Nonnegative Functions In Weighted Hardy Spaces

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Abstract. Let $W$ be a nonnegative summable function whose logarithm is also summable with respect to the Lebesgue measure on the unit circle. For $0 < p < \infty$, $H^p(W)$ denotes a weighted Hardy space on the unit circle. When $W \equiv 1$, $H^p(W)$ is the usual Hardy space $H^p$. We are interested in $H^p(W)_+$ the set of all nonnegative functions in $H^p(W)$. If $p \geq 1/2$, $H^p_{+}$ consists of constant functions. However $H^p(W)_+$ contains a nonconstant nonnegative function for some weight $W$. In this paper, if $p \geq 1/2$ we determine $W$ and describe $H^p(W)_+$ when the linear span of $H^p(W)_+$ is of finite dimension. Moreover we show that the linear span of $H^p(W)_+$ is of infinite dimension for arbitrary weight $W$ when $0 < p < 1/2$. 
§1. Introduction and preliminaries

Let $W$ be a nonnegative function in \( L^1 = L^1(d\theta/2\pi, \partial D) \) where $D$ is the open unit disc and $\partial D$ is its boundary. For $0 < p < \infty$, a weighted Hardy space $H^p(W)$ denotes the closure of all analytic polynomials in $L^p(W) = L^p(d\theta/2\pi, \partial D)$. We may assume that $\log W$ is in $L^1$. For otherwise $H^p(W) = L^p(W)$. If $W \equiv 1$, then $H^p(W)$ is the usual Hardy space $H^p$. Let $N_*$ be the Smirnov class (see [1]), then $H^p = N_* \cap L^p$. A function $h$ in $N_*$ is called outer if it is invertible in $N_*$ and a function $q$ in $N_*$ is called inner if $|q| = 1$ a.e. on $\partial D$. If $W = |h|^p$ and $h$ is an outer function in $H^p$, it is known that $H^p(W) = h^{-1}H^p$ and so $H^p(W)$ is a subset of $N_*$. Put

\[ H^p(W)_+ = \{ s \in H^p(W) ; s \geq 0 \text{ a.e. on } \partial D \}. \]

If $W = |h|^p$, then $hH^p(W)_+ = \{ g \in H^p : \arg g = \arg h \text{ a.e. on } \partial D \}$. In this paper, the dimension of $H^p(W)_+$ means that of the linear span of $H^p(W)_+$ in $H^p(W)$.

J. Neuwirth and D. J. Newman [7] showed that $H^{1/2}_+ = R_+ = \{ p \in \mathbb{R} \}$ is the set of all nonnegative real constants. Hence $H^p_+ = R_+$ if $p \geq 1/2$. However $H^p_+ \neq R_+$ if $0 < p < 1/2$ because $z/(1 + z)^2$ belongs to $\bigcap_{p < 1/2} H^p$. $H^1(W)_+$ has been studied because $hH^1(W)_+ = \{ g \in H^1 : \arg g = \arg h \text{ a.e. on } \partial D \}$ when $W = |h|$. In fact, this set is related to the set of extremal functions of a well known linear extremal problem in $H^1$. (cf. [5],[6],[3],[4]). Hence $H^1(W)_+$ is known enough. However $H^p(W)_+$ has not studied before when $p \neq 1$. In this paper, we study $H^p(W)_+$ for arbitrary $p$. In §2, we show that $H^p(W)_+$ is of infinite dimension when $0 < p < 1/2$ and $W$ is arbitrary. In §3, we describe $H^p(W)_+$ when $p \geq 1/2$ and $H^p(W)_+$ is of finite dimension. In §4, we show that $H^p(W)_+$ is of finite dimension if $p \geq 1/2$ and $W^{-1}$ is locally in $L^{1/(2p-1)}$ except at a finite set.

Now we give a general result about $H^p(W)_+$ for $0 < p < \infty$. For any nonzero function $f$ in $H^p$, put $S^f_p = \{ g \in H^p ; \arg f = \arg g \text{ a.e. on } \partial D \}$ for $0 < p < \infty$. Put $(N_*)_+ = \{ s \in N_* ; s \geq 0 \text{ a.e. on } \partial D \}$.

