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Nonnegative Functions In Weighted Hardy Spaces

by

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and

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Abstract. Let $W$ be a nonnegative summable function whose logarithm is also summable with respect to the Lebesgue measure on the unit circle. For $0 < p < \infty$, $H^p(W)$ denotes a weighted Hardy space on the unit circle. When $W \equiv 1$, $H^p(W)$ is the usual Hardy space $H^p$. We are interested in $H^p(W)_+$ the set of all nonnegative functions in $H^p(W)$. If $p \geq 1/2$, $H^p_{+}$ consists of constant functions. However $H^p(W)_+$ contains a nonconstant nonnegative function for some weight $W$. In this paper, if $p \geq 1/2$ we determine $W$ and describe $H^p(W)_+$ when the linear span of $H^p(W)_+$ is of finite dimension. Moreover we show that the linear span of $H^p(W)_+$ is of infinite dimension for arbitrary weight $W$ when $0 < p < 1/2$. 
§1. Introduction and preliminaries

Let $W$ be a nonnegative function in $L^1 = L^1(d\theta/2\pi, \partial D)$ where $D$ is the open unit disc and $\partial D$ is its boundary. For $0 < p < \infty$, a weighted Hardy space $H^p(W)$ denotes the closure of all analytic polynomials in $L^p(W) = L^p(Wd\theta/2\pi, \partial D)$. We may assume that $\log W$ is in $L^1$. For otherwise $H^p(W) = L^p(W)$. If $W \equiv 1$, then $H^p(W)$ is the usual Hardy space $H^p$. Let $N_*$ be the Smirnov class (see [1]), then $H^p = N_* \cap L^p$. A function $h$ in $N_*$ is called outer if it is invertible in $N_*$ and a function $q$ in $N_*$ is called inner if $|q| = 1$ a.e. on $\partial D$. If $W = |h|^p$ and $h$ is an outer function in $H^p$, it is known that $H^p(W) = h^{-1}H^p$ and so $H^p(W)$ is a subset of $N_*$. Put

$$H^p(W)_+ = \{ s \in H^p(W) ; s \geq 0 \text{ a.e. on } \partial D \}.$$

If $W = |h|^p$, then $hH^p(W)_+ = \{ g \in H^p : \arg g = \arg h \text{ a.e. on } \partial D \}$. In this paper, the dimension of $H^p(W)_+$ means that of the linear span of $H^p(W)_+$ in $H^p(W)$.

J. Neuwirth and D. J. Newman [7] showed that $H^{1/2}_+ = R_+ = \{ s \in H^1 : \arg s = \arg h \text{ a.e. on } \partial D \}$ when $W = |h|$. In fact, this set is related to the set of extremal functions of a well known linear extremal problem in $H^1$. (cf. [5],[6],[3],[4]). Hence $H^1(W)_+$ is known enough. However $H^p(W)_+$ has not studied before when $p \neq 1$ and $W \neq 1$. In this paper, we study $H^p(W)_+$ for arbitrary $p$. In §2, we show that $H^p(W)_+$ is of infinite dimension when $0 < p < 1/2$ and $W$ is arbitrary. In §3, we describe $H^p(W)_+$ when $p \geq 1/2$ and $H^p(W)_+$ is of finite dimension. In §4, we show that $H^p(W)_+$ is of finite dimension if $p \geq 1/2$ and $W^{-1}$ is locally in $L^{1/2p-1}$ except at a finite set.

Now we give a general result about $H^p(W)_+$ for $0 < p < \infty$. For any nonzero function $f$ in $H^p$, put $S^h_p = \{ g \in H^p : \arg g = \arg f \text{ a.e. on } \partial D \}$ for $0 < p < \infty$. Put $(N_*)_+ = \{ s \in N_* ; s \geq 0 \text{ a.e. on } \partial D \}$.

**Proposition 1.** Let $0 < p < \infty$. Suppose $W = |h|^p$ and $h$ is an outer function in $H^p$.

1. $H^p(W)_+ = h^{-1}S^h_p$ and $R_+ \subseteq H^p(W)_+ \subseteq (N_*)_+$.
   
2. If $W$ is in $L^\infty$ then $H^p(W)_+ \supseteq H^p_+$.
   
3. If $W^{-1}$ is in $L^\infty$ then $H^p(W)_+ \subseteq H^p_+$.

Proof. Since $H^p(W) = h^{-1}H^p$, (1) follows from the definition of $S^h_p$. (2) and (3) are clear.

