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**SPACE-TIME L^2 -ESTIMATES AND LIFE-SPAN OF
THE KLAINERMAN-MACHEDON RADIAL SOLUTIONS TO
SOME SEMI-LINEAR WAVE EQUATIONS**

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Abstract. We consider the Cauchy problem for some semi-linear wave equations in three space dimensions and prove global or almost global existence of the Klainerman-Machedon radial solutions. The proof is carried out by a contraction-mapping argument based on a refined version of the Keel-Smith-Sogge estimate, together with the Morawetz-type inequality.

1. INTRODUCTION

In this paper we are concerned with the long-time existence of low-regularity small solutions to some semi-linear wave equation in three space dimensions of the form

$$\square u = Q(\partial u), \quad 0 < t < T, \quad x \in \mathbb{R}^3 \quad (1.1)$$

or

$$\square u = G(u, \partial u), \quad 0 < t < T, \quad x \in \mathbb{R}^3 \quad (1.2)$$

subject to the initial data

$$u(0) = f, \quad \partial_t u(0) = g. \quad (1.3)$$

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Here $\square = \partial_t^2 - \Delta$, $u : (0, T) \times \mathbb{R}^3 \rightarrow \mathbb{R}$, $\partial u = (\partial_t u, \nabla u)$ and

$$Q(\partial u) = \sum_{a,b=0}^3 C_{ab} \partial_a u \partial_b u, \quad (1.4)$$

$$G(u, \partial u) = \sum_{a,b=0}^3 C_{ab} u^{l_{ab}} \partial_a u \partial_b u \quad (1.5)$$

for $C_{ab} \in \mathbb{R}$, $l_{ab} \in \{1, 2, \dots\}$. Here and in what follows, $\partial_0 = \partial/\partial_t$, $\partial_j = \partial/\partial_{x_j}$.

To review some results on classical solutions, let us suppose $(f, g) \in H^s \times H^{s-1}$ and $s > 7/2$ first. Then the Cauchy problem for a quasi-linear wave equation has a unique classical solution locally in time. Moreover, by giving initial data of the form

$$u(0) = \varepsilon f, \quad \partial_t u(0) = \varepsilon g \quad (1.6)$$

for small $\varepsilon > 0$ instead of (1.3), we can investigate the influence of the ‘‘amplitude’’ ε on the existence time. In fact, it is well-known that the maximal existence time of smooth solutions to (1.1) – (1.6) (the life-span) can be estimated from below by $\exp[c/\varepsilon]$ as $\varepsilon \rightarrow 0$, if f, g are sufficiently smooth functions which decay sufficiently fast at the spatial infinity. (See John-Klainerman [6], where they discussed more general nonlinearities including some second derivatives.) It is also known that the smooth solutions to (1.1) in general blow up in finite time, however small the initial data are. More precisely, Rammaha [19] showed that upper bounds of the life-span for $Q(\partial u) = |\nabla_x u|^2$ are given by $\exp[c'/\varepsilon]$ as $\varepsilon \rightarrow 0$ (See also [18]). In addition, we note that quite sharp estimates on the life-span are established at least for certain nonlinearities. See p.128 of [4] and references cited there. See also [5]. When speaking of (1.2), it is well-known that the Cauchy problem for (1.2) or more general cubic nonlinearities has unique global in time solutions if f, g in (1.3) are sufficiently small and smooth data, decaying sufficiently fast at spatial infinity. See Klainerman [8], Shatah [20], and Klainerman-Ponce [12].

The purpose of this paper is to study the same problems for rougher initial data. It follows from the standard energy method that both the Cauchy problem for (1.1) and that for (1.2) are well-posed locally in time in $H^s \times H^{s-1}$ if $s > 5/2$. Ponce and Sideris [17] showed that the Cauchy problems for (1.1) and (1.2) are both locally well-posed in $H^s \times H^{s-1}$ with $s > 2$ by applying the Strichartz estimate. This result is sharp in general, because Lindblad [13] proved the ill-posedness in $H^2 \times H^1$ for

the equation

$$\square u = (\partial_1 u - \partial_t u)^2. \quad (1.7)$$

However, Klainerman and Machedon showed in [9] that if the nonlinearities satisfy the null condition, then the well-posedness for $s = 2$ can be achieved. They also showed that as far as spherically symmetric (in x) data f , g and quadratic nonlinear terms are concerned, the well-posedness holds true for $s = 2$. (It should be noted that the nonlinearity of (1.7) does not satisfy the null condition. We also keep in mind that the equation (1.7) possesses no spherically symmetric solutions.) It is therefore worth while to address the question as to whether these $H^2 \times H^1$ -solutions exist for long time or not.

In the present paper, we investigate the Klainerman-Machedon radial $H^2 \times H^1$ -solutions as the size of data goes to zero. For this study of the equation (1.1), let us give initial data (1.6) for small $\varepsilon > 0$. Compared with the classical results explained above, our $H^2 \times H^1$ -data may have much less regularity and slower decay at the spatial infinity. We therefore need some new techniques to measure the life-span of our small $H^2 \times H^1$ -radial solutions. The key to our results is an effective use of some space-time L^2 -estimates such as the Morawetz-type inequality. Our space-time L^2 -estimates are almost scale-invariant, and they are in fact sharper versions of the inequalities due to Keel, Smith, and Sogge [7]. This refinement comes from a simple but crucial idea of carrying out estimates for $|x| < 1$ and $|x| > 1$ separately. This device, which is essentially based on smoothing property of solutions, enables us to get over the singularity at $r = 0$ caused by the Sobolev-type inequality (2.3), and we can obtain a priori bounds on the radial solution in H^2 . It is then proved that the maximal existence time of our radial $H^2 \times H^1$ -solutions can be estimated from below by $\exp[c/\varepsilon]$ as $\varepsilon \rightarrow 0$, as in the case of smooth, rapidly decaying solutions. In addition, we also study the equation (1.2) with small $H^2 \times H^1$ -radial initial data. We show the global existence of small $H^2 \times H^1$ -radial solutions, as in the case of smooth, rapidly decaying solutions.

We finally make two remarks. Firstly, we mention the case where the nonlinearity satisfies the null condition. In this case Georgiev and Schirmer [2] showed the global existence of $H^2 \times H^1$ -solutions for all data which are small in a weighted $H^2 \times H^1$ -norm. We also note that the local well-posedness in $H^s \times H^{s-1}$ for this nonlinearity

was improved to $s > 3/2$ by Klainerman and Machedon [10, 11]. The second is a remark on the space-time L^2 -estimate

$$\|\langle x \rangle^{-1/2} \partial u\|_{L^2((0,T) \times \mathbb{R}^n)} \leq C \sqrt{\log(2+T)} \left(\|\partial u(0)\|_{L^2} + \int_0^T \|\square u(\tau)\|_{L^2} d\tau \right) \quad (1.8)$$

of Keel, Smith and Sogge [7]. This sort of estimate was discovered earlier by Morawetz [16] and Strauss [21] during the studies on decay properties of the local energy. In [7] they appropriately localized the energy estimates and summed them up using the Huygens' principle, to obtain (1.8). Recently, Alinhac [1] revived and refined the multiplier method by Morawetz [16] from geometric point of view. His approach makes possible curved-background versions of (1.8). For another geometric approach, see Metcalfe and Sogge [15], who obtained an analog of the Keel-Smith-Sogge estimate for the perturbed Dirichlet-wave equation in an exterior to a star-shaped obstacle.