**Proposition 1.** Let $0 < p < \infty$. Suppose $W = |h|^p$ and $h$ is an outer function in $H^p$.

1. $H^p(W)_+ = h^{-1}S^h_p$ and $R_+ \subseteq H^p(W)_+ \subseteq (N_*)_+$.
2. If $W$ is in $L^\infty$ then $H^p(W)_+ \supseteq H^p_+$.
3. If $W^{-1}$ is in $L^\infty$ then $H^p(W)_+ \subseteq H^p_+$.

Proof. Since $H^p(W) = h^{-1}H^p$, (1) follows from the definition of $S^h_p$. (2) and (3) are clear.

**Lemma 1.** If $f$ and $g$ are nonzero functions in $N_*$ such that $f/g$ is nonnegative and nonconstant almost everywhere on $\partial D$, then $f + g$ is not outer.
Proof. Put $h = f + g$, then $f/h + g/h = 1$, $0 \leq f/h \leq 1$ and $0 \leq g/h \leq 1$ on $\partial D$. If $h$ is outer, then both $f/h$ and $g/h$ belong to $N_s \cap L^\infty = H^\infty$. Thus both $f/h$ and $g/h$ are constant and so $f/g$ is constant. This contradiction shows the lemma.

Proposition 2. Let $0 < p < \infty$. If $H^p(W)_+ \not= R_+$, then there exists a function in $H^p(W)_+$ which is not outer. Hence the dimension of $H^p(W)_+$ is bigger than or equal to three.

Proof. If $s \in H^p(W)_+$ is nonconstant, by Lemma 1 $s + 1$ is not outer and so this implies the first part. Hence $qg = s + 1$ belongs to $H^p(W)_+$ where $q$ is a nonconstant inner part of $s + 1$. Since both $(1 + q)^2 g$ and $1$ belong to $H^p(W)_+$, this implies the second part.

§2. General weights for $0 < p < 1/2$.

If $0 < p < 1/2$, then $(z - b)(1 - \overline{b}z)/(z - a)(1 - \overline{a}z)$ belongs to $H^p$ where $|a| = 1$ and $|b| \leq 1$ and so $H^p(z)$ is of infinite dimension. In this section, we show that $H^p(W)_+$ is of infinite dimension for arbitrary weight $W$.

Lemma 2. If $0 < p < 1/2$ and if $h$ is a function in $H^p$, $h(z)/(z - e^{it})(1 - e^{-it}z)$ belongs to $H^p$ for a.e. $e^{it}$.

Proof. If $k(z) = z/(1 - z)^2$, $|k|^p$ and $|h|^p$ belongs to $L^1$ and hence $|k|^p * |h|^p \in L^1$.

$$|k|^p * |h|^p(e^{it}) = \int_0^{2\pi} |k(e^{i(t-\theta)}h(e^{i\theta}))|^p d\theta / 2\pi \leq \int_0^{2\pi} \left| \frac{e^{it}h(e^{i\theta})}{(e^{it} - e^{i\theta})(1 - e^{it}e^{i\theta})} \right|^p d\theta / 2\pi < \infty$$

This implies that $h(z)/(z - e^{it})(1 - e^{-it}z)$ belongs to $H^p$ for a.e. $e^{it}$

Theorem 1. For arbitrary weight $W$, $H^p(W)_+$ is of infinite dimension.

Proof. Suppose that $W = |h|^p$ and $h$ is an outer function in $H^p$. Since $H^p(W)_+ = h^{-1}S^h_p$ by (1) of Proposition 1, it is enough to prove that $S^h_p$ is of infinite dimension. By Lemma 2, for any finite $n$, $h(z)\prod_{j=1}^n(z - b_j)(1 - \overline{b}_jz)/\prod_{j=1}^n(z - a_j)(1 - \overline{a}_jz)$ belongs to $H^p$ where $|b_j| < 1$ and $|a_j| = 1$ for $1 \leq j \leq n$. Since $\prod_{j=1}^n(z - b_j)(1 - \overline{b}_jz)/\prod_{j=1}^n(z - a_j)(1 - \overline{a}_jz)$ is nonnegative on $\partial D$, $S^h_p$ is of infinite dimension.
§3. General weights for $1/2 \leq p < \infty$

Unlike when $0 < p < 1/2$, if $1/2 \leq p < \infty$, then $H^p(W)_+$ may be of finite dimension for some weight $W$. In this section, we describe $H^p(W)_+$ when $H^p(W)_+$ is of finite dimension.

