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Stability of Discrete Ground State

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Abstract

We present new criteria for a self-adjoint operator to have a ground state. As an application, we consider models of “quantum particles” coupled to a massive Bose field and prove the existence of a ground state of them, where the particle Hamiltonian does not necessarily have compact resolvent.

Key words: Ground state; discrete ground state; generalized spin-boson model; Fock space; Derežiński-Gérard model.

1 INTRODUCTION

Let $T$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$, and bounded from below. We say that $T$ has a discrete ground state if the bottom of the spectrum of $T$ is an isolated eigenvalue of $T$. In that case a non-zero vector

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in \( \ker(T - E_0(T)) \) is called a ground state of \( T \). Let \( S \) be a symmetric operator on \( \mathcal{H} \). Suppose that \( T \) has a discrete ground state and \( S \) is \( T \)-bounded. By the regular perturbation theory \([8, XII]\) it is already known that \( T + \lambda S \) has a discrete ground state for “sufficiently small” \( \lambda \in \mathbb{R} \). Our aim is to present new criteria for \( T + \lambda S \) to have a ground state.

In Section 2, we prove an existence theorem of a ground state which is useful to show the existence of a ground state of models of quantum particles coupled to a massive Bose field.

In Section 3, we consider the GSB model \([2]\) with a self-interaction term of a Bose field, which we call the GSB + \( \phi^2 \) model. We consider only the case where the Bose field is massive. The GSB model — an abstract system of quantum particles coupled to a Bose field — was proposed in \([2]\). In \([2]\) A. Arai and M. Hirokawa proved the existence and uniqueness of the GSB model in the case where the particle Hamiltonian \( A \) has compact resolvent. Shortly after that, they proved the existence of a ground state of the GSB model in the case where \( A \) does not have necessarily compact resolvent \([4, 3]\). In this paper, using a theorem in Section 2, we prove the existence of a ground state of the GSB + \( \phi^2 \) model in the case where \( A \) does not necessarily have compact resolvent.

In Section 4, we consider an extended version of the Nelson type model, which we call the Dereziński-Gérard model \([5]\). The Dereziński-Gérard model introduced in \([5]\) and J. Dereziński and C. Gérard prove an existence of a ground state for their model under some conditions including that \( A \) has compact resolvent. In Section 4, we prove the existence of a ground state of the Dereziński-Gérard model in the case where \( A \) does not have compact resolvent. Our strategy to establish a ground state is the same as in Section 3.

2 BASIC RESULTS

Let \( \mathcal{H} \) be a separable complex Hilbert space. We denote by \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \) the scalar product on Hilbert space \( \mathcal{H} \) and by \( \|\cdot\|_{\mathcal{H}} \) the associated norm. Scalar product \( \langle f, g \rangle_{\mathcal{H}} \) is linear in \( g \) and antilinear in \( f \). We omit \( \mathcal{H} \) in \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \) and \( \|\cdot\|_{\mathcal{H}} \), respectively if there is no danger of confusion. For a linear operator \( T \) in Hilbert space, we denote by \( D(T) \) and \( \sigma(T) \) the domain and the spectrum of \( T \) respectively. If \( T \) is self-adjoint and bounded from below, then we define

\[
E_0(T) := \inf \sigma(T), \quad \Sigma(T) := \inf \sigma_{\text{ess}}(T),
\]
where \( \sigma_{\text{ess}}(T) \) is the essential spectrum of \( T \). If \( T \) has no essential spectrum, then we set \( \Sigma(T) = \infty \). For a self-adjoint operator \( T \), we denote the form domain of \( T \) by \( Q(T) \). In this paper, an eigenvector of a self-adjoint operator \( T \) with eigenvalue \( E_0(T) \) is called a ground state of \( T \) (if it exists). We say that \( T \) has a ground state if \( \dim \ker(T - E_0(T)) > 0 \).

The basic results are as follows:

**Theorem 2.1.** Let \( H \) be a self-adjoint operator on \( \mathcal{H} \), and bounded from below. Suppose that there exists a self-adjoint operator \( V \) on \( \mathcal{H} \) satisfying the following conditions (i)-(iii):

(i) \( D(H) \subset D(V) \).
(ii) \( V \) is bounded from below, and \( \Sigma(V) > 0 \).
(iii) \( H - E_0(H) \geq V \) on \( D(H) \).

Then \( H \) has purely discrete spectrum in the interval \([E_0(H), E_0(H) + \Sigma(V)]\). In particular, \( H \) has a ground state.

**Proof.** For all \( u_1, \ldots, u_{n-1} \in \mathcal{H} \), we have

\[
\inf_{\Psi \in \text{L.h.}, \frac{\Psi}{\|\Psi\|} \in D(H)} \langle \Psi, H\Psi \rangle - E_0(H) \geq \inf_{\Psi \in \text{L.h.}, \frac{\Psi}{\|\Psi\|} \in D(V)} \langle \Psi, V\Psi \rangle,
\]

where \( \text{L.h.}[\cdots] \) denotes the linear hull of the vectors in \([\cdots]\). Since \( D(H) \subset D(V) \), we have that

\[
\inf_{\Psi \in \text{L.h.}, \frac{\Psi}{\|\Psi\|} \in D(H)} \langle \Psi, V\Psi \rangle \geq \inf_{\Psi \in \text{L.h.}, \frac{\Psi}{\|\Psi\|} \in D(V)} \langle \Psi, V\Psi \rangle.
\]

Hence, for all \( n \in \mathbb{N} \)

\[
\mu_n(H) - E_0(H) \geq \mu_n(V).
\]

where

\[
\mu_n(H) := \sup_{u_1, \ldots, u_{n-1} \in \mathcal{H}} \inf_{\Psi \in \text{L.h.}, \frac{\Psi}{\|\Psi\|} \in D(H)} \langle \Psi, H\Psi \rangle.
\]

By the min-max principle ([8, Theorem XIII.1]), \( \lim_{n \to \infty} \mu_n(H) = \Sigma(H) \) and \( \lim_{n \to \infty} \mu_n(V) = \Sigma(V) \). Therefore we obtain

\[
\Sigma(H) - E_0(H) \geq \Sigma(V) > 0.
\]

This means that \( H \) has purely discrete spectrum in \([E_0(H), E_0(H) + \Sigma(V)]\).
Theorem 2.2. Let $H$ be a self-adjoint operator on $\mathcal{H}$, and bounded from below. Suppose that there exists a self-adjoint operator $V$ on $\mathcal{H}$ satisfying the following conditions (i)-(iii):

(i) $Q(H) \subset Q(V)$.
(ii) $V$ is bounded from below, and $\Sigma(V) > 0$.
(iii) $H - E_0(H) \geq V$ on $Q(H)$.

Then $H$ has purely discrete spectrum in the interval $[E_0(H), E_0(H) + \Sigma(V))$. In particular, $H$ has a ground state.

Proof. Similar to the proof of Theorem 2.1.

We apply Theorems 2.1 and 2.2 to a perturbation problem of a self-adjoint operator.

Theorem 2.3. Let $A$ be a self-adjoint operator on $\mathcal{H}$ with $E_0(A) = 0$, and let $B$ be a symmetric operator on $D(A)$. Suppose that $A + B$ is self-adjoint on $D(A)$ and that there exist constants $a \in [0, 1)$ and $b \geq 0$ such that

$$|\langle \psi, B\psi \rangle| \leq a \langle \psi, A\psi \rangle + b \|\phi\|^2, \quad \psi \in D(A).$$

Assume

$$\frac{b + E_0(A + B)}{1 - a} < \Sigma(A). \quad (1)$$

Then $A + B$ has purely discrete spectrum in $[E_0(A + B), (1 - a)\Sigma(A) - b]$. In particular, $A + B$ has a ground state.

Proof. By the assumption we have

$$A + B - E_0(A + B) \geq (1 - a)A - b - E_0(A + B)$$
on $D(A)$, and $(1 - a)\Sigma(A) - b - E_0(A + B) > 0$. Hence we can apply Theorem 2.1, to conclude that $A + B$ has purely discrete spectrum in $[E_0(A + B), (1 - a)\Sigma(A) - b]$. In particular, $A + B$ has a ground state.

Remark. It is easily to see that $-b \leq E_0(A + B) \leq b$. Therefore condition (1) is satisfied if

$$\frac{2b}{1 - a} < \Sigma(A).$$
**Theorem 2.4.** Let $\mathcal{H}, \mathcal{K}$ be complex separable Hilbert spaces. Let $A$ and $B$ be self-adjoint operators on $\mathcal{H}$ and $\mathcal{K}$ respectively. Suppose that $E_0(A) = E_0(B) = 0$. We set

$$T_0 := A \otimes I + I \otimes B.$$ 

Let $Z$ be a symmetric sesquilinear form on $Q(T_0)$, and assume that there exist constants $a_1 \in [0, 1), a_2 \in [0, 1)$ and $b \geq 0$ such that, for all $\Psi \in Q(T_0)$

$$|Z(\Psi, \Psi)| \leq a_1 \langle \Psi, A \otimes I \Psi \rangle_{\text{form}} + a_2 \langle \Psi, I \otimes B \Psi \rangle_{\text{form}} + b\|\Psi\|^2,$$

where $\langle \Psi, A \otimes I \Psi \rangle_{\text{form}} = \|A^{1/2} \otimes I\Psi\|^2$. Therefore, by the KLMN theorem there exists a unique self-adjoint operator $T$ on $\mathcal{H} \otimes \mathcal{K}$ such that $Q(T) = Q(T_0)$ and $T = T_0 + Z$ in the sense of sesquilinear form on $Q(T_0)$. We set

$$s := \min\{(1 - a_1)\Sigma(A), (1 - a_2)\Sigma(B)\}.$$ 

Assume

$$s > b + E_0(T).$$

Then, $T$ has purely discrete spectrum in the interval $[E_0(T), s - b)$. In particular, $T$ has a ground state.

