Nonnegative Functions In Weighted Hardy Spaces

by

Jyunji Inoue

and

Takahiko Nakazi *

*This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education

2000 Mathematics Subject Classifications: Primary 30 D 55, 46 J 15

Abstract. Let W be a nonnegative summable function whose logarithm is also summable with respect to the Lebesgue measure on the unit circle. For $0 < p < \infty$, $H^p(W)$ denotes a weighted Hardy space on the unit circle. When $W \equiv 1$, $H^p(W)$ is the usual Hardy space H^p. We are interested in $H^p(W)_+$ the set of all nonnegative functions in $H^p(W)$. If $p \geq 1/2$, H^p_+ consists of constant functions. However $H^p(W)_+$ contains a nonconstant nonnegative function for some weight W. In this paper, if $p \geq 1/2$ we determine W and describe $H^p(W)_+$ when the linear span of $H^p(W)_+$ is of finite dimension. Moreover we show that the linear span of $H^p(W)_+$ is of infinite dimension for arbitrary weight W when $0 < p < 1/2$.

§1. Introduction and preliminaries

Let W be a nonnegative function in $L^1 = L^1(d\theta/2\pi,\ \partial D)$ where D is the open unit disc and ∂D is its boundary. For $0 < p < \infty$, a weighted Hardy space $H^p(W)$ denotes the closure of all analytic polynomials in $L^p(W) = L^p\ (W d\theta/2\pi,\ \partial D)$. We may assume that $\log W$ is in L^1. For otherwise $H^p(W) = L^p(W)$. If $W \equiv 1$, then $H^p(W)$ is the usual Hardy space H^p. Let N_* be the Smirnov class (see [1]), then $H^p = N_* \cap L^p$. A function h in N_* is called outer if it is invertible in N_* and a function q in N_* is called inner if $|q| = 1$ a.e. on ∂D. If $W = |h|^p$ and h is an outer function in H^p, it is known that $H^p(W) = h^{-1}H^p$ and so $H^p(W)$ is a subset of N_*. Put

$$H^p(W)_+ = \{s \in H^p(W)\ ;\ s \geq 0 \ \text{ a.e. on } \partial D\}.$$

If $W = |h|^p$, then $hH^p(W)_+ = \{g \in H^p : \arg g = \arg h \ \text{ a.e. on } \partial D\}$. In this paper, the dimension of $H^p(W)_+$ means that of the linear span of $H^p(W)_+$ in $H^p(W)$.

J. Neuwirth and D. J. Newman [7] showed that $H^{1/2}_+ = R_+ =$ the set of all nonnegative real constants. Hence $H^p_+ = R_+$ if $p \geq 1/2$. However $H^p_+ \neq R_+$ if $0 < p < 1/2$ because $z/(1+z)^2$ belongs to $\bigcap_{p<1/2} H^p$. $H^1(W)_+$ has been studied because $hH^1(W)_+ = \{g \in H^1 : \arg g = \arg h \ \text{ a.e. on } \partial D\}$ when $W = |h|$. In fact, this set is related to the set of extremal functions of a well known linear extremal problem in H^1. (cf. [5],[6],[3],[4]). Hence $H^1(W)_+$ is known enough. However $H^p(W)_+$ has not studied before when $p \neq 1$ and $W \not\equiv 1$. In this paper, we study $H^p(W)_+$ for arbitrary p. In §2, we show that $H^p(W)_+$ is of infinite dimension when $0 < p < 1/2$ and W is arbitrary. In §3, we describe $H^p(W)_+$ when $p \geq 1/2$ and $H^p(W)_+$ is of finite dimension. In §4, we show that $H^p(W)_+$ is of finite dimension if $p \geq 1/2$ and W^{-1} is locally in $L^{1/2p-1}$ except at a finite set.

Now we give a general result about $H^p(W)_+$ for $0 < p < \infty$. For any nonzero function f in H^p, put $\mathcal{S}^f_p = \{g \in H^p\ ;\ \arg f = \arg g \ \ a.e. \text{ on } \partial D\}$ for $0 < p < \infty$. Put $(N_*)_+ = \{s \in N_*\ ;\ s \geq 0\ a.e. \text{ on } \partial D\}$.

Proposition 1. *Let $0 < p < \infty$. Suppose $W = |h|^p$ and h is an outer function in H^p.*

(1) $H^p(W)_+ = h^{-1}\mathcal{S}^h_p$ and $R_+ \subseteq H^p(W)_+ \subseteq (N_*)_+$.

