Abstract

We present new criteria for a self-adjoint operator to have a ground state. As an application, we consider models of “quantum particles” coupled to a massive Bose field and prove the existence of a ground state of them, where the particle Hamiltonian does not necessarily have compact resolvent.

Key words: Ground state; discrete ground state; generalized spin-boson model; Fock space; Dereziński-Gérard model.

1 INTRODUCTION

Let $T$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$, and bounded from below. We say that $T$ has a discrete ground state if the bottom of the spectrum of $T$ is an isolated eigenvalue of $T$. In that case a non-zero vector
in $\ker(T - E_0(T))$ is called a ground state of $T$. Let $S$ be a symmetric operator on $\mathcal{H}$. Suppose that $T$ has a discrete ground state and $S$ is $T$-bounded. By the regular perturbation theory\cite{8, XII} it is already known that $T + \lambda S$ has a discrete ground state for “sufficiently small” $\lambda \in \mathbb{R}$. Our aim is to present new criteria for $T + \lambda S$ to have a ground state.

In Section 2, we prove an existence theorem of a ground state which is useful to show the existence of a ground state of models of quantum particles coupled to a massive Bose field.

In Section 3, we consider the GSB model\cite{2} with a self-interaction term of a Bose field, which we call the GSB + $\phi^2$ model. We consider only the case where the Bose field is massive. The GSB model — an abstract system of quantum particles coupled to a Bose field — was proposed in\cite{2}. In\cite{2} A. Arai and M. Hirokawa proved the existence and uniqueness of the GSB model in the case where the particle Hamiltonian $A$ has compact resolvent. Shortly after that, they proved the existence of a ground state of the GSB model in the case where $A$ does not have necessarily compact resolvent\cite{4, 3}. In this paper, using a theorem in Section 2, we prove the existence of a ground state of the GSB + $\phi^2$ model in the case where $A$ does not necessarily have compact resolvent.

In Section 4, we consider an extended version of the Nelson type model, which we call the Dereziński-Gérard model\cite{5}. The Dereziński-Gérard model introduced in\cite{5} and J. Dereziński and C. Gérard prove an existence of a ground state for their model under some conditions including that $A$ has compact resolvent. In Section 4, we prove the existence of a ground state of the Dereziński-Gérard model in the case where $A$ does not have compact resolvent. Our strategy to establish a ground state is the same as in Section 3.

2 BASIC RESULTS

Let $\mathcal{H}$ be a separable complex Hilbert space. We denote by $\langle \cdot, \cdot \rangle_\mathcal{H}$ the scalar product on Hilbert space $\mathcal{H}$ and by $\|\cdot\|_\mathcal{H}$ the associated norm. Scalar product $\langle f, g \rangle_\mathcal{H}$ is linear in $g$ and antilinear in $f$. We omit $\mathcal{H}$ in $\langle \cdot, \cdot \rangle_\mathcal{H}$ and $\|\cdot\|_\mathcal{H}$, respectively if there is no danger of confusion. For a linear operator $T$ in Hilbert space, we denote by $D(T)$ and $\sigma(T)$ the domain and the spectrum of $T$ respectively. If $T$ is self-adjoint and bounded from below, then we define $E_0(T) := \inf \sigma(T)$, $\Sigma(T) := \inf \sigma_{ess}(T)$,
where $\sigma_{\text{ess}}(T)$ is the essential spectrum of $T$. If $T$ has no essential spectrum, then we set $\Sigma(T) = \infty$. For a self-adjoint operator $T$, we denote the form domain of $T$ by $Q(T)$. In this paper, an eigenvector of a self-adjoint operator $T$ with eigenvalue $E_0(T)$ is called a ground state of $T$ (if it exists). We say that $T$ has a ground state if $\dim \ker (T - E_0(T)) > 0$.

The basic results are as follows:

**Theorem 2.1.** Let $H$ be a self-adjoint operator on $\mathcal{H}$, and bounded from below. Suppose that there exists a self-adjoint operator $V$ on $\mathcal{H}$ satisfying the following conditions (i)-(iii):

(i) $D(H) \subset D(V)$.
(ii) $V$ is bounded from below, and $\Sigma(V) > 0$.
(iii) $H - E_0(H) \geq V$ on $D(H)$.

Then $H$ has purely discrete spectrum in the interval $[E_0(H), E_0(H) + \Sigma(V))$. In particular, $H$ has a ground state.

**Proof.** For all $u_1, \ldots, u_{n-1} \in \mathcal{H}$, we have

$$\inf_{\Psi \in \text{L.h.} | u_1, \ldots, u_{n-1} \rangle} \langle \Psi, H \Psi \rangle - E_0(H) \geq \inf_{\Psi \in \text{L.h.} | u_1, \ldots, u_{n-1} \rangle} \langle \Psi, V \Psi \rangle,$$

where L.h. denotes the linear hull of the vectors in $[\cdots]$. Since $D(H) \subset D(V)$, we have that

$$\inf_{\Psi \in \text{L.h.} | u_1, \ldots, u_{n-1} \rangle \mid \| \Psi \| = 1, \Psi \in D(H)} \langle \Psi, V \Psi \rangle \geq \inf_{\Psi \in \text{L.h.} | u_1, \ldots, u_{n-1} \rangle \mid \| \Psi \| = 1, \Psi \in D(V)} \langle \Psi, V \Psi \rangle.$$

Hence, for all $n \in \mathbb{N}$

$$\mu_n(H) - E_0(H) \geq \mu_n(V).$$

where

$$\mu_n(H) := \sup_{u_1, \ldots, u_{n-1} \in \mathcal{H}} \inf_{\Psi \in \text{L.h.} | u_1, \ldots, u_{n-1} \rangle \mid \| \Psi \| = 1, \Psi \in D(H)} \langle \Psi, H \Psi \rangle.$$

By the min-max principle (cf. Theorem XIII.1), $\lim_{n \to \infty} \mu_n(H) = \Sigma(H)$ and $\lim_{n \to \infty} \mu_n(V) = \Sigma(V)$. Therefore we obtain

$$\Sigma(H) - E_0(H) \geq \Sigma(V) > 0.$$

This means that $H$ has purely discrete spectrum in $[E_0(H), E_0(H) + \Sigma(V))$. \qed
Theorem 2.2. Let $H$ be a self-adjoint operator on $\mathcal{H}$, and bounded from below. Suppose that there exists a self-adjoint operator $V$ on $\mathcal{H}$ satisfying the following conditions (i)-(iii):

(i) $Q(H) \subset Q(V)$.
(ii) $V$ is bounded from below, and $\Sigma(V) > 0$.
(iii) $H - E_0(H) \geq V$ on $Q(H)$.

Then $H$ has purely discrete spectrum in the interval $[E_0(H), E_0(H) + \Sigma(V))$. In particular, $H$ has a ground state.

Proof. Similar to the proof of Theorem 2.1.

We apply Theorems 2.1 and 2.2 to a perturbation problem of a self-adjoint operator.

Theorem 2.3. Let $A$ be a self-adjoint operator on $\mathcal{H}$ with $E_0(A) = 0$, and let $B$ be a symmetric operator on $D(A)$. Suppose that $A + B$ is self-adjoint on $D(A)$ and that there exist constants $a \in [0, 1)$ and $b \geq 0$ such that

$$|\langle \psi, B\psi \rangle| \leq a \langle \psi, A\psi \rangle + b\|\phi\|^2, \quad \psi \in D(A).$$

Assume

$$\frac{b + E_0(A + B)}{1 - a} < \Sigma(A). \quad (1)$$

Then $A + B$ has purely discrete spectrum in $[E_0(A + B), (1 - a)\Sigma(A) - b)$. In particular, $A + B$ has a ground state.

Proof. By the assumption we have

$$A + B - E_0(A + B) \geq (1 - a)A - b - E_0(A + B)$$

on $D(A)$, and $(1-a)\Sigma(A) - b - E_0(A+B) > 0$. Hence we can apply Theorem 2.1, to conclude that $A+B$ has purely discrete spectrum in $[E_0(A+B), (1-a)\Sigma(A) - b)$. In particular, $A+B$ has a ground state.