**Lemma 1.** If $f$ and $g$ are nonzero functions in $N_*$ such that $f/g$ is nonnegative and nonconstant almost everywhere on $\partial D$, then $f + g$ is not outer.
Proof. Put \( h = f + g \), then \( f/h + g/h = 1 \), \( 0 \leq f/h \leq 1 \) and \( 0 \leq g/h \leq 1 \) on \( \partial D \). If \( h \) is outer, then both \( f/h \) and \( g/h \) belong to \( N_\infty \cap L^\infty = H^\infty \). Thus both \( f/h \) and \( g/h \) are constant and so \( f/g \) is constant. This contradiction shows the lemma.

**Proposition 2.** Let \( 0 < p < \infty \). If \( H^p(W)_+ \neq R_+ \), then there exists a function in \( H^p(W)_+ \) which is not outer. Hence the dimension of \( H^p(W)_+ \) is bigger than or equal to three.

Proof. If \( s \in H^p(W)_+ \) is nonconstant, by Lemma 1 \( s + 1 \) is not outer and so this implies the first part. Hence \( qs = s + 1 \) belongs to \( H^p(W)_+ \) where \( q \) is a nonconstant inner part of \( s + 1 \). Since both \((1 + q)g\) and 1 belong to \( H^p(W)_+ \), this implies the second part.

\[ \]$
\textbf{§2. General weights for } 0 < p < 1/2.$

If \( 0 < p < 1/2 \), then \((z - b)(1 - \overline{b}z)/(z - a)(1 - \overline{a}z)\) belongs to \( H^p \) where \(|a| = 1\) and \(|b| \leq 1\) and so \( H^p_+ \) is of infinite dimension. In this section, we show that \( H^p(W)_+ \) is of infinite dimension for arbitrary weight \( W \).

**Lemma 2.** Let \( 0 < p < 1/2 \) and if \( h \) is a function in \( H^p \), \( h(z)/(1 - e^{it})/(1 - \overline{e}^t z) \) belongs to \( H^p \) for a.e. \( e^{it} \).

Proof. If \( k(z) = z/(1 - z)^2 \), \(|k|^p \) and \(|h|^p \) belongs to \( L^1 \) and hence \(|k|^p * |h|^p \in L^1 \).

\[
|k|^p * |h|^p(e^{it}) = \int_0^{2\pi} |k(e^{i(t-\theta)}h(e^{i\theta})|^p d\theta / 2\pi = \int_0^{2\pi} \left| \frac{e^{i\theta}h(e^{i\theta})}{(e^{i\theta} - e^{it})(1 - e^{it}e^{i\theta})} \right|^p d\theta / 2\pi < \infty
\]

This implies that \( h(z)/(1 - e^{it})(1 - \overline{e}^t z) \) belongs to \( H^p \) for a.e. \( e^{it} \)

**Theorem 1.** For arbitrary weight \( W \), \( H^p(W)_+ \) is of infinite dimension.

Proof. Suppose that \( W = |h|^p \) and \( h \) is an outer function in \( H^p \). Since \( H^p(W)_+ = h^{-1}S^h \) by (1) of Proposition 1, it is enough to prove that \( S^h_p \) is of infinite dimension. By Lemma 2, for any finite \( n \), \( h(z)\prod_{j=1}^{n}(z - b_j)/(1 - \overline{b}_j z)\prod_{j=1}^{n}(z - a_j)/(1 - \overline{a}_j z) \) belongs to \( H^p \) where \(|b_j| < 1 \) and \(|a_j| = 1 \) for \( 1 \leq j \leq n \). Since \( \prod_{j=1}^{n}(z - b_j)/(1 - \overline{b}_j z)\prod_{j=1}^{n}(z - a_j)/(1 - \overline{a}_j z) \) is nonnegative on \( \partial D \), \( S^h_p \) is of infinite dimension.
§3. General weights for $1/2 \leq p < \infty$

Unlike when $0 < p < 1/2$, if $1/2 \leq p < \infty$, then $H^p(W)_+$ may be of finite dimension for some weight $W$. In this section, we describe $H^p(W)_+$ when $H^p(W)_+$ is of finite dimension.