**Proof.** Similar to the proof of Theorem 2.3. 

**Remark.** It is easy to see that $-b \leq E_0(T) \leq b$. Therefore the condition (2) is satisfied if

$$s > 2b.$$ 

**Remark.** Theorem 2.4 is essentially same as [4, Theorem B.1] But our proof is very simple.

### 3 Ground States of a General Class of Quantum Field Hamiltonians

We consider a model which is an abstract unification of some quantum field models of particles interacting with a Bose field. It is the GSB model [2] with a self-interaction term of the field.

Let $\mathcal{H}$ be a separable complex Hilbert space and $\mathcal{F}_b$ be the Boson Fock space over $L^2(\mathbb{R}^d)$:

$$\mathcal{F}_b := \bigoplus_{n=0}^{\infty} \bigotimes_{s=1}^{n} L^2(\mathbb{R}^d).$$
The Hilbert space of the quantum field model we consider is
\[ \mathcal{F} := \mathcal{H} \otimes \mathcal{F}_b. \]

Let \( \omega : \mathbb{R}^d \to [0, \infty) \) be Borel measurable such that \( 0 < \omega(k) < \infty \) for all most everywhere (a.e.) \( k \in \mathbb{R}^d \). We denote the multiplication operator by the function \( \omega \) acting in \( L^2(\mathbb{R}^d) \) by the same symbol \( \omega \). We set
\[ H_b := d\Gamma_b(\omega), \]
the second quantization of \( \omega \) (e.g. Section X.7). We denote by \( a(f) \), \( f \in L^2(\mathbb{R}^d) \), the smeared annihilation operators on \( \mathcal{F}_b \). It is a densely defined closed linear operator on \( \mathcal{F}_b \) (e.g. Section X.7). The adjoint \( a(f)^* \), called the creation operator, and the annihilation operator \( a(g) \), \( g \in L^2(\mathbb{R}^d) \) obey the canonical commutation relations
\[ [a(f), a(g)^*] = \langle f, g \rangle, \quad [a(f), a(g)] = 0, \quad [a(f)^*, a(g)^*] = 0 \]
for all \( f, g \in L^2(\mathbb{R}^d) \) on the dense subspace
\[ \mathcal{F}_0 := \{ \psi = (\psi^{(n)})_{n=0}^\infty \in \mathcal{F}_b | \text{there exists a number } n_0 \text{ such that} \psi^{(n)} = 0 \text{ for all } n \geq n_0 \}, \]
where \([X, Y] = XY - YX\). The symmetric operator
\[ \phi(f) := \frac{1}{\sqrt{2}} [a(f)^* + a(f)], \]
called the Segal field operator, is essentially self-adjoint on \( \mathcal{F}_0 \) (e.g. Section X.7). We denote its closure by the same symbol. Let \( A \) be a positive self-adjoint operator on \( \mathcal{H} \) with \( E_0(A) = 0 \). Then, the unperturbed Hamiltonian of the model is defined by
\[ H_0 := A \otimes I + I \otimes H_b \]
with domain \( D(H_0) = D(A \otimes I) \cap D(I \otimes H_b) \). For \( g_j, f_j \in L^2(\mathbb{R}^d) \) \( j = 1, \ldots, J \), and \( B_j(j = 1, \ldots, J) \) a symmetric operator on \( \mathcal{H} \), we define a symmetric operator
\[ H_1 := \sum_{j=1}^J B_j \otimes \phi(g_j), \]
\[ H_2 := \sum_{j=1}^J I \otimes \phi(f_j)^2. \]
The Hamiltonian of the model we consider is of the form

\[ H(\lambda, \mu) := H_0 + \lambda H_1 + \mu H_2, \]

where \( \lambda \in \mathbb{R} \) and \( \mu \geq 0 \) are coupling parameters.

For \( H(\lambda, \mu) \) to be self-adjoint, we shall need the following conditions [H.1]-[H.3]:

[H.1] \( g_j \in D(\omega^{-1/2}) \),  \( f_j \in D(\omega^{1/2}) \cap D(\omega^{-1/2}) \),  \( j = 1, \ldots, J \).

[H.2] \( D(A^{1/2}) \subset \cap_{j=1}^J D(B_j) \) and there exist constants \( a_j \geq 0 \),  \( b_j \geq 0 \),  \( j = 1, \ldots, J \), such that,

\[ \|B_j u\| \leq a_j \|A^{1/2} u\| + b_j \|u\|, \quad u \in D(A^{1/2}). \]

[H.3] \( |\lambda| \sum_{j=1}^J a_j \|g_j/\sqrt{\omega}\| < 1 \).

Proposition 3.1. Assume [H.1], [H.2] and [H.3]. Then, \( H(\lambda, \mu) \) is self-adjoint with \( D(H(\lambda, \mu)) = D(H_0) \subset D(H_1) \cap D(H_2) \) and bounded from below. Moreover, \( H(\lambda, \mu) \) is essentially self-adjoint on every core of \( H_0 \).

Remark. This proposition has no restriction of the coupling parameter \( \mu \geq 0 \).

* * *

To perform a finite volume approximation, we need an additional condition:

[H.4] The function \( \omega(k) \ (k \in \mathbb{R}^d) \) is continuous with

\[ \lim_{|k| \to \infty} \omega(k) = \infty, \]

and there exist constants \( \gamma > 0 \),  \( C > 0 \) such that

\[ |\omega(k) - \omega(k')| \leq C|k - k'|^{\gamma} [1 + \omega(k) + \omega(k')], \quad k, k' \in \mathbb{R}^d. \]

Let

\[ m := \inf_{k \in \mathbb{R}^d} \omega(k). \quad (3) \]

If \( A \) has compact resolvent, we can prove the extension of the previous theorem \[ \text{Theorem 1.2} \]
Theorem 3.2. Consider the case $m > 0$. Suppose that $A$ has entire purely discrete spectrum. Assume Hypotheses [H.1]-[H.4]. Then, $H(\lambda, \mu)$ has purely discrete spectrum in the interval $[E_0(H(\lambda, \mu)), E_0(H(\lambda, \mu)) + m]$. In particular, $H(\lambda, \mu)$ has a ground state.

Remark. This theorem has no restriction of the coupling parameter $\mu \geq 0$.

Remark. In the case $m > 0$, the condition [H.1] equivalent to the following:

$$g_j \in L^2(\mathbb{R}^d), \quad f_j \in D(\sqrt{\omega}), \quad j = 1, \ldots, J.$$  

For a vector $v = (v_1, \ldots, v_J) \in \mathbb{R}^J$ and $h = (h_1, \ldots, h_J) \in \bigoplus_{j=1}^J L^2(\mathbb{R}^d)$, we define

$$M_v(h) = \sum_{j=1}^J v_j \|h_j\|.$$  

We set

$$g = (g_1, \ldots, g_J) \in \bigoplus_{j=1}^J L^2(\mathbb{R}^d), \quad f = (f_1, \ldots, f_J) \in \bigoplus_{j=1}^J L^2(\mathbb{R}^d),$$

and

$$a = (a_1, \ldots, a_J), \quad b = (b_1, \ldots, b_J).$$

For $\theta$, $\epsilon$, $\epsilon'$, we introduce the following constants:

$$C_{\theta, \epsilon} := \theta M_a(g/\sqrt{\omega}) + \epsilon M_a(g),$$

$$D_{\theta, \epsilon'} := M_a(g/\sqrt{\omega})/2\theta + \epsilon' M_b(g/\sqrt{\omega}),$$

$$E_{\epsilon, \epsilon'} := M_a(g)/2\epsilon + a M_b(g/\sqrt{\omega})/2\epsilon' + b M_b(g)/\sqrt{2}.$$  

Let the condition [H.3] be satisfied. Then, we define

$$I_{\lambda, g} := \left\{ \begin{array}{ll} \frac{|\lambda| M_a(g/\sqrt{\omega})}{2}, & |\lambda| M_a(g/\sqrt{\omega}) \neq 0 \\ 1, & |\lambda| M_a(g/\sqrt{\omega}) = 0 \end{array} \right. \quad \text{for } |\lambda| M_a(g/\sqrt{\omega}) \neq 0$$

It is easy to see that $[1/2, 1] \subset I_{\lambda, g}$. Therefore, for all $\theta \in I_{\lambda, g}$,

$$1 - \theta |\lambda| M_a(g/\sqrt{\omega}) > 0,$$

$$1 - \frac{|\lambda| M_a(g/\sqrt{\omega})}{2\theta} > 0.$$
We define for \( \theta \in I_{\lambda,g} \),
\[
S_\theta := \{ (\epsilon, \epsilon') | |\epsilon|, |\epsilon'| > 0, |\lambda|C_{\theta,\epsilon} < 1, |\lambda|D_{\theta,\epsilon'} < 1 \}.
\]
Next we set
\[
\tau_{\theta,\epsilon,\epsilon'} := (1 - |\lambda|C_{\theta,\epsilon})\Sigma(A) - |\lambda|E_{\epsilon,\epsilon'},
\]
and
\[
T := \{ (\theta, \epsilon, \epsilon') \in \mathbb{R}^3 | \theta \in I_{\lambda,g}, (\epsilon, \epsilon') \in S_\theta, \tau_{\theta,\epsilon,\epsilon'} > E_0(H(\lambda, \mu)) \}.
\]

**Theorem 3.3.** Consider the case \( m > 0 \). Suppose that \( \sigma_{ess}(A) \neq \emptyset \). Assume Hypothesis \([\text{H.1}]-[\text{H.4}]\), and \( T \neq \emptyset \). Then, \( H(\lambda, \mu) \) has purely discrete spectrum in the interval
\[
[E_0(H(\lambda, \mu)), \min\{ m + E_0(H(\lambda, \mu)), \sup_{(\theta,\epsilon,\epsilon') \in T} \tau_{\theta,\epsilon,\epsilon'} \}].
\]

In particular, \( H(\lambda, \mu) \) has a ground state.

