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Inverse Problem for the Nonselfadjoint Schrödinger Operator with Energy Dependent Potential in Two Dimensions

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Abstract

In this paper we study the inverse scattering problem of determining the potential for the two dimensional Schrödinger operator of the form

$$-\Delta u(x) + i\sqrt{E}b(x)u(x) = Eu(x), \quad E > 0,$$

which is derived from the dissipative wave equation

$$w_{tt}(x, t) - \Delta w(x, t) + b(x)w_t = 0.$$

The uniqueness theorem will be shown without assuming the smallness condition on $b(x)$ under the low energy.

1 Introduction

1.1 Main results

Let us consider the two dimensional Schrödinger equation of the form

$$-\Delta u(x) + i\sqrt{E}b(x)u(x) = Eu(x), \quad x \in \mathbf{R}^2 \quad (1.1)$$

with asymptotic expansion

$$u(x) = e^{i\sqrt{E}\omega \cdot x} + \frac{e^{i\sqrt{E}r}}{r^{1/2}}A(E, \theta, \omega) + o(r^{-1/2}), \quad |r| \rightarrow \infty, \quad (1.2)$$

where $E > 0$, $r = |x|$, $\theta = x/|x|$ and $\omega \in S^1$. $A(E, \theta, \omega)$ is called the scattering amplitude. The equation (1.1) is derived from the dissipative wave equation

$$w_{tt}(x, t) - \Delta w(x, t) + b(x)w_t(x, t) = 0 \quad (1.3)$$

by considering the time harmonic solution of the form $w(x, t) = e^{i\sqrt{E}t}u(x)$.

In this paper we shall consider the inverse scattering problem of determining $b(x)$ from the scattering data $A(E, \theta, \omega)$ at fixed energy $E > 0$. The uniqueness theorem will be shown without assuming the smallness condition on $b(x)$ under the low energy.

Let us state our main results.

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Theorem 1.1. *Assume that $b(x)$ is a function on \mathbf{R}^2 satisfying*

$$|b(x)| \leq C(1 + |x|)^{-1-\varepsilon}, \quad C > 0, \quad \varepsilon > 0. \quad (1.4)$$

Then there exist $E > 0$ such that the equation (1.1) has a solution with asymptotic expansion (1.2).

Put

$$L(\lambda) = -\Delta + i\lambda b - \lambda^2. \quad (1.5)$$

where b is the multiplication operator by a function $b(x)$. Throughout this paper we shall say that λ is an eigenvalue of $L(\lambda)$ provided that there exists a nontrivial solution v of the boundary value problem

$$\begin{cases} L(\lambda)v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

and denote the set of eigenvalues by $\sigma_\Omega(L)$.

Let $A_j(E, \theta, \omega)$ be the scattering amplitude corresponding to the potentials $b_j(x)$, $j = 1, 2$. We shall denote by $W^{l,p}$ the usual Sobolev space of order l in L^p and denote by B_R the disk with radius $R > 0$.

Theorem 1.2. *Let $b_1(x), b_2(x) \in W^{1,p}(\mathbf{R}^2)$ for some $p > 2$ and $\text{supp } b_1, \text{supp } b_2 \subset B_R$ for some $R > 0$. Assume that*

$$\|b_j\|_{W^{1,p}(\mathbf{R}^2)} \leq M, \quad j = 1, 2.$$

Then there exists a positive number $N = N(M, R, p)$ depending on M, R, p such that if

$$A_1(E, \theta, \omega) = A_2(E, \theta, \omega), \quad \text{for all } \theta, \omega \in S^1$$

holds at fixed $E > 0$ which satisfies $E < N$ and $E \notin \sigma_{B_R}(L)$, then we have

$$b_1(x) = b_2(x) \quad \text{in } \mathbf{R}^2.$$

1.2 Known results

Uniqueness results for the coefficient b of the wave equation (1.3) from the Neumann to Dirichlet map were studied by Nakamura ([14], [15] and [16]). In these results reconstruction procedure of $b(x)$ is not given.

On the other hand, The formulation of the inverse scattering problem for the wave equation (1.3) and reconstruction procedure of $b(x)$ from the scattering amplitude was given by Mochizuki [8].

We shall here review the short history of the scattering and inverse scattering problem for the wave equation (1.3). In multidimensional case $n \geq 3$, the existence of the scattering state was proved by Mochizuki [9] in the dissipative case $b(x) \geq 0$. In the stationary problem (1.1) Nakazawa [17] proved the limiting absorption principle for the operator $L(\lambda)$ for small $b(x)$. Later, for complex valued small function $b(x)$, the completeness of wave operators and expression of the scattering matrix were proved by Mochizuki [8]. Moreover, in

[8] a reconstruction procedure of complex valued small potential $b(x)$ from the scattering amplitude at fixed energy was given.

In two dimensions limiting absorption principle, existence and completeness of wave operators were proved by Nakazawa [18] for small $b(x)$. The expression of the scattering amplitude has not shown yet.

It should be noted that a reconstruction procedure by using the scattering amplitude for all high energies is not valid for our equation (1.1). For the Schrödinger equation

$$-\Delta u(x) + V(x)u(x) = Eu(x)$$

where $V(x)$ has compact support for simplicity, the scattering amplitude is expressed as

$$A(E, \theta, \omega) = \int_{\mathbf{R}^n} e^{-i\sqrt{E}(\theta-\omega)\cdot x} V(x) dx - \int_{\mathbf{R}^n} e^{-i\sqrt{E}\theta\cdot x} V(x) R(E+i0) (V(\cdot) e^{i\sqrt{E}\cdot}) dx \quad (1.6)$$

where $R(E+i0) = \lim_{\varepsilon \rightarrow 0} (-\Delta + V - E + i\varepsilon)^{-1}$, $\varepsilon > 0$. It is well known that for $s > 1/2, 0 < \lambda_0 \leq \lambda$ we have

$$\|R(E+i0)\|_{\mathcal{B}(L^{2,s}, L^{2,-s})} \leq \frac{C}{\sqrt{E}}.$$

Since for $n \geq 2$ and any given $\xi \in \mathbf{R}^n$ we can take θ, ω and E such that $\sqrt{E}(\theta - \omega) = \xi$, it is found that (1.6) tends to the Fourier transform of $V(x)$

$$A(E, \theta, \omega) \longrightarrow \hat{V}(\xi), \quad \text{as } E \rightarrow \infty.$$

In our case, $V(x) = i\sqrt{E}b(x)$, this method is not applicable. Since $V(x)$ depends on E it is not clear that the second integral in (1.6) tends to zero.

As for the fixed energy inverse problem, the method of proof in Mochizuki [8] relies on the direction dependent Green operator. It seems not to be applicable to two dimensional inverse problem.

And further, we must remember that the inverse scattering problem for wave equation (1.3) is reduced to the problem for the stationary equation (1.1) and (1.2).

Put $v = \{\sqrt{-\Delta}w, w_t\}$. Then the wave equation (1.3) is rewritten in the form

$$\frac{1}{i}v_t = Hv = (H_0 + V^b)v \quad (1.7)$$

where

$$H_0 = \begin{pmatrix} 0 & \sqrt{-\Delta} \\ -\sqrt{-\Delta} & 0 \end{pmatrix}, \quad V^b = \frac{1}{i} \begin{pmatrix} 0 & 0 \\ 0 & -b(x) \end{pmatrix}.$$

Mochizuki [8] applied the scattering theory for the schrödinger equation to the equation (1.7) and obtained the expression of the scattering amplitude in the following form

$$2\pi i A(\lambda) = F_0(\lambda) \{V^b - V^b R(\lambda + i0) V^b\} F_0^*(\lambda), \quad (1.8)$$

where

$$F_0(\lambda) = \begin{cases} \mathcal{F}_0(\lambda) \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} & \text{for } \lambda > 0 \\ \mathcal{F}_0(-\lambda) \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} & \text{for } \lambda < 0 \end{cases}$$

and $\mathcal{F}_0(\lambda)$ is the Fourier transform. The scattering amplitude $A(\lambda)$ is an operator matrix. However, noting the form of V^b it is reduced to the scalar case. This means that the problem which determines the $b(x)$ in the equation (1.3) from the scattering amplitude (1.8) is reduced to the stationary problem which determines $b(x)$ in the equation (1.1) from the scattering amplitude (1.2). This is a reason why we consider the inverse problem for the equation (1.1).

1.3 Methods

In the present paper the proof is based on the so-called $\bar{\partial}$ -method. First we reduce the inverse scattering problem to the inverse boundary value problem. In this step the key point is that under the low energy, we can define $R(E + i0)$ satisfying the resolvent equation. (See Section 2.). After that we apply the Kang-Uhlmann [6] approach for small potentials, which relied on a reduction to a first order system.