**Lemma 3.** Let $1/2 \leq p < \infty$. If $W = W_1W_0$, and $W_1 = \prod_{j=1}^n |z - a_j|^{2p}$ where $|a_j| = 1$ $(1 \leq j \leq n)$ and $H^p(W_0)_+ = R_+$, then $H^p(W)_+ \subseteq H^p(W_1)_+$. If $W_0$ is in $L^\infty$, then $H^p(W)_+ = H^p(W_1)_+$.

Proof. Let $h_1 = \prod_{j=1}^n (z - a_j)(1 - \bar{a}_j z)$ and $h_0$ an outer function in $H^p$ with $|h_0|^p = W_0$. Put $h = h_1h_0$, then $W = |h|^p = |h_1|^p \times |h_0|^p = W_1W_0$. Since $\bar{z}_n \prod_{j=1}^n (z - a_j)(1 - \bar{a}_j z) = \prod_{j=1}^n |z - a_j|^2$, $|h|/h = \bar{z}_n|h_0|/h_0$. Hence

$$H^p(W)_+ = h^{-1}S^h_p = h^{-1}S^{z^nh_0}_p$$

and $S^{z^nh_0}_p = \{ \gamma h_0 : \gamma \in R_+ \}$. Now we will prove that $S^{z^nh_0}_p \subseteq S^{z^n}_p \times h_0$. Then the lemma follows.

We will prove it by induction on $n$. It is clear when $n = 0$. We assume that $S^{z^jh_0}_p \subseteq S^{z^j}_p \times h_0$ for $j = 0, 1, \ldots, n - 1$. If $f \in S^{z^nh_0}_p$ and $f$ is not a scalar multiple of $z^nh_0$, then by Lemma 1 $z^nh_0 + f$ has the form $qg$ for some nonconstant inner function $q$.

By a theorem of Frostman, there exists a sequence $\{d_\ell\}$ in $D$ such that $d_\ell \to 0$ ($\ell \to \infty$) and $(q - d_\ell)/(1 - \bar{d}_\ell q)$ is a Blaschke product for each $\ell$. Then $(q - d_\ell)(1 - \bar{d}_\ell q)g \in S^{z^nh_0}_p$ because $qg \in S^{z^nh_0}_p$. If $(q - d_\ell)(1 - \bar{d}_\ell q)g$ belongs to $S^{z^n}_p \times h_0$, then $qg \in S^{z^n}_p \times h_0$ as $\ell \to \infty$. Since $z^nh_0 \in S^{z^n}_p \times h_0$, $f$ belongs to $S^{z^n}_p \times h_0$ because $z^nh_0 + f = qg$ and so $S^{z^nh_0}_p \subseteq S^{z^n}_p \times h_0$. Now we will show that $(q - d_\ell)(1 - \bar{d}_\ell q)g$ belongs to $S^{z^n}_p \times h_0$ for all $\ell$. For each $\ell$, there exist a complex constant $z_\ell \in D$

$$(q - d_\ell)(1 - \bar{d}_\ell q)g = (z - z_\ell)(1 - \bar{z}_\ell z)Q_\ell(1 - \bar{z}_\ell z)^{-2}(1 - \bar{d}_\ell q)^2 g$$

and $Q_\ell = (z - z_\ell)/(z - z_\ell)$ is a Blaschke product. Since $(z - z_\ell)(1 - \bar{z}_\ell z)/z \geq 0$, $Q_\ell(1 - \bar{z}_\ell z)^{-2}(1 - \bar{d}_\ell q)^2 g \in S^{z^{n-1}h_0}_p$. By hypothesis on the induction, $Q_\ell(1 - \bar{z}_\ell z)^{-2}(1 - \bar{d}_\ell q)^2 g \in S^{z^n}_p \times h_0$ and so $(q - d_\ell)(1 - \bar{d}_\ell q)g$ belongs to $S^{z^n}_p \times h_0$ and so $S^{z^nh_0}_p \subseteq S^{z^n}_p \times h_0$.

**Lemma 4.** Let $1/2 \leq p < \infty$. If $H^p(W)_+$ is of finite dimension $n$ the inner part of any nonzero function $s$ in $H^p(W)_+$ is a finite Blaschke product of degree $\leq n - 1$.  