**Remark.** \( T \neq \emptyset \) is necessary condition for \( A \) to have a discrete ground state. Conversely, if \( A \) has a discrete ground state, then \( T \neq \emptyset \) holds for sufficiently small \( \lambda, \mu \). Therefore the condition \( T \neq \emptyset \) is a restriction for the coupling constants \( \lambda, \mu \).

* * *

3.1 Proof of Proposition 3.1

In what follows, we write simply
\[
H := H(\lambda, \mu).
\]

For \( D \) a dense subspace of \( L^2(\mathbb{R}^d) \), we define
\[
\mathcal{F}_{\text{fin}}(D) := L^1[\Omega, a(h_1) \cdots a(h_n)^* \Omega | n \in \mathbb{N}, h_j \in D, j = 1, \ldots, n],
\]
where \( \Omega := (1, 0, 0, \ldots) \) is the Fock vacuum in \( \mathcal{F}_h \). We introduce a dense subspace in \( \mathcal{F} \)
\[
\mathcal{D}_\omega := D(A) \hat{\otimes} \mathcal{F}_{\text{fin}}(D(\omega)),
\]
where \( \hat{\otimes} \) denotes algebraic tensor product. The subspace \( \mathcal{D}_\omega \) is a core of \( H_0 \).
Let 

\[ H_{\text{GSB}} := H_0 + \lambda H_1 \]

be a GSB Hamiltonian. The Hamiltonian \( H \) and \( H_{\text{GSB}} \) has the following relation:

**Proposition 3.4.** Let \( D(A) \subset D(B_j), j = 1, \ldots, J \) and \( f_j \in D(\omega^{1/2}) \). Assume that \( H_{\text{GSB}} \) is bounded from below. Then, for all \( \Psi \in D_{\omega} \),

\[
\| (H_{\text{GSB}} - E_0)\Psi \|^2 + \| \mu H_2 \Psi \|^2 \leq \| (H - E_0)\Psi \|^2 + D \| \Psi \|^2,
\]

where \( D = \mu \sum_{j=1}^J \| \omega^{1/2} f_j \|^2 \) and \( E_0 := \inf_{\Psi \in D(H_{\text{GSB}})} \langle \Psi, H_{\text{GSB}} \Psi \rangle \).

**Proof.** It is enough to show (5) the case \( \lambda = \mu = 1 \). First we consider the case where \( f_j \in D(\omega) \). Inequality (5) is equivalent to

\[
-2 \text{Re} \langle (H_{\text{GSB}} - E_0)\Psi, H_2 \Psi \rangle \leq D \| \Psi \|^2.
\]

By \( H_{\text{GSB}} - E_0 \geq 0 \), we have

\[
\langle (H_{\text{GSB}} - E_0)\Psi, I \otimes \phi(f_j)^2 \Psi \rangle = \langle [I \otimes \phi(f_j), (H_{\text{GSB}} - E_0)]\Psi, I \otimes \phi(f_j) \Psi \rangle + \langle (H_{\text{GSB}} - E_0) I \otimes \phi(f_j) \Psi, I \otimes \phi(f_j) \Psi \rangle \geq \langle [I \otimes \phi(f_j), H_{\text{GSB}} - E_0] \Psi, I \otimes \phi(f_j) \Psi \rangle.
\]

Therefore we have

\[
2 \text{Re} \langle (H_{\text{GSB}} - E_0)\Psi, \phi(f_j)^2 \Psi \rangle \geq -\| \sqrt{\omega} f_j \|^2 \| \Psi \|^2.
\]

This means inequality (6). Next, we set \( f_j \in D(\sqrt{\omega}) \). Then, there exists a sequence \( \{ f_{jn} \}_{n=0}^\infty \subset D(\omega) \) such that \( f_{jn} \to f_j, \omega^{1/2} f_{jn} \to \omega^{1/2} f_j \) \((n \to \infty)\).

By limiting argument, (6) holds with \( f_j \in D(\omega^{1/2}) \).

**Lemma 3.5.** Suppose that \( H_{\text{GSB}} \) is self-adjoint with \( D(H_{\text{GSB}}) = D(H_0) \), essentially self-adjoint on \( D_{\omega} \), and bounded from below. Let \( f_j \in D(\omega^{1/2}) \cap D(\omega^{-1/2}) \). Then \( H \) is self-adjoint with \( D(H) = D(H_0) \) and essentially self-adjoint on any core of \( H_{\text{GSB}} \) with

\[
\| (H_{\text{GSB}} - E_0)\Psi \|^2 + \| \mu H_2 \Psi \|^2 \leq \| (H - E_0)\Psi \|^2 + D \| \Psi \|^2, \quad \Psi \in D(H_0).
\]
Proof. It is well known that $D(H_b) \subset D(\phi(f_j)^2)$, and $\phi(f_j)^2$ is $H_b$-bounded (e.g. [1, Lemma 13-16]). Namely, there exist constants $\eta \geq 0$, $\theta \geq 0$ such that

$$\left\| \sum_{j=1}^{J} \phi(f_j)^2 \psi \right\| \leq \eta \|H_b\psi\| + \theta \|\psi\|, \quad \psi \in D(H_b).$$  \tag{7}$$

Since $H_{GSB}$ is self-adjoint on $D(H_0)$, by the closed graph theorem, we have

$$\|H_0\Psi\| \leq \lambda \|H_{GSB}\Psi\| + \nu \|\Psi\|, \quad \Psi \in D(H_0),$$  \tag{8}$$

where $\lambda$ and $\nu$ are non-negative constant independent of $\Psi$. Hence

$$\|H_2\Psi\| \leq \eta \lambda \|H_{GSB}\Psi\| + (\eta \nu + \theta) \|\Psi\|, \quad \Psi \in D(H_0).$$

We fix a positive number $\mu_0$ such that $\mu_0 < 1/(\mu \lambda)$. Then, by the Kato-Rellich theorem, $H(\lambda, \mu_0)$ is self-adjoint on $D(H_{GSB})$, bounded from below and essentially self-adjoint on any core of $H_{GSB}$. For a constant $a \in (0 < a < 1)$, we set $\mu_n := (1 + a)^n \mu_0$. Since $H_{GSB}$ is self-adjoint on $D(H_0)$, for each $j = 1, \ldots, J$ we have $D(A) \subset D(B)$. Thus by Proposition 3.4, for all $\Psi \in D_\omega$

$$\|(H_{GSB} - E_0)\Psi\|^2 + \|\mu_n H_2 \Psi\|^2 \leq \|(H(\lambda, \mu_n) - E_0)\Psi\|^2 + D\|\Psi\|^2.$$

If $H(\lambda, \mu_n)$ is self-adjoint on $D(H_{GSB})$, bounded from below and essentially self-adjoint on any core of $H_{GSB}$, then $H(\lambda, \mu_{n+1})$ has the same property. On the other hand, we have $\mu_n \to \infty$ ($n \to \infty$). Hence we conclude that $H$ is self-adjoint with $D(H) = D(H_{GSB})$, bounded from below and essentially self-adjoint on any core of $H_{GSB}$. \hfill \Box

Now, we assume conditions [H.1],[H.2] and [H.3].

Then $H_{GSB}$ is self-adjoint on $D(H_0)$, bounded from below and essentially self-adjoint on any core of $H_0$ (see [2]). Hence, the assumptions of Lemma 3.5 hold. Thus Proposition 3.1 follows. \hfill \Box

3.2 Proofs of Theorems 3.2 and 3.3

Throughout this subsection, we assume Hypotheses [H.1]-[H.4] and $m > 0$. 

