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Nonnegative Functions In Weighted Hardy Spaces

by

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Abstract. Let W be a nonnegative summable function whose logarithm is also summable with respect to the Lebesgue measure on the unit circle. For $0 < p < \infty$, $H^p(W)$ denotes a weighted Hardy space on the unit circle. When $W \equiv 1$, $H^p(W)$ is the usual Hardy space H^p . We are interested in $H^p(W)_+$ the set of all nonnegative functions in $H^p(W)$. If $p \geq 1/2$, H^p_+ consists of constant functions. However $H^p(W)_+$ contains a nonconstant nonnegative function for some weight W . In this paper, if $p \geq 1/2$ we determine W and describe $H^p(W)_+$ when the linear span of $H^p(W)_+$ is of finite dimension. Moreover we show that the linear span of $H^p(W)_+$ is of infinite dimension for arbitrary weight W when $0 < p < 1/2$.

§1. Introduction and preliminaries

Let W be a nonnegative function in $L^1 = L^1(d\theta/2\pi, \partial D)$ where D is the open unit disc and ∂D is its boundary. For $0 < p < \infty$, a weighted Hardy space $H^p(W)$ denotes the closure of all analytic polynomials in $L^p(W) = L^p(Wd\theta/2\pi, \partial D)$. We may assume that $\log W$ is in L^1 . For otherwise $H^p(W) = L^p(W)$. If $W \equiv 1$, then $H^p(W)$ is the usual Hardy space H^p . Let N_* be the Smirnov class (see [1]), then $H^p = N_* \cap L^p$. A function h in N_* is called outer if it is invertible in N_* and a function q in N_* is called inner if $|q| = 1$ a.e. on ∂D . If $W = |h|^p$ and h is an outer function in H^p , it is known that $H^p(W) = h^{-1}H^p$ and so $H^p(W)$ is a subset of N_* . Put

$$H^p(W)_+ = \{s \in H^p(W) ; s \geq 0 \text{ a.e. on } \partial D\}.$$

If $W = |h|^p$, then $hH^p(W)_+ = \{g \in H^p : \arg g = \arg h \text{ a.e. on } \partial D\}$. In this paper, the dimension of $H^p(W)_+$ means that of the linear span of $H^p(W)_+$ in $H^p(W)$.

J. Neuwirth and D. J. Newman [7] showed that $H_+^{1/2} = R_+$ = the set of all nonnegative real constants. Hence $H_+^p = R_+$ if $p \geq 1/2$. However $H_+^p \neq R_+$ if $0 < p < 1/2$ because $z/(1+z)^2$ belongs to $\bigcap_{p < 1/2} H^p$. $H^1(W)_+$ has been studied because $hH^1(W)_+ =$

$\{g \in H^1 : \arg g = \arg h \text{ a.e. on } \partial D\}$ when $W = |h|$. In fact, this set is related to the set of extremal functions of a well known linear extremal problem in H^1 . (cf. [5],[6],[3],[4]). Hence $H^1(W)_+$ is known enough. However $H^p(W)_+$ has not studied before when $p \neq 1$ and $W \not\equiv 1$. In this paper, we study $H^p(W)_+$ for arbitrary p . In §2, we show that $H^p(W)_+$ is of infinite dimension when $0 < p < 1/2$ and W is arbitrary. In §3, we describe $H^p(W)_+$ when $p \geq 1/2$ and $H^p(W)_+$ is of finite dimension. In §4, we show that $H^p(W)_+$ is of finite dimension if $p \geq 1/2$ and W^{-1} is locally in $L^{1/2p-1}$ except at a finite set.

Now we give a general result about $H^p(W)_+$ for $0 < p < \infty$. For any nonzero function f in H^p , put $\mathcal{S}_p^f = \{g \in H^p ; \arg f = \arg g \text{ a.e. on } \partial D\}$ for $0 < p < \infty$. Put $(N_*)_+ = \{s \in N_* ; s \geq 0 \text{ a.e. on } \partial D\}$.

Proposition 1. *Let $0 < p < \infty$. Suppose $W = |h|^p$ and h is an outer function in H^p .*

- (1) $H^p(W)_+ = h^{-1}\mathcal{S}_p^h$ and $R_+ \subseteq H^p(W)_+ \subseteq (N_*)_+$.
- (2) If W is in L^∞ then $H^p(W)_+ \supseteq H_+^p$.
- (3) If W^{-1} is in L^∞ then $H^p(W)_+ \subseteq H_+^p$.

