Stability of Discrete Ground State

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Abstract

We present new criteria for a self-adjoint operator to have a ground state. As an application, we consider models of "quantum particles" coupled to a massive Bose field and prove the existence of a ground state of them, where the particle Hamiltonian does not necessarily have compact resolvent.

Key words: Ground state; discrete ground state; generalized spin-boson model; Fock space; Dereziński-Gérard model.

1 INTRODUCTION

Let T be a self-adjoint operator on a Hilbert space $\mathcal{H}$, and bounded from below. We say that T has a discrete ground state if the bottom of the spectrum of T is an isolated eigenvalue of T. In that case a non-zero vector

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in $\ker(T - E_0(T))$ is called a ground state of T. Let S be a symmetric operator on $\mathcal{H}$. Suppose that T has a discrete ground state and S is T-bounded. By the regular perturbation theory [8, XII], it is already known that $T + \lambda S$ has a discrete ground state for "sufficiently small" $\lambda \in \mathbb{R}$. Our aim is to present new criteria for $T + \lambda S$ to have a ground state.

In Section 2, we prove an existence theorem of a ground state which is useful to show the existence of a ground state of models of quantum particles coupled to a massive Bose field.

In Section 3, we consider the GSB model [2] with a self-interaction term of a Bose field, which we call the GSB + ϕ^2 model. We consider only the case where the Bose field is massive. The GSB model — an abstract system of quantum particles coupled to a Bose field — was proposed in [2]. In [2], A. Arai and M. Hirokawa proved the existence and uniqueness of the GSB model in the case where the particle Hamiltonian A has compact resolvent. Shortly after that, they proved the existence of a ground state of the GSB model in the case where A does not have necessarily compact resolvent [4, 3]. In this paper, using a theorem in Section 2, we prove the existence of a ground state of the GSB + ϕ^2 model in the case where A does not necessarily have compact resolvent.

In Section 4, we consider an extended version of the Nelson type model, which we call the Dereziński-Gérard model [5]. The Dereziński-Gérard model introduced in [5], and J. Dereziński and C. Gérard prove an existence of a ground state for their model under some conditions including that A has compact resolvent. In Section 4, we prove the existence of a ground state of the Dereziński-Gérard model in the case where A does not have compact resolvent. Our strategy to establish a ground state is the same as in Section 3.

2 BASIC RESULTS

Let $\mathcal{H}$ be a separable complex Hilbert space. We denote by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ the scalar product on Hilbert space $\mathcal{H}$ and by $\|\cdot\|_{\mathcal{H}}$ the associated norm. Scalar product $\langle f, g \rangle_{\mathcal{H}}$ is linear in g and antilinear in f. We omit $\mathcal{H}$ in $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$, respectively if there is no danger of confusion. For a linear operator T in Hilbert space, we denote by $D(T)$ and $\sigma(T)$ the domain and the spectrum of T respectively. If T is self-adjoint and bounded from below, then we define

$$E_0(T) := \inf \sigma(T), \quad \Sigma(T) := \inf \sigma_{\mathrm{ess}}(T),$$

where $\sigma_{\mathrm{ess}}(T)$ is the essential spectrum of T. If T has no essential spectrum, then we set $\Sigma(T) = \infty$. For a self-adjoint operator T, we denote the form domain of T by $Q(T)$. In this paper, an eigenvector of a self-adjoint operator T with eigenvalue $E_0(T)$ is called a ground state of T (if it exists). We say that T has a ground state if $\dim\ker(T - E_0(T)) > 0$.

The basic results are as follows:

Theorem 2.1. *Let H be a self-adjoint operator on $\mathcal{H}$, and bounded from below. Suppose that there exists a self-adjoint operator V on $\mathcal{H}$ satisfying the following conditions* (i)-(iii):

(i) $D(H) \subset D(V)$.
(ii) *V is bounded from below, and* $\Sigma(V) > 0$.
(iii) $H - E_0(H) \geq V$ *on* $D(H)$.

Then H has purely discrete spectrum in the interval $[E_0(H), E_0(H)+\Sigma(V))$. In particular, H has a ground state.

Proof. For all $u_1, \dots, u_{n-1} \in \mathcal{H}$, we have

$$\inf_{\substack{\Psi \in \mathrm{L.h.}[u_1,\dots,u_{n-1}]^{\perp} \\ \|\Psi\|=1, u \in D(H)}} \langle \Psi, H\Psi \rangle - E_0(H) \geq \inf_{\substack{\Psi \in \mathrm{L.h.}[u_1,\dots,u_{n-1}]^{\perp} \\ \|\Psi\|=1, u \in D(H)}} \langle \Psi, V\Psi \rangle,$$

where $\mathrm{L.h.}[\cdots]$ denotes the linear hull of the vectors in $[\cdots]$. Since $D(H) \subset D(V)$, we have that

$$\inf_{\substack{\Psi \in \mathrm{L.h.}[u_1,\dots,u_{n-1}]^{\perp} \\ \|\Psi\|=1, \Psi \in D(H)}} \langle \Psi, V\Psi \rangle \geq \inf_{\substack{\Psi \in \mathrm{L.h.}[u_1,\dots,u_{n-1}]^{\perp} \\ \|\Psi\|=1, \Psi \in D(V)}} \langle \Psi, V\Psi \rangle.$$

Hence, for all $n \in \mathbb{N}$

$$\mu_n(H) - E_0(H) \geq \mu_n(V).$$

where

$$\mu_n(H) := \sup_{u_1,\dots,u_{n-1} \in \mathcal{H}} \inf_{\substack{\Psi \in \mathrm{L.h.}[u_1,\dots,u_{n-1}]^{\perp} \\ \|\Psi\|=1, \Psi \in D(H)}} \langle \Psi, H\Psi \rangle.$$

By the min-max principle ([8, Theorem XIII.1]), $\lim_{n\to\infty} \mu_n(H) = \Sigma(H)$ and $\lim_{n\to\infty} \mu_n(V) = \Sigma(V)$. Therefore we obtain

$$\Sigma(H) - E_0(H) \geq \Sigma(V) > 0.$$

This means that H has purely discrete spectrum in $[E_0(H), E_0(H)+\Sigma(V))$. ∎

Theorem 2.2. *Let H be a self-adjoint operator on $\mathcal{H}$, and bounded from below. Suppose that there exists a self-adjoint operator V on $\mathcal{H}$ satisfying the following conditions* (i)-(iii):

(i) $Q(H) \subset Q(V)$.
(ii) *V is bounded from below, and $\Sigma(V) > 0$.*
(iii) $H - E_0(H) \geq V$ *on* $Q(H)$.

Then H has purely discrete spectrum in the interval $[E_0(H), E_0(H)+\Sigma(V))$. In particular, H has a ground state.

Proof. Similar to the proof of Theorem 2.1. ∎

We apply Theorems 2.1 and 2.2 to a perturbation problem of a self-adjoint operator.

Theorem 2.3. *Let A be a self-adjoint operator on $\mathcal{H}$ with $E_0(A) = 0$, and let B be a symmetric operator on $D(A)$. Suppose that $A+B$ is self-adjoint on $D(A)$ and that there exist constants $a \in [0,1)$ and $b \geq 0$ such that*

$$|\langle\psi, B\psi\rangle| \leq a\langle\psi, A\psi\rangle + b\|\phi\|^2, \quad \psi \in D(A).$$

Assume

$$\frac{b + E_0(A+B)}{1-a} < \Sigma(A). \tag{1}$$

Then $A+B$ has purely discrete spectrum in $[E_0(A+B), (1-a)\Sigma(A) - b)$. In particular, $A+B$ has a ground state.

Proof. By the assumption we have

$$A + B - E_0(A+B) \geq (1-a)A - b - E_0(A+B)$$

on $D(A)$, and $(1-a)\Sigma(A) - b - E_0(A+B) > 0$. Hence we can apply Theorem 2.1, to conclude that $A+B$ has purely discrete spectrum in $[E_0(A+B), (1-a)\Sigma(A) - b)$. In particular, $A+B$ has a ground state. ∎

Remark. It is easily to see that $-b \leq E_0(A+B) \leq b$. Therefore condition (1) is satisfied if

$$\frac{2b}{1-a} < \Sigma(A).$$

Theorem 2.4. *Let $\mathcal{H}, \mathcal{K}$ be complex separable Hilbert spaces. Let A and B be self-adjoint operators on $\mathcal{H}$ and $\mathcal{K}$ respectively. Suppose that $E_0(A) = E_0(B) = 0$. We set*

$$T_0 := A \otimes I + I \otimes B.$$

Let Z be a symmetric sesquilinear form on $Q(T_0)$, and assume that there exist constants $a_1 \in [0, 1)$, $a_2 \in [0, 1)$ and $b \geq 0$ such that, for all $\Psi \in Q(T_0)$

$$|Z(\Psi, \Psi)| \leq a_1 \langle \Psi, A \otimes I \Psi \rangle_{\text{form}} + a_2 \langle \Psi, I \otimes B \Psi \rangle_{\text{form}} + b \|\Psi\|^2,$$

where $\langle \Psi, A \otimes I \Psi \rangle_{\text{form}} = \|A^{1/2} \otimes I \Psi\|^2$. Therefore, by the KLMN theorem there exists a unique self-adjoint operator T on $\mathcal{H} \otimes \mathcal{K}$ such that $Q(T) = Q(T_0)$ and $T = T_0 + Z$ in the sense of sesquilinear form on $Q(T_0)$. We set

$$s := \min\{(1 - a_1)\Sigma(A), (1 - a_2)\Sigma(B)\}.$$

Assume

$$s > b + E_0(T). \tag{2}$$

Then, T has purely discrete spectrum in the interval $[E_0(T), s - b)$. In particular, T has a ground state.

Proof. Similar to the proof of Theorem 2.3. ∎

Remark. It is easy to see that $-b \leq E_0(T) \leq b$. Therefore the condition (2) is satisfied if

$$s > 2b.$$

Remark. Theorem 2.4 is essentially same as [4, Theorem B.1]. But our proof is very simple.