Here we explain the notation used in this paper. For a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ we denote $\partial_x^\alpha := \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$. We also use the notation $\partial_t^\alpha := \partial_t^{\alpha_0} \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$ for a multi-index $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$. For $r := |x|$, the operator $\partial_r = (x_1/r)\partial_1 + (x_2/r)\partial_2 + (x_3/r)\partial_3$ will be used frequently. Let us define the homogeneous Sobolev space $\dot{H}_2^s(\mathbb{R}^n)$ ($n \geq 2$, $1/2 < s < n/2$) as

$$\dot{H}_2^s(\mathbb{R}^n) = \{v \in \mathcal{S}'(\mathbb{R}^n) : \|v\|_{\dot{H}_2^s} < \infty\},$$

where $\|v\|_{\dot{H}_2^s} := \|\ |\xi|^s \hat{v} \|_{L^2}$ and $\hat{v}(\xi)$ is the Fourier transform of $v(x)$. The set of all the radial elements of $\dot{H}_2^s(\mathbb{R}^n)$ is denoted as $\dot{H}_{\text{rad}}^s(\mathbb{R}^n)$. We note that $\dot{H}_{\text{rad}}^s(\mathbb{R}^n)$ is a closed subset of $\dot{H}_2^s(\mathbb{R}^n)$. The set of all the spherically symmetric elements of the standard Sobolev space $H^s(\mathbb{R}^n)$ is also denoted as $H_{\text{rad}}^s(\mathbb{R}^n)$.

Before the statement of our main results, let us limit the class of nonlinear terms because we are concerned only with spherically symmetric solutions. Let U be an arbitrary element of $SO(3)$. We naturally assume that the nonlinear term Q of (1.4) satisfies

$$Q(\partial u) = Q(\tilde{\partial} \tilde{u}) \quad (1.9)$$

for any $u \in C^1(\mathbb{R} \times \mathbb{R}^3)$, where $\tilde{u}(t, y) = u(t, Uy)$ and $\tilde{\partial} = (\partial_t, \partial_{y_1}, \partial_{y_2}, \partial_{y_3})$. Similarly,

$$G(u, \partial u) = G(\tilde{u}, \tilde{\partial} u) \quad (1.10)$$

is assumed for the nonlinear term G of (1.5). It is easy to see that the nonlinear terms can be written as

$$Q(\partial u) = a(\partial_t u)^2 + b|\partial_x u|^2, \quad G(u, \partial u) = \sum_{k=1}^N u^k \{a_k(\partial_t u)^2 + b_k|\partial_x u|^2\}, \quad (1.11)$$

where a, b, a_k, b_k are real constants.

We are in a position to state the main results in this paper.

Theorem 1.1. *Suppose that $f \in H_{\text{rad}}^2(\mathbb{R}^3)$ and $g \in H_{\text{rad}}^1(\mathbb{R}^3)$. There exist positive constants ε_0, A and C with the following property: For any $0 < \varepsilon \leq \varepsilon_0$ there exists a unique solution to (1.1) with data $(\varepsilon f, \varepsilon g)$ at $t = 0$ satisfying*

$$u \in \bigcap_{j=0}^2 C^j([0, T_\varepsilon]; H^{2-j}(\mathbb{R}^3)), \quad (1.12)$$

$$\sum_{\substack{0 \leq |\alpha| \leq 2, 0 \leq j \leq 2 \\ 1 \leq |\alpha| + j \leq 2}} \sup_{0 < t < T_\varepsilon} \|\partial_x^\alpha \partial_t^j u(t)\|_{L^2(\mathbb{R}^3)} \quad (1.13)$$

$$+ \sum_{\substack{0 \leq |\alpha| \leq 2, 0 \leq j \leq 2 \\ 1 \leq |\alpha| + j \leq 2}} (\log(2 + T_\varepsilon))^{-1/2} \| |x|^{-1/4} \langle x \rangle^{-1/4} \partial_x^\alpha \partial_t^j u \|_{L^2((0, T_\varepsilon) \times \mathbb{R}^3)}$$

$$+ \sum_{|\alpha| + j = 1} \| |x|^{-5/4} \partial_x^\alpha \partial_t^j u \|_{L^2((0, T_\varepsilon) \times B_4)} \leq C\varepsilon \quad (B_4 := \{x \in \mathbb{R}^3 : |x| < 4\}),$$

where $2 + T_\varepsilon = \exp[A/\varepsilon]$.

Theorem 1.2. *Suppose that $f \in H_{\text{rad}}^2(\mathbb{R}^3)$ and $g \in H_{\text{rad}}^1(\mathbb{R}^3)$. There exist positive constants ε and C with the following property: If*

$$\sum_{1 \leq |\alpha| \leq 2} \|\partial_x^\alpha f\|_{L^2} + \|g\|_{H^1} \leq \varepsilon, \quad (1.14)$$

then the Cauchy problem (1.2) with data (f, g) at $t = 0$ admits a unique solution satisfying

$$u \in \bigcap_{j=0}^2 C^j([0, \infty); H^{2-j}(\mathbb{R}^3)), \quad (1.15)$$

$$\sum_{\substack{0 \leq |\alpha| \leq 2, 0 \leq j \leq 2 \\ 1 \leq |\alpha| + j \leq 2}} \sup_{t > 0} \|\partial_x^\alpha \partial_t^j u(t)\|_{L^2(\mathbb{R}^3)} \quad (1.16)$$

$$+ \sum_{\substack{0 \leq |\alpha| \leq 2, 0 \leq j \leq 2 \\ 1 \leq |\alpha| + j \leq 2}} \| |x|^{-1/4} \langle x \rangle^{-\delta} \partial_x^\alpha \partial_t^j u \|_{L^2((0, \infty) \times \mathbb{R}^3)}$$

$$+ \sum_{|\alpha| + j = 1} \| |x|^{-5/4} \partial_x^\alpha \partial_t^j u \|_{L^2((0, \infty) \times B_4)} \leq C\varepsilon$$

for a constant $\delta > 1/4$.

2. PRELIMINARIES

We prove some lemmas which play important roles in the proof of our main theorems.

Lemma 2.1. *Let $v \in C^2(\mathbb{R}^3)$. If v is spherically symmetric, the inequalities*

$$|\partial_j v(x)| \leq |\partial_r v(x)| \leq \sum_{i=1}^3 |\partial_i v(x)|, \quad (2.1)$$

$$|\partial_j \partial_r v(x)| \leq C |\partial_x^2 v(x)| + \frac{C}{|x|} |\partial_x v(x)| \quad (2.2)$$

hold.

Proof. Writing $v(x) = \phi(r)$, we have for $j = 1, 2, 3$

$$|\partial_j v(x)| = |\phi'(r) \frac{x_j}{r}| \leq |\phi'(r)| = |\partial_r v(x)|,$$

which proves the first inequality of (2.1). The other is obvious. Since

$$\partial_j \partial_i v(x) = \partial_j \left(\phi'(r) \frac{x_i}{r} \right) = \partial_j \phi'(r) \frac{x_i}{r} - \phi'(r) \frac{x_j x_i}{r^3} + \delta_{ij} \phi'(r) \frac{1}{r},$$

we have

$$|\partial_j \phi'(r)| \leq \left(\sum_{i=1}^3 \left| \partial_j \phi'(r) \frac{x_i}{r} \right|^2 \right)^{1/2} \leq C |\partial_x^2 v(x)| + \frac{C}{r} |\phi'(r)|.$$

Thus we get (2.2). \square

Lemma 2.2. *Let $v \in \dot{H}_{\text{rad}}^1(\mathbb{R}^3)$. The inequality*

$$|x|^{1/2} |v(x)| \leq C \|v\|_{\dot{H}^1(\mathbb{R}^3)} \quad (2.3)$$

holds.

