Uniqueness on identification of cubic convolution nonlinearity

Hironobu Sasaki*and Michiyuki Watanabe†

Abstract

We shall consider the inverse scattering problem for time dependent version of Hartree-Fock equation and nonlinear Klein-Gordon equation. The uniqueness theorem on identifying the cubic convolution nonlinearity from the knowledge of the scattering operator will be shown.

1 Introduction

In this paper we shall consider the inverse scattering problem of determining the nonlinearity for Hartree-Fock equation

$$i\frac{\partial \mathbf{u}}{\partial t} = -\Delta \mathbf{u} + \int_{\mathbf{R}^n} V(x-y)U(x,y,t)\overline{\mathbf{u}(y,t)}dy \tag{1.1}$$

and Klein-Gordon equation with cubic convolution nonlinearity

$$\frac{\partial^2 w}{\partial t^2} = \Delta w + w + (V * w^2)w \tag{1.2}$$

for $(x.t) \in \mathbf{R}^n \times \mathbf{R}$, where $\mathbf{u} = {}^t(u_1, \cdots, u_N)$,

$$U(x,y,t) = (U_{jk}(x,y,t)) \qquad N \times N \text{ matrix}$$
$$U_{jk}(x,y,t) = u_j(x,t)u_k(y,t) - u_k(x,t)u_j(y,t)$$

*Department of Mathematics, Hokkaido University Kita 10, Nishi 8, Kita-ku, Sapporo, Hokkaido, 060-0810, Japan. (E-mail: hisasaki@math.sci.hokudai.ac.jp).

†Department of Mathematics, Hokkaido University Kita 10, Nishi 8, Kita-ku, Sapporo, Hokkaido, 060-0810, Japan. (E-mail: m-watanabe@math.sci.hokudai.ac.jp) Supported by 21st Century COE Program, “Mathematics of Non-Linear Structure via Singularity”.

and

$$(V * w^2)w = \left(\int_{\mathbf{R}^n} V(x-y)w(y)^2 dy\right) w.$$

The uniqueness theorem on identification of $V(x)$ from the scattering operator will be shown.

Hartree-Fock equation (1.1) is derived in order to obtain an approximate solution of an N-body Schrödinger equation

$$i\frac{\partial \mathbf{\Phi}}{\partial t} = -\sum_{j=1}^{N} \Delta_j \mathbf{\Phi} + \sum_{i<j} V(x_i - x_j)\mathbf{\Phi},$$

where $x_j = (x_j^1, \cdots, x_j^n) \in \mathbf{R}^n$, and $\Delta_j = \sum_{i=1}^n (\partial/\partial x_j^i)^2$. $V(x)$ represents the interaction potential acting on between particles. In more details, we refer to Isozaki [1, Introduction]. We would like to determine the interaction potential $V(x)$ from the knowledge of the scattering operator.

Hartree-Fock equation (1.1) is a kind of nonlinear Schrödinger equation. So let us here review the inverse scattering problem for nonlinear equation in short. The power type nonlinearity case was initially studied by Morawetz-Strauss [3] and Strauss [4, pp 64-70]. For example, consider the nonlinear Schrödinger equation

$$i\frac{\partial u}{\partial t} = H_0 u + V(x)|u|^{p-1}u, \qquad (t,x) \in \mathbf{R} \times \mathbf{R}^n,$$

where $H_0 = -\Delta$. Suppose that p is an integer, $p \geq 3$ if $n \geq 3$, $p > 3$ if $n = 2$, $p > 4$ if $n = 1$ and $V(x)$ is real valued continuous, bounded, whose derivatives up to order $l > 3n/4$ are bounded. Then the scattering operator S

$$(S\phi)(x) = \phi(x) + \frac{1}{i}\int_{\mathbf{R}} \left(e^{itH_0} V|u|^{p-1}u\right)(t,x)\, dt$$

is well defined. It was shown that $V(x)$ is recovered from the scattering operator S as follows. For any $\phi \in H^1 \cap L^{1+1/p}$,

$$V(x_0) = \frac{\lim_{\lambda\to\infty} \lambda^{n+2} I[\phi_\lambda]}{\int_{\mathbf{R}} \|e^{-itH_0}\phi\|_{L^{p+1}}^{p+1}\, dt}, \tag{1.3}$$

where $\phi_\lambda(x) = \phi(\lambda(x - x_0))$, x, $x_0 \in \mathbf{R}^n$ and

$$I[\phi] := \lim_{\varepsilon \to 0} \frac{i}{\varepsilon^p}((S - I)(\varepsilon\phi), \phi). \tag{1.4}$$

Later by Weder [7] above results was extended to more general cases.