**Lemma 3.** Let $1/2 \leq p < \infty$. If $W = W_1W_0$, and $W_1 = \prod_{j=1}^{n} |z - a_j|^{2p}$ where $|a_j| = 1$ ($1 \leq j \leq n$) and $H^p(W_0)_+ = R_+$, then $H^p(W)_+ \subseteq H^p(W_1)_+$. If $W_0$ is in $L^\infty$, then $H^p(W)_+ = H^p(W_1)_+$.

Proof. Let $h_1 = \prod_{j=1}^{n} (z - a_j)(1 - \bar{a}_jz)$ and $h_0$ an outer function in $H^p$ with $|h_0|^p = W_0$. Put $h = h_1h_0$, then $W = |h|^p = |h_1|^p \times |h_0|^p = W_1W_0$. Since $\bar{z}^n\prod_{j=1}^{n} (z - a_j)(1 - \bar{a}_jz) =$ $\prod_{j=1}^{n} |z - a_j|^2$, $|h|/h = \bar{z}^n|h_0|/h_0$. Hence

$$H^p(W)_+ = h^{-1}S^h_p = h^{-1}S^{z^n}_p h_0$$

and $S^{z^n}_p h_0 = \{ \gamma h_0 : \gamma \in R_+ \}$. Now we will prove that $S^{z^n}_p h_0 \subseteq S^{z^n}_p \times h_0$. Then the lemma follows.

We will prove it by induction on $n$. It is clear when $n = 0$. We assume that $S^{z^n}_p h_0 \subseteq S^{z^n}_p \times h_0$ for $j = 0, 1, \cdots, n - 1$. If $f \in S^{z^n}_p h_0$ and $f$ is not a scalar multiple of $z^n h_0$, then by Lemma 1 $z^n h_0 + f$ has the form $qg$ for some nonconstant inner function $q$. By a theorem of Frostman, there exists a sequence $\{d_\ell\}$ in $D$ such that $d_\ell \to 0$ ($\ell \to \infty$) and $(q - d_\ell)/(1 - \bar{d}_\ell q)$ is a Blaschke product for each $\ell$. Then $(q - d_\ell)(1 - \bar{d}_\ell q)g \in S^{z^n}_p h_0$ because $qg \in S^{z^n}_p h_0$. If $(q - d_\ell)(1 - \bar{d}_\ell q)g$ belongs to $S^{z^n}_p \times h_0$, then $qg \in S^{z^n}_p \times h_0$ as $\ell \to \infty$. Since $z^n h_0 \in S^{z^n}_p \times h_0$, $f$ belongs to $S^{z^n}_p \times h_0$ because $z^n h_0 + f = qg$ and so $S^{z^n}_p h_0 \subseteq S^{z^n}_p \times h_0$. Now we will show that $(q - d_\ell)(1 - \bar{d}_\ell q)g$ belongs to $S^{z^n}_p \times h_0$ for all $\ell$. For each $\ell$, there exist a complex constant $z_\ell \in D$

$$(q - d_\ell)(1 - \bar{d}_\ell q)g = (z - z_\ell)(1 - \bar{z}_\ell z)Q_\ell(1 - \bar{z}_\ell z)^{-2}(1 - \bar{d}_\ell q)^2 g$$

and $Q_\ell = \frac{1 - \bar{z}_\ell z}{z - z_\ell} - d_\ell$ is a Blaschke product. Since $(z - z_\ell)(1 - \bar{z}_\ell z)/z \geq 0$, $Q_\ell(1 - \bar{z}_\ell z)^{-2}(1 - \bar{d}_\ell q)^2 g \in S^{z^n-1}_p \times h_0$. By hypothesis on the induction, $Q_\ell(1 - \bar{z}_\ell z)^{-2}(1 - \bar{d}_\ell q)^2 g \in S^{z^n}_p \times h_0$ and so $(q - d_\ell)(1 - \bar{d}_\ell q)g$ belongs to $S^{z^n}_p \times h_0$ and so $S^{z^n}_p h_0 \subseteq S^{z^n}_p \times h_0$.