Proof. This inequality is well-known. For the sake completeness we prove this inequality. We may assume $v \in \mathcal{S}(\mathbb{R}^3)$. Writing $v(x) = \phi(r)$, we see

$$\begin{aligned} r v^2(x) &= r \phi^2(r) = \int_0^r (\rho \phi^2(\rho))' d\rho \leq \int_0^r \phi^2(\rho) d\rho + 2 \int_0^r \rho \phi(\rho) \phi'(\rho) d\rho \\ &\leq C \left(\left\| \frac{1}{|x|} v \right\|_{L^2(\mathbb{R}^3)}^2 + \left\| \frac{1}{|x|} v \right\|_{L^2(\mathbb{R}^3)} \|\nabla v\|_{L^2(\mathbb{R}^3)} \right) \leq C \|\nabla v\|_{L^2(\mathbb{R}^3)}^2. \end{aligned}$$

At the last inequality we have used the Hardy inequality. \square

Lemma 2.3. *Let $v \in H_{\text{rad}}^1(\mathbb{R}^3)$. The inequality*

$$|x|v(x) \leq C\|v\|_{H^1(\mathbb{R}^3)} \quad (2.4)$$

holds.

Proof. This inequality is also well-known. We prove it for the sake of completeness. Writing $v(x) = \phi(r)$, we have by the Sobolev embedding on $(0, \infty)$

$$\begin{aligned} r|\phi(r)| &\leq C\|r\phi\|_{H^1(0,\infty)} \\ &\leq C\left(\|v\|_{L^2(\mathbb{R}^3)} + \left\|\frac{1}{|x|}v\right\|_{L^2(\mathbb{R}^3)} + \|\nabla v\|_{L^2(\mathbb{R}^3)}\right) \leq C\|v\|_{H^1(\mathbb{R}^3)}. \end{aligned}$$

At the last inequality we have used the Hardy inequality. \square

Lemma 2.4. *Assume $v \in \dot{H}_2^1(\mathbb{R}^3)$ with $\partial_x^\alpha v \in L^2(\mathbb{R}^3)$ for any $|\alpha| = 2$. The inequality*

$$|v(x)| \leq C\left(\sum_{1 \leq |\gamma| \leq 2} \|\partial_x^\gamma v\|_{L^2(\mathbb{R}^3)}\right) \quad (2.5)$$

holds.

Proof. We may assume $v \in \mathcal{S}(\mathbb{R}^3)$. Using the Fourier transform, we observe

$$\begin{aligned} |v(x)| &\leq C\left(\int_{|\xi|<1} |\hat{f}(\xi)|d\xi + \int_{|\xi|>1} |\hat{f}(\xi)|d\xi\right) \\ &\leq C\left(\|\|\xi\|\hat{f}\|_{L^2(\{\xi \in \mathbb{R}^3: |\xi|<1\})} + \|\|\xi\|^2\hat{f}\|_{L^2(\{\xi \in \mathbb{R}^3: |\xi|>1\})}\right) \\ &\leq C\left(\sum_{1 \leq |\gamma| \leq 2} \|\partial_x^\gamma f\|_{L^2(\mathbb{R}^3)}\right), \end{aligned}$$

which yields the result. \square

Lemma 2.5. *Let $n \geq 3$, $\delta > 0$ and $T > 0$. Suppose that u solves the Cauchy problem $\square u = F$ with data $(u(0), \partial_t u(0)) = (f, g) \in \dot{H}_2^1 \times L^2$ and $F \in L^1((0, T); L^2(\mathbb{R}^n))$.*

The inequality

$$\begin{aligned} &\|\langle x \rangle^{-\delta} |x|^{-(3/2)+\delta} u\|_{L^2((0, T) \times \mathbb{R}^n)} \\ &+ \sum_{a=0}^n \|\langle x \rangle^{-\delta} |x|^{-(1/2)+\delta} \partial_a u\|_{L^2((0, T) \times \mathbb{R}^n)} \\ &\leq C\sqrt{\log(2+T)}\left(\|\nabla f\|_{L^2} + \|g\|_{L^2} + \int_0^T \|F(\tau)\|_{L^2(\mathbb{R}^n)} d\tau\right) \end{aligned} \quad (2.6)$$

holds, where C is a constant independent of T . If $\delta' > 0$, then the inequality

$$\begin{aligned} & \|\langle x \rangle^{-(\delta+\delta')} |x|^{-(3/2)+\delta} u\|_{L^2((0,T)\times\mathbb{R}^n)} \\ & + \sum_{a=0}^n \|\langle x \rangle^{-(\delta+\delta')} |x|^{-(1/2)+\delta} \partial_a u\|_{L^2((0,T)\times\mathbb{R}^n)} \\ & \leq C \left(\|\nabla f\|_{L^2} + \|g\|_{L^2} + \int_0^T \|F(\tau)\|_{L^2(\mathbb{R}^n)} d\tau \right) \end{aligned} \quad (2.7)$$

also holds.

Proof. With the weights $\langle x \rangle^{-\delta} |x|^{-(3/2)+\delta}$ and $\langle x \rangle^{-\delta} |x|^{-(1/2)+\delta}$ replaced by $\langle x \rangle^{-3/2}$ and $\langle x \rangle^{-1/2}$ respectively, the estimate (2.6) was proved by Keel, Smith and Sogge [7] for $n = 3$. Our sharper version, which is essential in the proof of Theorems 1.1–1.2, follows from suitable modifications of the argument in Metcalfe [14]. For the complete proof of (2.6), see the Appendix B of [3]. The proof of (2.7) is quite similar. \square

Remark. (1) Without the first norm

$$\|\langle x \rangle^{-\delta} |x|^{-(3/2)+\delta} u\|_{L^2((0,T)\times\mathbb{R}^n)} \quad (\text{resp. } \|\langle x \rangle^{-(\delta+\delta')} |x|^{-(3/2)+\delta} u\|_{L^2((0,T)\times\mathbb{R}^n)})$$

on the left-hand side, the estimate (2.6) (resp. (2.7)) remains true for $n = 1, 2$. See Appendix B of [3].

(2) Compared with the Hardy inequality

$$\||x|^{-1} v\|_{L^2(\mathbb{R}^n)} \leq C \|\nabla v\|_{L^2(\mathbb{R}^n)} \quad (n \geq 3),$$

the estimates (2.6)–(2.7) manifest that the integration in time yields the gain of smoothness of solutions. This gain plays an essential role in our proof of Theorems 1.1–1.2.

The standard dyadic decomposition will also be an ingredient in our estimates of the nonlinear terms.

Lemma 2.6. *There exists a spherically symmetric function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that*

$$\text{supp } \varphi \subset \{x \in \mathbb{R}^n : \frac{1}{2} \leq |x| \leq 2\}, \quad \varphi(x) > 0 \text{ for } \frac{1}{2} < |x| < 2, \quad (2.8)$$

$$\sum_{j=-\infty}^{\infty} \varphi(x/2^j) = 1, \quad x \in \mathbb{R}^n \setminus \{0\}. \quad (2.9)$$

Moreover, for every $j \in \mathbb{Z}$

$$\varphi(x/2^{j-1}) + \varphi(x/2^j) + \varphi(x/2^{j+1}) \equiv 1 \text{ for all } x \in \text{supp } \varphi(\cdot/2^j). \quad (2.10)$$

Proof. The existence and properties (2.8)–(2.10) of such a function φ is well-known in connection with the dyadic decomposition in the theory of Littlewood-Paley. \square

Remark. This dyadic decomposition will often appear in Sections 3 and 4, where we shall denote $\varphi(x/2^j)$ by $\varphi_j(x)$. The auxiliary function $\varphi(x/2^{j-1}) + \varphi(x/2^j) + \varphi(x/2^{j+1})$ will also be denoted by $\varphi_j^*(x)$.

