Duality Theorem for Stochastic Optimal Control Problem *

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Abstract

We give a duality theorem for the stochastic optimal control problem with a convex cost function and show that the minimizer can be characterized by a class of forward-backward stochastic differential equations. As an application, we give an approach, from the duality theorem, to h-path processes for diffusion processes.

1 Introduction.

Let P_0 and P_1 be Borel probability measures on $\mathbf{R}^d$ and let $L(t,x;u) : [0,1]\times \mathbf{R}^d \times \mathbf{R}^d \mapsto [0,\infty)$ be measurable and convex in u. In the present paper we prove the duality theorem for the following problem

$$\begin{aligned} V(P_0,P_1) \ &:= \ \inf\Big\{E\Big[\int_0^1 L(t,X(t);\beta_X(t,X))dt\Big]\Big| \\ &\qquad PX(t)^{-1} = P_t(t=0,1), X\in\mathcal{A}\Big\}. \qquad (1.1)\end{aligned}$$

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The set $\mathcal{A}$ will be given a precise definition below. For the moment let us just say that $X \in \mathcal{A}$ implies that $\{W_X(t) := X(t) - X(0) - \int_0^t \beta_X(s, X)ds\}_{0\le t\le 1}$ is a $\sigma[X(s) : 0 \le s \le t]$-Brownian motion.
When L depends only on u, that is, when $L(t, x, u) = \ell(u)$, the study of a minimizer of the following $\mathcal{T}(P_0, P_1)$ can be considered as a special case of the Monge-Kantorovich problem which has been studied by many authors and which has been applied to many fields (see [1, 2, 6, 10, 12, 15, 25, 28] and the references therein):

$$\begin{aligned}\mathcal{T}(P_0, P_1) \quad := \quad \inf\Big\{ E\Big[\int_0^1 \ell\Big(\frac{d\phi(t)}{dt}\Big)dt\Big]\Big| P\phi(t)^{-1} = P_t (t = 0, 1), \\ t \mapsto \phi(t) \text{ is absolutely continuous}\Big\}. \qquad (1.2)\end{aligned}$$

The duality theorem for $\mathcal{T}(P_0, P_1)$ has been proved for a wide class of functions $\ell(\cdot)$ (see [2, 20, 25] and also [6, 10] for applications to limit theorems). They say that the duality theorem for $\mathcal{T}(P_0, P_1)$ holds if the following is true:

$$\mathcal{T}(P_0, P_1) = \sup\Big\{\int_{\mathbf{R}^d} \psi(1, x)P_1(dx) - \int_{\mathbf{R}^d} \psi(0, x)P_0(dx)\Big\}, \qquad (1.3)$$

where the supremum is taken over all continuous viscosity solutions ψ to the following Hamilton-Jacobi equation:

$$\frac{\partial\psi(t, x)}{\partial t} + \ell^*(D_x\psi(t, x)) = 0 \quad ((t, x) \in (0, 1) \times \mathbf{R}^d) \qquad (1.4)$$

(see [11, Chap. 3]). Here $D_x := (\partial/\partial x_i)_{i=1}^d$ and for $z \in \mathbf{R}^d$,

$$\ell^*(z) := \sup_{u\in\mathbf{R}^d} \{< z, u > -\ell(u)\}$$

and $< \cdot, \cdot >$ denotes the inner product in $\mathbf{R}^d$.
As a counterpart in the stochastic optimal control theory, we study the duality theorem for $V(P_0, P_1)$. More precisely, we prove the following (see Theorem 2.1):

$$V(P_0, P_1) = \sup\Big\{\int_{\mathbf{R}^d} \varphi(1, y)P_1(dy) - \int_{\mathbf{R}^d} \varphi(0, x)P_0(dx)\Big\}, \qquad (1.5)$$

where the supremum is taken over all classical solutions φ, to the following Hamilton-Jacobi-Bellman equation (HJB for short), for which $\varphi(1,\cdot) \in C_b^\infty(\mathbf{R}^d)$:

$$\frac{\partial\varphi(t,x)}{\partial t} + \frac{1}{2}\triangle\varphi(t,x) + H(t,x;D_x\varphi(t,x)) = 0 \quad ((t,x) \in (0,1) \times \mathbf{R}^d) \quad (1.6)$$

(see Lemma 3.3). Here $\triangle := \sum_{i=1}^d \partial^2/\partial x_i^2$ and for $(t,x,z) \in (0,1) \times \mathbf{R}^d \times \mathbf{R}^d$,

$$H(t,x;z) := \sup_{u \in \mathbf{R}^d} \{< z, u > -L(t,x;u)\}. \quad (1.7)$$

As the set $\mathcal{A}$ over which the infimum is taken in (1.1), we consider the set of all $\mathbf{R}^d$-valued, continuous semimartingales $\{X(t)\}_{0\le t\le 1}$ on a probability space $(\Omega_X, \mathbf{B}_X, P_X)$ such that there exists a Borel measurable $\beta_X : [0,1] \times C([0,1]) \mapsto \mathbf{R}^d$ for which
(i) $\omega \mapsto \beta_X(t,\omega)$ is $\mathcal{B}(C([0,t]))_+$-measurable for all $t \in [0,1]$, where $\mathcal{B}(C([0,t]))$ denotes the Borel σ-field of $C([0,t])$,
(ii) $\{W_X(t) := X(t) - X(0) - \int_0^t \beta_X(s,X)ds\}_{0\le t\le 1}$ is a $\sigma[X(s) : 0 \le s \le t]$-Brownian motion (see [20]).

We explain why this is appropriate. Let $(\Omega, \mathbf{B}, P)$ be a probability space, $\{\mathbf{B}_t\}_{t\ge 0}$ be a nondecreasing family of sub σ-fields of $\mathbf{B}$, X_o be a $(\mathbf{B}_0)$-adapted random variable for which $PX_o^{-1} = P_0$, and $\{W(t)\}_{t\ge 0}$ denote a d-dimensional $(\mathbf{B}_t)$-Brownian motion for which $W(0) = o$ (see e.g., [16] or [20]). For a $\mathbf{R}^d$-valued, $(\mathbf{B}_t)$-progressively measurable stochastic process $\{u(t)\}_{0\le t\le 1}$, put

$$X^u(t) = X_o + \int_0^t u(s)ds + W(t) \quad (t \in [0,1]). \quad (1.8)$$

If $E[\int_0^1 |u(t)|dt]$ is finite, then $\{X^u(t)\}_{0\le t\le 1} \in \mathcal{A}$ and

$$\beta_{X^u}(t,X^u) = E[u(t)|X^u(s), 0 \le s \le t] \quad (1.9)$$

(see [20, p. 270]). Besides, by Jensen's inequality,

$$E\Big[\int_0^1 L(t,X^u(t);u(t))dt\Big] \ge E\Big[\int_0^1 L(t,X^u(t);\beta_{X^u}(t,X^u))dt\Big]. \quad (1.10)$$

As is well known in the optimal transportation problems, the quadratic case plays a special role as is stated in the following

Proposition 1.1 *Suppose that $L = |u|^2$, and P_1 is absolutely continuous w.r.t. Lebesgue measure with $p_1(x) := P_1(x)/dx$. Let us also assume that*

$$\begin{aligned}\int_{\mathbf{R}^d} |x|^2 (P_0(dx) + P_1(dx)) &< \infty, \\ \int_{\mathbf{R}^d} p_1(x) \log p_1(x) dx &< \infty.\end{aligned}$$

Then $V(P_0, P_1)$ is finite, there exists a unique minimizer which is an h-path process $\{X_h(t)\}_{0 \le t \le 1}$ for Brownian motion and (1.5) holds.