In the above mentioned result the method to obtain the reconstruction formula (1.3) is not applicable to our cubic convolution case. The essential point to prove the formula (1.3) is the change of variables in the following integral

$$I[\phi_\lambda] = \int_{\mathbf{R}} \int_{\mathbf{R}^n} V(x)|e^{-itH_0}\phi_\lambda|^{p+1} dx dt.$$

Changing variable x by $\lambda(x - x_0)$, we have

$$I[\phi_\lambda] = \lambda^{-n-2} \int_{\mathbf{R}} \int_{\mathbf{R}^n} V\left(x_0 + \frac{x}{\lambda}\right) |e^{-itH_0}\phi|^{p+1} dx dt.$$

Therefore, as $\lambda \to \infty$, we can take the value $V(x_0)$ from the inside integral.

Applying the same method to cubic convolution nonlinearity we obtain

$$\begin{aligned} I[\phi_\lambda] &= \int_{\mathbf{R}} \int_{\mathbf{R}^n} (V * |e^{-itH_0}\phi_\lambda|^2)|e^{-itH_0}\phi_\lambda|^2 dx dt \\ &= \lambda^{-2n-2} \int_{\mathbf{R}} \int_{\mathbf{R}^n} (V\left(\frac{\cdot}{\lambda}\right) * |e^{-itH_0}\phi|^2)|e^{-itH_0}\phi|^2 dx dt. \end{aligned}$$

Since the integral

$$\int_{\mathbf{R}} \int_{\mathbf{R}^n} (V(0) * |e^{-itH_0}\phi|^2)|e^{-itH_0}\phi|^2 dx dt$$

does not converge, we can not make λ tend to infinity. Thus, to identify the cubic convolution nonlinearity is not clear. Reconstruction formula was given only in the special case $V(x) = \beta|x|^{-\sigma}$, which was proved in [6].

Our purpose of this paper is to study the identification of cubic convolution nonlinearity which dose not directly apply the method in the case of the power type nonlinearity but which is physically important.

Notations : Let us introduce some notations used throughout this paper. Let $H^{s,q}(\mathbf{R}^n)$ be the usual Sobolev space of order s in $L^q(\mathbf{R}^n)$. Especially we use the abbreviation $H^s = H^{s,2}(\mathbf{R}^n)$. We shall denote by $L^p(\mathbf{R}; Z)$ the set of Z-valued $L^p(\mathbf{R})$ functions. A set of bounded continuous functions whose derivatives up to the mth order are bounded continuous will be denoted by $\mathcal{B}^m(\mathbf{R}^n)$. The function space S is indefinitely differentiable on $\mathbf{R}^n$ and all of whose derivatives remain bounded when multiplied by polynominals. For a Banach space $\mathcal{H}$ we shall denote $\mathcal{H} \times \mathcal{H}$ by $[\mathcal{H}]^2$.

Fourier transform of ϕ is denoted by $\hat{\phi}$ or $\mathcal{F}\phi$. For a vector valued function $\mathbf{f} = (f_1, \cdots, f_N)$ we put

$$\|\mathbf{f}\|_X = \|f_1\|_X + \cdots + \|f_N\|_X.$$

The plan of this paper is as follows. In Section 2 we shall define the scattering operator for the evolution equation. The inverse problem for the Hartree-Fock equation will be considered in Section 3, which is the main part of this paper. Nonlinear Klein-Gordon case will be treated in Section 4. In Section 5 we will give a remark for Hartree type equation in the scalar case.

2 Scattering

Before consider the inverse problem we have to solve the direct problem. In this section we shall mention the scattering problem for the evolution equation

$$i\frac{\partial \boldsymbol{v}}{\partial t}(t) = J\boldsymbol{v}(t) + \mathbf{F}(\boldsymbol{v}(t)), \tag{2.1}$$

where J is a selfadjoint operator in Hilbert space X with dense domain, $\boldsymbol{v}$ is a X-valued unknown function over $\mathbf{R}$ and $\mathbf{F}$ is a map form a suitable subspace of X into X.