**Lemma 4.** Let $1/2 \leq p < \infty$. If $H^p(W)_+$ is of finite dimension $n$ the inner part of any nonzero function $s$ in $H^p(W)_+$ is a finite Blaschke product of degree $\leq n - 1$.}

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Proof. Suppose that \( 0 \neq s \in H^p(W)_+ \) and \( s = qk \) where \( q \) is inner. If \( q = \prod_{j=1}^{m} q_j \) and \( q_j (1 \leq j \leq m) \) is a nonconstant inner function then put for each \( \ell \leq m \)

\[
s_\ell = \left(1 + \prod_{j=1}^{\ell} q_j\right)^2 \left(q \prod_{j=1}^{\ell} \bar{q}_j\right) k.
\]

Then \( \{s_j\}_{j=1}^{m} \) is a system of independent elements in \( H^p(W)_+ \). This implies that \( \dim H^p(W)_+ \geq m + 1 \).

**Theorem 2.** Suppose \( H^p(W)_+ (1/2 \leq p < \infty) \) is of finite dimension \( n \). Then,

1. \( W = W_1W_0, \ W_1 = \prod_{j=1}^{n} |z - a_j|^{2p} \) where \( |a_j| = 1 \) (1 \leq j \leq n) and \( H^p(W_0)_+ = R_+ \).

2. \( H^p(W)_+ = \left\{ \gamma \prod_{j=1}^{n} (z - b_j)(1 - \bar{b}_jz) ; \gamma > 0 \text{ and } |b_j| \leq 1 \ (j = 1, \cdots, n) \right\} \).

Proof. Suppose \( W = |h|^p \) and \( h \) is an outer function in \( H^p \). By Lemma 4, for any function \( f \in S^h_p \) the inner part \( q \) is a finite Blaschke product of degree \( \leq \dim H^p(W)_+ - 1 \) because \( H^p(W)_+ = h^{-1}S^h_p \). Then there exists at least a function in \( S^h_p \) such that the degree of the inner part \( q \) is the largest \( n \). If \( qk \in S^h_p \) then there exists a function \( h_0 \in H^p \) such that \( z^n h_0 \in S^h_p \). Then

\[
S^h_p = S^{z^n h_0}_p \text{ and } S^{h_0}_p = \{\gamma h_0 ; \gamma \in R_+\}.
\]

In fact, if there exists a function \( g \in S^{h_0}_p \setminus \{\gamma h_0 ; \gamma \in R_+\} \), then by Lemma 1 \( g + h_0 = q_1 h_1 \) where \( q_1 \) is a nontrivial inner function. Then \( z^n q_1 h_1 \in S^h_p \) and by Lemma 4 this contradicts the definition of \( n \). By the proof of Lemma 3, \( S^{z^n h_0}_p \subseteq S^{z^n}_p \times h_0 \). By a lemma of H. Helson and D. Sarason [2], because \( p \geq 1/2 \)

\[
S^{z^n h_0}_p = \left\{ \gamma \prod_{j=1}^{n} (z - b_j)(1 - \bar{b}_jz) ; |b_j| \leq 1 \ (1 \leq j \leq n), \gamma \in R_+ \right\}.
\]

This implies that

\[
S^{z^n h_0}_p = S^{z^n}_p \times h_0 = S^h_p
\]

and \( h = \gamma \prod_{j=1}^{n} (z - a_j)(1 - \bar{a}_jz) h_0 \) where \( |a_j| = 1 \) (1 \leq j \leq n) and \( \gamma \in R_+ \). Put \( W_1 = \prod_{j=1}^{n} |z - a_j|^{2p} \) and \( W_0 = |h_0|^2 \), then (1) follows. By Proposition 1, \( H^p(W)_+ = h^{-1}S^h_p \) and so (2) follows.
§4. Special weights for \(1/2 \leq p < \infty\)

It is known that a nonnegative function in \(H^{1/2}\) is constant almost everywhere on \(\partial D\). Lemma 5 is a generalization of it and known in [4]. This is important in the proof of Theorem 3. If \(W^{-1}\) is in \(L^\infty\) by Proposition 1 \(H^{1/2}(W)_+ = R_+\). In this section, we study \(H^{1/2}(W)_+\) when \(W^{-1}\) locally belongs to \(L^\infty\) except a finite set. By [4, Theorem 1] and Proposition 1, \(H^1(W)_+\) is of finite dimension when \(W^{-1}\) locally belongs to \(L^1\) except a finite set.

**Lemma 5.** Let \(F\) be a function in \(N\), such that
(a) \(F\) belongs to \(H^p\) for some \(p > 0\),
(b) \(F\) locally belongs to \(H^{1/2}\) except a finite set of \(\partial D\),
(c) \(F\) is outer and \(F\) is nonnegative a.e. on \(\partial D\).

