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# Stability of Discrete Ground State

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## Abstract

We present new criteria for a self-adjoint operator to have a ground state. As an application, we consider models of “quantum particles” coupled to a massive Bose field and prove the existence of a ground state of them, where the particle Hamiltonian does not necessarily have compact resolvent.

*Key words:* Ground state; discrete ground state; generalized spin-boson model; Fock space; Dereziński-Gérard model.

## 1 INTRODUCTION

Let  $T$  be a self-adjoint operator on a Hilbert space  $\mathcal{H}$ , and bounded from below. We say that  $T$  has a discrete ground state if the bottom of the spectrum of  $T$  is an isolated eigenvalue of  $T$ . In that case a non-zero vector

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in  $\ker(T - E_0(T))$  is called a ground state of  $T$ . Let  $S$  be a symmetric operator on  $\mathcal{H}$ . Suppose that  $T$  has a discrete ground state and  $S$  is  $T$ -bounded. By the regular perturbation theory [ 8 , XII ] it is already known that  $T + \lambda S$  has a discrete ground state for “sufficiently small”  $\lambda \in \mathbb{R}$ . Our aim is to present new criteria for  $T + \lambda S$  to have a ground state.

In Section 2, we prove an existence theorem of a ground state which is useful to show the existence of a ground state of models of quantum particles coupled to a massive Bose field.

In Section 3, we consider the GSB model [ 2 ] with a self-interaction term of a Bose field, which we call the GSB +  $\phi^2$  model. We consider only the case where the Bose field is massive. The GSB model — an abstract system of quantum particles coupled to a Bose field — was proposed in [ 2 ]. In [ 2 ] A. Arai and M. Hirokawa proved the existence and uniqueness of the GSB model in the case where the particle Hamiltonian  $A$  has compact resolvent. Shortly after that, they proved the existence of a ground state of the GSB model in the case where  $A$  does not have necessarily compact resolvent [ 4 , 3 ]. In this paper, using a theorem in Section 2, we prove the existence of a ground state of the GSB +  $\phi^2$  model in the case where  $A$  does not necessarily have compact resolvent.

In Section 4, we consider an extended version of the Nelson type model, which we call the Dereziński-Gérard model [ 5 ]. The Dereziński-Gérard model introduced in [ 5 ] and J. Dereziński and C. Gérard prove an existence of a ground state for their model under some conditions including that  $A$  has compact resolvent. In Section 4, we prove the existence of a ground state of the Dereziński-Gérard model in the case where  $A$  does not have compact resolvent. Our strategy to establish a ground state is the same as in Section 3.

## 2 BASIC RESULTS

Let  $\mathcal{H}$  be a separable complex Hilbert space. We denote by  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  the scalar product on Hilbert space  $\mathcal{H}$  and by  $\|\cdot\|_{\mathcal{H}}$  the associated norm. Scalar product  $\langle f, g \rangle_{\mathcal{H}}$  is linear in  $g$  and antilinear in  $f$ . We omit  $\mathcal{H}$  in  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  and  $\|\cdot\|_{\mathcal{H}}$ , respectively if there is no danger of confusion. For a linear operator  $T$  in Hilbert space, we denote by  $D(T)$  and  $\sigma(T)$  the domain and the spectrum of  $T$  respectively. If  $T$  is self-adjoint and bounded from below, then we define

$$E_0(T) := \inf \sigma(T), \quad \Sigma(T) := \inf \sigma_{\text{ess}}(T),$$

where  $\sigma_{\text{ess}}(T)$  is the essential spectrum of  $T$ . If  $T$  has no essential spectrum, then we set  $\Sigma(T) = \infty$ . For a self-adjoint operator  $T$ , we denote the form domain of  $T$  by  $Q(T)$ . In this paper, an eigenvector of a self-adjoint operator  $T$  with eigenvalue  $E_0(T)$  is called a ground state of  $T$  (if it exists). We say that  $T$  has a ground state if  $\dim \ker(T - E_0(T)) > 0$ .

The basic results are as follows:

**Theorem 2.1.** *Let  $H$  be a self-adjoint operator on  $\mathcal{H}$ , and bounded from below. Suppose that there exists a self-adjoint operator  $V$  on  $\mathcal{H}$  satisfying the following conditions (i)-(iii):*

- (i)  $D(H) \subset D(V)$ .
- (ii)  $V$  is bounded from below, and  $\Sigma(V) > 0$ .
- (iii)  $H - E_0(H) \geq V$  on  $D(H)$ .

*Then  $H$  has purely discrete spectrum in the interval  $[E_0(H), E_0(H) + \Sigma(V)]$ . In particular,  $H$  has a ground state.*

*Proof.* For all  $u_1, \dots, u_{n-1} \in \mathcal{H}$ , we have

$$\inf_{\substack{\Psi \in \text{L.h.}[u_1, \dots, u_{n-1}]^\perp \\ \|\Psi\|=1, \Psi \in D(H)}} \langle \Psi, H\Psi \rangle - E_0(H) \geq \inf_{\substack{\Psi \in \text{L.h.}[u_1, \dots, u_{n-1}]^\perp \\ \|\Psi\|=1, \Psi \in D(H)}} \langle \Psi, V\Psi \rangle,$$

where  $\text{L.h.}[\dots]$  denotes the linear hull of the vectors in  $[\dots]$ . Since  $D(H) \subset D(V)$ , we have that

$$\inf_{\substack{\Psi \in \text{L.h.}[u_1, \dots, u_{n-1}]^\perp \\ \|\Psi\|=1, \Psi \in D(H)}} \langle \Psi, V\Psi \rangle \geq \inf_{\substack{\Psi \in \text{L.h.}[u_1, \dots, u_{n-1}]^\perp \\ \|\Psi\|=1, \Psi \in D(V)}} \langle \Psi, V\Psi \rangle.$$

Hence, for all  $n \in \mathbb{N}$

$$\mu_n(H) - E_0(H) \geq \mu_n(V).$$

where

$$\mu_n(H) := \sup_{u_1, \dots, u_{n-1} \in \mathcal{H}} \inf_{\substack{\Psi \in \text{L.h.}[u_1, \dots, u_{n-1}]^\perp \\ \|\Psi\|=1, \Psi \in D(H)}} \langle \Psi, H\Psi \rangle.$$

By the min-max principle ([8, Theorem XIII.1]),  $\lim_{n \rightarrow \infty} \mu_n(H) = \Sigma(H)$  and  $\lim_{n \rightarrow \infty} \mu_n(V) = \Sigma(V)$ . Therefore we obtain

$$\Sigma(H) - E_0(H) \geq \Sigma(V) > 0.$$

This means that  $H$  has purely discrete spectrum in  $[E_0(H), E_0(H) + \Sigma(V)]$ . ■

**Theorem 2.2.** *Let  $H$  be a self-adjoint operator on  $\mathcal{H}$ , and bounded from below. Suppose that there exists a self-adjoint operator  $V$  on  $\mathcal{H}$  satisfying the following conditions (i)-(iii):*

- (i)  $Q(H) \subset Q(V)$ .
- (ii)  $V$  is bounded from below, and  $\Sigma(V) > 0$ .
- (iii)  $H - E_0(H) \geq V$  on  $Q(H)$ .

*Then  $H$  has purely discrete spectrum in the interval  $[E_0(H), E_0(H) + \Sigma(V)]$ . In particular,  $H$  has a ground state.*

*Proof.* Similar to the proof of Theorem 2.1. ■

We apply Theorems 2.1 and 2.2 to a perturbation problem of a self-adjoint operator.

**Theorem 2.3.** *Let  $A$  be a self-adjoint operator on  $\mathcal{H}$  with  $E_0(A) = 0$ , and let  $B$  be a symmetric operator on  $D(A)$ . Suppose that  $A + B$  is self-adjoint on  $D(A)$  and that there exist constants  $a \in [0, 1)$  and  $b \geq 0$  such that*

$$|\langle \psi, B\psi \rangle| \leq a\langle \psi, A\psi \rangle + b\|\phi\|^2, \quad \psi \in D(A).$$

*Assume*

$$\frac{b + E_0(A + B)}{1 - a} < \Sigma(A). \quad (1)$$

*Then  $A + B$  has purely discrete spectrum in  $[E_0(A + B), (1 - a)\Sigma(A) - b]$ . In particular,  $A + B$  has a ground state.*

*Proof.* By the assumption we have

$$A + B - E_0(A + B) \geq (1 - a)A - b - E_0(A + B)$$

on  $D(A)$ , and  $(1 - a)\Sigma(A) - b - E_0(A + B) > 0$ . Hence we can apply Theorem 2.1, to conclude that  $A + B$  has purely discrete spectrum in  $[E_0(A + B), (1 - a)\Sigma(A) - b]$ . In particular,  $A + B$  has a ground state. ■

*Remark.* It is easily to see that  $-b \leq E_0(A + B) \leq b$ . Therefore condition (1) is satisfied if

$$\frac{2b}{1 - a} < \Sigma(A).$$

**Theorem 2.4.** Let  $\mathcal{H}, \mathcal{K}$  be complex separable Hilbert spaces. Let  $A$  and  $B$  be self-adjoint operators on  $\mathcal{H}$  and  $\mathcal{K}$  respectively. Suppose that  $E_0(A) = E_0(B) = 0$ . We set

$$T_0 := A \otimes I + I \otimes B.$$

Let  $Z$  be a symmetric sesquilinear form on  $Q(T_0)$ , and assume that there exist constants  $a_1 \in [0, 1)$ ,  $a_2 \in [0, 1)$  and  $b \geq 0$  such that, for all  $\Psi \in Q(T_0)$

$$|Z(\Psi, \Psi)| \leq a_1 \langle \Psi, A \otimes I \Psi \rangle_{\text{form}} + a_2 \langle \Psi, I \otimes B \Psi \rangle_{\text{form}} + b \|\Psi\|^2,$$

where  $\langle \Psi, A \otimes I \Psi \rangle_{\text{form}} = \|A^{1/2} \otimes I \Psi\|^2$ . Therefore, by the KLMN theorem there exists a unique self-adjoint operator  $T$  on  $\mathcal{H} \otimes \mathcal{K}$  such that  $Q(T) = Q(T_0)$  and  $T = T_0 + Z$  in the sense of sesquilinear form on  $Q(T_0)$ . We set

$$s := \min\{(1 - a_1)\Sigma(A), (1 - a_2)\Sigma(B)\}.$$

Assume

$$s > b + E_0(T). \quad (2)$$

Then,  $T$  has purely discrete spectrum in the interval  $[E_0(T), s - b)$ . In particular,  $T$  has a ground state.