In order to define the scattering operator, we refer the integral equation

$$\boldsymbol{v}(t) = e^{-itJ}\boldsymbol{\phi}_- + \frac{1}{i}\int_{-\infty}^{t} e^{-i(t-s)J}\mathbf{F}(\boldsymbol{v}(s))ds. \tag{2.2}$$

and put

$$X_\rho := \{\boldsymbol{\phi} \in X; \|\boldsymbol{\phi}\|_X \le \rho\}.$$

Definition 2.1. Assume that there exist some $\rho > 0$ and some $W \subset L^\infty(\mathbf{R}; X)$ such that for all $\boldsymbol{\phi}_- \in X_\rho$, there exists a unique solution $\boldsymbol{v} \in W$ to (2.2) satisfying

$$\|\boldsymbol{v}(t) - e^{-itJ}(\boldsymbol{\phi}_-)\|_X \longrightarrow 0, \qquad \text{as } t \to -\infty.$$

Then the operator $\mathcal{S}$ satisfying following properties is called the scattering operator for (2.2)

$$\begin{aligned} &(1)\ \mathcal{S} : X_\rho \ni \boldsymbol{\phi} \mapsto \boldsymbol{\phi} + \frac{1}{i}\int_{\mathbf{R}} e^{itJ}\mathbf{F}(\boldsymbol{v}(t))dt \in X, \\ &(2)\ \|\boldsymbol{v}(t) - e^{-itJ}\mathcal{S}(\boldsymbol{\phi}_-)\|_X \longrightarrow 0, \qquad \text{as } t \to \infty. \end{aligned} \tag{2.3}$$

3 Hartree-Fock equation

3.1 Scattering operator

We shall see that the scattering operator for Hartree-Fock equation (1.1) is well defined. Global existence of a solution for Hartree-Fock equation was studied in [1]. Scattering problem was considered in Wada [5]. In this paper we will define the scattering operator based on results which was proved by Mochizuki [2].

We denote that

$$\mathbf{F}(\mathbf{u}(t)) = \int_{\mathbf{R}^n} V(x-y)U(x,y,t)\overline{\mathbf{u}(y,t)}dy \tag{3.1}$$

and $J = H_0 := -\Delta$. Assume that $V(x)$ is real valued function on $\mathbf{R}^n$ and satisfies

$$|V(x)| \le C|x|^{-\sigma}, \qquad \text{for } 2 \le \sigma \le 4 \text{ and } \sigma < n. \tag{3.2}$$

Let $X = H^1$ and $W := L^3(\mathbf{R} : H^{1,q}) \cap L^\infty(\mathbf{R} : H^1)$, where $1/q = 1/2 - 2/(3n)$.

Setting $G(f, g, h) = [V * (fg)](x)h(x)$, we have

$$\begin{aligned} \Big\| \int_{-\infty}^t e^{-i(t-s)H_0}\{G(u_1,u_2,u_3) - G(v_1,v_2,v_3)\}(\cdot, s)ds \Big\|_W \\ \le C(\|u_1 - v_1\|_W\|u_2\|_W\|u_3\|_W \\ + \|u_2 - v_2\|_W\|v_1\|_W\|u_3\|_W \\ + \|u_3 - v_3\|_W\|v_1\|_W\|v_2\|_W). \end{aligned} \tag{3.3}$$

Using this estimate (3.3) it follows that

$$\Big\| \int_{-\infty}^{t} e^{-i(t-s)H_0}\{F(u) - F(v)\}(s)\,ds \Big\|_W \le C\|u - v\|_W(\|u\|_W^2 + \|v\|_W^2). \qquad (3.4)$$

According to the argument in Mochizuki [2], with the help of the estimate (3.4) the scattering operator (2.3) in Definition 2.1 is well defined.

3.2 Inverse scattering problem

We shall now consider the inverse scattering problem of determining V from the scattering operator $\mathcal{S}$ defined by (2.3).

Assumption : We assume that V satisfies following conditions.

(i) $V \in \mathcal{B}^1(\mathbf{R}^n)$ such that

$$|V(x)| \le C|x|^{-\sigma}, \qquad \text{for } 2 \le \sigma \le 4 \text{ and } \sigma < n.$$

(ii) $V(x)$ is the real valued and radial symmetric function; $V(x) = V(|x|)$.

It is noted that $\hat{V}(\xi)$ is also real valued and radial symmetric function in view of assumption (ii). Our main result is the following

Theorem 3.1. *Let $\mathcal{S}_j$ be the scattering operator corresponding to $V_j(x)$, $j = 1, 2$. If $\mathcal{S}_1 = \mathcal{S}_2$ then we have*

$$V_1 = V_2 \qquad \text{in } \mathbf{R}^n.$$

3.3 Lemma

For simplicity, we will prove Theorem 3.1 in the case of $N = 2$. To denote the nonlinearity $F(u)$ more simple, we shall employ the following symbols. For a vector $a = {}^t(a_1, a_2)$, set $a^\perp = {}^t(a_2, -a_1)$ and

$$[{}^t(a^\perp) \otimes \bar{a}^\perp]a := \begin{bmatrix} a_2\bar{a}_2 & -a_1\bar{a}_2 \\ -a_2\bar{a}_1 & a_1\bar{a}_1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