Proof. Since $H^p(W) = h^{-1}H^p$, (1) follows from the definition of \mathcal{S}_p^h . (2) and (3) are clear.

Lemma 1. *If f and g are nonzero functions in N_* such that f/g is nonnegative and nonconstant almost everywhere on ∂D , then $f + g$ is not outer.*

Proof. Put $h = f + g$, then $f/h + g/h = 1$, $0 \leq f/h \leq 1$ and $0 \leq g/h \leq 1$ on ∂D . If h is outer, then both f/h and g/h belong to $N_* \cap L^\infty = H^\infty$. Thus both f/h and g/h are constant and so f/g is constant. This contradiction shows the lemma.

Proposition 2. *Let $0 < p < \infty$. If $H^p(W)_+ \neq R_+$, then there exists a function in $H^p(W)_+$ which is not outer. Hence the dimension of $H^p(W)_+$ is bigger than or equal to three.*

Proof. If $s \in H^p(W)_+$ is nonconstant, by Lemma 1 $s + 1$ is not outer and so this implies the first part. Hence $qg = s + 1$ belongs to $H^p(W)_+$ where q is a nonconstant inner part of $s + 1$. Since both $(1 + q)^2g$ and 1 belong to $H^p(W)_+$, this implies the second part.

§2. General weights for $0 < p < 1/2$.

If $0 < p < 1/2$, then $(z - b)(1 - \bar{b}z)/(z - a)(1 - \bar{a}z)$ belongs to H^p where $|a| = 1$ and $|b| \leq 1$ and so H_+^p is of infinite dimension. In this section, we show that $H^p(W)_+$ is of infinite dimension for arbitrary weight W .

Lemma 2. *If $0 < p < 1/2$ and if h is a function in H^p , $h(z)/(z - e^{it})(1 - \bar{e}^{it}z)$ belongs to H^p for a.e. e^{it} .*

Proof. If $k(z) = z/(1 - z)^2$, $|k|^p$ and $|h|^p$ belongs to L^1 and hence $|k|^p * |h|^p \in L^1$.

$$\begin{aligned} |k|^p * |h|^p(e^{it}) &= \int_0^{2\pi} |k(e^{i(t-\theta)})h(e^{i\theta})|^p d\theta / 2\pi \\ &= \int_0^{2\pi} \left| \frac{e^{i\theta}h(e^{i\theta})}{(e^{i\theta} - e^{it})(1 - \bar{e}^{it}e^{i\theta})} \right|^p d\theta / 2\pi < \infty \end{aligned}$$

This implies that $h(z)/(z - e^{it})(1 - \bar{e}^{it}z)$ belongs to H^p for a.e. e^{it}

Theorem 1. *For arbitrary weight W , $H^p(W)_+$ is of infinite dimension.*

Proof. Suppose that $W = |h|^p$ and h is an outer function in H^p . Since $H^p(W)_+ = h^{-1}\mathcal{S}_p^h$ by (1) of Proposition 1, it is enough to prove that \mathcal{S}_p^h is of infinite dimension. By Lemma 2, for any finite n , $h(z) \prod_{j=1}^n (z - b_j)(1 - \bar{b}_j z) / \prod_{j=1}^n (z - a_j)(1 - \bar{a}_j z)$ belongs to H^p where $|b_j| < 1$ and $|a_j| = 1$ for $1 \leq j \leq n$. Since $\prod_{j=1}^n (z - b_j)(1 - \bar{b}_j z) / \prod_{j=1}^n (z - a_j)(1 - \bar{a}_j z)$ is nonnegative on ∂D , \mathcal{S}_p^h is of infinite dimension.

§3. General weights for $1/2 \leq p < \infty$

Unlike when $0 < p < 1/2$, if $1/2 \leq p < \infty$, then $H^p(W)_+$ may be of finite dimension for some weight W . In this section, we describe $H^p(W)_+$ when $H^p(W)_+$ is of finite dimension.