*Proof.* Similar to the proof of Theorem 2.3. ■

*Remark.* It is easy to see that  $-b \leq E_0(T) \leq b$ . Therefore the condition (2) is satisfied if

$$s > 2b.$$

*Remark.* Theorem 2.4 is essentially same as [4, Theorem B.1]. But our proof is very simple.

### 3 Ground States of a General Class of Quantum Field Hamiltonians

We consider a model which is an abstract unification of some quantum field models of particles interacting with a Bose field. It is the GSB model [2] with a self-interaction term of the field.

Let  $\mathcal{H}$  be a separable complex Hilbert space and  $\mathcal{F}_b$  be the Boson Fock space over  $L^2(\mathbb{R}^d)$  :

$$\mathcal{F}_b := \bigoplus_{n=0}^{\infty} \left[ \bigotimes_s^n L^2(\mathbb{R}^d) \right].$$

The Hilbert space of the quantum field model we consider is

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_b.$$

Let  $\omega : \mathbb{R}^d \rightarrow [0, \infty)$  be Borel measurable such that  $0 < \omega(k) < \infty$  for all most everywhere (a.e.)  $k \in \mathbb{R}^d$ . We denote the multiplication operator by the function  $\omega$  acting in  $L^2(\mathbb{R}^d)$  by the same symbol  $\omega$ . We set

$$H_b := d\Gamma_b(\omega)$$

the second quantization of  $\omega$  (e.g. [7, Section X.7]). We denote by  $a(f)$ ,  $f \in L^2(\mathbb{R}^d)$ , the smeared annihilation operators on  $\mathcal{F}_b$ . It is a densely defined closed linear operator on  $\mathcal{F}_b(\mathbb{R}^d)$  (e.g. [7, Section X.7]). The adjoint  $a(f)^*$ , called the creation operator, and the annihilation operator  $a(g)$ ,  $g \in L^2(\mathbb{R}^d)$  obey the canonical commutation relations

$$[a(f), a(g)^*] = \langle f, g \rangle, \quad [a(f), a(g)] = 0, \quad [a(f)^*, a(g)^*] = 0$$

for all  $f, g \in L^2(\mathbb{R}^d)$  on the dense subspace

$$\mathcal{F}_0 := \{ \psi = (\psi^{(n)})_{n=0}^\infty \in \mathcal{F}_b \mid \text{there exists a number } n_0 \text{ such that} \\ \psi^{(n)} = 0 \text{ for all } n \geq n_0 \},$$

where  $[X, Y] = XY - YX$ . The symmetric operator

$$\phi(f) := \frac{1}{\sqrt{2}} [a(f)^* + a(f)],$$

called the Segal field operator, is essentially self-adjoint on  $\mathcal{F}_0$  (e.g. [7, Section X.7]). We denote its closure by the same symbol. Let  $A$  be a positive self-adjoint operator on  $\mathcal{H}$  with  $E_0(A) = 0$ . Then, the unperturbed Hamiltonian of the model is defined by

$$H_0 := A \otimes I + I \otimes H_b$$

with domain  $D(H_0) = D(A \otimes I) \cap D(I \otimes H_b)$ . For  $g_j, f_j \in L^2(\mathbb{R}^d)$   $j = 1, \dots, J$ , and  $B_j (j = 1, \dots, J)$  a symmetric operator on  $\mathcal{H}$ , we define a symmetric operator

$$H_1 := \sum_{j=1}^J B_j \otimes \phi(g_j), \\ H_2 := \sum_{j=1}^J I \otimes \phi(f_j)^2.$$

The Hamiltonian of the model we consider is of the form

$$H(\lambda, \mu) := H_0 + \lambda H_1 + \mu H_2,$$

where  $\lambda \in \mathbb{R}$  and  $\mu \geq 0$  are coupling parameters.

For  $H(\lambda, \mu)$  to be self-adjoint, we shall need the following conditions [H.1]-[H.3]:

$$[\text{H.1}] \quad g_j \in D(\omega^{-1/2}), \quad f_j \in D(\omega^{1/2}) \cap D(\omega^{-1/2}), \quad j = 1, \dots, J.$$

$$[\text{H.2}] \quad D(A^{1/2}) \subset \cap_{j=1}^J D(B_j) \text{ and there exist constants } a_j \geq 0, b_j \geq 0, \\ j = 1, \dots, J, \text{ such that,}$$

$$\|B_j u\| \leq a_j \|A^{1/2} u\| + b_j \|u\|, \quad u \in D(A^{1/2}).$$

$$[\text{H.3}] \quad |\lambda| \sum_{j=1}^J a_j \|g_j / \sqrt{\omega}\| < 1.$$

**Proposition 3.1.** *Assume [H.1], [H.2] and [H.3]. Then,  $H(\lambda, \mu)$  is self-adjoint with  $D(H(\lambda, \mu)) = D(H_0) \subset D(H_1) \cap D(H_2)$  and bounded from below. Moreover,  $H(\lambda, \mu)$  is essentially self-adjoint on every core of  $H_0$ .*

*Remark.* This proposition has no restriction of the coupling parameter  $\mu \geq 0$ .

\* \* \*

To perform a finite volume approximation, we need an additional condition:

$$[\text{H.4}] \quad \text{The function } \omega(k) \text{ } (k \in \mathbb{R}^d) \text{ is continuous with}$$

$$\lim_{|k| \rightarrow \infty} \omega(k) = \infty,$$

and there exist constants  $\gamma > 0, C > 0$  such that

$$|\omega(k) - \omega(k')| \leq C |k - k'|^\gamma [1 + \omega(k) + \omega(k')], \quad k, k' \in \mathbb{R}^d.$$

Let

$$m := \inf_{k \in \mathbb{R}^d} \omega(k). \quad (3)$$

If  $A$  has compact resolvent, we can prove the extension of the previous theorem [ 2 , Theorem 1.2 ]



**Theorem 3.2.** *Consider the case  $m > 0$ . Suppose that  $A$  has entire purely discrete spectrum. Assume Hypotheses [H.1]-[H.4]. Then,  $H(\lambda, \mu)$  has purely discrete spectrum in the interval  $[E_0(H(\lambda, \mu)), E_0(H(\lambda, \mu)) + m)$ . In particular,  $H(\lambda, \mu)$  has a ground state.*

*Remark.* This theorem has no restriction of the coupling parameter  $\mu \geq 0$ .

*Remark.* In the case  $m > 0$ , the condition [H.1] equivalent to the following:

$$g_j \in L^2(\mathbb{R}^d), \quad f_j \in D(\sqrt{\omega}), \quad j = 1, \dots, J.$$

For a vector  $v = (v_1, \dots, v_J) \in \mathbb{R}^J$  and  $h = (h_1, \dots, h_J) \in \bigoplus_{j=1}^J L^2(\mathbb{R}^d)$ , we define

$$M_v(h) = \sum_{j=1}^J v_j \|h_j\|.$$

We set

$$g = (g_1, \dots, g_J) \in \bigoplus_{j=1}^J L^2(\mathbb{R}^d), \quad f = (f_1, \dots, f_J) \in \bigoplus_{j=1}^J L^2(\mathbb{R}^d),$$

and

$$a = (a_1, \dots, a_J), \quad b = (b_1, \dots, b_J).$$

For  $\theta, \epsilon, \epsilon'$ , we introduce the following constants:

$$\begin{aligned} C_{\theta, \epsilon} &:= \theta M_a(g/\sqrt{\omega}) + \epsilon M_a(g), \\ D_{\theta, \epsilon'} &:= M_a(g/\sqrt{\omega})/2\theta + \epsilon' M_b(g/\sqrt{\omega}), \\ E_{\epsilon, \epsilon'} &:= M_a(g)/8\epsilon + M_b(g/\sqrt{\omega})/2\epsilon' + M_b(g)/\sqrt{2}. \end{aligned}$$

Let the condition [H.3] be satisfied. Then, we define

$$I_{\lambda, g} := \begin{cases} \left( \frac{|\lambda| M_a(g/\sqrt{\omega})}{2}, \frac{1}{|\lambda| M_a(g/\sqrt{\omega})} \right), & |\lambda| M_a(g/\sqrt{\omega}) \neq 0 \\ [0, \infty], & |\lambda| M_a(g/\sqrt{\omega}) = 0 \end{cases}$$

It is easy to see that  $[1/2, 1] \subset I_{\lambda, g}$ . Therefore, for all  $\theta \in I_{\lambda, g}$ ,

$$\begin{aligned} 1 - \theta |\lambda| M_a(g/\sqrt{\omega}) &> 0, \\ 1 - \frac{|\lambda| M_a(g/\sqrt{\omega})}{2\theta} &> 0. \end{aligned}$$

We define for  $\theta \in I_{\lambda,g}$ ,

$$\mathbf{S}_\theta := \{(\epsilon, \epsilon') | \epsilon, \epsilon' > 0, |\lambda|C_{\theta,\epsilon} < 1, |\lambda|D_{\theta,\epsilon'} < 1\}.$$

Next we set

$$\tau_{\theta,\epsilon,\epsilon'} := (1 - |\lambda|C_{\theta,\epsilon})\Sigma(A) - |\lambda|E_{\epsilon,\epsilon'},$$

and

$$\mathbf{T} := \{(\theta, \epsilon, \epsilon') \in \mathbb{R}^3 | \theta \in I_{\lambda,g}, (\epsilon, \epsilon') \in \mathbf{S}_\theta, \tau_{\theta,\epsilon,\epsilon'} > E_0(H(\lambda, \mu))\}.$$

**Theorem 3.3.** *Consider the case  $m > 0$ . Suppose that  $\sigma_{\text{ess}}(A) \neq \emptyset$ . Assume Hypothesis [H.1]-[H.4], and  $\mathbf{T} \neq \emptyset$ . Then,  $H(\lambda, \mu)$  has purely discrete spectrum in the interval*

$$[E_0(H(\lambda, \mu)), \min\{m + E_0(H(\lambda, \mu)), \sup_{(\theta,\epsilon,\epsilon') \in \mathbf{T}} \tau_{\theta,\epsilon,\epsilon'}\}]. \quad (4)$$

*In particular,  $H(\lambda, \mu)$  has a ground state.*

*Remark.*  $\mathbf{T} \neq \emptyset$  is necessary condition for  $A$  to have a discrete ground state. Conversely, if  $A$  has a discrete ground state, then  $\mathbf{T} \neq \emptyset$  holds for sufficiently small  $\lambda, \mu$ . Therefore the condition  $\mathbf{T} \neq \emptyset$  is a restriction for the coupling constants  $\lambda, \mu$ .

