Stability of Discrete Ground State

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Abstract

We present new criteria for a self-adjoint operator to have a ground state. As an application, we consider models of “quantum particles” coupled to a massive Bose field and prove the existence of a ground state of them, where the particle Hamiltonian does not necessarily have compact resolvent.

Key words: Ground state; discrete ground state; generalized spin-boson model; Fock space; Dereziński-Gérard model.

1 INTRODUCTION

Let $T$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$, and bounded from below. We say that $T$ has a discrete ground state if the bottom of the spectrum of $T$ is an isolated eigenvalue of $T$. In that case a non-zero vector
in \( \ker(T - E_0(T)) \) is called a ground state of \( T \). Let \( S \) be a symmetric operator on \( \mathcal{H} \). Suppose that \( T \) has a discrete ground state and \( S \) is \( T \)-bounded. By the regular perturbation theory \( [8, \text{XII}] \) it is already known that \( T + \lambda S \) has a discrete ground state for “sufficiently small” \( \lambda \in \mathbb{R} \). Our aim is to present new criteria for \( T + \lambda S \) to have a ground state.

In Section 2, we prove an existence theorem of a ground state which is useful to show the existence of a ground state of models of quantum particles coupled to a massive Bose field.

In Section 3, we consider the GSB model \([2]\) with a self-interaction term of a Bose field, which we call the GSB + \( \phi^2 \) model. We consider only the case where the Bose field is massive. The GSB model — an abstract system of quantum particles coupled to a Bose field — was proposed in \([2]\). In \([2]\) A. Arai and M. Hirokawa proved the existence and uniqueness of the GSB model in the case where the particle Hamiltonian \( A \) has compact resolvent. Shortly after that, they proved the existence of a ground state of the GSB model in the case where \( A \) does not have necessarily compact resolvent \([4,3]\). In this paper, using a theorem in Section 2, we prove the existence of a ground state of the GSB + \( \phi^2 \) model in the case where \( A \) does not necessarily have compact resolvent.

In Section 4, we consider an extended version of the Nelson type model, which we call the Dereziński-Gérard model \([5]\). The Dereziński-Gérard model introduced in \([5]\) and J. Dereziński and C. Gérard prove an existence of a ground state for their model under some conditions including that \( A \) has compact resolvent. In Section 4, we prove the existence of a ground state of the Dereziński-Gérard model in the case where \( A \) does not have compact resolvent. Our strategy to establish a ground state is the same as in Section 3.

## 2 BASIC RESULTS

Let \( \mathcal{H} \) be a separable complex Hilbert space. We denote by \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \) the scalar product on Hilbert space \( \mathcal{H} \) and by \( \| \cdot \|_{\mathcal{H}} \) the associated norm. Scalar product \( \langle f, g \rangle_{\mathcal{H}} \) is linear in \( g \) and antilinear in \( f \). We omit \( \mathcal{H} \) in \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \) and \( \| \cdot \|_{\mathcal{H}}, \) respectively if there is no danger of confusion. For a linear operator \( T \) in Hilbert space, we denote by \( D(T) \) and \( \sigma(T) \) the domain and the spectrum of \( T \) respectively. If \( T \) is self-adjoint and bounded from below, then we define

\[
E_0(T) := \inf \sigma(T), \quad \Sigma(T) := \inf \sigma_{\text{ess}}(T),
\]
where $\sigma_{\text{ess}}(T)$ is the essential spectrum of $T$. If $T$ has no essential spectrum, then we set $\Sigma(T) = \infty$. For a self-adjoint operator $T$, we denote the form domain of $T$ by $Q(T)$. In this paper, an eigenvector of a self-adjoint operator $T$ with eigenvalue $E_0(T)$ is called a ground state of $T$ (if it exists). We say that $T$ has a ground state if $\dim \ker(T - E_0(T)) > 0$.

The basic results are as follows:

**Theorem 2.1.** Let $H$ be a self-adjoint operator on $\mathcal{H}$, and bounded from below. Suppose that there exists a self-adjoint operator $V$ on $\mathcal{H}$ satisfying the following conditions (i)-(iii):

(i) $D(H) \subset D(V)$.
(ii) $V$ is bounded from below, and $\Sigma(V) > 0$.
(iii) $H - E_0(H) \geq V$ on $D(H)$.

Then $H$ has purely discrete spectrum in the interval $[E_0(H), E_0(H) + \Sigma(V))$. In particular, $H$ has a ground state.

**Proof.** For all $u_1, \ldots, u_{n−1} \in \mathcal{H}$, we have

$$\inf_{\Psi \in \text{L.h.}[u_1, \ldots, u_{n−1}]^\perp, \|\Psi\|=1} \langle \Psi, H\Psi \rangle - E_0(H) \geq \inf_{\Psi \in \text{L.h.}[u_1, \ldots, u_{n−1}]^\perp, \|\Psi\|=1} \langle \Psi, V\Psi \rangle,$$

where L.h.[· · ·] denotes the linear hull of the vectors in [· · ·]. Since $D(H) \subset D(V)$, we have that

$$\inf_{\Psi \in \text{L.h.}[u_1, \ldots, u_{n−1}]^\perp, \|\Psi\|=1} \langle \Psi, V\Psi \rangle \geq \inf_{\Psi \in \text{L.h.}[u_1, \ldots, u_{n−1}]^\perp, \|\Psi\|=1} \langle \Psi, V\Psi \rangle.$$

Hence, for all $n \in \mathbb{N}$

$$\mu_n(H) - E_0(H) \geq \mu_n(V).$$

where

$$\mu_n(H) := \sup_{u_1, \ldots, u_{n−1} \in \mathcal{H}} \inf_{\Psi \in \text{L.h.}[u_1, \ldots, u_{n−1}]^\perp, \|\Psi\|=1} \langle \Psi, H\Psi \rangle.$$

By the min-max principle (\textit{ibid.}, Theorem XIII.1) and $\lim_{n \to \infty} \mu_n(H) = \Sigma(H)$ and $\lim_{n \to \infty} \mu_n(V) = \Sigma(V)$. Therefore we obtain

$$\Sigma(H) - E_0(H) \geq \Sigma(V) > 0.$$ 

This means that $H$ has purely discrete spectrum in $[E_0(H), E_0(H) + \Sigma(V))$. 

$$\Box$$
Theorem 2.2. Let $H$ be a self-adjoint operator on $\mathcal{H}$, and bounded from below. Suppose that there exists a self-adjoint operator $V$ on $\mathcal{H}$ satisfying the following conditions (i)-(iii):

(i) $Q(H) \subset Q(V)$.
(ii) $V$ is bounded from below, and $\Sigma(V) > 0$.
(iii) $H - E_0(H) \geq V$ on $Q(H)$.

Then $H$ has purely discrete spectrum in the interval $[E_0(H), E_0(H) + \Sigma(V))$. In particular, $H$ has a ground state.

Proof. Similar to the proof of Theorem 2.1.

We apply Theorems 2.1 and 2.2 to a perturbation problem of a self-adjoint operator.

Theorem 2.3. Let $A$ be a self-adjoint operator on $\mathcal{H}$ with $E_0(A) = 0$, and let $B$ be a symmetric operator on $D(A)$. Suppose that $A + B$ is self-adjoint on $D(A)$ and that there exist constants $a \in [0, 1)$ and $b \geq 0$ such that

$$|\langle \psi, B\psi \rangle| \leq a\langle \psi, A\psi \rangle + b\|\phi\|^2, \quad \psi \in D(A).$$

Assume

$$\frac{b + E_0(A + B)}{1 - a} < \Sigma(A).$$

(1)

Then $A + B$ has purely discrete spectrum in $[E_0(A + B), (1 - a)\Sigma(A) - b)$. In particular, $A + B$ has a ground state.

Proof. By the assumption we have

$$A + B - E_0(A + B) \geq (1 - a)A - b - E_0(A + B)$$
on $D(A)$, and $(1 - a)\Sigma(A) - b - E_0(A + B) > 0$. Hence we can apply Theorem 2.1, to conclude that $A + B$ has purely discrete spectrum in $[E_0(A + B), (1 - a)\Sigma(A) - b)$. In particular, $A + B$ has a ground state.

Remark. It is easily to see that $-b \leq E_0(A + B) \leq b$. Therefore condition (1) is satisfied if

$$\frac{2b}{1 - a} < \Sigma(A).$$
Theorem 2.4. Let $\mathcal{H}, \mathcal{K}$ be complex separable Hilbert spaces. Let $A$ and $B$ be self-adjoint operators on $\mathcal{H}$ and $\mathcal{K}$ respectively. Suppose that $E_0(A) = E_0(B) = 0$. We set
\[ T_0 := A \otimes I + I \otimes B. \]
Let $Z$ be a symmetric sesquilinear form on $Q(T_0)$, and assume that there exist constants $a_1 \in [0, 1)$, $a_2 \in [0, 1)$ and $b \geq 0$ such that, for all $\Psi \in Q(T_0)$
\[ |Z(\Psi, \Psi)| \leq a_1 \langle \Psi, A \otimes I \Psi \rangle_{\text{form}} + a_2 \langle \Psi, I \otimes B \Psi \rangle_{\text{form}} + b \|\Psi\|^2, \]
where $\langle \Psi, A \otimes I \Psi \rangle_{\text{form}} = \|A^{1/2} \otimes I \Psi\|^2$. Therefore, by the KLMN theorem there exists a unique self-adjoint operator $T$ on $\mathcal{H} \otimes \mathcal{K}$ such that $Q(T) = Q(T_0)$ and $T = T_0 + Z$ in the sense of sesquilinear form on $Q(T_0)$. We set
\[ s := \min\{(1 - a_1)\Sigma(A), (1 - a_2)\Sigma(B)\}. \]
Assume
\[ s > b + E_0(T). \]
Then, $T$ has purely discrete spectrum in the interval $[E_0(T), s - b)$. In particular, $T$ has a ground state.