For the proof of this proposition we refer the reader to [24, Lemma 3.4], [14] and [29] and also [5] and [27]. In Appendix in section 5 we give a brief description of an h-path process $\{X_h(t)\}_{0 \le t \le 1}$.

Since we fix initial and terminal distributions of semimartingales under consideration, known approach is not useful (see [13]). Our proof relies on the Legendre duality of a lower semicontinuous convex function of Borel probability measures on $\mathbf{R}^d$ (see the proof of Theorem 2.1).

When $D_u^2 L(t, x; u)$ and $D_u^2 L(t, x; u)^{-1}$ exist and are bounded, we show, from (1.5), that the minimizer of $V(P_0, P_1)$ can be characterized by a forward-backward stochastic differential equation (FBSDE for short). This enables us to show the existence of a solution to a class of FBSDEs (see Theorem 2.2). As the second application, we give an approach, from the duality theorem, to h-path processes for diffusion processes (see Corollary 2.2).
Our result in this paper is a stepping stone to generalize [24] and study the Monge-Kantorovich problem $\mathcal{T}(P_0, P_1)$ as the zero noise limit of $V(P_0, P_1)$. Indeed, in [24] we gave a new proof for the existence of a deterministic minimizer of $\mathcal{T}(P_0, P_1)$ when $\ell(u) = |u|^2$, by proving that the zero noise limit of $\{X_h(t)\}_{0 \le t \le 1}$ exists, is deterministic and is a minimizer of $\mathcal{T}(P_0, P_1)$ (here we say that a stochastic process $\{X(t)\}_{0 \le t \le 1}$ is deterministic if $X(t)$ is a function of t and $X(0)$). As far as this future application is concerned, we can always assume that $V(P_0, P_1)$ is finite (see the proof of the main result in [24]).

In section 2 we state our result which will be proved in section 4. Technical lemmas are given in section 3. Section 5 is Appendix.

2 Duality Theorem and Applications.

We recall that our minimization problem is

$$
\begin{aligned}
V(P_0,P_1) \quad := \quad \inf\Big\{ E\Big[\int_0^1 L(t,X(t);\beta_X(t,X))dt\Big]\Big| \\
PX(t)^{-1}=P_t(t=0,1), X\in\mathcal{A}\Big\}. \qquad (2.1)
\end{aligned}
$$

Here $\mathcal{A}$ denote the set of all $\mathbf{R}^d$-valued, continuous semimartingales $\{X(t)\}_{0\le t\le 1}$ on a probability space $(\Omega_X,\mathbf{B}_X,P_X)$ such that there exists a Borel measurable $\beta_X:[0,1]\times C([0,1])\mapsto\mathbf{R}^d$ for which
(i) $\omega\mapsto\beta_X(t,\omega)$ is $\mathcal{B}(C([0,t]))_+$-measurable for all $t\in[0,1]$, where $\mathcal{B}(C([0,t]))$ denotes the Borel σ-field of $C([0,t])$,
(ii) $\{W_X(t):=X(t)-X(0)-\int_0^t\beta_X(s,X)ds\}_{0\le t\le 1}$ is a $\sigma[X(s):0\le s\le t]$-Brownian motion.
We also assume the following
(A.0) (i) P_0 and P_1 are Borel probability measures on $\mathbf{R}^d$,
(ii) $L(t,x;u):[0,1]\times\mathbf{R}^d\times\mathbf{R}^d\mapsto[0,\infty)$ is measurable and is convex in u,
(iii) $V(P_0,P_1)$ is finite.

In the present paper we will use the following notation when we refer to properties of L.
(A.1). There exists $\delta>1$ such that

$$\liminf_{|u|\to\infty}\frac{essinf\{L(t,x;u):(t,x)\in[0,1]\times\mathbf{R}^d\}}{|u|^\delta}>0.$$

(A.2).

$$\Delta L(\varepsilon_1,\varepsilon_2):=\sup\frac{L(t,x;u)-L(s,y;u)}{1+L(s,y;u)}\to 0\quad\text{as } \varepsilon_1,\,\varepsilon_2\to 0,$$

where the supremum is taken over all (t,x) and (s,y), $\in[0,1]\times\mathbf{R}^d$, for which $|t-s|\le\varepsilon_1$, $|x-y|<\varepsilon_2$ and all $u\in\mathbf{R}^d$.
(A.3). (i)$L(t,x;u)\in C^3([0,1]\times\mathbf{R}^d\times\mathbf{R}^d)$,
(ii) $D_u^2L(t,x;u)$ is positive definite for all $(t,x,u)\in[0,1]\times\mathbf{R}^d\times\mathbf{R}^d$,
(iii) $\sup\{L(t,x;o):(t,x)\in[0,1]\times\mathbf{R}^d\}$ is finite,
(iv) $|D_xL(t,x;u)|/(1+L(t,x;u))$ is bounded,
(v) $\sup\{|D_uL(t,x;u)|:(t,x)\in[0,1]\times\mathbf{R}^d,|u|\le R\}$ is finite for all $R>0$.

(A.4). (i) $\Delta L(0,\infty)$ is finite, or (ii) $\delta = 2$ in (A.1).

Remark 2.1 (i) *(A.0, ii) and (A.2) imply that* $L \in C([0,1]\times \mathbf{R}^d \times \mathbf{R}^d)$. (ii) *(A.1) and (A.3, i, ii) imply that for any* $(t,x) \in [0,1]\times \mathbf{R}^d$, $H(t,x;\cdot) \in C^3(\mathbf{R}^d)$ *and for any* u *and* $z \in \mathbf{R}^d$,

$$z = D_u L(t,x;u) \quad \textit{if and only if} \quad u = D_z H(t,x;z),$$

$$D_u^2 L(t,x;u) = D_z^2 H(t,x;z)^{-1} \quad \textit{if } u = D_z H(t,x;z)$$

(see [28, 2.1.3]).

We give a result on the existence of a minimizer of $V(P_0,P_1)$.

Proposition 2.1 *Suppose that (A.0)-(A.2) hold. Then* $V(P_0,P_1)$ *has a minimizer.*

The following is our main result.

Theorem 2.1 (Duality Theorem) *Suppose that (A.0)-(A.4) hold. Then (1.5) holds, namely*

$$V(P_0,P_1) = \sup\Big\{\int_{\mathbf{R}^d} \varphi(1,y)P_1(dy) - \int_{\mathbf{R}^d} \varphi(0,x)P_0(dx)\Big\}, \qquad (2.2)$$

where the supremum is taken over all classical solutions φ, *to the following HJB equation, for which* $\varphi(1,\cdot) \in C_b^\infty(\mathbf{R}^d)$:

$$\frac{\partial \varphi(t,x)}{\partial t} + \frac{1}{2}\triangle\varphi(t,x) + H(t,x;D_x\varphi(t,x)) = 0 \quad ((t,x) \in (0,1)\times \mathbf{R}^d) \quad (2.3)$$

Corollary 2.1 *Suppose that (A.0)-(A.4) hold. Then for any minimizer* $\{X(t)\}_{0\le t\le 1}$ *of* $V(P_0,P_1)$, *there exists a sequence of classical solutions* $\{\varphi_n\}_{n\ge 1}$, *of the HJB equation (2.3), such that* $\varphi_n(1,\cdot) \in C_b^\infty(\mathbf{R}^d)$ $(n \ge 1)$ *and that the following holds:*

$$\begin{aligned} &\beta_X(t,X) = b_X(t,X(t)) := E[\beta_X(t,X)|(t,X(t))] \qquad (2.4)\\ = &\lim_{n\to\infty} D_z H(t,X(t);D_x\varphi_n(t,X(t))) \quad dtdPX(\cdot)^{-1} - a.e.. \end{aligned}$$

Next we only study the case where (A.4, ii) holds.