(2) If W is in L^∞ then $H^p(W)_+ \supseteq H^p_+$.

(3) If W^{-1} is in L^∞ then $H^p(W)_+ \subseteq H^p_+$.

Proof. Since $H^p(W) = h^{-1}H^p$, (1) follows from the definition of $\mathcal{S}^h_p$. (2) and (3) are clear.

Lemma 1. *If f and g are nonzero functions in N_* such that f/g is nonnegative and nonconstant almost everywhere on ∂D, then $f + g$ is not outer.*

Proof. Put $h = f + g$, then $f/h + g/h = 1$, $0 \le f/h \le 1$ and $0 \le g/h \le 1$ on ∂D. If h is outer, then both f/h and g/h belong to $N_* \cap L^\infty = H^\infty$. Thus both f/h and g/h are constant and so f/g is constant. This contradiction shows the lemma.

Proposition 2. *Let $0 < p < \infty$. If $H^p(W)_+ \neq R_+$, then there exists a function in $H^p(W)_+$ which is not outer. Hence the dimension of $H^p(W)_+$ is bigger than or equal to three.*

Proof. If $s \in H^p(W)_+$ is nonconstant, by Lemma 1 $s + 1$ is not outer and so this implies the first part. Hence $qg = s + 1$ belongs to $H^p(W)_+$ where q is a nonconstant inner part of $s+1$. Since both $(1+q)^2 g$ and 1 belong to $H^p(W)_+$, this implies the second part.

§2. General weights for $0 < p < 1/2$.

If $0 < p < 1/2$, then $(z - b)(1 - \bar{b}z)/(z - a)(1 - \bar{a}z)$ belongs to H^p where $|a| = 1$ and $|b| \le 1$ and so H^p_+ is of infinite dimension. In this section, we show that $H^p(W)_+$ is of infinite dimension for arbitrary weight W.

Lemma 2. *If $0 < p < 1/2$ and if h is a function in H^p, $h(z)/(z - e^{it})(1 - \overline{e^{it}}z)$ belongs to H^p for* a.e. e^{it}.

Proof. If $k(z) = z/(1 - z)^2$, $|k|^p$ and $|h|^p$ belongs to L^1 and hence $|k|^p * |h|^p \in L^1$.

$$\begin{aligned} |k|^p * |h|^p(e^{it}) &= \int_0^{2\pi} |k(e^{i(t-\theta)})h(e^{i\theta})|^p d\theta/2\pi \\ &= \int_0^{2\pi} \left| \frac{e^{i\theta}h(e^{i\theta})}{(e^{i\theta} - e^{it})(1 - \overline{e^{it}}e^{i\theta})} \right|^p d\theta/2\pi < \infty \end{aligned}$$

This implies that $h(z)/(z - e^{it})(1 - \overline{e^{it}}z)$ belongs to H^p for a.e. e^{it}

Theorem 1. *For arbitrary weight W, $H^p(W)_+$ is of infinite dimension.*

Proof. Suppose that $W = |h|^p$ and h is an outer function in H^p. Since $H^p(W)_+ = h^{-1}\mathcal{S}_p^h$ by (1) of Proposition 1, it is enough to prove that $\mathcal{S}_p^h$ is of infinite dimension. By Lemma 2, for any finite $n, h(z)\prod_{j=1}^n (z - b_j)(1 - \bar{b}_j z)/\prod_{j=1}^n (z - a_j)(1 - \bar{a}_j z)$ belongs to H^p where $|b_j| < 1$ and $|a_j| = 1$ for $1 \le j \le n$. Since $\prod_{j=1}^n (z - b_j)(1 - \bar{b}_j z)/\prod_{j=1}^n (z - a_j)(1 - \bar{a}_j z)$ is nonnegative on ∂D, S_p^h is of infinite dimension.

§3. General weights for $1/2 \leq p < \infty$

Unlike when $0 < p < 1/2$, if $1/2 \leq p < \infty$, then $H^p(W)_+$ may be of finite dimension for some weight W. In this section, we describe $H^p(W)_+$ when $H^p(W)_+$ is of finite dimension.