3. PROOF OF THEOREM 1.1

The proof is based on the contraction mapping principle. We define for $T > 0$

$$\begin{aligned} X_T = & \{ u \in C([0, T]; \dot{H}_{\text{rad}}^1(\mathbb{R}^3)) : \\ & \partial_x^\alpha \partial_t^j u \in C([0, T]; L^2(\mathbb{R}^3)), \ 1 \leq |\alpha| + j \leq 2, \ 0 \leq |\alpha| \leq 2, \ 0 \leq j \leq 2, \\ & M_T(u) < \infty \}, \end{aligned} \quad (3.1)$$

where

$$\begin{aligned} M_T(u) = & \sum_{\substack{0 \leq |\alpha| \leq 2, 0 \leq j \leq 2 \\ 1 \leq |\alpha| + j \leq 2}} \left(\|\partial_x^\alpha \partial_t^j u\|_{L^\infty(0, T; L^2(\mathbb{R}^3))} \right. \\ & \left. + (\log(2 + T))^{-1/2} \| |x|^{-1/4} \langle x \rangle^{-1/4} \partial_x^\alpha \partial_t^j u \|_{L^2((0, T) \times \mathbb{R}^3)} \right) \\ & + \sum_{|\alpha| + j = 1} \| |x|^{-5/4} \partial_x^\alpha \partial_t^j u \|_{L^2((0, T) \times B_4)} \quad (B_4 = \{x \in \mathbb{R}^3 : |x| < 4\}). \end{aligned} \quad (3.2)$$

For $R > 0$ we define

$$X_{R, T} = \{ u \in X_T : M_T(u) \leq R \}. \quad (3.3)$$

$X_{R, T}$ is complete with respect to the metric defined as $\rho(u, v) = M_T(u - v)$.

For any fixed $(f, g) \in H_{\text{rad}}^2 \times H_{\text{rad}}^1$ we define the mapping $\Phi : u \mapsto v$ for $u \in X_{R, T}$ by solving the Cauchy problem $\square v = Q(\partial u)$ with data $(\varepsilon f, \varepsilon g)$ at $t = 0$. Φ is written as

$$\Phi[u](t) = \varepsilon u_0(t) + \int_0^t \frac{\sin \omega(t - \tau)}{\omega} Q(\partial u(\tau)) d\tau. \quad (3.4)$$

Here $u_0(t) = (\cos \omega t)f + (\omega^{-1} \sin \omega t)g$ ($\omega = \sqrt{-\Delta}$). Obviously, $u_0(t)$ is spherically symmetric in x . We start with the estimate of u_0 .

Proposition 3.1. *The estimate*

$$M_T(\varepsilon u_0) \leq C_0 \varepsilon \left(\sum_{1 \leq |\alpha| \leq 2} \|\partial_x^\alpha f\|_{L^2} + \|g\|_{H^1} \right) \quad (3.5)$$

holds for a constant $C_0 > 0$ independent of $T > 0$.

Remark. As is obvious from the proof below, the result of this proposition is valid even for non-radial data.

Proof. Since $\square \partial^\alpha u_0 = 0$, it follows from the standard energy inequality and (2.6) with $\delta = 1/4$ that

$$\begin{aligned} & \|\partial \partial^\alpha u_0\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} + (\log(2+T))^{-1/2} \| |x|^{-1/4} \langle x \rangle^{-1/4} \partial \partial^\alpha u_0 \|_{L^2((0,T) \times \mathbb{R}^3)} \\ & + \| |x|^{-5/4} \partial^\alpha u_0 \|_{L^2((0,T) \times B_4)} \leq C \|\partial \partial^\alpha u_0(0)\|_{L^2(\mathbb{R}^3)} \end{aligned}$$

for $|\alpha| \leq 1$. Since $\partial_x \partial_x^\beta u_0(0) = \partial_x \partial_x^\beta f$, $\partial_t \partial_x^\beta u_0(0) = \partial_x^\beta g$ and $\partial_t^2 u_0(0) = \Delta f$, we get

$$\begin{aligned} M_T(\varepsilon u_0) &= \varepsilon M_T(u_0) \\ &\leq C_0 \varepsilon \sum_{|\alpha| \leq 1} \|\partial \partial^\alpha u_0(0)\|_{L^2(\mathbb{R}^3)} \\ &\leq C_0 \varepsilon \left(\sum_{1 \leq |\alpha| \leq 2} \|\partial^\alpha f\|_{L^2(\mathbb{R}^3)} + \|g\|_{H^1(\mathbb{R}^3)} \right). \end{aligned}$$

□

Denoting

$$I[F](t) = \int_0^t \frac{\sin \omega(t-\tau)}{\omega} F(\tau) d\tau, \quad (3.6)$$

we prove the following proposition.

Proposition 3.2. *Assume $\partial^\alpha F \in C([0,T]; L^2(\mathbb{R}^3))$ for $|\alpha| \leq 1$. Then we have*

$$M_T(I[F]) \leq C \left(\|F(0)\|_{L^2(\mathbb{R}^3)} + \sum_{|\alpha| \leq 1} \int_0^T \|\partial^\alpha F(\tau)\|_{L^2(\mathbb{R}^3)} d\tau \right). \quad (3.7)$$

Proof. Let $|\alpha| \leq 1$. By the energy inequality and (2.6) with $\delta = 1/4$,

$$\begin{aligned} & \|\partial \partial^\alpha I[F]\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \\ & + (\log(2+T))^{-1/2} \| |x|^{-1/4} \langle x \rangle^{-1/4} \partial \partial^\alpha I[F] \|_{L^2((0,T) \times \mathbb{R}^3)} \\ & + \| |x|^{-5/4} \partial^\alpha I[F] \|_{L^2((0,T) \times B_4)} \\ & \leq C \left(\|\partial \partial^\alpha I[F](0)\|_{L^2(\mathbb{R}^3)} + \int_0^T \|\square(\partial^\alpha I[F])(\tau)\|_{L^2(\mathbb{R}^3)} d\tau \right). \end{aligned}$$

Noting $I[F](0) = \partial_t I[F](0) = 0$ and $\partial_t^2 I[F](0) = \square I[F](0) = F(0)$, we get

$$\begin{aligned} M_T(I[F]) &\leq C \sum_{|\alpha| \leq 1} \left(\|\partial \partial^\alpha I[F](0)\|_{L^2(\mathbb{R}^3)} + \int_0^T \|\partial^\alpha \square I[F](\tau)\|_{L^2(\mathbb{R}^3)} d\tau \right) \\ &\leq C \left(\|\partial_t^2 I[F](0)\|_{L^2(\mathbb{R}^3)} + \sum_{|\alpha| \leq 1} \int_0^T \|\partial^\alpha F(\tau)\|_{L^2(\mathbb{R}^3)} d\tau \right) \\ &\leq C \left(\|F(0)\|_{L^2(\mathbb{R}^3)} + \sum_{|\alpha| \leq 1} \int_0^T \|\partial^\alpha F(\tau)\|_{L^2(\mathbb{R}^3)} d\tau \right), \end{aligned}$$

which completes the proof of the proposition. \square

Here and in what follows, we use the notation

$$B_a = \{x \in \mathbb{R}^3 : |x| < a\}, \quad \Omega_a = \{x \in \mathbb{R}^3 : |x| > a\}$$

for $a > 0$.