Lemma 3. *Let $1/2 \leq p < \infty$. If $W = W_1W_0$, and $W_1 = \prod_{j=1}^n |z - a_j|^{2p}$ where $|a_j| = 1$ ($1 \leq j \leq n$) and $H^p(W_0)_+ = R_+$, then $H^p(W)_+ \subseteq H^p(W_1)_+$. If W_0 is in L^∞ , then $H^p(W)_+ = H^p(W_1)_+$.*

Proof. Let $h_1 = \prod_{j=1}^n (z - a_j)(1 - \bar{a}_j z)$ and h_0 an outer function in H^p with $|h_0|^p = W_0$. Put $h = h_1 h_0$, then $W = |h|^p = |h_1|^p \times |h_0|^p = W_1 W_0$. Since $\bar{z}^n \prod_{j=1}^n (z - a_j)(1 - \bar{a}_j z) = \prod_{j=1}^n |z - a_j|^2$, $|h|/h = \bar{z}^n |h_0|/h_0$. Hence

$$H^p(W)_+ = h^{-1} \mathcal{S}_p^h = h^{-1} \mathcal{S}_p^{z^n h_0}$$

and $\mathcal{S}_p^{h_0} = \{\gamma h_0 ; \gamma \in R_+\}$. Now we will prove that $\mathcal{S}_p^{z^n h_0} \subseteq \mathcal{S}_p^{z^n} \times h_0$. Then the lemma follows.

We will prove it by induction on n . It is clear when $n = 0$. We assume that $\mathcal{S}_p^{z^j h_0} \subseteq \mathcal{S}_p^{z^j} \times h_0$ for $j = 0, 1, \dots, n-1$. If $f \in \mathcal{S}_p^{z^n h_0}$ and f is not a scalar multiple of $z^n h_0$, then by Lemma 1 $z^n h_0 + f$ has the form qg for some nonconstant inner function q . By a theorem of Frostman, there exists a sequence $\{d_\ell\}$ in D such that $d_\ell \rightarrow 0$ ($\ell \rightarrow \infty$) and $(q - d_\ell)/(1 - \bar{d}_\ell q)$ is a Blaschke product for each ℓ . Then $(q - d_\ell)(1 - \bar{d}_\ell q)g \in \mathcal{S}_p^{z^n h_0}$ because $qg \in \mathcal{S}_p^{z^n h_0}$. If $(q - d_\ell)(1 - \bar{d}_\ell q)g$ belongs to $\mathcal{S}_p^{z^n} \times h_0$, then $qg \in \mathcal{S}_p^{z^n} \times h_0$ as $\ell \rightarrow \infty$. Since $z^n h_0 \in \mathcal{S}_p^{z^n} \times h_0$, f belongs to $\mathcal{S}_p^{z^n} \times h_0$ because $z^n h_0 + f = qg$ and so $\mathcal{S}_p^{z^n h_0} \subseteq \mathcal{S}_p^{z^n} \times h_0$. Now we will show that $(q - d_\ell)(1 - \bar{d}_\ell q)g$ belongs to $\mathcal{S}_p^{z^n} \times h_0$ for all ℓ . For each ℓ , there exist a complex constant $z_\ell \in D$

$$(q - d_\ell)(1 - \bar{d}_\ell q)g = (z - z_\ell)(1 - \bar{z}_\ell z)Q_\ell(1 - \bar{z}_\ell z)^{-2}(1 - \bar{d}_\ell q)^2 g$$

and $Q_\ell = \frac{1 - \bar{z}_\ell z}{z - z_\ell} \frac{q - d_\ell}{1 - \bar{d}_\ell q}$ is a Blaschke product. Since $(z - z_\ell)(1 - \bar{z}_\ell z)/z \geq 0$, $Q_\ell(1 - \bar{z}_\ell z)^{-2}(1 - \bar{d}_\ell q)^2$ belongs to $\mathcal{S}_p^{z^{n-1} h_0}$. By hypothesis on the induction, $Q_\ell(1 - \bar{z}_\ell z)^{-2}(1 - \bar{d}_\ell q)^2 g \in \mathcal{S}_p^{z^{n-1}} \times h_0$ and so $(q - d_\ell)(1 - \bar{d}_\ell q)g$ belongs to $\mathcal{S}_p^{z^n} \times h_0$ and so $\mathcal{S}_p^{z^n h_0} \subseteq \mathcal{S}_p^{z^n} \times h_0$.

