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BACKWARD SHIFT INVARIANT SUBSPACES IN THE BIDISC III

KEIJI IZUCHI, TAKAHIKO NAKAZI, AND MICHIO SETO

Abstract. In the previous paper, we give a characterization of backward shift invariant subspaces of the Hardy space in the bidisc which satisfy the doubly commuting condition $S_z S_w^* = S_w^* S_z$ for the compression operators $S_z$ and $S_w$. In this paper, we give a characterization of backward shift invariant subspaces satisfying $S_z^2 S_w^* = S_w^* S_z^2$.

1. Introduction.

Let $\Gamma^2$ be the 2-dimensional unit torus. We denote by $(z, w) = (e^{i\theta}, e^{i\phi})$ the variables in $\Gamma^2 = \Gamma_z \times \Gamma_w$. Let $L^2 = L^2(\Gamma^2)$ be the usual Lebesgue space on $\Gamma^2$ with the norm $\|f\|_2 = (\int_{\Gamma^2} |f(e^{i\theta}, e^{i\phi})|^2 d\theta d\phi/(2\pi)^2)^{1/2}$. Then $L^2$ is a Hilbert space with the usual inner product. For $f \in L^2$, the Fourier coefficients are given by

$$\hat{f}(n, m) = \int_{\Gamma^2} f(e^{i\theta}, e^{i\phi}) e^{-in\theta} e^{-im\phi} d\theta d\phi/(2\pi)^2 = \langle f, z^n w^m \rangle.$$

Let $H^2 = H^2(\Gamma^2)$ be the Hardy space on $\Gamma^2$, that is,

$$H^2 = \{ f \in L^2; \hat{f}(n, m) = 0 \text{ if } n < 0 \text{ or } m < 0 \}.$$

For $f \in H^2$, we can write $f$ as

$$f = \sum_{i, j=0}^{\infty} \oplus a_{i,j} z^i w^j, \text{ where } \sum_{i, j=0}^{\infty} |a_{i,j}|^2 < \infty.$$ 

Let $P$ be the orthogonal projection from $L^2$ onto $H^2$. For a closed subspace $M$ of $L^2$, we denote by $P_M$ the orthogonal projection from $L^2$ onto $M$. For a function $\psi \in L^\infty$, let $L_\psi f = \psi f$ for $f \in L^2$. The Toeplitz operator $T_\psi$ is defined by $T_\psi f = PL_\psi f$ for $f \in H^2$. It is well known that $T_\psi^* = T_{\overline{\psi}}$ and $T_n^* T_m = T_{nm}$ for $n, m \geq 1$. A function $f \in H^2$ is called inner if $|f| = 1$ on $\Gamma^2$ almost everywhere. A closed subspace $M$ of $H^2$ is called invariant if $z M \subset M$ and $w M \subset M$.

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case, an invariant subspace $M$ of $H^2(\Gamma)$ has a form $M = qH^2(\Gamma)$, where $q$ is inner. This is the well known Beurling theorem [B]. In two variable case, the structure of invariant subspaces of $H^2$ is very complicated, see [AC], [DY], [Na1], and [R].

Let $M$ be an invariant subspace of $H^2$. Then $T_z^*(H^2 \ominus M) \subset (H^2 \ominus M)$ and $T_w^*(H^2 \ominus M) \subset (H^2 \ominus M)$. A closed subspace $N$ of $H^2$ is called backward shift invariant if $H^2 \ominus N$ is invariant. In [IY], the first author and Yang studied backward shift invariant subspaces $N$ on which $T_z^*$ is strictly contractive. See [CR] and [S] for studies of backward shift invariant subspaces on the unit circle $\Gamma$.

Let $M$ be an invariant subspace of $H^2$ and $\psi \in L^\infty$. Let $V_\psi$ be the operator on $M$ defined by $V_\psi = P_ML_\psi|_M$. Then $V_\psi = T_\psi$ and $V_\psi^* = V_\psi$ on $M$. In [M], Mandrekar proved that $V_\psi V_\psi^* = V_\psi^* V_\psi$ on $M$ if and only if $M$ is Beurling type, that is, $M = qH^2$ for some inner function $q$ in $H^\infty$, see also [CS] and [Na2].

For $\psi \in L^\infty$, let $S_\psi = P_NL_\psi|_N$. Then we have $S_\psi^* = S_\psi^*$ and $S_\psi = T_\psi^*$ on $N$. In the previous paper [INS], we characterized backward shift invariant subspaces $N$ which satisfy the condition $S_z S_w^* = S_w^* S_z$ on $N$ as follows.

**Theorem A.** Let $N$ be a backward shift invariant subspace of $H^2$ and $N \neq H^2$. Then $S_z S_w^* = S_w^* S_z$ on $N$ if and only if $N$ has one of the following forms;

(i) $N = H^2 \ominus q_1(z)H^2$,
(ii) $N = H^2 \ominus q_2(w)H^2$,
(iii) $N = (H^2 \ominus q_1(z)H^2) \cap (H^2 \ominus q_2(w)H^2)$,

where $q_1(z)$ and $q_2(w)$ are one variable inner functions.

This paper is a continuation of previous papers [IN] and [INS]. Here we are interesting in backward shift invariant subspaces $N$ of $H^2$ satisfying that $S_{z^n} S_{w^m}^* = S_{w^m}^* S_{z^n}$ for given positive integers $n$ and $m$. Up to now, we can not give a complete characterization of $N$ satisfying $S_{z^n} S_{w^m}^* = S_{w^m}^* S_{z^n}$. If $S_z S_w^* = S_w^* S_z$, then trivially $S_{z^n} S_{w^m}^* = S_{w^m}^* S_{z^n}$. But the converse is not true. In this paper, we concentrated on the case $m = 1$. One reason is the work of this paper deeply concerns with the problem of the normality of the operators $S_{z^n}$. For many backward shift invariant subspaces $N$, $S_{z^n}$ are not normal operators, see [Y]. If $S_{z^n}$ is normal, since $S_{z^n} S_w = S_w S_{z^n}$, by the Fuglede-Putnam theorem we have $S_{z^n} S_w^* = S_w^* S_{z^n}$. So, the results obtained in this paper will be a big help for the above mentioned problem. We will discuss on this subject in the forthcoming paper.
In Section 2, we give characterizations of $N$ satisfying $S_{zn}S_{wm}^* = 0$ and $S_{wm}^*S_{zn} = 0$, respectively. If $S_{zn}S_{wm}^* = 0$, then $S_{wm}^*S_{zn} = 0$. The converse is not true. In Section 3, we study $N$ satisfying $S_{zn}S_{wm}^* = S_{wm}^*S_{zn}$, and give a necessary condition for $S_{zn}S_{wm}^* = S_{wm}^*S_{zn}$. In Section 4, we study $N$ satisfying $S_{zn}S_{zm}^* = S_{zm}^*S_{zn}$. We gave a complete characterization of such $N$. In [INS], we gave two different type of proofs of Theorem A. One of them is used in Section 3 and the other is used in Section 4.

Let $H^2(\Gamma_z)$ and $H^2(\Gamma_w)$ be the Hardy spaces on the unit circle in variables $z$ and $w$, respectively. We think that $H^2(\Gamma_z)$ and $H^2(\Gamma_w)$. For a subset $E$ of $H^2$, we denote by $[E]$ the closed linear span of $E$. A function $b(z) = (z - \alpha)/(1 - \overline{\alpha}z), |\alpha| < 1$, is called a simple Blaschke product.

2. $S_{zn}S_{wm}^* = 0$ or $S_{wm}^*S_{zn} = 0$.

Let $n$ and $m$ be positive integers. In this section, we study backward shift invariant subspaces $N$ of $H^2$ satisfying $S_{zn}S_{wm}^* = 0$ and $S_{wm}^*S_{zn} = 0$, respectively.