Proposition 2.2 *(i) Suppose that (A.0)-(A.2) and (A.4, ii) hold. Then* $V(P_0, P_1)$ *has a Markovian minimizer.*
(ii) Suppose in addition that for any $(t, x) \in [0, 1] \times \mathbf{R}^d$, $L(t, x; u)$ *is strictly convex in* u. *Then the minimizer is unique.*

We now introduce the additional assumption:
(A.5). $D_u^2 L(t, x; u)$ is bounded,
and we give a characterization of a minimizer of $V(P_0, P_1)$ by a FBSDE.

Theorem 2.2 *Suppose that (A.0)-(A.3), (A.4, ii) and (A.5) hold. Then, for the unique minimizer* $\{X(t)\}_{0\le t\le 1}$ *of* $V(P_0, P_1)$, *there exist* $f(\cdot) \in L^1(\mathbf{R}^d, P_1(dx))$ *and a* $\sigma[X(s) : 0 \le s \le t]$*- continuous semimartingale* $\{Y(t)\}_{0\le t\le 1}$ *such that*

$$\{(X(t), Y(t), Z(t) := D_u L(t, X(t); b_X(t, X(t))))\}_{0\le t\le 1}$$

satisfies the following FBSDE in a weak sense: for $t \in [0, 1]$,

$$\begin{aligned} X(t) &= X(0) + \int_0^t D_z H(s, X(s); Z(s))ds + W(t), \qquad (2.5)\\ Y(t) &= f(X(1)) - \int_t^1 L(s, X(s); D_z H(s, X(s); Z(s)))ds \\ &\quad - \int_t^1 < Z(s), dW(s) > . \end{aligned}$$

Besides, there exist $f_0(\cdot) \in L^1(\mathbf{R}^d, P_0(dx))$ *and* $\varphi(\cdot, \cdot) \in L^1([0, 1]\times\mathbf{R}^d, P((t, X(t)) \in dtdx))$ *such that* $Y(0) = f_0(X(0))$ *and such that*

$$Y(t) - Y(0) = \varphi(t, X(t)) - \varphi(0, X(0)) \quad dtdPX(\cdot)^{-1} - a.e., \qquad (2.6)$$

that is, $Y(t)$ *is a continuous version of* $\varphi(t, X(t)) - \varphi(0, X(0)) + f_0(X(0))$.

Remark 2.2 (i) *(A.4, ii) and (A.5) is appropriate in our approach. Indeed, suppose that* $L = |u|^\delta$ *and* $E[\int_0^1 |b_X(s, X(s))|^\delta ds]$ *is finite. Then it should be true that* $\delta \ge 2$ *so that* $P((X(0), (t, X(t))) \in dxdtdy)$ *is absolutely continuous*

with respect to $P(X(0) \in dx)P((t, X(t)) \in dtdy)$ *(see the proof of Theorem 2.2). It should also be true that* $\delta \leq 2$ *so that* $\{\int_0^t < Z(s), dW(s) >\}_{0 \leq t \leq 1}$ *is a square integrable martingale.*
(ii) *The existence of a solution to (2.5) can not be proved by the known result since assumptions in Theorem 2.2 do not imply the Lipschitz continuity of* $z \mapsto L(s, x; D_z H(s, x; z))$ *(see [7]). Indeed, (A.1) and (A.3, i, ii) imply the following (see (ii) in Remark 2.1):*

$$
\begin{aligned}
& D_z\{L(s, x; D_z H(s, x; z))\} \\
= \; & D_z^2 H(s, x; z) D_u L(s, x; D_z H(s, x; z)) = D_z^2 H(s, x; z) z.
\end{aligned}
$$

Besides, $f(x)$ *is not always smooth even when* $L = |u|^2$ *(see Appendix in section 5).*
(iii) *If* $\varphi(t, x)$ *is sufficiently smooth, then it can be considered as a solution of the HJB equation (2.3) (see Appnedix in section 5).*

As an application of Theorem 2.1, we consider h-path processes. We shall refer here to
(A.6). There exist bounded, uniformly continuous functions $\xi : [0, 1] \times \mathbf{R}^d \mapsto \mathbf{R}^d$ and $c : [0, 1] \times \mathbf{R}^d \mapsto [0, \infty)$ such that

$$
L(t, x; u) = \frac{1}{2}|u - \xi(t, x)|^2 + c(t, x) \quad ((t, x; u) \in [0, 1] \times \mathbf{R}^d \times \mathbf{R}^d).
$$

Let $\{\mathbf{X}(t)\}_{0 \leq t \leq 1}$ be a unique weak solution, to the following SDE, which can be constructed by the change of measure (see (1.8) for notation and [20]): for $t \in [0, 1]$,

$$
\mathbf{X}(t) = X_o + \int_0^t \xi(s, \mathbf{X}(s)) ds + W(t). \tag{2.7}
$$

As a corollary to Theorem 2.2, we obtain an approach to the h-path process for $\{\mathbf{X}(t)\}_{0 \leq t \leq 1}$ by the duality theorem.

Corollary 2.2 *Suppose that (A.0, i, iii), (A.3, i, iv) and (A.6) hold. Then for the unique minimizer* $\{X(t)\}_{0 \leq t \leq 1}$ *of* $V(P_0, P_1)$*, there exist* $f_t \in L^1(\mathbf{R}^d, P_t(dx))$ $(t = 0,\, 1)$ *such that the following holds: for any Borel set* $A \subset C([0, 1])$,

$$
\begin{aligned}
P(X(\cdot) \in A) \quad = \quad & E\Big[\exp\Big\{ f_1(\mathbf{X}(1)) - f_0(\mathbf{X}(0)) \\
& \qquad - \int_0^1 c(t, \mathbf{X}(t))dt \Big\} : \mathbf{X}(\cdot) \in A \Big].
\end{aligned} \tag{2.8}
$$

Remark 2.3 *Corollary 2.2 is known (see [24]). In Corollary 2.2,* $V(P_0, P_1)$ *is finite if there exists* $X \in \mathcal{A}$ *for which* $PX(t)^{-1} = P_t$ *(*$t = 0,\ 1$*) and for which the relative entropy of* $PX(\cdot)^{-1}$ *with respect to* $P\mathbf{X}(\cdot)^{-1}$ *on* $C([0,1])$ *is finite (see [20]).*

3 Lemmas.

In this section we give technical lemmas.

The following two lemmas on the property of $V(\cdot,\cdot)$ will play a crucial role in the sequel.

Lemma 3.1 *Suppose that (A.0, ii), (A.1) and (A.2) hold. Then* $(Q,P)\mapsto V(Q,P)$ *is lower semicontinuous.*

(Proof) Suppose that Q_n and P_n weakly converges to Q and P as $n\to\infty$, respectively, and that $\{V(Q_n,P_n)\}_{n\ge 1}$ is bounded. Then we can take $\{X_n(t)\}_{n\ge 1}\subset \mathcal{A}$ such that $PX_n(0)^{-1}=Q_n$ and $PX_n(1)^{-1}=P_n$ $(n\ge 1)$ and that

$$0\le E\Big[\int_0^1 L(t,X_n(t);\beta_{X_n}(t,X_n))dt\Big]-V(Q_n,P_n)\to 0\quad \text{as } n\to\infty.$$

It is easy to see that $\{(X_n(t),\int_0^t\beta_{X_n}(s,X_n)ds):t\in[0,1]\}_{n\ge 1}$ is tight in $C([0,1];\mathbf{R}^{2d})$ from (A.1) (see [30, Theorem 3]).