Lemma 4. *Let $1/2 \leq p < \infty$. If $H^p(W)_+$ is of finite dimension n the inner part of any nonzero function s in $H^p(W)_+$ is a finite Blaschke product of degree $\leq n - 1$.*

Proof. Suppose that $0 \neq s \in H^p(W)_+$ and $s = qk$ where q is inner. If $q = \prod_{j=1}^m q_j$ and q_j ($1 \leq j \leq m$) is a nonconstant inner function then put for each $\ell \leq m$

$$s_\ell = \left(1 + \prod_{j=1}^{\ell} q_j\right)^2 \left(q \prod_{j=1}^{\ell} \bar{q}_j\right) k.$$

Then $\{s_j\}_{j=1}^m$ is a system of independent elements in $H^p(W)_+$. This implies that $\dim H^p(W)_+ \geq m + 1$.

Theorem 2. *Suppose $H^p(W)_+$ ($1/2 \leq p < \infty$) is of finite dimension n . Then,*

(1) $W = W_1 W_0$, $W_1 = \prod_{j=1}^n |z - a_j|^{2p}$ where $|a_j| = 1$ ($1 \leq j \leq n$) and $H^p(W_0)_+ = R_+$.

(2) $H^p(W)_+ = \left\{ \gamma \prod_{j=1}^n \frac{(z - b_j)(1 - \bar{b}_j z)}{(z - a_j)(1 - \bar{a}_j z)} ; \gamma > 0 \text{ and } |b_j| \leq 1 \text{ (} j = 1, \dots, n \text{)} \right\}$.

Proof. Suppose $W = |h|^p$ and h is an outer function in H^p . By Lemma 4, for any function $f \in \mathcal{S}_p^h$ the inner part q is a finite Blaschke product of degree $\leq \dim H^p(W)_+ - 1$ because $H^p(W)_+ = h^{-1} \mathcal{S}_p^h$. Then there exists at least a function in \mathcal{S}_p^h such that the degree of the inner part q is the largest n . If $qk \in \mathcal{S}_p^h$ then there exists a function $h_0 \in H^p$ such that $z^n h_0 \in \mathcal{S}_p^h$. Then

$$\mathcal{S}_p^h = \mathcal{S}_p^{z^n h_0} \text{ and } \mathcal{S}_p^{h_0} = \{\gamma h_0 ; \gamma \in R_+\}.$$

In fact, if there exists a function g in $\mathcal{S}_p^{h_0} \setminus \{\gamma h_0 ; \gamma \in R_+\}$, then by Lemma 1 $g + h_0 = q_1 h_1$ where q_1 is a nontrivial inner function. Then $z^n q_1 h_1 \in \mathcal{S}_p^h$ and by Lemma 4 this contradicts the definition of n . By the proof of Lemma 3, $\mathcal{S}_p^{z^n h_0} \subseteq \mathcal{S}_p^{z^n} \times h_0$. By a lemma of H. Helson and D. Sarason [2], because $p \geq 1/2$

$$\mathcal{S}_p^{z^n} = \left\{ \gamma \prod_{j=1}^n (z - b_j)(1 - \bar{b}_j z) ; |b_j| \leq 1 \text{ (} 1 \leq j \leq n \text{)}, \gamma \in R_+ \right\}.$$

This implies that

$$\mathcal{S}_p^{z^n h_0} = \mathcal{S}_p^{z^n} \times h_0 = \mathcal{S}_p^h$$

and $h = \gamma \prod_{j=1}^n (z - a_j)(1 - \bar{a}_j z) h_0$ where $|a_j| = 1$ ($1 \leq j \leq n$) and $\gamma \in R_+$. Put $W_1 = \prod_{j=1}^n |z - a_j|^{2p}$ and $W_0 = |h_0|^{2p}$, then (1) follows. By Proposition 1, $H^p(W)_+ = h^{-1} \mathcal{S}_p^h$ and so (2) follows.

§4. Special weights for $1/2 \leq p < \infty$

It is known that a nonnegative function in $H^{1/2}$ is constant almost everywhere on ∂D . Lemma 5 is a generalization of it and known in [4]. This is important in the proof of Theorem 3. If W^{-1} is in L^∞ by Proposition 1 $H^{1/2}(W)_+ = R_+$. In this section, we study $H^{1/2}(W)_+$ when W^{-1} locally belongs to L^∞ except a finite set. By [4, Theorem 1] and Proposition 1, $H^1(W)_+$ is of finite dimension when W^{-1} locally belongs to L^1 except a finite set.