Setting

$$q_E(x) = \sqrt{E}(ib(x) - \sqrt{E}),$$

then q_E is small when $E > 0$ is small. Hence we can apply the Kang-Uhlmann's approach to our equation (1.1) without assuming the smallness condition on $b(x)$, if we restrict the low energy case.

To reconstruct the potential, $\bar{\partial}$ -equation is one of the most important properties of the complex geometrical optics solution. In our case Nachman's approach is not applicable directly since exponentially growing solutions for complex potentials do not hold the $\bar{\partial}$ -equation of the form $\bar{\partial}u = au$. From this point of view it seems that the first order $\bar{\partial}$ -system is useful not only for relaxing the regularity assumption on potentials but also for complex valued potentials.

On the other hand, it will be remarked in Section 5 that if $b(x)$ is real valued then special solutions (Faddeev solutions) of the equation (1.1) satisfies the $\bar{\partial}$ -equation of the form $\bar{\partial}u = au$. This suggests that in our case if $b(x)$ is real valued, then the inverse scattering problem for the equation (1.1) will be solved as the same way as the case of real valued potentials. (See Novikov [20] and Grinevich [4]).

We here note that uniqueness results for complex coefficient in two dimensional inverse boundary value problem was studied by Kang [7] and Francini [3].

1.4 Notations

Throughout this paper we will use usual notations for function spaces. We denote the set of bounded continuous functions by $\mathcal{B}^0(\mathbf{R}^n)$. For $s \in \mathbf{R}$ the set of weighted L^p space is denoted by $L^{p,s}$. Let $C^{m,\alpha}(\Omega)$ be the Hölder space with index $0 < \alpha \leq 1$ in a closed domain. For the Banach space X we denote by $\mathcal{B}(X)$ the set of all bounded operators on X .

For $z \in \mathbf{C} \setminus \mathbf{R}$ the resolvents of $H_0 = -\Delta$ and $H = H_0 + V$ will be denoted by $R_0(z)$ and $R(z)$, respectively.

After Section 4 complex notations will be used. For $z = x + iy$, we put

$$\begin{aligned} x = z_{\mathbf{R}} = \Re z, & \quad y = z_{\mathbf{I}} = \Im z, \\ \partial = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_x - i\partial_y), & \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial_x + i\partial_y). \end{aligned}$$

1.5 The structure of this paper

In Section 2 we prove Theorem 1.1. Relationship between the inverse scattering problem and the inverse boundary value problem is considered in Section 3. In Section 4 the inverse boundary value problem is reviewed and Theorem 1.2 is proved. A remark on Novikov's exponentially growing solution is given in Section 5.

2 Proof of Theorem 1.1

In this section we shall prove the Theorem 1.1. Put $H_0 = -\Delta$ and $H = H_0 + i\sqrt{E}b$. Then the equation (1.1) is rewritten in the form

$$Hu = Eu. \quad (2.1)$$

Let us consider the solution in the form $u(x) = e^{i\sqrt{E}\omega \cdot x} + v(x, \omega, E)$. Then v satisfies the following equation

$$(H - E)v = -i\sqrt{E}b(x)e^{i\sqrt{E}\omega \cdot x}. \quad (2.2)$$

We shall see that v can be written by the resolvent of $H - E$ in the following form

$$v(x, \omega, E) = -i\sqrt{E}[R(E + i0)(b(\cdot)e^{i\sqrt{E}\omega \cdot})](x) \quad (2.3)$$

where

$$R(E + i0) = \lim_{\varepsilon \rightarrow 0} R(E + i\varepsilon).$$

If we can define $R(E + i0)$ as satisfying the resolvent equation

$$R(E + i0) = R_0(E + i0) - i\sqrt{E}R_0(E + i0)bR(E + i0), \quad (2.4)$$

then it is found that v is a solution of the equation (2.2) and we will see later that u have the asymptotic expansion (1.2).

It is well know that for any $E > 0$ and $s > 1/2$, the following limit exist.

$$R_0(E + i0) = \lim_{\varepsilon \rightarrow 0} R_0(E + i\varepsilon) \in \mathcal{B}(L^{2,s}, L^{2,-s}).$$

We shall see that the operator $I + i\sqrt{E}R_0(E + i0)b$ is invertible on $L^{2,-s}(\mathbf{R}^2)$, $s > 1/2$ for sufficiently small $E > 0$.

If $b(x)$ satisfies the assumption (1.4) in Theorem 1.1, then we see that

$$R_0(E + i0)b \in \mathcal{B}(L^{2,-s}), \quad s > 1/2.$$

It is well known that $R_0(E + i0)f(x)$ is represented by the formula

$$R_0(E + i0)f(x) = \int_{\mathbf{R}^2} G_0(x - y)f(y) dy, \quad (2.5)$$

$$G_0(x) = \frac{i}{4} H_0^{(1)}(\sqrt{E}|x|) \quad (2.6)$$

where $H_0^{(1)}(z)$ is the Hankel function. (Titchmarsh [21, p. 79]). To see the behavior of $\sqrt{E}R_0(E + i0)$ at small $E > 0$, we shall remember that the Hankel function is represented by the sum of the Bessel function $J_0(z)$ and the Neumann function $N_0(z)$.

$$H_0^{(1)}(z) = J_0(z) + iN_0(z) \quad (2.7)$$

where

$$J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n! \Gamma(n+1)},$$

$$N_0(z) = \frac{2i}{\pi} \left\{ J_0(z) (\gamma + \log \frac{z}{2}) - \sum_{n=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{z}{2}\right)^{2k} \sum_{m=1}^k \frac{1}{m} \right\},$$

and γ is a Euler constant. Noting that $x^p \log x \rightarrow 0$, as $x \rightarrow 0$ ($x > 0, p > 0$), in view of the formula (2.7) we find that there exist a small $E > 0$ such that the operator norm of $i\sqrt{E}R_0(E + i0)b$ is less than 1;

$$\|i\sqrt{E}R_0(E + i0)b\|_{\mathcal{B}(L^2, -s)} < 1.$$

Hence $[I + i\sqrt{E}R_0(E + i0)b]^{-1}$ exist on $L^{2, -s}(\mathbf{R}^2)$, $s > 1/2$ for sufficiently small $E > 0$ by means of the convergence of Neumann series.

Let us now define $R(E + i0)$ by

$$R(E + i0) = [I + i\sqrt{E}R_0(E + i0)b]^{-1} R_0(E + i0)$$

and define $v(x, \omega, E)$ by (2.3). Then v satisfies the equation (2.2). Putting $u(x) = e^{i\sqrt{E}\omega \cdot x} + v(x, \omega, E)$ then we find that this $u(x)$ is the solution of the equation (1.1).

We shall now prove that a solution $u(x)$ constructed above have the asymptotic expansion (1.2). Recalling the asymptotic expansion formula for the Hankel function (Watson [22, pp. 196-198])

$$H_0^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{i(z - \pi/4)}, \quad |z| \rightarrow \infty,$$

we obtain

$$R_0(E + i0)f(x) \sim \frac{i}{4} \sqrt{\frac{2}{\pi}} E^{-1/4} \int_{\mathbf{R}^2} \frac{e^{i(\sqrt{E}|x-y| - \pi/4)}}{\sqrt{|x-y|}} f(y) dy, \quad |x| \rightarrow \infty. \quad (2.8)$$

Taking into account that

$$|x - y| = r - \omega \cdot y + O(r^{-1}), \quad \frac{1}{\sqrt{|x-y|}} = \frac{1}{r^{1/2}} + \frac{\omega \cdot y}{2} \frac{1}{r^{3/2}} + O(r^{-5/2})$$

and

$$\frac{e^{i\sqrt{E}|x-y|}}{\sqrt{|x-y|}} = \frac{e^{i\sqrt{E}r}}{r^{1/2}} e^{-i\sqrt{E}\omega \cdot y} + O(r^{-3/2}), \quad r \rightarrow \infty,$$

where $r = |x|$ and $\omega = x/|x|$, we see that the asymptotic expansion (2.8) is written in the form

$$R_0(E + i0)f(x) = 2\pi C(E) \frac{e^{i\sqrt{E}r}}{r^{1/2}} \hat{f}(\sqrt{E}\omega) + O(r^{-3/2}), \quad r \rightarrow \infty$$

where

$$C(E) = \frac{i}{4} \sqrt{\frac{2}{\pi}} E^{-1/4} e^{-i\pi/4}.$$

By means of the resolvent equation (2.4) we obtain the asymptotic expansion of v ;

$$\begin{aligned} v(x, \omega, E) &= -i\sqrt{E}R(E + i0)(b(\cdot)e^{i\sqrt{E}\omega \cdot})(x) \\ &= -i\sqrt{E}R_0(E + i0)\{b(\cdot)e^{i\sqrt{E}\omega \cdot} - b(\cdot)R(E + i0)(i\sqrt{E}b(\cdot)e^{i\sqrt{E}\omega \cdot})\}(x) \\ &= -C(E) \frac{e^{i\sqrt{E}r}}{r^{1/2}} \int_{\mathbf{R}^2} e^{-i\sqrt{E}\theta \cdot x} i\sqrt{E}b(x)\{e^{i\sqrt{E}\omega \cdot x} + v(x, \omega, E)\}dx + O(r^{-3/2}) \\ &= -C(E) \frac{e^{i\sqrt{E}r}}{r^{1/2}} \int_{\mathbf{R}^2} e^{-i\sqrt{E}\theta \cdot x} i\sqrt{E}b(x)u(x)dx + O(r^{-3/2}), \quad r \rightarrow \infty. \end{aligned}$$

Hence we arrive at the asymptotic expansion of $u(x)$

$$u(x) = e^{i\sqrt{E}\omega \cdot x} - C(E) \frac{e^{i\sqrt{E}r}}{r^{1/2}} A(E, \theta, \omega) + O(r^{-3/2}), \quad r \rightarrow \infty$$

where

$$A(E, \theta, \omega) = \int_{\mathbf{R}^n} e^{-i\sqrt{E}\theta \cdot x} i\sqrt{E}b(x)u(x)dx. \quad (2.9)$$

Thus, Theorem 1.1 has been completely proved.