\* \* \*

### 3.1 Proof of Proposition 3.1

In what follows, we write simply

$$H := H(\lambda, \mu).$$

For  $\mathcal{D}$  a dense subspace of  $L^2(\mathbb{R}^d)$ , we define

$$\mathcal{F}_{\text{fin}}(\mathcal{D}) := \text{L.h}\{[\Omega, a(h_1)^* \cdots a(h_n)^* \Omega | n \in \mathbb{N}, h_j \in \mathcal{D}, j = 1, \dots, n]\},$$

where  $\Omega := (1, 0, 0, \dots)$  is the Fock vacuum in  $\mathcal{F}_{\text{b}}$ . We introduce a dense subspace in  $\mathcal{F}$

$$\mathcal{D}_\omega := D(A) \hat{\otimes} \mathcal{F}_{\text{fin}}(D(\omega)),$$

where  $\hat{\otimes}$  denotes algebraic tensor product. The subspace  $\mathcal{D}_\omega$  is a core of  $H_0$ .

Let

$$H_{\text{GSB}} := H_0 + \lambda H_1$$

be a GSB Hamiltonian. The Hamiltonian  $H$  and  $H_{\text{GSB}}$  has the following relation:

**Proposition 3.4.** *Let  $D(A) \subset D(B_j)$ ,  $j = 1, \dots, J$  and  $f_j \in D(\omega^{1/2})$ . Assume that  $H_{\text{GSB}}$  is bounded from below. Then, for all  $\Psi \in D_\omega$ ,*

$$\|(H_{\text{GSB}} - E_0)\Psi\|^2 + \|\mu H_2\Psi\|^2 \leq \|(H - E_0)\Psi\|^2 + D\|\Psi\|^2, \quad (5)$$

where  $D = \mu \sum_{j=1}^J \|\omega^{1/2} f_j\|^2$  and

$$E_0 := \inf_{\substack{\Psi \in D(H_{\text{GSB}}) \\ \|\Psi\|=1}} \langle \Psi, H_{\text{GSB}}\Psi \rangle.$$

*Proof.* It is enough to show (5) the case  $\lambda = \mu = 1$ . First we consider the case where  $f_j \in D(\omega)$ . Inequality (5) is equivalent to

$$-2 \operatorname{Re} \langle (H_{\text{GSB}} - E_0)\Psi, H_2\Psi \rangle \leq D\|\Psi\|^2. \quad (6)$$

By  $H_{\text{GSB}} - E_0 \geq 0$ , we have

$$\begin{aligned} \langle (H_{\text{GSB}} - E_0)\Psi, I \otimes \phi(f_j)^2\Psi \rangle &= \langle [I \otimes \phi(f_j), (H_{\text{GSB}} - E_0)]\Psi, I \otimes \phi(f_j)\Psi \rangle \\ &\quad + \langle (H_{\text{GSB}} - E_0)I \otimes \phi(f_j)\Psi, I \otimes \phi(f_j)\Psi \rangle \\ &\geq \langle [I \otimes \phi(f_j), H_{\text{GSB}} - E_0]\Psi, I \otimes \phi(f_j)\Psi \rangle. \end{aligned}$$

Therefore we have

$$2 \operatorname{Re} \langle (H_{\text{GSB}} - E_0)\Psi, \phi(f_j)^2\Psi \rangle \geq -\|\sqrt{\omega} f_j\|^2 \|\Psi\|^2.$$

This means inequality (6). Next, we set  $f_j \in D(\sqrt{\omega})$ . Then, there exists a sequence  $\{f_{jn}\}_{n=0}^\infty \subset D(\omega)$  such that  $f_{jn} \rightarrow f_j$ ,  $\omega^{1/2} f_{jn} \rightarrow \omega^{1/2} f_j$  ( $n \rightarrow \infty$ ). By limiting argument, (6) holds with  $f_j \in D(\omega^{1/2})$ .  $\blacksquare$

**Lemma 3.5.** *Suppose that  $H_{\text{GSB}}$  is self-adjoint with  $D(H_{\text{GSB}}) = D(H_0)$ , essentially self-adjoint on  $\mathcal{D}_\omega$ , and bounded from below. Let  $f_j \in D(\omega^{1/2}) \cap D(\omega^{-1/2})$ . Then  $H$  is self-adjoint with  $D(H) = D(H_0)$  and essentially self-adjoint on any core of  $H_{\text{GSB}}$  with*

$$\|(H_{\text{GSB}} - E_0)\Psi\|^2 + \|\mu H_2\Psi\|^2 \leq \|(H - E_0)\Psi\|^2 + D\|\Psi\|^2, \quad \Psi \in D(H_0).$$

*Proof.* It is well known that  $D(H_b) \subset D(\phi(f_j)^2)$ , and  $\phi(f_j)^2$  is  $H_b$ -bounded (e.g. [1, Lemma 13-16]). Namely, there exist constants  $\eta \geq 0$ ,  $\theta \geq 0$  such that

$$\left\| \sum_{j=1}^J \phi(f_j)^2 \psi \right\| \leq \eta \|H_b \psi\| + \theta \|\psi\|, \quad \psi \in D(H_b). \quad (7)$$

Since  $H_{\text{GSB}}$  is self-adjoint on  $D(H_0)$ , by the closed graph theorem, we have

$$\|H_0 \Psi\| \leq \lambda \|H_{\text{GSB}} \Psi\| + \nu \|\Psi\|, \quad \Psi \in D(H_0), \quad (8)$$

where  $\lambda$  and  $\nu$  are non-negative constant independent of  $\Psi$ . Hence

$$\|H_2 \Psi\| \leq \eta \lambda \|H_{\text{GSB}} \Psi\| + (\eta \nu + \theta) \|\Psi\|, \quad \Psi \in D(H_0).$$

We fix a positive number  $\mu_0$  such that  $\mu_0 < 1/(\mu\lambda)$ . Then, by the Kato-Rellich theorem,  $H(\lambda, \mu_0)$  is self-adjoint on  $D(H_{\text{GSB}})$ , bounded from below and essentially self-adjoint on any core of  $H_{\text{GSB}}$ . For a constant  $a$  ( $0 < a < 1$ ), we set  $\mu_n := (1+a)^n \mu_0$ . Since  $H_{\text{GSB}}$  is self-adjoint on  $D(H_0)$ , for each  $j = 1, \dots, J$  we have  $D(A) \subset D(B)$ . Thus by Proposition 3.4, for all  $\Psi \in \mathcal{D}_\omega$

$$\|(H_{\text{GSB}} - E_0)\Psi\|^2 + \|\mu_n H_2 \Psi\|^2 \leq \|(H(\lambda, \mu_n) - E_0)\Psi\|^2 + D\|\Psi\|^2.$$

If  $H(\lambda, \mu_n)$  is self-adjoint on  $D(H_{\text{GSB}})$ , bounded from below and essentially self-adjoint on any core of  $H_{\text{GSB}}$ , then  $H(\lambda, \mu_{n+1})$  has the same property. On the other hand, we have  $\mu_n \rightarrow \infty$  ( $n \rightarrow \infty$ ). Hence we conclude that  $H$  is self-adjoint with  $D(H) = D(H_{\text{GSB}})$ , bounded from below and essentially self-adjoint on any core of  $H_{\text{GSB}}$ .  $\blacksquare$

Now, we assume conditions [H.1],[H.2] and [H.3].

Then  $H_{\text{GSB}}$  is self-adjoint on  $D(H_0)$ , bounded from below and essentially self-adjoint on any core of  $H_0$  (see [2]). Hence, the assumptions of Lemma 3.5 hold. Thus Proposition 3.1 follows.  $\blacksquare$

### 3.2 Proofs of Theorems 3.2 and 3.3

Throughout this subsection, we assume Hypotheses [H.1]-[H.4] and  $m > 0$ .