Proof. Similar to the proof of Theorem 2.3. □

Remark. It is easy to see that $-b \leq E_0(T) \leq b$. Therefore the condition (2) is satisfied if
\[ s > 2b. \]

Remark. Theorem 2.4 is essentially same as [4, Theorem B.1]. But our proof is very simple.

3 Ground States of a General Class of Quantum Field Hamiltonians

We consider a model which is an abstract unification of some quantum field models of particles interacting with a Bose field. It is the GSB model [2] with a self-interaction term of the field.

Let $\mathcal{H}$ be a separable complex Hilbert space and $\mathcal{F}_b$ be the Boson Fock space over $L^2(\mathbb{R}^d)$:
\[ \mathcal{F}_b := \bigoplus_{n=0}^{\infty} \bigotimes_{s=1}^{n} L^2(\mathbb{R}^d). \]
The Hilbert space of the quantum field model we consider is

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_b.$$ 

Let $\omega : \mathbb{R}^d \to [0, \infty)$ be Borel measurable such that $0 < \omega(k) < \infty$ for all most everywhere (a.e.) $k \in \mathbb{R}^d$. We denote the multiplication operator by the function $\omega$ acting in $L^2(\mathbb{R}^d)$ by the same symbol $\omega$. We set

$$H_b := d\Gamma_b(\omega)$$

the second quantization of $\omega$ (e.g. Section X.7). We denote by $a(f)$, $f \in L^2(\mathbb{R}^d)$, the smeared annihilation operators on $\mathcal{F}_b$. It is a densely defined closed linear operator on $\mathcal{F}_b$ (e.g. Section X.7). The adjoint $a(f)^*$, called the creation operator, and the annihilation operator $a(g)$, $g \in L^2(\mathbb{R}^d)$ obey the canonical commutation relations

$$[a(f), a(g)^*] = \langle f, g \rangle, \quad [a(f), a(g)] = 0, \quad [a(f)^*, a(g)^*] = 0$$

for all $f, g \in L^2(\mathbb{R}^d)$ on the dense subspace

$$\mathcal{F}_0 := \{\psi = (\psi^{(n)})_{n=0}^{\infty} \in \mathcal{F}_b | \text{there exists a number } n_0 \text{ such that } \psi^{(n)} = 0 \text{ for all } n \geq n_0\},$$

where $[X, Y] = XY - YX$. The symmetric operator

$$\phi(f) := \frac{1}{\sqrt{2}}[a(f)^* + a(f)],$$

called the Segal field operator, is essentially self-adjoint on $\mathcal{F}_0$ (e.g. Section X.7). We denote its closure by the same symbol. Let $A$ be a positive self-adjoint operator on $\mathcal{H}$ with $E_0(A) = 0$. Then, the unperturbed Hamiltonian of the model is defined by

$$H_0 := A \otimes I + I \otimes H_b$$

with domain $D(H_0) = D(A \otimes I) \cap D(I \otimes H_b)$. For $g_j, f_j \in L^2(\mathbb{R}^d)$ $j = 1, \ldots, J$, and $B_j(j = 1, \ldots, J)$ a symmetric operator on $\mathcal{H}$, we define a symmetric operator

$$H_1 := \sum_{j=1}^{J} B_j \otimes \phi(g_j),$$

$$H_2 := \sum_{j=1}^{J} I \otimes \phi(f_j)^2.$$
The Hamiltonian of the model we consider is of the form
\[ H(\lambda, \mu) := H_0 + \lambda H_1 + \mu H_2, \]
where \( \lambda \in \mathbb{R} \) and \( \mu \geq 0 \) are coupling parameters.

For \( H(\lambda, \mu) \) to be self-adjoint, we shall need the following conditions [H.1]-[H.3]:

[H.1] \( g_j \in D(\omega^{-1/2}), f_j \in D(\omega^{1/2}) \cap D(\omega^{-1/2}), j = 1, \ldots, J. \)

[H.2] \( D(A^{1/2}) \subset \cap_{j=1}^J D(B_j) \) and there exist constants \( a_j \geq 0, b_j \geq 0, j = 1, \ldots, J, \) such that,
\[ \|B_j u\| \leq a_j \|A^{1/2} u\| + b_j \|u\|, \quad u \in D(A^{1/2}). \]

[H.3] \( |\lambda| \sum_{j=1}^J a_j \|g_j/\sqrt{\omega}\| < 1. \)

**Proposition 3.1.** Assume [H.1], [H.2] and [H.3]. Then, \( H(\lambda, \mu) \) is self-adjoint with \( D(H(\lambda, \mu)) = D(H_0) \subset D(H_1) \cap D(H_2) \) and bounded from below. Moreover, \( H(\lambda, \mu) \) is essentially self-adjoint on every core of \( H_0. \)

**Remark.** This proposition has no restriction of the coupling parameter \( \mu \geq 0. \)

**To perform a finite volume approximation, we need an additional condition:**

[H.4] The function \( \omega(k) \) \( (k \in \mathbb{R}^d) \) is continuous with
\[ \lim_{|k| \to \infty} \omega(k) = \infty, \]
and there exist constants \( \gamma > 0, C > 0 \) such that
\[ |\omega(k) - \omega(k')| \leq C|k - k'|^\gamma [1 + \omega(k) + \omega(k')], \quad k, k' \in \mathbb{R}^d. \]

Let
\[ m := \inf_{k \in \mathbb{R}^d} \omega(k). \]

If \( A \) has compact resolvent, we can prove the extension of the previous theorem [2, Theorem 1.2].
Theorem 3.2. Consider the case $m > 0$. Suppose that $A$ has entire purely discrete spectrum. Assume Hypotheses [H.1]-[H.4]. Then, $H(\lambda, \mu)$ has purely discrete spectrum in the interval $[E_0(H(\lambda, \mu)), E_0(H(\lambda, \mu)) + m)$. In particular, $H(\lambda, \mu)$ has a ground state.

Remark. This theorem has no restriction of the coupling parameter $\mu \geq 0$.

Remark. In the case $m > 0$, the condition [H.1] equivalent to the following:

$$g_j \in L^2(\mathbb{R}^d), \quad f_j \in D(\sqrt{\omega}), \quad j = 1, \ldots, J.$$  

For a vector $v = (v_1, \ldots, v_J) \in \mathbb{R}^J$ and $h = (h_1, \ldots, h_J) \in \bigoplus_{j=1}^J L^2(\mathbb{R}^d)$, we define

$$M_v(h) = \sum_{j=1}^J v_j \|h_j\|.$$  

We set

$$g = (g_1, \ldots, g_J) \in \bigoplus_{j=1}^J L^2(\mathbb{R}^d), \quad f = (f_1, \ldots, f_J) \in \bigoplus_{j=1}^J L^2(\mathbb{R}^d),$$  

and

$$a = (a_1, \ldots, a_J), \quad b = (b_1, \ldots, b_J).$$  

For $\theta$, $\epsilon$, $\epsilon'$, we introduce the following constants:

$$C_{\theta, \epsilon} := \theta M_a(g/\sqrt{\omega}) + \epsilon M_a(g),$$  

$$D_{\theta, \epsilon'} := M_a(g/\sqrt{\omega})/2\theta + \epsilon' M_b(g/\sqrt{\omega}),$$  

$$E_{\epsilon, \epsilon'} := M_a(g)/8\epsilon + M_b(g/\sqrt{\omega})/2\epsilon' + M_b(g)/\sqrt{2}.$$  

Let the condition [H.3] be satisfied. Then, we define

$$I_{\lambda, g} := \begin{cases} 
\frac{|\lambda|M_a(g/\sqrt{\omega})}{2}, & |\lambda|M_a(g/\sqrt{\omega}) \neq 0 \\
[0, \infty], & |\lambda|M_a(g/\sqrt{\omega}) = 0
\end{cases}$$  

It is easy to see that $[1/2, 1] \subset I_{\lambda, g}$. Therefore, for all $\theta \in I_{\lambda, g}$,

$$1 - \theta |\lambda|M_a(g/\sqrt{\omega}) > 0,$$

$$1 - \frac{|\lambda|M_a(g/\sqrt{\omega})}{2\theta} > 0.$$  

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We define for $\theta \in I_{\lambda,g}$,

$$S_{\theta} := \{(\epsilon, \epsilon')|\epsilon, \epsilon' > 0, |\lambda|C_{\theta,\epsilon} < 1, |\lambda|D_{\theta,\epsilon'} < 1\}.$$ 

Next we set

$$\tau_{\theta,\epsilon,\epsilon'} := (1 - |\lambda|C_{\theta,\epsilon})\Sigma(A) - |\lambda|E_{\epsilon,\epsilon'},$$

and

$$T := \{(\theta, \epsilon, \epsilon') \in \mathbb{R}^3|\theta \in I_{\lambda,g}, (\epsilon, \epsilon') \in S_{\theta}, \tau_{\theta,\epsilon,\epsilon'} > E_0(H(\lambda, \mu))\}.$$ 

**Theorem 3.3.** Consider the case $m > 0$. Suppose that $\sigma_{ess}(A) \neq \emptyset$. Assume Hypothesis [H.1]-[H.4], and $T \neq \emptyset$. Then, $H(\lambda, \mu)$ has purely discrete spectrum in the interval

$$[E_0(H(\lambda, \mu)), \min\{m + E_0(H(\lambda, \mu)), \sup_{(\theta,\epsilon,\epsilon') \in T} \tau_{\theta,\epsilon,\epsilon'}\}].$$

(4)

In particular, $H(\lambda, \mu)$ has a ground state.