Proof. Suppose that $0 \neq s \in H^p(W)_+$ and $s = qk$ where $q$ is inner. If $q = \prod_{j=1}^m q_j$ and $q_j (1 \leq j \leq m)$ is a nonconstant inner function then put for each $\ell \leq m$

$$s_\ell = \left(1 + \prod_{j=1}^\ell q_j \right)^2 \left( q \prod_{j=1}^\ell \bar{q}_j \right) k.$$ Then $\{s_j\}_{j=1}^m$ is a system of independent elements in $H^p(W)_+$. This implies that $\dim H^p(W)_+ \geq m + 1$.

**Theorem 2.** Suppose $H^p(W)_+ (1/2 \leq p < \infty)$ is of finite dimension $n$. Then,

1. $W = W_1 W_0$, $W_1 = \prod_{j=1}^n |z - a_j|^{2p}$ where $|a_j| = 1$ ($1 \leq j \leq n$) and $H^p(W_0)_+ = R_+$.

2. $H^p(W)_+ = \left\{ \gamma \prod_{j=1}^n (z - b_j)(1 - \bar{b}_jz) \mid \gamma > 0 \text{ and } |b_j| \leq 1 \ (j = 1, \ldots, n) \right\}$.

Proof. Suppose $W = |h|^p$ and $h$ is an outer function in $H^p$. By Lemma 4, for any function $f \in S^h_p$ the inner part $q$ is a finite Blaschke product of degree $\leq \dim H^p(W)_+ - 1$ because $H^p(W)_+ = h^{-1}S_p^h$. Then there exists at least a function in $S_p^h$ such that the degree of the inner part $q$ is the largest $n$. If $qk \in S_p^h$ then there exists a function $h_0 \in H^p$ such that $z^n h_0 \in S_p^h$. Then

$$S_p^h = S_p^{z^n h_0} \text{ and } S_p^{h_0} = \{ \gamma h_0 \mid \gamma \in R_+ \}.$$ In fact, if there exists a function $g$ in $S_p^{h_0} \setminus \{ \gamma h_0 \mid \gamma \in R_+ \}$, then by Lemma 1 $g + h_0 = q_1 h_1$ where $q_1$ is a nontrivial inner function. Then $z^n q_1 h_1 \in S_p^h$ and by Lemma 4 this contradicts the definition of $n$. By the proof of Lemma 3, $S_p^{z^n h_0} \subseteq S_p^{z^n} \times h_0$. By a lemma of H. Helson and D. Sarason [2], because $p \geq 1/2$

$$S_p^{z^n} = \left\{ \gamma \prod_{j=1}^n (z - b_j)(1 - \bar{b}_j z) \mid |b_j| \leq 1 (1 \leq j \leq n), \gamma \in R_+ \right\}.$$ This implies that

$$S_p^{z^n h_0} = S_p^{z^n} \times h_0 = S_p^h$$ and $h = \gamma \prod_{j=1}^n (z - a_j)(1 - \bar{a}_j z) h_0$ where $|a_j| = 1$ ($1 \leq j \leq n$) and $\gamma \in R_+$. Put $W_1 = \prod_{j=1}^n |z - a_j|^{2p}$ and $W_0 = |h_0|^{2p}$, then (1) follows. By Proposition 1, $H^p(W)_+ = h^{-1}S_p^h$ and so (2) follows.
§4. Special weights for $1/2 < p < \infty$

It is known that a nonnegative function in $H^{1/2}$ is constant almost everywhere on $\partial D$. Lemma 5 is a generalization of it and known in [4]. This is important in the proof of Theorem 3. If $W^{-1}$ is in $L^\infty$ by Proposition 1 $H^{1/2}(W)_+ = R_+$. In this section, we study $H^{1/2}(W)_+$ when $W^{-1}$ locally belongs to $L^\infty$ except a finite set. By [4, Theorem 1] and Proposition 1, $H^1(W)_+$ is of finite dimension when $W^{-1}$ locally belongs to $L^1$ except a finite set.

Lemma 5. Let $F$ be a function in $N$, such that
(a) $F$ belongs to $H^p$ for some $p > 0$,
(b) $F$ locally belongs to $H^{1/2}$ except a finite set of $\partial D$,
(c) $F$ is outer and $F$ is nonnegative a.e. on $\partial D$.

Then $F$ can be extended to a rational function.