3 Scattering amplitude and Dirichlet to Neumann map

In this section we shall summarize the relationship between the scattering amplitude (2.9) and the Dirichlet to Neumann map. This results are well known for the the real valued potential (see for instance [13], [11] and [5]). Their method is also applicable to our case without no difficulty provided that $b(x)$ has compact support. Throughout this section and next section we shall denote that $\Omega = B_R$, $\Omega^e = \mathbf{R}^2 \setminus \bar{B}_R$ and $\alpha = (p - 2)/p$.

Let us consider the Dirichlet boundary value problem

$$\begin{cases} -\Delta u(x) + i\sqrt{E}b(x)u(x) = Eu(x) & \text{in } \Omega \\ u(x) = f(x) & \text{on } \partial\Omega \end{cases} \quad (3.1)$$

Let $\mathbf{b} \in L^p(\Omega)$, $p > 2$, $f \in C^{1,\alpha}(\partial\Omega)$ and $E \notin \sigma_\Omega(L)$. Then there exists a unique solution $\mathbf{u} \in C^{1,\alpha}(\bar{\Omega})$ of the boundary value problem (3.1). For this solution $\mathbf{u}(\mathbf{x})$ we shall define the Dirichlet to Neumann map $\Lambda_{\mathbf{b}} : C^{1,\alpha}(\partial\Omega) \rightarrow C^{0,\alpha}(\partial\Omega)$ by

$$\Lambda_{\mathbf{b}} f = \left. \frac{\partial \mathbf{u}}{\partial \nu} \right|_{\partial\Omega},$$

where ν is the outer unit normal to $\partial\Omega$.

Theorem 3.1. *Assume that $\mathbf{b}_j \in L^p(\mathbf{R}^2)$ and $\text{supp } \mathbf{b}_j \subset \Omega$, $j = 1, 2$. If*

$$A_1(E, \theta, \omega) = A_2(E, \theta, \omega) \quad \text{for all } \theta, \omega \in S^1$$

holds at $E > 0$ satisfying $E \notin \sigma_\Omega(L)$, then we have

$$\Lambda_{\mathbf{b}_1} = \Lambda_{\mathbf{b}_2}.$$

We shall summarize the proof.

3.1 Near field and Dirichlet to Neumann map

First We shall remember that the Dirichlet to Neumann map $\Lambda_{\mathbf{b}}$ can be uniquely determined from the near field data S_E .

We shall define the exterior Dirichlet to Neumann map by

$$\Lambda^e f = \left. \frac{\partial w}{\partial \nu} \right|_{\Omega},$$

where w is a outgoing solution to the exterior Dirichlet problem

$$\begin{cases} (\Delta + E)w = 0 & \text{in } \Omega^e \\ w(\mathbf{x}) = f(\mathbf{x}) & \text{on } \partial\Omega. \end{cases}$$

Let us denote by S_E the single layer operator on $\partial\Omega$

$$S_E f(\mathbf{x}) = \int_{\partial\Omega} G_E(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mathbf{y},$$

where $G_E(\mathbf{x}, \mathbf{y})$ is the Green function which is the kernel of $R(E + i0)$ defined in Section 2. Then in view of the resolvent equation (2.4) we see that a singularity of $R(E + i0)$ is the same as that of $R_0(E + i0)$. Hence it found by the same argument Colton-Kress [2, pp. 51–106] S_E is the bounded operator from $C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$ and jump relation for the double layer potential holds. According to the result of Isakov-Nachman [5] we see that S_E is invertible and the following relation holds.

Lemma 3.1. *Let $E \notin \sigma_\Omega(L)$. Then we have*

$$S_E^{-1} = \Lambda_{\mathbf{b}} - \Lambda^e.$$

3.2 Scattering amplitude and near field data

To see that the scattering amplitude $A(E, \theta, \omega)$ uniquely determines the near field data S_E , we shall rewrite the scattering amplitude by using the far field operator. According to Nachman [11] let us define the far field operator $F_E : C^{1,\alpha}(\partial\Omega) \rightarrow L^2(S^1)$ by

$$F_E f(\theta) = w_\infty(\theta),$$

where w_∞ is the far field pattern of the solution to the exterior Dirichlet problem

$$\begin{cases} (\Delta + E)w(x) = 0 & \text{in } \Omega^e \\ w(x) = f(x) & \text{on } \partial\Omega \\ w(x) = \frac{e^{i\sqrt{E}r}}{r^{1/2}}w_\infty(\theta) + o(r^{-1/2}), & r = |x| \rightarrow \infty, \theta = x/r. \end{cases} \quad (3.2)$$

Then we can consider the scattering amplitude $A(E, \theta, \omega)$ to be the far field pattern of the exterior Dirichlet problem (3.2) with boundary value $f(x) = u(x, \omega, E) - e^{i\sqrt{E}\omega \cdot x}$ provided that $\text{supp } b(x) \subset \Omega$, where $u(x, \omega, E)$ is a solution of the equation (1.1) with (1.2). Making use of this far field operator F_E we have

$$\begin{aligned} -C(E)A(E, \omega, \theta) &= [F_E\{u(x, \omega, E) - e^{i\sqrt{E}\omega \cdot x}\}](\theta) \\ &= [F_E u(\cdot, \omega, E)](\theta) - A_\Omega(E, \theta, \omega), \end{aligned} \quad (3.3)$$

where $A_\Omega(E, \theta, \omega)$ is the far field pattern of the solution of (3.2) with boundary value $f(x) = e^{i\sqrt{E}\omega \cdot x}$.

We shall see that following identity holds on $\partial\Omega$:

$$u(x, \omega, E) = S_E \left(\frac{\partial \phi^e}{\partial \nu}(\cdot, \omega, E) \right) (x), \quad \text{on } \partial\Omega, \quad (3.4)$$

where $\phi^e(x, \omega, E)$ is a solution of the exterior Dirichlet problem

$$\begin{cases} (\Delta + E)\phi^e(x, \omega, E) = 0 & \text{in } \Omega^e \\ \phi^e(x, \omega, E) = 0 & \text{on } \partial\Omega \end{cases}$$

with $\phi^e(x, \omega, E) - e^{i\sqrt{E}\omega \cdot x}$ outgoing. Since both $u(x, \omega, E) - e^{i\sqrt{E}\omega \cdot x}$ and $\phi^e(x, \omega, E) - e^{i\sqrt{E}\omega \cdot x}$ are outgoing solutions, we see that $u(x, \omega, E) - \phi^e(x, \omega, E)$ is also outgoing solution of exterior Dirichlet problem. Recalling the definition of Dirichlet to Neumann map Λ_b and Λ^e we have

$$\Lambda^e u = \frac{\partial(u - \phi^e)}{\partial \nu} = \Lambda_b u - \frac{\partial \phi^e}{\partial \nu}.$$

Applying S_E to both sides and making use of Theorem 3.1 we have

$$\begin{aligned} S_E \left(\frac{\partial \phi^e}{\partial \nu}(\cdot, \omega, E) \right) (x) &= [S_E(\Lambda_b - \Lambda^e)u(\cdot, \omega, E)](x) \\ &= u(x, \omega, E). \end{aligned}$$

Thus we obtain (3.4). In view of these equalities (3.3) and (3.4) we have

$$-C(E)A(E, \theta, \omega) = \left[F_E S_E \frac{\partial \phi^e}{\partial \nu}(\cdot, \omega, E) \right] (\theta) - A_\Omega(E, \theta, \omega).$$

Let us define the operator $\tilde{F}_E : L^2(S^1) \rightarrow C^{0,\alpha}(\partial\Omega)$ by

$$(\tilde{F}_E g)(x) = \int_{S^1} \frac{\partial \phi^e}{\partial \nu}(x, \omega, E) g(\omega) d\omega.$$

Then we have

Lemma 3.2. *Let $A(E)$ and $A_\Omega(E)$ be integral operators on $L^2(S^1)$ with kernel $C(E)A(E, \omega, \theta)$, $A_\Omega(E, \omega, \theta)$ respectively. Then we have*

$$A_\Omega(E) - A(E) = F_E S_E \tilde{F}_E.$$

Theorem 3.1 is immediately obtained from Lemma 3.1, Lemma 3.2 and the injectivity of F_E and \tilde{F}_E . (See [11], [5].)