Remark. It is easily to see that $-b \leq E_0(A + B) \leq b$. Therefore condition (1) is satisfied if

$$\frac{2b}{1 - a} < \Sigma(A).$$
Theorem 2.4. Let $\mathcal{H}, \mathcal{K}$ be complex separable Hilbert spaces. Let $A$ and $B$ be self-adjoint operators on $\mathcal{H}$ and $\mathcal{K}$ respectively. Suppose that $E_0(A) = E_0(B) = 0$. We set

$$T_0 := A \otimes I + I \otimes B.$$ 

Let $Z$ be a symmetric sesquilinear form on $Q(T_0)$, and assume that there exist constants $a_1 \in [0, 1), a_2 \in [0, 1)$ and $b \geq 0$ such that, for all $\Psi \in Q(T_0)$

$$|Z(\Psi, \Psi)| \leq a_1 \langle \Psi, A \otimes I \Psi \rangle_{\text{form}} + a_2 \langle \Psi, I \otimes B \Psi \rangle_{\text{form}} + b \|\Psi\|^2,$$

where $\langle \Psi, A \otimes I \Psi \rangle_{\text{form}} = \|A^{1/2} \otimes I \Psi\|^2$. Therefore, by the KLMN theorem there exists a unique self-adjoint operator $T$ on $\mathcal{H} \otimes \mathcal{K}$ such that $Q(T) = Q(T_0)$ and $T = T_0 + Z$ in the sense of sesquilinear form on $Q(T_0)$. We set

$$s := \min\{(1 - a_1)\Sigma(A), (1 - a_2)\Sigma(B)\}.$$ 

Assume

$$s > b + E_0(T).$$

Then, $T$ has purely discrete spectrum in the interval $[E_0(T), s - b)$. In particular, $T$ has a ground state.

Proof. Similar to the proof of Theorem 2.3.

Remark. It is easy to see that $-b \leq E_0(T) \leq b$. Therefore the condition (2) is satisfied if

$$s > 2b.$$ 

Remark. Theorem 2.4 is essentially same as [4, Theorem B.1]. But our proof is very simple.

3 Ground States of a General Class of Quantum Field Hamiltonians

We consider a model which is an abstract unification of some quantum field models of particles interacting with a Bose field. It is the GSB model [2] with a self-interaction term of the field.

Let $\mathcal{H}$ be a separable complex Hilbert space and $\mathcal{F}_b$ be the Boson Fock space over $L^2(\mathbb{R}^d)$:

$$\mathcal{F}_b := \bigoplus_{n=0}^{\infty} \bigotimes_{s}^{n} L^2(\mathbb{R}^d).$$
The Hilbert space of the quantum field model we consider is
\[ \mathcal{F} := \mathcal{H} \otimes \mathcal{F}_b. \]
Let \( \omega : \mathbb{R}^d \to [0, \infty) \) be Borel measurable such that \( 0 < \omega(k) < \infty \) for all most everywhere (a.e.) \( k \in \mathbb{R}^d \). We denote the multiplication operator by the function \( \omega \) acting in \( L^2(\mathbb{R}^d) \) by the same symbol \( \omega \). We set
\[ \mathcal{H}_b := d\Gamma_b(\omega) \]
the second quantization of \( \omega \) (e.g. [7, Section X.7]). We denote by \( a(f), f \in L^2(\mathbb{R}^d) \), the smeared annihilation operators on \( \mathcal{F}_b \). It is a densely defined closed linear operator on \( \mathcal{F}_b(\mathbb{R}^d) \) (e.g. [7, Section X.7]). The adjoint \( a(f)^* \), called the creation operator, and the annihilation operator \( a(g), g \in L^2(\mathbb{R}^d) \) obey the canonical commutation relations
\[ [a(f), a(g)^*] = \langle f, g \rangle, \quad [a(f), a(g)] = 0, \quad [a(f)^*, a(g)^*] = 0 \]
for all \( f, g \in L^2(\mathbb{R}^d) \) on the dense subspace
\[ \mathcal{F}_0 := \{ \psi = (\psi^{(n)})_{n=0}^\infty \in \mathcal{F}_b | \text{there exists a number } n_0 \text{ such that} \psi^{(n)} = 0 \text{ for all } n \geq n_0 \}, \]
where \([X,Y] = XY - YX\). The symmetric operator
\[ \phi(f) := \frac{1}{\sqrt{2}} [a(f)^* + a(f)], \]
called the Segal field operator, is essentially self-adjoint on \( \mathcal{F}_0 \) (e.g. [7, Section X.7]). We denote its closure by the same symbol. Let \( A \) be a positive self-adjoint operator on \( \mathcal{H} \) with \( E_0(A) = 0 \). Then, the unperturbed Hamiltonian of the model is defined by
\[ H_0 := A \otimes I + I \otimes H_b \]
with domain \( D(H_0) = D(A \otimes I) \cap D(I \otimes H_b) \). For \( g_j, f_j \in L^2(\mathbb{R}^d) j = 1, \ldots, J \), and \( B_j(j = 1, \ldots, J) \) a symmetric operator on \( \mathcal{H} \), we define a symmetric operator
\[ H_1 := \sum_{j=1}^J B_j \otimes \phi(g_j), \]
\[ H_2 := \sum_{j=1}^J I \otimes \phi(f_j)^2. \]
The Hamiltonian of the model we consider is of the form
\[ H(\lambda, \mu) := H_0 + \lambda H_1 + \mu H_2, \]
where \( \lambda \in \mathbb{R} \) and \( \mu \geq 0 \) are coupling parameters.

For \( H(\lambda, \mu) \) to be self-adjoint, we shall need the following conditions [H.1]-[H.3]:

[H.1] \( g_j \in \mathcal{D}(\omega^{-1/2}), f_j \in \mathcal{D}(\omega^{1/2}) \cap \mathcal{D}(\omega^{-1/2}), j = 1, \ldots, J. \)

[H.2] \( \mathcal{D}(A^{1/2}) \subset \cap_{j=1}^J D(B_j) \) and there exist constants \( a_j \geq 0, b_j \geq 0, j = 1, \ldots, J, \) such that,
\[ \|B_j u\| \leq a_j \|A^{1/2} u\| + b_j \|u\|, \quad u \in \mathcal{D}(A^{1/2}). \]

[H.3] \( |\lambda| \sum_{j=1}^J a_j \|g_j/\sqrt{\omega}\| < 1. \)

**Proposition 3.1.** Assume [H.1], [H.2] and [H.3]. Then, \( H(\lambda, \mu) \) is self-adjoint with \( \mathcal{D}(H(\lambda, \mu)) = \mathcal{D}(H_0) \subset \mathcal{D}(H_1) \cap \mathcal{D}(H_2) \) and bounded from below. Moreover, \( H(\lambda, \mu) \) is essentially self-adjoint on every core of \( H_0. \)

**Remark.** This proposition has no restriction of the coupling parameter \( \mu \geq 0. \)

***

To perform a finite volume approximation, we need an additional condition:

[H.4] The function \( \omega(k) \) \( (k \in \mathbb{R}^d) \) is continuous with
\[ \lim_{|k| \to \infty} \omega(k) = \infty, \]
and there exist constants \( \gamma > 0, C > 0 \) such that
\[ |\omega(k) - \omega(k')| \leq C|k - k'|^{\gamma} [1 + \omega(k) + \omega(k')], \quad k, k' \in \mathbb{R}^d. \]

Let
\[ m := \inf_{k \in \mathbb{R}^d} \omega(k). \quad (3) \]

If \( A \) has compact resolvent, we can prove the extension of the previous theorem [2, Theorem 1.2].
Theorem 3.2. Consider the case $m > 0$. Suppose that $A$ has entire purely discrete spectrum. Assume Hypotheses [H.1]-[H.4]. Then, $H(\lambda, \mu)$ has purely discrete spectrum in the interval $[E_0(H(\lambda, \mu)), E_0(H(\lambda, \mu)) + m)$. In particular, $H(\lambda, \mu)$ has a ground state.

Remark. This theorem has no restriction of the coupling parameter $\mu \geq 0$.