Lemma 3.3. *Let $u \in X_T$. Then the inequality*

$$\begin{aligned} &\|\partial_a u(t) \partial_b \partial^\alpha u(t)\|_{L^2(\mathbb{R}^3)} \tag{3.8} \\ &\leq C \left(\sum_{|\gamma| \leq 1} \| |x|^{-1/4} \langle x \rangle^{-1/4} \partial \partial_x^\gamma u(t) \|_{L^2(\mathbb{R}^3)} + \| |x|^{-5/4} \partial u(t) \|_{L^2(B_4)} \right) \\ &\quad \times \| |x|^{-1/4} \langle x \rangle^{-1/4} \partial_b \partial^\alpha u(t) \|_{L^2(\mathbb{R}^3)} \end{aligned}$$

holds for $|\alpha| \leq 1$.

Proof. Over the set B_1 we have

$$\begin{aligned} &\|\partial_a u(t) \partial_b \partial^\alpha u(t)\|_{L^2(B_1)}^2 \leq \sum_{j \leq 0} \|(\varphi_j^* \partial_a u(t)) (\varphi_j^{1/2} \partial_b \partial^\alpha u(t))\|_{L^2(\mathbb{R}^3)}^2 \\ &\leq C \sum_{j \leq 0} 2^{-j} \| |x|^{1/2} \varphi_j^* \partial_{t,r} u(t) \|_{L^\infty(\mathbb{R}^3)}^2 \|\varphi_j^{1/2} \partial_b \partial^\alpha u(t)\|_{L^2(\mathbb{R}^3)}^2 \\ &\leq C \sum_{j \leq 0} 2^{-j/2} \|\varphi_j^* \partial_{t,r} u(t)\|_{H^1(\mathbb{R}^3)}^2 \| |x|^{-1/4} \varphi_j^{1/2} \partial_b \partial^\alpha u(t) \|_{L^2(\mathbb{R}^3)}^2 \end{aligned}$$

where Lemma 2.1 has been used to show $|\partial_a u| \leq |\partial_r u|$ for $1 \leq a \leq 3$. We have also employed Lemma 2.2. Moreover,

$$\begin{aligned}
& 2^{-j/4} \|\varphi_j^* \partial_{t,r} u(t)\|_{\dot{H}^1(\mathbb{R}^3)} \\
& \leq C 2^{-j/4} \left(\sum_{|j'-j| \leq 1} \|2^{-j'} (\nabla \varphi)(\cdot/2^{j'}) \partial_{t,r} u(t)\|_{L^2(\mathbb{R}^3)} + \|\varphi_j^* \nabla \partial_{t,r} u(t)\|_{L^2(\mathbb{R}^3)} \right) \\
& \leq C \left(\| |x|^{-5/4} \partial_{t,r} u(t) \|_{L^2(B_4)} + \| |x|^{-1/4} \nabla \partial_{t,r} u(t) \|_{L^2(B_4)} \right) \\
& \leq C \left(\| |x|^{-5/4} \partial u(t) \|_{L^2(B_4)} + \| |x|^{-1/4} \nabla \partial u(t) \|_{L^2(B_4)} \right),
\end{aligned}$$

where we have used Lemma 2.1. Therefore, it follows that

$$\begin{aligned}
& \|\partial_a u(t) \partial_b \partial^\alpha u(t)\|_{L^2(B_1)} \\
& \leq C \left(\| |x|^{-5/4} \partial u(t) \|_{L^2(B_4)} + \| |x|^{-1/4} \nabla \partial u(t) \|_{L^2(B_4)} \right) \| |x|^{-1/4} \partial_b \partial^\alpha u(t) \|_{L^2(B_2)}.
\end{aligned}$$

On the other hand, we obtain over the set Ω_1

$$\begin{aligned}
& \|\partial_a u(t) \partial_b \partial^\alpha u(t)\|_{L^2(\Omega_1)}^2 \\
& \leq \sum_{j \geq 0} \|(\varphi_j^* \partial_a u(t)) (\varphi_j^{1/2} \partial_b \partial^\alpha u(t))\|_{L^2(\mathbb{R}^3)}^2 \\
& \leq C \sum_{j \geq 0} 2^{-2j} \| |x| \varphi_j^* \partial_{t,r} u(t) \|_{L^\infty(\mathbb{R}^3)}^2 \| \varphi_j^{1/2} \partial_b \partial^\alpha u(t) \|_{L^2(\mathbb{R}^3)}^2 \\
& \leq C \sum_{j \geq 0} 2^{-j} \| \varphi_j^* \partial_{t,r} u(t) \|_{H^1(\mathbb{R}^3)}^2 \| |x|^{-1/2} \varphi_j^{1/2} \partial_b \partial^\alpha u(t) \|_{L^2(\mathbb{R}^3)}^2,
\end{aligned}$$

where Lemma 2.3 has been used. Moreover,

$$\begin{aligned}
& 2^{-j/2} \|\varphi_j^* \partial_{t,r} u(t)\|_{H^1(\mathbb{R}^3)} \\
& \leq C 2^{-j/2} \left(\|\varphi_j^* \partial_{t,r} u(t)\|_{L^2(\mathbb{R}^3)} + \sum_{|j'-j| \leq 1} \|(\nabla \varphi)(\cdot/2^{j'}) \partial_{t,r} u(t)\|_{L^2(\mathbb{R}^3)} \right. \\
& \quad \left. + \|\varphi_j^* \nabla \partial_{t,r} u(t)\|_{L^2(\mathbb{R}^3)} \right) \\
& \leq C \left(\| |x|^{-1/2} \partial u(t) \|_{L^2(\Omega_{1/4})} + \| |x|^{-1/2} \nabla \partial u(t) \|_{L^2(\Omega_{1/4})} \right).
\end{aligned}$$

Hence we get

$$\begin{aligned}
& \|\partial_a u(t) \partial_b \partial^\alpha u(t)\|_{L^2(\Omega_1)} \\
& \leq C \left(\sum_{|\gamma| \leq 1} \| |x|^{-1/2} \partial \partial_x^\gamma u(t) \|_{L^2(\Omega_{1/4})} \right) \| |x|^{-1/2} \partial_b \partial^\alpha u(t) \|_{L^2(\Omega_{1/2})}.
\end{aligned}$$