3 Ground States of a General Class of Quantum Field Hamiltonians

We consider a model which is an abstract unification of some quantum field models of particles interacting with a Bose field. It is the GSB model [2] with a self-interaction term of the field.

Let $\mathcal{H}$ be a separable complex Hilbert space and $\mathcal{F}_{\text{b}}$ be the Boson Fock space over $L^2(\mathbb{R}^d)$:

$$\mathcal{F}_{\text{b}} := \bigoplus_{n=0}^{\infty} \left[\bigotimes_s^n L^2(\mathbb{R}^d) \right].$$

The Hilbert space of the quantum field model we consider is

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_{\mathrm{b}}.$$

Let $\omega : \mathbb{R}^d \to [0, \infty)$ be Borel measurable such that $0 < \omega(k) < \infty$ for all most everywhere (a.e.)$k \in \mathbb{R}^d$. We denote the multiplication operator by the function ω acting in $L^2(\mathbb{R}^d)$ by the same symbol ω. We set

$$H_{\mathrm{b}} := \mathrm{d}\Gamma_{\mathrm{b}}(\omega)$$

the second quantization of ω (e.g. [7, Section X.7]). We denote by $a(f)$, $f \in L^2(\mathbb{R}^d)$, the smeared annihilation operators on $\mathcal{F}_{\mathrm{b}}$. It is a densely defined closed linear operator on $\mathcal{F}_{\mathrm{b}}(\mathbb{R}^d)$ (e.g. [7, Section X.7]). The adjoint $a(f)^*$, called the creation operator, and the annihilation operator $a(g)$, $g \in L^2(\mathbb{R}^d)$ obey the canonical commutation relations

$$[a(f), a(g)^*] = \langle f, g\rangle, \quad [a(f), a(g)] = 0, \quad [a(f)^*, a(g)^*] = 0$$

for all $f, g \in L^2(\mathbb{R}^d)$ on the dense subspace

$$\begin{aligned}\mathcal{F}_0 := &\{\psi = (\psi^{(n)})_{n=0}^{\infty} \in \mathcal{F}_{\mathrm{b}} | \text{there exists a number } n_0 \text{ such that} \\ &\psi^{(n)} = 0 \text{ for all } n \geq n_0\},\end{aligned}$$

where $[X, Y] = XY - YX$. The symmetric operator

$$\phi(f) := \frac{1}{\sqrt{2}}[a(f)^* + a(f)],$$

called the Segal field operator, is essentially self-adjoint on $\mathcal{F}_0$(e.g. [7, Section X.7]). We denote its closure by the same symbol. Let A be a positive self-adjoint operator on $\mathcal{H}$ with $E_0(A) = 0$. Then, the unperturbed Hamiltonian of the model is defined by

$$H_0 := A \otimes I + I \otimes H_{\mathrm{b}}$$

with domain $D(H_0) = D(A \otimes I) \cap D(I \otimes H_{\mathrm{b}})$. For g_j, $f_j \in L^2(\mathbb{R}^d)$ $j = 1, \ldots, J$, and $B_j (j = 1, \ldots, J)$ a symmetric operator on $\mathcal{H}$, we define a symmetric operator

$$H_1 := \sum_{j=1}^{J} B_j \otimes \phi(g_j),$$

$$H_2 := \sum_{j=1}^{J} I \otimes \phi(f_j)^2.$$

The Hamiltonian of the model we consider is of the form

$$H(\lambda, \mu) := H_0 + \lambda H_1 + \mu H_2,$$

where $\lambda \in \mathbb{R}$ and $\mu \geq 0$ are coupling parameters.

For $H(\lambda, \mu)$ to be self-adjoint, we shall need the following conditions [H.1]-[H.3]:

[H.1] $g_j \in D(\omega^{-1/2})$, $f_j \in D(\omega^{1/2}) \cap D(\omega^{-1/2})$, $j = 1, \ldots, J$.

[H.2] $D(A^{1/2}) \subset \cap_{j=1}^{J} D(B_j)$ and there exist constants $a_j \geq 0$, $b_j \geq 0$, $j = 1, \ldots, J$, such that,

$$\|B_j u\| \leq a_j \|A^{1/2} u\| + b_j \|u\|, \quad u \in D(A^{1/2}).$$

[H.3] $|\lambda| \sum_{j=1}^{J} a_j \|g_j / \sqrt{\omega}\| < 1.$

Proposition 3.1. *Assume* [H.1], [H.2] *and* [H.3]. *Then,* $H(\lambda, \mu)$ *is self-adjoint with* $D(H(\lambda, \mu)) = D(H_0) \subset D(H_1) \cap D(H_2)$ *and bounded from below. Moreover,* $H(\lambda, \mu)$ *is essentially self-adjoint on every core of* H_0.

Remark. This proposition has no restriction of the coupling parameter $\mu \geq 0$.

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To perform a finite volume approximation, we need an additional condition:

[H.4] The function $\omega(k)$ ($k \in \mathbb{R}^d$) is continuous with

$$\lim_{|k| \to \infty} \omega(k) = \infty,$$

and there exist constants $\gamma > 0$, $C > 0$ such that

$$|\omega(k) - \omega(k')| \leq C|k - k'|^{\gamma} [1 + \omega(k) + \omega(k')], \quad k, k' \in \mathbb{R}^d.$$

Let

$$m := \inf_{k \in \mathbb{R}^d} \omega(k). \tag{3}$$

If A has compact resolvent, we can prove the extension of the previous theorem [2, Theorem 1.2].

Theorem 3.2. *Consider the case $m > 0$. Suppose that A has entire purely discrete spectrum. Assume Hypotheses [H.1]-[H.4]. Then, $H(\lambda, \mu)$ has purely discrete spectrum in the interval $[E_0(H(\lambda,\mu)), E_0(H(\lambda,\mu)) + m)$. In particular, $H(\lambda, \mu)$ has a ground state.*

Remark. This theorem has no restriction of the coupling parameter $\mu \geq 0$.

Remark. In the case $m > 0$, the condition [H.1] equivalent to the following:

$$g_j \in L^2(\mathbb{R}^d), \quad f_j \in D(\sqrt{\omega}), \quad j = 1, \ldots, J.$$

For a vector $v = (v_1, \ldots, v_J) \in \mathbb{R}^J$ and $h = (h_1, \ldots, h_J) \in \oplus_{j=1}^J L^2(\mathbb{R}^d)$, we define

$$M_v(h) = \sum_{j=1}^J v_j \|h_j\|.$$

We set

$$g = (g_1, \ldots, g_J) \in \bigoplus_{j=1}^J L^2(\mathbb{R}^d), \quad f = (f_1, \ldots, f_J) \in \bigoplus_{j=1}^J L^2(\mathbb{R}^d),$$

and

$$a = (a_1, \ldots, a_J), \quad b = (b_1, \ldots, b_J).$$

For θ, ϵ, ϵ', we introduce the following constants:

$$\begin{aligned}
C_{\theta,\epsilon} &:= \theta M_a(g/\sqrt{\omega}) + \epsilon M_a(g), \\
D_{\theta,\epsilon'} &:= M_a(g/\sqrt{\omega})/2\theta + \epsilon' M_b(g/\sqrt{\omega}), \\
E_{\epsilon,\epsilon'} &:= M_a(g)/8\epsilon + M_b(g/\sqrt{\omega})/2\epsilon' + M_b(g)/\sqrt{2}.
\end{aligned}$$

Let the condition [H.3] be satisfied. Then, we define

$$I_{\lambda,g} := \begin{cases} \left(\dfrac{|\lambda| M_a(g\sqrt{\omega})}{2}, \dfrac{1}{|\lambda| M_a(g/\sqrt{\omega})} \right), & |\lambda| M_a(g/\sqrt{\omega}) \neq 0 \\ [0, \infty], & |\lambda| M_a(g/\sqrt{\omega}) = 0 \end{cases}$$

It is easy to see that $[1/2, 1] \subset I_{\lambda,g}$. Therefore, for all $\theta \in I_{\lambda,g}$,

$$\begin{aligned}
1 - \theta |\lambda| M_a(g/\sqrt{\omega}) &> 0, \\
1 - \frac{|\lambda| M_a(g/\sqrt{\omega})}{2\theta} &> 0.
\end{aligned}$$

We define for $\theta \in I_{\lambda,g}$,

$$\mathsf{S}_\theta := \{(\epsilon,\epsilon')|\epsilon,\epsilon' > 0,\ |\lambda|C_{\theta,\epsilon} < 1,\ |\lambda|D_{\theta,\epsilon'} < 1\}.$$

Next we set

$$\tau_{\theta,\epsilon,\epsilon'} := (1-|\lambda|C_{\theta,\epsilon})\Sigma(A) - |\lambda|E_{\epsilon,\epsilon'},$$

and

$$\mathsf{T} := \big\{(\theta,\epsilon,\epsilon') \in \mathbb{R}^3 | \theta \in I_{\lambda,g},\ (\epsilon,\epsilon') \in \mathsf{S}_\theta,\ \tau_{\theta,\epsilon,\epsilon'} > E_0(H(\lambda,\mu))\big\}.$$

Theorem 3.3. *Consider the case $m > 0$. Suppose that $\sigma_{\mathrm{ess}}(A) \neq \emptyset$. Assume Hypothesis [H.1]-[H.4], and $\mathsf{T} \neq \emptyset$. Then, $H(\lambda,\mu)$ has purely discrete spectrum in the interval*

$$\big[E_0(H(\lambda,\mu)), \min\{m + E_0(H(\lambda,\mu)), \sup_{(\theta,\epsilon,\epsilon')\in\mathsf{T}} \tau_{\theta,\epsilon,\epsilon'}\}\big). \tag{4}$$

In particular, $H(\lambda,\mu)$ has a ground state.