Lemma 5. *Let F be a function in N_* such that*

- (a) F belongs to H^p for some $p > 0$,
- (b) F locally belongs to $H^{1/2}$ except a finite set of ∂D ,
- (c) F is outer and F is nonnegative a.e. on ∂D .

Then F can be extended to a rational function.

Theorem 3. *Let $p \geq 1/2$. Suppose W^{-1} is in L^q for some $q > 0$. If W^{-1} belongs locally to $L^{1/(2p-1)}$ except a finite set A of ∂D , then $H^p(W)_+$ is of finite dimension and $\dim H^p(W)_+ = 2N + 1$ for some nonnegative integer N . If A is an empty set, then $H^p(W)_+ = R_+$. If $A = \{a_j ; j = 1, 2, \dots, n\}$ then there exist nonnegative integers m_1, \dots, m_n such that the following (1) and (2) are valid for $N = \sum_{j=1}^n m_j$.*

$$(1) \ W = W_1 W_0, \ W_1 = \prod_{j=1}^n |z - a_j|^{2m_j p}, \ H^p(W_1)_+ = H^p(W)_+ \text{ and } H^p(W_0)_+ = R_+.$$

$$(2) \ H^p(W)_+ = \left\{ \gamma \prod_{j=1}^N (z - b_j)(1 - \bar{b}_j z) / \prod_{j=1}^n (z - a_j)^{m_j} (1 - \bar{a}_j z)^{m_j} ; \ \gamma > 0 \text{ and } |b_j| \leq 1 \ (j = 1, \dots, N) \right\}.$$

Proof. At first we prove that $H^p(W)_+$ is of finite dimension. Since $H^p(W)_+ = h^{-1} \mathcal{S}_p^h$ where $W = |h|^p$ and h is an outer function in H^p , it is enough to show that \mathcal{S}_p^h is of finite dimension. If g is an outer function in \mathcal{S}_p^h , put $F = g/h$. Then $F \geq 0$ a.e. on ∂D , $F \in H^q$ for some $q > 0$. For a measurable subset $E \subset \partial D \setminus A$

$$\int_E |gh^{-1}|^{1/2} d\theta / 2\pi \leq \left(\int_E (|g|^{1/2})^\ell d\theta / 2\pi \right)^{1/\ell} \left(\int_E |h^{-1/2}|^k d\theta / 2\pi \right)^{1/k}$$

where $1/\ell + 1/k = 1$. If $\ell = 2p$, then $k = 2p/(2p-1)$ when $p \neq 1/2$, and $k = \infty$ when $p = 1/2$. If $k \neq \infty$, then

$$|h^{-1/2}|^k = |h|^{-\frac{p}{2p-1}} = W^{-\frac{1}{2p-1}}$$

and so F locally belongs to $H^{1/2}$ except A by hypothesis on W . Hence by Lemma 5 F is a rational function and

$$F(z) = \gamma \frac{\prod_{j=1}^N (z - b_j)(1 - \bar{b}_j z)}{\prod_{j=1}^n (z - a_j)^{m_j} (1 - \bar{a}_j z)^{m_j}}$$

where $\gamma > 0$, $N = m_1 + \cdots + m_n$, $|b_j| = 1$ ($1 \leq j \leq N$) and $|a_j| = 1$ ($1 \leq j \leq n$). Since h^{-1} locally belongs to $H^{p/(2p-1)}$, $\{a_j\}_{j=1}^n \subset A$. Therefore by finiteness of A and Theorem 2 there exists a positive integer $N_0 < \infty$ such that $N \leq N_0$. This implies that \mathcal{S}_p^h is of finite dimension. (1) and (2) are clear by the above proof and Theorem 2. If A is empty, then F belongs to $H^{1/2}$ by the proof above and so F is constant. This implies that $\mathcal{S}_p^h = \{\gamma h ; \gamma \in R_+\}$ and so $\dim H^p(W)_+ = 1$.