Theorem 3. Let $p \geq 1/2$. Suppose $W^{-1}$ is in $L^q$ for some $q > 0$. If $W^{-1}$ belongs locally to $L^{1/(2p-1)}$ except a finite set $A$ of $\partial D$, then $H^p(W)_+$ is of finite dimension and $\dim H^p(W)_+ = 2N + 1$ for some nonnegative integer $N$. If $A$ is an empty set, then $H^p(W)_+ = R_+$. If $A = \{a_j; j = 1, 2, \ldots, n\}$ then there exist nonnegative integers $m_1, \ldots, m_n$ such that the following (1) and (2) are valid for $N = \sum_{j=1}^n m_j$.

\begin{enumerate}
  \item[(1)] $W = W_1W_0$, $W_1 = \prod_{j=1}^n |z - a_j|^{2m_j}$, $H^p(W_1)_+ = H^p(W)_+$ and $H^p(W_0)_+ = R_+$.
  \item[(2)] $H^p(W)_+ = \{\gamma \prod_{j=1}^n (z - b_j)(1 - \bar{b}_jz)/\prod_{j=1}^n (z - a_j)^{m_j}(1 - \bar{a}_jz)^{m_j}; \gamma > 0 \text{ and } |b_j| \leq 1 \ (j = 1, \ldots, N)\}$.
\end{enumerate}

Proof. At first we prove that $H^p(W)_+$ is of finite dimension. Since $H^p(W)_+ = h^{-1}S^h_p$ where $W = |h|^p$ and $h$ is an outer function in $H^p$, it is enough to show that $S^h_p$ is of finite dimension. If $g$ is an outer function in $S^h_p$, put $F = g/h$. Then $F \geq 0$ a.e. on $\partial D$, $F \in H^q$ for some $q > 0$. For a measurable subset $E \subset \partial D \setminus A$

\[
\int_E |g|^{-1/2} |d\theta|/2\pi \leq \left(\int_E (|g|^{-1/2} |d\theta|/2\pi)^{\ell/\ell} \left(\int_E |h^{-1/2} |d\theta|/2\pi\right)^{k/k}\right)^{1/\ell}
\]

where $1/\ell + 1/k = 1$. If $\ell = 2p$, then $k = 2p/(2p - 1)$ when $p \neq 1/2$, and $k = \infty$ when $p = 1/2$. If $k \neq \infty$, then

\[
|h^{-1/2}|^k = |h|^{-\frac{p}{p-1}} = W^{-\frac{1}{p-1}}
\]

and so $F$ locally belongs to $H^{1/2}$ except $A$ by hypothesis on $W$. Hence by Lemma 5 $F$ is a rational function and

\[
F(z) = \frac{\prod_{j=1}^N (z - b_j)(1 - \bar{b}_jz)}{\prod_{j=1}^N (z - a_j)^{m_j}(1 - \bar{a}_jz)^{m_j}}
\]
where \( \gamma > 0 \), \( N = m_1 + \cdots + m_n \), \( |b_j| = 1 \) \((1 \leq j \leq N)\) and \( |a_j| = 1 \) \((1 \leq j \leq n)\).

Since \( h^{-1} \) locally belongs to \( H^{p/(2p-1)} \), \( \{a_j\}_{j=1}^n \subset A \). Therefore by finiteness of \( A \) and Theorem 2 there exists a positive integer \( N_0 < \infty \) such that \( N \leq N_0 \). This implies that \( S_p^h \) is of finite dimension. \((1) \) and \((2) \) follows from Theorem 3.

Theorem 2 there exists a positive integer \( N \) such that
\[
S \text{ is of finite dimension. If } A \text{ is empty, then } F \text{ belongs to } H^{1/2} \text{ by the proof above and so } F \text{ is constant. This implies that } S_p^h = \{\gamma h ; \gamma \in R_+\} \text{ and so dim } H^p(W)_+ = 1.
\]

**Corollary 1.** Let \( \frac{1}{2} \leq p < \infty \). Suppose that \( W = |h|^p \) and \( h \) is a rational outer function, that is,
\[
h(z) = \prod_{j=1}^n (z - a_j)^{m_j} / \prod_{j=1}^k (z - c_j)^{k_j}
\]
where \( |a_j| = 1 \) \((1 \leq j \leq n)\), \( a_j \neq a_i \) \((j \neq i)\) and \( |c_j| = 1 \) \((1 \leq j \leq k)\), \( c_j \neq c_i \) \((j \neq i)\), \( 0 < pk_j < 1 \) \((1 \leq j \leq k)\). Then \( H^p(W)_+ \) is of finite dimension. Moreover there exist nonnegative integers \( \ell_1, \cdots, \ell_n \) such that
\[
\frac{1}{2} (m_j + \frac{1}{p}) - 1 < \ell_j \leq \frac{1}{2} (m_j + \frac{1}{p}) \quad (1 \leq j \leq n)
\]
and \( N = \sum_{j=1}^n \ell_j \), and the following \((1) \) and \((2) \) are valid.