Take a weakly convergent subsequence $\{(X_{n_k}(t),\int_0^t\beta_{X_{n_k}}(s,X_{n_k})ds):t\in[0,1]\}_{k\ge 1}$ so that

$$\lim_{k\to\infty}E\Big[\int_0^1 L(t,X_{n_k}(t);\beta_{X_{n_k}}(t,X_{n_k}))dt\Big]=\liminf_{n\to\infty}V(Q_n,P_n)<\infty. \tag{3.1}$$

Let $\{(X(t),A(t))\}_{t\in[0,1]}$ denote the limit of $\{(X_{n_k}(t),\int_0^t\beta_{X_{n_k}}(s,X_{n_k})ds):t\in[0,1]\}_{k\ge 1}$ as $k\to\infty$.

Then $\{X(t)-X(0)-A(t)\}_{t\in[0,1]}$ is a standard Brownian motion and $\{A(t)\}_{t\in[0,1]}$ is absolutely continuous (see [30, Theorem 5]).

We can also prove, in the same way as in the proof of [22, (3.17)], the following: from (A.0, ii) and (A.2),

$$\begin{aligned}&\lim_{k\to\infty}E\Big[\int_0^1 L(t,X_{n_k}(t);\beta_{X_{n_k}}(t,X_{n_k}))dt\Big] \\ \ge\ & E\Big[\int_0^1 L\Big(t,X(t);\frac{dA(t)}{dt}\Big)dt\Big].\end{aligned} \tag{3.2}$$

In the same way as in (1.10), the proof is complete since

$$PX(t)^{-1} = \lim_{k\to\infty} PX_{n_k}(t)^{-1} \quad (0 \le t \le 1).$$

Q. E. D.

Lemma 3.2 *Suppose that (A.0, i, ii), (A.1), (A.2), (A.3, iii) and (A.4) hold. Then* $P \mapsto V(P_0, P)$ *is convex.*

(Proof). Take $X_i \in \mathcal{A}$ $(i = 1, 2)$ for which $PX_i(0)^{-1} = P_0$ and

$$\sum_{j=1}^{2} E\Big[\int_0^1 L(t, X_j(t); \beta_{X_j}(t, X_j))dt\Big] < \infty. \tag{3.3}$$

For $i = 1, 2$, $n \ge 1$, $t \in [0, 1]$ and $\omega \in C([0, 1])$, put

$$\begin{aligned} u_{n,i}(t, \omega) &:= 1_{[0,n]}(|\beta_{X_i}(t, \omega)|)\beta_{X_i}(t, \omega), & (3.4)\\ X_{n,i}(t) &:= X_i(0) + \int_0^t u_{n,i}(s, X_i)ds + W_{X_i}(t), & (3.5)\end{aligned}$$

where 1_A denotes the indicator function of A.

Then $\{X_{n,i}(t)\}_{0\le t\le 1} \in \mathcal{A}$ since $u_{n,i}$ $(i = 1, 2)$ are bounded for each $n \ge 1$. In particular, we can assume that on the same probability space $(\Omega, \mathbf{B}, P)$ $\{X_{n,i}(t)\}_{0\le t\le 1}$ $(i = 1, 2)$ are defined by the change of measures (see [20, p. 279]): for $n \ge 1$ and $t \in [0, 1]$,

$$X_{n,i}(t) = X_o + \int_0^t \beta_{X_{n,i}}(s, X_{n,i})ds + W(t) \tag{3.6}$$

(see (1.8) for notation). More precisely, for any $B \in \mathcal{B}(C([0, 1]))$,

$$PX_{n,i}(\cdot)^{-1}(B) = E[M_{n,i}(1, X_o + W(\cdot)) : X_o + W(\cdot) \in B], \tag{3.7}$$

where

$$M_{n,i}(t, \omega) := \exp\Big(\int_0^t \beta_{X_{n,i}}(s, \omega)d\omega(s) - \frac{\int_0^t |\beta_{X_{n,i}}(s, \omega)|^2 ds}{2}\Big).$$

By Itô's formula, we can show that for any $\lambda \in (0, 1)$, $\lambda PX_{n,1}(\cdot)^{-1} + (1 - \lambda)PX_{n,2}(\cdot)^{-1}$ is a distribution of $\{Z_{n,\lambda}(t)\}_{0\le t\le 1}, \in \mathcal{A}$, such that for $t \in [0, 1]$,

$$\beta_{Z_{n,\lambda}}(t,\omega) = \frac{\lambda\beta_{X_{n,1}}(t,\omega)M_{n,1}(t,\omega) + (1-\lambda)\beta_{X_{n,2}}(t,\omega)M_{n,2}(t,\omega)}{\lambda M_{n,1}(t,\omega) + (1-\lambda)M_{n,2}(t,\omega)}. \tag{3.8}$$

Hence, from (A.0, ii), (3.7) and (3.8),

$$\begin{aligned}
& E\Big[\int_0^1 L(t, Z_{n,\lambda}(t); \beta_{Z_{n,\lambda}}(t, Z_{n,\lambda}))dt\Big] \qquad (3.9) \\
= \ & E\Big[\int_0^1 L(t, X_o + W(t); \beta_{Z_{n,\lambda}}(t, X_o + W(\cdot))) \\
& \quad \times\{\lambda M_{n,1}(t, X_o + W(\cdot)) + (1-\lambda)M_{n,2}(t, X_o + W(\cdot))\}dt\Big] \\
\leq \ & \lambda E\Big[\int_0^1 L(t, X_{n,1}(t); \beta_{X_{n,1}}(t, X_{n,1}))dt\Big] \\
& \quad +(1-\lambda)E\Big[\int_0^1 L(t, X_{n,2}(t); \beta_{X_{n,2}}(t, X_{n,2}))dt\Big].
\end{aligned}$$

First we consider the left hand side of (3.9). In the same way as in the proof of Lemma 3.1, we can show that the liminf of the left hand side of (3.9) as $n \to \infty$ is greater than or equal to $V(P_0, \lambda PX_1(1)^{-1} + (1-\lambda)PX_2(1)^{-1})$ since, from (3.3), (3.4), (1.9), (3.7), (3.8) and (A.1), by Hölder's inequality,

$$\begin{aligned}
& E\Big[\int_0^1 |\beta_{Z_{n,\lambda}}(s, Z_{n,\lambda})|^\delta ds\Big] \\
\leq \ & \lambda E\Big[\int_0^1 |\beta_{X_{n,1}}(s, X_{n,1})|^\delta ds\Big] + (1-\lambda)E\Big[\int_0^1 |\beta_{X_{n,2}}(s, X_{n,2})|^\delta ds\Big] \\
\leq \ & \lambda E\Big[\int_0^1 |u_{n,1}(s, X_1)|^\delta ds\Big] + (1-\lambda)E\Big[\int_0^1 |u_{n,2}(s, X_2)|^\delta ds\Big] \\
\leq \ & \lambda E\Big[\int_0^1 |\beta_{X_1}(s, X_1)|^\delta ds\Big] + (1-\lambda)E\Big[\int_0^1 |\beta_{X_2}(s, X_2)|^\delta ds\Big] < \infty.
\end{aligned}$$

Next we consider the right hand side of (3.9). We first consider the case where (A.4, i) holds. For $i = 1$ and 2, by Jensen's inequality, from (1.9),

$$\begin{aligned}
& \int_0^1 E[L(t, X_{n,i}(t); \beta_{X_{n,i}}(t, X_{n,i}))]dt \qquad (3.10) \\
\leq \ & \int_0^1 E[L(t, X_{n,i}(t); u_{n,i}(t, X_i))]dt \to \int_0^1 E[L(t, X_i(t); \beta_{X_i}(t, X_i))]dt
\end{aligned}$$

as $n \to \infty$ from (i) in Remark 2.1, (A.3, iii) and (3.3), by the dominated convergence theorem. Indeed, from (A.4, i) and (3.4),

$$\begin{aligned}
0 &\le L(t, X_{n,i}(t); u_{n,i}(t, X_i)) \\
&\le (1 + \Delta L(0, \infty))(L(t, X_i(t); u_{n,i}(t, X_i)) + 1) \\
&\le (1 + \Delta L(0, \infty))\{L(t, X_i(t); \beta_{X_i}(t, X_i)) + L(t, X_i(t); o) + 1\}.
\end{aligned}$$

(3.9)-(3.10) imply that $P \mapsto V(P_0, P)$ is convex.

