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}_i z), 0 < |\alpha_i| < 1, \) such that \( N \subset H^2 \oplus b_1(z)b_2(z)H^2 \) and \( \alpha_1 + \alpha_2 = 0 \).
\end{theorem}

A backward shift invariant subspace \( N \) satisfying condition (i) is given by Theorem A. Also \( N \) satisfying condition (ii) is given by Theorem 2.2.

\begin{proof}[Proof of Theorem 4.10]
Suppose that \( S_{\tilde{z}} S_{\tilde{w}}^* = S_{\tilde{w}}^* S_{\tilde{z}} \). Moreover suppose that \( S_{\tilde{z}} S_{\tilde{w}}^* \neq S_{\tilde{w}}^* S_{\tilde{z}} \). By Theorem 3.1, \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \) for an inner function \( q_1(z) \) such that either \( q_1(z) = b_1(z) \) or \( q_1(z) = b_1(z)b_2(z), \) where \( b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}_i z), |\alpha_i| < 1. \) If \( q_1(z) = b_1(z), \) by Corollary 4.6, (i) holds.

Suppose that \( q_1(z) = b_1(z)b_2(z). \) Moreover suppose that \( \alpha_1 + \alpha_2 = 0. \) Then \( q_1(z) = z^2 \). Hence \( z^2 H^2 \subset M, \) so that \( N \subset H^2(\Gamma_w) \oplus z H^2(\Gamma_w). \) Then by Theorem 2.2, \( S_{\tilde{z}} S_{\tilde{w}}^* = 0. \) Thus (ii) holds.

Suppose that \( \alpha_1 \neq 0 \) and \( \alpha_2 = 0. \) Then \( q_1(z) = zb_1(z). \) By Corollary 4.7, (i) holds.

Suppose that \( \alpha_1 \neq 0 \) and \( \alpha_2 \neq 0. \) Then by Corollary 4.9(i), we get \( \alpha_1 + \alpha_2 = 0. \) Hence (iii) holds.
\end{proof}
Next, we prove the converse assertion. Suppose that one of conditions (i), (ii), and (iii) holds. If (i) holds, then trivially \( S_z S_w^* = S_w^* S_z \).

If (ii) holds, by Corollary 2.3 we have \( S_z S_w^* = S_w^* S_z \).

Suppose that (iii) holds. Then \( b_1(z) b_2(z) H^2(\Gamma_z) \subset M \). Hence \( M \cap H^2(\Gamma_z) \) equals to either \( b_1(z) H^2(\Gamma_z) \), or \( b_2(z) H^2(\Gamma_z) \), or \( b_1(z) b_2(z) H^2(\Gamma_z) \).

By Corollary 4.6, \( S_z S_w^* = S_w^* S_z \) for the first two cases. Hence \( S_z S_w^* = S_w^* S_z \). For the last case, by Corollary 4.9(ii), \( S_z S_w^* = S_w^* S_z \).

\[ \square \]

**Example 4.11.** We give an example of a backward shift invariant subspace \( N \) of \( H^2 \) satisfying \( S_z S_w^* = S_w^* S_z \), \( S_z S_w^* \neq S_w^* S_z \), and \( S_z S_w^* \neq 0 \). Let \( q_1(z) = b_1(z) b_2(z) \), where \( b_i(z), i = 1, 2 \), are \( b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha_i} z) \), \( \alpha_1 + \alpha_2 = 0 \), and \( \alpha_1 \neq 0 \). Let \( q_2(w) \) be a non-constant inner function. Let \( M = q_1(z) H^2 \oplus b_1(z) b_2(z) q_2(w) H^2(\Gamma_w) \).

Then \( M \) is an invariant subspace and \( M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z) \).

Let \( N = H^2 \oplus M \). By Theorem 4.10, \( S_z S_w^* = S_w^* S_z \). We have \( \tilde{M} = M \oplus q_1(z) H^2 = b_1(z) b_2(z) q_2(w) H^2(\Gamma_w) \).

Since \( T_z^* b_1(z) b_2(z) q_2(w) = (1 - |\alpha_1|^2) b_1(z) q_2(w) + \overline{\alpha_1} b_1(z) b_2(z) q_2(w) \), \( T_z^* b_1(z) b_2(z) q_2(w) \notin \tilde{M} \). By Theorem 4.4, \( S_z S_w^* \neq S_w^* S_z \). By Theorem 2.2, \( S_z S_w^* \neq 0 \).

We leave the following problem for the reader.

**Problem 4.12.** Characterize backward invariant subspaces \( N \) of \( H^2 \) satisfying \( S_z^* S_w^* = S_w^* S_z^* \) for \( n \geq 3 \).

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