**Remark.** $T \neq \emptyset$ is necessary condition for $A$ to have a discrete ground state. Conversely, if $A$ has a discrete ground state, then $T \neq \emptyset$ holds for sufficiently small $\lambda, \mu$. Therefore the condition $T \neq \emptyset$ is a restriction for the coupling constants $\lambda, \mu$.

**3.1 Proof of Proposition 3.1**

In what follows, we write simply

$$H := H(\lambda, \mu).$$

For $D$ a dense subspace of $L^2(\mathbb{R}^d)$, we define

$$\mathcal{F}_{\text{fin}}(D) := \text{L}h[\Omega, a(h_1)^* \cdots a(h_n)^*\Omega|n \in \mathbb{N}, h_j \in D, j = 1, \ldots, n],$$

where $\Omega := (1, 0, 0, \ldots)$ is the Fock vacuum in $\mathcal{F}_h$. We introduce a dense subspace in $\mathcal{F}$

$$D_{\omega} := D(A)\hat{\otimes}\mathcal{F}_{\text{fin}}(D(\omega)),$$

where $\hat{\otimes}$ denotes algebraic tensor product. The subspace $D_{\omega}$ is a core of $H_0$. 

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Let

\[ H_{\text{GSB}} := H_0 + \lambda H_1 \]

be a GSB Hamiltonian. The Hamiltonian \( H \) and \( H_{\text{GSB}} \) has the following relation:

**Proposition 3.4.** Let \( D(A) \subset D(B_j), j = 1, \ldots, J \) and \( f_j \in D(\omega^{1/2}) \). Assume that \( H_{\text{GSB}} \) is bounded from below. Then, for all \( \Psi \in D_\omega \),

\[
\| (H_{\text{GSB}} - E_0) \Psi \|^2 + \| \mu H_2 \Psi \|^2 \leq \| (H - E_0) \Psi \|^2 + D \| \Psi \|^2, \tag{5}
\]

where \( D = \mu \sum_{j=1}^{J} \| \omega^{1/2} f_j \|^2 \) and

\[
E_0 := \inf_{\Psi \in D(H_{\text{GSB}})} \langle \Psi, H_{\text{GSB}} \Psi \rangle.
\]

**Proof.** It is enough to show (5) the case \( \lambda = \mu = 1 \). First we consider the case where \( f_j \in D(\omega) \). Inequality (5) is equivalent to

\[
-2 \text{Re} \langle (H_{\text{GSB}} - E_0) \Psi, H_2 \Psi \rangle \leq D \| \Psi \|^2.
\] \( (6) \)

By \( H_{\text{GSB}} - E_0 \geq 0 \), we have

\[
\langle (H_{\text{GSB}} - E_0) \Psi, I \otimes \phi(f_j)^2 \Psi \rangle = \langle (I \otimes \phi(f_j), (H_{\text{GSB}} - E_0)) \Psi, I \otimes \phi(f_j) \Psi \rangle \\
+ \langle (H_{\text{GSB}} - E_0) I \otimes \phi(f_j) \Psi, I \otimes \phi(f_j) \Psi \rangle \\
\geq \langle (I \otimes \phi(f_j), H_{\text{GSB}} - E_0) \Psi, I \otimes \phi(f_j) \Psi \rangle.
\]

Therefore we have

\[
2 \text{Re} \langle (H_{\text{GSB}} - E_0) \Psi, \phi(f_j)^2 \Psi \rangle \geq -\| \sqrt{\omega} f_j \|^2 \| \Psi \|^2.
\]

This means inequality (6). Next, we set \( f_j \in D(\sqrt{\omega}) \). Then, there exists a sequence \( \{f_{j_n}\}_{n=0}^\infty \subset D(\omega) \) such that \( f_{j_n} \to f_j, \omega^{1/2} f_{j_n} \to \omega^{1/2} f_j \) \((n \to \infty)\).

By limiting argument, (6) holds with \( f_j \in D(\omega^{1/2}) \).

**Lemma 3.5.** Suppose that \( H_{\text{GSB}} \) is self-adjoint with \( D(H_{\text{GSB}}) = D(H_0) \), essentially self-adjoint on \( D_\omega \), and bounded from below. Let \( f_j \in D(\omega^{1/2}) \cap D(\omega^{-1/2}) \). Then \( H \) is self-adjoint with \( D(H) = D(H_0) \) and essentially self-adjoint on any core of \( H_{\text{GSB}} \) with

\[
\| (H_{\text{GSB}} - E_0) \Psi \|^2 + \| \mu H_2 \Psi \|^2 \leq \| (H - E_0) \Psi \|^2 + D \| \Psi \|^2, \quad \Psi \in D(H_0).
\]
Proof. It is well known that $D(H_b) \subset D(\phi(f_j)^2)$, and $\phi(f_j)^2$ is $H_b$-bounded (e.g. [1, Lemma 13-16]). Namely, there exist constants $\eta \geq 0$, $\theta \geq 0$ such that

$$\left\| \sum_{j=1}^{J} \phi(f_j)^2 \psi \right\| \leq \eta \|H_b \psi\| + \theta \|\psi\|, \quad \psi \in D(H_b). \quad (7)$$

Since $H_{GSB}$ is self-adjoint on $D(H_0)$, by the closed graph theorem, we have

$$\|H_0 \Psi\| \leq \lambda \|H_{GSB} \Psi\| + \nu \|\Psi\|, \quad \Psi \in D(H_0), \quad (8)$$

where $\lambda$ and $\nu$ are non-negative constant independent of $\Psi$. Hence

$$\|H_2 \Psi\| \leq \eta \lambda \|H_{GSB} \Psi\| + (\eta \nu + \theta) \|\Psi\|, \quad \Psi \in D(H_0).$$

We fix a positive number $\mu_0$ such that $\mu_0 < 1/(\mu \lambda)$. Then, by the Kato-Rellich theorem, $H(\lambda, \mu_0)$ is self-adjoint on $D(H_{GSB})$, bounded from below and essentially self-adjoint on any core of $H_{GSB}$. For a constant $a$ ($0 < a < 1$), we set $\mu_n := (1 + a)^n \mu_0$. Since $H_{GSB}$ is self-adjoint on $D(H_0)$, for each $j = 1, \ldots, J$ we have $D(A) \subset D(B)$. Thus by Proposition 3.4, for all $\Psi \in D_\omega$

$$||(H_{GSB} - E_0) \Psi||^2 + \|\mu_n H_2 \Psi\|^2 \leq ||(H(\lambda, \mu_n) - E_0) \Psi||^2 + D\|\Psi\|^2.$$ 

If $H(\lambda, \mu_n)$ is self-adjoint on $D(H_{GSB})$, bounded from below and essentially self-adjoint on any core of $H_{GSB}$, then $H(\lambda, \mu_{n+1})$ has the same property. On the other hand, we have $\mu_n \rightarrow \infty$ ($n \rightarrow \infty$). Hence we conclude that $H$ is self-adjoint with $D(H) = D(H_{GSB})$, bounded from below and essentially self-adjoint on any core of $H_{GSB}$. 

Now, we assume conditions [H.1],[H.2] and [H.3].

Then $H_{GSB}$ is self-adjoint on $D(H_0)$, bounded from below and essentially self-adjoint on any core of $H_0$(see [2]). Hence, the assumptions of Lemma 3.5 hold. Thus Proposition 3.1 follows.

3.2 Proofs of Theorems 3.2 and 3.3

Throughout this subsection, we assume Hypotheses [H.1]-[H.4] and $m > 0$. 