Next we consider the case where (A.4, ii) holds. In this case we can let $n \to \infty$ from the beginning since $PX_i(\cdot)^{-1}$ ($i = 1, 2$) are absolutely continuous with respect to $P(X_o + W(\cdot))^{-1}$ (see [20]). Hence (3.9) immediately implies that $P \mapsto V(P_0, P)$ is convex.

Q. E. D.

In the same way as to $\mathcal{A}$, we define the set of semimartingales $\mathcal{A}_t$ in $C([t, 1])$. Let us recall the following result which relies on the fact that (A.3, ii) implies that for any $(t, x) \in [0, 1] \times \mathbf{R}^d$, $L(t, x; u)$ is strictly convex in u.

Lemma 3.3 ([13, p. 210, Remark 11.2]) *Suppose that (A.1) and (A.3) hold. Then for any $f \in C_b^\infty(\mathbf{R}^d)$, the HJB equation (2.3) with $\varphi(1, \cdot) = f$ has a unique solution φ, $\in C^{1,2}([0, 1] \times \mathbf{R}^d)$, which can be written as follows:*

$$\begin{aligned}
\varphi(t, x) &= \sup_{X \in \mathcal{A}_t} \Big\{ E[\varphi(1, X(1)) | X(t) = x] \\
&\qquad - E\Big[\int_t^1 L(s, X(s); \beta_X(s, X)) ds \Big| X(t) = x \Big] \Big\},
\end{aligned} \tag{3.11}$$

where for the minimizer $X \in \mathcal{A}_t$, the following holds:

$$\beta_X(s, X) = D_x H(s, X(s); D_x \varphi(s, X(s))).$$

Next we state and prove lemmas which will be used in the proof of Proposition 2.2. The following lemma which can be proved easily from [4], [14] and [21] slightly improves [3, 4].

Lemma 3.4 *Suppose that* $\{P(t,dx)\}_{t\in[0,1]}$ *is a family of Borel probability measures on* $\mathbf{R}^d$ *such that*
(i) $p(t,x) := P(t,dx)/dx$ *exists for all* $t \in (0,1]$,
(ii) there exists $b(t,x) : [0,1]\times\mathbf{R}^d \mapsto \mathbf{R}^d$ *which satisfies the following:*

$$\frac{\partial P(t,dx)}{\partial t} = \frac{1}{2}\triangle P(t,dx) - div(b(t,x)P(t,dx))\,(\textit{in dist. sense}), \tag{3.12}$$

$$\int_0^1 dt \int_{\mathbf{R}^d} |b(t,x)|^2 P(t,dx) < \infty. \tag{3.13}$$

Then (1) $\int_t^1 ds \int_{\mathbf{R}^d} |D_x \log p(s,x)|^2 p(s,x)dx$ *is finite for all* $t \in (0,1]$, *and (2) there exists a unique weak solution* $\{X(t)\}_{0\le t\le 1}$ *to the following (see (1.8) for notation): for* $t \in [0,1]$,

$$X(t) = X(0) + \int_0^t b(s,X(s))ds + W(t), \tag{3.14}$$

$$P(X(t)\in dx) = P(t,dx). \tag{3.15}$$

(Proof) From [21], there exists a measurable $\tilde{b} : [0,1]\times\mathbf{R}^d \mapsto \mathbf{R}^d$ for which (3.14)-(3.15) hold if we replace b by $\tilde{b}$. Hence, from [14], (1) holds, which implies (2) by [4].

Q.E.D.

Put

$$\underline{V}(P_0,P_1) := \inf \int_0^1 \int_{\mathbf{R}^d} L(t,x;b(t,x))P(t,dx)dt, \tag{3.16}$$

where the infimum is taken over all $(b(t,x),P(t,dx))$ for which $P(t,dx)$ $(0 \le t \le 1)$ are Borel probability measures, on $\mathbf{R}^d$, such that (i) in Lemma 3.4 and (3.12) hold and that $P(t,dx) = P_t$ $(t = 0,1)$.

The following which can be proved from Lemma 3.4 can be considered as a generalization of [21, Lemma 2.5] which is a stochastic control counterpart of [2] (see also [28, p. 239]) when $L(t,x;u) = |u|^2$.

Lemma 3.5 *Suppose that (A.0)-(A.1) and (A.4, ii) hold. Then* $V(P_0,P_1) = \underline{V}(P_0,P_1)$.

(Proof) We first prove

$$V(P_0, P_1) \geq \underline{V}(P_0, P_1). \tag{3.17}$$

Take $X \in \mathcal{A}$ such that $E[\int_0^1 L(t, X(t); \beta_X(t, X))dt]$ is finite and that $PX(t)^{-1} = P_t$ $(t = 0, 1)$. Set $b_X(t, X(t)) := E[\beta_X(t, X)|(t, X(t))]$.

Then $P(X(t) \in dx)/dx$ exists for all $t \in (0, 1]$ since $PX(\cdot)^{-1}$ is absolutely continuous with respect to $P(X_o + W(\cdot))^{-1}$ from (A.1) and (A.4, ii) (see (1.8) for notation and [20]). $(b_X(t, x), P(X(t) \in dx))$ satisfies (3.12). Indeed, for any $f \in C_o^\infty(\mathbf{R}^d)$ and $t \in [0, 1]$, by Itô's formula,

$$\begin{aligned}
&\int_{\mathbf{R}^d} f(x) P(X(t) \in dx) - \int_{\mathbf{R}^d} f(x) P(X(0) \in dx) \qquad (3.18)\\
=\ & E[f(X(t)) - f(X(0))]\\
=\ & \int_0^t ds E\Big[\frac{1}{2} \triangle f(X(s)) + < \beta_X(s, X), Df(X(s)) > \Big]\\
=\ & \int_0^t ds E\Big[\frac{1}{2} \triangle f(X(s)) + < E[\beta_X(s, X)|(s, X(s))], Df(X(s)) > \Big]\\
=\ & \int_0^t ds \int_{\mathbf{R}^d} \Big(\frac{1}{2} \triangle f(x) + < b_X(s, x), Df(x) > \Big) P(X(s) \in dx).
\end{aligned}$$

Hence, from (A.0, ii), by Jensen's inequality,

$$\begin{aligned}
&E\Big[\int_0^1 L(t, X(t); \beta_X(t, X))dt\Big] \qquad (3.19)\\
\geq\ & E\Big[\int_0^1 L(t, X(t); b_X(t, X(t)))dt\Big]\\
=\ & \int_0^1 dt \int_{\mathbf{R}^d} L(t, x; b_X(t, x)) P(X(t) \in dx) \geq \underline{V}(P_0, P_1),
\end{aligned}$$

which implies (3.17).

Next we prove the opposite inequality of (3.17). Take $(b(t, x), P(t, dx))$ for which $P(t, dx)$ are Borel probability measures on $\mathbf{R}^d$ $(0 \leq t \leq 1)$, (i) in Lemma 3.4 and (3.12) hold and $P(t, dx) = P_t(dx)$ $(t = 0, 1)$ and for which $\int_0^1 dt \int_{\mathbf{R}^d} L(t, x; b(t, x)) P(t, dx)$ is finite.