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For a parameter $V > 0$, we define the set of lattice points by
\[
\Gamma_V := \frac{2\pi \mathbb{Z}^d}{V} := \left\{ k = (k_1, \ldots, k_d) \middle| k_j = \frac{2\pi n_j}{V}, n_j \in \mathbb{Z}, j = 1, \ldots, d \right\}
\]
and we denote by $l^2(\Gamma_V)$ the set of $l^2$ sequences over $\Gamma_V$. For each $k \in \Gamma_V$ we introduce
\[
C(k, V) := [k_1 - \frac{\pi}{V}, k_1 + \frac{\pi}{V}) \times \cdots \times [k_d - \frac{\pi}{V}, k_d + \frac{\pi}{V}) \subset \mathbb{R}^d,
\]
the cube centered about $k$. By the map
\[
U : l^2(\Gamma_V) \ni \{h_l\}_{l \in \Gamma_V} \mapsto (V/2\pi)^{d/2} \sum_{l \in \Gamma_V} h_l \chi_{l,V}(\cdot) \in L^2(\mathbb{R}^d),
\]
we identify $l^2(\Gamma_V)$ with a subspace in $L^2(\mathbb{R}^d)$, where $\chi_{l,V}(\cdot)$ is the characteristic function of the cube $C(l, V) \subset \mathbb{R}^d$. It is easy to see that $l^2(\Gamma_V)$ is a closed subspace of $L^2(\mathbb{R}^d)$. Let
\[
\mathcal{F}_{b,V} := \mathcal{F}_b(l^2(\Gamma_V)) = \bigoplus_{n=0}^{\infty} \bigotimes_{s}^n l^2(\Gamma_V),
\]
the boson Fock space over $l^2(\Gamma_V)$. We can identify $\mathcal{F}_{b,V}$ the closed subspace of $\mathcal{F}_b$ by the operator $\Gamma(U) := \bigoplus_{n=0}^{\infty} \bigotimes^n U$, where we define $\bigotimes^0 U = 0$. For each $k \in \mathbb{R}^d$, there exists a unique point $k_V \in \Gamma_V$ such that $k \in C(k_V, V)$. Let
\[
\omega_V(k) := \omega(k_V), \quad k \in \mathbb{R}^d
\]
be a lattice approximate function of $\omega(k)$ and let
\[
H_{b,V} := d\Gamma(\omega_V)
\]
be the second quantization of $\omega_V$. We define a constant
\[
C_V := C d^\gamma \left( \frac{\pi}{V} \right) \left( \frac{1}{2m} + 1 \right),
\]
where $C$ and $\gamma$ were defined in [H.4]. In what follows we assume that
\[
C_V < 1.
\]
This is satisfied for all sufficiently large $V$. 

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Lemma 3.6. (Lemma 3.1). We have

\[ D(H_{b,V}) = D(H_b), \]

and

\[ \| (H_b - H_{b,V}) \Psi \| = \frac{2C_V}{1 - C_V} \| H_b \Psi \|, \quad \Psi \in D(H_b). \]

First we consider the case where \( g_j \)'s and \( f_j \)'s are continuous, and finally, by limiting argument, we treat a general case. For a constant \( K > 0 \), we define \( g_{j,K}, f_{j,K}, g_{j,K,V}, f_{j,K,V} \) as follows:

\[
g_{j,K}(k) := \chi_K(k_1) \cdots \chi_K(k_d)g_j(k), \quad g_{j,K,V}(k) := \sum_{\ell \in \Gamma_V, |\ell_i| < K} g_j(\ell)\chi_{\ell,V}(k),
\]

\[
f_{j,K}(k) := \chi_K(k_1) \cdots \chi_K(k_d)f_j(k), \quad f_{j,K,V}(k) := \sum_{\ell \in \Gamma_V, |\ell_i| < K} f_j(\ell)\chi_{\ell,V}(k),
\]

where \( \chi_K \) denotes the characteristic function of \([-K, K]\).

Lemma 3.7. For all \( j = 1, \ldots, J \),

\[
\lim_{V \to \infty} \| g_{j,K,V} - g_{j,K} \| = 0, \quad \lim_{V \to \infty} \| g_{j,K,V}/\sqrt{\omega_V} - g_{j,K}/\sqrt{\omega} \| = 0,
\]

\[
\lim_{K \to \infty} \| g_{j,K} - g_j \| = 0, \quad \lim_{K \to \infty} \| g_{j,K}/\sqrt{\omega} - g_j/\sqrt{\omega} \| = 0,
\]

\[
\lim_{V \to \infty} \| f_{j,K,V} - f_{j,K} \| = 0, \quad \lim_{V \to \infty} \| f_{j,K,V}/\sqrt{\omega_V} - f_{j,K}/\sqrt{\omega} \| = 0,
\]

\[
\lim_{K \to \infty} \| f_{j,K} - f_j \| = 0, \quad \lim_{K \to \infty} \| f_{j,K}/\sqrt{\omega} - f_j/\sqrt{\omega} \| = 0,
\]

\[
\lim_{K \to \infty} \| \sqrt{\omega}f_{j,K} - \sqrt{\omega}f_j \| = 0, \quad \lim_{V \to \infty} \| \sqrt{\omega_V} f_{j,K,V} - \sqrt{\omega} f_{j,K,V} \| = 0.
\]

Proof. Similar to the proof of Lemma 3.10.

We introduce a new operator:

\[
H_{0,V} := A \otimes I + I \otimes H_{b,V},
\]

\[
H_{1,K} := \sum_{j=1}^{J} B_j \otimes \phi(g_{j,K}),
\]

\[
H_{1,K,V} := \sum_{j=1}^{J} B_j \otimes \phi(g_{j,K,V}),
\]

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and define
\[ H_K := H_0 + \lambda H_{1,K} + \mu H_{2,K}, \]
\[ H_{K,V} := H_{0,V} + \lambda H_{1,K,V} + \mu H_{2,K,V}. \]

**Lemma 3.8.**
(i) \( H_K \) is self-adjoint with \( D(H_K) = D(H_0) \subseteq D(H_{1,K}) \cap D(H_{2,K}) \), bounded from below, and essentially self-adjoint on any core of \( H_0 \).
(ii) For all large \( V \), \( H_{K,V} \) is self-adjoint with \( D(H_{K,V}) = D(H_0) \subseteq D(H_{1,K,V}) \cap D(H_{2,K,V}) \), bounded from below, and essentially self-adjoint on any core of \( H_{0,V} \).

**Proof.** Similar to the proof of Proposition 3.1.

**Lemma 3.9.** For all \( z \in \mathbb{C} \setminus \mathbb{R} \), and \( K > 0 \),
\[ \lim_{K \to \infty} \| (H_K - z)^{-1} - (H - z)^{-1} \| = 0, \]
\[ \lim_{V \to \infty} \| (H_{K,V} - z)^{-1} - (H_K - z)^{-1} \| = 0. \]

**Proof.** Similar to the proof of [2, Lemma 3.5]

The following fact is well known:

**Lemma 3.10.** The operator \( H_{b,V} \) is reduced by \( \mathcal{F}_{b,V} \) and \( H_{b,V} \mathcal{F}_{b,V} \) equal to the second quantization of \( \omega_V |t^2(\Gamma_V) \) on \( \mathcal{F}_{b,V} \).

**Lemma 3.11.** \( H_{K,V} \) is reduced by \( \mathcal{F}_V \).

**Proof.** Similar to the proof of [2, Lemma 3.7]

**Lemma 3.12.** We have
\[ H_{K,V} \mathcal{F}_V \geq E_0(H_{K,V}) + m. \]
The following operator inequalities hold:

**Lemma 3.14.** Proof. Similar to the calculation of $2$, Lemma 3.10.

**Lemma 3.13.** Let $T_n$ and $T$ be a self-adjoint operators on a separable Hilbert space and bounded from below. Suppose that $T_n \to T$ in norm resolvent sense as $n \to \infty$ and $T_n$ has purely discrete spectrum in the interval $[E_0(T_n), E_0(T_n) + c_n]$ with some constant $c_n$. If $\epsilon := \limsup_{n \to \infty} c_n > 0$, then $T$ has purely discrete spectrum in $[E_0(T), E_0(T) + \epsilon]$.

**Proof.** There exists a sequence $\{c_n\}_{n=1}^{\infty} \subset \{c_n\}_{n=1}^{\infty}$ so that $c_{n_j} \to c(j \to \infty)$. So, for all $\epsilon > 0$ and for sufficiently large $j$, the spectrum of $T_{n_j}$ in $[E_0(T_{n_j}), E_0(T_{n_j}) + \epsilon]$, $\epsilon$ is discrete. Therefore, applying $[2$, Lemma 3.12$]$ we find that the spectrum of $T$ in $[E_0(T), E_0(T) + \epsilon]$ is discrete. Since $\epsilon > 0$ is arbitrary, we get the conclusion.

Now, if $A$ has compact resolvent, by a method similar to the proof of $[2$, Theorem 1.2$]$ we can prove Theorem 3.2. Therefore, we only prove Theorem 3.3.