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For a parameter \( V > 0 \), we define the set of lattice points by
\[
\Gamma_V := \frac{2\pi \mathbb{Z}^d}{V} := \left\{ k = (k_1, \ldots, k_d) \Big| k_j = \frac{2\pi n_j}{V}, n_j \in \mathbb{Z}, j = 1, \ldots, d \right\}
\]
and we denote by \( l^2(\Gamma_V) \) the set of \( l^2 \) sequences over \( \Gamma_V \). For each \( k \in \Gamma_V \) we introduce
\[
C(k, V) := \left[ k_1 - \frac{\pi}{V}, k_1 + \frac{\pi}{V} \right) \times \cdots \times \left[ k_d - \frac{\pi}{V}, k_d + \frac{\pi}{V} \right) \subset \mathbb{R}^d,
\]
the cube centered about \( k \). By the map
\[
U : l^2(\Gamma_V) \ni \{ h_l \}_{l \in \Gamma_V} \mapsto (V/2\pi)^{d/2} \sum_{l \in \Gamma_V} h_l \chi_{l,V}(\cdot) \in L^2(\mathbb{R}^d),
\]
we identify \( l^2(\Gamma_V) \) with a subspace in \( L^2(\mathbb{R}^d) \), where \( \chi_{l,V}(\cdot) \) is the characteristic function of the cube \( C(l,V) \subset \mathbb{R}^d \). It is easy to see that \( l^2(\Gamma_V) \) is a closed subspace of \( L^2(\mathbb{R}^d) \). Let
\[
\mathcal{F}_{b,V} := \mathcal{F}_b(l^2(\Gamma_V)) = \bigoplus_{n=0}^{\infty} \bigotimes_{s}^{n} l^2(\Gamma_V),
\]
the boson Fock space over \( l^2(\Gamma_V) \). We can identify \( \mathcal{F}_{b,V} \) the closed subspace of \( \mathcal{F}_b \) by the operator \( \Gamma(U) := \bigoplus_{n=0}^{\infty} \bigotimes^{n} U \), where we define \( \bigotimes^{0} U = 0 \). For each \( k \in \mathbb{R}^d \), there exists a unique point \( k_V \in \Gamma_V \) such that \( k \in C(k_V, V) \). Let
\[
\omega_V(k) := \omega(k_V), \quad k \in \mathbb{R}^d
\]
be a lattice approximate function of \( \omega(k) \) and let
\[
H_{b,V} := d\Gamma(\omega_V)
\]
be the second quantization of \( \omega_V \). We define a constant
\[
C_V := C d\gamma \left( \frac{\pi}{V} \right) \left( \frac{1}{2m} + 1 \right),
\]
where \( C \) and \( \gamma \) were defined in [H.4]. In what follows we assume that
\[
C_V < 1.
\]
This is satisfied for all sufficiently large \( V \).
Lemma 3.6. \( \{2, \text{Lemma 3.1}\} \). \( D(H_{b,V}) = D(H_b) \), and

\[
\|(H_b - H_{b,V})\Psi\| = \frac{2C_V}{1 - C_V} \|H_b\Psi\|, \quad \Psi \in D(H_b).
\]

First we consider the case where \( g_j \)'s and \( f_j \)'s are continuous, and finally, by limiting argument, we treat a general case. For a constant \( K > 0 \), we define \( g_j,K, f_j,K, g_j,K,V, f_j,K,V \) as follows:

\[
g_{j,K}(k) := \chi_K(k_1) \cdots \chi_K(k_d)g_j(k), \quad g_{j,K,V}(k) := \sum_{\ell \in \Gamma_{V,|\ell| < K}} g_j(\ell)\chi_{\ell,V}(k),
\]

\[
f_{j,K}(k) := \chi_K(k_1) \cdots \chi_K(k_d)f_j(k), \quad f_{j,K,V}(k) := \sum_{\ell \in \Gamma_{V,|\ell| < K}} f_j(\ell)\chi_{\ell,V}(k),
\]

where \( \chi_K \) denotes the characteristic function of \([-K, K]\).

Lemma 3.7. For all \( j = 1, \ldots, J \),

\[
\lim_{V \to \infty} \|g_{j,K,V} - g_{j,K}\| = 0, \quad \lim_{V \to \infty} \|g_{j,K,V}/\sqrt{\omega V} - g_{j,K}/\sqrt{\omega}\| = 0,
\]

\[
\lim_{K \to \infty} \|g_{j,K} - g_j\| = 0, \quad \lim_{K \to \infty} \|g_{j,K}/\sqrt{\omega} - g_j/\sqrt{\omega}\| = 0,
\]

\[
\lim_{V \to \infty} \|f_{j,K,V} - f_{j,K}\| = 0, \quad \lim_{V \to \infty} \|f_{j,K,V}/\sqrt{\omega V} - f_{j,K}/\sqrt{\omega}\| = 0,
\]

\[
\lim_{K \to \infty} \|f_{j,K} - f_j\| = 0, \quad \lim_{K \to \infty} \|f_{j,K}/\sqrt{\omega} - f_j/\sqrt{\omega}\| = 0,
\]

\[
\lim_{K \to \infty} \|\sqrt{\omega}f_{j,K} - \sqrt{\omega}f_j\| = 0, \quad \lim_{V \to \infty} \|\sqrt{\omega V}f_{j,K,V} - \sqrt{\omega f_{j,K}}\| = 0.
\]

Proof. Similar to the proof of \[2, \text{Lemma 3.10}\]

We introduce a new operator:

\[
H_{0,V} := A \otimes I + I \otimes H_{b,V},
\]

\[
H_{1,K} := \sum_{j=1}^J B_j \otimes \phi(g_{j,K}),
\]

\[
H_{1,K,V} := \sum_{j=1}^J B_j \otimes \phi(g_{j,K,V}),
\]
\[ H_{2,K} := \sum_{j=1}^{J} I \otimes \phi(f_{j,K})^2, \]
\[ H_{2,K,V} := \sum_{j=1}^{J} I \otimes \phi(f_{j,K,V})^2, \]

and define
\[ H_K := H_0 + \lambda H_{1,K} + \mu H_{2,K}, \]
\[ H_{K,V} := H_{0,V} + \lambda H_{1,K,V} + \mu H_{2,K,V}. \]

**Lemma 3.8.** (i) \( H_K \) is self-adjoint with \( D(H_K) = D(H_0) \subset D(H_{1,K}) \cap D(H_{2,K}) \), bounded from below, and essentially self-adjoint on any core of \( H_0 \).

(ii) For all large \( V \), \( H_{K,V} \) is self-adjoint with \( D(H_{K,V}) = D(H_0) \subset D(H_{1,K,V}) \cap D(H_{2,K,V}) \), bounded from below, and essentially self-adjoint on any core of \( H_{0,V} \).

**Proof.** Similar to the proof of Proposition 3.1.

**Lemma 3.9.** For all \( z \in \mathbb{C}\setminus\mathbb{R} \), and \( K > 0 \),
\[ \lim_{K \to \infty} \| (H_K - z)^{-1} - (H - z)^{-1} \| = 0, \]
\[ \lim_{V \to \infty} \| (H_{K,V} - z)^{-1} - (H_K - z)^{-1} \| = 0. \]

**Proof.** Similar to the proof of Lemma 3.5.

The following fact is well known:

**Lemma 3.10.** The operator \( H_{b,V} \) is reduced by \( F_{b,V} \) and \( H_{b,V}[F_{b,V}] \) equal to the second quantization of \( \omega_V |1^2(\Gamma_V) \) on \( F_{b,V} \).

**Lemma 3.11.** \( H_{K,V} \) is reduced by \( F_V \).

**Proof.** Similar to the proof of Lemma 3.7.

**Lemma 3.12.** We have
\[ H_{K,V}[F_V] \geq E_0(H_{K,V}) + m. \]
Lemma 3.13. Let $T_n$ and $T$ be a self-adjoint operators on a separable Hilbert space and bounded from below. Suppose that $T_n \to T$ in norm resolvent sense as $n \to \infty$ and $T_n$ has purely discrete spectrum in the interval $[E_0(T_n), E_0(T_n) + c_n]$ with some constant $c_n$. If $c := \limsup_{n \to \infty} c_n > 0$, then $T$ has purely discrete spectrum in $[E_0(T), E_0(T) + c]$.

Proof. There exists a sequence $\{c_{n_j}\}_{j=1}^\infty \subset \{c_n\}_{n=1}^\infty$ so that $c_{n_j} \to c (j \to \infty)$. So, for all $\epsilon > 0$ and for sufficiently large $j$, the spectrum of $T_{n_j}$ in $[E_0(T_{n_j}), E_0(T_{n_j}) + c - \epsilon)$ is discrete. Therefore, applying Lemma 3.12 we find that the spectrum of $T$ in $[E_0(T), E_0(T) + c - \epsilon)$ is discrete. Since $\epsilon > 0$ is arbitrary, we get the conclusion.

Now, if $A$ has compact resolvent, by a method similar to the proof of Theorem 1.2 we can prove Theorem 3.2. Therefore, we only prove Theorem 3.3.