Thus the inverse scattering problem is reduced to the inverse boundary value problem.

4 Inverse boundary value problem

In this section we shall consider the inverse boundary value problem for the equation (3.1).

Theorem 4.1. *Let $b_j(x) \in W^{1,p}(\Omega)$ for some $p > 2$ and $\text{supp } b_j(x) \subset \Omega$, $j = 1, 2$. Assume that*

$$\|b_j\|_{W^{1,p}(\mathbb{R}^2)} \leq M, \quad j = 1, 2.$$

Then for a positive number E such that $E < N(M, p, R)$ and $E \notin \sigma_\Omega(L)$ if

$$\Lambda_{b_1} = \Lambda_{b_2},$$

then we have

$$b_1(x) = b_2(x) \quad \text{in } \Omega.$$

Noting that if we set $q = (i\sqrt{E}b - E)$, q is small when E is small and that $\bar{\partial}$ -equation is also valid for complex valued potentials (see Lemma 4.3), it is found that the proof is almost same as in Kang-Uhlmann [6]. So we shall here review the outline of the proof. In subsection 4.1, 4.2 and 4.3 we will review the property of complex geometrical optics solution for the first order $\bar{\partial}$ -system. Smallness condition on E is needed when we construct the complex geometrical optics solution. (Lemma 4.1.) The proof of Theorem 4.1 will be done as follows. First we know in subsection 4.4 that $\Lambda_{b_1} = \Lambda_{b_2}$ implies $S_1(k) = S_2(k)$, where $S_j(k)$, $j = 1, 2$ is the scattering transform (4.13). After that, in subsection 4.5 it will be shown that the scattering transform $S(k)$ determines uniquely the complex geometrical optics solution $M(z, k)$. In this step we need to use the smallness condition on E again. Finally, it will be seen that the complex geometrical optics solution $M(z, k)$ determines uniquely the potential Q .

4.1 First order system

We shall reduce the second order equation (1.1) into a first order $\bar{\partial}$ -system. Using complex notations the equation (1.1) is rewritten in the form

$$(\bar{\partial}\partial - q_E)u(z) = 0, \quad (4.1)$$

where

$$q_E(z) = \frac{1}{4}(i\sqrt{E}\bar{b}(z) - E).$$

Let

$$D = \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q_E \\ 1 & 0 \end{pmatrix}.$$

Then the equation (4.1) is rewritten in the first order system

$$(D - Q)v = 0,$$

where $v = {}^t(\partial u, u)$ and u is the solution of the equation (4.1).

4.2 Complex geometrical optics solution

We shall construct wave function of first order system

$$(D - Q)\Phi(z, k) = 0 \quad \text{in } \Omega \quad (4.2)$$

of the form

$$\Phi(z, k) = M(z, k) \begin{pmatrix} e^{ikz} & 0 \\ 0 & e^{-ik\bar{z}} \end{pmatrix} \quad (4.3)$$

with the property

$$M(z, k) \longrightarrow I \quad \text{as } |k| \rightarrow \infty \quad (4.4)$$

for $k \in \mathbf{C}$, where $M(z, k)$ is the 2×2 matrix and I is the 2×2 identity matrix.

According to Kang-Uhlmann [6] we see that there is a unique solution $\Phi(z, k)$ of the equation (4.2) with (4.4). We shall summarize the proof.

Let us denote that

$$e_k(z) = e^{i(kz + \bar{k}\bar{z})}, \quad \tilde{e}_k(z) = e^{i(\bar{k}z + k\bar{z})}$$

and

$$E_k A = \begin{pmatrix} a_{11} & \tilde{e}_{-k}(z)a_{12} \\ e_k(z)a_{21} & a_{22} \end{pmatrix},$$

where $A = \{a_{ij}\}_{1 \leq i, j \leq 2}$ is any 2×2 matrices. Then it is found that $M(z, k)$ satisfies

$$(D_k - Q)M(z, k) = 0, \quad \text{in } \Omega, \quad (4.5)$$

where $D_k A = E_k^{-1} D E_k A$ for any 2×2 matrices A .

Lemma 4.1. *Let $\varepsilon > 0$ be sufficiently small. Then $I - D_k^{-1}Q$ is invertible on $L^\infty(\Omega)$*

Proof. Let $\mathbf{u} = \{u_{ij}\}_{1 \leq i, j \leq 2}$ satisfy

$$\mathbf{u} - D_k^{-1}Q\mathbf{u} = 0.$$

Then we have (remember that $\Omega = B_R$)

$$\begin{aligned} \|u_{11}\|_{L^\infty(\Omega)} &\leq C_1 \sqrt{\varepsilon} \left(\|b\|_{L^p(\Omega)} + \sqrt{\varepsilon}(\pi R^2)^{1/p} \right) \|u_{21}\|_{L^\infty(\Omega)}, \\ \|u_{21}\|_{L^\infty(\Omega)} &\leq C_1 (\pi R^2)^{1/p} \|u_{11}\|_{L^\infty(\Omega)}, \end{aligned}$$

where $C_1 > 0$ is a constant depending on Ω and p . If ε is small enough to satisfy

$$C_1^2 (\pi R^2)^{1/p} \sqrt{\varepsilon} \left(\|b\|_{L^p(\Omega)} + \sqrt{\varepsilon}(\pi R^2)^{1/p} \right) < 1,$$

then we have $u_{11} = u_{21} = 0$. Thus we find the injectivity of $I - D_k^{-1}Q$. According to the proof in Kang-Uhlmann [6, Lemma 2.2] we see the invertibility of $I - D_k^{-1}Q$. \square

Let us now define

$$M(z, k) = (I - D_k^{-1}Q)^{-1}[I]. \quad (4.6)$$

Then it is found that $M(z, k)$ satisfies the equation (4.5). We shall show that this $M(z, k)$ has property (4.4). Let us denote the components of $M(z, k)$ by $m_{ij}(z, k)$, $1 \leq i, j \leq 2$. Then the asymptotics (4.4) follows from the following lemma (See [6, Theorem 3.1]).

Lemma 4.2. *For sufficiently large $|k|$ we have*

$$\begin{aligned} \|m_{11}(\cdot, k) - 1\|_{L^\infty(\Omega)} + \|m_{22}(\cdot, k) - 1\|_{L^\infty(\Omega)} &\leq \frac{C}{|k|} \|q_E\|_{L^p(\Omega)} \\ \|m_{12}(\cdot, k)\|_{L^\infty(\Omega)} + \|m_{21}(\cdot, k)\|_{L^\infty(\Omega)} &\leq \frac{C}{|k|} \end{aligned}$$

4.3 $\bar{\partial}$ -equation

We shall show that $M(z, k)$ defined by (4.6) satisfies $\bar{\partial}$ -equation. First, we calculate $\frac{\partial M}{\partial k}(z, k)$ formally. In view of the definition of $M(z, k)$ (4.6) we have

$$\begin{cases} m_{11} + \bar{\partial}^{-1}(q_E m_{21}) = 1 \\ m_{21} - e_{-k} \bar{\partial}^{-1}(e_k m_{11}) = 0, \end{cases} \quad \begin{cases} m_{12} + \tilde{e}_k \bar{\partial}^{-1}(\tilde{e}_{-k} q_E m_{22}) = 0 \\ m_{22} - \bar{\partial}^{-1} m_{12} = 1 \end{cases} \quad (4.7)$$

and

$$m_{11} = [1 + \bar{\partial}^{-1} q_E e_{-k} \bar{\partial}^{-1} e_k]^{-1}(1), \quad (4.8)$$

$$m_{21} = [1 + e_{-k} \bar{\partial}^{-1} e_k \bar{\partial}^{-1} q_E]^{-1}(e_{-k} \bar{\partial}^{-1} e_k), \quad (4.9)$$

$$m_{12} = [1 + \tilde{e}_k \bar{\partial}^{-1} \tilde{e}_{-k} q_E \bar{\partial}^{-1}]^{-1}(-\tilde{e}_k \bar{\partial}^{-1} \tilde{e}_{-k} q_E), \quad (4.10)$$

$$m_{22} = [1 + \bar{\partial}^{-1} \tilde{e}_k \bar{\partial}^{-1} \tilde{e}_{-k} q_E]^{-1}(1). \quad (4.11)$$