Remark. $\mathsf{T} \neq \emptyset$ is necessary condition for A to have a discrete ground state. Conversely, if A has a discrete ground state, then $\mathsf{T} \neq \emptyset$ holds for sufficiently small λ, μ. Therefore the condition $\mathsf{T} \neq \emptyset$ is a restriction for the coupling constants λ, μ.

* * *

3.1 Proof of Proposition 3.1

In what follows, we write simply

$$H := H(\lambda,\mu).$$

For $\mathcal{D}$ a dense subspace of $L^2(\mathbb{R}^d)$, we define

$$\mathcal{F}_{\mathrm{fin}}(\mathcal{D}) := \mathrm{L.h}[\{\Omega, a(h_1)^* \cdots a(h_n)^*\Omega | n \in \mathbb{N}, h_j \in \mathcal{D}, j = 1,\dots,n\}],$$

where $\Omega := (1,0,0,\dots)$ is the Fock vacuum in $\mathcal{F}_{\mathrm{b}}$. We introduce a dense subspace in $\mathcal{F}$

$$\mathcal{D}_\omega := D(A)\hat{\otimes}\mathcal{F}_{\mathrm{fin}}(D(\omega)),$$

where $\hat{\otimes}$ denotes algebraic tensor product. The subspace $\mathcal{D}_\omega$ is a core of H_0.

Let

$$H_{\mathrm{GSB}} := H_0 + \lambda H_1$$

be a GSB Hamiltonian. The Hamiltonian H and H_{GSB} has the following relation:

Proposition 3.4. *Let $D(A) \subset D(B_j)$, $j = 1, \ldots, J$ and $f_j \in D(\omega^{1/2})$. Assume that H_{GSB} is bounded from below. Then, for all $\Psi \in D_\omega$,*

$$\|(H_{\mathrm{GSB}} - E_0)\Psi\|^2 + \|\mu H_2 \Psi\|^2 \leq \|(H - E_0)\Psi\|^2 + D\|\Psi\|^2, \tag{5}$$

where $D = \mu \sum_{j=1}^{J} \|\omega^{1/2} f_j\|^2$ and

$$E_0 := \inf_{\substack{\Psi \in D(H_{\mathrm{GSB}}) \\ \|\Psi\|=1}} \langle \Psi, H_{\mathrm{GSB}} \Psi \rangle.$$

Proof. It is enough to show (5) the case $\lambda = \mu = 1$. First we consider the case where $f_j \in D(\omega)$. Inequality (5) is equivalent to

$$-2\,\mathrm{Re}\big\langle (H_{\mathrm{GSB}} - E_0)\Psi, H_2 \Psi \big\rangle \leq D\|\Psi\|^2. \tag{6}$$

By $H_{\mathrm{GSB}} - E_0 \geq 0$, we have

$$\begin{aligned} \big\langle (H_{\mathrm{GSB}} - E_0)\Psi, I \otimes \phi(f_j)^2 \Psi \big\rangle =& \big\langle [I \otimes \phi(f_j), (H_{\mathrm{GSB}} - E_0)]\Psi, I \otimes \phi(f_j)\Psi \big\rangle \\ &+ \big\langle (H_{\mathrm{GSB}} - E_0) I \otimes \phi(f_j)\Psi, I \otimes \phi(f_j)\Psi \big\rangle \\ \geq& \big\langle [I \otimes \phi(f_j), H_{\mathrm{GSB}} - E_0]\Psi, I \otimes \phi(f_j)\Psi \big\rangle. \end{aligned}$$

Therefore we have

$$2\,\mathrm{Re}\big\langle (H_{\mathrm{GSB}} - E_0)\Psi, \phi(f_j)^2 \Psi \big\rangle \geq -\|\sqrt{\omega} f_j\|^2 \|\Psi\|^2.$$

This means inequality (6). Next, we set $f_j \in D(\sqrt{\omega})$. Then, there exists a sequence $\{f_{jn}\}_{n=0}^{\infty} \subset D(\omega)$ such that $f_{jn} \to f_j$, $\omega^{1/2} f_{jn} \to \omega^{1/2} f_j$ $(n \to \infty)$. By limiting argument, (6) holds with $f_j \in D(\omega^{1/2})$. ∎

Lemma 3.5. *Suppose that H_{GSB} is self-adjoint with $D(H_{\mathrm{GSB}}) = D(H_0)$, essentially self-adjoint on $\mathcal{D}_\omega$, and bounded from below. Let $f_j \in D(\omega^{1/2}) \cap D(\omega^{-1/2})$. Then H is self-adjoint with $D(H) = D(H_0)$ and essentially self-adjoint on any core of H_{GSB} with*

$$\|(H_{\mathrm{GSB}} - E_0)\Psi\|^2 + \|\mu H_2 \Psi\|^2 \leq \|(H - E_0)\Psi\|^2 + D\|\Psi\|^2, \quad \Psi \in D(H_0).$$

Proof. It is well known that $D(H_{\rm b}) \subset D(\phi(f_j)^2)$, and $\phi(f_j)^2$ is $H_{\rm b}$-bounded (e.g. [1, Lemma 13-16]). Namely, there exist constants $\eta \geq 0$, $\theta \geq 0$ such that

$$\Big\|\sum_{j=1}^{J} \phi(f_j)^2\psi\Big\| \leq \eta\|H_{\rm b}\psi\| + \theta\|\psi\|, \quad \psi \in D(H_{\rm b}). \tag{7}$$

Since $H_{\rm GSB}$ is self-adjoint on $D(H_0)$, by the closed graph theorem, we have

$$\|H_0\Psi\| \leq \lambda\|H_{\rm GSB}\Psi\| + \nu\|\Psi\|,\ \Psi \in D(H_0), \tag{8}$$

where λ and ν are non-negative constant independent of Ψ. Hence

$$\|H_2\Psi\| \leq \eta\lambda\|H_{\rm GSB}\Psi\| + (\eta\nu + \theta)\|\Psi\|, \quad \Psi \in D(H_0).$$

We fix a positive number μ_0 such that $\mu_0 < 1/(\mu\lambda)$. Then, by the Kato-Rellich theorem, $H(\lambda, \mu_0)$ is self-adjoint on $D(H_{\rm GSB})$, bounded from below and essentially self-adjoint on any core of $H_{\rm GSB}$. For a constant a $(0 < a < 1)$, we set $\mu_n := (1 + a)^n\mu_0$. Since $H_{\rm GSB}$ is self-adjoint on $D(H_0)$, for each $j = 1, \ldots, J$ we have $D(A) \subset D(B)$. Thus by Proposition 3.4, for all $\Psi \in \mathcal{D}_\omega$

$$\|(H_{\rm GSB} - E_0)\Psi\|^2 + \|\mu_n H_2\Psi\|^2 \leq \|(H(\lambda, \mu_n) - E_0)\Psi\|^2 + D\|\Psi\|^2.$$

If $H(\lambda, \mu_n)$ is self-adjoint on $D(H_{\rm GSB})$, bounded from below and essentially self-adjoint on any core of $H_{\rm GSB}$, then $H(\lambda, \mu_{n+1})$ has the same property. On the other hand, we have $\mu_n \to \infty$ $(n \to \infty)$. Hence we conclude that H is self-adjoint with $D(H) = D(H_{\rm GSB})$, bounded from below and essentially self-adjoint on any core of $H_{\rm GSB}$. ∎

Now, we assume conditions [H.1],[H.2] and [H.3].

Then $H_{\rm GSB}$ is self-adjoint on $D(H_0)$, bounded from below and essentially self-adjoint on any core of H_0(see [2]). Hence, the assumptions of Lemma 3.5 hold. Thus Proposition 3.1 follows. ∎

3.2 Proofs of Theorems 3.2 and 3.3

Throughout this subsection, we assume Hypotheses [H.1]-[H.4] and $m > 0$.

For a parameter $V > 0$, we define the set of lattice points by

$$\Gamma_V := \frac{2\pi\mathbb{Z}^d}{V} := \left\{ k = (k_1, \ldots, k_d) \middle| k_j = \frac{2\pi n_j}{V}, n_j \in \mathbb{Z}, j = 1, \ldots, d \right\}$$

and we denote by $l^2(\Gamma_V)$ the set of l^2 sequences over Γ_V. For each $k \in \Gamma_V$ we introduce

$$C(k, V) := \left[k_1 - \frac{\pi}{V}, k_1 + \frac{\pi}{V} \right) \times \cdots \times \left[k_d - \frac{\pi}{V}, k_d + \frac{\pi}{V} \right) \subset \mathbb{R}^d,$$

the cube centered about k. By the map

$$U : l^2(\Gamma_V) \ni \{h_l\}_{l \in \Gamma_V} \mapsto (V/2\pi)^{d/2} \sum_{l \in \Gamma_V} h_l \chi_{l,V}(\cdot) \in L^2(\mathbb{R}^d),$$

we identify $l^2(\Gamma_V)$ with a subspace in $L^2(\mathbb{R}^d)$, where $\chi_{l,V}(\cdot)$ is the characteristic function of the cube $C(l, V) \subset \mathbb{R}^d$. It is easy to see that $l^2(\Gamma_V)$ is a closed subspace of $L^2(\mathbb{R}^d)$. Let

$$\mathcal{F}_{\mathrm{b,V}} := \mathcal{F}_{\mathrm{b}}(l^2(\Gamma_V)) = \bigoplus_{n=0}^{\infty} \left[\bigotimes_s^n l^2(\Gamma_V) \right],$$

the boson Fock space over $l^2(\Gamma_V)$. We can identify $\mathcal{F}_{\mathrm{b,V}}$ the closed subspace of $\mathcal{F}_{\mathrm{b}}$ by the operator $\Gamma(U) := \oplus_{n=0}^{\infty} \otimes^n U$, where we define $\otimes^0 U = 0$. For each $k \in \mathbb{R}^d$, there exists a unique point $k_V \in \Gamma_V$ such that $k \in C(k_V, V)$. Let

$$\omega_V(k) := \omega(k_V), \quad k \in \mathbb{R}^d$$

be a lattice approximate function of $\omega(k)$ and let

$$H_{\mathrm{b,V}} := \mathrm{d}\Gamma(\omega_V)$$

be the second quantization of ω_V. We define a constant

$$C_V := C d^\gamma \left(\frac{\pi}{V} \right) \left(\frac{1}{2m} + 1 \right),$$

where C and γ were defined in [H.4]. In what follows we assume that

$$C_V < 1.$$

This is satisfied for all sufficiently large V.