The following inequality is known $[2$, (2.12)$]$

$$|\langle \Psi, H_1 \Psi \rangle| \leq C_{\theta, \epsilon}(\Psi, A \otimes I \Psi) + D_{\theta, \epsilon}(\Psi, I \otimes H_b(\Psi)) + E_{\epsilon, \epsilon'}||\Psi||^2,$$

where $\Psi \in D(H_0)$ is arbitrary. Thus we have,

$$H \geq (1 - |\lambda|C_{\theta, \epsilon})A \otimes I + (1 - |\lambda|D_{\theta, \epsilon})I \otimes H_b + \mu H_2 - |\lambda|E_{\epsilon, \epsilon'}.$$

Let $I_{\lambda, \xi}(K)$, $C_{\theta, \epsilon}(K)$, $D_{\theta, \epsilon}(K)$ and $E_{\epsilon, \epsilon'}(K)$ are $I_{\lambda, \xi}$, $C_{\theta, \epsilon}$, $D_{\theta, \epsilon}$, $E_{\epsilon, \epsilon'}$ with $g_j$, $f_j$ replaced by $g_j, K$, $f_j, K$ respectively, and let $I_{\lambda, \xi}(K, V)$, $C_{\theta, \epsilon}(K, V)$, $D_{\theta, \epsilon}(K, V)$ and $E_{\epsilon, \epsilon'}(K, V)$ are $I_{\lambda, \xi}$, $C_{\theta, \epsilon}$, $D_{\theta, \epsilon}$, $E_{\epsilon, \epsilon'}$ with $g_j$, $f_j$ and $\omega$ replaced by $g_j, K, V$, $f_j, K, V$ and $\omega, V$ respectively. Then we have

**Lemma 3.14.** The following operator inequalities hold:

$$H_K \geq (1 - |\lambda|C_{\theta, \epsilon}(K))A \otimes I + (1 - |\lambda|D_{\theta, \epsilon}(K))I \otimes H_b$$

$$+ \mu H_2 - |\lambda|E_{\epsilon, \epsilon'}(K) \text{ on } D(H_0),$$

$$H_{K, V} \geq (1 - |\lambda|C_{\theta, \epsilon}(K, V))A \otimes I + (1 - |\lambda|D_{\theta, \epsilon}(K, K))I \otimes H_b, V$$

$$+ \mu H_2 - |\lambda|E_{\epsilon, \epsilon'}(K, V) \text{ on } D(H_0).$$

**Proof.** Similar to the calculation of $[2$, (2.12)$]$. 

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By Lemma 3.7, we have
\[
\lim_{V \to \infty} C_{\theta, \epsilon}(K, V) = C_{\theta, \epsilon}(K), \quad \lim_{K \to \infty} C_{\theta, \epsilon}(K) = C_{\theta, \epsilon}, \quad (9)
\]
\[
\lim_{V \to \infty} D_{\theta, \epsilon'}(K, V) = D_{\theta, \epsilon'}(K), \quad \lim_{K \to \infty} D_{\theta, \epsilon'}(K) = D_{\theta, \epsilon'}, \quad (10)
\]
\[
\lim_{V \to \infty} E_{\epsilon, \epsilon'}(K, V) = E_{\epsilon, \epsilon'}(K), \quad \lim_{K \to \infty} E_{\epsilon, \epsilon'}(K) = E_{\epsilon, \epsilon'}. \quad (11)
\]

Let \((\theta, \epsilon, \epsilon') \in T\), namely
\[
\tau_{\theta, \epsilon, \epsilon'} = (1 - |\lambda|C_{\theta, \epsilon})\Sigma(A) - |\lambda|E_{\epsilon, \epsilon'} > E_0(H).
\]

Formulas (9)-(11) and Lemma 3.9 imply that for all large \(V\) there exists a constant \(K_0 > 0\) such that for all \(K > K_0\),
\[
(1 - |\lambda|C_{\theta, \epsilon}(K, V))\Sigma(A) - |\lambda|E_{\epsilon, \epsilon'}(K, V) > E_0(H_{K, V}), \quad (12)
\]
\[
|\lambda|C_{\theta, \epsilon}(K, V) < 1, \quad |\lambda|D_{\theta, \epsilon'}(K, V) < 1. \quad (13)
\]

By Lemma 3.11, \(H_{K, V}\) is reduced by \(F_V\). Therefore, \(H_{K, V}\) satisfies the following inequality:
\[
H_{K, V}[F_V \geq (1 - |\lambda|C_{\theta, \epsilon}(K, V))A \otimes I[F_V + (1 - |\lambda|D_{\theta, \epsilon'}(K, V))I \otimes H_{b, V}[F_V
\]
\[
- |\lambda|E_{\epsilon, \epsilon'}(K, V). \quad (14)
\]

Since \(H_{b, V}[F_{V}\) has compact resolvent, the bottom of essential spectrum of the right hand side of (14) is equal to
\[
(1 - |\lambda|C_{\theta, \epsilon}(K, V))\Sigma(A) - |\lambda|E_{\epsilon, \epsilon'}(K, V).
\]

By Lemma 3.12, we have \(E_0(H_{K, V}[F_V] = E_0(H_{K, V}). \) Thus, applying Theorem 2.1 with \(H_{K, V}[F_V, \) we have that \(H_{K, V}[F_V\) has purely discrete spectrum in \([E_0(H_{K, V}), (1 - |\lambda|C_{\theta, \epsilon}(K, V))\Sigma(A) - E_{\epsilon, \epsilon'}(K, V)).\) Since this fact and Lemma 3.12, \(H_{K, V}\) has purely discrete spectrum in
\[
[E_0(H_{K, V}), \min\{E_0(H_{K, V}) + m, (1 - |\lambda|C_{\theta, \epsilon}(K, V))\Sigma(A) - E_{\epsilon, \epsilon'}(K, V)).\]

By Lemma 3.9 and Lemma 3.13, we have that for all sufficiently large \(K > 0, H_K\) has purely discrete spectrum in \([E_0(H_K), \min\{E_0(H_K) + m, (1 - |\lambda|C_{\theta, \epsilon}(K))\Sigma(A) - |\lambda|E_{\epsilon, \epsilon'}(K))\}].\) Similarly, \(H\) has purely discrete spectrum in \([E_0(H(\lambda, \mu)), \min\{m + E_0(H(\lambda, \mu)), \tau_{\theta, \epsilon, \epsilon'})\}).\) Since \((\theta, \epsilon, \epsilon') \in T\) is arbitrary, \(H\) has purely discrete spectrum in (4). Finally, we have to consider the case where \(g_j's\) and \(f_j's\) are not necessarily continuous. But, that argument were already discussed in [4.] So we skip that argument. \(\blacksquare\)
4 Ground State of the Dereziński-Gérard Model

We consider a model discussed by J. Dereziński and C. Gérard. We take the Hilbert space of the particle system is taken to be

$$\mathcal{H} = L^2(\mathbb{R}^N).$$

The Hilbert space for the Dereziński-Gérard (DG) model is given by

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_b(\mathbb{L}^2(\mathbb{R}^d)).$$

We identify $$\mathcal{F}$$ as

$$\bigoplus_{n=0}^{\infty} \mathcal{H} \otimes \bigotimes_{s}^{n} \mathbb{L}^2(\mathbb{R}^d).$$

Hence, if we denote that $$\Psi \in (\Psi^{(n)})_{n=0}^{\infty} \in \mathcal{F}$$, each $$\Psi^{(n)}$$ belongs to $$\mathcal{H} \otimes [\bigotimes_{s}^{n} \mathbb{L}^2(\mathbb{R}^d)]$$. We denote by $$\mathcal{B}(\mathcal{K}, \mathcal{J})$$ the set of bounded linear operators from $$\mathcal{K}$$ to $$\mathcal{J}$$. For $$v \in \mathcal{B}(\mathcal{H}, \mathcal{H} \otimes \mathbb{L}^2(\mathbb{R}^d))$$, we define an operator $$\tilde{a}^*(v)$$ by

$$\langle \tilde{a}^*(v) \Psi \rangle^{(0)} := 0,$$

$$\langle \tilde{a}^*(v) \Psi \rangle^{(n)} := \sqrt{n} (I_{\mathcal{H}} \otimes S_n)(v \otimes I_{\bigotimes_{s}^{n-1} \mathbb{L}^2(\mathbb{R}^d)}) \Psi^{(n-1)}, \quad (n \geq 1),$$

$$\Psi \in D(\tilde{a}^*(v)) := \left\{ \Psi = (\Psi^{(n)})_{n=0}^{\infty} \in \mathcal{F} \bigg| \sum_{n=0}^{\infty} \| (\tilde{a}^*(v) \Psi)^{(n)} \|^2 < \infty \right\}.$$

We set

$$D_0 := \left\{ \Psi = (\Psi^{(n)})_{n=0}^{\infty} \in \mathcal{F} \big| \text{there exists a constant } n_0 \in \mathbb{N}, \text{ such that, for all } n \geq n_0, \Psi^{(n)} = 0 \right\}.$$

Throughout this section, we write simply $$I_n := I_{\bigotimes_{s}^{n} \mathbb{L}^2(\mathbb{R}^d)}$$. It is easy to see that:

**Proposition 4.1.** $$\tilde{a}^*(v)$$ is a closed linear operator and $$D_0$$ is a core of $$\tilde{a}^*(v).$$

So we set

$$\tilde{a}(v) := (\tilde{a}^*(v))^*$$

the adjoint operator of $$\tilde{a}^*(v).$$
Proposition 4.2. The operator \( \tilde{a}(v) \) has the following properties:

\[
D(\tilde{a}(v)) = \left\{ \Psi = (\Psi^{(n)})_{n=0}^{\infty} \bigg| \sum_{n=0}^{\infty} (n+1)\|(I_{H} \otimes S_{n})(v^{*} \otimes I_{n})\Psi^{(n+1)}\|^{2} < \infty \right\}
\]

(15)

\[
(\tilde{a}(v)\Psi)^{(n)} = \sqrt{n + 1}I_{H} \otimes S_{n}(v^{*} \otimes I_{n})\Psi^{(n+1)}, \quad \Psi \in D(\tilde{a}(v)),
\]

(16)

and \( D_{0} \) is a core of \( \tilde{a}(v) \).