**Lemma 2.1.** Let $N$ be a backward shift invariant subspace of $H^2$. Then we have the following.

(i) $S_n^z = S_{zn}$.

(ii) $S_{wm}^*S_{zn} = S_{zn}S_{wm}^*$ and $S_{zn}^*S_{wm} = S_{zm}^*S_{zn}^*$.

(iii) If $S_{zn}^*S_{wm}^*N \neq \{0\}$, then there exists $f \in N$ such that $(S_{zn}^*S_{wm}f)(0,0) \neq 0$.

**Proof.** All assertions are not difficult to prove.

The following theorem says that the structure of backward shift invariant subspaces satisfying $S_{zn}S_{wm}^* = 0$ is simple.

**Theorem 2.2.** Let $N$ be a backward shift invariant subspace of $H^2$. Then $S_{zn}S_{wm}^* = 0$ if and only if $N$ satisfies one of the following conditions;

(i) $N \subset \sum_{i=0}^{m-1} \oplus H^2(\Gamma_z)$.

(ii) $N \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)$.

**Proof.** Suppose that $S_{zn}S_{wm}^* = 0$. Then

\[ S_{wm}^*N \perp S_{zn}^*N. \]

Since $N$ is backward shift invariant, if $S_{wm}^*N = \{0\}$ then $N$ satisfies condition (i). If $S_{zn}^*N = \{0\}$, then $N$ satisfies (ii).

Next, suppose that

\[ S_{wm}^*N \neq \{0\} \quad \text{and} \quad S_{zn}^*N \neq \{0\}. \]
We shall lead a contradiction. By (2.1), \( S_{w^m}^* S_{z^n}^* N \perp S_{z^n}^* S_{w^m}^* N \). By Lemma 2.1(ii), \( S_{w^m}^* S_{z^n}^* N = S_{z^n}^* S_{w^m}^* N = \{0\} \). Then

\[
(2.3) \quad S_{z^n}^* N \subset \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)
\]

and

\[
(2.4) \quad S_{w^m}^* N \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w).
\]

By (2.2) and (2.3), there exists a nonnegative integer \( j, 0 \leq j \leq m - 1 \), such that

\[
(2.5) \quad \{0\} \neq S_{w^j} S_{z^n}^* N \subset H^2(\Gamma_z).
\]

By Lemma 2.1(iii), there exists \( g \in N \) such that

\[
(2.6) \quad (S_{w^j} S_{z^n}^* g)(0,0) \neq 0.
\]

Also by (2.2) and (2.4), there exist \( f \in N \) and a nonnegative integer \( i, 0 \leq i \leq n - 1 \), such that

\[
(2.7) \quad S_{z^i} S_{w^m}^* f \in H^2(\Gamma_w)
\]

and

\[
(2.8) \quad (S_{z^i} S_{w^m}^* f)(0,0) \neq 0.
\]

Then

\[
0 = \langle S_{w^m}^* S_{z^n}^* f, S_{z^n}^* S_{w^j}^* g \rangle \quad \text{by (2.1)}
\]

\[
= \langle S_{z^i} S_{w^m}^* f, S_{w^j} S_{z^n}^* g \rangle \quad \text{by Lemma 2.1(ii)}
\]

\[
= \langle S_{z^i} S_{w^m}^* f \rangle \langle 0,0 \rangle (S_{w^j} S_{z^n}^* g)(0,0) \quad \text{by (2.5) and (2.7)}
\]

\[
\neq 0 \quad \text{by (2.6) and (2.8)}.
\]

This is a desired contradiction. The converse is trivial. \( \square \)

**Corollary 2.3.** Let \( N \) be a backward shift invariant subspace of \( H^2 \). Then \( S_{z^n} S_{w^m}^* = 0 \) if and only if either \( S_{z^n} = 0 \) or \( S_{w^m}^* = 0 \). Hence if \( S_{z^n}^* S_{w^m}^* = 0 \), then \( S_{w^m}^* S_{z^n} = 0 \).

**Lemma 2.4.** Let \( M_1 \) and \( M_2 \) be closed subspaces of \( H^2 \) such that

\[
M_1 \subset \sum_{j=0}^{m} \oplus w^j H^2(\Gamma_z) \quad \text{and} \quad M_2 \subset \sum_{i=1}^{n} \oplus z^i H^2(\Gamma_w).
\]

Then \( M_1 + M_2 \) is closed.
Proof. We denote by \((z^jw^j)_{M_1}\) and \((z^jw^j)_{M_2}\) the orthogonal projections of \(z^jw^j\) to the spaces \(M_1\) and \(M_2\), respectively. Let

\[
M'_1 = M_1 \ominus \left( \{ (z^jw^j)_{M_1}; 0 \leq i \leq n, 0 \leq j \leq m \} \right)
\]

and

\[
M'_2 = M_2 \ominus \left( \{ (z^jw^j)_{M_2}; 0 \leq i \leq n, 0 \leq j \leq m \} \right).
\]

Then \(M'_1\) and \(M'_2\) are closed subspaces of \(H^2\),

\[
M'_1 \subset z^{n+1} \left( \sum_{j=0}^{m} \oplus w^j H^2(\Gamma_z) \right), \quad M'_2 \subset w^{m+1} \left( \sum_{i=0}^{n} \oplus z^i H^2(\Gamma_w) \right),
\]

and

\[
M'_1 + M'_2 \perp \left( \{ z^jw^j; 0 \leq i \leq n, 0 \leq j \leq m \} \right).
\]

Since

\[
z^{n+1} \left( \sum_{j=0}^{m} \oplus w^j H^2(\Gamma_z) \right) \perp w^{m+1} \left( \sum_{i=0}^{n} \oplus z^i H^2(\Gamma_w) \right),
\]

\(M'_1 + M'_2 = M'_1 \oplus M'_2\) is closed. Hence

\[
M_1 + M_2 = M'_1 + M'_2 + \left( \{ (z^jw^j)_{M_1}, (z^jw^j)_{M_2}; 0 \leq i \leq n, 0 \leq j \leq m \} \right)
\]

is closed.

Theorem 2.5. Let \(N\) be a backward shift invariant subspace of \(H^2\). Then \(S^*_wS^*_z = 0\) if and only if

(i) \(N \ominus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \subset N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \).

Proof. Suppose that \(S^*_wS^*_z = 0\). Then \(S^*_zN \subset \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z)\). Since \(S^*_zS^*_w = 0\), \(S^*_wN \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\). First, we prove the following

\[
(2.9) \quad N = \left( N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \right) + \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right).
\]

Let

\[
(2.10) \quad K = \left( N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \right) + \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right),
\]

By Lemma 2.4, \(K\) is closed and \(N = K \oplus (N \ominus K)\). To prove (i), it is sufficient to prove \(N \ominus K = \{0\}\). We have \(N \ominus K \subset N \cap (S^*_zN) \perp (S^*_wN) \perp \). Let \(f \in N \ominus K\). Then \(f \perp S^*_zN\), so that \(S^*_z f \perp N\). Since
\[ S^*_n f \in N, \quad S^*_m f = 0. \] Hence \( f \in \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \). By (2.10), \( f \in K \).

This shows \( f = 0 \), so that \( N \oplus K = \{0\} \). Thus we get (2.9).

Let

\begin{equation}
(2.11) \quad N_1 = \left\{ f \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z); \ f \perp N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right\}.
\end{equation}

Then \( N_1 \) is a closed subspace and

\[ N_1 \oplus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \subset N. \]

If the equality holds in the above, (i) holds. So we assume that

\[ N_1 \oplus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \not\subset N. \]

We shall lead a contradiction. Let

\begin{equation}
(2.12) \quad N_2 = N \oplus \left( N_1 \oplus \left( N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right).
\end{equation}

Then \( N_2 \neq \{0\} \) and \( N = N_1 \oplus N_2 \oplus (N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)). \)

Let \( g \in N_2 \) be such that \( g \neq 0 \). We shall prove that

\begin{equation}
(2.13) \quad g \notin N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \quad \text{and} \quad g \notin N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z).
\end{equation}

The fact \( g \notin N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \) is trivial. Suppose that \( g \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \). By (2.11), \( g \in N_1 \). Since \( g \in N_2 \), by (2.12) we have \( g \perp N_1 \). Hence \( g = 0 \). This is a contradiction. Thus we get (2.13).