Lemma 3.6. ([2, Lemma 3.1]). *We have*

$$D(H_{\mathrm{b,V}}) = D(H_{\mathrm{b}}),$$

and

$$\|(H_{\mathrm{b}} - H_{\mathrm{b,V}})\Psi\| = \frac{2C_V}{1 - C_V}\|H_{\mathrm{b}}\Psi\|, \quad \Psi \in D(H_{\mathrm{b}}).$$

First we consider the case where g_j's and f_j's are continuous, and finally, by limiting argument, we treat a general case. For a constant $K > 0$, we define $g_{j,K}$, $f_{j,K}$, and $g_{j,K,V}$, $f_{j,K,V}$ as follows:

$$g_{j,K}(k) := \chi_K(k_1)\cdots\chi_K(k_d)g_j(k), \quad g_{j,K,V}(k) := \sum_{\substack{\ell\in\Gamma_V, |\ell_i|<K\\ i=1,\ldots,d}} g_j(\ell)\chi_{\ell,V}(k),$$

$$f_{j,K}(k) := \chi_K(k_1)\cdots\chi_K(k_d)f_j(k), \quad f_{j,K,V}(k) := \sum_{\substack{\ell\in\Gamma_V, |\ell_i|<K\\ i=1,\ldots,d}} f_j(\ell)\chi_{\ell,V}(k),$$

where χ_K denotes the characteristic function of $[-K, K]$.

Lemma 3.7. *For all* $j = 1, \ldots, J$,

$$\begin{aligned}
\lim_{V\to\infty} \|g_{j,K,V} - g_{j,K}\| &= 0, & \lim_{V\to\infty} \|g_{j,K,V}/\sqrt{\omega_V} - g_{j,K}/\sqrt{\omega}\| &= 0,\\
\lim_{K\to\infty} \|g_{j,K} - g_j\| &= 0, & \lim_{K\to\infty} \|g_{j,K}/\sqrt{\omega} - g_j/\sqrt{\omega}\| &= 0,\\
\lim_{V\to\infty} \|f_{j,K,V} - f_{j,K}\| &= 0, & \lim_{V\to\infty} \|f_{j,K,V}/\sqrt{\omega_V} - f_{j,K}/\sqrt{\omega}\| &= 0,\\
\lim_{K\to\infty} \|f_{j,K} - f_j\| &= 0, & \lim_{K\to\infty} \|f_{j,K}/\sqrt{\omega} - f_j/\sqrt{\omega}\| &= 0,\\
\lim_{K\to\infty} \|\sqrt{\omega} f_{j,K} - \sqrt{\omega} f_j\| &= 0, & \lim_{V\to\infty} \|\sqrt{\omega_V} f_{j,K,V} - \sqrt{\omega} f_{j,K}\| &= 0.
\end{aligned}$$

Proof. Similar to the proof of [2, Lemma 3.10]. ∎

We introduce a new operator:

$$\begin{aligned}
H_{0,V} &:= A \otimes I + I \otimes H_{\mathrm{b,V}},\\
H_{1,K} &:= \sum_{j=1}^{J} B_j \otimes \phi(g_{j,K}),\\
H_{1,K,V} &:= \sum_{j=1}^{J} B_j \otimes \phi(g_{j,K,V}),
\end{aligned}$$

$$H_{2,K} := \sum_{j=1}^{J} I \otimes \phi(f_{j,K})^2,$$
$$H_{2,K,V} := \sum_{j=1}^{J} I \otimes \phi(f_{j,K,V})^2,$$

and define

$$\begin{aligned} H_K :=& H_0 + \lambda H_{1,K} + \mu H_{2,K}, \\ H_{K,V} :=& H_{0,V} + \lambda H_{1,K,V} + \mu H_{2,K,V}. \end{aligned}$$

Lemma 3.8. (i) *H_K is self-adjoint with $D(H_K) = D(H_0) \subset D(H_{1,K}) \cap D(H_{2,K})$, bounded from below, and essentially self-adjoint on any core of H_0.*

(ii) *For all large V, $H_{K,V}$ is self-adjoint with $D(H_{K,V}) = D(H_0) \subset D(H_{1,K,V}) \cap D(H_{2,K,V})$, bounded from below, and essentially self-adjoint on any core of $H_{0,V}$.*

Proof. Similar to the proof of Proposition 3.1. ∎

Lemma 3.9. *For all $z \in \mathbb{C}\backslash\mathbb{R}$, and $K > 0$,*

$$\lim_{K\to\infty} \|(H_K - z)^{-1} - (H - z)^{-1}\| = 0,$$
$$\lim_{V\to\infty} \|(H_{K,V} - z)^{-1} - (H_K - z)^{-1}\| = 0.$$

Proof. Similar to the proof of [2, Lemma 3.5]. ∎

The following fact is well known:

Lemma 3.10. *The operator $H_{\mathrm{b,V}}$ is reduced by $\mathcal{F}_{\mathrm{b,V}}$ and $H_{\mathrm{b,V}}\lceil\mathcal{F}_{\mathrm{b,V}}$ equal to the second quantization of $\omega_V\lceil l^2(\Gamma_V)$ on $\mathcal{F}_{\mathrm{b,V}}$.*

Lemma 3.11. *$H_{K,V}$ is reduced by $\mathcal{F}_V$.*

Proof. Similar to the proof of [2, Lemma 3.7]. ∎

Lemma 3.12. *We have*

$$H_{K,V}\lceil\mathcal{F}_V^{\perp} \geq E_0(H_{K,V}) + m.$$

Proof. Similar to the proof of [2, Lemma 3.10]. ∎

Lemma 3.13. *Let T_n and T be a self-adjoint operators on a separable Hilbert space and bounded from below. Suppose that $T_n \to T$ in norm resolvent sense as $n \to \infty$ and T_n has purely discrete spectrum in the interval $[E_0(T_n), E_0(T_n) + c_n)$ with some constant c_n. If $c := \limsup_{n\to\infty} c_n > 0$, then T has purely discrete spectrum in $[E_0(T), E_0(T) + c)$.*

Proof. There exists a sequence $\{c_{n_j}\}_{j=1}^{\infty} \subset \{c_n\}_{n=1}^{\infty}$ so that $c_{n_j} \to c (j \to \infty)$. So, for all $\epsilon > 0$ and for sufficiently large j, the spectrum of T_{n_j} in $[E_0(T_{n_j}), E_0(T_{n_j}) + c - \epsilon)$ is discrete. Therefore, applying [2, Lemma 3.12], we find that the spectrum of T in $[E_0(T), E_0(T) + c - \epsilon)$ is discrete. Since $\epsilon > 0$ is arbitrary, we get the conclusion. ∎

Now, if A has compact resolvent, by a method similar to the proof of [2, Theorem 1.2], we can prove Theorem 3.2. Therefore, we only prove Theorem 3.3.

The following inequality is known [2, (2.12)]:

$$|\langle \Psi, H_1 \Psi \rangle| \leq C_{\theta,\epsilon} \langle \Psi, A \otimes I \Psi \rangle + D_{\theta,\epsilon} \langle \Psi, I \otimes H_{\mathrm{b}} \Psi \rangle + E_{\epsilon,\epsilon'} \|\Psi\|^2,$$

where $\Psi \in D(H_0)$ is arbitrary. Thus we have,

$$H \geq (1 - |\lambda| C_{\theta,\epsilon}) A \otimes I + (1 - |\lambda| D_{\theta,\epsilon'}) I \otimes H_{\mathrm{b}} + \mu H_2 - |\lambda| E_{\epsilon,\epsilon'}.$$

Let $I_{\lambda,g}(K)$, $C_{\theta,\epsilon}(K)$, $D_{\theta,\epsilon}(K)$ and $E_{\epsilon,\epsilon'}(K)$ are $I_{\lambda,g}$, $C_{\theta,\epsilon}$, $D_{\theta,\epsilon}$, $E_{\epsilon,\epsilon'}$ with g_j, f_j replaced by $g_{j,K}$, $f_{j,K}$ respectively, and let $I_{\lambda,g}(K,V)$, $C_{\theta,\epsilon}(K,V)$, $D_{\theta,\epsilon}(K,V)$ and $E_{\epsilon,\epsilon'}(K,V)$ are $I_{\lambda,g}$, $C_{\theta,\epsilon}$, $D_{\theta,\epsilon}$, $E_{\epsilon,\epsilon'}$ with g_j, f_j and ω replaced by $g_{j,K,V}$, $f_{j,K,V}$ and ω_V respectively. Then we have

Lemma 3.14. *The following operator inequalities hold:*

$$\begin{aligned} H_K \geq & (1 - |\lambda| C_{\theta,\epsilon}(K)) A \otimes I + (1 - |\lambda| D_{\theta,\epsilon'}(K)) I \otimes H_{\mathrm{b}} \\ & + \mu H_{2,K} - |\lambda| E_{\epsilon,\epsilon'}(K) \quad \text{on} \quad D(H_0), \\ H_{K,V} \geq & (1 - |\lambda| C_{\theta,\epsilon}(K,V)) A \otimes I + (1 - |\lambda| D_{\theta,\epsilon'}(K,K)) I \otimes H_{\mathrm{b,V}} \\ & + \mu H_{2,K,V} - |\lambda| E_{\epsilon,\epsilon'}(K,V) \quad \text{on} \quad D(H_0). \end{aligned}$$