Remark. In the case $m > 0$, the condition [H.1] equivalent to the following:

$$g_j \in L^2(\mathbb{R}^d), \quad f_j \in D(\sqrt{\omega}), \quad j = 1, \ldots, J.$$ 

For a vector $v = (v_1, \ldots, v_J) \in \mathbb{R}^J$ and $h = (h_1, \ldots, h_J) \in \bigoplus_{j=1}^{J} L^2(\mathbb{R}^d)$, we define

$$M_v(h) = \sum_{j=1}^{J} v_j \|h_j\|.$$ 

We set

$$g = (g_1, \ldots, g_J) \in \bigoplus_{j=1}^{J} L^2(\mathbb{R}^d), \quad f = (f_1, \ldots, f_J) \in \bigoplus_{j=1}^{J} L^2(\mathbb{R}^d),$$

and

$$a = (a_1, \ldots, a_J), \quad b = (b_1, \ldots, b_J).$$

For $\theta, \epsilon, \epsilon'$, we introduce the following constants:

$$C_{\theta, \epsilon} := \theta M_a(g/\sqrt{\omega}) + \epsilon M_a(g),$$

$$D_{\theta, \epsilon'} := M_a(g/\sqrt{\omega})/2\theta + \epsilon' M_b(g/\sqrt{\omega}),$$

$$E_{\epsilon, \epsilon'} := M_a(g)/8\epsilon + M_b(g/\sqrt{\omega})/2\epsilon' + M_b(g)/\sqrt{2}.$$ 

Let the condition [H.3] be satisfied. Then, we define

$$I_{\lambda, g} := \left\{ \begin{array}{ll} 1 - \frac{|\lambda|M_a(g/\sqrt{\omega})}{2}, & |\lambda|M_a(g/\sqrt{\omega}) \neq 0 \\ 0, & |\lambda|M_a(g/\sqrt{\omega}) = 0 \end{array} \right.$$ 

It is easy to see that $[1/2, 1] \subset I_{\lambda, g}$. Therefore, for all $\theta \in I_{\lambda, g}$,

$$1 - \theta|\lambda|M_a(g/\sqrt{\omega}) > 0,$$

$$1 - \frac{|\lambda|M_a(g/\sqrt{\omega})}{2\theta} > 0.$$ 

8
We define for $\theta \in I_{\lambda,g}$,

$$S_\theta := \{ (\epsilon, \epsilon') | \epsilon, \epsilon' > 0, |\lambda| C_{\theta, \epsilon} < 1, |\lambda| D_{\theta, \epsilon'} < 1 \}.$$ 

Next we set

$$\tau_{\theta, \epsilon, \epsilon'} := (1 - |\lambda| C_{\theta, \epsilon}) \Sigma(A) - |\lambda| E_{\epsilon, \epsilon'},$$

and

$$T := \{ (\theta, \epsilon, \epsilon') \in \mathbb{R}^3 | \theta \in I_{\lambda,g}, (\epsilon, \epsilon') \in S_\theta, \tau_{\theta, \epsilon, \epsilon'} > E_0(H(\lambda, \mu)) \}.$$ 

**Theorem 3.3.** Consider the case $m > 0$. Suppose that $\sigma_{\text{ess}}(A) \neq \emptyset$. Assume Hypothesis [H.1]-[H.4], and $T \neq \emptyset$. Then, $H(\lambda, \mu)$ has purely discrete spectrum in the interval

$$\left[ E_0(H(\lambda, \mu)), \min \{ m + E_0(H(\lambda, \mu)), \sup_{(\theta, \epsilon, \epsilon') \in T} \tau_{\theta, \epsilon, \epsilon'} \} \right]. \quad (4)$$

In particular, $H(\lambda, \mu)$ has a ground state.

**Remark.** $T \neq \emptyset$ is necessary condition for $A$ to have a discrete ground state. Conversely, if $A$ has a discrete ground state, then $T \neq \emptyset$ holds for sufficiently small $\lambda, \mu$. Therefore the condition $T \neq \emptyset$ is a restriction for the coupling constants $\lambda, \mu$.

* * *

3.1 Proof of Proposition 3.1

In what follows, we write simply

$$H := H(\lambda, \mu).$$

For $D$ a dense subspace of $L^2(\mathbb{R}^d)$, we define

$$\mathcal{F}_{\text{fin}}(D) := \text{L.h}[\{ \Omega, a(h_1)^* \cdots a(h_n)^* \Omega | n \in \mathbb{N}, h_j \in D, j = 1, \ldots, n \}],$$

where $\Omega := (1, 0, 0, \ldots)$ is the Fock vacuum in $\mathcal{F}_h$. We introduce a dense subspace in $\mathcal{F}$

$$\mathcal{D}_\omega := D(A) \hat{\otimes} \mathcal{F}_{\text{fin}}(D(\omega)),$$

where $\hat{\otimes}$ denotes algebraic tensor product. The subspace $\mathcal{D}_\omega$ is a core of $H_0$. 

9
Let
\[ H_{\text{GSB}} := H_0 + \lambda H_1 \]
be a GSB Hamiltonian. The Hamiltonian \( H \) and \( H_{\text{GSB}} \) has the following relation:

**Proposition 3.4.** Let \( D(A) \subset D(B_j), \ j = 1, \ldots, J \) and \( f_j \in D(\omega^{1/2}) \).

Assume that \( H_{\text{GSB}} \) is bounded from below. Then, for all \( \Psi \in D_\omega \),
\[
\| (H_{\text{GSB}} - E_0)\Psi \|^2 + \| \mu H_2 \Psi \|^2 \leq \| (H - E_0)\Psi \|^2 + D\| \Psi \|^2, \tag{5}
\]
where \( D = \mu \sum_{j=1}^J \| \omega^{1/2} f_j \|^2 \) and
\[
E_0 := \inf_{\Psi \in D(H_{\text{GSB}})} \langle \Psi, H_{\text{GSB}}\Psi \rangle.
\]

**Proof.** It is enough to show (5) the case \( \lambda = \mu = 1 \). First we consider the case where \( f_j \in D(\omega) \).

Inequality (5) is equivalent to
\[
-2 \text{Re} \langle (H_{\text{GSB}} - E_0)\Psi, H_2 \Psi \rangle \leq D\| \Psi \|^2. \tag{6}
\]

By \( H_{\text{GSB}} - E_0 \geq 0 \), we have
\[
\langle (H_{\text{GSB}} - E_0)\Psi, I \otimes \phi(f_j)^2 \Psi \rangle = \langle \left[ I \otimes \phi(f_j), (H_{\text{GSB}} - E_0) \right] \Psi, I \otimes \phi(f_j) \Psi \rangle + \langle (H_{\text{GSB}} - E_0)I \otimes \phi(f_j) \Psi, I \otimes \phi(f_j) \Psi \rangle \\
\geq \langle \left[ I \otimes \phi(f_j), H_{\text{GSB}} - E_0 \right] \Psi, I \otimes \phi(f_j) \Psi \rangle.
\]

Therefore we have
\[
2 \text{Re} \langle (H_{\text{GSB}} - E_0)\Psi, \phi(f_j)^2 \Psi \rangle \geq -\| \sqrt{\omega} f_j \|^2 \| \Psi \|^2.
\]

This means inequality (6). Next, we set \( f_j \in D(\sqrt{\omega}) \). Then, there exists a sequence \( \{f_{jn}\}_{n=0}^\infty \subset D(\omega) \) such that \( f_{jn} \to f_j, \omega^{1/2} f_{jn} \to \omega^{1/2} f_j \ (n \to \infty) \).

By limiting argument, (6) holds with \( f_j \in D(\omega^{1/2}) \).

**Lemma 3.5.** Suppose that \( H_{\text{GSB}} \) is self-adjoint with \( D(H_{\text{GSB}}) = D(H_0) \), essentially self-adjoint on \( D_\omega \), and bounded from below. Let \( f_j \in D(\omega^{1/2}) \) \( \cap \) \( D(\omega^{-1/2}) \). Then \( H \) is self-adjoint with \( D(H) = D(H_0) \) and essentially self-adjoint on any core of \( H_{\text{GSB}} \) with
\[
\| (H_{\text{GSB}} - E_0)\Psi \|^2 + \| \mu H_2 \Psi \|^2 \leq \| (H - E_0)\Psi \|^2 + D\| \Psi \|^2, \quad \Psi \in D(H_0).
\]
Proof. It is well known that $D(H_b) \subset D(\phi(f_j)^2)$, and $\phi(f_j)^2$ is $H_b$-bounded (e.g. [1, Lemma 13-16]). Namely, there exist constants $\eta \geq 0$, $\theta \geq 0$ such that

$$\left\| \sum_{j=1}^{J} \phi(f_j)^2 \psi \right\| \leq \eta \| H_b \psi \| + \theta \| \psi \|, \quad \psi \in D(H_b). \quad (7)$$

Since $H_{GSB}$ is self-adjoint on $D(H_0)$, by the closed graph theorem, we have

$$\| H_0 \Psi \| \leq \lambda \| H_{GSB} \Psi \| + \nu \| \Psi \|, \quad \Psi \in D(H_0), \quad (8)$$

where $\lambda$ and $\nu$ are non-negative constant independent of $\Psi$. Hence

$$\| H_2 \Psi \| \leq \eta \lambda \| H_{GSB} \Psi \| + (\eta \nu + \theta) \| \Psi \|, \quad \Psi \in D(H_0).$$