Then we can write the nonlinearity in the form

$$F(u) = [V * \{{}^t(u^\perp) \otimes \bar{u}^\perp\}]u.$$

For two vectors $\mathbf{a} = {}^t(a_1, a_2)$ and $\mathbf{b} = {}^t(b_1, b_2)$ we shall write $\mathbf{ab} = {}^t(a_1 b_1, a_2 b_2)$. The following lemma is essentially due to Strauss [4, pp. 65-66]

Lemma 3.1. *Let $\varepsilon > 0$. Then for any $\boldsymbol{\phi} \in [H^1]^2$ we have*

$$\lim_{\varepsilon \to 0} \frac{i}{\varepsilon^3}([\mathcal{S} - I](\varepsilon \boldsymbol{\phi}), \boldsymbol{\phi}) = \int_{\mathbf{R}} \int_{\mathbf{R}^n} \mathsf{F}(e^{-itH_0} \boldsymbol{\phi}) \overline{e^{-itH_0} \boldsymbol{\phi}(x)} dx dt. \tag{3.5}$$

Proof. Let $\mathbf{u}_\varepsilon$ be a solution of (2.2) with initial data $\varepsilon \boldsymbol{\phi}$. Then we have

$$\begin{aligned} \mathsf{F}(\mathbf{u}_\varepsilon) = [V * \{{}^t(\mathbf{u}_\varepsilon^\perp) \otimes \bar{\mathbf{u}}_\varepsilon^\perp\}] \boldsymbol{w}_\varepsilon \\ + [V * \{{}^t(\mathbf{u}_\varepsilon^\perp) \otimes \bar{\boldsymbol{w}}_\varepsilon^\perp\}] \mathbf{u}_0 \\ + [V * \{{}^t(\boldsymbol{w}_\varepsilon^\perp) \otimes \bar{\mathbf{u}}_0^\perp\}] \mathbf{u}_0 + \mathsf{F}(\mathbf{u}_0), \end{aligned}$$

where $\boldsymbol{w}_\varepsilon = \mathbf{u}_\varepsilon - \mathbf{u}_0$ and $\mathbf{u}_0 = e^{-itH_0}(\varepsilon \boldsymbol{\phi})$. Making use of the estimate (3.4) we obtain

$$\begin{aligned} \|\boldsymbol{w}_\varepsilon\|_W &\le C \|\mathbf{u}_\varepsilon\|_W^3, \\ \|\mathbf{u}_\varepsilon\|_W &\le C\varepsilon \|\boldsymbol{\phi}\|_{H^1}. \end{aligned}$$

According to the proof of Strauss [4, pp. 65-66] the formula (3.5) follows. □

3.4 Proof of Theorem 3.1

By means of the Plancherel's theorem the formula (3.5) is rewritten in the form

$$\mathbf{T}[\boldsymbol{\phi}] = \int_{\mathbf{R}} \int_{\mathbf{R}^n} \begin{bmatrix} \hat{V}(\xi) \{\mathcal{F}(|u_0^{(2)}|^2) \overline{\mathcal{F}(|u_0^{(1)}|^2)} - |\mathcal{F}(u_0^{(1)} \bar{u}_0^{(2)})|^2\} \\ \hat{V}(\xi) \{\mathcal{F}(|u_0^{(1)}|^2) \overline{\mathcal{F}(|u_0^{(2)}|^2)} - |\mathcal{F}(u_0^{(2)} \bar{u}_0^{(1)})|^2\} \end{bmatrix} d\xi dt, \tag{3.6}$$

where $\mathbf{T}[\boldsymbol{\phi}] = \lim_{\varepsilon \to 0} \frac{i}{\varepsilon^3}([\mathcal{S} - I](\varepsilon \boldsymbol{\phi}), \boldsymbol{\phi})$ and $u_0^{(j)} = e^{-itH_0} \phi_j$, $j = 1, 2$. Since $\mathcal{S}_1 = \mathcal{S}_2$ we have $\mathbf{T}[\boldsymbol{\phi}] = \mathbf{0}$. Putting $w(\xi) = \hat{V}_1(\xi) - \hat{V}_2(\xi)$ then it follows by adding the first component and the second component in the relation (3.6) that for any $\boldsymbol{\phi} \in [H^1]^2$

$$0 = \int_{\mathbf{R}} \int_{\mathbf{R}^n} w(\xi) H(t, \xi) d\xi dt, \tag{3.7}$$

where

$$H(t,\xi) = \mathrm{Re}\{\mathcal{F}(|u_0^{(2)}|^2)\overline{\mathcal{F}(|u_0^{(1)}|^2)}\} - \{|\mathcal{F}(u_0^{(1)}\bar{u}_0^{(2)})|^2 + |\mathcal{F}(u_0^{(2)}\bar{u}_0^{(1)})|^2\}.$$

First we shall prove $w(0) = 0$. Suppose that $w(0) > 0$. Then by virtue of the continuity of $w(\xi)$ there exists $\varepsilon > 0$ such that

$$w(\xi) > 0 \qquad \text{in } B_\varepsilon,$$

where $B_\varepsilon = \{\xi \in \mathbf{R}^n ; |\xi| < \varepsilon\}$.