Since

$$\tilde{e}_k \bar{\partial}^{-1}(\tilde{e}_{-k} f)(z) = -\frac{1}{\pi} \int_{\Omega} \frac{\tilde{e}_{-k}(\zeta - z)}{\zeta - z} f(\zeta) d\zeta_R d\zeta_I,$$

we have formally (namely, if $\frac{\partial}{\partial \bar{k}}$ and $\bar{\partial}^{-1}$ are commutable,)

$$\begin{aligned} \frac{\partial m_{12}}{\partial \bar{k}} &= -\frac{\partial}{\partial \bar{k}} \left\{ \tilde{e}_k(\bar{\partial}^{-1} \tilde{e}_{-k} q_E m_{22})(z, k) \right\} \\ &= \frac{1}{\pi} \left\{ -i \int_{\Omega} \tilde{e}_{-k}(\zeta - z) q_E(\zeta) m_{22}(\zeta, k) d\zeta_R d\zeta_I \right. \\ &\quad \left. + \int_{\Omega} \frac{\tilde{e}_{-k}(\zeta - z)}{\zeta - z} q_E(\zeta) \frac{\partial m_{22}}{\partial \bar{k}}(\zeta, k) d\zeta_R d\zeta_I \right\} \\ &= -\frac{i}{\pi} \tilde{e}_k \int_{\Omega} \tilde{e}_{-k}(\zeta) q_E(\zeta) m_{22}(\zeta, k) d\zeta_R d\zeta_I \\ &\quad - \tilde{e}_k \bar{\partial}^{-1} \left(\tilde{e}_{-k} q_E \frac{\partial m_{22}}{\partial \bar{k}} \right) (z, k). \end{aligned}$$

In view of the relation (4.7) it follows that

$$\frac{\partial m_{12}}{\partial \bar{k}} = i \tilde{e}_k(z) s_{12}(k) - \tilde{e}_k \bar{\partial}^{-1} \left(\tilde{e}_{-k} q_E \bar{\partial}^{-1} \frac{\partial m_{12}}{\partial \bar{k}} \right) (z, k),$$

where

$$s_{12}(k) = -\frac{1}{\pi} \int_{\Omega} \tilde{e}_{-k}(\zeta) q_E(\zeta) m_{22}(\zeta, k) d\zeta_R d\zeta_I.$$

Noting that $s_{12}(k)$ is a function of k independent of z , it may be written in the form

$$\frac{\partial m_{12}}{\partial \bar{k}} = i s_{12}(k) [1 + \tilde{e}_k \bar{\partial}^{-1} \tilde{e}_{-k} q_E \bar{\partial}^{-1}]^{-1}(\tilde{e}_k).$$

Put $v = [1 + \tilde{e}_k \bar{\partial}^{-1} \tilde{e}_{-k} q_E \bar{\partial}^{-1}]^{-1}(\tilde{e}_k)$. Then v is rewritten in the form

$$v = \tilde{e}_k [1 + \bar{\partial}^{-1} \tilde{e}_{-k} q_E \bar{\partial}^{-1} \tilde{e}_k]^{-1}(1).$$

By virtue of relations $\tilde{e}_k(z) = e_{\bar{k}}(z)$, $\tilde{e}_{-k}(z) = e_{-\bar{k}}(z)$ and (4.8) it follows that

$$\begin{aligned} v &= \tilde{e}_k [1 + \bar{\partial}^{-1} e_{-\bar{k}} q_E \bar{\partial}^{-1} e_{\bar{k}}]^{-1}(1) \\ &= \tilde{e}_k m_{11}(z, \bar{k}) \end{aligned}$$

Thus, we obtain

$$\frac{\partial m_{12}}{\partial \bar{k}} = i s_{12}(k) \tilde{e}_k(z) m_{11}(z, \bar{k}).$$

Furthermore, we obtain from (4.7)

$$\begin{aligned} \frac{\partial m_{22}}{\partial \bar{k}} &= \bar{\partial}^{-1} \frac{\partial m_{12}}{\partial \bar{k}} \\ &= i s_{12}(k) \bar{\partial}^{-1} \tilde{e}_k(z) m_{11}(z, \bar{k}) \\ &= i s_{12}(k) \tilde{e}_k m_{21}(z, \bar{k}). \end{aligned}$$

In an analogous way the following formula is derived.

$$\begin{aligned}\frac{\partial \mathbf{m}_{21}}{\partial \bar{k}} &= i s_{21}(k) e_{-k}(z) \mathbf{m}_{22}(z, \bar{k}) \\ \frac{\partial \mathbf{m}_{11}}{\partial \bar{k}} &= i s_{21}(k) e_{-k}(z) \mathbf{m}_{12}(z, \bar{k}),\end{aligned}$$

where

$$s_{21}(k) = \frac{1}{\pi} \int_{\Omega} e_k(\zeta) \mathbf{m}_{11}(\zeta, k) d\zeta_{\mathbb{R}} d\zeta_{\mathbb{I}}.$$

To verify these computation let us show that $\partial/\partial \bar{k}$ and $\bar{\partial}^{-1}$ are commutable. Noting that the identity

$$\tilde{e}_k \bar{\partial}^{-1} \tilde{e}_{-k} = (\bar{\partial} - ik)^{-1}$$

holds and according to Nachman [12, Lemma 2.2], it is enough to show that

$$\frac{\partial}{\partial k_j} (\bar{\partial} - ik)^{-1} f = D_j(k) f(z), \quad j = 1, 2, \quad (4.12)$$

where

$$\begin{aligned}D_j(k) f(z) &= -\frac{1}{\pi} \int_{\Omega} \frac{f(\zeta)}{\zeta - z} \frac{\partial}{\partial k_j} \tilde{e}_{-k}(\zeta) d\zeta_{\mathbb{R}} d\zeta_{\mathbb{I}} \\ &= -\frac{2i(-1)^j}{\pi} \int_{\Omega} \frac{y_j - x_j}{\zeta - z} \frac{\partial}{\partial k_j} \tilde{e}_{-k}(\zeta) f(\zeta) d\zeta_{\mathbb{R}} d\zeta_{\mathbb{I}},\end{aligned}$$

$k = k_1 + ik_2$, $\zeta = y_1 + iy_2$ and $z = x_1 + ix_2$. Obviously it follows that $\|D_j(k) f\|_{L^\infty(\Omega)} \leq C \|f\|_{L^\infty(\Omega)}$.

Simple computation shows that

$$\left| \frac{(\bar{\partial} - i(k+h))^{-1} - (\bar{\partial} - ik)^{-1}}{h} f(z) - D_1(k) f(z) \right| \leq \int_{\Omega} |F_h(z, \zeta) f(\zeta)| d\zeta_{\mathbb{R}} d\zeta_{\mathbb{I}},$$

where

$$F_h(z, \zeta) = \frac{\tilde{e}_{-k}(\zeta - z)}{\zeta - z} \left(\frac{\tilde{e}_{-h}(\zeta - z) - 1}{h} - 2i(y_1 - x_1) \right).$$

Since $\lim_{h \rightarrow \infty} F_h(z, \zeta) = 0$ and $|F_h(z, \zeta) f(\zeta)| \leq C |f(\zeta)|$, we have (4.12) for $j = 1$ by means of Lebesgue's convergence theorem. For $j = 2$ it follows in the same way. Thus we see that the map $k \rightarrow (\bar{\partial} - ik)^{-1}$ is differentiable on \mathbf{C} in the strong operator topology $L^\infty(\Omega) \rightarrow L^\infty(\Omega)$.

Let us now define matrixes $S(k)$ and J by

$$S(k) = -\frac{1}{\pi} \begin{pmatrix} 0 & i s_{12}(k) \\ -i s_{21}(k) & 0 \end{pmatrix}, \quad J\mathcal{A} = \begin{pmatrix} 0 & i a_{12} \\ -i a_{21} & 0 \end{pmatrix}.$$

Then we can write

$$S(k) = -\frac{1}{\pi} J \int_{\Omega} E_k Q(z) M(z, k) dz_{\mathbb{R}} dz_{\mathbb{I}}. \quad (4.13)$$

$M(z, k)$ defined as (4.6) satisfies the following equation so called the $\bar{\partial}$ -equation.

Lemma 4.3. *For each z we have*

$$\frac{\partial M}{\partial \bar{k}}(z, k) = M(z, \bar{k})\Gamma_k(z)S(k)$$

in $L^\infty(\Omega)$, where

$$\Gamma_k(z) = \begin{pmatrix} \tilde{e}_k(z) & 0 \\ 0 & e_{-k}(z) \end{pmatrix}.$$

4.4 Relation Λ_b and $S(k)$.

Lemma 4.4. *Let $\Lambda_{b_1} = \Lambda_{b_2}$. Then we have*

$$S_1(k) = S_2(k)$$

for all $k \in \mathbf{C}$.