Corollary 1. Let $\frac{1}{2} \leq p < \infty$. Suppose that $W = |h|^p$ and h is a rational outer function, that is,

$$h(z) = \prod_{j=1}^n (z - a_j)^{m_j} / \prod_{j=1}^k (z - c_j)^{k_j}$$

where $|a_j| = 1$ ($1 \leq j \leq n$), $a_j \neq a_i$ ($j \neq i$) and $|c_j| = 1$ ($1 \leq j \leq k$), $c_j \neq c_i$ ($j \neq i$), $0 < pk_j < 1$ ($1 \leq j \leq k$). Then $H^p(W)_+$ is of finite dimension. Moreover there exist nonnegative integers ℓ_1, \dots, ℓ_n such that

$$\frac{1}{2}(m_j + \frac{1}{p}) - 1 < \ell_j \leq \frac{1}{2}(m_j + \frac{1}{p}) \quad (1 \leq j \leq n)$$

and $N = \sum_{j=1}^n \ell_j$, and the following (1) and (2) are valid.

$$(1) W = W_1 W_0, \quad W_1 = \prod_{j=1}^n |z - a_j|^{2\ell_j p}, \quad W_0 = \prod_{j=1}^n |z - a_j|^{m_j p - 2\ell_j p} / \prod_{j=1}^k |z - c_j|^{pk_j}.$$

$$(2) H^p(W)_+ = \left\{ \gamma \prod_{j=1}^N (z - b_j)(1 - \bar{b}_j z) / \prod_{j=1}^n (z - a_j)^{\ell_j} (1 - \bar{a}_j z)^{\ell_j} ; \gamma > 0, |b_j| \leq 1 \right. \\ \left. (j = 1, \dots, N) \right\}$$

Proof. Since W^{-1} belongs locally to L^∞ except $\{a_j\}_{j=1}^n$, by Theorem 3 $H^p(W)_+$ is of finite dimension. If ℓ_1, \dots, ℓ_n are nonnegative integers such that

$$\frac{1}{2}(m_j + \frac{1}{p}) - 1 < \ell_j \leq \frac{1}{2}(m_j + \frac{1}{p}) \quad (1 \leq j \leq n)$$

and $N = \sum_{j=1}^n \ell_j$, then put $W_1 = \prod_{j=1}^n |z - a_j|^{2\ell_j p}$ and $W_0 = W/W_1$. Then by Theorem 3 $H^p(W_0)_+ = R_+$. For $W_0^{-1} \in L^{1/(2p-1)}$ because $-1 < (2\ell_j p - m_j p)/(2p-1) < 1/(2p-1)$. Hence (1) and (2) follows from Theorem 3.

Remark. (1) Let $p \geq 1/2$. Suppose W^{-1} is in L^q for some $q > 0$ and W^{-1} belongs locally to $L^{1/2p-1}$ except at $z = -1$. Then by Theorem 3 $H^p(W)_+$ is of finite dimension. Moreover, $\dim H^p(W)_+ = 2N + 1$ if and only if $W|1+z|^{-2pN} \in L^1$ and $W|1+z|^{-2p(N+1)} \notin L^1$. (2) Let $p \geq 1/2$. Suppose $W = |1+z|^\alpha$ and $-1 < \alpha < \infty$. Then the following

(i) \sim (iii) are equivalent. (i) $\dim H^p(W)_+ = 2N + 1$. (ii) $\max((2N)p - 1, -1) < \alpha < (2N + 3)p - 1$. (iii) $\max(\frac{\alpha+1}{2N+3}, \frac{1}{2}) \leq p < \frac{\alpha+1}{2N}$. For by Theorem 3, $\dim H^p(W)_+ = 2N + 1$ for some nonnegative integer N and $H^p(W)_+ = \{\gamma \prod_{j=1}^N (z - b_j)(1 - \bar{b}_j z) / (1+z)^{2N} ; \gamma > 0 \text{ and } |b_j| \leq 1 (j = 1, \dots, n)\}$. Since $hH^p(W)_+ = \mathcal{S}_p^h$ and $h = (1+z)^{\frac{\alpha}{p}}$, $(1+z)^{\frac{\alpha}{p}} H^p(W)_+ \subset H^p$ and so $(1+z)^{\frac{\alpha}{p} - 2N}$ belongs to H^p . Hence $\alpha - (2N)p > -1$ and so $\alpha > \max((2N)p - 1, -1)$. Since $\dim H^p(W)_+ = 2N + 1$, $\alpha \leq (2N + 1)p - 1$.

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