On the other hand, since $H(t, 0) > 0$ for any $0 \not\equiv \boldsymbol{\phi} \in [H^1]^2$ and $t \in \mathbf{R}$, with the help of continuity of $H(t, \xi)$ we have

$$H(t,\xi) > 0 \qquad \text{in } B_{\varepsilon'}$$

for some $\varepsilon' > 0$.

Let $\mathcal{D}_\varepsilon := \{\boldsymbol{\phi} \in [S]^2 ; \phi_1 \not\equiv \phi_2, \quad \mathrm{supp}\,\hat{\phi}_j \subset B_\varepsilon, j = 1, 2\}$. Obviously we have $\mathcal{D}_\varepsilon \subset H^1$. Moreover it is noticed that there exists $\boldsymbol{\psi} \in \mathcal{D}_\eta$ satisfying $\mathrm{supp}_\xi\, H(t,\xi) \subset B_\eta$, where $\eta = \min\{\varepsilon, \varepsilon'\}$. Consequently, for this $\boldsymbol{\psi} \in \mathcal{D}_\eta$ we have

$$\int_{\mathbf{R}}\int_{\mathbf{R}^n} w(\xi)H(t,\xi)d\xi dt > 0.$$

This contradicts with (3.7). Hence it follows that $w(0) = 0$.

Nest we shall prove that $w(\xi) = 0$ in a neighborhood of $\xi = 0$. Assume that $w(\xi) > 0$ in $B_\varepsilon \setminus \{0\}$. Then we see the contradiction from the previous argument. Because of the radial symmetry of $w(\xi)$ and $w(0) = 0$ the function $w(\xi)$ must be zero in B_η for some $\eta > 0$.

Changing variable ξ by $\xi - \eta$ in the integral (3.7) the same argument applicable. Hence we know that

$$w(\xi) = 0, \qquad \text{in } 0 < |\xi| \le \eta'$$

for some $\eta' > \eta$. Repeating this argument it follows that $w(\xi) = 0$ in $\mathbf{R}^n$. Thus the proof has been completed.

4 Nonlinear Klein-Gordon

Let

$$J = i\begin{pmatrix} 0 & 1 \\ \Delta - 1 & 0 \end{pmatrix} \qquad \text{and} \qquad f(u) = (V * u^2)u.$$

Then we can rewrite the equation (1.2) into (2.1), where $\boldsymbol{v} = {}^t(u, v) = {}^t(u, \partial_t u)$ and $\mathbf{F}(\boldsymbol{v}) = {}^t(0, if(u))$.

Assume that

$$|V(x)| \leq C|x|^{-\sigma}, \qquad 2 \leq \sigma \leq 4 \text{ and } \sigma < n. \tag{4.1}$$

In Definition 2.1, taking $X = H^1 \times L^2$ and $W := L^3(\mathbf{R} : H^{1,q}) \cap L^\infty(\mathbf{R} : H^1)$, where $1/q = 1/2 - 2/3(n-1+\theta)$ with some $0 \leq \theta \leq 1$ which depends on σ. it follows from results of Mochizuki [2] that the scattering operator (2.3) is well defined.

4.1 Inverse scattering problem

Inverse scattering problem for the nonlinear Klein-Gordon equation was initially studied by Morawetz-Strauss [3]. For the following equation

$$u_{tt} - \Delta u + m^2 u + g u^3 = 0,$$

it was proved that the scattering operator S uniquely determines the coupling constant g. Later in [4] it was proved that a function $g = g(x)$ is uniquely determined by the scattering operator. As we mentioned in Section 1, the method for power type nonlinearity is not applicable to our problem of identifying $V(x)$ in the equation (1.2).

In this section we will show that the uniqueness on identifying $V(x)$ also holds for the nonlinear Klein-Gordon equation.

Let $\mathcal{S}_j$, $j = 1, 2$ be the scattering operator corresponding to V_j.

Assumption : We assume that V_j, $j = 1, 2$ satisfy (4.1) and following conditions.

(i) V is a real valued radial symmetric function.