Lemma 3. *Let $1/2 \leq p < \infty$. If $W = W_1 W_0$, and $W_1 = \prod_{j=1}^{n} |z - a_j|^{2p}$ where $|a_j| = 1$ $(1 \leq j \leq n)$ and $H^p(W_0)_+ = R_+$, then $H^p(W)_+ \subseteq H^p(W_1)_+$. If W_0 is in L^∞, then $H^p(W)_+ = H^p(W_1)_+$.*

Proof. Let $h_1 = \prod_{j=1}^{n}(z - a_j)(1 - \bar{a}_j z)$ and h_0 an outer function in H^p with $|h_0|^p = W_0$. Put $h = h_1 h_0$, then $W = |h|^p = |h_1|^p \times |h_0|^p = W_1 W_0$. Since $\bar{z}^n \prod_{j=1}^{n}(z - a_j)(1 - \bar{a}_j z) = \prod_{j=1}^{n} |z - a_j|^2$, $|h|/h = \bar{z}^n |h_0|/h_0$. Hence

$$H^p(W)_+ = h^{-1}\mathcal{S}_p^h = h^{-1}\mathcal{S}_p^{z^n h_0}$$

and $\mathcal{S}_p^{h_0} = \{\gamma h_0 \; ; \; \gamma \in R_+\}$. Now we will prove that $\mathcal{S}_p^{z^n h_0} \subseteq \mathcal{S}_p^{z^n} \times h_0$. Then the lemma follows.

We will prove it by induction on n. It is clear when $n = 0$. We assume that $\mathcal{S}_p^{z^j h_0} \subseteq \mathcal{S}_p^{z^j} \times h_0$ for $j = 0, 1, \cdots, n-1$. If $f \in \mathcal{S}_p^{z^n h_0}$ and f is not a scalar multiple of $z^n h_0$, then by Lemma 1 $z^n h_0 + f$ has the form qg for some nonconstant inner function q. By a theorem of Frostman, there exists a sequence $\{d_\ell\}$ in D such that $d_\ell \to 0$ $(\ell \to \infty)$ and $(q - d_\ell)/(1 - \bar{d}_\ell q)$ is a Blaschke product for each ℓ. Then $(q - d_\ell)(1 - \bar{d}_\ell q)g \in \mathcal{S}_p^{z^n h_0}$ because $qg \in \mathcal{S}_p^{z^n h_0}$. If $(q - d_\ell)(1 - \bar{d}_\ell q)g$ belongs to $\mathcal{S}_p^{z^n} \times h_0$, then $qg \in \mathcal{S}_p^{z^n} \times h_0$ as $\ell \to \infty$. Since $z^n h_0 \in \mathcal{S}_p^{z^n} \times h_0$, f belongs to $\mathcal{S}_p^{z^n} \times h_0$ because $z^n h_0 + f = qg$ and so $\mathcal{S}_p^{z^n h_0} \subseteq \mathcal{S}_p^{z^n} \times h_0$. Now we will show that $(q - d_\ell)(1 - \bar{d}_\ell q)g$ belongs to $\mathcal{S}_p^{z^n} \times h_0$ for all ℓ. For each ℓ, there exist a complex constant $z_\ell \in D$

$$(q - d_\ell)(1 - \bar{d}_\ell q)g = (z - z_\ell)(1 - \bar{z}_\ell z) Q_\ell (1 - \bar{z}_\ell z)^{-2}(1 - \bar{d}_\ell q)^2 g$$

and $Q_\ell = \dfrac{1 - \bar{z}_\ell z}{z - z_\ell} \; \dfrac{q - d_\ell}{1 - \bar{d}_\ell q}$ is a Blaschke product. Since $(z - z_\ell)(1 - \bar{z}_\ell z)/z \geq 0$, $Q_\ell (1 - \bar{z}_\ell z)^{-2}(1 - \bar{d}_\ell q)^2$ belongs to $\mathcal{S}_p^{z^{n-1} h_0}$. By hypothesis on the induction, $Q_\ell (1 - \bar{z}_\ell z)^{-2}(1 - \bar{d}_\ell q)^2 g \in \mathcal{S}_p^{z^{n-1}} \times h_0$ and so $(q - d_\ell)(1 - \bar{d}_\ell q)g$ belongs to $\mathcal{S}_p^{z^n} \times h_0$ and so $\mathcal{S}_p^{z^n h_0} \subseteq \mathcal{S}_p^{z^n} \times h_0$.

Lemma 4. *Let $1/2 \leq p < \infty$. If $H^p(W)_+$ is of finite dimension n the inner part of any nonzero function s in $H^p(W)_+$ is a finite Blaschke product of degree $\leq n - 1$.*

Proof. Suppose that $0 \neq s \in H^p(W)_+$ and $s = qk$ where q is inner. If $q = \prod_{j=1}^{m} q_j$ and q_j $(1 \leq j \leq m)$ is a nonconstant inner function then put for each $\ell \leq m$

$$s_\ell = \left(1 + \prod_{j=1}^{\ell} q_j\right)^2 \left(q \prod_{j=1}^{\ell} \bar{q}_j\right) k.$$

Then $\{s_j\}_{j=1}^m$ is a system of independent elements in $H^p(W)_+$. This implies that $\dim H^p(W)_+ \geq m+1$.