Proof. For \( \Phi \in \mathcal{F}, \Psi \in D(\tilde{a}^{*}(v)) \),

\[
\langle \Phi, \tilde{a}^{*}(v)\Psi \rangle = \sum_{n=1}^{\infty} \langle \Phi^{(n)}, \sqrt{n}(I_{H} \otimes S_{n})(v \otimes I_{n-1})\Psi^{(n-1)} \rangle
\]

\[
= \sum_{n=0}^{\infty} \sqrt{n + 1}\langle v^{*} \otimes I_{n}\Phi^{(n+1)}, \Psi^{(n)} \rangle
\]

\[
= \sum_{n=0}^{\infty} \langle \sqrt{n + 1}(I_{H} \otimes S_{n})(v^{*} \otimes I_{n})\Phi^{(n+1)}, \Phi^{(n)} \rangle.
\]

This implies (15) and (16). It is easy to prove that \( D_{0} \) is a core of \( \tilde{a}(v) \). \qed

An analogue of the Segal field operator is defined by

\[
\tilde{\phi}(v) := \frac{1}{\sqrt{2}}(\tilde{a}(v) + \tilde{a}^{*}(v)).
\]

Let \( A \) be a non-negative self-adjoint operator on \( H \) with \( E_{0}(A) = 0 \). Then the Hamiltonian of the DG model is defined by

\[
H_{DG} := A \otimes I + I \otimes H_{b} + \tilde{\phi}(v).
\]

We call it the Dereziński-Gérard Hamiltonian. Here \( H_{b} \) is the second quantization of \( \omega \) introduce in Section 3. Let

\[
H_{0} := A \otimes I + I \otimes H_{b}.
\]

Throughout this section we assume the following conditions:

[DG.1] There is a Borel measurable function \( v(x,k) \in \mathbb{C}, (x \in \mathbb{R}^{N}, k \in \mathbb{R}^{d}) \), such that

\[
(vf)(x,k) = v(x,k)f(x), \quad f \in L^{2}(\mathbb{R}^{d}).
\]
We need also the following assumption:

\[ \text{DG.2} \]
\[
\text{ess sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} \left| \frac{v(x, k)}{\sqrt{\omega(k)}} \right|^2 \, dk < \infty.
\]

**Proposition 4.3.** Assume [DG.1] and [DG.2]. Then \( H_{DG} \) is self-adjoint with \( D(H_{DG}) = D(H_0) \), and essentially self-adjoint on any core of \( H_0 \).

For a finite volume approximation, we introduce the following hypotheses:

\[ \text{DG.3} \]
There exists a nonnegative function \( \tilde{v} \in L^2(\mathbb{R}^d) \) and function \( \tilde{o} : \mathbb{R} \to \mathbb{R} \), such that
\[
\text{ess sup}_{x \in \mathbb{R}^n} \left| v(x, k) - v(x, \ell) \right| \leq \tilde{v}(k)\tilde{o}(|k - \ell|), \quad \text{a.e. } k, \ell \in \mathbb{R}^d
\]
\[
\lim_{t \to 0} \tilde{o}(t) = 0.
\]

\[ \text{DG.4} \]
\[
\text{ess sup}_{x \in \mathbb{R}^n} \int_{([-K,K]^d)^c} |v(x, k)|^2 \, dk = o(K^0).
\]

where
\[
([-K,K]^d)^c := \mathbb{R}^d \setminus (I \times \cdots \times I), \quad I := [-K, K]
\]
and, \( o(t^0) \) satisfies \( \lim_{t \to 0} o(t^0) = 0 \).

Let \( m \) be defined by (3). Let
\[
D := \frac{1}{2} \inf_{0 < \epsilon' < \frac{\|v\|}{\|v/\sqrt{\omega}\|}} \left( \epsilon' + \frac{1}{\epsilon'} \right).
\]
(17)

Here, \( v/\sqrt{\omega} \) is a multiplication operator by the function \( v(x, k)/\sqrt{\omega(k)} \) from \( L^2(\mathbb{R}^N) \) to \( L^2(\mathbb{R}^N) \otimes L^2(\mathbb{R}^d) \). In the case \( m > 0 \), we can establish the existence of a ground state of \( H_{DG} \):

**Theorem 4.4.** Let \( m > 0 \). Suppose that [DG.1]-[DG.4] and [H.4] hold, and suppose
\[
\Sigma(A) - \|v\|D - E_0(H_{DG}) > 0.
\]
Then, $H_{DG}$ has purely discrete spectrum in

$$[E_0(H_{DG}), \min\{E_0(H_{DG}) + m, \Sigma(A) - \|v\|D\}].$$

In particular $H_{DG}$ has a ground state.

Remark. In the case where $A$ has compact resolvent, this theorem has been proved in [5]. A new aspect here is in that $A$ does not necessarily have compact resolvent. Also our method is different from that in [5].

4.1 Proof of Proposition 4.3

Lemma 4.5. Let

$$M(x) = \left(\int_{\mathbb{R}^d} |v(x, k)|^2 dk\right)^{1/2}, \quad x \in \mathbb{R}^N$$

and $M : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ be a multiplication operator by the function $M(x)$. Then

$$\|vf\| = \|Mf\|, \quad f \in L^2(\mathbb{R}^N).$$

In particular, $\|v\| = \|M\| = (\text{ess. sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} |v(x, k)|^2 dk)^{1/2}$ hold.

Proof. By the Fubini’s theorem, we have

$$\|vf\|^2 = \int_{\mathbb{R}^d} dk \int_{\mathbb{R}^N} dx |v(x, k)|^2 |f(x)|^2 = \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^d} |f(x)|^2 \int_{\mathbb{R}^d} |v(x, k)|^2 dk \right) dx.$$

This means the result.

The adjoint $v^*$ has the following form:

Lemma 4.6. For all $g \in \mathcal{H} \otimes L^2(\mathbb{R}^d)$,

$$(v^*g)(x) = \int_{\mathbb{R}^d} v(x, k)^* g(x, k) dk, \quad \text{a.e.} \ x \in \mathbb{R}^d. \quad (18)$$

Proof. For all $f \in \mathcal{H}$, we have

$$\langle g, vf \rangle = \int dx \int dk g(x, k)^* v(x, k) f(x) = \int dx \left( \int g(x, k)^* v(x, k) dk \right) f(x).$$

Since $f$ is arbitrary, this proves (18).
Lemma 4.7. \( \tilde{a}(v) \) is
\[
D(\tilde{a}(v)) = \left\{ \Psi \in \mathcal{F} \left| \sum_{n=0}^{\infty} (n+1) \int_{\mathbb{R}^{d+n}} dx dk_1 \cdots dk_n \right. \right.
\]
\[ \left. \left| \int_{\mathbb{R}^d} dk v(k,x)^* \Psi^{(n+1)}(x, k, k_1, \ldots, k_n) \right|^2 < \infty \right\}
\]
\[
(\tilde{a}(v)\Psi)^{(n)}(x, k_1, \ldots, k_n) = \sqrt{n+1} \int_{\mathbb{R}^d} v(x, k)^* \Psi^{(n+1)}(x, k, k_1, \ldots, k_n), \quad \text{a.e.} \quad (\Psi \in D(\tilde{a}(v)))
\]

Proof. Using Lemma 4.6, we have
\[
(v^* \otimes I_n)\Psi^{(n+1)}(x, k_1, \ldots, k_n) = \int_{\mathbb{R}^d} v^*(x, k)\Psi^{(n+1)}(x, k, k_1, \ldots, k_n) dk.
\]
This is invariant for all permutations of \( k_1, \ldots, k_n \). Therefore, using Proposition 4.2, we get
\[
(\tilde{a}(v)\Psi)^{(n)}(x, k_1, \ldots, k_n) = \sqrt{n+1} \int_{\mathbb{R}^d} v(x, k)^* \Psi^{(n+1)}(x, k, k_1, \ldots, k_n) dk.
\]

Lemma 4.8. Suppose that [DG.1] and [DG.2] hold. Then, \( D(\tilde{a}(v)) \supset D(I \otimes H^{1/2}_b) \) and
\[
\|\tilde{a}(v)\Phi\| \leq \|v/\sqrt{\omega}\|_2 \|I \otimes H^{1/2}_b\|, \quad \Phi \in D(I \otimes H^{1/2}_b).
\]

Proof. By (19), we have for all \( \Phi \in D(\tilde{a}(v)) \)
\[
\| (\tilde{a}(v)\Phi)^{(n)} \|^2 = (n+1) \int_{\mathbb{R}^{d+n}} dx dk_1 \cdots dk_n \left| \int_{\mathbb{R}^d} \frac{1}{\sqrt{\omega(k)}} v(x, k)^* \Phi^{(n+1)}(x, k, k_1, \ldots, k_n) dk \right|^2.
\]
Using the Schwarz inequality, one has
\[
\left| \int_{\mathbb{R}^d} \frac{1}{\sqrt{\omega(k)}} v(x, k)^* \Phi^{(n+1)}(x, k, k_1, \ldots, k_n) dk \right|^2 \leq \int_{\mathbb{R}^d} \frac{|v(x, k)^*|^2}{\sqrt{\omega(k)}} dk \cdot \int_{\mathbb{R}^d} \omega(k)|\Phi^{(n+1)}(x, k, k_1, \ldots, k_n)|^2 dk.
\]
Hence, for every $\Phi \in D_0 \cap D(I \otimes H_b^{1/2})$, we have

$$
\| (\tilde{a}(v)\Phi)^{(n)} \|^2 \\
\leq \left( \text{ess.sup} \int_{\mathbb{R}^d} \left| \frac{v(x,k)^*}{\sqrt{\omega(k)}} \right|^2 dk \right) (n+1) \times \\
\int_{\mathbb{R}^{d+n+N}} dx dk_1 \cdots dk_{n} dk \omega(k) |\Phi^{(n+1)}(x,k,k_1,\ldots,k_n)|^2 \\
= \left( \text{ess.sup} \int_{\mathbb{R}^d} \left| \frac{v(x,k)^*}{\sqrt{\omega(k)}} \right|^2 dk \right) \times \\
\int_{\mathbb{R}^{d+n+N}} dx dk_1 \cdots dk_{n+1} \sum_{j=1}^{n+1} \omega(k_j) |\Phi^{(n+1)}(x,k_1,\ldots,k_{n+1})|^2 \\
= \left\| \frac{v}{\sqrt{\omega}} \right\| \left\| (I \otimes H_b^{1/2}\Phi)^{(n+1)} \right\|^2.
$$