This completes the proof of the lemma. \square

Proposition 3.4. *Let $u \in X_T$. Then we have*

$$M_T(I[Q(\partial u)]) \leq C_1 M_T(u)^2 \log(2 + T). \quad (3.9)$$

Proof. By Proposition 3.2, we get

$$M_T(I[Q(\partial u)]) \leq C \left(\|Q(\partial u)(0)\|_{L^2(\mathbb{R}^3)} + \sum_{|\alpha| \leq 1} \int_0^T \|\partial^\alpha Q(\partial u)(\tau)\|_{L^2(\mathbb{R}^3)} d\tau \right).$$

$$\text{Since } Q(\partial u) = \sum_{a,b=0}^3 C_{ab} \partial_a u \partial_b u,$$

$$\begin{aligned} \|Q(\partial u)(0)\|_{L^2(\mathbb{R}^3)} &\leq C \| |x| \partial u(0) \|_{L^\infty(\mathbb{R}^3)} \| |x|^{-1} \partial u(0) \|_{L^2(\mathbb{R}^3)} \\ &\leq C \| \partial u(0) \|_{H^1(\mathbb{R}^3)} \| \nabla \partial u(0) \|_{L^2(\mathbb{R}^3)} \leq C M_T(u)^2, \end{aligned}$$

where we have used Lemma 2.4 and the Hardy inequality. Moreover, Lemma 3.3 yields

$$\begin{aligned} \sum_{|\alpha| \leq 1} \int_0^T \|\partial^\alpha Q(\partial u)(\tau)\|_{L^2(\mathbb{R}^3)} d\tau &\leq C \sum_{a,b=0}^3 \sum_{|\alpha| \leq 1} \int_0^T \|\partial_a u(\tau) \partial_b \partial^\alpha u(\tau)\|_{L^2(\mathbb{R}^3)} d\tau \\ &\leq C \int_0^T \left(\sum_{|\alpha| \leq 1} \| |x|^{-1/4} \langle x \rangle^{-1/4} \partial \partial^\alpha u(\tau) \|_{L^2(\mathbb{R}^3)} + \| |x|^{-5/4} \partial u(\tau) \|_{L^2(B_4)} \right) \\ &\quad \times \sum_{|\alpha| \leq 1} \| |x|^{-1/4} \langle x \rangle^{-1/4} \partial \partial^\alpha u(\tau) \|_{L^2(\mathbb{R}^3)} d\tau, \end{aligned}$$

so we obtain

$$\sum_{|\alpha| \leq 1} \int_0^T \|\partial^\alpha Q(\partial u)(\tau)\|_{L^2(\mathbb{R}^3)} d\tau \leq C M_T(u)^2 \log(2 + T)$$

by the Schwarz inequality in time. \square

Repeating the argument in the proof of Proposition 3.4, we have the following proposition.

Proposition 3.5. *Let $u, v \in X_T$. Then the inequality*

$$\begin{aligned} \rho(\Phi[u], \Phi[v]) &= M_T(I[Q(\partial u) - Q(\partial v)]) \\ &\leq C_2 \log(2 + T) (M_T(u) + M_T(v)) \rho(u, v) \end{aligned} \quad (3.10)$$

holds.

We are ready to complete the proof of Theorem 1.1. We define

$$\Lambda := \sum_{1 \leq |\gamma| \leq 2} \|\partial_x^\gamma f\|_{L^2(\mathbb{R}^3)} + \|g\|_{H^1(\mathbb{R}^3)}, \quad (3.11)$$

$$R_\varepsilon := 2C_0\Lambda\varepsilon, \quad (3.12)$$

where C_0 is the constant in Proposition 3.1. Then it follows from Propositions 3.1 and 3.4 that if $u \in X_{R_\varepsilon, T}$,

$$\begin{aligned} M_T(\Phi[u]) &\leq M_T(\varepsilon u_0) + M_T(I[Q(\partial u)]) \\ &\leq C_0\Lambda\varepsilon + C_1R_\varepsilon^2 \log(2+T) \\ &\leq (1 + 4C_0C_1\Lambda\varepsilon \log(2+T))C_0\Lambda\varepsilon. \end{aligned} \quad (3.13)$$

We also obtain by Proposition 3.5

$$\rho(\Phi[u], \Phi[v]) \leq 4C_0C_2\Lambda\varepsilon \log(2+T)\rho(u, v). \quad (3.14)$$

We define T_ε as

$$\log(2+T_\varepsilon) = \min\left\{\frac{1}{8C_0C_1\Lambda\varepsilon}, \frac{1}{8C_0C_2\Lambda\varepsilon}\right\}, \quad (3.15)$$

where $\varepsilon > 0$ should be taken sufficiently small so that the right-hand side is greater than $\log 2$. Then we find by (3.13) that Φ maps $X_{R_\varepsilon, T_\varepsilon}$ into itself. Moreover, (3.14) implies that Φ is a contraction mapping of $X_{R_\varepsilon, T_\varepsilon}$. The unique fixed point is the solution we seek. The proof has been completed.

4. PROOF OF THEOREM 1.2

For the positive constant δ to be chosen later, we define

$$\begin{aligned} Y &= \{u \in C([0, \infty); \dot{H}_{\text{rad}}^1(\mathbb{R}^3)) : \\ &\quad \partial_x^\alpha \partial_t^j u \in C([0, \infty); L^2(\mathbb{R}^3)), \quad 1 \leq |\alpha| + j \leq 2, \quad 0 \leq |\alpha| \leq 2, \quad 0 \leq j \leq 2, \\ &\quad N_T(u) < \infty\}, \end{aligned} \quad (4.1)$$

where

$$\begin{aligned}
N(u) = & \sum_{\substack{0 \leq |\alpha| \leq 2, 0 \leq j \leq 2 \\ 1 \leq |\alpha| + j \leq 2}} \left(\|\partial_x^\alpha \partial_t^j u\|_{L^\infty(0, \infty; L^2(\mathbb{R}^3))} \right. \\
& \left. + \||x|^{-1/4} \langle x \rangle^{-\delta} \partial_x^\alpha \partial_t^j u\|_{L^2((0, \infty) \times \mathbb{R}^3)} \right) \\
& + \sum_{|\alpha| + j = 1} \||x|^{-5/4} \partial_x^\alpha \partial_t^j u\|_{L^2((0, \infty) \times B_4)} \quad (B_4 := \{x \in \mathbb{R}^3 : |x| < 4\}).
\end{aligned} \tag{4.2}$$

For $R > 0$ let us define

$$Y_R = \{u \in Y : N(u) \leq R\}.$$

Y_R is complete with the metric defined as $d(u, v) = N(u - v)$.

As in the previous section, for any fixed $(f, g) \in H_{\text{rad}}^2(\mathbb{R}^3) \times H_{\text{rad}}^1(\mathbb{R}^3)$ we define the mapping $\Psi : u \mapsto v$ for $u \in Y_R$ by solving the Cauchy problem $\square v = G(u, \partial u)$ with data (f, g) at $t = 0$. Denoting $u_0(t) = (\cos \omega t)f + (\omega^{-1} \sin \omega t)g$, we start with the estimate of u_0 .

Proposition 4.1. *If $\delta > 1/4$, then there exists a constant $C_3 > 0$ depending on δ such that the estimate*

$$N(u_0) \leq C_3 \left(\sum_{1 \leq |\gamma| \leq 2} \|\partial_x^\gamma f\|_{L^2(\mathbb{R}^3)} + \|g\|_{H^1(\mathbb{R}^3)} \right) \tag{4.3}$$

holds.

Proof. Following the argument in the proof of Proposition 3.1, we obtain (4.3) by (2.7). We may omit the details. \square

Recall that $I[F]$ was defined in (3.6) by

$$I[F](t) = \int_0^t \frac{\sin \omega(t - \tau)}{\omega} F(\tau) d\tau.$$

We get the following proposition by the same argument as in Proposition 3.2.

Proposition 4.2. *Assume $\partial^\alpha F \in C([0, \infty); L^2(\mathbb{R}^3))$ for $|\alpha| \leq 1$. Then we have*

$$N(I[F]) \leq C \left(\|F(0)\|_{L^2(\mathbb{R}^3)} + \sum_{|\alpha| \leq 1} \int_0^\infty \|\partial^\alpha F(\tau)\|_{L^2(\mathbb{R}^3)} d\tau \right). \tag{4.4}$$

We next prove some estimates for cubic nonlinear terms by suitably modifying the proof of Lemma 3.3.