Proof. Similar to the calculation of [2, (2.12)]. ∎

By Lemma 3.7, we have

$$\lim_{V\to\infty} C_{\theta,\epsilon}(K,V) = C_{\theta,\epsilon}(K), \qquad \lim_{K\to\infty} C_{\theta,\epsilon}(K) = C_{\theta,\epsilon}, \tag{9}$$

$$\lim_{V\to\infty} D_{\theta,\epsilon'}(K,V) = D_{\theta,\epsilon'}(K), \qquad \lim_{K\to\infty} D_{\theta,\epsilon'}(K) = D_{\theta,\epsilon'}, \tag{10}$$

$$\lim_{V\to\infty} E_{\epsilon,\epsilon'}(K,V) = E_{\epsilon,\epsilon'}(K), \qquad \lim_{K\to\infty} E_{\epsilon,\epsilon'}(K) = E_{\epsilon,\epsilon'}. \tag{11}$$

Let $(\theta,\epsilon,\epsilon') \in \mathsf{T}$, namely

$$\tau_{\theta,\epsilon,\epsilon'} = (1-|\lambda|C_{\theta,\epsilon})\Sigma(A) - |\lambda|E_{\epsilon,\epsilon'} > E_0(H).$$

Formulas (9)-(11) and Lemma 3.9 imply that for all large V there exists a constant $K_0 > 0$ such that for all $K > K_0$,

$$(1-|\lambda|C_{\theta,\epsilon}(K,V))\Sigma(A) - |\lambda|E_{\epsilon,\epsilon'}(K,V) > E_0(H_{K,V}), \tag{12}$$

$$|\lambda|C_{\theta,\epsilon}(K,V) < 1, \quad |\lambda|D_{\theta,\epsilon'}(K,V) < 1. \tag{13}$$

By Lemma 3.11, $H_{K,V}$ is reduced by $\mathcal{F}_V$. Therefore, $H_{K,V}$ satisfies the following inequality:

$$\begin{aligned} H_{K,V}\lceil\mathcal{F}_V \geq & (1-|\lambda|C_{\theta,\epsilon}(K,V))A\otimes I\lceil\mathcal{F}_V \\ & + (1-|\lambda|D_{\theta,\epsilon'}(K,V))I\otimes H_{\mathrm{b,V}}\lceil\mathcal{F}_V \\ & - |\lambda|E_{\epsilon,\epsilon'}(K,V). \end{aligned} \tag{14}$$

Since $H_{\mathrm{b,V}}\lceil\mathcal{F}_{\mathrm{b,V}}$ has compact resolvent, the bottom of essential spectrum of the right hand side of (14) is equal to

$$(1-|\lambda|C_{\theta,\epsilon}(K,V))\Sigma(A) - |\lambda|E_{\epsilon,\epsilon'}(K,V).$$

By Lemma 3.12, we have $E_0(H_{K,V}\lceil\mathcal{F}_V) = E_0(H_{K,V})$. Thus, applying Theorem 2.1 with $H_{K,V}\lceil\mathcal{F}_V$, we have that $H_{K,V}\lceil\mathcal{F}_V$ has purely discrete spectrum in $[E_0(H_{K,V}), (1-|\lambda|C_{\theta,\epsilon}(K,V))\Sigma_A - E_{\epsilon,\epsilon'}(K,V)\,)$. Since this fact and Lemma 3.12, $H_{K,V}$ has purely discrete spectrum in

$$[E_0(H_{K,V}), \min\{E_0(H_{K,V}) + m, (1-|\lambda|C_{\theta,\epsilon}(K,V))\Sigma_A - E_{\epsilon,\epsilon'}(K,V)\}).$$

By Lemma 3.9 and Lemma 3.13, we have that for all sufficiently large $K > 0$, H_K has purely discrete spectrum in $[E_0(H_K), \min\{E_0(H_K)+m, (1-|\lambda|C_{\theta,\epsilon}(K))\Sigma(A) - |\lambda|E_{\epsilon,\epsilon'}(K)\})$. Similarly, H has purely discrete spectrum in $[E_0(H(\lambda,\mu)), \min\{m + E_0(H(\lambda,\mu)), \tau_{\theta,\epsilon,\epsilon'}\})$. Since $(\theta,\epsilon,\epsilon') \in \mathsf{T}$ is arbitrary, H has purely discrete spectrum in (4). Finally, we have to consider the case where g_j's and f_j's are not necessarily continuous. But, that argument were already discussed in [4]. So we skip that argument. ∎

4 Ground State of the Dereziński-Gérard Model

We consider a model discussed by J. Dereziński and C. Gérard [5]. We take the Hilbert space of the particle system is taken to be

$$\mathcal{H} = L^2(\mathbb{R}^N).$$

The Hilbert space for the Dereziński-Gérard (DG) model is given by

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^d)).$$

We identify $\mathcal{F}$ as

$$\bigoplus_{n=0}^{\infty} \left[\mathcal{H} \otimes \bigotimes_s^n L^2(\mathbb{R}^d) \right].$$

Hence, if we denote that $\Psi \in (\Psi^{(n)})_{n=0}^{\infty} \in \mathcal{F}$, each $\Psi^{(n)}$ belongs to $\mathcal{H} \otimes [\otimes_s^n L^2(\mathbb{R}^d)]$. We denote by $\mathsf{B}(\mathcal{K}, \mathcal{J})$ the set of bounded linear operators from $\mathcal{K}$ to $\mathcal{J}$. For $v \in \mathsf{B}(\mathcal{H}, \mathcal{H} \otimes L^2(\mathbb{R}^d))$, we define an operator $\widetilde{a}^*(v)$ by

$$\begin{aligned}
&(\widetilde{a}^*(v)\Psi)^{(0)} := 0, \\
&(\widetilde{a}^*(v)\Psi)^{(n)} := \sqrt{n}(I_{\mathcal{H}} \otimes S_n)(v \otimes I_{\otimes_s^{n-1} L^2(\mathbb{R}^d)})\Psi^{(n-1)}, \quad (n \geq 1), \\
&\Psi \in D(\widetilde{a}^*(v)) := \left\{ \Psi = (\Psi^{(n)})_{n=0}^{\infty} \in \mathcal{F} \middle| \sum_{n=0}^{\infty} \|(\widetilde{a}^*(v)\Psi)^{(n)}\|^2 < \infty \right\}.
\end{aligned}$$

We set

$$\begin{aligned}
\mathcal{D}_0 := \{\Psi = (\Psi^{(n)})_{n=0}^{\infty} \in \mathcal{F} | &\text{there exists a constant } n_0 \in \mathbb{N}, \\
&\text{such that, for all } n \geq n_0,\ \Psi^{(n)} = 0\}.
\end{aligned}$$

Throughout this section, we write simply $I_n := I_{\otimes_s^n L^2(\mathbb{R}^d)}$. It is easy to see that:

Proposition 4.1. *$\widetilde{a}^*(v)$ is a closed linear operator and $\mathcal{D}_0$ is a core of $\widetilde{a}^*(v)$.*

So we set

$$\widetilde{a}(v) := (\widetilde{a}^*(v))^*$$

the adjoint operator of $\widetilde{a}^*(v)$.

Proposition 4.2. *The operator $\widetilde{a}(v)$ has the following properties:*

$$D(\widetilde{a}(v)) = \left\{ \Psi = (\Psi^{(n)})_{n=0}^{\infty} \Big| \sum_{n=0}^{\infty} (n+1) \|(I_{\mathcal{H}} \otimes S_n)(v^* \otimes I_n)\Psi^{(n+1)}\|^2 < \infty \right\} \tag{15}$$

$$(\widetilde{a}(v)\Psi)^{(n)} = \sqrt{n+1} I_{\mathcal{H}} \otimes S_n(v^* \otimes I_n)\Psi^{(n+1)}, \quad \Psi \in D(\widetilde{a}(v)), \tag{16}$$

and $\mathcal{D}_0$ is a core of $\widetilde{a}(v)$.

Proof. For $\Phi \in \mathcal{F}$, $\Psi \in D(\widetilde{a}^*(v))$,

$$\begin{aligned} \langle \Phi, \widetilde{a}^*(v)\Psi \rangle &= \sum_{n=1}^{\infty} \langle \Phi^{(n)}, \sqrt{n}(I_{\mathcal{H}} \otimes S_n)(v \otimes I_{n-1})\Psi^{(n-1)} \rangle \\ &= \sum_{n=0}^{\infty} \sqrt{n+1} \langle v^* \otimes I_n \Phi^{(n+1)}, \Psi^{(n)} \rangle \\ &= \sum_{n=0}^{\infty} \big\langle \sqrt{n+1}(I_{\mathcal{H}} \otimes S_n)(v^* \otimes I_n)\Phi^{(n+1)}, \Psi^{(n)} \big\rangle. \end{aligned}$$

This implies (15) and (16). It is easy to prove that $\mathcal{D}_0$ is a core of $\widetilde{a}(v)$. ∎

An analogue of the Segal field operator is defined by

$$\widetilde{\phi}(v) := \frac{1}{\sqrt{2}}(\widetilde{a}(v) + \widetilde{a}^*(v)).$$

Let A be a non-negative self-adjoint operator on $\mathcal{H}$ with $E_0(A) = 0$. Then the Hamiltonian of the DG model is defined by

$$H_{\mathrm{DG}} := A \otimes I + I \otimes H_{\mathrm{b}} + \widetilde{\phi}(v).$$

We call it the *Dereziński-Gérard Hamiltonian*. Here H_{b} is the second quantization of ω introduce in Section 3. Let

$$H_0 := A \otimes I + I \otimes H_{\mathrm{b}}.$$

Throughout this section we assume the following conditions:

[DG.1] There is a Borel measurable function $v(x,k) \in \mathbb{C}$, $(x \in \mathbb{R}^N, k \in \mathbb{R}^d)$, such that

$$(vf)(x,k) = v(x,k)f(x), \quad f \in L^2(\mathbb{R}^d).$$

We need also the following assumption:

[DG.2]

$$\operatorname*{ess.sup}_{x\in\mathbb{R}^N}\int_{\mathbb{R}^d}\left|\frac{v(x,k)}{\sqrt{\omega(k)}}\right|^2 \mathrm{d}k<\infty.$$

Proposition 4.3. *Assume* [DG.1] *and* [DG.2]*. Then H_{DG} is self-adjoint with $D(H_{\mathrm{DG}})=D(H_0)$, and essentially self-adjoint on any core of H_0.*

For a finite volume approximation, we introduce the following hypotheses:

[DG.3] There exists a nonnegative function $\widetilde{v}\in L^2(\mathbb{R}^d)$ and function $\widetilde{o}:\mathbb{R}\to\mathbb{R}$, such that

$$\operatorname*{ess.sup}_{x\in\mathbb{R}^n}|v(x,k)-v(x,\ell)|\le\widetilde{v}(k)\widetilde{o}(|k-\ell|),\quad \text{a.e. } k,\ell\in\mathbb{R}^d$$

$$\lim_{t\downarrow 0}\widetilde{o}(t)=0.$$

[DG.4]

$$\operatorname*{ess.sup}_{x\in\mathbb{R}^n}\int_{([-K,K]^d)^{\mathrm{c}}}|v(x,k)|^2\mathrm{d}k=\mathsf{o}(K^0).$$

where

$$([-K,K]^d)^{\mathrm{c}}:=\mathbb{R}^d\setminus(I\times\cdots\times I),\quad I:=[-K,K]$$

and, $\mathsf{o}(t^0)$ satisfies $\lim_{t\to 0}\mathsf{o}(t^0)=0$.