Therefore

$$
\| \tilde{a}(v)\Phi \| \leq \left\| \frac{v}{\sqrt{\omega}} \right\| \left\| (I \otimes H_b^{1/2}\Phi) \right\|^2.
$$

Since, $D_0 \cap D(I \otimes H_b^{1/2})$ is a core of $I \otimes H_b^{1/2}$, one can extend this inequality to all $\Phi \in D(I \otimes H_b^{1/2})$, and $D(I \otimes H_b^{1/2}) \subset D(\tilde{a}(v))$ holds.

**Lemma 4.9.** On $D_0$, $\tilde{a}(v)$ and $\tilde{a}^*(v)$ satisfy the following commutation relation:

$$
[\tilde{a}(v), \tilde{a}(v)^*] = \int_{\mathbb{R}^d} |v(\cdot,k)|^2 dk.
$$

where the right hand side is a multiplication operator by the function $x \mapsto \int_{\mathbb{R}^d} |v(x,k)|^2 dk$.

**Proof.** Let $\Phi \in D_0$. By the definition of $\tilde{a}^*(v)$, and using Proposition 4.2, we get

$$
([\tilde{a}^*(v), \tilde{a}(v)]\Phi)^{(n)} = (\tilde{a}(v)\tilde{a}(v)^*\Phi)^{(n)} - (\tilde{a}(v)^*\tilde{a}(v)\Phi)^{(n)} \\
= \sqrt{n+1} I_{\mathcal{H}} \otimes S_n(v^* \otimes I_n)(\tilde{a}(v)^*\Phi)^{(n+1)} \\
- \sqrt{n}(I \otimes S_n)(v \otimes I_{n-1})(\tilde{a}(v)\Phi)^{(n-1)}.
$$

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Hence, we have
\[
([\tilde{a}^*(v), \tilde{a}(v)]\Phi)^{(n)}(x, k_1, \ldots, k_n) = (n + 1) \int_{\mathbb{R}^d} v(x, k)^*(I \otimes S_{n+1}(v \otimes I_{n-1})\Phi^{(n)})(x, k, k_1, \ldots, k_n) \, dk 
- \frac{1}{n} \sum_{j=1}^{n} v(x, k_j)(v^* \otimes I_{n-1}\Phi^{(n)})(x, k_1, \ldots, \hat{k}_j, \ldots, k_n) 
= \int_{\mathbb{R}^d} \, \text{dk} \, v(x, k)^*(v(x, k)\Phi^{(n)}(x, k, k_1, \ldots, k_n) 
+ \sum_{j=1}^{n} \, \text{dk} \, \Phi^{(n)}(x, k, k_1, \ldots, \hat{k}_j, \ldots, k_n) 
- \sum_{j=1}^{n} \, \text{dk} \, \Phi^{(n)}(x, k, k_1, \ldots, \hat{k}_j, \ldots, k_n) 
= \left( \int_{\mathbb{R}^d} |v(x, k)|^2 \right) \Phi(x, k_1, \ldots, k_n).
\]

Here ‘\(\hat{\cdot}\)’ indicates the omission of the object wearing the hat.

Lemma 4.10. Assume, [DG.1] and [DG.2]. Then \(D(I \otimes H^{1/2}_b) \subset D(\tilde{a}^*(v))\) and for all \(\Phi \in D(I \otimes H^{1/2}_b)\),
\[
\|\tilde{a}^*(v)\Phi\|^2 \leq \|v/\sqrt{\omega}\|^2\|I \otimes H^{1/2}_b\Phi\|^2 + \|v\|^2\|\Phi\|^2. 
\] (20)

Proof. For all \(\Phi \in D_0 \cap D(I \otimes H^{1/2}_b)\), we have
\[
\|\tilde{a}^*(v)\Phi\|^2 = \langle \Phi, \tilde{a}(v)\tilde{a}^*(v)\Phi \rangle = \langle \Phi, \tilde{a}^*(v)\tilde{a}(v)\Phi \rangle + \left( \int_{\mathbb{R}^d} |v(\cdot, k)|^2 \right) \Phi, \Phi \rangle 
\leq \|\tilde{a}(v)\Phi\|^2 + \|v\|^2\|\Phi\|^2.
\]
Thus we can apply Lemma 4.8 to obtain the result.

Proof of Proposition 4.3. By Lemma 4.8 and 4.10, the operator \(\tilde{\phi}(v)\) is \(I \otimes H^{1/2}_b\)-bounded. Hence \(\tilde{\phi}(v)\) is infinitesimally small with respect to \(I \otimes H_b\). Namely, for all \(\epsilon > 0\), there exists a constant \(c_\epsilon > 0\), such that,
\[
\|\tilde{\phi}(v)\Phi\| \leq \epsilon\|I \otimes H_b\Phi\| + c_\epsilon\|\Phi\|, \quad \Phi \in D(I \otimes H_b).
\]
Since \( A \geq 0 \), we have
\[
\| \tilde{\phi}(v) \| \leq \epsilon \| H_0 \| + c \| \Phi \|, \quad \Phi \in D(H_0).
\]
Thus we can apply the Kato-Rellich theorem to obtain the conclusion of Proposition 4.3.

### 4.2 Proof of Theorem 4.4

In this subsection we suppose that the assumption of Theorem 4.4 holds.

Let \( F b, V, \omega V, H b, V, H_0, V, F V, \Gamma V, \chi_{\ell V}(k) \) be an object already defined in Section 3, respectively. Suppose that \( \chi_K \) is a characteristic function of \([-K, K]\).

For a parameter \( K > 0 \), we define \( v_K \in B(H, H \otimes L^2(\mathbb{R}^d)) \) by
\[
(v_K f)(x, k) := \chi_{[-K,K]}(k)v(x, k)f(x).
\]
and \( v_{K,V} \in B(H, H \otimes L^2(\mathbb{R}^d)) \) by
\[
(v_{K,V} f)(x, k) := \sum_{\ell \in \Gamma V, |\ell| < K} \chi_{\ell,V}(k)v(x, \ell)f(x).
\]

**Lemma 4.11.** The following hold:
\[
\|v_K - v_{K,V}\| \to 0 \quad (V \to \infty), \quad \|v_K - v\| \to 0 \quad (K \to \infty). \quad (21)
\]
\[
\|v_K - v_{K,V}\| \to 0 \quad (V \to \infty), \quad \|v_K - v_{K,V}\| \to 0 \quad (K \to \infty). \quad (22)
\]

**Proof.** By [DG.3] and [DG.4], we have
\[
\|v_K - v_{K,V}\|^2 = \text{ess sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} \left| \chi_K(k)v(x, k) - \sum_{\ell \in \Gamma V, |\ell| < K} v(x, \ell)\chi_{\ell,V}(k) \right|^2 dk
\]
\[
= \text{ess sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} \sum_{\ell \in \Gamma V, |\ell| < K} \chi_{\ell,V}(k)|v(x, k) - v(x, \ell)|^2 dk
\]
\[
\leq \text{ess sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} \sum_{\ell \in \Gamma V, |\ell| < K} \chi_{\ell,V}(k)|\tilde{v}(k)|^2\tilde{\phi}(|k - \ell|)^2 dk
\]
\[
\leq \int_{\mathbb{R}^d} \sum_{\ell \in \Gamma V, |\ell| < K} \chi_{\ell,V}(k)|\tilde{v}(k)|^2\tilde{\phi}(|k - \ell|)^2 dk.
\]
It follows from the property of $\tilde{o}$ that for every $\epsilon > 0$, there exists a constant $V_0 > 0$ such that, for all $V > V_0$,

$$\chi_{\ell,V}(k)\tilde{o}(|k-\ell|)^2 \leq \epsilon \chi_{\ell,V}(k).$$

Therefore,

$$\|v_K - v_{K,V}\|^2 \leq \epsilon \int_{\mathbb{R}^d} \sum_{|\ell| < K} \chi_{\ell,V}(k)|\tilde{v}(k)|^2\, dk = \epsilon \|\tilde{v}\|^2_{L^2(\mathbb{R}^d)}.$$ 

Hence the first one of (21) holds. The second one is a direct result of condition [DG.4]:

$$\|v_K - v\|^2 = \text{ess.sup } \int_{\mathbb{R}^d} |\chi_K(k) - 1|^2 |v(x,k)|^2\, dk$$

$$= \text{ess.sup } \int_{([-K,K]^d)^c} |v(x,k)|^2\, dk = o(K^0) \to 0 (K \to \infty).$$ 

Using [H.4], one can easily check (22).