(ii) $\hat{V}(\xi)$ is a bounded continuous function.

We here note that if $V(x)$ is radial symmetric then $\hat{V}(\xi)$ is also radial symmetric.

The result is the following.

Theorem 4.1. *If $\mathcal{S}_1 = \mathcal{S}_2$ then we have*

$$V_1 = V_2 \qquad \textit{in } \mathbf{R}^n.$$

The policy of proof is almost same as the Hartree-Fock equation case. But we pay attention to how to choose the test function.

4.2 Proof of Theorem 4.1

According to the proof of Strauss [4, pp. 65-66] and using estimates proved by Mochizuki [2] we have

$$K[\boldsymbol{\phi}, \boldsymbol{\psi}] = -i\int_{\mathbf{R}} (\mathbf{F}(e^{-itJ}\boldsymbol{\phi}), e^{-itJ}\boldsymbol{\psi})_X dt \tag{4.2}$$

for any $\boldsymbol{\phi}$, $\boldsymbol{\psi} \in X = H^1 \times L^2$, where

$$K[\boldsymbol{\phi}, \boldsymbol{\psi}] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^3}([\mathcal{S} - I](\varepsilon\boldsymbol{\phi}), \boldsymbol{\psi}).$$

Noting that $B\boldsymbol{\phi} \in X$, for any $\boldsymbol{\phi} \in X$, where

$$B = \begin{bmatrix} 0 & -\omega^{-2} \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \omega = \sqrt{1-\Delta},$$

it follows from the formula (4.2) and Plancherel's theorem that

$$K[\boldsymbol{\phi}, B\boldsymbol{\phi}] = \int_{\mathbf{R}}\int_{\mathbf{R}^n} \hat{V}(\xi)|\hat{u}_0(t,\xi)|^2 d\xi dt \tag{4.3}$$

for any $\boldsymbol{\phi} \in [H^1]^2$, where $u_0(t,\xi) = \cos t\omega\phi_1 + \omega^{-1}\sin t\omega\phi_2$.

Let η be a positive number satisfying $\eta < |\xi_0|/2$ for $0 \not\equiv \xi_0 \in \mathbf{R}^n$. It is denoted by $B_R(a) = \{x \in \mathbf{R}^n; |x-a| \le R\}$. We shall define the real valued function $\phi^\eta_{\xi_0} \in S$ as follows.

(i) $\hat{\phi}^\eta_{\xi_0}(\xi)$ is real valued function such that $\hat{\phi}^\eta_{\xi_0}(\xi/2) \neq 0$.

(ii) $\operatorname{supp}\hat{\phi}^\eta_{\xi_0} \subset B_{\eta/2}(\xi_0/2) \cup B_{\eta/2}(-\xi_0/2)$.

(iii) $\phi^\eta_{\xi_0}(x) = \phi^\eta_{\xi_0}(-x)$.

(iv) $\hat{\phi}^\eta_{\xi_0}(\xi_0/2 - \xi) = \hat{\phi}^\eta_{\xi_0}(\xi_0/2 + \xi)$ if $\xi \in B_{\eta/2}(0)$.

Putting $u^\eta_{\xi_0}(t,x) = \cos t\omega\phi^\eta_{\xi_0}(x)$ then it is found that

$$\operatorname{supp}_\xi(\hat{\phi}^\eta_{\xi_0}(\xi))^2 \subset B_\eta(\xi_0) \cup B_\eta(0) \cup B_\eta(-\xi_0).$$

Let us now prove Theorem 4.1. There are three steps. We denote that $V(x) = V_1(x) - V_2(x)$.

Step 1 : First we shall prove $\hat{V}(0) = 0$. Assume that $\hat{V}(0) > 0$, then by means of continuity of $\hat{V}(\xi)$ we see that there exists $\eta > 0$ satisfying

$$\hat{V}(\xi) > 0, \qquad \text{in } |\xi| < \eta. \tag{4.4}$$

Put $\widetilde{\boldsymbol{\phi}} = {}^t(\phi_0^\eta\ ,\ 0)$. Obviously we have $\widetilde{\boldsymbol{\phi}} \in [H^1]^2$. Hence applying the formula (4.3) for this $\widetilde{\boldsymbol{\phi}}$ it follows by assumption $\mathcal{S}_1 = \mathcal{S}_2$ that

$$0 = \int_{|\xi|<\eta} \int_{\mathbf{R}} \hat{V}(\xi)\{\mathcal{F}(v_0^2)(t,\xi)\}^2 dt d\xi, \tag{4.5}$$

where $v_0(t,x) = \cos t\omega\phi_0^\eta$. Noting that $|\mathcal{F}(v_0^2)(0,0)|^2 = \|\phi_0^\eta\|_{L^2}^4 > 0$ and (4.4) we see that the integral in the right hand side of (4.5) is positive. This is contradiction. Thus, we have $\hat{V}(0) = 0$.