The following inequality is known [2, (2.12)]

$$|\langle \Psi, H \Psi \rangle| \leq C_{\theta,\epsilon}(\Psi, A \otimes I \Psi) + D_{\theta,\epsilon}(\Psi, I \otimes H_0(\Psi)) + E_{\epsilon,\epsilon'}\|\Psi\|^2,$$

where $\Psi \in D(H_0)$ is arbitrary. Thus we have,

$$H \geq (1 - |\lambda|C_{\theta,\epsilon})A \otimes I + (1 - |\lambda|D_{\theta,\epsilon})I \otimes H_0 + \mu H_2 - |\lambda|E_{\epsilon,\epsilon'}.$$

Let $I_{\lambda,g}(K)$, $C_{\theta,\epsilon}(K)$, $D_{\theta,\epsilon}(K)$ and $E_{\epsilon,\epsilon'}(K)$ are $I_{\lambda,g}$, $C_{\theta,\epsilon}$, $D_{\theta,\epsilon}$, $E_{\epsilon,\epsilon'}$ with $g_j$, $f_j$ replaced by $g_{j,K}$, $f_{j,K}$ respectively, and let $I_{\lambda,g}(K, V)$, $C_{\theta,\epsilon}(K, V)$, $D_{\theta,\epsilon}(K, V)$ and $E_{\epsilon,\epsilon'}(K, V)$ are $I_{\lambda,g}$, $C_{\theta,\epsilon}$, $D_{\theta,\epsilon}$, $E_{\epsilon,\epsilon'}$ with $g_j$, $f_j$ and $\omega$ replaced by $g_{j,K,V}$, $f_{j,K,V}$ and $\omega_V$ respectively. Then we have

Lemma 3.14. The following operator inequalities hold:

$$H_K \geq (1 - |\lambda|C_{\theta,\epsilon}(K))A \otimes I + (1 - |\lambda|D_{\theta,\epsilon}(K))I \otimes H_0 + \mu H_{2,K} - |\lambda|E_{\epsilon,\epsilon'}(K) \quad \text{on} \quad D(H_0),$$

$$H_{K,V} \geq (1 - |\lambda|C_{\theta,\epsilon}(K, V))A \otimes I + (1 - |\lambda|D_{\theta,\epsilon}(K, K))I \otimes H_0 + \mu H_{2,K,V} - |\lambda|E_{\epsilon,\epsilon'}(K, V) \quad \text{on} \quad D(H_0).$$

Proof. Similar to the calculation of [2, (2.12)].
By Lemma 3.7, we have
\[
\lim_{V \to \infty} C_{\theta,\epsilon}(K, V) = C_{\theta,\epsilon}(K), \quad \lim_{K \to \infty} C_{\theta,\epsilon}(K) = C_{\theta,\epsilon}, \quad (9)
\]
\[
\lim_{V \to \infty} D_{\theta,\epsilon'}(K, V) = D_{\theta,\epsilon'}(K), \quad \lim_{K \to \infty} D_{\theta,\epsilon'}(K) = D_{\theta,\epsilon'}, \quad (10)
\]
\[
\lim_{V \to \infty} E_{\epsilon,\epsilon'}(K, V) = E_{\epsilon,\epsilon'}(K), \quad \lim_{K \to \infty} E_{\epsilon,\epsilon'}(K) = E_{\epsilon,\epsilon'}. \quad (11)
\]
Let \((\theta, \epsilon, \epsilon') \in T\), namely
\[
\tau_{\theta,\epsilon,\epsilon'} = (1 - |\lambda|)C_{\theta,\epsilon}(K, V) - |\lambda|E_{\epsilon,\epsilon'} > E_0(H).
\]
Formulas (9)-(11) and Lemma 3.9 imply that for all large \(V\) there exists a constant \(K_0 > 0\) such that for all \(K > K_0\),
\[
(1 - |\lambda|C_{\theta,\epsilon}(K, V))\Sigma(A) - |\lambda|E_{\epsilon,\epsilon'}(K, V) > E_0(H_{K,V}), \quad (12)
\]
\[
|\lambda|C_{\theta,\epsilon}(K, V) < 1, \quad |\lambda|D_{\theta,\epsilon'}(K, V) < 1. \quad (13)
\]
By Lemma 3.11, \(H_{K,V}\) is reduced by \(F_V\). Therefore, \(H_{K,V}\) satisfies the following inequality:
\[
H_{K,V}[F_V] \geq (1 - |\lambda|C_{\theta,\epsilon}(K, V))A \otimes I[F_V] + (1 - |\lambda|D_{\theta,\epsilon'}(K, V))I \otimes H_{b,V}[F_V] - |\lambda|E_{\epsilon,\epsilon'}(K, V). \quad (14)
\]
Since \(H_{b,V}[F_V]\) has compact resolvent, the bottom of essential spectrum of the right hand side of (14) is equal to
\[
(1 - |\lambda|C_{\theta,\epsilon}(K, V))\Sigma(A) - |\lambda|E_{\epsilon,\epsilon'}(K, V).
\]
By Lemma 3.12, we have \(E_0(H_{K,V}[F_V]) = E_0(H_{K,V})\). Thus, applying Theorem 2.1 with \(H_{K,V}[F_V]\), we have that \(H_{K,V}[F_V]\) satisfies a purely discrete spectrum in \([E_0(H_{K,V}), (1 - |\lambda|C_{\theta,\epsilon}(K, V))\Sigma_A - E_{\epsilon,\epsilon'}(K, V)]\). Since this fact and Lemma 3.12, \(H_{K,V}\) has purely discrete spectrum in
\[
[E_0(H_{K,V}), \min\{E_0(H_{K,V}) + m, (1 - |\lambda|C_{\theta,\epsilon}(K, V))\Sigma_A - E_{\epsilon,\epsilon'}(K, V)\}].
\]
By Lemma 3.9 and Lemma 3.13, we have that for all sufficiently large \(K > 0\), \(H_K\) has purely discrete spectrum in \([E_0(H_K), \min\{E_0(H_K) + m, (1 - |\lambda|C_{\theta,\epsilon}(K))\Sigma(A) - |\lambda|E_{\epsilon,\epsilon'}(K)\}].\) Similarly, \(H\) has purely discrete spectrum in \([E_0(H(\lambda, \mu)), \min\{m + E_0(H(\lambda, \mu)), \tau_{\theta,\epsilon,\epsilon'}\}].\) Since \((\theta, \epsilon, \epsilon') \in T\) is arbitrary, \(H\) has purely discrete spectrum in (4). Finally, we have to consider the case where \(g_j\)’s and \(f_j\)’s are not necessarily continuous. But, that argument were already discussed in [4] So we skip that argument. \(\blacksquare\)
4 Ground State of the Dereźniński-Gérard Model

We consider a model discussed by J. Dereźniński and C. Gérard [5]. We take the Hilbert space of the particle system is taken to be

$$\mathcal{H} = L^2(\mathbb{R}^N).$$

The Hilbert space for the Dereźniński-Gérard (DG) model is given by

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_b(L^2(\mathbb{R}^d)).$$

We identify $\mathcal{F}$ as

$$\bigoplus_{n=0}^{\infty} \left( \mathcal{H} \otimes \bigotimes_{s} L^2(\mathbb{R}^d) \right).$$

Hence, if we denote that $\Psi \in (\Psi^{(n)})_{n=0}^{\infty} \in \mathcal{F}$, each $\Psi^{(n)}$ belongs to $\mathcal{H} \otimes [\otimes_{s} L^2(\mathbb{R}^d)]$. We denote by $B(K, \mathcal{J})$ the set of bounded linear operators from $K$ to $\mathcal{J}$. For $v \in B(\mathcal{H}, \mathcal{H} \otimes L^2(\mathbb{R}^d))$, we define an operator $\tilde{a}^*(v)$ by

$$(\tilde{a}^*(v)\Psi)(0) := 0,$$

$$(\tilde{a}^*(v)\Psi)^{(n)} := \sqrt{n}(I_{\mathcal{H}} \otimes S_n)(v \otimes I_{\otimes_{s}^{n-1} L^2(\mathbb{R}^d)})\Psi^{(n-1)}, \quad (n \geq 1),$$

$$\Psi \in D(\tilde{a}^*(v)) := \left\{ \Psi = (\Psi^{(n)})_{n=0}^{\infty} \in \mathcal{F} \Big| \sum_{n=0}^{\infty} \| (\tilde{a}^*(v)\Psi)^{(n)} \|^2 < \infty \right\}.$$

We set

$$D_0 := \{ \Psi = (\Psi^{(n)})_{n=0}^{\infty} \in \mathcal{F} | \text{there exists a constant } n_0 \in \mathbb{N},$$

such that, for all $n \geq n_0$, $\Psi^{(n)} = 0 \}.$$

Throughout this section, we write simply $I_n := I_{\otimes_{s}^{n} L^2(\mathbb{R}^d)}$. It is easy to see that:

**Proposition 4.1.** $\tilde{a}^*(v)$ is a closed linear operator and $D_0$ is a core of $\tilde{a}^*(v)$.

So we set

$$\tilde{a}(v) := (\tilde{a}^*(v))^*$$

the adjoint operator of $\tilde{a}^*(v)$. 17
Proposition 4.2. The operator $\tilde{a}(v)$ has the following properties:

$$D(\tilde{a}(v)) = \left\{ \Psi = (\Psi^{(n)})_{n=0}^\infty : \sum_{n=0}^\infty (n+1) \|(I_H \otimes S_n) (v^* \otimes I_n) \Psi^{(n+1)}\|^2 < \infty \right\}$$

(15)

$$\tilde{a}(v) \Psi = \sqrt{n+1} (I_H \otimes S_n) (v^* \otimes I_n) \Psi^{(n+1)}, \quad \Psi \in D(\tilde{a}(v)),$$

(16)

and $D_0$ is a core of $\tilde{a}(v)$.

Proof. For $\Phi \in \mathcal{F}$, $\Psi \in D(\tilde{a}^*(v))$,

$$\langle \Phi, \tilde{a}^* (v) \Psi \rangle = \sum_{n=1}^\infty \langle \Phi^{(n)}, \sqrt{n} (I_H \otimes S_n) (v \otimes I_{n-1}) \Psi^{(n-1)} \rangle$$

$$= \sum_{n=0}^\infty \sqrt{n+1} \langle v^* \otimes I_n \Phi^{(n+1)}, \Psi^{(n)} \rangle$$

$$= \sum_{n=0}^\infty \langle \sqrt{n+1} (I_H \otimes S_n) (v^* \otimes I_n) \Phi^{(n+1)}, \Psi^{(n)} \rangle.$$ 

This implies (15) and (16). It is easy to prove that $D_0$ is a core of $\tilde{a}(v)$. 