Then, from (A.1) and (A.4, ii), (3.13) holds. From Lemma 3.4, there exists a Markov process $\{X(t)\}_{0 \leq t \leq 1}$ for which (3.14)-(3.15) hold. In particular, we have

$$
\begin{aligned}
&\int_0^1 dt \int_{\mathbf{R}^d} L(t,x;b(t,x))P(t,dx) \qquad (3.20)\\
= \quad & E\Big[\int_0^1 L(t,X(t);b(t,X(t)))dt\Big] \geq V(P_0,P_1).
\end{aligned}
$$

Q. E. D.

4 Proof of our result.

In this section we give the proof of our result.

Proposition 2.1 can be proved in the same way as in Lemma 3.1 and we omit the proof.

Since $P \mapsto V(P_0, P)$ is lower semicontinuous and convex from Lemmas 3.1 and 3.2, we can reduce the proof of Theorem 2.1 to the fact that $V(P_0, \cdot)^{**}(P) = V(P_0, P)$.

(Proof of Theorem 2.1). From Lemmas 3.1 and 3.2 and [8, Theorem 2.2.15 and Lemma 3.2.3],

$$V(P_0, P_1) = \sup_{f \in C_b(\mathbf{R}^d)} \left\{ \int_{\mathbf{R}^d} f(x) P_1(dx) - V_{P_0}^*(f) \right\}, \tag{4.1}$$

where for $f \in C_b(\mathbf{R}^d)$,

$$V_{P_0}^*(f) := \sup_{P \in \mathcal{M}_1(\mathbf{R}^d)} \left\{ \int_{\mathbf{R}^d} f(x) P(dx) - V(P_0, P) \right\},$$

and $\mathcal{M}_1(\mathbf{R}^d)$ denotes the complete separable metric space, with a weak topology, of Borel probability measures on $\mathbf{R}^d$.

Take $\Phi \in C_o^\infty([-1,1]^d; [0,\infty))$ for which $\int_{\mathbf{R}^d} \Phi(x)dx = 1$, and for $\varepsilon > 0$, put

$$\Phi_\varepsilon(x) := \varepsilon^{-d} \Phi(x/\varepsilon).$$

We prove the following which implies (2.2):

$$V(P_0, P_1) \geq \mathcal{V}(P_0, P_1) \geq \frac{V(\Phi_\varepsilon * P_0, \Phi_\varepsilon * P_1)}{1 + \Delta L(0, \varepsilon)} - \Delta L(0, \varepsilon), \tag{4.2}$$

where $*$ denotes the convolution of two measures and should be distinguished from * in (4.1). Indeed, from (A.2), Lemma 3.1 and (4.2), we have (2.2).

The first inequality in (4.2) can be proved from (4.1) and (4.3) below: for any $f \in C_b^\infty(\mathbf{R}^d)$, from Lemma 3.3,

$$V_{P_0}^*(f) = \sup \left\{ E[f(X(1))] - E\left[\int_0^1 L(t, X(t); \beta_X(t, X)) dt \right] : \right. \tag{4.3}$$

$$
\begin{aligned}
&\qquad X\in\mathcal{A}, PX(0)^{-1}=P_0\Big\}\\
&= \int_{\mathbf{R}^d}\varphi_f(0,x)P_0(dx),
\end{aligned}
$$

where φ_f denotes the unique classical solution to the HJB equation (2.3) with $\varphi(1,\cdot)=f(\cdot)$.

We prove the second inequality in (4.2). For $f\in C_b(\mathbf{R}^d)$, put

$$
f_\varepsilon(x):=\int_{\mathbf{R}^d}f(y)\Phi_\varepsilon(x-y)dy. \tag{4.4}
$$

Then $f_\varepsilon\in C_b^\infty(\mathbf{R}^d)$ and, from (4.3),

$$
\begin{aligned}
&\mathcal{V}(P_0,P_1) &(4.5)\\
\ge\ & \int_{\mathbf{R}^d}f_\varepsilon(x)P_1(dx)-V_{P_0}^*(f_\varepsilon)\\
\ge\ & \int_{\mathbf{R}^d}f(x)\Phi_\varepsilon*P_1(dx)-\frac{(V_{\Phi_\varepsilon*P_0})^*((1+\Delta L(0,\varepsilon))f)}{1+\Delta L(0,\varepsilon)}-\Delta L(0,\varepsilon).
\end{aligned}
$$

Indeed, for any $X\in\mathcal{A}$, from (A.2),

$$
\begin{aligned}
&E[f_\varepsilon(X(1))]-E\Big[\int_0^1 L(t,X(t);\beta_X(t,X))dt\Big]\\
=\ & \int_{\mathbf{R}^d}\Phi(z)dzE[f(X(1)-\varepsilon z)]-E\Big[\int_0^1 L(t,X(t);\beta_X(t,X))dt\Big]\\
\le\ & \int_{\mathbf{R}^d}\Phi(z)dz\Big\{E[f(X(1)-\varepsilon z)]-E\Big[\int_0^1\frac{L(t,X(t)-\varepsilon z;\beta_X(t,X))}{1+\Delta L(0,\varepsilon)}dt\Big]\Big\}\\
&\quad +\Delta L(0,\varepsilon).
\end{aligned}
$$

(4.1) and (4.5) imply the second inequality in (4.2).

Q. E. D.

(Proof of Corollary 2.1). Identity (2.2) implies, by Itô's formula, that there exists a sequence $\{\varphi_n\}_{n\ge1}$ of classical solutions, to the HJB equation (2.3), such that for any minimizer $\{X(t)\}_{0\le t\le1}$ of $V(P_0,P_1)$,

$$
\begin{aligned}
& E\Big[\int_0^1 L(t,X(t);\beta_X(t,X))dt\Big] && (4.6)\\
= \; & \lim_{n\to\infty} E\Big[\int_0^1 \{<\beta_X(t,X), D_x\varphi_n(t,X(t))> \\
& \qquad -H(t,X(t);D_x\varphi_n(t,X(t)))\}dt\Big].
\end{aligned}
$$

Since (A.3, i, ii) imply that $L(t,x;u)$ is of class C^3 and is strictly convex in u for any $(t,x)\in[0,1]\times\mathbf{R}^d$, (4.6) completes the proof.

Indeed, from (A.0, ii), the following holds (see e.g., [28]): for any $(t,x)\in[0,1]\times\mathbf{R}^d$,

$$
L(t,x;u) := \sup_{z\in\mathbf{R}^d}\{<z,u> -H(t,x;z)\}. \tag{4.7}
$$

Therefore (4.6) is equivalent to

$$
\begin{aligned}
0 \;=\; & \lim_{n\to\infty} E\Big[\int_0^1 |L(t,X(t);\beta_X(t,X)) && (4.8)\\
& -\{<\beta_X(t,X), D_x\varphi_n(t,X(t))> -H(t,X(t);D_x\varphi_n(t,X(t)))\}|dt\Big],
\end{aligned}
$$

which implies that there exists a subsequence $\{n_k\}_{k\ge 1}$ for which

$$
\begin{aligned}
& L(t,X(t);\beta_X(t,X)) && (4.9)\\
= \; & \lim_{k\to\infty}\{<\beta_X(t,X), D_x\varphi_{n_k}(t,X(t))> -H(t,X(t);D_x\varphi_{n_k}(t,X(t)))\}
\end{aligned}
$$

$dtdPX(\cdot)^{-1}$-a.e..

Q.E.D.

From Proposition 2.1, Lemmas 3.4 and 3.5, we prove Proposition 2.2.