Next, we shall prove that

\begin{equation}
(2.14) \quad S^*_n g \nmid N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z).
\end{equation}

To prove this, suppose not. Then \( S^*_n g \perp N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \).

By (2.10), \( g = g_1 + g_2 \), where \( g_1 \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \) and \( g_2 \in N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \). Then \( S^*_n g = S^*_n g_1 \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \). Therefore \( S^*_n g = 0 \), so that \( g \in N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \). This contradicts (2.13). Thus we get (2.14).

By (2.14), there exists \( h_0 \) such that \( h_0 \in N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \) and \( \langle g, S^*_n h_0 \rangle = \langle S^*_n g, h_0 \rangle \neq 0 \). Since \( S^*_n h_0 \in N \), by (2.12) we have \( S^*_n h_0 = h_1 \oplus h_2 \oplus h_3 \), where \( h_1 \in N_1, h_2 \in N_2, \) and \( h_3 \in \).
Theorem 2.6. Let \( n \) satisfying (assumption. Thus we have (i). Thus we get \( h_n^0 \) and \( m \).

Thus we get \( h_n^0 \) and \( m \). By (2.13) and \( h_2 \in N_2 \), \( h_2 \notin \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \). This implies that \( S_{w,m}^* h_2 \neq 0 \). Hence \( S_{w,m}^* S_{z^n} \neq 0 \). This contradicts our starting assumption. Thus we have (i).

To prove the converse, suppose that condition (i) holds. Then \( N = (N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)) \oplus L \), where \( L \subset N \cap \sum_{j=0}^{m-1} \oplus w^j H^2(\Gamma_z) \). Let \( F = F_1 + F_2 \in N \), where \( F_1 \in N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \) and \( F_2 \in L \). Since \( z^n F \perp N \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \), \( S_{z^n} F \in L \). Hence \( S_{w,m}^* S_{z^n} F = 0 \). Thus we get \( S_{w,m}^* S_{z^n} = 0 \).

By Theorem 2.2, the structure of backward shift invariant subspaces \( N \) satisfying \( S_{z^n} S_{w,m}^* = 0 \) is simple. By Theorem 2.5, the structure of backward shift invariant subspaces \( N \) satisfying \( S_{w,m}^* S_{z^n} = 0 \) is not so simple. When \( n = m = 1 \), we have the following.

**Theorem 2.6.** Let \( N \) be a backward shift invariant subspace of \( H^2 \).

Then \( S_{w}^* S_{z} = 0 \) if and only if \( N \) has one of the following forms;

(i) \( N = H^2(\Gamma_z) \ominus q(z) H^2(\Gamma_z) \) for some inner function \( q(z) \).

(ii) \( N = H^2(\Gamma_w) \ominus q(w) H^2(\Gamma_w) \) for some inner function \( q(w) \).

(iii) Either \( N = H^2(\Gamma_z) \oplus q_2(w) H^2(\Gamma_w) \), or \( N = H^2(\Gamma_z) \oplus q_2(w) H^2(\Gamma_w) \), or \( N = (H^2(\Gamma_z) \ominus q_2(w) H^2(\Gamma_w)) \) and \( q_1(z) \) and \( q_2(w) \) are inner functions.

(iv) \( N = (H^2(\Gamma_z) \ominus q_1(z) H^2(\Gamma_z)) \oplus (H^2(\Gamma_w) \ominus q_2(w) H^2(\Gamma_w)) \), where \( q_1(z) \) and \( q_2(w) \) are nonconstant inner functions and \( \hat{q}_1(0) \hat{q}_2(0) = 0 \).

In (iii) and (iv), since \( 1 \in N \), we may take \( q_1(z) \) and \( q_2(w) \) as \( \hat{q}_1(0) = \hat{q}_2(0) = 0 \).

**Proof.** By Theorem 2.2, \( S_{z} S_{w}^* = 0 \) if and only if either (i) or (ii) holds. By Theorem 2.5, \( S_{w}^* S_{z} = 0 \) if and only if

\[
N \ominus (N \cap H^2(\Gamma_w)) \subset N \cap H^2(\Gamma_z).
\]

If either (i) or (ii) holds, by Corollary 2.3 we have \( S_{w}^* S_{z} = 0 \). Suppose that \( N \) satisfies either (iii) or (iv). Then clearly \( 1 \in N \). Since \( N \) has a special form, it is not difficult to see that

\[
N \ominus (N \cap H^2(\Gamma_w)) = \{ f \in N \cap H^2(\Gamma_z); \hat{f}(0) = 0 \}.
\]

Hence (2.15) holds.
Next, suppose that (2.15) holds. Then we have
\[(2.16) \quad N = (N \cap H^2(\Gamma_z)) + (N \cap H^2(\Gamma_w)).\]
If either \(N \cap H^2(\Gamma_z) = \{0\}\) or \(N \cap H^2(\Gamma_w) = \{0\}\), then \(S_zS_w^* = 0\), and by Corollary 2.3, \(S_z^*S_w = 0\). Hence either (i) or (ii) holds. Suppose that \(N \cap H^2(\Gamma_z) \neq \{0\}\) and \(N \cap H^2(\Gamma_w) \neq \{0\}\). We shall prove \(1 \in N\). To prove this, suppose that \(1 \notin N\). Let \(1_w\) be the orthogonal projection of \(1\) to \(N \cap H^2(\Gamma_w)\). Then \(1_w \notin H^2(\Gamma_z)\). Since \(N \cap H^2(\Gamma_z) \neq \{0\}\), there exists \(f \in N \cap H^2(\Gamma_z)\) such that \(\hat{f}(0) \neq 0\). Let \(f_1 = f - \hat{f}(0)1_w \in N\). Then \(f_1 \notin H^2(\Gamma_z)\). Let \(h \in N \cap H^2(\Gamma_w)\). Since \(f \in H^2(\Gamma_z), f - \hat{f}(0) \perp h\). Since \(1 - 1_w \perp N \cap H^2(\Gamma_w)\),
\[\langle f_1, h \rangle = \langle \hat{f}(0)(1 - 1_w) + f - \hat{f}(0), h \rangle = 0.\]
Hence \(f_1 \in N \ominus (N \cap H^2(\Gamma_w))\). Thus (2.15) does not hold. Therefore \(1 \in N\). Since \(N \cap H^2(\Gamma_z)\) and \(N \cap H^2(\Gamma_w)\) are nonzero backward shift invariant subspaces, by (2.16) \(N\) has one of forms in (iii) and (iv).

\[3. \quad S_zS_w^* = S_w^*S_z.\]

The following is the main theorem in this section.

**Theorem 3.1.** Let \(N\) be a backward shift invariant subspace of \(H^2\), \(N \neq \{0\}\), and \(N \neq H^2\). Let \(M = H^2 \ominus N\) and \(n \geq 2\) be a positive integer. If \(S_zS_w^* = S_w^*S_z\), then one of the following conditions holds:

(i) \(S_zS_w^* = S_w^*S_z\),

(ii) \(M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)\) for an inner function \(q_1(z)\) satisfying \(q_1(z) = \prod_{j=1}^{k} b_j(z), 1 \leq k \leq n\), where \(b_i\) are simple Blaschke products.

The following lemma is a generalization of [IN], and the idea of the proof is the same.