Theorem 2. *Suppose $H^p(W)_+$ $(1/2 \leq p < \infty)$ is of finite dimension n. Then,*

(1) $W = W_1 W_0$, $W_1 = \prod_{j=1}^{n} |z - a_j|^{2p}$ where $|a_j| = 1$ $(1 \leq j \leq n)$ and $H^p(W_0)_+ = R_+$.

(2) $H^p(W)_+ = \left\{\gamma \prod_{j=1}^{n} \frac{(z - b_j)(1 - \bar{b}_j z)}{(z - a_j)(1 - \bar{a}_j z)} \ ; \ \gamma > 0 \text{ and } |b_j| \leq 1 \ (j = 1, \cdots, n)\right\}$.

Proof. Suppose $W = |h|^p$ and h is an outer function in H^p. By Lemma 4, for any function $f \in \mathcal{S}_p^h$ the inner part q is a finite Blaschke product of degree $\leq \dim H^p(W)_+ - 1$ because $H^p(W)_+ = h^{-1}\mathcal{S}_p^h$. Then there exists at least a function in $\mathcal{S}_p^h$ such that the degree of the inner part q is the largest n. If $qk \in \mathcal{S}_p^h$ then there exists a function $h_0 \in H^p$ such that $z^n h_0 \in \mathcal{S}_p^h$. Then

$$\mathcal{S}_p^h = \mathcal{S}_p^{z^n h_0} \text{ and } \mathcal{S}_p^{h_0} = \{\gamma h_0 \ ; \ \gamma \in R_+\}.$$

In fact, if there exists a function g in $\mathcal{S}_p^{h_0} \backslash \{\gamma h_0 \ ; \ \gamma \in R_+\}$, then by Lemma 1 $g + h_0 = q_1 h_1$ where q_1 is a nontrivial inner function. Then $z^n q_1 h_1 \in \mathcal{S}_p^h$ and by Lemma 4 this contradicts the definition of n. By the proof of Lemma 3, $\mathcal{S}_p^{z^n h_0} \subseteq \mathcal{S}_p^{z^n} \times h_0$. By a lemma of H. Helson and D. Sarason [2], because $p \geq 1/2$

$$\mathcal{S}_p^{z^n} = \left\{\gamma \prod_{j=1}^{n} (z - b_j)(1 - \bar{b}_j z) \ ; \ |b_j| \leq 1 \ (1 \leq j \leq n), \gamma \in R_+\right\}.$$

This implies that

$$\mathcal{S}_p^{z^n h_o} = \mathcal{S}_p^{z^n} \times h_0 = \mathcal{S}_p^h$$

and $h = \gamma \prod_{j=1}^{n} (z - a_j)(1 - \bar{a}_j z) h_0$ where $|a_j| = 1$ $(1 \leq j \leq n)$ and $\gamma \in R_+$. Put $W_1 = \prod_{j=1}^{n} |z - a_j|^{2p}$ and $W_0 = |h_0|^{2p}$, then (1) follows. By Proposition 1, $H^p(W)_+ = h^{-1}\mathcal{S}_p^h$ and so (2) follows.

§4. Special weights for $1/2 \le p < \infty$

It is known that a nonnegative function in $H^{1/2}$ is constant almost everywhere on ∂D. Lemma 5 is a generalization of it and known in [4]. This is important in the proof of Theorem 3. If W^{-1} is in L^∞ by Proposition 1 $H^{1/2}(W)_+ = R_+$. In this section, we study $H^{1/2}(W)_+$ when W^{-1} locally belongs to L^∞ except a finite set. By [4, Theorem 1] and Proposition 1, $H^1(W)_+$ is of finite dimension when W^{-1} locally belongs to L^1 except a finite set.

Lemma 5. *Let F be a function in N_* such that*
(a) F belongs to H^p for some $p > 0$,
(b) F locally belongs to $H^{1/2}$ except a finite set of ∂D,
(c) F is outer and F is nonnegative a.e. *on ∂D.*
Then F can be extended to a rational function.