We fix a positive number $\mu_0$ such that $\mu_0 < 1/(\mu \lambda)$. Then, by the Kato-Rellich theorem, $H(\lambda, \mu_0)$ is self-adjoint on $D(H_{GSB})$, bounded from below and essentially self-adjoint on any core of $H_{GSB}$. For a constant $a$ ($0 < a < 1$), we set $\mu_n := (1 + a)^n \mu_0$. Since $H_{GSB}$ is self-adjoint on $D(H_0)$, for each $j = 1, \ldots, J$ we have $D(A) \subset D(B)$. Thus by Proposition 3.4, for all $\Psi \in D_\omega$

$$\| (H_{GSB} - E_0) \Psi \|^2 + \| \mu_n H_2 \Psi \|^2 \leq \| (H(\lambda, \mu_n) - E_0) \Psi \|^2 + D \| \Psi \|^2.$$

If $H(\lambda, \mu_n)$ is self-adjoint on $D(H_{GSB})$, bounded from below and essentially self-adjoint on any core of $H_{GSB}$, then $H(\lambda, \mu_{n+1})$ has the same property. On the other hand, we have $\mu_n \to \infty$ ($n \to \infty$). Hence we conclude that $H$ is self-adjoint with $D(H) = D(H_{GSB})$, bounded from below and essentially self-adjoint on any core of $H_{GSB}$.

Now, we assume conditions [H.1],[H.2] and [H.3]. Then $H_{GSB}$ is self-adjoint on $D(H_0)$, bounded from below and essentially self-adjoint on any core of $H_0$ (see [2]). Hence, the assumptions of Lemma 3.5 hold. Thus Proposition 3.1 follows.

3.2 Proofs of Theorems 3.2 and 3.3

Throughout this subsection, we assume Hypotheses [H.1]-[H.4] and $m > 0$. 
For a parameter $V > 0$, we define the set of lattice points by

$$
\Gamma_V := \frac{2\pi \mathbb{Z}^d}{V} := \left\{ k = (k_1, \ldots, k_d) \bigg| k_j = \frac{2\pi n_j}{V}, n_j \in \mathbb{Z}, j = 1, \ldots, d \right\}
$$

and we denote by $l^2(\Gamma_V)$ the set of $l^2$ sequences over $\Gamma_V$. For each $k \in \Gamma_V$ we introduce

$$
C(k, V) := \left[ k_1 - \frac{\pi}{V}, k_1 + \frac{\pi}{V} \right) \times \cdots \times \left[ k_d - \frac{\pi}{V}, k_d + \frac{\pi}{V} \right) \subset \mathbb{R}^d,
$$

the cube centered about $k$. By the map

$$
U : l^2(\Gamma_V) \ni \{ h_l \}_{l \in \Gamma_V} \mapsto (V/2\pi)^{d/2} \sum_{l \in \Gamma_V} h_l \chi_{l,V}(\cdot) \in L^2(\mathbb{R}^d),
$$

we identify $l^2(\Gamma_V)$ with a subspace in $L^2(\mathbb{R}^d)$, where $\chi_{l,V}(\cdot)$ is the characteristic function of the cube $C(l, V) \subset \mathbb{R}^d$. It is easy to see that $l^2(\Gamma_V)$ is a closed subspace of $L^2(\mathbb{R}^d)$. Let

$$
\mathcal{F}_{b,V} := \mathcal{F}_b(l^2(\Gamma_V)) = \bigoplus_{n=0}^{\infty} \bigotimes_{s} l^2(\Gamma_V),
$$

the boson Fock space over $l^2(\Gamma_V)$. We can identify $\mathcal{F}_{b,V}$ the closed subspace of $\mathcal{F}_b$ by the operator $\Gamma(U) := \bigoplus_{n=0}^{\infty} \otimes^n U$, where we define $\otimes^0 U = 0$. For each $k \in \mathbb{R}^d$, there exists a unique point $k_V \in \Gamma_V$ such that $k \in C(k_V, V)$. Let

$$
\omega_V(k) := \omega(k_V), \quad k \in \mathbb{R}^d
$$

be a lattice approximate function of $\omega(k)$ and let

$$
H_{b,V} := d\Gamma(\omega_V)
$$

be the second quantization of $\omega_V$. We define a constant

$$
C_V := Cd^\gamma \left( \frac{\pi}{V} \right) \left( \frac{1}{2m} + 1 \right),
$$

where $C$ and $\gamma$ were defined in [H.4]. In what follows we assume that

$$
C_V < 1.
$$

This is satisfied for all sufficiently large $V$. 

12
Lemma 3.6. [2, Lemma 3.1]. We have
\[ D(H_{b,V}) = D(H_b), \]
and
\[ \| (H_b - H_{b,V}) \Psi \| = \frac{2C_V}{1 - C_V} \| H_b \Psi \|, \quad \Psi \in D(H_b). \]

First we consider the case where \( g_j \)'s and \( f_j \)'s are continuous, and finally, by limiting argument, we treat a general case. For a constant \( K > 0 \), we define \( g_j,K, f_j,K, \) and \( g_j,K,V, f_j,K,V \) as follows:
\[ g_j,K(k) := \chi_K(k_1) \cdots \chi_K(k_d) g_j(k), \quad g_j,K,V(k) := \sum_{\ell \in \Gamma_V, |\ell_i| < K} g_j(\ell) \chi_{\ell,V}(k), \]
\[ f_j,K(k) := \chi_K(k_1) \cdots \chi_K(k_d) f_j(k), \quad f_j,K,V(k) := \sum_{\ell \in \Gamma_V, |\ell_i| < K} f_j(\ell) \chi_{\ell,V}(k), \]
where \( \chi_K \) denotes the characteristic function of \( [-K, K] \).

Lemma 3.7. For all \( j = 1, \ldots, J \),
\[ \lim_{V \to \infty} \| g_j,K, V - g_j,K \| = 0, \quad \lim_{V \to \infty} \| g_j,K,V / \sqrt{\omega_V} - g_j,K / \sqrt{\omega} \| = 0, \]
\[ \lim_{K \to \infty} \| g_j,K - g_j \| = 0, \quad \lim_{K \to \infty} \| g_j,K / \sqrt{\omega} - g_j / \sqrt{\omega} \| = 0, \]
\[ \lim_{V \to \infty} \| f_j,K, V - f_j,K \| = 0, \quad \lim_{V \to \infty} \| f_j,K,V / \sqrt{\omega_V} - f_j,K / \sqrt{\omega} \| = 0, \]
\[ \lim_{K \to \infty} \| f_j,K - f_j \| = 0, \quad \lim_{K \to \infty} \| f_j,K / \sqrt{\omega} - f_j / \sqrt{\omega} \| = 0, \]
\[ \lim_{K \to \infty} \| \sqrt{\omega_V} f_j,K, V - \sqrt{\omega} f_j,K \| = 0, \quad \lim_{V \to \infty} \| \sqrt{\omega_V} f_j,K,V - \sqrt{\omega} f_j,K \| = 0. \]

Proof. Similar to the proof of [2, Lemma 3.10] \( \blacksquare \)

We introduce a new operator:
\[ H_{0,V} := A \otimes I + I \otimes H_{b,V}, \]
\[ H_{1,K} := \sum_{j=1}^J B_j \otimes \phi(g_j,K), \]
\[ H_{1,K,V} := \sum_{j=1}^J B_j \otimes \phi(g_j,K,V), \]
\[ H_{2,K} := \sum_{j=1}^{J} I \otimes \phi(f_{j,K})^2, \]
\[ H_{2,K,V} := \sum_{j=1}^{J} I \otimes \phi(f_{j,K,V})^2, \]

and define
\[ H_K := H_0 + \lambda H_1 + \mu H_2, \]
\[ H_{K,V} := H_{0,V} + \lambda H_{1,K,V} + \mu H_{2,K,V}. \]

**Lemma 3.8.** (i) \( H_K \) is self-adjoint with \( D(H_K) = D(H_0) \cap D(H_2) \), bounded from below, and essentially self-adjoint on any core of \( H_0 \).

(ii) For all large \( V \), \( H_{K,V} \) is self-adjoint with \( D(H_{K,V}) = D(H_0) \cap D(H_2) \), bounded from below, and essentially self-adjoint on any core of \( H_{0,V} \).