Proof. Noting that $\Lambda_{b_1} = \Lambda_{b_2}$ implies $\partial u_1 = \partial u_2$ on $\partial\Omega$, the proof is the same as Kang-Uhlmann [6, Lemma 4.3]. \square

4.5 Relation $S(k)$ and $M(z, k)$

We shall review that $S(k)$ uniquely determine $M(z, k)$. By means of $\bar{\partial}$ -equation we see that m_{11} and m_{12} satisfy following integral equations

$$\begin{cases} m_{11}(z, k) = i\bar{\partial}_k^{-1}\{e_{-k}s_{21}(k)m_{12}(z, \bar{k})\} \\ m_{12}(z, k) = i\bar{\partial}_k^{-1}\{\tilde{e}_k s_{12}(k)m_{11}(z, \bar{k})\}, \end{cases} \quad (4.14)$$

where

$$\bar{\partial}_k^{-1}f(\omega) = -\frac{1}{\pi} \int_{\mathbf{R}^2} \frac{f(k)}{k - \omega} dk_R dk_I.$$

Note that

$$\begin{aligned} |s_{12}(k)| &\leq \frac{C}{1 + |k|} \|q_E\|_{W^{1,p}(\Omega)} \\ |s_{21}(k)| &\leq \frac{C}{1 + |k|} (\|q_E\|_{L^p(\Omega)} + 1) \end{aligned}$$

for all $k \in \mathbf{C}$. (See [6, Lemma 3.2].) Taking into account that $\bar{\partial}_k^{-1}$ is a bounded operator : $L^{2,\delta+1} \rightarrow L^{2,\delta}$, $-1 < \delta < 0$ (see [12, Lemma 2.4]), we have

$$\begin{aligned} \|m_{11}(\cdot, z)\|_{L^{2,\delta}(\mathbf{C})} &\leq C(\|q_E\|_{L^p(\Omega)} + 1) \|m_{12}(\cdot, z)\|_{L^{2,\delta}(\mathbf{C})} \\ \|m_{12}(\cdot, z)\|_{L^{2,\delta}(\mathbf{C})} &\leq C\|q_E\|_{W^{1,p}(\Omega)} \|m_{11}(\cdot, z)\|_{L^{2,\delta}(\mathbf{C})}. \end{aligned}$$

If $E > 0$ is small enough such that

$$C(\|q_E\|_{L^p(\Omega)} + 1)\|q_E\|_{W^{1,p}(\Omega)} < 1,$$

then by the contraction mapping principle we see the unique solvability of the equations (4.14) in $L^{2,\delta}(\mathbf{C})$. Hence, to construct \mathfrak{m}_{11} and \mathfrak{m}_{12} for each $z \in \Omega$ we may solve the system of integral equation

$$\begin{cases} \mathfrak{m}_{11}(z, k) = -\frac{i}{\pi} \int_{\mathbf{R}^2} \frac{e_{-\bar{\omega}}(z)}{\bar{\omega} - k} s_{21}(\bar{\omega}) \mathfrak{m}_{12}(z, \omega) d\omega_{\mathbf{R}} d\omega_{\mathbf{I}} \\ \mathfrak{m}_{12}(z, k) = -\frac{i}{\pi} \int_{\mathbf{R}^2} \frac{\tilde{e}_{-\bar{\omega}}(z)}{\bar{\omega} - k} s_{12}(\bar{\omega}) \mathfrak{m}_{11}(z, \omega) d\omega_{\mathbf{R}} d\omega_{\mathbf{I}}, \end{cases}$$

here we change variable $\omega_{\mathbf{I}}$ by $-\omega_{\mathbf{I}}$. In the same way we can obtain \mathfrak{m}_{21} and \mathfrak{m}_{22} from $S(k)$. This implies that

Lemma 4.5. *Let $S_1(k) = S_2(k)$ for all $k \in \mathbf{C}$. Then we have*

$$M^1(z, k) = M^2(z, k) \quad \text{in } \Omega$$

for all $k \in \mathbf{C}$.

4.6 Relation $Q(z)$ and $M(z, k)$

It is well known that $Q(z)$ is recovered from $M(z, k)$.

Lemma 4.6. *For any $\rho > 0$ we have*

$$Q(z) = \frac{1}{\pi\rho^2} \lim_{k_0 \rightarrow \infty} \int_{|k-k_0| < \rho} D_k M(z, k) dk_{\mathbf{R}} dk_{\mathbf{I}}.$$

The proof will be found in [1, p1024, Theorem 5.2].

Theorem 4.1 now follows from Lemma 4.4, Lemma 4.5 and Lemma 4.6. Thus, combining Theorem 3.1 with Theorem 4.1, Theorem 1.2 has been completely proved.

5 Remark

In this section we give a remark on the $\bar{\partial}$ -equation for special solutions developed by Novikov [20].

Let us consider the Schrödinger equation in \mathbf{R}^2

$$(-4\partial\bar{\partial} + V)u(\lambda, z) = Eu(\lambda, z), \quad E > 0, \quad \lambda \in \mathbf{C}. \quad (5.1)$$

It is known (see for instance [4]) that if $V(z)$ satisfies

$$|V(z)| \leq C_0(1 + |z|)^{-2-\varepsilon}$$

for small $C_0 > 0$ and $\varepsilon > 0$, then there exist a unique solution of the form

$$u(\lambda, z) = e^{\frac{i\sqrt{E}}{2}(\lambda\bar{z} + \frac{1}{\lambda}z)} v(\lambda, z) \quad (5.2)$$

with asymptotic behavior

$$v(\lambda, z) \longrightarrow 1, \quad \text{as } |z| \rightarrow \infty. \quad (5.3)$$

Moreover, this solution $\mathbf{u}(\lambda, z)$ satisfies the so-called $\bar{\partial}$ -equation

$$\frac{\partial \mathbf{u}}{\partial \bar{\lambda}}(\lambda, z) = \begin{cases} T(\lambda) \mathbf{u}\left(-\frac{1}{\bar{\lambda}}, z\right) & \text{if } V \text{ is complex valued} \\ T(\lambda) \overline{\mathbf{u}(\lambda, z)} & \text{if } V \text{ is real valued} \end{cases} \quad (5.4)$$

where

$$T(\lambda) = \frac{\operatorname{sgn}(\lambda \bar{\lambda} - 1)}{\bar{\lambda}} S(\lambda), \quad (5.5)$$

and $S(\lambda)$ is called the scattering transform

$$S(\lambda) = \frac{1}{4\pi} \int_{\mathbf{R}^2} e^{\frac{i\sqrt{E}}{2}(\bar{\lambda}z + \frac{1}{\bar{\lambda}}\bar{z})} V(z) \mathbf{u}(\lambda, z) dz_{\mathbf{R}} dz_{\mathbf{I}}. \quad (5.6)$$

We shall remember that functions satisfying the generalized Cauchy-Riemann equation

$$\bar{\partial} \mathbf{u} = \mathbf{a} \mathbf{u} + \mathbf{b} \bar{\mathbf{u}}$$

are called generalized analytic functions (or pseudo-analytic functions). From the property (5.4) we can say that $\mathbf{u}(\lambda, z)$ is a generalized analytic function with respect to λ if $V(z)$ is real valued functions, but is not for complex valued case.

Let us return to our case $V(z) = i\sqrt{E}b(z)$. If $b(z)$ is the real valued function then we will see that $\mathbf{u}(\lambda, z)$ is the generalized analytic functions with respect to λ although $V(z)$ is the complex valued.

Theorem 5.1. *Let $b(z)$ be the real valued function such that*

$$|b(z)| \leq M(1 + |z|)^{-\alpha}, \quad M > 0 \quad (5.7)$$

for $\alpha > 7/2$. Put $V(z) = i\sqrt{E}b(z)$. If $E > 0$ is sufficiently small then there exist a unique solution $\mathbf{u}(\lambda, z)$ of the equation (5.1) such that (5.2) and (5.3). Moreover, this solution $\mathbf{u}(\lambda, z)$ satisfies the following equation

$$\frac{\partial \mathbf{u}}{\partial \bar{\lambda}}(\lambda, z) = T(\lambda) \overline{\mathbf{u}(\lambda, z)}, \quad |\lambda| \neq 1 \quad (5.8)$$

where $T(\lambda)$ is (5.5).

In this theorem the decay condition of $b(z)$ is rough. It is for simplification and is not so important. What to assert are that smallness of $b(z)$ is not necessary for our case and that exponentially growing solution for purely imaginary valued potential satisfy as the same $\bar{\partial}$ -equation as that for real valued potential.

The rest of this paper we shall prove Theorem 5.1.