Theorem 3. *Let $p \ge 1/2$. Suppose W^{-1} is in L^q for some $q > 0$. If W^{-1} belongs locally to $L^{1/(2p-1)}$ except a finite set A of ∂D, then $H^p(W)_+$ is of finite dimension and* $\dim H^p(W)_+ = 2N + 1$ *for some nonnegative integer N. If A is an empty set, then $H^p(W)_+ = R_+$. If $A = \{a_j \ ; \ j = 1, 2, \cdots, n\}$ then there exist nonnegative integers $m_1, \cdots, m_n$ such that the following (1) and (2) are valid for $N = \sum_{j=1}^{n} m_j$.*

(1) $W = W_1 W_0,\ W_1 = \prod_{j=1}^{n} |z - a_j|^{2m_j p},\ H^p(W_1)_+ = H^p(W)_+$ *and* $H^p(W_0)_+ = R_+$.

(2) $H^p(W)_+ = \{\gamma \prod_{j=1}^{N} (z - b_j)(1 - \bar{b}_j z) / \prod_{j=1}^{n} (z - a_j)^{m_j} (1 - \bar{a}_j z)^{m_j} \ ; \ \gamma > 0$ *and* $|b_j| \le 1 \ \ (j = 1, \cdots, N)\}$.

Proof. At first we prove that $H^p(W)_+$ is of finite dimension. Since $H^p(W)_+ = h^{-1}\mathcal{S}_p^h$ where $W = |h|^p$ and h is an outer function in H^p, it is enough to show that $\mathcal{S}_p^h$ is of finite dimension. If g is an outer function in $\mathcal{S}_p^h$, put $F = g/h$. Then $F \ge 0$ a.e. on ∂D, $F \in H^q$ for some $q > 0$. For a measurable subset $E \subset \partial D \backslash A$

$$\int_E |gh^{-1}|^{1/2} d\theta/2\pi \le \left(\int_E (|g|^{1/2})^\ell d\theta/2\pi\right)^{1/\ell} \left(\int_E |h^{-1/2}|^k d\theta/2\pi\right)^{1/k}$$

where $1/\ell + 1/k = 1$. If $\ell = 2p$, then $k = 2p/(2p-1)$ when $p \ne 1/2$, and $k = \infty$ when $p = 1/2$. If $k \ne \infty$, then

$$|h^{-\frac{1}{2}}|^k = |h|^{-\frac{p}{2p-1}} = W^{-\frac{1}{2p-1}}$$

and so F locally belongs to $H^{1/2}$ except A by hypothesis on W. Hence by Lemma 5 F is a rational function and

$$F(z) = \gamma \frac{\prod_{j=1}^N (z - b_j)(1 - \bar{b}_j z)}{\prod_{j=1}^n (z - a_j)^{m_j} (1 - \bar{a}_j z)^{m_j}}$$

where $\gamma > 0$, $N = m_1 + \cdots + m_n$, $|b_j| = 1$ $(1 \le j \le N)$ and $|a_j| = 1$ $(1 \le j \le n)$. Since h^{-1} locally belongs to $H^{p/(2p-1)}$, $\{a_j\}_{j=1}^n \subset A$. Therefore by finiteness of A and Theorem 2 there exists a positive integer $N_0 < \infty$ such that $N \le N_0$. This implies that $\mathcal{S}_p^h$ is of finite dimension. (1) and (2) are clear by the above proof and Therem 2. If A is empty, then F belongs to $H^{1/2}$ by the proof above and so F is constant. This implies that $\mathcal{S}_p^h = \{\gamma h \ ; \ \gamma \in R_+\}$ and so $\dim H^p(W)_+ = 1$.

Corollary 1. *Let* $\frac{1}{2} \le p < \infty$. *Suppose that* $W = |h|^p$ *and* h *is a rational outer function, that is,*

$$h(z) = \prod_{j=1}^n (z - a_j)^{m_j} / \prod_{j=1}^k (z - c_j)^{k_j}$$

where $|a_j| = 1$ $(1 \le j \le n)$, $a_j \ne a_i$ $(j \ne i)$ *and* $|c_j| = 1$ $(1 \le j \le k)$, $c_j \ne c_i$ $(j \ne i)$, $0 < pk_j < 1$ $(1 \le j \le k)$. *Then* $H^p(W)_+$ *is of finite dimension. Moreover there exist nonnegative integers* $\ell_1, \cdots, \ell_n$ *such that*

$$\frac{1}{2}(m_j + \frac{1}{p}) - 1 < \ell_j \le \frac{1}{2}(m_j + \frac{1}{p}) \quad (1 \le j \le n)$$

and $N = \sum_{j=1}^n \ell_j$, *and the following (1) and (2) are valid.*

(1) $W = W_1 W_0$, $W_1 = \prod_{j=1}^n |z - a_j|^{2\ell_j p}$, $W_0 = \prod_{j=1}^n |z - a_j|^{m_j p - 2\ell_j p} / \prod_{j=1}^k |z - c_j|^{pk_j}$.