For a parameter  $V > 0$ , we define the set of lattice points by

$$\Gamma_V := \frac{2\pi\mathbb{Z}^d}{V} := \left\{ k = (k_1, \dots, k_d) \mid k_j = \frac{2\pi n_j}{V}, n_j \in \mathbb{Z}, j = 1, \dots, d \right\}$$

and we denote by  $l^2(\Gamma_V)$  the set of  $l^2$  sequences over  $\Gamma_V$ . For each  $k \in \Gamma_V$  we introduce

$$C(k, V) := \left[ k_1 - \frac{\pi}{V}, k_1 + \frac{\pi}{V} \right) \times \cdots \times \left[ k_d - \frac{\pi}{V}, k_d + \frac{\pi}{V} \right) \subset \mathbb{R}^d,$$

the cube centered about  $k$ . By the map

$$U : l^2(\Gamma_V) \ni \{h_l\}_{l \in \Gamma_V} \mapsto (V/2\pi)^{d/2} \sum_{l \in \Gamma_V} h_l \chi_{l,V}(\cdot) \in L^2(\mathbb{R}^d),$$

we identify  $l^2(\Gamma_V)$  with a subspace in  $L^2(\mathbb{R}^d)$ , where  $\chi_{l,V}(\cdot)$  is the characteristic function of the cube  $C(l, V) \subset \mathbb{R}^d$ . It is easy to see that  $l^2(\Gamma_V)$  is a closed subspace of  $L^2(\mathbb{R}^d)$ . Let

$$\mathcal{F}_{b,V} := \mathcal{F}_b(l^2(\Gamma_V)) = \bigoplus_{n=0}^{\infty} \left[ \bigotimes_s^n l^2(\Gamma_V) \right],$$

the boson Fock space over  $l^2(\Gamma_V)$ . We can identify  $\mathcal{F}_{b,V}$  the closed subspace of  $\mathcal{F}_b$  by the operator  $\Gamma(U) := \bigoplus_{n=0}^{\infty} \bigotimes^n U$ , where we define  $\bigotimes^0 U = 0$ . For each  $k \in \mathbb{R}^d$ , there exists a unique point  $k_V \in \Gamma_V$  such that  $k \in C(k_V, V)$ . Let

$$\omega_V(k) := \omega(k_V), \quad k \in \mathbb{R}^d$$

be a lattice approximate function of  $\omega(k)$  and let

$$H_{b,V} := d\Gamma(\omega_V)$$

be the second quantization of  $\omega_V$ . We define a constant

$$C_V := Cd^\gamma \left( \frac{\pi}{V} \right) \left( \frac{1}{2m} + 1 \right),$$

where  $C$  and  $\gamma$  were defined in [H.4]. In what follows we assume that

$$C_V < 1.$$

This is satisfied for all sufficiently large  $V$ .

**Lemma 3.6.** ([ 2 , Lemma 3.1 ]). *We have*

$$D(H_{b,V}) = D(H_b),$$

and

$$\|(H_b - H_{b,V})\Psi\| = \frac{2C_V}{1 - C_V} \|H_b\Psi\|, \quad \Psi \in D(H_b).$$

First we consider the case where  $g_j$ 's and  $f_j$ 's are continuous, and finally, by limiting argument, we treat a general case. For a constant  $K > 0$ , we define  $g_{j,K}$ ,  $f_{j,K}$ , and  $g_{j,K,V}$ ,  $f_{j,K,V}$  as follows:

$$g_{j,K}(k) := \chi_K(k_1) \cdots \chi_K(k_d) g_j(k), \quad g_{j,K,V}(k) := \sum_{\substack{\ell \in \Gamma_V, |\ell_i| < K \\ i=1, \dots, d}} g_j(\ell) \chi_{\ell,V}(k),$$

$$f_{j,K}(k) := \chi_K(k_1) \cdots \chi_K(k_d) f_j(k), \quad f_{j,K,V}(k) := \sum_{\substack{\ell \in \Gamma_V, |\ell_i| < K \\ i=1, \dots, d}} f_j(\ell) \chi_{\ell,V}(k),$$

where  $\chi_K$  denotes the characteristic function of  $[-K, K]$ .

**Lemma 3.7.** *For all  $j = 1, \dots, J$ ,*

$$\begin{aligned} \lim_{V \rightarrow \infty} \|g_{j,K,V} - g_{j,K}\| &= 0, & \lim_{V \rightarrow \infty} \|g_{j,K,V}/\sqrt{\omega_V} - g_{j,K}/\sqrt{\omega}\| &= 0, \\ \lim_{K \rightarrow \infty} \|g_{j,K} - g_j\| &= 0, & \lim_{K \rightarrow \infty} \|g_{j,K}/\sqrt{\omega} - g_j/\sqrt{\omega}\| &= 0, \\ \lim_{V \rightarrow \infty} \|f_{j,K,V} - f_{j,K}\| &= 0, & \lim_{V \rightarrow \infty} \|f_{j,K,V}/\sqrt{\omega_V} - f_{j,K}/\sqrt{\omega}\| &= 0, \\ \lim_{K \rightarrow \infty} \|f_{j,K} - f_j\| &= 0, & \lim_{K \rightarrow \infty} \|f_{j,K}/\sqrt{\omega} - f_j/\sqrt{\omega}\| &= 0, \\ \lim_{K \rightarrow \infty} \|\sqrt{\omega} f_{j,K} - \sqrt{\omega} f_j\| &= 0, & \lim_{V \rightarrow \infty} \|\sqrt{\omega_V} f_{j,K,V} - \sqrt{\omega} f_{j,K}\| &= 0. \end{aligned}$$

*Proof.* Similar to the proof of [ 2 , Lemma 3.10 ] ■

We introduce a new operator:

$$H_{0,V} := A \otimes I + I \otimes H_{b,V},$$

$$H_{1,K} := \sum_{j=1}^J B_j \otimes \phi(g_{j,K}),$$

$$H_{1,K,V} := \sum_{j=1}^J B_j \otimes \phi(g_{j,K,V}),$$

$$H_{2,K} := \sum_{j=1}^J I \otimes \phi(f_{j,K})^2,$$

$$H_{2,K,V} := \sum_{j=1}^J I \otimes \phi(f_{j,K,V})^2,$$

and define

$$H_K := H_0 + \lambda H_{1,K} + \mu H_{2,K},$$

$$H_{K,V} := H_{0,V} + \lambda H_{1,K,V} + \mu H_{2,K,V}.$$

- Lemma 3.8.** (i)  $H_K$  is self-adjoint with  $D(H_K) = D(H_0) \subset D(H_{1,K}) \cap D(H_{2,K})$ , bounded from below, and essentially self-adjoint on any core of  $H_0$ .
- (ii) For all large  $V$ ,  $H_{K,V}$  is self-adjoint with  $D(H_{K,V}) = D(H_0) \subset D(H_{1,K,V}) \cap D(H_{2,K,V})$ , bounded from below, and essentially self-adjoint on any core of  $H_{0,V}$ .

*Proof.* Similar to the proof of Proposition 3.1. ■

**Lemma 3.9.** For all  $z \in \mathbb{C} \setminus \mathbb{R}$ , and  $K > 0$ ,

$$\lim_{K \rightarrow \infty} \|(H_K - z)^{-1} - (H - z)^{-1}\| = 0,$$

$$\lim_{V \rightarrow \infty} \|(H_{K,V} - z)^{-1} - (H_K - z)^{-1}\| = 0.$$

*Proof.* Similar to the proof of [2, Lemma 3.5] ■

The following fact is well known:

**Lemma 3.10.** The operator  $H_{b,V}$  is reduced by  $\mathcal{F}_{b,V}$  and  $H_{b,V} \upharpoonright \mathcal{F}_{b,V}$  equal to the second quantization of  $\omega_V \upharpoonright l^2(\Gamma_V)$  on  $\mathcal{F}_{b,V}$ .

**Lemma 3.11.**  $H_{K,V}$  is reduced by  $\mathcal{F}_V$ .

*Proof.* Similar to the proof of [2, Lemma 3.7] ■

**Lemma 3.12.** We have

$$H_{K,V} \upharpoonright \mathcal{F}_V^\perp \geq E_0(H_{K,V}) + m.$$

*Proof.* Similar to the proof of [ 2 , Lemma 3.10 ] ■

**Lemma 3.13.** *Let  $T_n$  and  $T$  be a self-adjoint operators on a separable Hilbert space and bounded from below. Suppose that  $T_n \rightarrow T$  in norm resolvent sense as  $n \rightarrow \infty$  and  $T_n$  has purely discrete spectrum in the interval  $[E_0(T_n), E_0(T_n) + c_n)$  with some constant  $c_n$ . If  $c := \limsup_{n \rightarrow \infty} c_n > 0$ , then  $T$  has purely discrete spectrum in  $[E_0(T), E_0(T) + c)$ .*

*Proof.* There exists a sequence  $\{c_{n_j}\}_{j=1}^\infty \subset \{c_n\}_{n=1}^\infty$  so that  $c_{n_j} \rightarrow c (j \rightarrow \infty)$ . So, for all  $\epsilon > 0$  and for sufficiently large  $j$ , the spectrum of  $T_{n_j}$  in  $[E_0(T_{n_j}), E_0(T_{n_j}) + c - \epsilon)$  is discrete. Therefore, applying [ 2 , Lemma 3.12 ], we find that the spectrum of  $T$  in  $[E_0(T), E_0(T) + c - \epsilon)$  is discrete. Since  $\epsilon > 0$  is arbitrary, we get the conclusion. ■

Now, if  $A$  has compact resolvent, by a method similar to the proof of [ 2 , Theorem 1.2 ] we can prove Theorem 3.2. Therefore, we only prove Theorem 3.3.

The following inequality is known [ 2 , (2.12) ]

$$|\langle \Psi, H_1 \Psi \rangle| \leq C_{\theta, \epsilon} \langle \Psi, A \otimes I \Psi \rangle + D_{\theta, \epsilon} \langle \Psi, I \otimes H_b \Psi \rangle + E_{\epsilon, \epsilon'} \|\Psi\|^2,$$

where  $\Psi \in D(H_0)$  is arbitrary. Thus we have,

$$H \geq (1 - |\lambda| C_{\theta, \epsilon}) A \otimes I + (1 - |\lambda| D_{\theta, \epsilon'}) I \otimes H_b + \mu H_2 - |\lambda| E_{\epsilon, \epsilon'}.$$

Let  $I_{\lambda, g}(K)$ ,  $C_{\theta, \epsilon}(K)$ ,  $D_{\theta, \epsilon}(K)$  and  $E_{\epsilon, \epsilon'}(K)$  are  $I_{\lambda, g}$ ,  $C_{\theta, \epsilon}$ ,  $D_{\theta, \epsilon}$ ,  $E_{\epsilon, \epsilon'}$  with  $g_j$ ,  $f_j$  replaced by  $g_{j, K}$ ,  $f_{j, K}$  respectively, and let  $I_{\lambda, g}(K, V)$ ,  $C_{\theta, \epsilon}(K, V)$ ,  $D_{\theta, \epsilon}(K, V)$  and  $E_{\epsilon, \epsilon'}(K, V)$  are  $I_{\lambda, g}$ ,  $C_{\theta, \epsilon}$ ,  $D_{\theta, \epsilon}$ ,  $E_{\epsilon, \epsilon'}$  with  $g_j$ ,  $f_j$  and  $\omega$  replaced by  $g_{j, K, V}$ ,  $f_{j, K, V}$  and  $\omega_V$  respectively. Then we have