5.1 Properties of Green function $G(\lambda, z)$

We first note that the Fourier transform

$$\hat{f}(\xi) = \frac{1}{2\pi} \int_{\mathbf{R}^2} e^{-i\mathbf{x} \cdot \xi} f(\mathbf{x}) d\mathbf{x}$$

can be written in the form

$$\hat{f}(\mathbf{k}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-\frac{i}{2}(\bar{k}z + k\bar{z})} f(z) dz_{\mathbb{R}} dz_{\mathbb{I}},$$

where $z = x_1 + x_2, \mathbf{k} = \xi_1 + \xi_2$. It is easy to see that

$$\widehat{\partial} f(\mathbf{k}) = \frac{i}{2} \bar{k} \hat{f}(\mathbf{k}), \quad \widehat{\partial} f(\mathbf{k}) = \frac{i}{2} k \hat{f}(\mathbf{k}), \quad \widehat{\partial \partial} f(\mathbf{k}) = -\frac{1}{4} k \bar{k} \hat{f}(\mathbf{k}).$$

Let us return to the equation (5.1). Substituting (5.2) into the equation (5.1) we have

$$(P_0(\lambda) + V)v(\lambda, z) = 0, \quad (5.9)$$

where

$$P_0(\lambda) = -4\partial\bar{\partial} - 2i\sqrt{E}\lambda\partial - \frac{2i\sqrt{E}}{\lambda}\bar{\partial}.$$

Let us consider the equation

$$P_0(\lambda)\phi(\lambda, z) = f(z).$$

Applying the Fourier transform to the both side we have

$$(k\bar{k} + \sqrt{E}\lambda\bar{k} + \frac{\sqrt{E}}{\lambda}k)\hat{\phi}(\lambda, \mathbf{k}) = \hat{f}(\mathbf{k}).$$

Defining the Green function $G(\lambda, z)$ by

$$G(\lambda, z) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{e^{\frac{i}{2}(\bar{k}z + k\bar{z})}}{k\bar{k} + \sqrt{E}\lambda\bar{k} + \frac{\sqrt{E}}{\lambda}k} dk_{\mathbb{R}} dk_{\mathbb{I}}$$

and Green operator $G_0(\lambda)$ by

$$G_0(\lambda)f(z) = \int_{\mathbb{R}^2} G(\lambda, z - \mathbf{k})f(\mathbf{k}) dk_{\mathbb{R}} dk_{\mathbb{I}},$$

$\phi(\lambda, z)$ is written in the form

$$\phi(\lambda, z) = G_0(\lambda)f(z).$$

Thus, making use of the Green operator we can rewrite the equation (5.9) into the integral equation

$$v(\lambda, z) = 1 - G_0(\lambda)Vv(\lambda, z), \quad |\lambda| \neq 1. \quad (5.10)$$

We shall remember the properties of Green function $G(\lambda, z)$.

Proposition 5.1 ([4], [20]). *If $|\lambda| \neq 1$ then $G(\lambda, z)$ has the following properties.*

$$|G(\lambda, z)| \leq \frac{C}{\sqrt[4]{E}\sqrt{|z|(|\lambda| + |1/\lambda|)}}. \quad (5.11)$$

$$G\left(-\frac{1}{\lambda}, z\right) = e^{\frac{i\sqrt{E}}{2}\{(\bar{\lambda}+1/\lambda)z + (\lambda+1/\bar{\lambda})\bar{z}\}} G(\lambda, z). \quad (5.12)$$

$$\frac{\partial G}{\partial \lambda}(\lambda, z) = -\frac{\operatorname{sgn}(\bar{\lambda}\lambda - 1)}{4\pi\bar{\lambda}} e^{-\frac{i\sqrt{E}}{2}\{(\bar{\lambda}+1/\lambda)z + (\lambda+1/\bar{\lambda})\bar{z}\}}. \quad (5.13)$$

Proof. (5.11) will be found in Novikov [20, p.424]. From the definition of $G(\lambda, z)$ we have

$$G\left(-\frac{1}{\bar{\lambda}}, z\right) = \frac{1}{4\pi^2} \int_{\mathbf{R}^2} \frac{e^{\frac{1}{2}(\bar{k}z + kz)}}{k\bar{k} - \frac{\sqrt{E}}{\lambda}\bar{k} - \sqrt{E}\lambda k} dk_R dk_I.$$

Changing the variable k by $k + \sqrt{E}(\lambda + 1/\bar{\lambda})$ we obtain (5.12).

Let us prove (5.13). We denote by K_λ the set of zero points of $p(\lambda, k)$

$$K_\lambda = \{k \in \mathbf{C}; p(\lambda, k) = 0\},$$

where

$$p(\lambda, k) = k\bar{k} + \sqrt{E}\lambda\bar{k} + \frac{\sqrt{E}}{\lambda}k.$$

It is easy to see that K_λ consists of two points $k = 0, -\sqrt{E}(\lambda + 1/\bar{\lambda})$.

First let us consider the function $G_\varepsilon(\lambda, z)$

$$G_\varepsilon(\lambda, z) = \frac{1}{4\pi^2} \int_{\mathbf{R}^2} \frac{\overline{p(\lambda, k)}}{|p(\lambda, k)|^2 + \varepsilon} e_z(k) dk_R dk_I,$$

where $e_z(k) = e^{i(\bar{k}z + kz)/2}$. Since

$$\frac{\partial}{\partial \bar{\lambda}} \left(\frac{\bar{p}}{|p|^2 + \varepsilon} \right) (\lambda, z) = \frac{\varepsilon}{(|p(\lambda, z)|^2 + \varepsilon)^2} \frac{\partial \bar{p}}{\partial \bar{\lambda}} (\lambda, z),$$

we have

$$\frac{\partial}{\partial \bar{\lambda}} G_\varepsilon(\lambda, z) = \frac{1}{4\pi^2} \left(\int_{(K_\lambda^\delta)^c} + \int_{K_\lambda^\delta} \right) \frac{\varepsilon e_z(k)}{(|p(\lambda, k)|^2 + \varepsilon)^2} \frac{\partial \bar{p}}{\partial \bar{\lambda}} (\lambda, k) dk_R dk_I, \quad (5.14)$$

where K_λ^δ is the δ -neighborhood of K_λ and $(K_\lambda^\delta)^c$ is its complement. We denote $\eta_1 = \Re(p(\lambda, k))$, $\eta_2 = \Im(p(\lambda, k))$ and $\eta = (\eta_1, \eta_2)$. Changing the variable k_R and k_I by η_1 and η_2 in the integral we have

$$\int_{K_\lambda^\delta} \frac{\varepsilon e_z(k)}{(|p(\lambda, k)|^2 + \varepsilon)^2} \frac{\partial \bar{p}}{\partial \bar{\lambda}} (\lambda, k) dk_R dk_I = \int_{|\eta| < \delta} \frac{\varepsilon e_z(\eta)}{(|\eta|^2 + \varepsilon)^2} \frac{\partial \bar{p}}{\partial \bar{\lambda}} (\lambda, \eta) |J(\eta)| d\eta,$$

where

$$J(\eta) = \begin{vmatrix} \partial\eta_1/\partial k_R & \partial\eta_2/\partial k_R \\ \partial\eta_1/\partial k_I & \partial\eta_2/\partial k_I \end{vmatrix}^{-1}.$$

Changing the variable η_1 and η_2 by $\sqrt{E}\eta_1$ and $\sqrt{E}\eta_2$ the second integral in (5.14) tends to

$$e_z(\eta) |J(\eta)| \frac{\partial \bar{p}}{\partial \bar{\lambda}} (\lambda, \eta) \Big|_{\eta=0} \times \int_{\mathbf{R}^2} \frac{1}{(|\eta|^2 + 1)^2} d\eta$$

when $\varepsilon \rightarrow 0$. Since the first integral in (5.14) tends to 0 as $\varepsilon \rightarrow 0$ and $\eta = 0$ means $k \in K_\lambda$, we obtain

$$\frac{\partial G}{\partial \bar{\lambda}} (\lambda, z) = \frac{1}{4\pi} \frac{\partial \bar{p}}{\partial \bar{\lambda}} (\lambda, k) e_z(k) |J(\lambda, k)| \Big|_{K_\lambda}. \quad (5.15)$$

Let us calculate the right hand side of equation (5.15). Since

$$\frac{\partial \bar{p}}{\partial \bar{\lambda}}(\lambda, k) = \sqrt{E} \left(k - \frac{\bar{k}}{\bar{\lambda}^2} \right),$$

we have

$$\frac{\partial \bar{p}}{\partial \bar{\lambda}}(\lambda, 0) = 0, \quad \frac{\partial \bar{p}}{\partial \bar{\lambda}}(\lambda, -\sqrt{E}(\lambda + \frac{1}{\bar{\lambda}})) = -\frac{E}{\bar{\lambda}} \frac{|\lambda|^4 - 1}{|\lambda|^2}.$$

Moreover simple computation shows that

$$e_z(-\sqrt{E}(\lambda + 1/\bar{\lambda})) = e^{-\frac{i\sqrt{E}}{2}\{(\bar{\lambda}+1/\lambda)z+(\lambda+1/\bar{\lambda})\bar{z}\}}$$

$$|J(\lambda, -\sqrt{E}(\lambda + 1/\bar{\lambda}))| = \left| \frac{E(|\lambda|^4 - 1)}{|\lambda|^2} \right|^{-1}.$$

Thus, we obtain

$$\frac{\partial \bar{p}}{\partial \bar{\lambda}}(\lambda, k) e_z(k) J(\lambda, k) \Big|_{k_\lambda} = \frac{\text{sgn}(\lambda \bar{\lambda} - 1)}{\bar{\lambda}} e^{-\frac{i\sqrt{E}}{2}\{(\bar{\lambda}+1/\lambda)z+(\lambda+1/\bar{\lambda})\bar{z}\}}.$$

This completes the proof. \square

5.2 Proof of Theorem 5.1

First we shall show that the integral equation (5.10) has a unique solution. If $[I + G_0(\lambda)V]^{-1}$ exists on the set of bounded continuous functions then the solution can be constructed in the form

$$v(\lambda, z) = [I + G_0(\lambda)V]^{-1}(1).$$

So let us prove the invertibility of $[I + G_0(\lambda)V]$.