(2) $H^p(W)_+ = \{\gamma \prod_{j=1}^N (z - b_j)(1 - \bar{b}_j z) / \prod_{j=1}^n (z - a_j)^{\ell_j} (1 - \bar{a}_j z)^{\ell_j} \ ; \ \gamma > 0, \ |b_j| \le 1 \ (j = 1, \cdots, N)\}$

Proof. Since W^{-1} belongs locally to L^∞ except $\{a_j\}_{j=1}^n$, by Theorem 3 $H^p(W)_+$ is of finite dimension. If $\ell_1, \cdots, \ell_n$ are nonnegative integers such that

$$\frac{1}{2}(m_j + \frac{1}{p}) - 1 < \ell_j \le \frac{1}{2}(m_j + \frac{1}{p}) \quad (1 \le j \le n)$$

and $N = \sum_{j=1}^n \ell_j$, then put $W_1 = \prod_{j=1}^n |z - a_j|^{2\ell_j p}$ and $W_0 = W/W_1$. Then by Theorem 3 $H^p(W_0)_+ = R_+$. For $W_0^{-1} \in L^{1/(2p-1)}$ because $-1 < (2\ell_j p - m_j p)/(2p-1) < 1/(2p-1)$. Hence (1) and (2) follows from Theorem 3.

Remark. (1) Let $p \ge 1/2$. Suppose W^{-1} is in L^q for some $q > 0$ and W^{-1} belongs locally to $L^{1/2p-1}$ except at $z = -1$. Then by Theorem 3 $H^p(W)_+$ is of finite dimension. Moreover, $\dim H^p(W)_+ = 2N+1$ if and only if $W|1+z|^{-2pN} \in L^1$ and $W|1+z|^{-2p(N+1)} \notin L^1$. (2) Let $p \ge 1/2$. Suppose $W = |1 + z|^\alpha$ and $-1 < \alpha < \infty$. Then the following

$(i) \sim (iii)$ are equivalent. $(i)\, \dim H^p(W)_+ = 2N+1$. $(ii)\, \max((2N)p-1, -1) < \alpha < (2N+3)p-1$. $(iii)\, \max(\frac{\alpha+1}{2N+3}, \frac{1}{2}) \le p < \frac{\alpha+1}{2N}$. For by Theorem 3, $\dim H^p(W)_+ = 2N+1$ for some nonnegative integer N and $H^p(W)_+ = \{\gamma \prod_{j=1}^{N}(z-b_j)(1-\bar{b}_j z)/(1+z)^{2N}\ ;\ \gamma > 0$ and $|b_j| \le 1\ (j = 1, \cdots, n)\}$. Since $hH^p(W)_+ = \mathcal{S}_p^h$ and $h = (1+z)^{\frac{\alpha}{p}}$, $(1+z)^{\frac{\alpha}{p}} H^p(W)_+ \subset H^p$ and so $(1+z)^{\frac{\alpha}{p}-2N}$ belongs to H^p. Hence $\alpha - (2N)p > -1$ and so $\alpha > \max((2N)p-1, -1)$. Since $\dim H^p(W)_+ = 2N+1$, $\alpha \le (2N+1)p-1$.

References

1. P.Duren, Theory of H^p Spaces (Academic Press, New York, 1970)
2. H.Helson and D.Sarason, Past and future, Math.Scand. 21(1967), 5-16.
3. E.Hayashi, The kernel of a Toeplitz operator, Integral Eq. Operator Theory, 9(1986), 588-591.
4. J.Inoue and T.Nakazi, Finite dimensional solution sets of extremal problems in H^1, Operator Theory : Adv.Appl.Vol. 62(1993), Birkhaüser Verlag, Basel, 115-124.
5. K.deLeeuw and W.Rudin, Extreme points and extremum problems in H^1, Pacific J.Math. 8(1958), 467-485.
6. T.Nakazi, Exposed points and extremal problems in H^1, J.Funct.Anal. 53(1983), 224-230.
7. J.Neuwirth and D.J.Newman, Positive $H^{1/2}$ functions are constants, Proc.Amer.Math.Soc. 18(1967), 958.