Lemma 4.3. *Let $u \in Y$. Then the inequality*

$$\begin{aligned} & \|\partial_a u(t) \partial^\alpha u(t) \partial_b \partial^\beta u(t)\|_{L^2(\mathbb{R}^3)} \\ & \leq C \|\partial u(t)\|_{H^1(\mathbb{R}^3)} \left(\sum_{|\gamma| \leq 1} \| |x|^{-1/4} \langle x \rangle^{-1/2} \partial \partial_x^\gamma u(t) \|_{L^2(\mathbb{R}^3)} \right. \\ & \quad \left. + \| |x|^{-5/4} \partial u(t) \|_{L^2(B_4)} \right)^2 \end{aligned} \quad (4.5)$$

holds for $|\alpha| + |\beta| = 1$.

Proof. As in the previous section, we divide the whole space \mathbb{R}^3 into the two pieces B_1 and Ω_1 . Over the set B_1 we have

$$\begin{aligned} & \|\partial_a u(t) \partial^\alpha u(t) \partial_b \partial^\beta u(t)\|_{L^2(B_1)}^2 \\ & \leq \|\partial_a u(t) u(t) \partial_b \partial u(t)\|_{L^2(B_1)}^2 + \|\partial_a u(t) \partial u(t) \partial_b u(t)\|_{L^2(B_1)}^2 \\ & \leq C \sum_{j \leq 0} 2^{-j} \| |x|^{1/2} (\varphi_j^* \partial_{t,r} u(t)) \|_{L^\infty(\mathbb{R}^3)}^2 (\|u(t)\|_{L^\infty(\mathbb{R}^3)}^2 \|\varphi_j^{1/2} \partial_b \partial u(t)\|_{L^2(\mathbb{R}^3)}^2 \\ & \quad + 2^{-j} \| |x|^{1/2} \partial_{t,r} u(t) \|_{L^\infty(\mathbb{R}^3)}^2 \|\varphi_j^{1/2} \partial_b u(t)\|_{L^2(\mathbb{R}^3)}^2) \\ & \leq C \sum_{j \leq 0} 2^{-j} \|\varphi_j^* \partial_{t,r} u(t)\|_{\dot{H}^1(\mathbb{R}^3)}^2 (\|\nabla u(t)\|_{H^1(\mathbb{R}^3)}^2 \|\varphi_j^{1/2} \partial_b \partial u(t)\|_{L^2(\mathbb{R}^3)}^2 \\ & \quad + 2^{-j} \|\partial_{t,r} u(t)\|_{\dot{H}^1(\mathbb{R}^3)}^2 \|\varphi_j^{1/2} \partial_b u(t)\|_{L^2(\mathbb{R}^3)}^2) \\ & \leq C \sum_{j \leq 0} (2^{-j})^{1/2} \|\varphi_j^* \partial_{t,r} u(t)\|_{\dot{H}^1(\mathbb{R}^3)}^2 (\|\nabla u(t)\|_{H^1(\mathbb{R}^3)}^2 \|\varphi_j^{1/2} |x|^{-1/4} \partial_b \partial u(t)\|_{L^2(\mathbb{R}^3)}^2 \\ & \quad + \|\partial u(t)\|_{H^1(\mathbb{R}^3)}^2 \|\varphi_j^{1/2} |x|^{-3/4} \partial_b u(t)\|_{L^2(\mathbb{R}^3)}^2) \end{aligned}$$

where we have employed Lemmas 2.2 and 2.4. As we observed in the proof of Lemma 3.3,

$$\begin{aligned} & (2^{-j})^{1/4} \|\varphi_j^* \partial_{t,r} u(t)\|_{\dot{H}^1(\mathbb{R}^3)} \\ & \leq C \| |x|^{-5/4} \partial u(t) \|_{L^2(B_4)} + C \sum_{|\gamma| \leq 1} \| |x|^{-1/4} \partial_x^\gamma \partial u(t) \|_{L^2(B_4)}. \end{aligned}$$

Thus we get

$$\begin{aligned} & \|\partial_a u(t) \partial^\alpha u(t) \partial_b \partial^\beta u(t)\|_{L^2(B_1)} \\ & \leq C \|\partial u(t)\|_{H^1(\mathbb{R}^3)} \left(\sum_{|\gamma| \leq 1} \| |x|^{-1/4} \partial^\gamma \partial u(t) \|_{L^2(B_4)} + \| |x|^{-5/4} \partial u(t) \|_{L^2(B_4)} \right)^2. \end{aligned}$$

On the other hand, we obtain over the set Ω_1

$$\begin{aligned}
& \|\partial_a u(t) \partial^\alpha u(t) \partial_b \partial^\beta u(t)\|_{L^2(\Omega_1)}^2 \\
& \leq \|\partial_a u(t) u(t) \partial_b \partial u(t)\|_{L^2(\Omega_1)}^2 + \|\partial_a u(t) \partial u(t) \partial_b u(t)\|_{L^2(\Omega_1)}^2 \\
& \leq C \sum_{j \geq 0} 2^{-3j} \| |x| (\varphi_j^* \partial_{t,r} u(t)) \|_{L^\infty(\mathbb{R}^3)}^2 \left(\| |x|^{1/2} \varphi_j^* u(t) \|_{L^\infty(\mathbb{R}^3)}^2 \| \varphi_j^{1/2} \partial_b \partial u(t) \|_{L^2(\mathbb{R}^3)}^2 \right. \\
& \quad \left. + \| |x|^{1/2} \varphi_j^* \partial_{t,r} u(t) \|_{L^\infty(\mathbb{R}^3)}^2 \| \varphi_j^{1/2} \partial_b u(t) \|_{L^2(\mathbb{R}^3)}^2 \right) \\
& \leq C \sum_{j \geq 0} 2^{-3j} \| \varphi_j^* \partial_{t,r} u(t) \|_{H^1(\mathbb{R}^3)}^2 \left(\| u(t) \|_{H^1(\mathbb{R}^3)}^2 \| \varphi_j^{1/2} \partial_b \partial u(t) \|_{L^2(\mathbb{R}^3)}^2 \right. \\
& \quad \left. + \| \partial_{t,r} u(t) \|_{H^1(\mathbb{R}^3)}^2 \| \varphi_j^{1/2} \partial_b u(t) \|_{L^2(\mathbb{R}^3)}^2 \right) \\
& \leq C \sum_{j \geq 0} 2^{-3j/2} \| \varphi_j^* \partial_{t,r} u(t) \|_{H^1(\mathbb{R}^3)}^2 \| \partial u(t) \|_{H^1(\mathbb{R}^3)}^2 \\
& \quad \times \sum_{|\gamma| \leq 1} \| \varphi_j^{1/2} |x|^{-3/4} \partial_b \partial^\gamma u(t) \|_{L^2(\mathbb{R}^3)}^2
\end{aligned}$$