Let $f \in C(\mathbf{C})$. Then making use of following estimates

$$\max_{z \in \mathbf{C}} |V(z)f(z)| \leq \sqrt{E}M \max_{z \in \mathbf{C}} |f(z)|$$

$$\int_{\mathbf{R}^2} |V(z)f(z)| dz_{\mathbf{R}} dz_{\mathbf{I}} \leq \sqrt{E}\tilde{M} \max_{z \in \mathbf{C}} |f(z)|,$$

we have $Vf \in C(\mathbf{C}) \cap L^1(\mathbf{C})$. Since

$$\int_{\mathbf{R}^2} \frac{|f(\zeta)|}{\sqrt{|z - \zeta|}} d\zeta_{\mathbf{R}} d\zeta_{\mathbf{I}} \leq \left(\int_{|z - \zeta| \geq 1} + \int_{|z - \zeta| \leq 1} \right) \frac{|f(\zeta)|}{\sqrt{|z - \zeta|}} d\zeta_{\mathbf{R}} d\zeta_{\mathbf{I}}$$

$$\leq \int_{\mathbf{R}^2} |f(z)| dz_{\mathbf{R}} dz_{\mathbf{I}} + \frac{4\pi}{3} \max_{z \in \mathbf{C}} |f(z)|,$$

we have by virtue of the estimate (5.11)

$$|G_0(\lambda)Vf(z)| \leq \frac{C}{\sqrt[4]{E}\sqrt{|\lambda| + |1/\lambda|}} \left\{ \int_{\mathbf{R}^2} |i\sqrt{E}b(z)f(z)| dz_{\mathbf{R}} dz_{\mathbf{I}} \right. \\ \left. + \frac{4\pi}{3} \max_{z \in \mathbf{C}} |i\sqrt{E}b(z)f(z)| \right\}$$

$$\leq \frac{C}{\sqrt{2}} \sqrt[4]{E} \left(\tilde{M} + \frac{4\pi}{3} M \right) \max_{z \in \mathbf{C}} |f(z)|.$$

If E is sufficiently small then the operator norm of $G_0(\lambda)V$ is less than 1. Hence $I + G_0(\lambda)V$ is invertible on the set of bounded continuous functions.

Next we shall see that $v(\cdot, \lambda) - 1 \in L^1(\mathbf{C})$. Let $f \in \mathcal{B}^0(\mathbf{C})$. Then in view of (5.11) we have

$$\|G_0(\lambda)Vf\|_{L^1(\mathbf{C})} \leq C \max_{z \in \mathbf{C}} |f(z)| \int_{\mathbf{R}^2} K g(x) dx,$$

where

$$Kg(x) = \int_{\mathbf{R}^2} K(x, y)g(y) dy,$$

$$K(x, y) = |x - y|^{-1/2}|y|^{-3/2}, \quad g(y) = (1 + |y|)^{-(\alpha-3/2)}.$$

If $\alpha > 7/2$ then $g \in L^1(\mathbf{R}^2)$. Since K is a bounded operator on $L^1(\mathbf{R}^2)$ (see [19, Lemma 2.1]), it is found that $G_0(\lambda)Vf \in L^1(\mathbf{C})$. Thus we obtain the property (5.3).

We shall prove the equation (5.8). Making use of the formula (5.13) it is found that

$$\frac{\partial v}{\partial \bar{\lambda}}(\lambda, z) = -\frac{\partial G_0(\lambda)Vv}{\partial \bar{\lambda}}(\lambda, z) \quad (5.16)$$

$$= -\int_{\mathbf{R}^2} \frac{\partial G}{\partial \bar{\lambda}}(\lambda, z - z')V(z')v(\lambda, z') dz_{\mathbf{R}} dz_{\mathbf{I}} - \left[G_0(\lambda)\left(V\frac{\partial v}{\partial \bar{\lambda}}\right) \right](\lambda, z) \quad (5.17)$$

$$= T(\lambda)\tilde{e}(z, \lambda) - \left[G_0(\lambda)\left(V\frac{\partial v}{\partial \bar{\lambda}}\right) \right](\lambda, z), \quad (5.18)$$

where

$$\tilde{e}(z, \lambda) = e^{-\frac{i\sqrt{E}}{2}\{(\bar{\lambda}+1/\lambda)z + (\lambda+1/\bar{\lambda})\bar{z}\}}.$$

In view of the formula (5.12) we obtain

$$v\left(-\frac{1}{\bar{\lambda}}, z\right) = 1 - \left[G_0\left(-\frac{1}{\bar{\lambda}}\right)(Vv) \right]\left(-\frac{1}{\bar{\lambda}}, z\right)$$

$$= 1 - \tilde{e}(-z, \lambda)[G_0(\lambda)(V\tilde{v})](\lambda, z),$$

where

$$\tilde{v}(\lambda, z) = \tilde{e}(z, \lambda)v\left(-\frac{1}{\bar{\lambda}}, z\right).$$

It is easy to see that this \tilde{v} satisfies the integral equation

$$\tilde{v}(\lambda, z) = \tilde{e}(z, \lambda) - [G_0(\lambda)(V\tilde{v})](\lambda, z).$$

Noting that $T(\lambda)$ is a function of λ independent of z and applying $T(\lambda)$ to both sides we have

$$T(\lambda)\tilde{v}(\lambda, z) = T(\lambda)\tilde{e}(z, \lambda) - \left[G_0(\lambda)(VT(\lambda)\tilde{v}) \right](\lambda, z).$$

This integral equation has a unique solution. Hence, comparing the equation (5.18) we obtain

$$\frac{\partial v}{\partial \bar{\lambda}}(\lambda, z) = T(\lambda)\tilde{v}(\lambda, z).$$

Moreover, since

$$\tilde{v}(\lambda, z) = u\left(-\frac{1}{\bar{\lambda}}, z\right) \quad \text{and} \quad \frac{\partial u}{\partial \bar{\lambda}} = \frac{\partial v}{\partial \bar{\lambda}},$$

we have

$$\frac{\partial u}{\partial \bar{\lambda}}(\lambda, z) = T(\lambda)u\left(-\frac{1}{\bar{\lambda}}, z\right).$$

We shall now prove

$$u\left(-\frac{1}{\bar{\lambda}}, z\right) = \overline{u(\lambda, z)}.$$

Recalling $V(z) = i\sqrt{E}b(z)$, (5.2) and (5.14), we see that

$$u(\lambda, z) = \dot{e}_\lambda(z) - i\sqrt{E}\dot{e}_\lambda(z)(G_0(\lambda)b\dot{e}_\lambda u)(\lambda, z), \quad (5.19)$$

where

$$\dot{e}_\lambda(z) = e^{\frac{i\sqrt{E}}{2}\left(\frac{1}{\lambda}z + \lambda\bar{z}\right)}.$$

Since $b(x)$ is real valued function, we have by means of (5.12)

$$\overline{u\left(-\frac{1}{\bar{\lambda}}, z\right)} = \dot{e}_\lambda(z) + i\sqrt{E}\dot{e}_\lambda(z)\left[G_0(\lambda)b\dot{e}_{-\lambda}u\left(-\frac{1}{\bar{\lambda}}, \cdot\right)\right](z), \quad (5.20)$$

Let $w(\lambda, z) = u(\lambda, z) - \overline{u(-1/\bar{\lambda}, z)}$. Then, in view of the equations (5.19) and (5.20) $w(\lambda, z)$ satisfy

$$w(\lambda, z) = -i\sqrt{E}\dot{e}_\lambda(z)[G_0(\lambda)b\dot{e}_{-\lambda}w](\lambda, z).$$

Hence we obtain

$$\left[[I + i\sqrt{E}G_0(\lambda)b](\dot{e}_{-\lambda}w)\right](\lambda, z) = 0$$

and it is found $w(\lambda, z) = 0$ because of invertibility of $[I + i\sqrt{E}G_0(\lambda)b]$. Thus, Theorem 5.1 has been completely proved.

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