We introduce two operators:

$$H_{DG}(K) := A \otimes I + I \otimes H_b + \tilde{\phi}(v_K),$$

$$H_{DG}(K, V) := A \otimes I + I \otimes H_{b,V} + \tilde{\phi}(v_{K,V}).$$

**Lemma 4.12.**

(i) $H_{DG}(K)$ is self-adjoint with $D(H_{DG}(K)) = D(H_0)$, bounded from below, and essentially self-adjoint on any core of $H_0$.

(ii) For sufficiently large $V > 0$, $H_{DG}(K, V)$ is self-adjoint with domain $D(H_{DG}(K, V)) = D(H_0)$, bounded from below, and essentially self-adjoint on any core of $H_0$.

**Proof.** Similar to the proof of Proposition 4.3.

**Lemma 4.13.** For all $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\lim_{V \to \infty} \|(H_{DG}(K, V) - z)^{-1} - (H_{DG}(K) - z)^{-1}\| = 0,$$

$$\lim_{K \to \infty} \|(H_{DG}(K) - z)^{-1} - (H_{DG} - z)^{-1}\| = 0.$$

**Proof.** Similar to the proof of Lemma 3.5
Lemma 4.14. The operator $H_{DG}(K, V)$ is reduced by $\mathcal{F}_V$.

Proof. We identify $v(x, \ell)$ with multiplication operator by $v(\cdot, \ell)$. By abuse of symbols, we denote $\chi_{\ell, V}(\cdot)$ by $\chi_{\ell,V}(k)$. Then

$$\left(\tilde{a}^*(v(x, \ell)\chi_{\ell,V}(k))\Phi\right)^{(n)} = \sqrt{n}(I \otimes S_n)(v(x, \ell)\chi_{\ell,V}(k) \otimes I)\Phi^{(n-1)}$$

$$= \sqrt{n}v(x, \ell)S_n(\chi_{\ell,V} \otimes \Phi^{(n-1)})$$

$$= \chi(x, \ell)\sqrt{n}S_n(\chi_{\ell,V} \otimes \Phi^{(n-1)}).$$

Hence, we have

$$\tilde{a}^*(v(x, \ell)\chi_{\ell,V}(k))\Phi = v(x, \ell) \otimes a^*(\chi_{\ell,V})\Phi.$$ 

Therefore, we get

$$\tilde{a}^*(v_{K,V}) = \sum_{\ell \in \Gamma_V \cap |\ell| < K} v(\cdot, \ell) \otimes a^*(\chi_{\ell,V}). \quad (23)$$

Hence, its adjoint is

$$\tilde{a}(v_{K,V}) = \sum_{\ell \in \Gamma_V \cap |\ell| < K} v(\cdot, \ell)^* \otimes a(\chi_{\ell,V}). \quad (24)$$

This means that the operator $H_{DG}(K, V)$ is a special case of the GSB Hamiltonian (see [2]). Hence, by [2, Lemma 3.7] $H_{DG}(K, V)$ is reduced by $\mathcal{F}_V$. \hfill \Box

Lemma 4.15. $H_{DG}(K, V)[\mathcal{F}_V^\perp] \geq E_0(H_{DG}(K, V)) + m$

Proof. Similar to the proof of [2, Lemma 3.10]. \hfill \Box

Lemma 4.16. For all $\Phi \in D(I \otimes H_b^{1/2})$, and for all $\epsilon' > 0$,

$$|\langle \Phi, \tilde{a}(v)\Phi \rangle| \leq \frac{\epsilon}{\|v\|} \|v\| \sqrt{\omega_n} \|I \otimes H_b^{1/2}\|^2 + \frac{\|v\|}{2} \left( \epsilon' + \frac{1}{\epsilon'} \right) \|\Phi\|^2.$$
Proof. For all $\Phi \in D(I \otimes H_0^{1/2})$, $\epsilon' > 0$,

$$
|\langle \Phi, \tilde{\phi}(v) \Phi \rangle| \leq \frac{1}{\sqrt{2}} \left( \epsilon \|a(v)\Phi\|^2 + \frac{1}{4\epsilon} \|\Phi\|^2 + \epsilon \|a^*(v)\Phi\|^2 + \frac{1}{4\epsilon} \|\Phi\|^2 \right)
$$

\leq \frac{1}{\sqrt{2}} \left( 2\epsilon \frac{\|v\|}{\sqrt{\omega}} \right)^2 \left( \|I \otimes H_0^{1/2} \Phi\|^2 + \epsilon \|v\|^2 \|\Phi\|^2 + \frac{1}{2\epsilon} \|\Phi\|^2 \right)

= \sqrt{2\epsilon} \left( \frac{v}{\sqrt{\omega}} \right)^2 \left( \|I \otimes H_0^{1/2} \Phi\|^2 + \frac{\|v\|}{2} \left( \sqrt{2\epsilon} \|v\| + \frac{1}{\sqrt{2\epsilon} \|v\|} \right) \right) \|\Phi\|^2,

where we have used Lemma 4.8 and 4.10. Let $\sqrt{2\epsilon} \|v\| =: \epsilon'$. Then, for all $\epsilon' > 0$, we have

$$
|\langle \Phi, \tilde{\phi}(v) \Phi \rangle| \leq \epsilon' \frac{\|v\|}{\sqrt{\omega}} \left( \|I \otimes H_0^{1/2} \Phi\|^2 + \frac{\|v\|}{2} \left( \epsilon' + \frac{1}{\epsilon'} \right) \|\Phi\|^2 \right).
$$

Proof of Theorem 4.4. From (23) and (24), $H_{DG}(K, V)$ is equal to the special case of the GSB model. Therefore, $H_{DG}(K, V)[F_{b, V}$ has the same form with $H_{DG}(K, V)$. Using Lemma 4.16 we have on $D(H_0) \cap F_{b, V}$

$$
H_{DG}(K, V)
= A \otimes I + I \otimes H_{b, V} + \tilde{\phi}(v_{K, V})
\geq A \otimes I + I \otimes H_{b, V} - \frac{\epsilon'}{\|v_{K, V}\|} \frac{\|v_{K, V}\|}{\sqrt{\omega V}} \|I \otimes H_{b, V} - \frac{\|v_{K, V}\|}{2} \left( \epsilon' + \frac{1}{\epsilon'} \right)
= A \otimes I + \left( 1 - \frac{\epsilon'}{\|v_{K, V}\|} \frac{\|v_{K, V}\|}{\sqrt{\omega V}} \right) I \otimes H_{b, V} - \frac{\|v_{K, V}\|}{2} \left( \epsilon' + \frac{1}{\epsilon'} \right),
$$

(25)

where $\epsilon' > 0$ is an arbitrary constant. By Lemma 3.10, $H_{b, V}[F_{b, V}$ has compact resolvent. Thus, for $\epsilon' > 0$ satisfying

$$
1 - \frac{\epsilon'}{\|v_{K, V}\|} \frac{\|v_{K, V}\|}{\sqrt{\omega V}} > 0,
$$

(26)

the bottom of the essential spectrum of (25) is equal to

$$
\Sigma(A) - \frac{\|v_{K, V}\|}{2} \left( \epsilon' + \frac{1}{\epsilon'} \right).
$$

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Let, $D_K$ and $D_{K,V}$ be $D$ with $v$ replaced by $v_K$, $v_{K,V}$, respectively. It is easy to see that
\[
\lim_{K \to \infty} D_K = D, \quad \lim_{V \to \infty} D_{K,V} = D_K.
\]
By Lemma 4.13, one has
\[
\lim_{K \to \infty} E_0(H_{DG}(K)) = E_0(H_{DG}), \quad \lim_{V \to \infty} E_0(H_{DG}(K,V)) = E_0(DG(K)).
\]
From the assumption of Theorem 4.4, for all $K > 0$, there exists a constant $V_0$ such that for $V > V_0$,
\[
\Sigma(A) - \frac{\|v_{K,V}\|}{2} D_{K,V} = E_0(0) > 0.
\]
By the definition of $D_{K,V}$, for all $K > 0$ and $V > V_0$, and for all $\epsilon'$ which satisfies (26), we have
\[
\Sigma(A) - \frac{\|v_{K,V}\|}{2} (\epsilon' + \frac{1}{\epsilon'}) > E_0(H_{DG}(K,V)).
\]
Therefore, by Theorem 2.1, we have that $H_{DG}(K,V)[\mathcal{F}_V]$ has purely discrete spectrum in
\[
[E_0(H_{DG}(K,V)), \Sigma(A) - \|v_{K,V}\|D_{K,V}].
\]
This fact and Lemma 4.15 mean that $H_{DG}(K,V)$ has purely discrete spectrum in
\[
[E_0(H_{DG}(K,V)), \min\{E_0(H_{DG}(K,V)) + m, \Sigma(A) - \|v_{K,V}\|D_{K,V}\}].
\]
Finally, we use Lemma 3.13 and Lemma 4.13, to conclude that $H_{DG}$ has purely discrete spectrum in the interval
\[
[E_0(H_{DG}), \min\{E_0(H_{DG}) + m, \Sigma(A) - \|v\|D\}]
\]

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References