Let m be defined by (3). Let

$$D:=\frac{1}{2}\inf_{0<\epsilon'<\frac{\|v\|}{\|v/\sqrt{\omega}\|^2}}\Big(\epsilon'+\frac{1}{\epsilon'}\Big). \tag{17}$$

Here, $v/\sqrt{\omega}$ is a multiplication operator by the function $v(x,k)/\sqrt{\omega(k)}$ from $L^2(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N)\otimes L^2(\mathbb{R}^d)$. In the case $m>0$, we can establish the existence of a ground state of H_{DG}:

Theorem 4.4. *Let $m>0$. Suppose that* [DG.1]-[DG.4] *and* [H.4] *hold, and suppose*

$$\Sigma(A)-\|v\|D-E_0(H_{\mathrm{DG}})>0.$$

Then, H_{DG} has purely discrete spectrum in

$$[E_0(H_{\mathrm{DG}}), \min\{E_0(H_{\mathrm{DG}}) + m, \Sigma(A) - \|v\|D\}).$$

In particular H_{DG} has a ground state.

Remark. In the case where A has compact resolvent, this theorem has been proved in [5]. A new aspect here is in that A does not necessarily have compact resolvent. Also our method is different from that in [5].

4.1 Proof of Proposition 4.3

Lemma 4.5. *Let $M(x) = (\int_{\mathbb{R}^d} |v(x,k)|^2 \mathrm{d}k)^{1/2}$, $x \in \mathbb{R}^N$ and $M : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ be a multiplication operator by the function $M(x)$. Then*

$$\|vf\|^2 = \|Mf\|^2, \quad f \in L^2(\mathbb{R}^N).$$

In particular, $\|v\| = \|M\| = (\mathrm{ess.sup}_{x\in\mathbb{R}^N} \int_{\mathbb{R}^d} |v(x,k)|^2 \mathrm{d}k)^{1/2}$ hold.

Proof. By the Fubini's theorem, we have

$$\|vf\|^2 = \int_{\mathbb{R}^d} \mathrm{d}k \int_{\mathbb{R}^N} \mathrm{d}x |v(x,k)|^2 |f(x)|^2 = \int_{\mathbb{R}^N} \left(|f(x)|^2 \int_{\mathbb{R}^d} |v(x,k)|^2 \mathrm{d}k \right) \mathrm{d}x.$$

This means the result. ∎

The adjoint v^* has the following form:

Lemma 4.6. *For all $g \in \mathcal{H} \otimes L^2(\mathbb{R}^d)$,*

$$(v^* g)(x) = \int_{\mathbb{R}^d} v(x,k)^* g(x,k) \mathrm{d}k, \quad \text{a.e.}\, x \in \mathbb{R}^d. \tag{18}$$

Proof. For all $f \in \mathcal{H}$, we have

$$\begin{aligned} \langle g, vf \rangle &= \int \mathrm{d}x \int \mathrm{d}k g(x,k)^* v(x,k) f(x) \\ &= \int \mathrm{d}x \Big(\int g(x,k)^* v(x,k) \mathrm{d}k \Big) f(x). \end{aligned}$$

Since f is arbitrary, this proves (18). ∎

Lemma 4.7. *$\widetilde{a}(v)$ is*

$$D(\widetilde{a}(v)) = \left\{\Psi \in \mathcal{F} \middle| \sum_{n=0}^{\infty}(n+1)\int_{\mathbb{R}^{N+dn}} \mathrm{d}x\mathrm{d}k_1\cdots\mathrm{d}k_n \right.$$
$$\left. \left|\int_{\mathbb{R}^d} \mathrm{d}k v(k,x)^*\Psi^{(n+1)}(x,k,k_1,\dots,k_n)\right|^2 < \infty\right\}$$
$$(\widetilde{a}(v)\Psi)^{(n)}(x,k_1,\dots,k_n)$$
$$= \sqrt{n+1}\int_{\mathbb{R}^d} v(x,k)^*\Psi^{(n+1)}(x,k,k_1,\dots,k_n), \quad \text{a.e.} \quad (\Psi \in D(\widetilde{a}(v)))$$

Proof. Using Lemma 4.6, we have

$$(v^*\otimes I_n)\Psi^{(n+1)}(x,k_1,\dots,k_n) = \int_{\mathbb{R}^d} v^*(x,k)\Psi^{(n+1)}(x,k,k_1,\dots,k_n)\mathrm{d}k. \tag{19}$$

This is invariant for all permutations of $k_1,\dots,k_n$. Therefore, using Proposition 4.2, we get

$$(\widetilde{a}(v)\Psi)^{(n)}(x,k_1,\dots,k_n) = \sqrt{n+1}\int_{\mathbb{R}^d} v(x,k)^*\Psi^{(n+1)}(x,k,k_1,\dots,k_n)\mathrm{d}k.$$

∎

Lemma 4.8. *Suppose that* [DG.1] *and* [DG.2] *hold. Then,* $D(\widetilde{a}(v)) \supset D(I\otimes H_{\mathrm{b}}^{1/2})$ *and*

$$\|\widetilde{a}(v)\Phi\| \le \|v/\sqrt{\omega}\|\|I\otimes H_{\mathrm{b}}^{1/2}\Phi\|, \quad \Phi \in D(I\otimes H_{\mathrm{b}}^{1/2}).$$

Proof. By(19), we have for all $\Phi \in D(\widetilde{a}(v))$

$$\|(\widetilde{a}(v)\Phi)^{(n)}\|^2 = (n+1)\int_{\mathbb{R}^{dn+N}} \mathrm{d}x\mathrm{d}k_1\cdots\mathrm{d}k_n\left|\int_{\mathbb{R}^d}\sqrt{\omega(k)}\right.$$
$$\left.\times\frac{1}{\sqrt{\omega(k)}}v(x,k)^*\Phi^{(n+1)}(x,k,k_1,\dots,k_n)\mathrm{d}k\right|^2.$$

Using the Schwarz inequality, one has

$$\left|\int_{\mathbb{R}^d}\sqrt{\omega(k)}\frac{1}{\sqrt{\omega(k)}}v(x,k)^*\Phi^{(n+1)}(x,k,k_1,\dots,k_n)\mathrm{d}k\right|^2$$
$$\le \int_{\mathbb{R}^d}\left|\frac{v(x,k)^*}{\sqrt{\omega(k)}}\right|^2\mathrm{d}k\cdot\int_{\mathbb{R}^d}\omega(k)|\Phi^{(n+1)}(x,k,k_1,\dots,k_n)|^2\mathrm{d}k.$$

Hence, for every $\Phi \in \mathcal{D}_0 \cap D(I \otimes H_{\mathrm{b}}^{1/2})$, we have

$$
\begin{aligned}
&\|(\widetilde{a}(v)\Phi)^{(n)}\|^2 \\
&\leq \left(\operatorname{ess.sup}_x \int_{\mathbb{R}^d} \Big| \frac{v(x,k)^*}{\sqrt{\omega(k)}} \Big|^2 \mathrm{d}k \right) (n+1) \times \\
&\quad \int_{\mathbb{R}^{dn+N}} \mathrm{d}x \mathrm{d}k_1 \cdots \mathrm{d}k_n \mathrm{d}k \omega(k) |\Phi^{(n+1)}(x,k,k_1,\ldots,k_n)|^2 \\
&= \left(\operatorname{ess.sup}_x \int_{\mathbb{R}^d} \Big| \frac{v(x,k)^*}{\sqrt{\omega(k)}} \Big|^2 \mathrm{d}k \right) \times \\
&\quad \int_{\mathbb{R}^{dn+N}} \mathrm{d}x \mathrm{d}k_1 \cdots \mathrm{d}k_{n+1} \sum_{j=1}^{n+1} \omega(k_j) |\Phi^{(n+1)}(x,k_1,\ldots,k_{n+1})|^2 \\
&= \Big\| \frac{v}{\sqrt{\omega}} \Big\| \big\| (I \otimes H_{\mathrm{b}}^{1/2} \Phi)^{(n+1)} \big\|^2.
\end{aligned}
$$

Therefore

$$
\|\widetilde{a}(v)\Phi\| \leq \Big\| \frac{v}{\sqrt{\omega}} \Big\| \big\| (I \otimes H_{\mathrm{b}}^{1/2} \Phi) \big\|^2.
$$

Since, $\mathcal{D}_0 \cap D(I \otimes H_{\mathrm{b}}^{1/2})$ is a core of $I \otimes H_{\mathrm{b}}^{1/2}$, one can extend this inequality to all $\Phi \in D(I \otimes H_{\mathrm{b}}^{1/2})$, and $D(I \otimes H_{\mathrm{b}}^{1/2}) \subset D(\widetilde{a}(v))$ holds. ∎

Lemma 4.9. *On $\mathcal{D}_0$, $\widetilde{a}(v)$ and $\widetilde{a}^*(v)$ satisfy the following commutation relation:*

$$
[\widetilde{a}(v), \widetilde{a}(v)^*] = \int_{\mathbb{R}^d} |v(\cdot,k)|^2 \mathrm{d}k.
$$

where the right hand side is a multiplication operator by the function : $x \mapsto \int_{\mathbb{R}^d} |v(x,k)|^2 \mathrm{d}k$.