Then \(F\) can be extended to a rational function.

**Theorem 3.** Let \(p \geq 1/2\). Suppose \(W^{-1}\) is in \(L^q\) for some \(q > 0\). If \(W^{-1}\) belongs locally to \(H^{1/(2p-1)}\) except a finite set \(A\) of \(\partial D\), then \(H^p(W)_+\) is of finite dimension and \(\dim H^p(W)_+ = 2N + 1\) for some nonnegative integer \(N\). If \(A\) is an empty set, then \(H^p(W)_+ = R_+\). Let \(A = \{a_j ; j = 1, 2, \ldots, n\}\) then there exist nonnegative integers \(m_1, \ldots, m_n\) such that the following (1) and (2) are valid for \(N = \sum_{j=1}^n m_j\).

1. \(W = W_1W_0\), \(W_1 = \prod_{j=1}^n |z-a_j|^{2m_j/p}\), \(H^p(W_1)_+ = H^p(W)_+\) and \(H^p(W_0)_+ = R_+\).
2. \(H^p(W)_+ = \{\gamma \prod_{j=1}^n (z-b_j)(1-\bar{b}_jz)/\prod_{j=1}^n (z-a_j)^{m_j}(1-\bar{a}_jz)^{m_j} ; \gamma > 0\) and \(|b_j| \leq 1\ (j = 1, \ldots, N)\}.

Proof. At first we prove that \(H^p(W)_+\) is of finite dimension. Since \(H^p(W)_+ = h^{-1}S^h_p\) where \(W = |h|^p\) and \(h\) is an outer function in \(H^p\), it is enough to show that \(S^h_p\) is of finite dimension. If \(g\) is an outer function in \(S^h_p\), put \(F = g/h\). Then \(F \geq 0\) a.e. on \(\partial D\), \(F \in H^q\) for some \(q > 0\). For a measurable subset \(E \subset \partial D\setminus A\)

\[
\int_E |gh^{-1}|^{1/2}d\theta/2\pi \leq \left(\int_E (|g|^{1/2})^{1/\ell}d\theta/2\pi\right)^{1/\ell} \left(\int_E |h^{-1/2}k|^{1/\ell}d\theta/2\pi\right)^{1/k}
\]

where \(1/\ell + 1/k = 1\). If \(\ell = 2p\), then \(k = 2p/(2p-1)\) when \(p \neq 1/2\), and \(k = \infty\) when \(p = 1/2\). If \(k \neq \infty\), then

\[
|h^{-1/2}|^k = |h|^{-2p/(2p-1)} = W^{-2p/(2p-1)}
\]

and so \(F\) locally belongs to \(H^{1/2}\) except \(A\) by hypothesis on \(W\). Hence by Lemma 5 \(F\) is a rational function and

\[
F(z) = \gamma \frac{\prod_{j=1}^N (z-b_j)(1-\bar{b}_jz)}{\prod_{j=1}^n (z-a_j)^{m_j}(1-\bar{a}_jz)^{m_j}}
\]
where \( \gamma > 0 \), \( N = m_1 + \cdots + m_n \), \( |b_j| = 1 \) \( (1 \leq j \leq N) \) and \( |a_j| = 1 \) \( (1 \leq j \leq n) \). Since \( h^{-1} \) locally belongs to \( H^{p/(2p-1)} \), \( \{a_j\}_{j=1}^n \subseteq A \). Therefore by finiteness of \( A \) and Theorem 2 there exists a positive integer \( N_0 < \infty \) such that \( N \leq N_0 \). This implies that \( \mathcal{S}_h^+ \) is of finite dimension. (1) and (2) are clear by the above proof and Theorem 2. If \( A \) is empty, then \( F \) belongs to \( H^{1/2} \) by the proof above and so \( F \) is constant. This implies that \( \mathcal{S}_h^+ = \{ \gamma h ; \gamma \in \mathbb{R}_+ \} \) and so \( \dim H^p(W)_+ = 1 \).