Step 2 : By virtue of the symmetry of $\hat{V}(\xi)$ and $\hat{V}(0) = 0$, we can show in the same way as the Hartree-Fock case that

$$\hat{V}(\xi) = 0 \qquad \text{in } B_\eta(0). \tag{4.6}$$

for some $\eta > 0$.

Step 3 : We shall complete the proof of Theorem 4.1. Assume that $\hat{V}(\xi) > 0$ in $|\xi_0 - \xi| < \eta$ for some $\xi_0 \in \mathbf{R}^n$. Put $u_{\xi_0}^\eta(t,x) = \cos t\omega\phi_{\xi_0}^\eta$. Then we have in view of definition of $\phi_{\xi_0}^\eta$

$$\begin{aligned} \mathcal{F}((u_{\xi_0}^\eta)^2)(0,\xi_0) &= \int_{|\xi_0/2-\xi|<\eta/2} \hat{\phi}_{\xi_0}^\eta(\xi_0 - \xi)\hat{\phi}_{\xi_0}^\eta(\xi) d\xi \\ &= \int_{|\xi|<\eta/2} (\hat{\phi}_{\xi_0}^\eta(\xi_0/2 - \xi))^2 d\xi > 0. \end{aligned} \tag{4.7}$$

Making use of the fact (4.6) and symmetry of $\hat{\phi}_{\xi_0}^\eta$ it follows that

$$\begin{aligned} &\int_{\mathbf{R}} \int_{\mathbf{R}^n} \hat{V}(\xi)\{\mathcal{F}((u_{\xi_0}^\eta)^2)(t,\xi)\}^2 d\xi dt \\ &\qquad = \left(\int_{B_\eta(\xi_0)} + \int_{B_\eta(0)} + \int_{B_\eta(-\xi_0)} \right) \int_{\mathbf{R}} \hat{V}(\xi)\{\mathcal{F}((u_{\xi_0}^\eta)^2)(t,\xi)\}^2 dt d\xi \\ &\qquad = 2\int_{B_\eta(\xi_0)} \int_{\mathbf{R}} \hat{V}(\xi)\{\mathcal{F}((u_{\xi_0}^\eta)^2)(t,\xi)\}^2 dt d\xi. \end{aligned} \tag{4.8}$$

Taking $\boldsymbol{\phi} = {}^t(\phi_{\xi_0}^\eta\ ,\ 0)$ in the identity (4.3) we see by assumption $\mathcal{S}_1 = \mathcal{S}_2$ that the right hand side of (4.8) is equal to zero. Hence taking into account the inequality (4.7) it must be that $\hat{V}(\xi_0^\eta) = 0$ at some point $\xi_0^\eta \in B_\eta(\xi_0)$. Since $\xi_0 \in \mathbf{R}^n$ is a arbitrary point we see that

$$\hat{V}(\xi) = 0 \qquad \text{in } \mathbf{R}^n.$$

Thus, Theorem 4.1 has been completely proved.

5 Remark

We shall give a remark for Hartree type equation

$$i\frac{\partial u}{\partial t} = H_0 u + (V * |u|^2)u, \tag{5.1}$$

where $H_0 = -\Delta$. In this scalar case, it will be shown that there is a formula (see (5.2)) to determine $\hat{V}(0)$, in other words a value of integral of $V(x)$, from the scattering operator. Formulae similar to (5.2) could not be obtained for Hartree-Fock equation (1.1) and Klein-Gordon equation (1.2).

Assume that $V(x)$ satisfy the condition (3.2). Then the scattering operator

$$(S\phi)(x) = \phi(x) + \frac{1}{i}\int_{-\infty}^{\infty} \left(e^{itH_0}(V * |u|^2)u\right)(t, x)\, dt$$

is well defined on a neighborhood of zero in H^1.