**Proof.** Similar to the proof of Proposition 3.1.

**Lemma 3.9.** For all \( z \in \mathbb{C} \setminus \mathbb{R}, \) and \( K > 0 \),
\[
\lim_{K \to \infty} \| (H_K - z)^{-1} - (H - z)^{-1} \| = 0,
\]
\[
\lim_{V \to \infty} \| (H_{K,V} - z)^{-1} - (H_K - z)^{-1} \| = 0.
\]

**Proof.** Similar to the proof of Lemma 3.5.

The following fact is well known:

**Lemma 3.10.** The operator \( H_{b,V} \) is reduced by \( \mathcal{F}_{b,V} \) and \( H_{b,V}[\mathcal{F}_{b,V}] \) equal to the second quantization of \( \omega_V |^2(\Gamma_V) \) on \( \mathcal{F}_{b,V} \).

**Lemma 3.11.** \( H_{K,V} \) is reduced by \( \mathcal{F}_V \).

**Proof.** Similar to the proof of Lemma 3.7.

**Lemma 3.12.** We have
\[ H_{K,V}[\mathcal{F}_V] \geq E_0(H_{K,V}) + m. \]
The following operator inequalities hold:

**Lemma 3.14.**

Similar to the calculation of \[ \text{Proof.} \]

**Lemma 3.13.** Let \( T_n \) and \( T \) be a self-adjoint operators on a separable Hilbert space and bounded from below. Suppose that \( T_n \rightarrow T \) in norm resolvent sense as \( n \rightarrow \infty \) and \( T_n \) has purely discrete spectrum in the interval \( [E_0(T_n), \epsilon + c_n] \) with some constant \( c_n \). If \( \epsilon := \limsup_{n \rightarrow \infty} c_n > 0 \), then \( T \) has purely discrete spectrum in \( [E_0(T), \epsilon + c] \).

**Proof.** There exists a sequence \( \{c_{n_j}\}_{j=1}^{\infty} \subset \{c_n\}_{n=1}^{\infty} \) so that \( c_{n_j} \rightarrow c (j \rightarrow \infty) \). So, for all \( \epsilon > 0 \) and for sufficiently large \( j \), the spectrum of \( T_{n_j} \) in \( [E_0(T_{n_j}), E_0(T_{n_j}) + c - \epsilon] \) is discrete. Therefore, applying [2, Lemma 3.12] we find that the spectrum of \( T \) in \( [E_0(T), E_0(T) + c - \epsilon] \) is discrete. Since \( \epsilon > 0 \) is arbitrary, we get the conclusion.

Now, if \( A \) has compact resolvent, by a method similar to the proof of [2, Theorem 1.2] we can prove Theorem 3.2. Therefore, we only prove Theorem 3.3.

The following inequality is known [2, (2.12)]

\[
|\langle \Psi, H_1|\Psi \rangle| \leq C_{\theta, \epsilon} |\Psi, A \otimes I| + D_{\theta, \epsilon} |\Psi, I \otimes H_b| + E_{\epsilon, \epsilon'} \|\Psi\|^2,
\]

where \( \Psi \in D(H_0) \) is arbitrary. Thus we have,

\[
H \geq (1 - |\lambda| C_{\theta, \epsilon}) A \otimes I + (1 - |\lambda| D_{\theta, \epsilon}) I \otimes H_b \mu H_2 - \lambda |E_{\epsilon, \epsilon'}|.
\]

Let \( I_{\lambda, g}(K), C_{\theta, \epsilon}(K), D_{\theta, \epsilon}(K) \) and \( E_{\epsilon, \epsilon'}(K) \) are \( I_{\lambda, g}, C_{\theta, \epsilon}, D_{\theta, \epsilon}, E_{\epsilon, \epsilon'} \) with \( g_j, f_j \) replaced by \( g_j, f_j \), respectively, and let \( I_{\lambda, 0}(K, V), C_{\theta, \epsilon}(K, V), D_{\theta, \epsilon}(K, V) \) and \( E_{\epsilon, \epsilon'}(K, V) \) are \( I_{\lambda, 0}, C_{\theta, \epsilon}, D_{\theta, \epsilon}, E_{\epsilon, \epsilon'} \) with \( g_j, f_j \) and \( \omega \) replaced by \( g_j, f_j, \omega \), respectively. Then we have

**Lemma 3.14.** The following operator inequalities hold:

\[
H_K \geq (1 - |\lambda| C_{\theta, \epsilon}(K)) A \otimes I + (1 - |\lambda| D_{\theta, \epsilon}(K)) I \otimes H_b \mu H_2, \quad \text{on} \quad D(H_0),
\]

\[
H_{K, V} \geq (1 - |\lambda| C_{\theta, \epsilon}(K, V)) A \otimes I + (1 - |\lambda| D_{\theta, \epsilon}(K, K)) I \otimes H_{b, V} \mu H_2, \quad \text{on} \quad D(H_0).
\]

**Proof.** Similar to the calculation of [2, (2.12)]
By Lemma 3.7, we have
\begin{align*}
\lim_{V \to \infty} C_{\theta, \epsilon}(K, V) &= C_{\theta, \epsilon}(K), \\
\lim_{V \to \infty} D_{\theta, \epsilon'}(K, V) &= D_{\theta, \epsilon'}(K), \\
\lim_{V \to \infty} E_{\epsilon, \epsilon'}(K, V) &= E_{\epsilon, \epsilon'}(K),
\end{align*}
(9), (10), (11)
By Lemma 3.11, we have
\[ H \text{ has purely discrete spectrum in } (14). \]

By Lemma 3.12, we have
\[ H \text{ has purely discrete spectrum in } (15). \]

Let \((\theta, \epsilon, \epsilon') \in \mathcal{T}, \) namely
\[ \tau_{\theta, \epsilon, \epsilon'} = (1 - |\lambda|C_{\theta, \epsilon}(K, V))\Sigma(A) - |\lambda|E_{\epsilon, \epsilon'} > E_0(H). \]

Formulas (9)-(11) and Lemma 3.9 imply that for all large \(V\) there exists a constant \(K_0 > 0\) such that for all \(K > K_0,\)
\begin{align*}
(1 - |\lambda|C_{\theta, \epsilon}(K, V))\Sigma(A) - |\lambda|E_{\epsilon, \epsilon'}(K, V) &> E_0(H_{K,V}), \\
|\lambda|C_{\theta, \epsilon}(K, V) &< 1, \\
|\lambda|D_{\theta, \epsilon'}(K, V) &< 1.
\end{align*}
(12), (13)
By Lemma 3.11, \(H_{K,V}\) is reduced by \(\mathcal{F}_V.\) Therefore, \(H_{K,V}\) satisfies the following inequality:
\begin{align*}
H_{K,V}[\mathcal{F}_V] &\geq (1 - |\lambda|C_{\theta, \epsilon}(K, V))A \otimes I[\mathcal{F}_V] \\
&+ (1 - |\lambda|D_{\theta, \epsilon'}(K, V))I \otimes H_{b,V}[\mathcal{F}_V] \\
&- |\lambda|E_{\epsilon, \epsilon'}(K, V).
\end{align*}
(14)
Since \(H_{b,V}[\mathcal{F}_V]\) has compact resolvent, the bottom of essential spectrum of the right hand side of (14) is equal to
\[ (1 - |\lambda|C_{\theta, \epsilon}(K, V))\Sigma(A) - |\lambda|E_{\epsilon, \epsilon'}(K, V). \]

By Lemma 3.12, we have \(E_0(H_{K,V}[\mathcal{F}_V]) = E_0(H_{K,V}).\) Thus, applying Theorem 2.1 with \(H_{K,V}[\mathcal{F}_V,\) we have that \(H_{K,V}[\mathcal{F}_V}\) has purely discrete spectrum in \([E_0(H_{K,V}), (1 - |\lambda|C_{\theta, \epsilon}(K, V))\Sigma_A - E_{\epsilon, \epsilon'}(K, V)].\) Since this fact and Lemma 3.12, \(H_{K,V}\) has purely discrete spectrum in
\[ [E_0(H_{K,V}), \min\{E_0(H_{K,V}) + m, (1 - |\lambda|C_{\theta, \epsilon}(K, V))\Sigma_A - E_{\epsilon, \epsilon'}(K, V)]).\]