Proof. Let $\Phi \in \mathcal{D}_0$. By the definition of $\widetilde{a}^*(v)$, and using Proposition 4.2, we get

$$
\begin{aligned}
([\widetilde{a}^*(v), \widetilde{a}(v)]\Phi)^{(n)} =& (\widetilde{a}(v)\widetilde{a}(v)^*\Phi)^{(n)} - (\widetilde{a}(v)^*\widetilde{a}(v)\Phi)^{(n)} \\
=& \sqrt{n+1} I_{\mathcal{H}} \otimes S_n (v^* \otimes I_n)(\widetilde{a}(v)^*\Phi)^{(n+1)} \\
& - \sqrt{n}(I \otimes S_n)(v \otimes I_{n-1})(\widetilde{a}(v)\Phi)^{(n-1)}.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&([\widetilde{a}^*(v),\widetilde{a}(v)]\Phi)^{(n)}(x,k_1,\ldots,k_n)\\
&= (n+1)\int_{\mathbb{R}^d} v(x,k)^*(I\otimes S_{n+1}(v\otimes I_{n-1})\Phi^{(n)})(x,k,k_1,\ldots,k_n)\mathrm{d}k\\
&\quad - n\frac{1}{n}\sum_{j=1}^n v(x,k_j)(v^*\otimes I_{n-1}\Phi^{(n)})(x,k_1,\ldots,\widehat{k_j},\ldots,k_n)\\
&= \int_{\mathbb{R}^d}\mathrm{d}k\, v(x,k)^*\Big(v(x,k)\Phi^{(n)}(x,k_1,\ldots,k_n)\\
&\quad + \sum_{j=1}^n v(x,k_j)\Phi^{(n)}(x,k,k_1,\ldots,\widehat{k_j},\ldots,k_n)\Big)\\
&\quad - \sum_{j=1}^n v(x,k_j)\int_{\mathbb{R}^d}\mathrm{d}k v(x,k)^*\Phi^{(n)}(x,k,k_1,\ldots,\widehat{k_j},\ldots,k_n)\\
&= \left(\int_{\mathbb{R}^d}|v(x,k)|^2\right)\Phi(x,k_1,\ldots,k_n).
\end{aligned}
$$

Here '$\widehat{\ }$' indicates the omission of the object wearing the hat. ∎

Lemma 4.10. *Assume,* [DG.1] *and* [DG.2]. *Then* $D(I\otimes H_{\mathrm{b}}^{1/2})\subset D(\widetilde{a}^*(v))$ *and for all* $\Phi\in D(I\otimes H_{\mathrm{b}}^{1/2})$,

$$
\|\widetilde{a}^*(v)\Phi\|^2 \le \|v/\sqrt{\omega}\|^2\|I\otimes H_{\mathrm{b}}^{1/2}\Phi\|^2 + \|v\|^2\|\Phi\|^2. \tag{20}
$$

Proof. For all $\Phi\in\mathcal{D}_0\cap D(I\otimes H_{\mathrm{b}}^{1/2})$, we have

$$
\begin{aligned}
\|\widetilde{a}^*(v)\Phi\|^2 &= \langle\Phi,\widetilde{a}(v)\widetilde{a}^*(v)\Phi\rangle = \langle\Phi,\widetilde{a}^*(v)\widetilde{a}(v)\Phi\rangle + \left\langle\left(\int_{\mathbb{R}^d}|v(\cdot,k)|^2\right)\Phi,\Phi\right\rangle\\
&\le \|\widetilde{a}(v)\Phi\|^2 + \|v\|^2\|\Phi\|^2.
\end{aligned}
$$

Thus we can apply Lemma 4.8 to obtain the result. ∎

Now we can prove Proposition 4.3:

Proof of Proposition 4.3. By Lemma 4.8 and 4.10, the operator $\widetilde{\phi}(v)$ is $I\otimes H_{\mathrm{b}}^{1/2}$-bounded. Hence $\widetilde{\phi}(v)$ is infinitesimally small with respect to $I\otimes H_{\mathrm{b}}$. Namely, for all $\epsilon>0$, there exists a constant $c_\epsilon>0$, such that,

$$
\|\widetilde{\phi}(v)\Phi\| \le \epsilon\|I\otimes H_{\mathrm{b}}\Phi\| + c_\epsilon\|\Phi\|, \quad \Phi\in D(I\otimes H_{\mathrm{b}}).
$$

Since $A \geq 0$, we have

$$\|\widetilde{\phi}(v)\Phi\| \leq \epsilon\|H_0\Phi\| + c\|\Phi\|, \quad \Phi \in D(H_0).$$

Thus we can apply the Kato-Rellich theorem to obtain the conclusion of Proposition 4.3. ∎

4.2 Proof of Theorem 4.4

In this subsection we suppose that the assumption of Theorem 4.4 holds. Let $\mathcal{F}_{\mathrm{b,V}}$, ω_V, $H_{\mathrm{b,V}}$, $H_{0,V}$, $\mathcal{F}_V$, Γ_V, $\chi_{\ell,V}(k)$ be an object already defined in Section 3, respectively. Suppose that χ_K is a characteristic function of $[-K, K]$.

For a parameter $K > 0$, we define $v_K \in \mathsf{B}(\mathcal{H}, \mathcal{H} \otimes L^2(\mathbb{R}^d))$ by

$$(v_K f)(x,k) := \chi_{[-K,K]}(k)v(x,k)f(x).$$

and $v_{K,V} \in \mathsf{B}(\mathcal{H}, \mathcal{H} \otimes L^2(\mathbb{R}^d))$ by

$$(v_{K,V} f)(x,k) := \sum_{\substack{\ell \in \Gamma_V, |\ell_i| < K \\ i=1,\ldots,d}} \chi_{\ell,V}(k)v(x,\ell)f(x).$$

Lemma 4.11. *The following hold:*

$$\|v_K - v_{K,V}\| \to 0\,(V \to \infty), \quad \|v_K - v\| \to 0\,(K \to \infty). \tag{21}$$

$$\left\|\frac{v_K}{\sqrt{\omega}} - \frac{v_{K,V}}{\sqrt{\omega_V}}\right\| \to 0\,(V \to \infty), \quad \left\|\frac{v}{\sqrt{\omega}} - \frac{v_K}{\sqrt{\omega}}\right\| \to 0\,(K \to \infty). \tag{22}$$

Proof. By [DG.3] and [DG.4], we have

$$\begin{aligned}
\|v_K - v_{K,V}\|^2 &= \operatorname*{ess.sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} \Big|\chi_K(k)v(x,k) - \sum_{\substack{\ell \in \Gamma_V \\ |\ell_i| < K}} v(x,\ell)\chi_{\ell,V}(k)\Big|^2 \mathrm{d}k \\
&= \operatorname*{ess.sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} \sum_{\substack{\ell \in \Gamma_V \\ |\ell_i| < K}} \chi_{\ell,V}(k)|v(x,k) - v(x,\ell)|^2 \mathrm{d}k \\
&\leq \operatorname*{ess.sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} \sum_{\substack{\ell \in \Gamma_V \\ |\ell_i| < K}} \chi_{\ell,V}(k)|\widetilde{v}(k)|^2 \widetilde{o}(|k-\ell|)^2 \mathrm{d}k \\
&\leq \int_{\mathbb{R}^d} \sum_{\substack{\ell \in \Gamma_V \\ |\ell_i| < K}} \chi_{\ell,V}(k)|\widetilde{v}(k)|^2 \widetilde{o}(|k-\ell|)^2 \mathrm{d}k.
\end{aligned}$$

It follows from the property of $\widetilde{o}$ that for every $\epsilon > 0$, there exists a constant $V_0 > 0$ such that, for all $V > V_0$,

$$\chi_{\ell,V}(k)\widetilde{o}(|k-\ell|)^2 \leq \epsilon\chi_{\ell,V}(k).$$

Therefore,

$$\|v_K - v_{K,V}\|^2 \leq \epsilon \int_{\mathbb{R}^d} \sum_{\substack{\ell\in\Gamma_V \\ |\ell_i|<K}} \chi_{\ell,V}(k)|\widetilde{v}(k)|^2 \mathrm{d}k = \epsilon\|\widetilde{v}\|^2_{L^2(\mathbb{R}^d)}.$$

Hence the first one of (21) holds. The second one is a direct result of condition [DG.4] :

$$\begin{aligned}\|v_K - v\|^2 &= \operatorname*{ess.sup}_x \int_{\mathbb{R}^d} |\chi_K(k)-1|^2|v(x,k)|^2\mathrm{d}k \\ &= \operatorname*{ess.sup}_x \int_{([-K,K]^d)^c} |v(x,k)|^2\mathrm{d}k = \mathsf{o}(K^0) \to 0\,(K\to\infty).\end{aligned}$$

Using [H.4], one can easily check (22). ∎

We introduce two operators:

$$\begin{aligned}H_{\mathrm{DG}}(K) &:= A\otimes I + I\otimes H_{\mathrm{b}} + \widetilde{\phi}(v_K), \\ H_{\mathrm{DG}}(K,V) &:= A\otimes I + I\otimes H_{\mathrm{b,V}} + \widetilde{\phi}(v_{K,V}).\end{aligned}$$