An analogue of the Segal field operator is defined by

$$\tilde{\phi}(v) := \frac{1}{\sqrt{2}} (\tilde{a}(v) + \tilde{a}^*(v)).$$

Let $A$ be a non-negative self-adjoint operator on $H$ with $E_0(A) = 0$. Then the Hamiltonian of the DG model is defined by

$$H_{DG} := A \otimes I + I \otimes H_b + \tilde{\phi}(v).$$

We call it the Dereźniski-Gérard Hamiltonian. Here $H_b$ is the second quantization of $\omega$ introduced in Section 3. Let

$$H_0 := A \otimes I + I \otimes H_b.$$

Throughout this section we assume the following conditions:

[DG.1] There is a Borel measurable function $v(x,k) \in \mathbb{C}$, $(x \in \mathbb{R}^N, k \in \mathbb{R}^d)$, such that

$$(vf)(x,k) = v(x,k)f(x), \quad f \in L^2(\mathbb{R}^d).$$
We need also the following assumption:

\[ \text{DG.2} \quad \text{ess.sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} \left| \frac{v(x, k)}{\sqrt{\omega(k)}} \right|^2 dk < \infty. \]

**Proposition 4.3.** Assume [DG.1] and [DG.2]. Then $H_{DG}$ is self-adjoint with $D(H_{DG}) = D(H_0)$, and essentially self-adjoint on any core of $H_0$.

For a finite volume approximation, we introduce the following hypotheses:

[DG.3] There exists a nonnegative function $\tilde{v} \in L^2(\mathbb{R}^d)$ and function $\tilde{o} : \mathbb{R} \to \mathbb{R}$, such that

\[ \text{ess.sup}_{x \in \mathbb{R}^n} |v(x, k) - v(x, \ell)| \leq \tilde{v}(k)\tilde{o}(|k - \ell|), \quad \text{a.e. } k, \ell \in \mathbb{R}^d \]

\[ \lim_{t \to 0} \tilde{o}(t) = 0. \]

[DG.4] 

\[ \text{ess.sup}_{x \in \mathbb{R}^n} \int_{([-K,K]^d)^c} |v(x, k)|^2 dk = o(K^0). \]

where

\[ ([{-K,K}]^d)^c := \mathbb{R}^d \setminus (I \times \cdots \times I), \quad I := [-K, K] \]

and, $o(t^0)$ satisfies $\lim_{t \to 0} o(t^0) = 0$.

Let $m$ be defined by (3). Let

\[ D := \frac{1}{2} \inf_{0 < \epsilon' < \|v\|_{\|v\|_{\sqrt{\omega}}}} \left( \frac{\epsilon'}{\epsilon} + \frac{1}{\epsilon'} \right). \tag{17} \]

Here, $v/\sqrt{\omega}$ is a multiplication operator by the function $v(x, k)/\sqrt{\omega(k)}$ from $L^2(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N) \otimes L^2(\mathbb{R}^d)$. In the case $m > 0$, we can establish the existence of a ground state of $H_{DG}$:

**Theorem 4.4.** Let $m > 0$. Suppose that [DG.1]-[DG.4] and [H.4] hold, and suppose

\[ \Sigma(A) - \|v\|D - E_0(H_{DG}) > 0. \]
Then, $H_{DG}$ has purely discrete spectrum in

$$[E_0(H_{DG}), \min\{E_0(H_{DG}) + m, \Sigma(A) - \|v\|D\}).$$

In particular $H_{DG}$ has a ground state.

Remark. In the case where $A$ has compact resolvent, this theorem has been proved in [5]. A new aspect here is in that $A$ does not necessarily have compact resolvent. Also our method is different from that in [5].

4.1 Proof of Proposition 4.3

Lemma 4.5. Let $M(x) = (\int_{\mathbb{R}^d} |v(x, k)|^2 \, dk)^{1/2}$, $x \in \mathbb{R}^N$ and $M : L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ be a multiplication operator by the function $M(x)$. Then

$$\|vf\|_2 = \|Mf\|_2, \quad f \in L^2(\mathbb{R}^N).$$

In particular, $\|v\| = \|M\| = (\text{ess.sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} |v(x, k)|^2 \, dk)^{1/2}$ hold.

Proof. By the Fubini’s theorem, we have

$$\|vf\|_2^2 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^N} \int_{\mathbb{R}^d} |v(x, k)|^2 |f(x)|^2 \, dx \, dk \, dk \, dx.$$

This means the result.

The adjoint $v^*$ has the following form:

Lemma 4.6. For all $g \in \mathcal{H} \otimes L^2(\mathbb{R}^d)$,

$$(v^*g)(x) = \int_{\mathbb{R}^d} v(x, k)^* g(x, k) \, dk, \quad \text{a.e. } x \in \mathbb{R}^d. \quad (18)$$

Proof. For all $f \in \mathcal{H}$, we have

$$\langle g, vf \rangle = \int dx \int dk g(x, k)^* v(x, k) f(x)$$

$$= \int dx \left( \int g(x, k)^* v(x, k) \, dk \right) f(x).$$

Since $f$ is arbitrary, this proves (18).
Lemma 4.7. \( \tilde{a}(v) \) is

\[
D(\tilde{a}(v)) = \left\{ \Psi \in \mathcal{F} \left| \sum_{n=0}^{\infty} (n+1) \int_{\mathbb{R}^{N+dn}} dx dk_1 \cdots dk_n \right. \right.
\left. \left| \int_{\mathbb{R}^d} dk v(k,x)^* \Psi^{(n+1)}(x,k,k_1,\ldots,k_n) \right|^2 < \infty \right\}
\]

\[
(\tilde{a}(v)\Psi)^{(n)}(x,k_1,\ldots,k_n) = \sqrt{n+1} \int_{\mathbb{R}^d} v(x,k)^* \Psi^{(n+1)}(x,k,k_1,\ldots,k_n), \quad \text{a.e.} \quad (\Psi \in D(\tilde{a}(v)))
\]

Proof. Using Lemma 4.6, we have

\[
(v^* \otimes I_n)\Psi^{(n+1)}(x,k_1,\ldots,k_n) = \int_{\mathbb{R}^d} v^*(x,k)\Psi^{(n+1)}(x,k,k_1,\ldots,k_n) dk. \tag{19}
\]

This is invariant for all permutations of \( k_1, \ldots, k_n \). Therefore, using Proposition 4.2, we get

\[
(\tilde{a}(v)\Psi)^{(n)}(x,k_1,\ldots,k_n) = \sqrt{n+1} \int_{\mathbb{R}^d} v(x,k)^* \Psi^{(n+1)}(x,k,k_1,\ldots,k_n) dk.
\]

Lemma 4.8. Suppose that \([DG.1]\) and \([DG.2]\) hold. Then, \( D(\tilde{a}(v)) \supset D(I \otimes H^{1/2}) \) and

\[
\|\tilde{a}(v)\Phi\| \leq \|v/\sqrt{\omega}\| \|I \otimes H^{1/2} \Phi\|, \quad \Phi \in D(I \otimes H^{1/2}).
\]

Proof. By (19), we have for all \( \Phi \in D(\tilde{a}(v)) \)

\[
\| (\tilde{a}(v)\Phi)^{(n)} \|^2 = (n+1) \int_{\mathbb{R}^{dn+N}} dx dk_1 \cdots dk_n \int_{\mathbb{R}^d} \sqrt{\omega(k)}
\times \frac{1}{\sqrt{\omega(k)}} v(x,k)^* \Phi^{(n+1)}(x,k,k_1,\ldots,k_n) dk \|^2.
\]

Using the Schwarz inequality, one has

\[
\left| \int_{\mathbb{R}^d} \frac{1}{\sqrt{\omega(k)}} v(x,k)^* \Phi^{(n+1)}(x,k,k_1,\ldots,k_n) dk \right|^2 \leq \int_{\mathbb{R}^d} \frac{|v(x,k)|^2}{\sqrt{\omega(k)}} dk \cdot \int_{\mathbb{R}^d} \omega(k) \Phi^{(n+1)}(x,k,k_1,\ldots,k_n)^2 dk.
\]
Hence, for every $\Phi \in D_0 \cap D(I \otimes H^{1/2}_b)$, we have
\[
\| (\tilde{a}(v)\Phi)^{(n)} \|^2 \\
\leq \left( \text{ess.sup} \int_{\mathbb{R}^d} \left| \frac{v(x,k)^*}{\sqrt{\omega(k)}} \right|^2 \, dk \right) \times (n + 1) \times \\
\int_{\mathbb{R}^{d+n+N}} dx dk_1 \cdots dk_n dk \omega(k) |\Phi^{(n+1)}(x,k,k_1,\ldots,k_n)|^2 \\
= \left( \text{ess.sup} \int_{\mathbb{R}^d} \left| \frac{v(x,k)^*}{\sqrt{\omega(k)}} \right|^2 \, dk \right) \times \\
\int_{\mathbb{R}^{d+n+N}} dx dk_1 \cdots dk_{n+1} \sum_{j=1}^{n+1} \omega(k_j) |\Phi^{(n+1)}(x,k_1,\ldots,k_{n+1})|^2 \\
= \left\| \frac{v}{\sqrt{\omega}} \right\| \left\| (I \otimes H^{1/2}_b\Phi)^{(n+1)} \right\|^2.
\]
Therefore
\[
\| \tilde{a}(v)\Phi \| \leq \left\| \frac{v}{\sqrt{\omega}} \right\| \left\| (I \otimes H^{1/2}_b\Phi) \right\|^2.
\]

Since, $D_0 \cap D(I \otimes H^{1/2}_b)$ is a core of $I \otimes H^{1/2}_b$, one can extend this inequality to all $\Phi \in D(I \otimes H^{1/2}_b)$, and $D(I \otimes H^{1/2}_b) \subset D(\tilde{a}(v))$ holds.