**Lemma 3.2.** Let \(N\) be a backward shift invariant subspace of \(H^2\) and \(M = H^2 \ominus N\). Let \(n\) be a positive integer. Then \(S_zS_w^* = S_w^*S_z\) if and only if
\[M \ominus \left(z^nM \oplus \left(M \cap \sum_{i=0}^{n-1} \oplus z^iH^2(\Gamma_w)\right)\right) \subset (M \cap H^2(\Gamma_z)) \oplus wM.\]

**Proof.** The operators \(T_{z^n}\) and \(T_{w^*}\) on \(H^2\) have the matrix forms as
\[T_{z^n} = \begin{pmatrix} * & P_MT_{z^n}|_N \\ 0 & S_{z^n} \end{pmatrix}, T_{w^*} = \begin{pmatrix} * & 0 \\ P_NT_{w^*}|_M & S_{w^*} \end{pmatrix} \text{ on } H^2 = \begin{pmatrix} M \\ \oplus & N \end{pmatrix}.\]
Set $A = P_MT_{w^n}|_N$ and $B = P_NT_{w^n}|_M$. Since $T_{w^n}T_{w^n}^* = T_{w^n}^*T_{w^n}$ on $H^2$, $S_{w^n}S_{w^n}^* = S_{w^n}^*S_{w^n}$ if and only if $BA = 0$. We have $T_{w^n}(M \ominus wM) \subset N$. For $f \in H^2$, $T_{w^n}f = 0$ if and only if $f \in H^2(\Gamma_z)$. Hence

$$\ker B = \{ f \in M; T_{w^n}f \in M \}$$

$$= \{ f \in M \ominus wM; T_{w^n}f = 0 \} \ominus wM$$

$$= (M \cap H^2(\Gamma_z)) \oplus wM.$$ We denote by $[\text{ran} \, A]$ the closed range of $A$. Let $A_1 = P_MT_{w^n}P_N$ on $H^2$. Then $[\text{ran} \, A] = [\text{ran} \, A_1]$. Since $A_1^* = P_NT_{w^n}^*P_M$, we get

$$\ker A_1^* = \{ f \in M; T_{w^n}^*f \in M \} \ominus N$$

$$= \left( M \cap \sum_{i=0}^{n-1} \ominus z^i H^2(\Gamma_w) \right) \ominus z^n M \ominus N.$$ Hence

$$[\text{ran} \, A] = [\text{ran} \, A_1] = H^2 \ominus \ker A_1^* = M \ominus \left( z^n M \ominus \left( M \cap \sum_{i=0}^{n-1} \ominus z^i H^2(\Gamma_w) \right) \right).$$

Since $BA = 0$ if and only if $[\text{ran} \, A] \subset \ker B$, we get our assertion. \qed

**Proof of Theorem 3.1.** Suppose that $S_{z^n}S_{w^n}^* = S_{w^n}^*S_{z^n}$. By Lemma 3.2,

$$M \ominus \left( z^n M \ominus \left( M \cap \sum_{i=0}^{n-1} \ominus z^i H^2(\Gamma_w) \right) \right) \subset (M \cap H^2(\Gamma_z)) \ominus wM.$$ Let

$$K_0 = M \ominus \left( z^n M \ominus \left( M \cap \sum_{i=0}^{n-1} \ominus z^i H^2(\Gamma_w) \right) \right).$$

Then

(3.1) $K_0 \subset (M \cap H^2(\Gamma_z)) \ominus wM$

and

(3.2) $M \ominus z^n M = K_0 \ominus \left( M \cap \sum_{i=0}^{n-1} \ominus z^i H^2(\Gamma_w) \right).$

Since $M = \sum_{s=0}^{\infty} \ominus z^{ns}(M \ominus z^n M)$,

(3.3) $K_0 \perp \sum_{s=0}^{\infty} \ominus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \ominus z^i H^2(\Gamma_w) \right)$

and

(3.4) $M = \left( \sum_{s=0}^{\infty} \ominus z^{ns}K_0 \right) \ominus \left( \sum_{s=0}^{\infty} \ominus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \ominus z^i H^2(\Gamma_w) \right) \right).$
First, suppose that $K_0 = \{0\}$. In this case, we shall show that condition (i) holds. By (3.2),

$$\sum_{i=0}^{n-1} \oplus z^i(M \ominus z^M) = M \ominus z^n M = M \cap \bigcup_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w).$$

Let $f \in M \ominus zM$. Then $f \in M \ominus z^n M$ and $f = \sum_{i=0}^{n-1} \oplus z^i h_i(w)$, where $h_i(w) \in H^2(\Gamma_w)$. Since $f \in M \ominus zM$,

$$z^{n-1}f = \sum_{i=0}^{n-1} \oplus z^{n-1+i} h_i(w) \in M \ominus z^n M.$$

Hence $f = h_0(w)$, so that $M \ominus zM = M \cap H^2(\Gamma_w)$. By Lemma 3.2 (for the case of $n = 1$), we get $S_z S^*_w = S^*_w S_z$.

Next, suppose that $K_0 \neq \{0\}$. The proof of this case is a little bit long, so we divide into several steps.

**Step 1.** In this step, we shall prove that there is a nonconstant inner function $q_1(z)$ such that

$$K_0 \subset q_1(z) H^2(\Gamma_z) \oplus wM. \tag{3.5}$$

First we prove that $K_0 \not\subset wM$. To prove this, suppose that $K_0 \subset wM$. Let $F \in K_0$. Then $F = w f$ for some $f \in M$. We shall prove that $f \in K_0$. We have

$$\langle f, \left(\sum_{s=1}^{\infty} \oplus z^{ns} K_0 \right) \oplus \left(\sum_{s=0}^{\infty} \oplus z^{ns} \left(M \cap \bigcup_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\right)\right)\rangle$$

$$= \langle w f, w \left(\sum_{s=1}^{\infty} \oplus z^{ns} K_0 \oplus \left(\sum_{s=0}^{\infty} \oplus z^{ns} \left(M \cap \bigcup_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w)\right)\right)\right)\rangle$$

$$= \langle F, z^n w \left(\sum_{s=1}^{\infty} \oplus z^{n(s-1)} K_0\right)\rangle \quad \text{by (3.3)}$$

$$= 0,$$

where the last equality follows from $w \sum_{s=1}^{\infty} \oplus z^{n(s-1)} K_0 \subset M$ and (3.2). Hence by (3.4), we get $f \in K_0$. Therefore for every positive integer $p$, we have $F = w^p f_p$ for some $f_p \in K_0$. This leads $F = 0$. This is a contradiction. Thus we get $K_0 \not\subset wM$.

Hence by (3.1), $M \cap H^2(\Gamma_z) \neq \{0\}$. By the Beurling theorem,

$$M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z), \tag{3.6}$$

where $q_1(z)$ is inner. Then $q_1(z) \in M$ and

$$q_1(z) H^2(\Gamma_z) \perp wM. \tag{3.7}$$
If \( q_1(z) \) is constant, we have \( M = H^2 \), so that \( N = \{0\} \). This contradicts our assumption. Hence \( q_1(z) \) is a nonconstant inner function. By (3.1) and (3.6), we get (3.5).