By Lemma 3.9 and Lemma 3.13, we have that for all sufficiently large \(K > 0,\) \(H_{K}\) has purely discrete spectrum in \([E_0(H_{K}), \min\{E_0(H_{K}) + m, (1 - |\lambda|C_{\theta, \epsilon}(K))\Sigma_A - |\lambda|E_{\epsilon, \epsilon'}(K)]).\) Similarly, \(H\) has purely discrete spectrum in \([E_0(H(\lambda, \mu)), \min\{m + E_0(H(\lambda, \mu)), \tau_{\theta, \epsilon, \epsilon'}\})\). Since \((\theta, \epsilon, \epsilon') \in \mathcal{T}\) is arbitrary, \(H\) has purely discrete spectrum in (4). Finally, we have to consider the case where \(g_j\)’s and \(f_j\)’s are not necessarily continuous. But, that argument were already discussed in [4] So we skip that argument. \(\square\)
4 Ground State of the Dereziński-Gérard Model

We consider a model discussed by J. Dereziński and C. Gérard. We take the Hilbert space of the particle system is taken to be
\[ \mathcal{H} = L^2(\mathbb{R}^N). \]

The Hilbert space for the Dereziński-Gérard (DG) model is given by
\[ \mathcal{F} := \mathcal{H} \otimes B_c(L^2(\mathbb{R}^d)). \]

We identify \( \mathcal{F} \) as
\[ \bigoplus_{n=0}^{\infty} [\mathcal{H} \otimes \bigotimes_{s}^{n} L^2(\mathbb{R}^d)]. \]

Hence, if we denote that \( \Psi \in (\Psi^{(n)})_{n=0}^{\infty} \in \mathcal{F}, \) each \( \Psi^{(n)} \) belongs to \( \mathcal{H} \otimes [\bigotimes_{s}^{n} L^2(\mathbb{R}^d)]. \) We denote by \( B(\mathcal{K}, \mathcal{F}) \) the set of bounded linear operators from \( \mathcal{K} \) to \( \mathcal{F}. \) For \( v \in B(\mathcal{H}, \mathcal{H} \otimes L^2(\mathbb{R}^d)), \) we define an operator \( \tilde{a}^*(v) \) by

\[
(\tilde{a}^*(v)\Psi)^{(0)} := 0,
(\tilde{a}^*(v)\Psi)^{(n)} := \sqrt{n}(I_{\mathcal{H}} \otimes S_n)(v \otimes I_{\bigotimes_{s}^{n-1} L^2(\mathbb{R}^d)})\Psi^{(n-1)}, \quad (n \geq 1),
\]

\[
\Psi \in D(\tilde{a}^*(v)) := \left\{ \Psi = (\Psi^{(n)})_{n=0}^{\infty} \in \mathcal{F} \left| \sum_{n=0}^{\infty} \| (\tilde{a}^*(v)\Psi)^{(n)} \|^2 < \infty \right. \right\}.
\]

We set
\[
D_0 := \{ \Psi = (\Psi^{(n)})_{n=0}^{\infty} \in \mathcal{F} | \text{there exists a constant } n_0 \in \mathbb{N}, \text{ such that, for all } n \geq n_0, \Psi^{(n)} = 0 \}.
\]

Throughout this section, we write simply \( I_n := I_{\bigotimes_{s}^{n} L^2(\mathbb{R}^d)}. \) It is easy to see that:

**Proposition 4.1.** \( \tilde{a}^*(v) \) is a closed linear operator and \( D_0 \) is a core of \( \tilde{a}^*(v). \)

So we set
\[ \tilde{a}(v) := (\tilde{a}^*(v))^* \]

the adjoint operator of \( \tilde{a}^*(v). \)
Proposition 4.2. The operator $\tilde{a}(v)$ has the following properties:

\[
D(\tilde{a}(v)) = \left\{ \Psi = (\Psi^{(n)})_{n=0}^{\infty} \bigg| \sum_{n=0}^{\infty} (n+1) \| (I_H \otimes S_n)(v^* \otimes I_n)\Psi^{(n+1)} \|^2 < \infty \right\}
\]

(15)

\[
(\tilde{a}(v)\Psi)^{(n)} = \sqrt{n+1} I_H \otimes S_n(v^* \otimes I_n)\Psi^{(n+1)}, \quad \Psi \in D(\tilde{a}(v)),
\]

and $D_0$ is a core of $\tilde{a}(v)$.

Proof. For $\Phi \in F$, $\Psi \in D(\tilde{a}^*(v))$,

\[
\langle \Phi, \tilde{a}^*(v)\Psi \rangle = \sum_{n=1}^{\infty} \langle \Phi^{(n)}, \sqrt{n} (I_H \otimes S_n)(v \otimes I_{n-1})\Psi^{(n-1)} \rangle
\]

\[
= \sum_{n=0}^{\infty} \sqrt{n+1} \langle v^* \otimes I_n \Phi^{(n+1)}, \Psi^{(n)} \rangle
\]

\[
= \sum_{n=0}^{\infty} \langle \sqrt{n+1} (I_H \otimes S_n)(v^* \otimes I_n)\Phi^{(n+1)}, \Psi^{(n)} \rangle.
\]

This implies (15) and (16). It is easy to prove that $D_0$ is a core of $\tilde{a}(v)$. □

An analogue of the Segal field operator is defined by

\[
\tilde{\phi}(v) := \frac{1}{\sqrt{2}} (\tilde{a}(v) + \tilde{a}^*(v)).
\]

Let $A$ be a non-negative self-adjoint operator on $H$ with $E_0(A) = 0$. Then the Hamiltonian of the DG model is defined by

\[
H_{DG} := A \otimes I + I \otimes H_b + \tilde{\phi}(v).
\]

We call it the Dereziński-Gérard Hamiltonian. Here $H_b$ is the second quantization of $\omega$ introduce in Section 3. Let

\[
H_0 := A \otimes I + I \otimes H_b.
\]

Throughout this section we assume the following conditions:

[DG.1] There is a Borel measurable function $v(x, k) \in \mathbb{C}$, $(x \in \mathbb{R}^N, k \in \mathbb{R}^d)$, such that

\[
(vf)(x, k) = v(x, k)f(x), \quad f \in L^2(\mathbb{R}^d).
\]
We need also the following assumption:

\[ \text{DG.2} \]

\[
\text{ess. sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} \left| \frac{v(x, k)}{\sqrt{\omega(k)}} \right|^2 \, dk < \infty.
\]

**Proposition 4.3.** Assume [DG.1] and [DG.2]. Then \( H_{\text{DG}} \) is self-adjoint with \( D(H_{\text{DG}}) = D(H_0) \), and essentially self-adjoint on any core of \( H_0 \).

For a finite volume approximation, we introduce the following hypotheses:

\[ \text{DG.3} \]

There exists a nonnegative function \( \tilde{v} \in L^2(\mathbb{R}^d) \) and function \( \tilde{\sigma} : \mathbb{R} \to \mathbb{R} \), such that

\[
\text{ess. sup}_{x \in \mathbb{R}^n} |v(x, k) - v(x, \ell)| \leq \tilde{v}(k) \tilde{\sigma}(|k - \ell|), \quad \text{a.e. } k, \ell \in \mathbb{R}^d
\]

\[
\lim_{t \downarrow 0} \tilde{\sigma}(t) = 0.
\]

\[ \text{DG.4} \]

\[
\text{ess. sup}_{x \in \mathbb{R}^n} \int_{([-K, K]^d)^c} |v(x, k)|^2 \, dk = o(K^0).
\]

where

\[
([K, K]^d)^c := \mathbb{R}^d \setminus (I \times \cdots \times I), \quad I := [-K, K]
\]

and, \( o(t^0) \) satisfies \( \lim_{t \to 0} o(t^0) = 0 \).

Let \( m \) be defined by (3). Let

\[
D := \frac{1}{2} \inf_{0 \leq \epsilon' \leq \frac{1}{\|v\|_{\infty}}} \left( \epsilon' + \frac{1}{\epsilon'} \right).
\]

(17)

Here, \( v/\sqrt{\omega} \) is a multiplication operator by the function \( v(x, k)/\sqrt{\omega(k)} \) from \( L^2(\mathbb{R}^N) \) to \( L^2(\mathbb{R}^N) \otimes L^2(\mathbb{R}^d) \). In the case \( m > 0 \), we can establish the existence of a ground state of \( H_{\text{DG}} \):

**Theorem 4.4.** Let \( m > 0 \). Suppose that [DG.1]-[DG.4] and [H.4] hold, and suppose

\[
\Sigma(A) - \|v\|D - E_0(H_{\text{DG}}) > 0.
\]
Then, $H_{DG}$ has purely discrete spectrum in
\[ [E_0(H_{DG}), \min\{E_0(H_{DG}) + m, \Sigma(A) - \|v\|D\}]. \]

In particular $H_{DG}$ has a ground state.