**Lemma 3.14.** *The following operator inequalities hold:*

$$\begin{aligned} H_K &\geq (1 - |\lambda| C_{\theta, \epsilon}(K)) A \otimes I + (1 - |\lambda| D_{\theta, \epsilon'}(K)) I \otimes H_b \\ &\quad + \mu H_{2, K} - |\lambda| E_{\epsilon, \epsilon'}(K) \quad \text{on } D(H_0), \\ H_{K, V} &\geq (1 - |\lambda| C_{\theta, \epsilon}(K, V)) A \otimes I + (1 - |\lambda| D_{\theta, \epsilon'}(K, K)) I \otimes H_{b, V} \\ &\quad + \mu H_{2, K, V} - |\lambda| E_{\epsilon, \epsilon'}(K, V) \quad \text{on } D(H_0). \end{aligned}$$

*Proof.* Similar to the calculation of [ 2 , (2.12) ] ■



By Lemma 3.7, we have

$$\lim_{V \rightarrow \infty} C_{\theta, \epsilon}(K, V) = C_{\theta, \epsilon}(K), \quad \lim_{K \rightarrow \infty} C_{\theta, \epsilon}(K) = C_{\theta, \epsilon}, \quad (9)$$

$$\lim_{V \rightarrow \infty} D_{\theta, \epsilon'}(K, V) = D_{\theta, \epsilon'}(K), \quad \lim_{K \rightarrow \infty} D_{\theta, \epsilon'}(K) = D_{\theta, \epsilon'}, \quad (10)$$

$$\lim_{V \rightarrow \infty} E_{\epsilon, \epsilon'}(K, V) = E_{\epsilon, \epsilon'}(K), \quad \lim_{K \rightarrow \infty} E_{\epsilon, \epsilon'}(K) = E_{\epsilon, \epsilon'}. \quad (11)$$

Let  $(\theta, \epsilon, \epsilon') \in \mathbb{T}$ , namely

$$\tau_{\theta, \epsilon, \epsilon'} = (1 - |\lambda|C_{\theta, \epsilon})\Sigma(A) - |\lambda|E_{\epsilon, \epsilon'} > E_0(H).$$

Formulas (9)-(11) and Lemma 3.9 imply that for all large  $V$  there exists a constant  $K_0 > 0$  such that for all  $K > K_0$ ,

$$(1 - |\lambda|C_{\theta, \epsilon}(K, V))\Sigma(A) - |\lambda|E_{\epsilon, \epsilon'}(K, V) > E_0(H_{K, V}), \quad (12)$$

$$|\lambda|C_{\theta, \epsilon}(K, V) < 1, \quad |\lambda|D_{\theta, \epsilon'}(K, V) < 1. \quad (13)$$

By Lemma 3.11,  $H_{K, V}$  is reduced by  $\mathcal{F}_V$ . Therefore,  $H_{K, V}$  satisfies the following inequality:

$$\begin{aligned} H_{K, V}[\mathcal{F}_V] \geq & (1 - |\lambda|C_{\theta, \epsilon}(K, V))A \otimes I[\mathcal{F}_V \\ & + (1 - |\lambda|D_{\theta, \epsilon'}(K, V))I \otimes H_{b, V}[\mathcal{F}_V \\ & - |\lambda|E_{\epsilon, \epsilon'}(K, V). \end{aligned} \quad (14)$$

Since  $H_{b, V}[\mathcal{F}_V]$  has compact resolvent, the bottom of essential spectrum of the right hand side of (14) is equal to

$$(1 - |\lambda|C_{\theta, \epsilon}(K, V))\Sigma(A) - |\lambda|E_{\epsilon, \epsilon'}(K, V).$$

By Lemma 3.12, we have  $E_0(H_{K, V}[\mathcal{F}_V]) = E_0(H_{K, V})$ . Thus, applying Theorem 2.1 with  $H_{K, V}[\mathcal{F}_V]$ , we have that  $H_{K, V}[\mathcal{F}_V$  has purely discrete spectrum in  $[E_0(H_{K, V}), (1 - |\lambda|C_{\theta, \epsilon}(K, V))\Sigma_A - E_{\epsilon, \epsilon'}(K, V)]$ . Since this fact and Lemma 3.12,  $H_{K, V}$  has purely discrete spectrum in

$$[E_0(H_{K, V}), \min\{E_0(H_{K, V}) + m, (1 - |\lambda|C_{\theta, \epsilon}(K, V))\Sigma_A - E_{\epsilon, \epsilon'}(K, V)\}].$$

By Lemma 3.9 and Lemma 3.13, we have that for all sufficiently large  $K > 0$ ,  $H_K$  has purely discrete spectrum in  $[E_0(H_K), \min\{E_0(H_K) + m, (1 - |\lambda|C_{\theta, \epsilon}(K))\Sigma(A) - |\lambda|E_{\epsilon, \epsilon'}(K)\}]$ . Similarly,  $H$  has purely discrete spectrum in  $[E_0(H(\lambda, \mu)), \min\{m + E_0(H(\lambda, \mu)), \tau_{\theta, \epsilon, \epsilon'}\}]$ . Since  $(\theta, \epsilon, \epsilon') \in \mathbb{T}$  is arbitrary,  $H$  has purely discrete spectrum in (4). Finally, we have to consider the case where  $g_j$ 's and  $f_j$ 's are not necessarily continuous. But, that argument were already discussed in [ 4 ] So we skip that argument. ■

## 4 Ground State of the Dereziński-Gérard Model

We consider a model discussed by J. Dereziński and C. Gérard [5]. We take the Hilbert space of the particle system is taken to be

$$\mathcal{H} = L^2(\mathbb{R}^N).$$

The Hilbert space for the Dereziński-Gérard (DG) model is given by

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_b(L^2(\mathbb{R}^d)).$$

We identify  $\mathcal{F}$  as

$$\bigoplus_{n=0}^{\infty} \left[ \mathcal{H} \otimes \bigotimes_s^n L^2(\mathbb{R}^d) \right].$$

Hence, if we denote that  $\Psi \in (\Psi^{(n)})_{n=0}^{\infty} \in \mathcal{F}$ , each  $\Psi^{(n)}$  belongs to  $\mathcal{H} \otimes [\bigotimes_s^n L^2(\mathbb{R}^d)]$ . We denote by  $\mathbf{B}(\mathcal{K}, \mathcal{J})$  the set of bounded linear operators from  $\mathcal{K}$  to  $\mathcal{J}$ . For  $v \in \mathbf{B}(\mathcal{H}, \mathcal{H} \otimes L^2(\mathbb{R}^d))$ , we define an operator  $\tilde{a}^*(v)$  by

$$\begin{aligned} (\tilde{a}^*(v)\Psi)^{(0)} &:= 0, \\ (\tilde{a}^*(v)\Psi)^{(n)} &:= \sqrt{n}(I_{\mathcal{H}} \otimes S_n)(v \otimes I_{\bigotimes_s^{n-1} L^2(\mathbb{R}^d)})\Psi^{(n-1)}, \quad (n \geq 1), \\ \Psi \in D(\tilde{a}^*(v)) &:= \left\{ \Psi = (\Psi^{(n)})_{n=0}^{\infty} \in \mathcal{F} \mid \sum_{n=0}^{\infty} \|(\tilde{a}^*(v)\Psi)^{(n)}\|^2 < \infty \right\}. \end{aligned}$$

We set

$$\begin{aligned} \mathcal{D}_0 &:= \{ \Psi = (\Psi^{(n)})_{n=0}^{\infty} \in \mathcal{F} \mid \text{there exists a constant } n_0 \in \mathbb{N}, \\ &\quad \text{such that, for all } n \geq n_0, \Psi^{(n)} = 0 \}. \end{aligned}$$

Throughout this section, we write simply  $I_n := I_{\bigotimes_s^n L^2(\mathbb{R}^d)}$ . It is easy to see that:

**Proposition 4.1.**  *$\tilde{a}^*(v)$  is a closed linear operator and  $\mathcal{D}_0$  is a core of  $\tilde{a}^*(v)$ .*

So we set

$$\tilde{a}(v) := (\tilde{a}^*(v))^*$$

the adjoint operator of  $\tilde{a}^*(v)$ .

**Proposition 4.2.** *The operator  $\tilde{a}(v)$  has the following properties:*

$$D(\tilde{a}(v)) = \left\{ \Psi = (\Psi^{(n)})_{n=0}^\infty \mid \sum_{n=0}^\infty (n+1) \|(I_{\mathcal{H}} \otimes S_n)(v^* \otimes I_n) \Psi^{(n+1)}\|^2 < \infty \right\} \quad (15)$$

$$(\tilde{a}(v)\Psi)^{(n)} = \sqrt{n+1} I_{\mathcal{H}} \otimes S_n(v^* \otimes I_n) \Psi^{(n+1)}, \quad \Psi \in D(\tilde{a}(v)), \quad (16)$$

and  $\mathcal{D}_0$  is a core of  $\tilde{a}(v)$ .