**Step 2.** In this step, we prove

\begin{equation}
K_0 \subset q_1(z) \left( \sum_{i=0}^{n-1} \bigoplus z^i H^2(\Gamma_w) \right).
\end{equation}

Let \( G \in K_0 \). Then by (3.5), \( G = q_1(z)h(z) \oplus wg \), where \( h(z) \in H^2(\Gamma_z) \) and \( g \in M \). Write

\[
h(z) = \left( \sum_{i=0}^{n-1} \bigoplus a_i z^i \right) \oplus z^n h_0(z).
\]

Then

\[
G = q_1(z) \left( \sum_{i=0}^{n-1} \bigoplus a_i z^i \right) \oplus q_1(z) z^n h_0(z) \oplus wg.
\]

By (3.6), \( q_1(z) z^n h_0(z) \in z^n M \). Since \( G \in K_0 \subset M \oplus z^n M \), we have \( h_0(z) = 0 \). Hence

\begin{equation}
G = q_1(z) \left( \sum_{i=0}^{n-1} \bigoplus a_i z^i \right) \oplus wg.
\end{equation}

Here we prove that

\begin{equation}
g \in K_0.
\end{equation}

Since \( G = q_1(z) h(z) \oplus wg \), we have

\[
\left\langle g, \sum_{s=1}^{\infty} \bigoplus z^{ns} K_0 \right\rangle = \left\langle wg, w \sum_{s=1}^{\infty} \bigoplus z^{ns} K_0 \right\rangle
\]

\[
= \left\langle wg, w \sum_{s=1}^{\infty} \bigoplus z^{ns} K_0 \right\rangle + \left\langle q_1 h, w \sum_{s=1}^{\infty} \bigoplus z^{ns} K_0 \right\rangle \quad \text{by (3.7)}
\]

\[
= \left\langle G, w \sum_{s=1}^{\infty} \bigoplus z^{ns} K_0 \right\rangle
\]

\[
= \left\langle G, z^n w \sum_{s=1}^{\infty} \bigoplus z^{n(s-1)} K_0 \right\rangle
\]

\[
= 0 \quad \text{by (3.2)}.
\]
We also have
\[ \left\langle g, \sum_{s=0}^{\infty} \oplus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right\rangle \]
\[ = \left\langle wg, w \left( \sum_{s=0}^{\infty} \oplus z^{ns} \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right) \right\rangle \]
\[ = \left\langle G, \sum_{s=0}^{\infty} \oplus z^{ns} w \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right) \right\rangle \text{ by (3.7)} \]
\[ = 0 \text{ by (3.3)}. \]
Hence by (3.4), we get (3.10).

Applying (3.9) and (3.10) infinitely many times, we have
\[ G = q_1(z) \left( \sum_{i=0}^{n-1} \oplus z^i \right) + wq_1(z) \left( \sum_{i=0}^{n-1} \oplus b_i z^i \right) + \cdots \]
\[ \in q_1(z) \left( \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right). \]
Therefore we get (3.8).

Step 3. In this step, we study functions in \( M \ominus zM \) and the inner function \( q_1(z) \). By (3.8), there is a closed subspace \( L \) such that \( K_0 = q_1(z)L \) and \( L \subset \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \). Then by (3.2),
\[ M \ominus z^n M = q_1(z)L \oplus \left( M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w) \right). \]
Since \( K_0 \neq \{0\} \), \( L \neq \{0\} \). We have \( M \ominus z^n M = \sum_{i=0}^{n-1} \oplus z^i(M \ominus z^i M) \).
Hence \( M \ominus zM \neq \{0\} \). Let \( F \in M \ominus zM \) be such that \( F \neq 0 \). Then
\[ F = \left( q_1 \sum_{i=0}^{n-1} \oplus z^i f_i \right) + \sum_{i=0}^{n-1} \oplus z^i g_i, \]
where \( f_i, g_i \in H^2(\Gamma_w) \).

(3.11)
\[ q_1 \sum_{i=0}^{n-1} \oplus z^i f_i \perp M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w), \]

(3.12)
and
\[ \sum_{i=0}^{n-1} \oplus z^i g_i \in M \cap \sum_{i=0}^{n-1} \oplus z^i H^2(\Gamma_w). \]

(3.13)
Since \( n \geq 2 \), \( zF \in \operatorname{M} \ominus z^n \operatorname{M} \), so that we have
\[
zF = q_1 \left( \sum_{i=0}^{n-1} \oplus z^i G_{1,i} \right) + \sum_{i=0}^{n-1} \oplus z^i H_{1,i},
\]
where \( G_{1,i}, H_{1,i} \in H^2(\Omega_w) \). Hence
(3.14)
\[
q_1 \left( z \left( \sum_{i=0}^{n-1} \oplus z^i f_i \right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i} \right) = \left( \sum_{i=0}^{n-1} \oplus z^i H_{1,i} \right) - z \sum_{i=0}^{n-1} \oplus z^i g_i.
\]
Here we divide into two subcases.

**Subcase 1.**
\[
z \left( \sum_{i=0}^{n-1} \oplus z^i f_i(w) \right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w) \neq 0.
\]
Then
\[
q_1(z) = \frac{\left( \sum_{i=0}^{n-1} \oplus z^i H_{1,i}(w) \right) - z \sum_{i=0}^{n-1} \oplus z^i g_i(w)}{z \left( \sum_{i=0}^{n-1} \oplus z^i f_i(w) \right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w)}.
\]
As proved in Step 1, \( q_1(z) \) is a nonconstant inner function. Then by the above, we have
(3.15)
\[
q_1(z) = \prod_{j=1}^{k} b_j(z), \quad 1 \leq k \leq n,
\]
where \( b_j \) are simple Blaschke products.

**Subcase 2.**
\[
z \left( \sum_{i=0}^{n-1} \oplus z^i f_i(w) \right) - \sum_{i=0}^{n-1} \oplus z^i G_{1,i}(w) = 0.
\]
Then by (3.14), \( f_{n-1} = g_{n-1} = 0 \), so that by (3.11)
\[
F = \left( q_1 \sum_{i=0}^{n-2} \oplus z^i f_i \right) + \sum_{i=0}^{n-2} \oplus z^i g_i.
\]
Since \( F \in \operatorname{M} \ominus z \operatorname{M} \),
\[
zF = \left( q_1 \sum_{i=0}^{n-2} \oplus z^{i+1} f_i \right) + \sum_{i=0}^{n-2} \oplus z^{i+1} g_i \in \operatorname{M} \ominus z^2 \operatorname{M}.
\]
In the same way as above, either (3.15) holds or \( f_{n-2} = g_{n-2} = 0 \). Repeat the same argument. Then either (3.15) holds or
\[
f_{n-1} = f_{n-2} = \cdots = f_1 = g_{n-1} = g_{n-2} = \cdots = g_1 = 0.
\]
Step 4. If (3.15) holds, then (ii) holds. So, suppose that (3.15) does not hold. Then by (3.11) and by the above fact, $F = q_1 f_0 \oplus g_0$, by (3.12) $q_1 f_0 \perp M \cap H^2(\Gamma_w)$, and by (3.13) $g_0 \in M \cap H^2(\Gamma_w)$ for every $F \in M \ominus zM$.

If $g_0 = 0$ for every $F \in M \ominus zM$, since $q_1(z) \in M$ it follows that $M \ominus zM \subset q_1(z)H^2(\Gamma_w) \subset M$. Since $M = \sum_{z \in \Gamma} \oplus z^4(M \ominus zM)$, we have $M = q_1(z)H^2$, so that $N = H^2 \ominus q_1(z)H^2$. By Theorem A, condition (i) holds.

So we assume that $g_0 \neq 0$ for some $F \in M \ominus zM$. We shall prove that

\begin{equation}
(M \ominus zM) \ominus (M \cap H^2(\Gamma_w)) \subset (M \cap H^2(\Gamma_w)) \ominus wM. 
\end{equation}

We may assume that $(M \ominus zM) \ominus (M \cap H^2(\Gamma_w)) \neq \{0\}$. Let $G \in (M \ominus zM) \ominus (M \cap H^2(\Gamma_w))$ be such that $G \neq 0$. Then $G = q_1(z)h_1(w) \ominus h_2(w)$, where $q_1(z)h_1(w) \perp M \cap H^2(\Gamma_w)$ and $h_2(w) \in M \cap H^2(\Gamma_w)$. Hence $h_2(w) = G - q_1(z)h_1(w) \perp M \cap H^2(\Gamma_w)$. Therefore $h_2(w) = 0$. Since $q_1(z) \in M$,

$$G = q_1(z)h_1(w) = h_1(0)q_1(z) \ominus wq_1(z) \frac{h_1(w) - h_1(0)}{w} \in M \cap H^2(\Gamma_w) \ominus wM.$$ 

Thus we get (3.16). By Lemma 3.2 (for $n = 1$), (i) holds.