Lemma 4.12. (i) *$H_{\mathrm{DG}}(K)$ is self-adjoint with $D(H_{\mathrm{DG}}(K)) = D(H_0)$, bounded from below, and essentially self-adjoint on any core of H_0.*

(ii) *For sufficiently large $V > 0$, $H_{\mathrm{DG}}(K,V)$ is self-adjoint with domain $D(H_{\mathrm{DG}}(K,V)) = D(H_0)$, bounded from below, and essentially self-adjoint on any core of H_0.*

Proof. Similar to the proof of Proposition 4.3. ∎

Lemma 4.13. *For all $z \in \mathbb{C}\backslash\mathbb{R}$,*

$$\begin{aligned}\lim_{V\to\infty} \|(H_{\mathrm{DG}}(K,V)-z)^{-1} - (H_{\mathrm{DG}}(K)-z)^{-1}\| = 0, \\ \lim_{K\to\infty} \|(H_{\mathrm{DG}}(K)-z)^{-1} - (H_{\mathrm{DG}}-z)^{-1}\| = 0.\end{aligned}$$

Proof. Similar to the proof of [2, Lemma 3.5]. ∎

Lemma 4.14. *The operator $H_{\mathrm{DG}}(K, V)$ is reduced by $\mathcal{F}_V$.*

Proof. We identify $v(x,\ell)$ with multiplication operator by $v(\cdot,\ell)$. By abuse of symbols, we denote $\chi_{\ell,V}(\cdot)$ by $\chi_{\ell,V}(k)$. Then

$$\begin{aligned}(\widetilde{a}^*(v(x,\ell)\chi_{\ell,V}(k))\Phi)^{(n)} &= \sqrt{n}(I\otimes S_n)(v(x,\ell)\chi_{\ell,V}(k)\otimes I)\Phi^{(n-1)}\\ &= \sqrt{n}v(x,\ell)S_n(\chi_{\ell,V}\otimes\Phi^{(n-1)})\\ &= \chi(x,\ell)\sqrt{n}S_n(\chi_{\ell,V}\otimes\Phi^{(n-1)}).\end{aligned}$$

Hence, we have

$$\widetilde{a}^*(v(x,\ell)\chi_{\ell,V}(k))\Phi = v(x,\ell)\otimes a^*(\chi_{\ell,V})\Phi.$$

Therefore, we get

$$\widetilde{a}^*(v_{K,V}) = \sum_{\substack{\ell\in\Gamma_V\\ |\ell_i|<K}} v(\cdot,\ell)\otimes a^*(\chi_{\ell,V}). \tag{23}$$

Hence, its adjoint is

$$\widetilde{a}(v_{K,V}) = \sum_{\substack{\ell\in\Gamma_V\\ |\ell_i|<K}} v(\cdot,\ell)^*\otimes a(\chi_{\ell,V}). \tag{24}$$

This means that the operator $H_{\mathrm{DG}}(K,V)$ is a special case of the GSB Hamiltonian(see [2]). Hence, by [2, Lemma 3.7], $H_{\mathrm{DG}}(K,V)$ is reduced by $\mathcal{F}_V$. ∎

Lemma 4.15. $H_{\mathrm{DG}}(K,V)\lceil\mathcal{F}_V^{\perp} \geq E_0(H_{\mathrm{DG}}(K,V)) + m$

Proof. Similar to the proof of [2, Lemma 3.10]. ∎

Lemma 4.16. *For all $\Phi\in D(I\otimes H_{\mathrm{b}}^{1/2})$, and for all $\epsilon'>0$,*

$$|\langle\Phi,\widetilde{\phi}(v)\Phi\rangle| \leq \frac{\epsilon'}{\|v\|}\left\|\frac{v}{\sqrt{\omega}}\right\|^2\|I\otimes H_{\mathrm{b}}^{1/2}\|^2 + \frac{\|v\|}{2}\left(\epsilon'+\frac{1}{\epsilon'}\right)\|\Phi\|^2.$$

Proof. For all $\Phi \in D(I \otimes H_{\mathrm{b}}^{1/2})$, $\epsilon' > 0$,

$$
\begin{aligned}
|\langle \Phi, \widetilde{\phi}(v)\Phi\rangle| &\le \frac{1}{\sqrt{2}}\left(\epsilon\|\widetilde{a}(v)\Phi\|^2 + \frac{1}{4\epsilon}\|\Phi\|^2 + \epsilon\|\widetilde{a}^*(v)\Phi\|^2 + \frac{1}{4\epsilon}\|\Phi\|^2\right)\\
&\le \frac{1}{\sqrt{2}}\left(2\epsilon\left\|\frac{v}{\sqrt{\omega}}\right\|^2\|I \otimes H_{\mathrm{b}}^{1/2}\Phi\|^2 + \epsilon\|v\|^2\|\Phi\|^2 + \frac{1}{2\epsilon}\|\Phi\|^2\right)\\
&= \sqrt{2}\epsilon\left\|\frac{v}{\sqrt{\omega}}\right\|^2\|I \otimes H_{\mathrm{b}}^{1/2}\Phi\|^2 + \frac{\|v\|}{2}\left(\sqrt{2}\epsilon\|v\| + \frac{1}{\sqrt{2}\epsilon\|v\|}\right)\|\Phi\|^2,
\end{aligned}
$$

where we have used Lemma 4.8 and 4.10. Let $\sqrt{2}\epsilon\|v\| =: \epsilon'$. Then, for all $\epsilon' > 0$, we have

$$
|\langle \Phi, \widetilde{\phi}(v)\Phi\rangle| \le \frac{\epsilon'}{\|v\|}\left\|\frac{v}{\sqrt{\omega}}\right\|^2\|I \otimes H_{\mathrm{b}}^{1/2}\Phi\|^2 + \frac{\|v\|}{2}\left(\epsilon' + \frac{1}{\epsilon'}\right)\|\Phi\|^2.
$$

∎

Proof of Theorem 4.4. From (23) and (24), $H_{\mathrm{DG}}(K,V)$ is equal to the special case of the GSB model. Therefore, $H_{\mathrm{DG}}(K,V)\lceil\mathcal{F}_V$ has the same form with $H_{\mathrm{DG}}(K,V)$. Using Lemma 4.16 we have on $D(H_0)\cap\mathcal{F}_V$

$$
\begin{aligned}
&H_{\mathrm{DG}}(K,V)\\
&= A \otimes I + I \otimes H_{\mathrm{b,V}} + \widetilde{\phi}(v_{K,V})\\
&\ge A \otimes I + I \otimes H_{\mathrm{b,V}} - \frac{\epsilon'}{\|v_{K,V}\|}\left\|\frac{v_{K,V}}{\sqrt{\omega_V}}\right\|^2 I \otimes H_{\mathrm{b,V}} - \frac{\|v_{K,V}\|}{2}\left(\epsilon' + \frac{1}{\epsilon'}\right)\\
&= A \otimes I + \left(1 - \frac{\epsilon'}{\|v_{K,V}\|}\left\|\frac{v_{K,V}}{\sqrt{\omega_V}}\right\|^2\right) I \otimes H_{\mathrm{b,V}} - \frac{\|v_{K,V}\|}{2}\left(\epsilon' + \frac{1}{\epsilon'}\right), \quad (25)
\end{aligned}
$$

where $\epsilon' > 0$ is an arbitrary constant. By Lemma 3.10, $H_{\mathrm{b,V}}\lceil\mathcal{F}_{\mathrm{b,V}}$ has compact resolvent. Thus, for $\epsilon' > 0$ satisfying

$$
1 - \frac{\epsilon'}{\|v_{K,V}\|}\left\|\frac{v_{K,V}}{\sqrt{\omega_V}}\right\|^2 > 0, \tag{26}
$$

the bottom of the essential spectrum of (25) is equal to

$$
\Sigma(A) - \frac{\|v_{k,V}\|}{2}\left(\epsilon' + \frac{1}{\epsilon'}\right).
$$

Let, D_K and $D_{K,V}$ be D with v replaced by v_K, $v_{K,V}$, respectively. It is easy to see that

$$\lim_{K\to\infty} D_K = D, \quad \lim_{V\to\infty} D_{K,V} = D_K.$$

By Lemma 4.13, one has

$$\lim_{K\to\infty} E_0(H_{\mathrm{DG}}(K)) = E_0(H_{\mathrm{DG}}), \quad \lim_{V\to\infty} E_0(H_{\mathrm{DG}}(K,V)) = E_0(\mathrm{DG}(K)).$$

From the assumption of Theorem 4.4, for all $K > 0$, there exists a constant V_0 such that for $V > V_0$,

$$\Sigma(A) - \frac{\|v_{K,V}\|}{2} D_{K,V} - E_0(H_{\mathrm{DG}}(K,V)) > 0.$$

By the definition of $D_{K,V}$, for all $K > 0$ and $V > V_0$, and for all ϵ' which satisfies (26), we have

$$\Sigma(A) - \frac{\|v_{K,V}\|}{2}\left(\epsilon' + \frac{1}{\epsilon'}\right) > E_0(H_{\mathrm{DG}}(K,V)).$$

Therefore, by Theorem 2.1, we have that $H_{\mathrm{DG}}(K,V)\lceil\mathcal{F}_V$ has purely discrete spectrum in

$$[E_0(H_{\mathrm{DG}}(K,V)), \Sigma(A) - \|v_{K,V}\|D_{K,V}).$$

This fact and Lemma 4.15 mean that $H_{\mathrm{DG}}(K,V)$ has purely discrete spectrum in

$$[E_0(H_{\mathrm{DG}}(K,V)), \min\{E_0(H_{\mathrm{DG}}(K,V)) + m,\ \Sigma(A) - \|v_{K,V}\|D_{K,V}\}).$$

Finally, we use Lemma 3.13 and Lemma 4.13, to conclude that H_{DG} has purely discrete spectrum in the interval

$$\big[E_0(H_{\mathrm{DG}}), \min\{E_0(H_{\mathrm{DG}}) + m, \Sigma(A) - \|v\|D\}\big)$$

∎

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