*Proof.* For  $\Phi \in \mathcal{F}$ ,  $\Psi \in D(\tilde{a}^*(v))$ ,

$$\begin{aligned} \langle \Phi, \tilde{a}^*(v)\Psi \rangle &= \sum_{n=1}^\infty \langle \Phi^{(n)}, \sqrt{n} (I_{\mathcal{H}} \otimes S_n)(v \otimes I_{n-1}) \Psi^{(n-1)} \rangle \\ &= \sum_{n=0}^\infty \sqrt{n+1} \langle v^* \otimes I_n \Phi^{(n+1)}, \Psi^{(n)} \rangle \\ &= \sum_{n=0}^\infty \langle \sqrt{n+1} (I_{\mathcal{H}} \otimes S_n)(v^* \otimes I_n) \Phi^{(n+1)}, \Psi^{(n)} \rangle. \end{aligned}$$

This implies (15) and (16). It is easy to prove that  $\mathcal{D}_0$  is a core of  $\tilde{a}(v)$ .  $\blacksquare$

An analogue of the Segal field operator is defined by

$$\tilde{\phi}(v) := \frac{1}{\sqrt{2}} (\tilde{a}(v) + \tilde{a}^*(v)).$$

Let  $A$  be a non-negative self-adjoint operator on  $\mathcal{H}$  with  $E_0(A) = 0$ . Then the Hamiltonian of the DG model is defined by

$$H_{\text{DG}} := A \otimes I + I \otimes H_{\text{b}} + \tilde{\phi}(v).$$

We call it the *Derezinski-Gérard Hamiltonian*. Here  $H_{\text{b}}$  is the second quantization of  $\omega$  introduced in Section 3. Let

$$H_0 := A \otimes I + I \otimes H_{\text{b}}.$$

Throughout this section we assume the following conditions:

[DG.1] There is a Borel measurable function  $v(x, k) \in \mathbb{C}$ , ( $x \in \mathbb{R}^N, k \in \mathbb{R}^d$ ), such that

$$(vf)(x, k) = v(x, k)f(x), \quad f \in L^2(\mathbb{R}^d).$$

We need also the following assumption:

[DG.2]

$$\operatorname{ess.\,sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} \left| \frac{v(x, k)}{\sqrt{\omega(k)}} \right|^2 dk < \infty.$$

**Proposition 4.3.** *Assume [DG.1] and [DG.2]. Then  $H_{\text{DG}}$  is self-adjoint with  $D(H_{\text{DG}}) = D(H_0)$ , and essentially self-adjoint on any core of  $H_0$ .*

For a finite volume approximation, we introduce the following hypotheses:

[DG.3] There exists a nonnegative function  $\tilde{v} \in L^2(\mathbb{R}^d)$  and function  $\tilde{o} : \mathbb{R} \rightarrow \mathbb{R}$ , such that

$$\begin{aligned} \operatorname{ess.\,sup}_{x \in \mathbb{R}^n} |v(x, k) - v(x, \ell)| &\leq \tilde{v}(k) \tilde{o}(|k - \ell|), \quad \text{a.e. } k, \ell \in \mathbb{R}^d \\ \lim_{t \downarrow 0} \tilde{o}(t) &= 0. \end{aligned}$$

[DG.4]

$$\operatorname{ess.\,sup}_{x \in \mathbb{R}^n} \int_{([-K, K]^d)^c} |v(x, k)|^2 dk = o(K^0).$$

where

$$([-K, K]^d)^c := \mathbb{R}^d \setminus (I \times \cdots \times I), \quad I := [-K, K]$$

and,  $o(t^0)$  satisfies  $\lim_{t \rightarrow 0} o(t^0) = 0$ .

Let  $m$  be defined by (3). Let

$$D := \frac{1}{2} \inf_{0 < \epsilon' < \frac{\|v\|}{\|v/\sqrt{\omega}\|^2}} \left( \epsilon' + \frac{1}{\epsilon'} \right). \quad (17)$$

Here,  $v/\sqrt{\omega}$  is a multiplication operator by the function  $v(x, k)/\sqrt{\omega(k)}$  from  $L^2(\mathbb{R}^N)$  to  $L^2(\mathbb{R}^N) \otimes L^2(\mathbb{R}^d)$ . In the case  $m > 0$ , we can establish the existence of a ground state of  $H_{\text{DG}}$ :

**Theorem 4.4.** *Let  $m > 0$ . Suppose that [DG.1]-[DG.4] and [H.4] hold, and suppose*

$$\Sigma(A) - \|v\|D - E_0(H_{\text{DG}}) > 0.$$

Then,  $H_{\text{DG}}$  has purely discrete spectrum in

$$[E_0(H_{\text{DG}}), \min\{E_0(H_{\text{DG}}) + m, \Sigma(A) - \|v\|D\}].$$

In particular  $H_{\text{DG}}$  has a ground state.

*Remark.* In the case where  $A$  has compact resolvent, this theorem has been proved in [ 5 ] A new aspect here is in that  $A$  does not necessarily have compact resolvent. Also our method is different from that in [ 5 ]

#### 4.1 Proof of Proposition 4.3

**Lemma 4.5.** Let  $M(x) = (\int_{\mathbb{R}^d} |v(x, k)|^2 dk)^{1/2}$ ,  $x \in \mathbb{R}^N$  and  $M : L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$  be a multiplication operator by the function  $M(x)$ . Then

$$\|vf\|^2 = \|Mf\|^2, \quad f \in L^2(\mathbb{R}^N).$$

In particular,  $\|v\| = \|M\| = (\text{ess. sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} |v(x, k)|^2 dk)^{1/2}$  hold.

*Proof.* By the Fubini's theorem, we have

$$\|vf\|^2 = \int_{\mathbb{R}^d} dk \int_{\mathbb{R}^N} dx |v(x, k)|^2 |f(x)|^2 = \int_{\mathbb{R}^N} dx \left( |f(x)|^2 \int_{\mathbb{R}^d} |v(x, k)|^2 dk \right) dx.$$

This means the result. ■

The adjoint  $v^*$  has the following form:

**Lemma 4.6.** For all  $g \in \mathcal{H} \otimes L^2(\mathbb{R}^d)$ ,

$$(v^*g)(x) = \int_{\mathbb{R}^d} v(x, k)^* g(x, k) dk, \quad \text{a.e. } x \in \mathbb{R}^d. \quad (18)$$

*Proof.* For all  $f \in \mathcal{H}$ , we have

$$\begin{aligned} \langle g, vf \rangle &= \int dx \int dk g(x, k)^* v(x, k) f(x) \\ &= \int dx \left( \int g(x, k)^* v(x, k) dk \right) f(x). \end{aligned}$$

Since  $f$  is arbitrary, this proves (18). ■

**Lemma 4.7.**  $\tilde{a}(v)$  is

$$\begin{aligned} D(\tilde{a}(v)) &= \left\{ \Psi \in \mathcal{F} \left| \sum_{n=0}^{\infty} (n+1) \int_{\mathbb{R}^{N+dn}} dx dk_1 \cdots dk_n \right. \right. \\ &\quad \left. \left| \int_{\mathbb{R}^d} dk v(k, x)^* \Psi^{(n+1)}(x, k, k_1, \dots, k_n) \right|^2 < \infty \right\} \\ &(\tilde{a}(v)\Psi)^{(n)}(x, k_1, \dots, k_n) \\ &= \sqrt{n+1} \int_{\mathbb{R}^d} v(x, k)^* \Psi^{(n+1)}(x, k, k_1, \dots, k_n), \quad \text{a.e.} \quad (\Psi \in D(\tilde{a}(v))) \end{aligned}$$

*Proof.* Using Lemma 4.6, we have

$$(v^* \otimes I_n) \Psi^{(n+1)}(x, k_1, \dots, k_n) = \int_{\mathbb{R}^d} v^*(x, k) \Psi^{(n+1)}(x, k, k_1, \dots, k_n) dk. \quad (19)$$

This is invariant for all permutations of  $k_1, \dots, k_n$ . Therefore, using Proposition 4.2, we get

$$(\tilde{a}(v)\Psi)^{(n)}(x, k_1, \dots, k_n) = \sqrt{n+1} \int_{\mathbb{R}^d} v(x, k)^* \Psi^{(n+1)}(x, k, k_1, \dots, k_n) dk.$$

■

**Lemma 4.8.** Suppose that [DG.1] and [DG.2] hold. Then,  $D(\tilde{a}(v)) \supset D(I \otimes H_b^{1/2})$  and

$$\|\tilde{a}(v)\Phi\| \leq \|v/\sqrt{\omega}\| \|I \otimes H_b^{1/2}\Phi\|, \quad \Phi \in D(I \otimes H_b^{1/2}).$$

*Proof.* By(19), we have for all  $\Phi \in D(\tilde{a}(v))$

$$\begin{aligned} \|(\tilde{a}(v)\Phi)^{(n)}\|^2 &= (n+1) \int_{\mathbb{R}^{dn+N}} dx dk_1 \cdots dk_n \left| \int_{\mathbb{R}^d} \sqrt{\omega(k)} \right. \\ &\quad \left. \times \frac{1}{\sqrt{\omega(k)}} v(x, k)^* \Phi^{(n+1)}(x, k, k_1, \dots, k_n) dk \right|^2. \end{aligned}$$

Using the Schwarz inequality, one has

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} \sqrt{\omega(k)} \frac{1}{\sqrt{\omega(k)}} v(x, k)^* \Phi^{(n+1)}(x, k, k_1, \dots, k_n) dk \right|^2 \\ &\leq \int_{\mathbb{R}^d} \left| \frac{v(x, k)^*}{\sqrt{\omega(k)}} \right|^2 dk \cdot \int_{\mathbb{R}^d} \omega(k) |\Phi^{(n+1)}(x, k, k_1, \dots, k_n)|^2 dk. \end{aligned}$$

Hence, for every  $\Phi \in \mathcal{D}_0 \cap D(I \otimes H_b^{1/2})$ , we have

$$\begin{aligned}
& \|(\tilde{a}(v)\Phi)^{(n)}\|^2 \\
& \leq \left( \operatorname{ess.\,sup}_x \int_{\mathbb{R}^d} \left| \frac{v(x, k)^*}{\sqrt{\omega(k)}} \right|^2 dk \right) (n+1) \times \\
& \quad \int_{\mathbb{R}^{dn+N}} dx dk_1 \cdots dk_n dk \omega(k) |\Phi^{(n+1)}(x, k, k_1, \dots, k_n)|^2 \\
& = \left( \operatorname{ess.\,sup}_x \int_{\mathbb{R}^d} \left| \frac{v(x, k)^*}{\sqrt{\omega(k)}} \right|^2 dk \right) \times \\
& \quad \int_{\mathbb{R}^{dn+N}} dx dk_1 \cdots dk_{n+1} \sum_{j=1}^{n+1} \omega(k_j) |\Phi^{(n+1)}(x, k_1, \dots, k_{n+1})|^2 \\
& = \left\| \frac{v}{\sqrt{\omega}} \right\| \left\| (I \otimes H_b^{1/2} \Phi)^{(n+1)} \right\|^2.
\end{aligned}$$

Therefore

$$\|\tilde{a}(v)\Phi\| \leq \left\| \frac{v}{\sqrt{\omega}} \right\| \left\| (I \otimes H_b^{1/2} \Phi) \right\|^2.$$

Since,  $\mathcal{D}_0 \cap D(I \otimes H_b^{1/2})$  is a core of  $I \otimes H_b^{1/2}$ , one can extend this inequality to all  $\Phi \in D(I \otimes H_b^{1/2})$ , and  $D(I \otimes H_b^{1/2}) \subset D(\tilde{a}(v))$  holds.  $\blacksquare$

**Lemma 4.9.** *On  $\mathcal{D}_0$ ,  $\tilde{a}(v)$  and  $\tilde{a}^*(v)$  satisfy the following commutation relation:*

$$[\tilde{a}(v), \tilde{a}(v)^*] = \int_{\mathbb{R}^d} |v(\cdot, k)|^2 dk.$$

where the right hand side is a multiplication operator by the function  $: x \mapsto \int_{\mathbb{R}^d} |v(x, k)|^2 dk$ .