4. $S_z S_w^* = S_w^* S_z \neq S_w^* S_z$.

Let $N$ be a backward shift invariant subspace of $H^2$ and let $n$ be a positive integer. Let $M = H^2 \ominus N$. Then $M$ is an invariant subspace. If both $S_z S_w^* = S_w^* S_z$ and $S_z S_w^* \neq S_w^* S_z$ hold, then by Theorem 3.1, $M \cap H^2(\Gamma_z) = q(z)H^2(\Gamma_z)$ for some nonconstant inner function $q(z)$.

In this section, we assume that $q_1(z)H^2 \subset M$ and $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$ for some nonconstant inner function $q_1(z)$. Let

$$\tilde{M} = M \ominus q_1(z)H^2 \subset M.$$ 

Then $H^2 \ominus q_1(z)H^2 = \tilde{M} \ominus N$ and $\tilde{M}$ is $w$-invariant. The following lemma is proved in [INS, Lemma 3.2].

**Lemma 4.1.** Let $f \in \tilde{M}$. Then $T_w^* f \in \tilde{M}$ if and only if $f \in w\tilde{M}$.

We denote by $P_{\perp}$ the orthogonal projection from $H^2$ onto $H^2 \ominus q_1(z)H^2$. Then we have a Toeplitz type operator $Q_{z^n}$ on $H^2 \ominus q_1(z)H^2$ such that

$$Q_{z^n}: H^2 \ominus q_1(z)H^2 \ni f \rightarrow P_{\perp}(T_{z^n} f) \in H^2 \ominus q_1(z)H^2.$$
Since $z^n M \subset M$, $Q_z M \subset M$ and $Q_z^n = Q_z$. Then $Q_z$ has the following matrix form:

$$Q_z = \begin{pmatrix} * & P_M T_z \vert N \\ 0 & S_z \end{pmatrix} \text{ on } H^2 \otimes q_1(z)H^2 = \begin{pmatrix} \tilde{M} \\ N \end{pmatrix}.$$ 

Since $H^2 \otimes q_1(z)H^2$ is backward shift invariant, $T_w^* (H^2 \otimes q_1(z)H^2) \subset H^2 \otimes q_1(z)H^2$. Since $T_w^* N \subset N$, the operator $T_w^*$ on $H^2 \otimes q_1(z)H^2$ has the following matrix form:

$$T_w^* = \begin{pmatrix} * & 0 \\ P_N T_w \vert M & S_w^* \end{pmatrix} \text{ on } H^2 \otimes q_1(z)H^2 = \begin{pmatrix} \tilde{M} \\ N \end{pmatrix}.$$

Set

(4.1) $A = P_M T_z \vert N$ and $B = P_N T_w \vert \tilde{M}$.

By [INS, Lemma 3.3], $T_w Q_z = Q_z T_w^*$ on $H^2 \otimes q_1(z)H^2$. Hence we have the following.

**Lemma 4.2.** $T_w^* Q_z = Q_z T_w^*$ on $H^2 \otimes q_1(z)H^2$.

**Lemma 4.3.** $S_w S_z^* = S_w^* S_z$ if and only if $BA = 0$.

**Proof.** By Lemma 4.2, $T_w^* Q_z = Q_z T_w^*$ on $H^2 \otimes q_1(z)H^2$. Then $BA + S_w^* S_z = S_w S_z^*$. Hence $S_w S_z^* = S_w^* S_z$ if and only if $BA = 0$. \(\square\)

The following is a slight generalization of [INS, Theorem 3.5].

**Theorem 4.4.** Let $N$ be a backward shift invariant subspace of $H^2$ and $M = H^2 \ominus N$. Suppose that $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$, where $q_1(z)$ is a nonconstant inner function. Let $\tilde{M} = M \ominus q_1(z)H^2$. Then the following conditions are equivalent.

(i) $S_z S_w^* = S_w^* S_z$ on $N$.

(ii) $\tilde{M} \cap \{ f \in \tilde{M}; T_z f \in \tilde{M} \} \subset \tilde{M}$.

(iii) $T_z \tilde{M} \subset \tilde{M}$.

**Proof.** The idea of the proof is similar to the one of [INS, Theorem 3.5]. For the sake of completeness, we give the proof.

(i) $\Leftrightarrow$ (ii) By Lemma 4.3, condition (i) is equivalent to $BA = 0$. By (4.1) and Lemma 4.1, $\ker B = \{ f \in \tilde{M}; T_z^* f \in \tilde{M} \} = w \tilde{M}$. Put $A_1 = P_M T_z \vert N$ on $\tilde{M} \ominus N$. Then $[\text{ran } A] = [\text{ran } A_1]$. Since $A_1^* = P_N T_z^* P_M$, $\ker A_1^* = N \oplus \{ f \in \tilde{M}; T_z^* f \in \tilde{M} \}$. Hence

$[\text{ran } A] = [\text{ran } A_1] = (\tilde{M} \ominus N) \ominus \ker A_1^* = \tilde{M} \ominus \{ f \in \tilde{M}; T_z^* f \in \tilde{M} \}$.

Therefore $BA = 0$ if and only if $\tilde{M} \ominus \{ f \in \tilde{M}; T_z^* f \in \tilde{M} \} \subset w \tilde{M}$. Thus we get (i) $\Leftrightarrow$ (ii).
(ii) $\Rightarrow$ (iii) Suppose that $\tilde{M} \supset \{f \in \tilde{M}; T_z^* f \in \tilde{M}\} \subset w\tilde{M}$. Since $\{f \in \tilde{M}; T_z^* f \in \tilde{M}\}$ is closed, $\tilde{M} \supset w\tilde{M} \subset \{f \in \tilde{M}; T_z^* f \in \tilde{M}\}$. Since $w\tilde{M} \subset \tilde{M}$, $\tilde{M} = \bigoplus_{j=0}^{\infty} w^j (\tilde{M} \ominus w\tilde{M})$. Since $T_z^* w^j f = w^j T_z^* f$ for $f \in H^2$, we have $T_z^* \tilde{M} \subset \tilde{M}$.

(iii) $\Rightarrow$ (ii) is trivial. \hfill \square

For $f \in H^2(\Gamma_z)$, write $f^*(z) = T_z^* f(z) = \varpi(f(z) - \hat{f}(0))$.

**Lemma 4.5.** Let $b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha_i} z)$, $|\alpha_i| < 1$, and $1 \leq i \leq n$. Then

(i) $T_z^* z = 1$, $T_z^* b_1 = b_1^*(z) = (1 - |\alpha_1|^2)/(1 - \overline{\alpha_1} z)$, and $T_z^* b_1(z) = \overline{\alpha_1} b_1^*(z)$.

(ii) $T_z^* (b_1(z) b_2^*(z)) = (1 - |\alpha_2|^2) b_1^*(z) + \overline{\alpha_2} b_1(z) b_2^*(z)$.

(iii) $H^2(\Gamma_z) \ominus (\bigoplus_{j=1}^{k} b_j(z) H^2(\Gamma_z)) = \bigoplus_{j=1}^{k} [b_1(z) \cdots \overline{b}_{j-1}(z) b_j^*(z)]$.

(iv) $H^2 \ominus (\bigoplus_{j=1}^{k} b_j(z) H^2(\Gamma_w)) = \bigoplus_{j=1}^{k} [b_1(z) \cdots \overline{b}_{j-1}(z) b_j^*(z) H^2(\Gamma_w)]$.