(Proof of Proposition 2.2) From Proposition 2.1, $V(P_0,P_1)$ has a minimizer. From Lemma 3.5, in the same way as in (3.19), we can prove that $\underline{V}(P_0,P_1)$ has a minimizer. Hence, from Lemmas 3.4 and 3.5, there exists a Markovian minimizer of $V(P_0,P_1)$.

If for any $(t,x)\in[0,1]\times\mathbf{R}^d$, $L(t,x;u)$ is strictly convex in u, then all minimizers of $V(P_0,P_1)$ are Markovian.

Indeed, Lemma 3.5 and (3.19) imply that if X is a minimizer of $V(P_0, P_1)$, then

$$\beta_X(t, X) = b_X(t, X(t)) \quad dtdPX(\cdot)^{-1} - a.e..$$

(A.4, ii) implies that $PX(\cdot)^{-1}$ is absolutely continuous with respect to $P(X_o + W(\cdot))^{-1}$ (see (1.8) for notation). Hence $\{X(t)\}_{0\le t\le 1}$ is Markovian.

In particular, from Lemmas 3.4 and 3.5, the set of all minimizers of $\underline{V}(P_0, P_1)$ is equal to that of all $\{(b_X(t, x), P(X(t) \in dx))\}_{0\le t\le 1}$ for the Markovian minimizers $\{X(t)\}_{0\le t\le 1}$ of $V(P_0, P_1)$.

Hence, to prove the uniqueness of a minimizer of $V(P_0, P_1)$, we only have to prove that of b for which there exists $\{P(t, dx)\}_{0\le t\le 1}$ such that $\{(b(t, x), P(t, dx))\}_{0\le t\le 1}$ is a minimizer of $\underline{V}(P_0, P_1)$.

Indeed, since $PX(\cdot)^{-1}$ is absolutely continuous with respect to $P(X_o + W(\cdot))^{-1}$ for a Markovian minimizer $\{X(t)\}_{0\le t\le 1}$ of $V(P_0, P_1)$, $\{b_X(t, x)\}_{0\le t\le 1}$ determines $PX(\cdot)^{-1}$.

Take minimizers $(b_i(t, x), P_i(t, dx))$ of $\underline{V}(P_0, P_1)$ $(i = 0,\ 1)$. For any $\lambda \in (0, 1)$, put $p_i(t, x) := P_i(t, dx)/dx$ and

$$b_\lambda(t, x) := \frac{(1-\lambda)b_0(t, x)p_0(t, x) + \lambda b_1(t, x)p_1(t, x)}{(1-\lambda)p_0(t, x) + \lambda p_1(t, x)} \quad (0 < t \le 1),$$

provided that the denominator is positive. Then

$$\begin{aligned}
& \underline{V}(P_0, P_1) \qquad (4.10)\\
\le\ & \int_0^1 dt \int_{\mathbf{R}^d} L(t, x; b_\lambda(t, x))((1-\lambda)p_0(t, x) + \lambda p_1(t, x))dx \\
\le\ & (1-\lambda)\int_0^1 ds \int_{\mathbf{R}^d} L(t, x; b_0(t, x))p_0(t, x)dx \\
& \quad +\lambda \int_0^1 ds \int_{\mathbf{R}^d} L(t, x; b_1(t, x))p_1(t, x)dx \\
=\ & \underline{V}(P_0, P_1).
\end{aligned}$$

Indeed,

$$\frac{\partial((1-\lambda)p_0(t, x) + \lambda p_1(t, x))}{\partial t}$$

$$= \frac{1}{2}\triangle((1-\lambda)p_0(t,x)+\lambda p_1(t,x)) \\ -div(b_\lambda(t,x)((1-\lambda)p_0(t,x)+\lambda p_1(t,x)))$$

in dist. sense.

From (4.10), by the strict convexity of $u \mapsto L(t,x;u)$ ($(t,x) \in [0,1]\times \mathbf{R}^d$),

$$b_0(t,x) = b_1(t,x) \quad \text{if} \quad p_0(t,x)p_1(t,x) > 0. \tag{4.11}$$

Putting $b_i(t,x) = b_j(t,x)$ if $p_i(t,x) = 0$ ($i,\ j = 0,\ 1,\ i \neq j$), the proof is over.
Q. E. D.

From Theorem 2.1 and Proposition 2.2, we prove Theorem 2.2.
(Proof of Theorem 2.2). Take $\{\varphi_n\}_{n\geq 1}$ in (4.8). Then for $t \in [0,1]$, by Itô's formula,

$$\begin{aligned} & \varphi_n(t,X(t)) - \varphi_n(0,X(0)) \qquad (4.12)\\ = & \int_0^t \{< b_X(s,X(s)), D_x\varphi_n(s,X(s)) > -H(s,X(s);D_x\varphi_n(s,X(s)))\}ds \\ & \quad + \int_0^t < D_x\varphi_n(s,X(s)), dW(s) > . \end{aligned}$$

By Doob's inequality (see [16]),

$$\begin{aligned} & E\Big[\sup_{0\leq t\leq 1}\Big|\int_0^t < D_x\varphi_n(s,X(s)), dW(s) > \qquad (4.13)\\ & \quad -\int_0^t < D_uL(s,X(s);b_X(s,X(s))), dW(s) >\Big|^2\Big] \\ \leq & \ 4E\Big[\int_0^1 |D_x\varphi_n(s,X(s)) - D_uL(s,X(s);b_X(s,X(s)))|^2 ds\Big] \\ \leq & \ 4CE\Big[\int_0^1 \{L(s,X(s);b_X(s,X(s))) - < b_X(s,X(s)), D_x\varphi_n(s,X(s)) > \\ & \quad +H(s,X(s);D_x\varphi_n(s,X(s)))\}ds\Big] \\ \to & \ 0 \quad \text{as } n \to \infty \end{aligned}$$

from (4.8), where

$C := 2\sup\{< D_u^2 L(t,x;u)z, z >: (t,x,u,z) \in [0,1]\times \mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R}^d, |z| = 1\}$.

Indeed, for a smooth, strictly convex function $f : \mathbf{R}^d \mapsto [0,\infty)$ for which $f(v)/|v| \to \infty$ as $|v| \to \infty$ and $(u,z) \in \mathbf{R}^d \times \mathbf{R}^d$, by Taylor's Theorem, there exists $\theta \in (0,1)$ such that

$$\begin{aligned} & f(u) - \{< u, z > - f^*(z)\} \\ = \ & f^*(z) - f^*(Df(u)) - < Df^*(Df(u)), z - Df(u) > \\ = \ & \frac{< D^2 f^*(z + \theta(z - Df(u)))(z - Df(u)), z - Df(u) >}{2}, \end{aligned}$$

and $D^2 f^*(z) = D^2 f(Df^*(z))^{-1}$ (see (ii) in Remark 2.1).

From (4.8), (4.12) and (4.13), $\varphi_n(1,y) - \varphi_n(0,x)$ and $\varphi_n(t,y) - \varphi_n(0,x)$ are convergent in $L^1(\mathbf{R}^d \times \mathbf{R}^d, P((X(0),X(1)) \in dxdy))$ and $L^1(\mathbf{R}^d \times [0,1] \times \mathbf{R}^d, P((X(0),(t,X(t))) \in dxdtdy))$, respectively.