\( (1) \) \( W = W_1 W_0 \), \( W_1 = \prod_{j=1}^n |z - a_j|^{2\ell_jp} \), \( W_0 = \prod_{j=1}^n |z - a_j|^{m_jp-2\ell_jp} / \prod_{j=1}^k |z - c_j|^{pk_j} \).

\( (2) \) \( H^p(W)_+ = \{\gamma \prod_{j=1}^n (z - b_j)(1 - \bar{b}_jz) / \prod_{j=1}^n (z - a_j)^{\ell_j}(1 - \bar{a}_jz)^{\ell_j} ; \gamma > 0, |b_j| \leq 1 \ (j = 1, \cdots, N)\} \)

Proof. Since \( W^{-1} \) belongs locally to \( L^\infty \) except \( \{a_j\}_{j=1}^n \), by Theorem 3 \( H^p(W)_+ \) is of finite dimension. If \( \ell_1, \cdots, \ell_n \) are nonnegative integers such that
\[
\frac{1}{2} (m_j + \frac{1}{p}) - 1 < \ell_j \leq \frac{1}{2} (m_j + \frac{1}{p}) \quad (1 \leq j \leq n)
\]
and \( N = \sum_{j=1}^n \ell_j \), then put \( W_1 = \prod_{j=1}^n |z - a_j|^{2\ell_jp} \) and \( W_0 = W/W_1 \). Then by Theorem 3 \( H^p(W_0)_+ = R_+ \). For \( W_0^{-1} \in L^{1/(2p-1)} \) because \(-1 < (2\ell_jp - m_jp)/(2p - 1) < 1/(2p - 1)\). Hence \((1) \) and \((2) \) follows from Theorem 3.

**Remark.** (1) Let \( p \geq 1/2 \). Suppose \( W^{-1} \) is in \( L^q \) for some \( q > 0 \) and \( W^{-1} \) belongs locally to \( L^{1/(2p-1)} \) except at \( z = \infty \). Then by Theorem 3 \( H^p(W)_+ \) is of finite dimension. Moreover, \( \dim H^p(W)_+ = 2N + 1 \) if and only if \( W[1 + z]^{-2pN} \in L^1 \) and \( W[1 + z]^{-2p(N+1)} \notin L^1 \). (2) Let \( p \geq 1/2 \). Suppose \( W = |1 + z|^\alpha \) and \(-1 < \alpha < \infty \). Then the following
(i) \sim (iii) are equivalent. (i) dim \( H^p(W)_+ \) = \( 2N + 1 \). (ii) \( \max((2N)p - 1, -1) < \alpha < (2N + 3)p - 1 \). (iii) \( \max(\frac{N+1}{2N+3}, \frac{1}{2}) \leq p < \frac{N+1}{2N} \). For by Theorem 3, dim \( H^p(W)_+ \) = \( 2N + 1 \) for some nonnegative integer \( N \) and \( H^p(W)_+ = \{ \gamma \prod_{j=1}^{N}(z-b_j)(1-\bar{b_j}z)/(1+z)^{2N} ; \gamma > 0 \) and \( |b_j| \leq 1 \ (j = 1, \cdots, n) \} \). Since \( hH^p(W)_+ = S^h_p \) and \( h = (1+z)\frac{\bar{\gamma}}{\gamma} \), \( (1+z)^{\frac{\bar{\gamma}}{\gamma}} H^p(W)_+ \subset H^p \) and so \( (1+z)^{\frac{\bar{\gamma}}{\gamma} - 2N} \) belongs to \( H^p \). Hence \( \alpha - (2N)p > -1 \) and so \( \alpha \geq \max((2N)p - 1, -1) \).

Since dim \( H^p(W)_+ \) = \( 2N + 1 \), \( \alpha \leq (2N + 1)p - 1 \).

References