**Proof.** It is not difficult to prove (i).

(ii) Since

$$\overline{\alpha} b_1(z) b_2^*(z) = \overline{\alpha} b_1(z) \frac{1 - |\alpha_2|^2}{1 - \overline{\alpha_2} z},$$

$$= (1 - |\alpha_2|^2) b_1^*(z) \left( \frac{\overline{\alpha}}{1 - \overline{\alpha} b_2} \right),$$

$$= (1 - |\alpha_2|^2) \overline{\alpha} b_1(z) + \overline{\alpha_2} b_1(z) b_2^*(z),$$

we get (ii).

(iii) See [Ni, p.33].

(iv) follows from (iii). \hfill \square

**Corollary 4.6.** Let $N$ be a backward shift invariant subspace of $H^2$ and $M = H^2 \ominus N$. Suppose that $M \cap H^2(\Gamma_z) = b_1(z) H^2(\Gamma_z)$, where $b_1(z)$ is a simple Blaschke product. Then $S_z S_w^* = S_w^* S_z$.

**Proof.** Let $b_1(z) = (z - \alpha)/(1 - \overline{\alpha} z)$, $|\alpha| < 1$, and $\tilde{M} = M \ominus b_1(z) H^2$. Since $b_1(z) \in M$, $b_1(z) H^2 \subset M$. By Lemma 4.5(iv), $\tilde{M} \subset b_1^*(z) H^2(\Gamma_w)$. By Lemma 4.5(i), $T_z^* (b_1^*(z) h(w)) = \overline{\alpha} b_1^*(z) h(w)$. Hence $T_z^* \tilde{M} \subset \tilde{M}$. By Theorem 4.4, $S_z S_w^* = S_w^* S_z$.

**Corollary 4.7.** Let $N$ be a backward shift invariant subspace of $H^2$ and $M = H^2 \ominus N$. Suppose that $M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z)$, where $q_1(z)$ is an inner function. Let $n, k$ be positive integers with $n \geq k + 1$. Moreover suppose that $q_1(z) = z^k b(z)$, where $b$ is a simple Blaschke product, $b(z) = (z - \alpha)/(1 - \overline{\alpha} z)$, and $\alpha \neq 0$. If $S_z S_w^* = S_w^* S_z^*$, then $S_z S_w^* = S_w^* S_z^k$. 
Proof. Let $\tilde{M} = M \ominus q_1(z)H^2$. If $\tilde{M} = \{0\}$, then $M = q_1(z)H^2$. By Theorem A, $S_zS_w^* = S_w^*S_z$. Suppose that $\tilde{M} \neq \{0\}$. By Lemma 4.5(iv),
\[ \tilde{M} \subset b^*(z)H^2(\Gamma_w) \oplus \left( \sum_{j=1}^{k} \oplus z^{j-1}b(z)H^2(\Gamma_w) \right). \]

Let $f \in \tilde{M}$. Then
\[ f = b^*(z)h_0(w) + \sum_{j=1}^{k} \oplus z^{j-1}b(z)h_j(w), \quad h_j(w) \in H^2(\Gamma_w). \]

By Lemma 4.5(i),
\[ T_{z^n}^*f = T_{z^{n-k}}^*(T_{z^k}^*f) = T_{z^{n-k}}^* \left( \sum_{j=0}^{k} \tilde{a}^{(k-j)}b^*(z)h_j(w) \right) = \tilde{a}^{(n-k)} \sum_{j=0}^{k} \tilde{a}^{(k-j)}b^*(z)h_j(w). \]

Since $S_{z^n}S_w^* = S_w^*S_{z^n}$, by Theorem 4.4 $T_{z^n}^*f \in \tilde{M}$. Since $\alpha \neq 0$,
\[ \sum_{j=0}^{k} \tilde{a}^{(k-j)}b^*(z)h_j(w) \in \tilde{M}. \]

Thus $T_{z^n}^*\tilde{M} \subset \tilde{M}$. By Theorem 4.4, $S_{z^n}S_w^* = S_w^*S_{z^n}$. \hfill $\square$

**Theorem 4.8.** Let $N$ be a backward shift invariant subspace of $H^2$ and $M = H^2 \ominus N$. Suppose that $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$, where $q_1(z)$ is an inner function. Moreover suppose that $q_1(z) = b_1(z)b_2(z)$, where $b_i(z), i = 1, 2$, are simple Blaschke products, $b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha_iz})$, and $\alpha_1\alpha_2 \neq 0$. Let $n \geq 2$ be a positive integer. Then we have the following.

(i) If $S_{z^n}S_w^* = S_w^*S_{z^n}$ and $S_{z^{n-1}}S_w^* \neq S_w^*S_{z^{n-1}}$, then $\alpha_1 = \alpha_2^n$ and $\alpha_1 \neq \alpha_2$.

(ii) If $\alpha_1^n = \alpha_2^n$ and $\alpha_1 \neq \alpha_2$, then $S_{z^n}S_w^* = S_w^*S_{z^n}$.

**Proof.** Let $\tilde{M} = M \ominus q_1(z)H^2$. Suppose that $S_{z^n}S_w^* = S_w^*S_{z^n}$ and $S_{z^{n-1}}S_w^* \neq S_w^*S_{z^{n-1}}$. By Theorem 4.4, $T_{z^n}^*\tilde{M} \subset \tilde{M}$ and $T_{z^{n-1}}^*\tilde{M} \not\subset \tilde{M}$. By Lemma 4.5(iv),
\[ \tilde{M} \subset b^*_1(z)H^2(\Gamma_w) \oplus b_1(z)b_2^*(z)H^2(\Gamma_w). \]

Then there exists $f_0 \in \tilde{M}$ such that $T_{z^{n-1}}^*f_0 \not\in \tilde{M}$, and
\[ f_0 = b_1^*(z)g_1(w) + b_1(z)b_2^*(z)g_2(w) \in b_1^*(z)H^2(\Gamma_w) \oplus b_1(z)b_2^*(z)H^2(\Gamma_w). \]
By Lemma 4.5,
\[ T_{z_{n-1}}^* f_0 = b_1^* \left( \alpha_1^{(n-1)} g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-2} \alpha_1^{(n-j)} \bar{\alpha}_2^j g_2 \right) \right) + \alpha_2^{(n-1)} b_1^* b_2^* g_2 \]
and
\[ T_{z_n}^* f_0 = b_1^* \left( \alpha_1^n g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \bar{\alpha}_2^j g_2 \right) \right) + \alpha_2^n b_1^* b_2^* g_2. \]

Since \( T_{z_{n-1}}^* f_0 \notin \tilde{M} \) and \( f_0 \in \tilde{M} \), \( T_{z_{n-1}}^* f_0 - \alpha_2^{n-1} f_0 \notin \tilde{M} \). Then
\[ b_1^* \left( (\alpha_1^{(n-1)} - \alpha_2^n) g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-2} \alpha_1^{(n-j)} \bar{\alpha}_2^j g_2 \right) \right) \notin \tilde{M}. \]

Hence
\[ \left( \sum_{j=0}^{n-2} \alpha_1^{(n-j)} \bar{\alpha}_2^j \right) b_1^* \left( (\alpha_1 - \alpha_2) g_1 + (1 - |\alpha_2|^2) g_2 \right) \notin \tilde{M}. \]

Since \( 0 \in \tilde{M} \), \( \sum_{j=0}^{n-2} \alpha_1^{(n-j)} \bar{\alpha}_2^j \neq 0 \), so that
\[ (4.3) \quad b_1^* \left( (\alpha_1 - \alpha_2) g_1 + (1 - |\alpha_2|^2) g_2 \right) \notin \tilde{M}. \]

Since \( T_{z_n}^* f_0 \in \tilde{M} \), \( T_{z_n}^* f_0 - \alpha_2^n f_0 \in \tilde{M} \). Then
\[ b_1^* \left( (\alpha_1^n - \alpha_2^n) g_1 + (1 - |\alpha_2|^2) \left( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \bar{\alpha}_2^j g_2 \right) \right) \in \tilde{M}. \]

Hence
\[ (4.4) \quad \left( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \bar{\alpha}_2^j \right) b_1^* \left( (\alpha_1 - \alpha_2) g_1 + (1 - |\alpha_2|^2) g_2 \right) \in \tilde{M}. \]

Now we prove (i). Suppose that \( \alpha_1 = \alpha_2 \). Then \( \alpha_1 = \alpha_2 \neq 0 \), so that \( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \bar{\alpha}_2^j \neq 0 \). By (4.3) and (4.4), \( b_1^* g_2 \notin \tilde{M} \) and \( b_1^* g_2 \in \tilde{M} \). This is a contradiction. Hence \( \alpha_1 \neq \alpha_2 \).