From (A.4, ii), $PX(\cdot)^{-1}$ is absolutely continuous with respect to $P(X_o + W(\cdot))^{-1}$ (see (1.8) for notation). In particular,

$$p(t,y) := P(X(t) \in dy)/dy \text{ exists } (t \in (0,1]),$$

$$p(0,x;t,y) := P(X(t) \in dy | X(0) = x)/dy \text{ exists } P_0(dx) - a.e. \ (t \in (0,1]).$$

Therefore $P((X(0),X(1)) \in dxdy)$ and $P((X(0),(t,X(t))) \in dxdtdy)$ are absolutely continuous with respect to $P_0(dx)P_1(dy)$ and $P_0(dx)P((t,X(t)) \in dtdy)$, respectively. Indeed,

$$P((X(0),X(1)) \in dxdy) = \frac{p(0,x;1,y)}{p(1,y)} P_0(dx)P_1(dy),$$

$$\begin{aligned} & P((X(0),(t,X(t))) \in dxdtdy) \\ = \ & \frac{p(0,x;t,y)}{p(t,y)} P_0(dx) P((t,X(t)) \in dtdy). \end{aligned}$$

Hence, from [26, Prop. 2], there exist $f \in L^1(\mathbf{R}^d, P_1(dx))$, $f_0 \in L^1(\mathbf{R}^d, P_0(dx))$, $\varphi_0 \in L^1(\mathbf{R}^d, P_0(dx))$ and $\varphi \in L^1([0,1] \times \mathbf{R}^d, P((t,X(t)) \in dtdy))$ such that

$$\lim_{n\to\infty} E[|\varphi_n(1,X(1)) - \varphi_n(0,X(0)) - \{f(X(1)) - f_0(X(0))\}|] = 0, \quad (4.14)$$

$$\lim_{n\to\infty} E\Big[\int_0^1 |\varphi_n(t,X(t)) - \varphi_n(0,X(0)) - \{\varphi(t,X(t)) - \varphi_0(X(0))\}|dt\Big] = 0. \quad (4.15)$$

Put

$$\begin{aligned} Y(t) \quad &:= \quad f_0(X(0)) + \int_0^t L(s,X(s);b_X(s,X(s)))ds \qquad (4.16) \\ &\quad + \int_0^t < D_u L(s,X(s);b_X(s,X(s))), dW(s) > . \end{aligned}$$

From (4.8) and (4.12)-(4.14), (2.5) holds.
From (4.8), (4.12)-(4.13) and (4.15)-(4.16), (2.6) holds.

Q. E. D.

We prove Corollary 2.2 from Theorem 2.2.
(Proof of Corollary 2.2). Assumptions in Corollary 2.2 imply those in Theorem 2.2.

When (A.6) holds, from (4.8) and (4.12)-(4.14),

$$\begin{aligned} & f(X(1)) - f_0(X(0)) - \int_0^t c(s,X(s))ds \qquad (4.17) \\ = & \int_0^1 < b_X(s,X(s)) - \xi(s,X(s)), dX(s) - \xi(s,X(s))ds > \\ & -\frac{1}{2}\int_0^1 |b_X(s,X(s)) - \xi(s,X(s))|^2 ds, \end{aligned}$$

which completes the proof (see [20]).

Q. E. D.

5 Appendix.

For the readers' convenience, we describe some properties of the h-path process $\{X_h(t)\}_{0\le t\le 1}$ introduced in section 1 and explain (iii) in Remark 2.2.

Suppose that $L = |u|^2$ and that (2.1) is finite. Then the probability law of $\{X_h(t)\}_{0\le t\le 1}$ is absolutely continuous with respect to that of $\{X_o + W(t)\}_{0\le t\le 1}$ (see (1.8) for notation). In particular, P_1 is absolutely continuous with respect to the Lesbegue measure dx (see [20]).

It is known that there exists a unique pair of nonnegative, σ-finite Borel measures (ν_0, ν_1) for which

$$\begin{cases} P_0(dx) = (\int_{\mathbf{R}^d} g_1(x-y)\nu_1(dy))\nu_0(dx), \\ P_1(dy) = (\int_{\mathbf{R}^d} g_1(x-y)\nu_0(dx))\nu_1(dy), \end{cases} \tag{5.1}$$

where for $x \in \mathbf{R}^d$ and $t > 0$,

$$g_t(x) := \frac{1}{(2\pi t)^{d/2}} \exp\Big(-\frac{|x|^2}{2t}\Big).$$

(5.1) is called Schrödinger's functional equation (see [17] and also [26] for the recent development).

For $x \in \mathbf{R}^d$, put

$$h(t,x) := \begin{cases} \int_{\mathbf{R}^d} g_{(1-t)}(x-y)\nu_1(dy) & (0 \le t < 1), \\ \frac{\nu_1(dx)}{dx} & (t = 1). \end{cases} \tag{5.2}$$

Then the h-path process $\{X_h(t)\}_{0\le t\le 1}$ is the unique weak solution to the following (see [18]): for $t \in [0,1]$,

$$X_h(t) = X_o + \int_0^t D_x \log h(s, X_h(s))ds + W(t). \tag{5.3}$$

It is known that for any Borel set $A \subset C([0,1])$,

$$P(X_h(\cdot) \in A) = E\Big[\frac{h(1, X_o + W(1))}{h(0, X_o)} : X_o + W(\cdot) \in A\Big]. \tag{5.4}$$

In particular,

$$P((X_h(0), X_h(1)) \in dxdy) = \nu_0(dx)g(x-y)\nu_1(dy). \tag{5.5}$$

From (5.1)-(5.2), $h(1,x)$ is not always smooth. But it is also known that $h \in C^{1,2}([0,1)\times \mathbf{R}^d)$ (see [18]) and $\varphi(t,x) := \log h(t,x)$ satisfies the HJB equation (2.3). Indeed, from [18],

$$\frac{\partial h(t,x)}{\partial t} + \frac{1}{2}\triangle h(t,x) = 0 \quad ((t,x)\in (0,1)\times \mathbf{R}^d).$$

From [23, Lemma 3.4], we also have

$$\begin{aligned} & V(P_0,P_1) \qquad (5.6)\\ = & \int_{\mathbf{R}^d} |x|^2 (P_0(dx)+P_1(dx)) - 2\log\Big\{ \iint_{\mathbf{R}^d\times\mathbf{R}^d} \exp(< x, z_1 > + < y, z_0 > \\ & - < z_0, z_1 >) P((X_h(0), X_h(1)) \in dz_0 dz_1)\Big\} \\ & +2\int_{\mathbf{R}^d} \Big(\log \frac{P_1(dx)}{dx}\Big) P_1(dx) + d\log(2\pi). \end{aligned}$$

Next we formally show that φ in Theorem 2.2 satisfies the HJB equation (2.3).

Suppose that $L(t,x;u)$ is differentiable in u for all $(t,x)\in[0,1]\times\mathbf{R}^d$ and that (2.5) holds for sufficiently smooth φ. Then

$$\frac{\partial \varphi(t,X(t))}{\partial t} + \frac{1}{2}\triangle\varphi(t,X(t)) + H(t,X(t);D_x\varphi(t,X(t))) = 0, \qquad (5.7)$$

$dtdPX(\cdot)^{-1}$-a.e.. Indeed, by Itô's formula,

$$\begin{aligned} & \frac{\partial \varphi(t,X(t))}{\partial t} + \frac{1}{2}\triangle\varphi(t,X(t)) + < b_X(t,X(t)), D_x\varphi(t,X(t)) > \\ = & L(t,X(t);b_X(t,X(t))), \end{aligned}$$

$$D_x\varphi(t,X(t)) = D_u L(t,X(t);b_X(t,X(t))),$$

$dtdPX(\cdot)^{-1}$-a.e.. In particular,

$$\begin{aligned} & < b_X(t,X(t)), D_x\varphi(t,X(t)) > - L(t,X(t);b_X(t,X(t))) \\ = & H(t,X(t);D_x\varphi(t,X(t))) \end{aligned}$$

(see e.g. [28, p. 55]).

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