**Remark.** In the case where $A$ has compact resolvent, this theorem has been proved in [5]. A new aspect here is in that $A$ does not necessarily have compact resolvent. Also our method is different from that in [5].

### 4.1 Proof of Proposition 4.3

**Lemma 4.5.** Let $M(x) = (\int_{\mathbb{R}^d} |v(x, k)|^2 dk)^{1/2}$, $x \in \mathbb{R}^N$ and $M : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ be a multiplication operator by the function $M(x)$. Then
\[ \|vf\|^2 = \|Mf\|^2, \quad f \in L^2(\mathbb{R}^N). \]

In particular, $\|v\| = \|M\| = (\text{ess.sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} |v(x, k)|^2 dk)^{1/2}$ hold.

**Proof.** By the Fubini’s theorem, we have
\[
\|vf\|^2 = \int_{\mathbb{R}^d} dk \int_{\mathbb{R}^N} dx |v(x, k)|^2 |f(x)|^2 = \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^d} |f(x)|^2 \int_{\mathbb{R}^d} |v(x, k)|^2 dk \right) dx.
\]

This means the result.

The adjoint $v^*$ has the following form:

**Lemma 4.6.** For all $g \in \mathcal{H} \otimes L^2(\mathbb{R}^d)$,
\[ (v^*g)(x) = \int_{\mathbb{R}^d} v(x, k)^* g(x, k) dk, \quad \text{a.e. } x \in \mathbb{R}^d. \]  

(18)

**Proof.** For all $f \in \mathcal{H}$, we have
\[
\langle g, vf \rangle = \int dx \int dk g(x, k)^* v(x, k) f(x) = \int dx \left( \int g(x, k)^* v(x, k) dk \right) f(x).
\]

Since $f$ is arbitrary, this proves (18).
Lemma 4.7. \( \tilde{a}(v) \) is
\[
D(\tilde{a}(v)) = \left\{ \Psi \in \mathcal{F} \Bigg| \sum_{n=0}^{\infty} (n+1) \int_{\mathbb{R}^{dN+dn}} dx dk_1 \cdots dk_n \right. 
\left. \left| \int_{\mathbb{R}^d} dk v(k,x)^* \Psi^{(n+1)}(x,k,k_1,\ldots,k_n) \right|^2 < \infty \right\}
\]
\[
(\tilde{a}(v)\Psi)^{(n)}(x,k_1,\ldots,k_n) = \sqrt{n+1} \int_{\mathbb{R}^d} v(x,k)^* \Psi^{(n+1)}(x,k,k_1,\ldots,k_n), \quad \text{a.e.} \quad (\Psi \in D(\tilde{a}(v)))
\]

Proof. Using Lemma 4.6, we have
\[
(v^* \otimes I_n)\Psi^{(n+1)}(x,k_1,\ldots,k_n) = \int_{\mathbb{R}^d} v^*(x,k)\Psi^{(n+1)}(x,k,k_1,\ldots,k_n) dk.
\]
This is invariant for all permutations of \( k_1,\ldots,k_n \). Therefore, using Proposition 4.2, we get
\[
(\tilde{a}(v)\Psi)^{(n)}(x,k_1,\ldots,k_n) = \sqrt{n+1} \int_{\mathbb{R}^d} v(x,k)^* \Psi^{(n+1)}(x,k,k_1,\ldots,k_n) dk.
\]

Lemma 4.8. Suppose that [DG.1] and [DG.2] hold. Then, \( D(\tilde{a}(v)) \subset D(I \otimes H^{1/2}_b) \) and
\[
\|\tilde{a}(v)\Phi\| \leq \|v/\sqrt{\omega}\| \|I \otimes H^{1/2}_b\|, \quad \Phi \in D(I \otimes H^{1/2}_b).
\]

Proof. By (19), we have for all \( \Phi \in D(\tilde{a}(v)) \)
\[
\|(\tilde{a}(v)\Phi)^{(n)}\|^2 = (n+1) \int_{\mathbb{R}^{dn+N}} dx dk_1 \cdots dk_n \left| \int_{\mathbb{R}^d} \frac{1}{\sqrt{\omega(k)}} v(x,k)^* \Phi^{(n+1)}(x,k,k_1,\ldots,k_n) dk \right|^2.
\]
Using the Schwarz inequality, one has
\[
\left| \int_{\mathbb{R}^d} \frac{1}{\sqrt{\omega(k)}} v(x,k)^* \Phi^{(n+1)}(x,k,k_1,\ldots,k_n) dk \right|^2 \leq \int_{\mathbb{R}^d} \frac{|v(x,k)|^2}{\omega(k)} dk \cdot \int_{\mathbb{R}^d} \omega(k)|\Phi^{(n+1)}(x,k,k_1,\ldots,k_n)|^2 dk.
\]
Hence, for every $\Phi \in D_0 \cap D(I \otimes H^1_{b/2})$, we have

$$\|\tilde{a}(v)\Phi^{(n)}\|^2 \leq \left( \text{ess.sup} \int \frac{|v(x,k)^*|^2}{\sqrt{\omega(k)}} \, dk \right) (n+1) \times$$

$$\int_{\mathbb{R}^{n+N}} dx dk_1 \cdots dk_n \, dk \omega(k) |\Phi^{(n+1)}(x,k,k_1,\ldots,k_n)|^2$$

$$= \left( \text{ess.sup} \int \frac{|v(x,k)^*|^2}{\sqrt{\omega(k)}} \, dk \right) \times$$

$$\int_{\mathbb{R}^{n+N}} dx dk_1 \cdots dk_{n+1} \sum_{j=1}^{n+1} \omega(k_j) |\Phi^{(n+1)}(x,k_1,\ldots,k_{n+1})|^2$$

$$= \left\| \frac{v}{\sqrt{\omega}} \right\| \| (I \otimes H^1_{b/2})^{(n+1)} \|^2.$$ 

Therefore

$$\|\tilde{a}(v)\Phi\| \leq \left\| \frac{v}{\sqrt{\omega}} \right\| \| (I \otimes H^1_{b/2}) \|^2.$$

Since, $D_0 \cap D(I \otimes H^1_{b/2})$ is a core of $I \otimes H^1_{b/2}$, one can extend this inequality to all $\Phi \in D(I \otimes H^1_{b/2})$, and $D(I \otimes H^1_{b/2}) \subset D(\tilde{a}(v))$ holds.

**Lemma 4.9.** On $D_0$, $\tilde{a}(v)$ and $\tilde{a}^*(v)$ satisfy the following commutation relation:

$$[\tilde{a}(v), \tilde{a}^*(v)] = \int \left| v(\cdot,k) \right|^2 \, dk.$$

where the right hand side is a multiplication operator by the function : $x \mapsto \int_{\mathbb{R}^{d}} |v(x,k)|^2 \, dk$.

**Proof.** Let $\Phi \in D_0$. By the definition of $\tilde{a}^*(v)$, and using Proposition 4.2, we get

$$([\tilde{a}^*(v), \tilde{a}(v)]\Phi)^{(n)} = (\tilde{a}(v)\tilde{a}(v)^*\Phi)^{(n)} - (\tilde{a}(v)^*\tilde{a}(v)\Phi)^{(n)}$$

$$= \sqrt{n+1} I_{\mathcal{H}} \otimes S_n (v^* \otimes I_n)(\tilde{a}(v)^*\Phi)^{(n+1)}$$

$$- \sqrt{n}(I \otimes S_n)(v \otimes I_{n-1})(\tilde{a}(v)\Phi)^{(n-1)}.$$
Hence, we have
\[
([\tilde{a}^*(v), \tilde{a}(v)]\Phi)(x, k_1, \ldots, k_n)
\]
\[
= (n + 1) \int_{\mathbb{R}^d} v(x, k)^* (I \otimes S_{n+1} (v \otimes I_{n-1}) \Phi(n)) (x, k, k_1, \ldots, k_n) \, dk
\]
\[
- \frac{1}{n} \sum_{j=1}^{n} v(x, k_j) (v^* \otimes I_{n-1} \Phi(n)) (x, k_1, \ldots, \hat{k}_j, \ldots, k_n)
\]
\[
= \int_{\mathbb{R}^d} dk \, v(x, k)^* \left( v(x, k) \Phi(n)(x, k_1, \ldots, k_n) + \sum_{j=1}^{n} v(x, k_j) \Phi(n)(x, k_1, \ldots, \hat{k}_j, \ldots, k_n) - \frac{1}{n} \sum_{j=1}^{n} v(x, k_j) \right)
\]
\[
- \frac{1}{n} \sum_{j=1}^{n} v(x, k_j) \int_{\mathbb{R}^d} dk v(x, k)^* \Phi(n)(x, k_1, \ldots, \hat{k}_j, \ldots, k_n)
\]
\[
= \left( \int_{\mathbb{R}^d} |v(x, k)|^2 \right) \Phi(x, k_1, \ldots, k_n).
\]
Here ‘\(\hat{\cdot}\)’ indicates the omission of the object wearing the hat.