Suppose that \( \alpha_1^n \neq \alpha_2^n \). Then \( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \bar{\alpha}_2^j \neq 0 \). By (4.4),
\[ b_1^* \left( (\alpha_1^n - \alpha_2^n) g_1 + (1 - |\alpha_2|^2) g_2 \right) \in \tilde{M}. \]

This contradicts (4.3), so that \( \alpha_1^n = \alpha_2^n \). Thus we get (i).

(ii) Suppose that \( \alpha_1^n = \alpha_2^n \) and \( \alpha_1 \neq \alpha_2 \). Then \( \sum_{j=0}^{n-1} \alpha_1^{(n-1-j)} \bar{\alpha}_2^j = 0 \).

Let \( f \in \tilde{M} \). Then by (4.2), \( f = b_1^* (z) h_1 (w) + b_1 (z) b_2^* (z) h_2 (w) \). Similarly,
we have
\[ T^*_n f - \bar{\alpha}_2^0 f = \left( \sum_{j=0}^{n-1} \bar{\alpha}_1^{(n-1-j)} \bar{\alpha}_2^j \right) b_1^* \left((\bar{\alpha}_1 - \bar{\alpha}_2)h_1 + (1-|\alpha_2|^2)h_2 \right). \]

Hence \( T^*_n f = \bar{\alpha}_2^0 f \in \tilde{M} \), so that we get \( T^*_n \tilde{M} \subset \tilde{M} \). By Theorem 4.4, \( S^n \tilde{M} = S^n \tilde{M} \).

**Corollary 4.9.** Let \( N \) be a backward shift invariant subspace of \( H^2 \) and \( M = H^2 \ominus N \). Suppose that \( M \cap H^2(\Gamma_z) = q(z)H^2(\Gamma_z) \), where \( q(z) \) is an inner function. Moreover suppose that \( q(z) = b_1(z)b_2(z) \), where \( b_i(z), i = 1, 2, \) are simple Blaschke products, \( b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}_i z) \), and \( \alpha_1 \alpha_2 \neq 0 \). Then we have the following.

(i) If \( S_z S^*_w = S^*_w S_z \) and \( S_z S^*_w \neq S^*_w S_z \), then \( \alpha_1 + \alpha_2 = 0 \).

(ii) If \( \alpha_1 + \alpha_2 = 0 \), then \( S_z S^*_w = S^*_w S_z \).

The following is the main theorem in this section.

**Theorem 4.10.** Let \( N \) be a backward shift invariant subspace of \( H^2 \), \( N \neq \{0\} \), and \( N \neq H^2 \). Then \( S_z S^*_w = S^*_w S_z \) if and only if one of the following conditions holds.

(i) \( S_z S^*_w = S^*_w S_z \).

(ii) \( S_z S^*_w = 0 \).

(iii) There exist two simple Blaschke products \( b_1(z) \) and \( b_2(z) \), \( b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}_i z) \), \( 0 < |\alpha_i| < 1 \), such that \( \alpha_1 \alpha_2 \neq 0 \). Then we have the following.

A backward shift invariant subspace \( N \) satisfying condition (i) is given by Theorem A. Also \( N \) satisfying condition (ii) is given by Theorem 2.2.

**Proof of Theorem 4.10.** Suppose that \( S_z S^*_w = S^*_w S_z \). Moreover suppose that \( S_z S^*_w \neq S^*_w S_z \). By Theorem 3.1, \( M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z) \) for an inner function \( q_1(z) \) such that either \( q_1(z) = b_1(z) \) or \( q_1(z) = b_1(z)b_2(z) \), where \( b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha}_i z) \), \( |\alpha_i| < 1 \). If \( q_1(z) = b_1(z) \), by Corollary 4.6, (i) holds.

Suppose that \( q_1(z) = b_1(z)b_2(z) \). Moreover suppose that \( \alpha_1 = \alpha_2 = 0 \). Then \( q_1(z) = z^2 \). Hence \( z^2 H^2 \subset M \), so that \( N \subset H^2(\Gamma_w) \oplus zH^2(\Gamma_w) \). Then by Theorem 2.2, \( S_z S^*_w = 0 \). Thus (ii) holds.

Suppose that \( \alpha_1 \neq 0 \) and \( \alpha_2 = 0 \). Then \( q_1(z) = zb_1(z) \). By Corollary 4.7, (i) holds.

Suppose that \( \alpha_1 \neq 0 \) and \( \alpha_2 \neq 0 \). Then by Corollary 4.9(i), we get \( \alpha_1 + \alpha_2 = 0 \). Hence (iii) holds.
Next, we prove the converse assertion. Suppose that one of conditions (i), (ii), and (iii) holds. If (i) holds, then trivially $S_2 S^*_w = S^*_w S_2$.

If (ii) holds, by Corollary 2.3 we have $S_2 S^*_w = S^*_w S_2$.

Suppose that (iii) holds. Then $b_1(z) b_2(z) H^2(\Gamma_z) \subset M$. Hence $M \cap H^2(\Gamma_z)$ equals to either $b_1(z) H^2(\Gamma_z)$, or $b_2(z) H^2(\Gamma_z)$, or $b_1(z) b_2(z) H^2(\Gamma_z)$. By Corollary 4.6, $S_2 S^*_w = S^*_w S_2$ for the first two cases. Hence $S_2 S^*_w = S^*_w S_2$. For the last case, by Corollary 4.9(ii), $S_2 S^*_w = S^*_w S_2$.

\[\Box\]

**Example 4.11.** We give an example of a backward shift invariant subspace $N$ of $H^2$ satisfying $S_2 S^*_w = S^*_w S_2$, $S_2 S^*_w \neq S^*_w S_2$, and $S_2 S^*_w \neq 0$. Let $q_1(z) = b_1(z) b_2(z)$, where $b_i(z)$, $i = 1, 2$, are $b_i(z) = (z - \alpha_i)/(1 - \overline{\alpha_i} z)$, $\alpha_1 + \alpha_2 = 0$, and $\alpha_1 \neq 0$. Let $q_2(z)$ be a non-constant inner function. Let $M = q_1(z) H^2 \oplus b_1(z) b_2^*(z) q_2(w) H^2(\Gamma_w)$. Then $M$ is an invariant subspace and $M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z)$.

Let $N = H^2 \ominus M$. By Theorem 4.10, $S_2 S^*_w = S^*_w S_2$. We have $\tilde{M} = M \ominus q_1(z) H^2 = b_1(z) b_2^*(z) q_2(w) H^2(\Gamma_w)$. Since

$$T^*_z b_1(z) b_2^*(z) q_2(w) = (1 - |\alpha_1|^2) b_1(z) q_2(w) + \overline{\alpha}_1 b_1(z) b_2^*(z) q_2(w),$$

$T^*_z b_1(z) b_2^*(z) q_2(w) \notin \tilde{M}$. By Theorem 4.4, $S_2 S^*_w \neq S^*_w S_2$. By Theorem 2.2, $S_2 S^*_w \neq 0$.

We leave the following problem for the reader.

**Problem 4.12.** Characterize backward invariant subspaces $N$ of $H^2$ satisfying $S_n S^*_w = S^*_w S^*_w$ for $n \geq 3$.

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