If we set $\phi_R(x) = \phi(x/R)$ and

$$T[\phi] = \lim_{\varepsilon \to 0} \frac{i}{\varepsilon^3}([S - I](\varepsilon\phi), \phi), \qquad \varepsilon > 0$$

then we have

Proposition 5.1. *Let $2 \le n \le 6$. Then for any $\phi \in H^1(\mathbf{R}^n)$ we have*

$$\hat{V}(0) = M \lim_{R \to \infty} \frac{T[\phi_R]}{R^{n+2}}, \tag{5.2}$$

where

$$M = \|e^{-itH_0}\phi\|^{-4}_{L^4(\mathbf{R}:L^4)} < \infty.$$

Proof. First we shall prove that if $2 \le n \le 6$ then the following estimate holds.

$$\|e^{-itH_0}\phi\|_{L^4(\mathbf{R},L^4)} \le C\|\phi\|_{H^1}, \qquad \text{for any } \phi \in H^1. \tag{5.3}$$

By means of Hölder's inequality we have

$$\int_{\mathbf{R}^n} |(e^{-itH_0}\phi)(x)|^4 dx \le \|e^{-itH_0}\phi\|_{L^{3p}}^3 \|e^{-itH_0}\phi\|_{L^{p'}}.$$

Making use of the Sobolev imbedding theorem we obtain

$$\begin{aligned} \|e^{-itH_0}\phi\|_{L^{p'}} &\le C\|e^{-itH_0}\phi\|_{H^1} = C\|\phi\|_{H^1}, \\ \|e^{-itH_0}\phi\|_{L^{3p}} &\le C\|e^{-itH_0}\phi\|_{H^{1,q}} \end{aligned}$$

if p and p' satisfy

$$\frac{1}{2} - \frac{1}{n} \le \frac{1}{p'} \le \frac{1}{2}, \tag{5.4}$$

$$\frac{1}{q} - \frac{1}{n} \le \frac{1}{3p} \le \frac{1}{q}. \tag{5.5}$$

Since

$$\frac{1}{p} + \frac{1}{p'} = 1 \quad \text{and} \quad \frac{1}{q} = \frac{1}{2} - \frac{2}{3n},$$

inequalities (5.4) and (5.5) can be rewritten in the form

$$\frac{1}{2} \le \frac{1}{p} \le \frac{1}{2} + \frac{1}{n}, \tag{5.6}$$

$$\frac{3}{2} - \frac{5}{n} \le \frac{1}{p} \le \frac{3}{2} - \frac{2}{n}. \tag{5.7}$$

Under the condition $2 \le n \le 6$ we can take p such that inequalities (5.6) and (5.7). Remembering that the following estimate holds for any $\phi \in H^1$ (see Mochizuki [2, p. 148, Proposition 3.3]);

$$\|e^{-itH_0}\phi\|_{L^3(\mathbf{R}:H^{1,q})} \le C\,\|\phi\|_{H^1},$$

where $1/q = 1/2 - 2/(3n)$, we obtain the estimate

$$\begin{aligned} \|e^{-itH_0}\phi\|_{L^4(\mathbf{R}:L^4)}^4 &\le C\|\phi\|_{H^1}\|e^{-itH_0}\phi\|_{L^3(\mathbf{R}:H^{1,q})}^3 \\ &\le C\|\phi\|_{H^1}^4. \end{aligned}$$

Let us return now to show the formula (5.2). Remember that free solutions of the Schrödinger equation are represented by the formula

$$(e^{-itH_0}f)(x) = \frac{1}{(4\pi it)^{n/2}} \int_{\mathbf{R}^n} e^{-\frac{|x-y|^2}{4it}} f(y)dy.$$

Then we have

$$(e^{-itH_0}\phi_R)(x) = \left(e^{-i\frac{t}{R^2}H_0}\phi\right)\left(\frac{x}{R}\right)$$

and

$$\mathcal{F}(|e^{-itH_0}\phi_R|^2)(\xi) = R^n \mathcal{F}(|e^{-i\frac{t}{R^2}H_0}\phi|^2)(R\xi). \tag{5.8}$$

Making use of the formula (5.8) it follows by virtue of the Parseval identity that

$$\begin{aligned} T[\phi_R] &= \int_{\mathbf{R}} \left(\hat{V}\mathcal{F}(|e^{-itH_0}\phi_R|^2), \mathcal{F}(|e^{-itH_0}\phi_R|^2)\right) dt \\ &= R^{2n} \int_{\mathbf{R}} \int_{\mathbf{R}^n} \hat{V}(\xi)|\mathcal{F}(|e^{-i\frac{t}{R^2}H_0}\phi|^2)(R\xi)|^2 d\xi dt. \end{aligned}$$

Consequently, we have

$$\frac{T[\phi_R]}{R^{n+2}} = \int_{\mathbf{R}} \int_{\mathbf{R}^n} \hat{V}\left(\frac{\eta}{R}\right) \left|\mathcal{F}(|e^{-isH_0}\phi|^2)(\eta)\right|^2 d\eta ds.$$

Since $\hat{V}(\xi)$ is bounded continuous function, we have the formula (5.2) by means of the estimate (5.3), Lebesgue's convergence theorem and Plancherel's theorem. □

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