*Proof.* Let  $\Phi \in \mathcal{D}_0$ . By the definition of  $\tilde{a}^*(v)$ , and using Proposition 4.2, we get

$$\begin{aligned}
([\tilde{a}^*(v), \tilde{a}(v)]\Phi)^{(n)} &= (\tilde{a}(v)\tilde{a}(v)^*\Phi)^{(n)} - (\tilde{a}(v)^*\tilde{a}(v)\Phi)^{(n)} \\
&= \sqrt{n+1} I_{\mathcal{H}} \otimes S_n(v^* \otimes I_n) (\tilde{a}(v)^*\Phi)^{(n+1)} \\
&\quad - \sqrt{n} (I \otimes S_n)(v \otimes I_{n-1}) (\tilde{a}(v)\Phi)^{(n-1)}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& ([\tilde{a}^*(v), \tilde{a}(v)]\Phi)^{(n)}(x, k_1, \dots, k_n) \\
&= (n+1) \int_{\mathbb{R}^d} v(x, k)^*(I \otimes S_{n+1}(v \otimes I_{n-1})\Phi^{(n)})(x, k, k_1, \dots, k_n) dk \\
&\quad - n \frac{1}{n} \sum_{j=1}^n v(x, k_j)(v^* \otimes I_{n-1}\Phi^{(n)})(x, k_1, \dots, \widehat{k}_j, \dots, k_n) \\
&= \int_{\mathbb{R}^d} dk v(x, k)^* \left( v(x, k)\Phi^{(n)}(x, k_1, \dots, k_n) \right. \\
&\quad \left. + \sum_{j=1}^n v(x, k_j)\Phi^{(n)}(x, k, k_1, \dots, \widehat{k}_j, \dots, k_n) \right) \\
&\quad - \sum_{j=1}^n v(x, k_j) \int_{\mathbb{R}^d} dk v(x, k)^*\Phi^{(n)}(x, k, k_1, \dots, \widehat{k}_j, \dots, k_n) \\
&= \left( \int_{\mathbb{R}^d} |v(x, k)|^2 \right) \Phi(x, k_1, \dots, k_n).
\end{aligned}$$

Here '\$\widehat{\phantom{x}}\$' indicates the omission of the object wearing the hat. ■

**Lemma 4.10.** *Assume, [DG.1] and [DG.2]. Then  $D(I \otimes H_b^{1/2}) \subset D(\tilde{a}^*(v))$  and for all  $\Phi \in D(I \otimes H_b^{1/2})$ ,*

$$\|\tilde{a}^*(v)\Phi\|^2 \leq \|v/\sqrt{\omega}\|^2 \|I \otimes H_b^{1/2}\Phi\|^2 + \|v\|^2 \|\Phi\|^2. \quad (20)$$

*Proof.* For all  $\Phi \in \mathcal{D}_0 \cap D(I \otimes H_b^{1/2})$ , we have

$$\begin{aligned}
\|\tilde{a}^*(v)\Phi\|^2 &= \langle \Phi, \tilde{a}(v)\tilde{a}^*(v)\Phi \rangle = \langle \Phi, \tilde{a}^*(v)\tilde{a}(v)\Phi \rangle + \left\langle \left( \int_{\mathbb{R}^d} |v(\cdot, k)|^2 \right) \Phi, \Phi \right\rangle \\
&\leq \|\tilde{a}(v)\Phi\|^2 + \|v\|^2 \|\Phi\|^2.
\end{aligned}$$

Thus we can apply Lemma 4.8 to obtain the result. ■

Now we can prove Proposition 4.3:

*Proof of Proposition 4.3.* By Lemma 4.8 and 4.10, the operator  $\tilde{\phi}(v)$  is  $I \otimes H_b^{1/2}$ -bounded. Hence  $\tilde{\phi}(v)$  is infinitesimally small with respect to  $I \otimes H_b$ . Namely, for all  $\epsilon > 0$ , there exists a constant  $c_\epsilon > 0$ , such that,

$$\|\tilde{\phi}(v)\Phi\| \leq \epsilon \|I \otimes H_b \Phi\| + c_\epsilon \|\Phi\|, \quad \Phi \in D(I \otimes H_b).$$



Since  $A \geq 0$ , we have

$$\|\tilde{\phi}(v)\Phi\| \leq \epsilon\|H_0\Phi\| + c\|\Phi\|, \quad \Phi \in D(H_0).$$

Thus we can apply the Kato-Rellich theorem to obtain the conclusion of Proposition 4.3.  $\blacksquare$

## 4.2 Proof of Theorem 4.4

In this subsection we suppose that the assumption of Theorem 4.4 holds. Let  $\mathcal{F}_{b,V}$ ,  $\omega_V$ ,  $H_{b,V}$ ,  $H_{0,V}$ ,  $\mathcal{F}_V$ ,  $\Gamma_V$ ,  $\chi_{\ell,V}(k)$  be an object already defined in Section 3, respectively. Suppose that  $\chi_K$  is a characteristic function of  $[-K, K]$ .

For a parameter  $K > 0$ , we define  $v_K \in \mathcal{B}(\mathcal{H}, \mathcal{H} \otimes L^2(\mathbb{R}^d))$  by

$$(v_K f)(x, k) := \chi_{[-K, K]}(k)v(x, k)f(x).$$

and  $v_{K,V} \in \mathcal{B}(\mathcal{H}, \mathcal{H} \otimes L^2(\mathbb{R}^d))$  by

$$(v_{K,V} f)(x, k) := \sum_{\substack{\ell \in \Gamma_V, |\ell_i| < K \\ i=1, \dots, d}} \chi_{\ell,V}(k)v(x, \ell)f(x).$$

**Lemma 4.11.** *The following hold:*

$$\|v_K - v_{K,V}\| \rightarrow 0 (V \rightarrow \infty), \quad \|v_K - v\| \rightarrow 0 (K \rightarrow \infty). \quad (21)$$

$$\left\| \frac{v_K}{\sqrt{\omega}} - \frac{v_{K,V}}{\sqrt{\omega_V}} \right\| \rightarrow 0 (V \rightarrow \infty), \quad \left\| \frac{v}{\sqrt{\omega}} - \frac{v_K}{\sqrt{\omega}} \right\| \rightarrow 0 (K \rightarrow \infty). \quad (22)$$

*Proof.* By [DG.3] and [DG.4], we have

$$\begin{aligned} \|v_K - v_{K,V}\|^2 &= \text{ess.sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} \left| \chi_K(k)v(x, k) - \sum_{\substack{\ell \in \Gamma_V \\ |\ell_i| < K}} v(x, \ell)\chi_{\ell,V}(k) \right|^2 dk \\ &= \text{ess.sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} \sum_{\substack{\ell \in \Gamma_V \\ |\ell_i| < K}} \chi_{\ell,V}(k) |v(x, k) - v(x, \ell)|^2 dk \\ &\leq \text{ess.sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} \sum_{\substack{\ell \in \Gamma_V \\ |\ell_i| < K}} \chi_{\ell,V}(k) |\tilde{v}(k)|^2 \tilde{\sigma}(|k - \ell|)^2 dk \\ &\leq \int_{\mathbb{R}^d} \sum_{\substack{\ell \in \Gamma_V \\ |\ell_i| < K}} \chi_{\ell,V}(k) |\tilde{v}(k)|^2 \tilde{\sigma}(|k - \ell|)^2 dk. \end{aligned}$$

It follows from the property of  $\tilde{\delta}$  that for every  $\epsilon > 0$ , there exists a constant  $V_0 > 0$  such that, for all  $V > V_0$ ,

$$\chi_{\ell,V}(k)\tilde{\delta}(|k - \ell|)^2 \leq \epsilon\chi_{\ell,V}(k).$$

Therefore,

$$\|v_K - v_{K,V}\|^2 \leq \epsilon \int_{\mathbb{R}^d} \sum_{\substack{\ell \in \Gamma_V \\ |\ell_i| < K}} \chi_{\ell,V}(k) |\tilde{v}(k)|^2 dk = \epsilon \|\tilde{v}\|_{L^2(\mathbb{R}^d)}^2.$$

Hence the first one of (21) holds. The second one is a direct result of condition [DG.4] :

$$\begin{aligned} \|v_K - v\|^2 &= \operatorname{ess.\,sup}_x \int_{\mathbb{R}^d} |\chi_K(k) - 1|^2 |v(x, k)|^2 dk \\ &= \operatorname{ess.\,sup}_x \int_{([-K, K]^d)^c} |v(x, k)|^2 dk = o(K^0) \rightarrow 0 \quad (K \rightarrow \infty). \end{aligned}$$

Using [H.4], one can easily check (22). ■

We introduce two operators:

$$\begin{aligned} H_{\text{DG}}(K) &:= A \otimes I + I \otimes H_b + \tilde{\phi}(v_K), \\ H_{\text{DG}}(K, V) &:= A \otimes I + I \otimes H_{b,V} + \tilde{\phi}(v_{K,V}). \end{aligned}$$

- Lemma 4.12.** (i)  $H_{\text{DG}}(K)$  is self-adjoint with  $D(H_{\text{DG}}(K)) = D(H_0)$ , bounded from below, and essentially self-adjoint on any core of  $H_0$ .  
(ii) For sufficiently large  $V > 0$ ,  $H_{\text{DG}}(K, V)$  is self-adjoint with domain  $D(H_{\text{DG}}(K, V)) = D(H_0)$ , bounded from below, and essentially self-adjoint on any core of  $H_0$ .