**Lemma 4.9.** On $D_0$, $\tilde{a}(v)$ and $\tilde{a}^*(v)$ satisfy the following commutation relation:
\[
[\tilde{a}(v), \tilde{a}^*(v)] = \int_{\mathbb{R}^d} |v(\cdot,k)|^2 \, dk.
\]
where the right hand side is a multiplication operator by the function $x \mapsto \int_{\mathbb{R}^d} |v(x,k)|^2 \, dk$.

**Proof.** Let $\Phi \in D_0$. By the definition of $\tilde{a}^*(v)$, and using Proposition 4.2, we get
\[
(\tilde{a}^*(v), \tilde{a}(v)|\Phi)^{(n)} = (\tilde{a}(v)\tilde{a}(v)^*\Phi)^{(n)} - (\tilde{a}(v)^*\tilde{a}(v)\Phi)^{(n)} \\
= \sqrt{n+1} I_{\mathcal{H}} \otimes S_n (v^* \otimes I_n)(\tilde{a}(v)^*\Phi)^{(n+1)} - \sqrt{n} (I \otimes S_n)(v \otimes I_{n-1})(\tilde{a}(v)\Phi)^{(n-1)}.
\]
Hence, we have

\[
([\tilde{a}^*(v), \tilde{a}(v)] \Phi)^{(n)}(x, k_1, \ldots, k_n) \\
= (n + 1) \int_{\mathbb{R}^d} v(x, k)^* (I \otimes S_{n+1}(v \otimes I_{n-1}) \Phi^{(n)})(x, k, k_1, \ldots, k_n) dk \\
- \frac{1}{n} \sum_{j=1}^{n} v(x, k_j) (v^* \otimes I_{n-1} \Phi^{(n)})(x, k_1, \ldots, \hat{k}_j, \ldots, k_n)
\]

\[
= \int_{\mathbb{R}^d} dk v(x, k)^* \left( v(x, k) \Phi^{(n)}(x, k_1, \ldots, k_n) \\
+ \sum_{j=1}^{n} v(x, k_j) \Phi^{(n)}(x, k, k_1, \ldots, \hat{k}_j, \ldots, k_n) \\
- \sum_{j=1}^{n} v(x, k_j) \int_{\mathbb{R}^d} dk v(x, k)^* \Phi^{(n)}(x, k, k_1, \ldots, \hat{k}_j, \ldots, k_n)
\]

\[
= \left( \int_{\mathbb{R}^d} |v(x, k)|^2 \right) \Phi(x, k_1, \ldots, k_n).
\]

Here ‘\(^\sim\)’ indicates the omission of the object wearing the hat.

**Lemma 4.10.** Assume, \([DG.1]\) and \([DG.2]\). Then \(D(I \otimes H_b^{1/2}) \subset D(\tilde{a}^*(v))\) and for all \(\Phi \in D(I \otimes H_b^{1/2})\),

\[
\|\tilde{a}^*(v) \Phi\|^2 \leq \|v/\sqrt{\omega}\|^2 \|I \otimes H_b^{1/2} \Phi\|^2 + \|v\|^2 \|\Phi\|^2.
\]  

(20)

**Proof.** For all \(\Phi \in D_0 \cap D(I \otimes H_b^{1/2})\), we have

\[
\|\tilde{a}^*(v) \Phi\|^2 = \langle \Phi, \tilde{a}(v) \tilde{a}^*(v) \Phi \rangle = \langle \Phi, \tilde{a}^*(v) \tilde{a}(v) \Phi \rangle + \left( \int_{\mathbb{R}^d} |v(\cdot, k)|^2 \right) \Phi, \Phi \rangle
\]

\[
\leq \|\tilde{a}(v) \Phi\|^2 + \|v\|^2 \|\Phi\|^2.
\]

Thus we can apply Lemma 4.8 to obtain the result.

Now we can prove Proposition 4.3:

**Proof of Proposition 4.3.** By Lemma 4.8 and 4.10, the operator \(\tilde{\phi}(v)\) is \(I \otimes H_b^{1/2}\)-bounded. Hence \(\tilde{\phi}(v)\) is infinitesimally small with respect to \(I \otimes H_b\). Namely, for all \(\epsilon > 0\), there exists a constant \(c_\epsilon > 0\), such that,

\[
\|\tilde{\phi}(v) \Phi\| \leq \epsilon \|I \otimes H_b \Phi\| + c_\epsilon \|\Phi\|, \quad \Phi \in D(I \otimes H_b).
\]

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Since $A \geq 0$, we have
$$\|\tilde{\phi}(v)\Phi\| \leq \epsilon\|H_0\Phi\| + c\|\Phi\|, \quad \Phi \in D(H_0).$$
Thus we can apply the Kato-Rellich theorem to obtain the conclusion of Proposition 4.3.

4.2 Proof of Theorem 4.4

In this subsection we suppose that the assumption of Theorem 4.4 holds. Let $F_{b,V}$, $H_{b,V}$, $H_{0,V}$, $F_V$, $V$, $\varphi_{b,V}$ be an object already defined in Section 3, respectively. Suppose that $\chi_K$ is a characteristic function of $[-K, K]$.

For a parameter $K > 0$, we define $v_K \in \mathcal{B}(\mathcal{H}, \mathcal{H} \otimes L^2(\mathbb{R}^d))$ by
$$(v_K f)(x, k) := \chi_{[-K,K]}(k)v(x, k)f(x).$$
and $v_{K,V} \in \mathcal{B}(\mathcal{H}, \mathcal{H} \otimes L^2(\mathbb{R}^d))$ by
$$(v_{K,V} f)(x, k) := \sum_{\ell \in \Gamma_V, |\ell| < K} \chi_{\ell,V}(k)v(x, \ell)f(x).$$

Lemma 4.11. The following hold:
$$\|v_K - v_{K,V}\| \to 0 \quad (V \to \infty),$$
$$\|\frac{v_K}{\sqrt{\omega}} - \frac{v_{K,V}}{\sqrt{\omega}}\| \to 0 \quad (V \to \infty),$$
$$\left\|\frac{v}{\sqrt{\omega}} - \frac{v_K}{\sqrt{\omega}}\right\| \to 0 \quad (K \to \infty).$$

Proof. By [DG.3] and [DG.4], we have
$$\|v_K - v_{K,V}\|^2 = \text{ess sup} \int_{\mathbb{R}^d} \left|\chi_K(k)v(x, k) - \sum_{\ell \in \Gamma_V, |\ell| < K} \chi_{\ell,V}(k)v(x, \ell)\right|^2 dk$$
$$= \text{ess sup} \int_{\mathbb{R}^d} \sum_{\ell \in \Gamma_V, |\ell| < K} \chi_{\ell,V}(k)|v(x, k) - v(x, \ell)|^2 dk$$
$$\leq \text{ess sup} \int_{\mathbb{R}^d} \sum_{\ell \in \Gamma_V, |\ell| < K} \chi_{\ell,V}(k)|\tilde{v}(k)|^2\tilde{\phi}(|k - \ell|)^2 dk$$
$$\leq \int_{\mathbb{R}^d} \sum_{\ell \in \Gamma_V, |\ell| < K} \chi_{\ell,V}(k)|\tilde{v}(k)|^2\tilde{\phi}(|k - \ell|)^2 dk.$$
It follows from the property of $\tilde{\phi}$ that for every $\epsilon > 0$, there exists a constant $V_0 > 0$ such that, for all $V > V_0$,

$$\chi_{\ell,V}(k)\tilde{\phi}(|k - \ell|)^2 \leq \epsilon \chi_{\ell,V}(k).$$

Therefore,

$$\|v_K - v_{K,V}\|^2 \leq \epsilon \int_{\mathbb{R}^d} \sum_{|\ell| \leq V} \chi_{\ell,V}(k)|\tilde{\phi}(k)|^2 \, dk = \epsilon\|\tilde{v}\|^2_{L^2(\mathbb{R}^d)}.$$

Hence the first one of (21) holds. The second one is a direct result of condition [DG.4]:

$$\|v_K - v\|^2 = \text{ess.sup} \int_{\mathbb{R}^d} |\chi_{K}(k) - 1|^2 |v(x,k)|^2 \, dk
\leq \text{ess.sup} \int_{([-K,K]^d)^c} |v(x,k)|^2 \, dk = o(K^0) \to 0 \ (K \to \infty).$$

Using [H.4], one can easily check (22).