**Corollary 1.** Let \( \frac{1}{2} \leq p < \infty \). Suppose that \( W = |h|^p \) and \( h \) is a rational outer function, that is,

\[
h(z) = \prod_{j=1}^n (z - a_j)^{m_j} / \prod_{j=1}^k (z - c_j)^{b_j}
\]

where \( |a_j| = 1 \) \( (1 \leq j \leq n) \), \( a_j \neq a_i \) \( (j \neq i) \) and \( |c_j| = 1 \) \( (1 \leq j \leq k) \), \( c_j \neq c_i \) \( (j \neq i) \), \( 0 < pk_j < 1 \) \( (1 \leq j \leq k) \). Then \( H^p(W)_+ \) is of finite dimension. Moreover there exist nonnegative integers \( \ell_1, \ldots, \ell_n \) such that

\[
\frac{1}{2}(m_j + \frac{1}{p}) - 1 < \ell_j \leq \frac{1}{2}(m_j + \frac{1}{p}) \quad (1 \leq j \leq n)
\]

and \( N = \sum_{j=1}^n \ell_j \), and the following (1) and (2) are valid.

1. \( W = W_1W_0 \), \( W_1 = \prod_{j=1}^n |z - a_j|^{\ell_jp} \), \( W_0 = \prod_{j=1}^n |z - a_j|^{m_jp - \ell_jp} / \prod_{j=1}^k |z - c_j|^{p\ell_j} \).

2. \( H^p(W)_+ = \{ \gamma \prod_{j=1}^{N_n} (z - b_j)/(1 - \bar{b}_jz) / \prod_{j=1}^n (z - a_j)^{\ell_j}/(1 - \bar{a}_jz)^{\ell_j} ; \gamma > 0, \ |b_j| \leq 1 \ (j = 1, \ldots, N) \} \)

Proof. Since \( W^{-1} \) belongs locally to \( L^\infty \) except \( \{a_j\}_{j=1}^n \), by Theorem 3 \( H^p(W)_+ \) is of finite dimension. If \( \ell_1, \ldots, \ell_n \) are nonnegative integers such that

\[
\frac{1}{2}(m_j + \frac{1}{p}) - 1 < \ell_j \leq \frac{1}{2}(m_j + \frac{1}{p}) \quad (1 \leq j \leq n)
\]

and \( N = \sum_{j=1}^n \ell_j \), then put \( W_1 = \prod_{j=1}^n |z - a_j|^{\ell_jp} \) and \( W_0 = W/W_1 \). Then by Theorem 3 \( H^p(W_0)_+ = R_+ \). For \( W_0^{-1} \in L^{1/(2p-1)} \) because \( -1 < (2\ell_jp - m_jp)/(2p - 1) < 1/(2p - 1) \). Hence (1) and (2) follows from Theorem 3.

**Remark.** (1) Let \( p \geq 1/2 \). Suppose \( W^{-1} \) is in \( L^q \) for some \( q > 0 \) and \( W^{-1} \) belongs locally to \( L^{1/(2p-1)} \) except at \( z = -1 \). Then by Theorem 3 \( H^p(W)_+ \) is of finite dimension. Moreover, \( \dim H^p(W)_+ = 2N + 1 \) if and only if \( W |1 + z|^{-2pN} \in L^1 \) and \( W |1 + z|^{-2p(N+1)} \notin L^1 \). (2) Let \( p \geq 1/2 \). Suppose \( W = |1 + z|^{\alpha} \) and \( -1 < \alpha < \infty \). Then the following
(i) $\sim (iii)$ are equivalent. (i) $\dim H^p(W)_+ = 2N + 1$. (ii) $\max((2N)p - 1, -1) < \alpha < (2N + 3)p - 1$. (iii) $\max(\frac{a+1}{2N+3}, \frac{1}{2}) \leq p < \frac{a+1}{2N}$. For by Theorem 3, $\dim H^p(W)_+ = 2N + 1$ for some nonnegative integer $N$ and $H^p(W)_+ = \{ \gamma \prod_{j=1}^{N}(z-b_j)(1-b_jz)/(1+z)^{2N}; \gamma > 0 \text{ and } |b_j| \leq 1 \ (j = 1, \cdots, n) \}$. Since $hH^p(W)_+ = S^h_p$ and $h = (1 + z)^{\frac{a}{p}}$, $(1 + z)^{\frac{a}{p}} H^p(W)_+ \subset H^p$ and so $(1+z)^{\frac{a}{p}-2N}$ belongs to $H^p$. Hence $\alpha - (2N)p > -1$ and so $\alpha > \max((2N)p - 1, -1)$. Since $\dim H^p(W)_+ = 2N + 1$, $\alpha \leq (2N + 1)p - 1$.

References