*Proof.* Similar to the proof of Proposition 4.3. ■

**Lemma 4.13.** For all  $z \in \mathbb{C} \setminus \mathbb{R}$ ,

$$\begin{aligned} \lim_{V \rightarrow \infty} \|(H_{\text{DG}}(K, V) - z)^{-1} - (H_{\text{DG}}(K) - z)^{-1}\| &= 0, \\ \lim_{K \rightarrow \infty} \|(H_{\text{DG}}(K) - z)^{-1} - (H_{\text{DG}} - z)^{-1}\| &= 0. \end{aligned}$$

*Proof.* Similar to the proof of [2, Lemma 3.5] ■

**Lemma 4.14.** *The operator  $H_{\text{DG}}(K, V)$  is reduced by  $\mathcal{F}_V$ .*

*Proof.* We identify  $v(x, \ell)$  with multiplication operator by  $v(\cdot, \ell)$ . By abuse of symbols, we denote  $\chi_{\ell, V}(\cdot)$  by  $\chi_{\ell, V}(k)$ . Then

$$\begin{aligned} (\tilde{a}^*(v(x, \ell)\chi_{\ell, V}(k))\Phi)^{(n)} &= \sqrt{n}(I \otimes S_n)(v(x, \ell)\chi_{\ell, V}(k) \otimes I)\Phi^{(n-1)} \\ &= \sqrt{n}v(x, \ell)S_n(\chi_{\ell, V} \otimes \Phi^{(n-1)}) \\ &= \chi(x, \ell)\sqrt{n}S_n(\chi_{\ell, V} \otimes \Phi^{(n-1)}). \end{aligned}$$

Hence, we have

$$\tilde{a}^*(v(x, \ell)\chi_{\ell, V}(k))\Phi = v(x, \ell) \otimes a^*(\chi_{\ell, V})\Phi.$$

Therefore, we get

$$\tilde{a}^*(v_{K, V}) = \sum_{\substack{\ell \in \Gamma_V \\ |\ell_i| < K}} v(\cdot, \ell) \otimes a^*(\chi_{\ell, V}). \quad (23)$$

Hence, its adjoint is

$$\tilde{a}(v_{K, V}) = \sum_{\substack{\ell \in \Gamma_V \\ |\ell_i| < K}} v(\cdot, \ell)^* \otimes a(\chi_{\ell, V}). \quad (24)$$

This means that the operator  $H_{\text{DG}}(K, V)$  is a special case of the GSB Hamiltonian(see [ 2 ]). Hence, by [ 2 , Lemma 3.7 ]  $H_{\text{DG}}(K, V)$  is reduced by  $\mathcal{F}_V$ . ■

**Lemma 4.15.**  $H_{\text{DG}}(K, V)[\mathcal{F}_V^\perp \geq E_0(H_{\text{DG}}(K, V)) + m$

*Proof.* Similar to the proof of [ 2 , Lemma 3.10 ] ■

**Lemma 4.16.** *For all  $\Phi \in D(I \otimes H_b^{1/2})$ , and for all  $\epsilon' > 0$ ,*

$$|\langle \Phi, \tilde{\phi}(v)\Phi \rangle| \leq \frac{\epsilon'}{\|v\|} \left\| \frac{v}{\sqrt{\omega}} \right\|^2 \|I \otimes H_b^{1/2}\|^2 + \frac{\|v\|}{2} \left( \epsilon' + \frac{1}{\epsilon'} \right) \|\Phi\|^2.$$

*Proof.* For all  $\Phi \in D(I \otimes H_b^{1/2})$ ,  $\epsilon' > 0$ ,

$$\begin{aligned} |\langle \Phi, \tilde{\phi}(v)\Phi \rangle| &\leq \frac{1}{\sqrt{2}} \left( \epsilon \|\tilde{a}(v)\Phi\|^2 + \frac{1}{4\epsilon} \|\Phi\|^2 + \epsilon \|\tilde{a}^*(v)\Phi\|^2 + \frac{1}{4\epsilon} \|\Phi\|^2 \right) \\ &\leq \frac{1}{\sqrt{2}} \left( 2\epsilon \left\| \frac{v}{\sqrt{\omega}} \right\|^2 \|I \otimes H_b^{1/2}\Phi\|^2 + \epsilon \|v\|^2 \|\Phi\|^2 + \frac{1}{2\epsilon} \|\Phi\|^2 \right) \\ &= \sqrt{2}\epsilon \left\| \frac{v}{\sqrt{\omega}} \right\|^2 \|I \otimes H_b^{1/2}\Phi\|^2 + \frac{\|v\|}{2} \left( \sqrt{2}\epsilon \|v\| + \frac{1}{\sqrt{2}\epsilon \|v\|} \right) \|\Phi\|^2, \end{aligned}$$

where we have used Lemma 4.8 and 4.10. Let  $\sqrt{2}\epsilon \|v\| =: \epsilon'$ . Then, for all  $\epsilon' > 0$ , we have

$$|\langle \Phi, \tilde{\phi}(v)\Phi \rangle| \leq \frac{\epsilon'}{\|v\|} \left\| \frac{v}{\sqrt{\omega}} \right\|^2 \|I \otimes H_b^{1/2}\Phi\|^2 + \frac{\|v\|}{2} \left( \epsilon' + \frac{1}{\epsilon'} \right) \|\Phi\|^2. \quad \blacksquare$$

*Proof of Theorem 4.4.* From (23) and (24),  $H_{\text{DG}}(K, V)$  is equal to the special case of the GSB model. Therefore,  $H_{\text{DG}}(K, V)[\mathcal{F}_V$  has the same form with  $H_{\text{DG}}(K, V)$ . Using Lemma 4.16 we have on  $D(H_0) \cap \mathcal{F}_V$

$$\begin{aligned} &H_{\text{DG}}(K, V) \\ &= A \otimes I + I \otimes H_{b,V} + \tilde{\phi}(v_{K,V}) \\ &\geq A \otimes I + I \otimes H_{b,V} - \frac{\epsilon'}{\|v_{K,V}\|} \left\| \frac{v_{K,V}}{\sqrt{\omega_V}} \right\|^2 I \otimes H_{b,V} - \frac{\|v_{K,V}\|}{2} \left( \epsilon' + \frac{1}{\epsilon'} \right) \\ &= A \otimes I + \left( 1 - \frac{\epsilon'}{\|v_{K,V}\|} \left\| \frac{v_{K,V}}{\sqrt{\omega_V}} \right\|^2 \right) I \otimes H_{b,V} - \frac{\|v_{K,V}\|}{2} \left( \epsilon' + \frac{1}{\epsilon'} \right), \quad (25) \end{aligned}$$

where  $\epsilon' > 0$  is an arbitrary constant. By Lemma 3.10,  $H_{b,V}[\mathcal{F}_{b,V}$  has compact resolvent. Thus, for  $\epsilon' > 0$  satisfying

$$1 - \frac{\epsilon'}{\|v_{K,V}\|} \left\| \frac{v_{K,V}}{\sqrt{\omega_V}} \right\|^2 > 0, \quad (26)$$

the bottom of the essential spectrum of (25) is equal to

$$\Sigma(A) - \frac{\|v_{K,V}\|}{2} \left( \epsilon' + \frac{1}{\epsilon'} \right).$$

Let,  $D_K$  and  $D_{K,V}$  be  $D$  with  $v$  replaced by  $v_K$ ,  $v_{K,V}$ , respectively. It is easy to see that

$$\lim_{K \rightarrow \infty} D_K = D, \quad \lim_{V \rightarrow \infty} D_{K,V} = D_K.$$

By Lemma 4.13, one has

$$\lim_{K \rightarrow \infty} E_0(H_{\text{DG}}(K)) = E_0(H_{\text{DG}}), \quad \lim_{V \rightarrow \infty} E_0(H_{\text{DG}}(K, V)) = E_0(\text{DG}(K)).$$

From the assumption of Theorem 4.4, for all  $K > 0$ , there exists a constant  $V_0$  such that for  $V > V_0$ ,

$$\Sigma(A) - \frac{\|v_{K,V}\|}{2} D_{K,V} - E_0(H_{\text{DG}}(K, V)) > 0.$$

By the definition of  $D_{K,V}$ , for all  $K > 0$  and  $V > V_0$ , and for all  $\epsilon'$  which satisfies (26), we have

$$\Sigma(A) - \frac{\|v_{K,V}\|}{2} \left( \epsilon' + \frac{1}{\epsilon'} \right) > E_0(H_{\text{DG}}(K, V)).$$

Therefore, by Theorem 2.1, we have that  $H_{\text{DG}}(K, V)[\mathcal{F}_V$  has purely discrete spectrum in

$$[E_0(H_{\text{DG}}(K, V)), \Sigma(A) - \|v_{K,V}\| D_{K,V}).$$

This fact and Lemma 4.15 mean that  $H_{\text{DG}}(K, V)$  has purely discrete spectrum in

$$[E_0(H_{\text{DG}}(K, V)), \min\{E_0(H_{\text{DG}}(K, V)) + m, \Sigma(A) - \|v_{K,V}\| D_{K,V}\}).$$

Finally, we use Lemma 3.13 and Lemma 4.13, to conclude that  $H_{\text{DG}}$  has purely discrete spectrum in the interval

$$[E_0(H_{\text{DG}}), \min\{E_0(H_{\text{DG}}) + m, \Sigma(A) - \|v\| D\}]$$

■

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