We introduce two operators:

$$H_{DG}(K) := A \otimes I + I \otimes H_b + \tilde{\phi}(v_K),$$

$$H_{DG}(K, V) := A \otimes I + I \otimes H_{b,V} + \tilde{\phi}(v_{K,V}).$$

**Lemma 4.12.** (i) $H_{DG}(K)$ is self-adjoint with $D(H_{DG}(K)) = D(H_0)$, bounded from below, and essentially self-adjoint on any core of $H_0$.

(ii) For sufficiently large $V > 0$, $H_{DG}(K, V)$ is self-adjoint with domain $D(H_{DG}(K, V)) = D(H_0)$, bounded from below, and essentially self-adjoint on any core of $H_0$.

**Proof.** Similar to the proof of Proposition 4.3.

**Lemma 4.13.** For all $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\lim_{V \to \infty} \|(H_{DG}(K, V) - z)^{-1} - (H_{DG}(K) - z)^{-1}\| = 0,$$

$$\lim_{K \to \infty} \|(H_{DG}(K) - z)^{-1} - (H_{DG} - z)^{-1}\| = 0.$$

**Proof.** Similar to the proof of [2, Lemma 3.5].
Lemma 4.14. The operator $H_{DG}(K, V)$ is reduced by $F_V$.

Proof. We identify $v(x, \ell)$ with multiplication operator by $v(\cdot, \ell)$. By abuse of symbols, we denote $\chi_{\ell,V}(\cdot)$ by $\chi_{\ell,V}(k)$. Then

$$\begin{align*}
(\tilde{a}^*(v(x, \ell)\chi_{\ell,V}(k))\Phi)^{(n)} &= \sqrt{n}(I \otimes S_n)(v(x, \ell)\chi_{\ell,V}(k) \otimes I)\Phi^{(n-1)} \\
&= \sqrt{n}v(x, \ell)S_n(\chi_{\ell,V} \otimes \Phi^{(n-1)}) \\
&= \chi(x, \ell)\sqrt{n}S_n(\chi_{\ell,V} \otimes \Phi^{(n-1)}).
\end{align*}$$

Hence, we have

$$\tilde{a}^*(v(x, \ell)\chi_{\ell,V}(k))\Phi = v(x, \ell) \otimes a^*(\chi_{\ell,V})\Phi.$$ 

Therefore, we get

$$\tilde{a}^*(v_{K,V}) = \sum_{\ell \in \Gamma_V \mid |\ell_i|<K} v(\cdot, \ell) \otimes a^*(\chi_{\ell,V}). \quad (23)$$

Hence, its adjoint is

$$\tilde{a}(v_{K,V}) = \sum_{\ell \in \Gamma_V \mid |\ell_i|<K} v(\cdot, \ell)^* \otimes a(\chi_{\ell,V}). \quad (24)$$

This means that the operator $H_{DG}(K, V)$ is a special case of the GSB Hamiltonian (see [2]). Hence, by [2, Lemma 3.7] $H_{DG}(K, V)$ is reduced by $F_V$. 

Lemma 4.15. $H_{DG}(K, V)[F_V^\perp \geq E_0(H_{DG}(K, V)) + m$

Proof. Similar to the proof of [2, Lemma 3.10].

Lemma 4.16. For all $\Phi \in D(I \otimes H_b^{1/2})$, and for all $\epsilon' > 0$,

$$|\langle \Phi, \tilde{a}(v)\Phi \rangle | \leq \frac{\epsilon'}{\|v\|} \left\| \frac{v}{\sqrt{\omega}} \right\|^2 \left\| I \otimes H_b^{1/2} \right\|^2 + \frac{\|v\|^2}{2} \left( \epsilon' + \frac{1}{\epsilon'} \right) \| \Phi \|^2.$$
Proof. For all $\Phi \in D(I \otimes H_{b}^{1/2})$, $\epsilon' > 0,$

$$
|\langle \Phi, \tilde{\phi}(v)\Phi \rangle| \leq \frac{1}{\sqrt{2}} \left( \epsilon \| \tilde{a}(v)\Phi \|^2 + \frac{1}{4\epsilon} \| \Phi \|^2 + \epsilon \| \tilde{a}^*(v)\Phi \|^2 + \frac{1}{4\epsilon} \| \Phi \|^2 \right)
$$

$$
\leq \frac{1}{\sqrt{2}} \left( 2\epsilon \left\| \frac{v}{\sqrt{\omega}} \right\|^2 \| I \otimes H_{b}^{1/2}\Phi \|^2 + \epsilon \| v \| \| \Phi \|^{2} + \frac{1}{2\epsilon} \| \Phi \|^2 \right)
$$

$$
= \sqrt{2\epsilon} \left\| \frac{v}{\sqrt{\omega}} \right\|^2 \| I \otimes H_{b}^{1/2}\Phi \|^2 + \frac{\| v \|}{\sqrt{2\epsilon}} \left( \sqrt{2\epsilon} \| v \| + \frac{1}{\sqrt{2\epsilon} \| v \|} \right) \| \Phi \|^2,
$$

where we have used Lemma 4.8 and 4.10. Let $\sqrt{2\epsilon} \| v \| =: \epsilon'$. Then, for all $\epsilon' > 0$, we have

$$
|\langle \Phi, \tilde{\phi}(v)\Phi \rangle| \leq \epsilon' \| v \| \left\| \frac{v}{\sqrt{\omega}} \right\|^2 \| I \otimes H_{b}^{1/2}\Phi \|^2 + \frac{\| v \|}{\sqrt{2\epsilon}} \left( \epsilon' + \frac{1}{\epsilon'} \right) \| \Phi \|^2.
$$

Proof of Theorem 4.4. From (23) and (24), $H_{DG}(K,V)$ is equal to the special case of the GSB model. Therefore, $H_{DG}(K,V)[\mathcal{F}_V$ has the same form with $H_{DG}(K,V)$. Using Lemma 4.16 we have on $D(H_0) \cap \mathcal{F}_V$

$$
H_{DG}(K,V)
$$

$$
= A \otimes I + I \otimes H_{b,V} + \tilde{\phi}(v_{K,V})
$$

$$
\geq A \otimes I + I \otimes H_{b,V} - \frac{\epsilon'}{\| v_{K,V} \| \sqrt{\omega_V}} \| I \otimes H_{b,V} - \| v_{K,V} \| \left( \epsilon' + \frac{1}{\epsilon'} \right)
$$

$$
= A \otimes I + \left( 1 - \frac{\epsilon'}{\| v_{K,V} \| \sqrt{\omega_V}} \right) \| I \otimes H_{b,V} - \frac{\| v_{K,V} \|}{2} \left( \epsilon' + \frac{1}{\epsilon'} \right), \quad (25)
$$

where $\epsilon' > 0$ is an arbitrary constant. By Lemma 3.10, $H_{b,V}[\mathcal{F}_{b,V}$ has compact resolvent. Thus, for $\epsilon' > 0$ satisfying

$$
1 - \frac{\epsilon'}{\| v_{K,V} \| \sqrt{\omega_V}} > 0, \quad (26)
$$

the bottom of the essential spectrum of (25) is equal to

$$
\Sigma(A) - \frac{\| v_{K,V} \|}{2} \left( \epsilon' + \frac{1}{\epsilon'} \right).
$$

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Let, $D_K$ and $D_{K,V}$ be $D$ with $v$ replaced by $v_K$, $v_{K,V}$, respectively. It is easy to see that

$$\lim_{K \to \infty} D_K = D, \quad \lim_{V \to \infty} D_{K,V} = D_K.$$  

By Lemma 4.13, one has

$$\lim_{K \to \infty} E_0(H_{DG}(K)) = E_0(H_{DG}), \quad \lim_{V \to \infty} E_0(H_{DG}(K,V)) = E_0(DG(K)).$$

From the assumption of Theorem 4.4, for all $K > 0$, there exists a constant $V_0$ such that for $V > V_0$,

$$\Sigma(A) - \frac{\|v_{K,V}\|}{2}D_{K,V} - E_0(H_{DG}(K,V)) > 0.$$  

By the definition of $D_{K,V}$, for all $K > 0$ and $V > V_0$, and for all $\epsilon'$ which satisfies (26), we have

$$\Sigma(A) - \frac{\|v_{K,V}\|}{2} \left( \epsilon' + \frac{1}{\epsilon'} \right) > E_0(H_{DG}(K,V)).$$

Therefore, by Theorem 2.1, we have that $H_{DG}(K,V)[F_V]$ has purely discrete spectrum in

$$[E_0(H_{DG}(K,V)), \Sigma(A) - \|v_{K,V}\|D_{K,V}).$$

This fact and Lemma 4.15 mean that $H_{DG}(K,V)$ has purely discrete spectrum in

$$[E_0(H_{DG}(K,V)), \min\{E_0(H_{DG}(K,V)) + m, \Sigma(A) - \|v_{K,V}\|D_{K,V}\}).$$

Finally, we use Lemma 3.13 and Lemma 4.13, to conclude that $H_{DG}$ has purely discrete spectrum in the interval

$$[E_0(H_{DG}), \min\{E_0(H_{DG}) + m, \Sigma(A) - \|v\|D\}).$$

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References