where Lemmas 2.2 and 2.3 have been used. Since

$$\begin{aligned}
& 2^{-3j/4} \| \varphi_j^* \partial_{t,r} u(t) \|_{H^1(\mathbb{R}^3)} \\
& \leq C 2^{-3j/4} \left(\sum_{|j'-j| \leq 1} \| 2^{-j'} (\nabla \varphi)(\cdot/2^{j'}) \partial_{t,r} u(t) \|_{L^2(\mathbb{R}^3)} \right. \\
& \quad \left. + \sum_{|\gamma| \leq 1} \| \varphi_j^* \partial_x^\gamma \partial_{t,r} u(t) \|_{L^2(\mathbb{R}^3)} \right) \\
& \leq C \sum_{|\gamma| \leq 1} \| |x|^{-3/4} \partial_x^\gamma \partial_{t,r} u(t) \|_{L^2(\Omega_{1/4})} \leq C \sum_{|\gamma| \leq 1} \| |x|^{-3/4} \partial_x^\gamma \partial u(t) \|_{L^2(\Omega_{1/4})}
\end{aligned}$$

for $j \geq 0$, we obtain

$$\begin{aligned}
& \|\partial_a u(t) \partial^\alpha u(t) \partial_b \partial^\beta u(t)\|_{L^2(\Omega_1)} \\
& \leq C \| \partial u(t) \|_{H^1(\mathbb{R}^3)} \left(\sum_{|\gamma| \leq 1} \| |x|^{-3/4} \partial \partial^\gamma u(t) \|_{L^2(\Omega_{1/4})} \right)^2.
\end{aligned}$$

This completes the proof of the lemma. \square

From now on we choose $\delta = 1/2$.

Proposition 4.4. *Let $R \leq 1$. For any $u \in Y_R$ the inequality*

$$N(I[G(u, \partial u)]) \leq C_4 N(u)^{l_*+2} \quad (4.6)$$

holds, where $l_* = \min\{l_{ab} : a, b = 0, \dots, 3\} \geq 1$.

Proof. By Proposition 4.2, we get

$$N(I[G(u, \partial u)]) \leq C \left(\|G(u, \partial u)(0)\|_{L^2(\mathbb{R}^3)} + \sum_{|\alpha| \leq 1} \int_0^\infty \|\partial^\alpha G(u, \partial u)(\tau)\|_{L^2(\mathbb{R}^3)} d\tau \right).$$

$$\text{Since } G(u, \partial u) = \sum_{a,b=0}^3 C_{ab} u^{l_{ab}} \partial_a u \partial_b u,$$

$$\begin{aligned} & \|G(u, \partial u)(0)\|_{L^2(\mathbb{R}^3)} \\ & \leq C \sum_{a,b} \|u(0)\|_{L^\infty(\mathbb{R}^3)}^{l_{ab}} \| |x| \partial u(0) \|_{L^\infty(\mathbb{R}^3)} \| |x|^{-1} \partial u(0) \|_{L^2(\mathbb{R}^3)} \\ & \leq C \| \nabla u(0) \|_{H^1(\mathbb{R}^3)}^{l_*} \| \partial u(0) \|_{H^1(\mathbb{R}^3)} \| \nabla \partial u(0) \|_{L^2(\mathbb{R}^3)} \leq CN(u)^{l_*+2}, \end{aligned}$$

where we have used Lemmas 2.3–2.4 and the Hardy inequality. Moreover, Lemma 4.3 yields

$$\begin{aligned} & \sum_{|\alpha| \leq 1} \int_0^\infty \|\partial^\alpha G(u, \partial u)(\tau)\|_{L^2(\mathbb{R}^3)} d\tau \\ & \leq C \sum_{a,b,c=0}^3 \int_0^\infty \left(\|u(\tau)^{l_{ab}-1} \partial_a u(\tau) \partial_b u(\tau) \partial_c u(\tau)\|_{L^2(\mathbb{R}^3)} \right. \\ & \quad \left. + \sum_{|\alpha| \leq 1} \|u(\tau)^{l_{ab}} \partial_a u(\tau) \partial_b \partial^\alpha u(\tau)\|_{L^2(\mathbb{R}^3)} \right) d\tau \\ & \leq C \sum_{a,b,c=0}^3 \|u\|_{L^\infty((0,\infty) \times \mathbb{R}^3)}^{l_{ab}-1} \int_0^\infty \left(\|\partial_a u(\tau) \partial_b u(\tau) \partial_c u(\tau)\|_{L^2(\mathbb{R}^3)} \right. \\ & \quad \left. + \sum_{|\alpha| \leq 1} \|u(\tau) \partial_a u(\tau) \partial_b \partial^\alpha u(\tau)\|_{L^2(\mathbb{R}^3)} \right) d\tau \\ & \leq C \left(\sum_{|\gamma| \leq 1} \|\partial^\gamma \partial u\|_{L^\infty((0,\infty); L^2(\mathbb{R}^3))} \right)^{l_*-1} \int_0^\infty \|\partial u(\tau)\|_{H^1(\mathbb{R}^3)} \\ & \quad \times \left(\sum_{|\gamma| \leq 1} \| |x|^{-1/4} \langle x \rangle^{-1/2} \partial^\gamma \partial u(\tau) \|_{L^2(\mathbb{R}^3)} + \| |x|^{-5/4} \partial u(\tau) \|_{L^2(B_4)} \right)^2 d\tau, \end{aligned}$$

so we obtain

$$\sum_{|\alpha| \leq 1} \int_0^\infty \|\partial^\alpha G(u, \partial u)(\tau)\|_{L^2(\mathbb{R}^3)} d\tau \leq CN(u)^{l_*+2}.$$

This completes the proof. \square

Repeating the argument in the proof of Proposition 4.4, we find

$$\begin{aligned} N(I[G(u, \partial u) - G(v, \partial v)]) &\leq CN(u)^{l_*+1}N(u-v) + CN(u)^{l_*}N(v)N(u-v) \\ &\quad + \cdots + CN(v)^{l_*+1}N(u-v) \end{aligned}$$

for $u, v \in Y_R$ ($R \leq 1$). Hence we obtain the following proposition.

Proposition 4.5. *Let $R \leq 1$. For any $u, v \in Y_R$ the inequality*

$$d(\Psi[u], \Psi[v]) \leq C_5(N(u)^{l_*+1} + N(v)^{l_*+1})d(u, v) \quad (4.7)$$

holds.

We define

$$\Lambda := \sum_{1 \leq |\gamma| \leq 2} \|\partial_x^\gamma f\|_{L^2(\mathbb{R}^3)} + \|g\|_{H^1(\mathbb{R}^3)}, \quad (4.8)$$

$$R_\Lambda := 2C_3\Lambda, \quad (4.9)$$

where C_3 is the constant in Proposition 4.1 and $\Lambda \geq 0$ is small enough so that $R_\Lambda \leq 1$. Then it follows from Propositions 4.1 and 4.4 that if $u \in Y_{R_\Lambda}$,

$$\begin{aligned} N(\Phi[u]) &\leq N(u_0) + N(I[G(u, \partial u)]) \\ &\leq C_3\Lambda + C_4R_\Lambda^{l_*+2} \\ &\leq (1 + 4C_3C_4\Lambda)C_3\Lambda. \end{aligned} \quad (4.10)$$

We also obtain by Proposition 4.5

$$d(\Phi[u], \Phi[v]) \leq 4C_3C_5\Lambda d(u, v). \quad (4.11)$$

Now we assume

$$4C_3(C_4 + C_5)\Lambda \leq \frac{1}{2}. \quad (4.12)$$

Then we find by (4.10) that Ψ maps Y_{R_Λ} into itself. Moreover, (4.11) implies that Ψ is a contraction mapping of Y_{R_Λ} . The unique fixed point is the solution we seek. The proof has been completed.

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