**Lemma 4.10.** Assume, [DG.1] and [DG.2]. Then \(D(I \otimes H_{b}^{1/2}) \subset D(\tilde{a}^*(v))\) and for all \(\Phi \in D(I \otimes H_{b}^{1/2})\),
\[
\|\tilde{a}^*(v)\Phi\|^2 \leq \|v/\sqrt{\omega}\|^2\|I \otimes H_{b}^{1/2}\Phi\|^2 + \|v\|^2\|\Phi\|^2. \tag{20}
\]

**Proof.** For all \(\Phi \in D_0 \cap D(I \otimes H_{b}^{1/2})\), we have
\[
\|\tilde{a}^*(v)\Phi\|^2 = \langle \Phi, \tilde{a}(v) \tilde{a}^*(v) \Phi \rangle = \langle \Phi, \tilde{a}^*(v) \tilde{a}(v) \Phi \rangle + \left( \int_{\mathbb{R}^d} |v(x, k)|^2 \right) \Phi, \Phi \rangle
\]
\[
\leq \|\tilde{a}(v)\Phi\|^2 + \|v\|^2\|\Phi\|^2.
\]
Thus we can apply Lemma 4.8 to obtain the result.

**Proof of Proposition 4.3.** By Lemma 4.8 and 4.10, the operator \(\tilde{\phi}(v)\) is \(I \otimes H_{b}^{1/2}\)-bounded. Hence \(\tilde{\phi}(v)\) is infinitesimally small with respect to \(I \otimes H_{b}\). Namely, for all \(\epsilon > 0\), there exists a constant \(c_\epsilon > 0\), such that,
\[
\|\tilde{\phi}(v)\Phi\| \leq \epsilon \|I \otimes H_{b}\Phi\| + c_\epsilon \|\Phi\|, \quad \Phi \in D(I \otimes H_{b}).
\]
Since $A \geq 0$, we have
\[ \|\tilde{\phi}(v)\Phi\| \leq \epsilon\|H_0\Phi\| + c\|\Phi\|, \quad \Phi \in D(H_0). \]
Thus we can apply the Kato-Rellich theorem to obtain the conclusion of Proposition 4.3.

4.2 Proof of Theorem 4.4

In this subsection we suppose that the assumption of Theorem 4.4 holds. Let $F_{b,V}$, $H_{b,V}$, $H_{0,V}$, $F_V$, $\Gamma_V$, $\chi_{\ell,V}(k)$ be an object already defined in Section 3, respectively. Suppose that $\chi_K$ is a characteristic function of $[-K,K]$.

For a parameter $K > 0$, we define $v_K \in B(H,H \otimes L^2(\mathbb{R}^d))$ by
\[ (v_K f)(x, k) := \chi_{[-K,K]}(k)v(x, k)f(x). \]
and $v_{K,V} \in B(H,H \otimes L^2(\mathbb{R}^d))$ by
\[ (v_{K,V} f)(x, k) := \sum_{\ell \in \Gamma_V, |\ell_i| < K} \chi_{\ell,V}(k)v(x, \ell)f(x). \]

Lemma 4.11. The following hold:
\[ \|v_K - v_{K,V}\| \to 0 \quad (V \to \infty), \quad \|v_K - v\| \to 0 \quad (K \to \infty). \]
\[ \left\| \frac{v_K}{\sqrt{\omega}} - \frac{v_{K,V}}{\sqrt{\omega}} \right\| \to 0 \quad (V \to \infty), \quad \left\| \frac{v}{\sqrt{\omega}} - \frac{v_K}{\sqrt{\omega}} \right\| \to 0 \quad (K \to \infty). \]

Proof. By [DG.3] and [DG.4], we have
\[ \|v_K - v_{K,V}\|^2 = \text{ess.sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} \left| \chi_K(k)v(x, k) - \sum_{\ell \in \Gamma_V, |\ell_i| < K} v(x, \ell)\chi_{\ell,V}(k) \right|^2 dk \]
\[ = \text{ess.sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} \sum_{\ell \in \Gamma_V, |\ell_i| < K} \chi_{\ell,V}(k)|v(x, k) - v(x, \ell)|^2 dk \]
\[ \leq \text{ess.sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} \sum_{\ell \in \Gamma_V, |\ell_i| < K} \chi_{\ell,V}(k)|\tilde{v}(k)|^2|\tilde{\phi}(|k - \ell|)|^2 dk \]
\[ \leq \int_{\mathbb{R}^d} \sum_{\ell \in \Gamma_V, |\ell_i| < K} \chi_{\ell,V}(k)|\tilde{v}(k)|^2|\tilde{\phi}(|k - \ell|)|^2 dk. \]

24
It follows from the property of \( \tilde{o} \) that for every \( \epsilon > 0 \), there exists a constant \( V_0 > 0 \) such that, for all \( V > V_0 \),

\[
\chi_{\ell,V}(k)\tilde{o}(|k - \ell|)^2 \leq \epsilon \chi_{\ell,V}(k).
\]

Therefore,

\[
\|v_K - v_{K,V}\|^2 \leq \epsilon \int_{\mathbb{R}^d} \sum_{|\ell| < K} \chi_{\ell,V}(k)\tilde{o}(k)^2 \, dk = \epsilon \|\tilde{v}\|^2_{L^2(\mathbb{R}^d)}.
\]

Hence the first one of (21) holds. The second one is a direct result of condition [DG.4] :

\[
\|v_K - v\|^2 = \text{ess.sup}_{x} \int_{\mathbb{R}^d} |\chi_K(k) - 1|^2 |v(x, k)|^2 \, dk = \text{ess.sup}_{x} \int_{([-K,K]^d)^c} |v(x, k)|^2 \, dk = o(K^0) \rightarrow 0 \ (K \rightarrow \infty).
\]

Using [H.4], one can easily check (22).

We introduce two operators:

\[
H_{DG}(K) := A \otimes I + I \otimes H_b + \tilde{\phi}(v_K),
\]

\[
H_{DG}(K, V) := A \otimes I + I \otimes H_{b,V} + \tilde{\phi}(v_{K,V}).
\]

Lemma 4.12. (i) \( H_{DG}(K) \) is self-adjoint with \( D(H_{DG}(K)) = D(H_0) \), bounded from below, and essentially self-adjoint on any core of \( H_0 \).

(ii) For sufficiently large \( V > 0 \), \( H_{DG}(K, V) \) is self-adjoint with domain \( D(H_{DG}(K,V)) = D(H_0) \), bounded from below, and essentially self-adjoint on any core of \( H_0 \).

Proof. Similar to the proof of Proposition 4.3.

Lemma 4.13. For all \( z \in \mathbb{C}\setminus\mathbb{R} \),

\[
\lim_{V \rightarrow \infty} \|(H_{DG}(K, V) - z)^{-1} - (H_{DG}(K) - z)^{-1}\| = 0,
\]

\[
\lim_{K \rightarrow \infty} \|(H_{DG}(K) - z)^{-1} - (H_{DG} - z)^{-1}\| = 0.
\]

Proof. Similar to the proof of [2 , Lemma 3.5]
Lemma 4.14. The operator $H_{DG}(K, V)$ is reduced by $F_V$.

Proof. We identify $v(x, \ell)$ with multiplication operator by $v(\cdot, \ell)$. By abuse of symbols, we denote $\chi_{\ell, V}(\cdot)$ by $\chi_{\ell, V}(k)$. Then

$$(\hat{a}^*(v(x, \ell)\chi_{\ell, V}(k))\Phi)^{(n)} = \sqrt{n}(I \otimes S_n)(v(x, \ell)\chi_{\ell, V}(k) \otimes I)\Phi^{(n-1)}$$

$$= \sqrt{n}v(x, \ell)S_n(\chi_{\ell, V} \otimes \Phi^{(n-1)})$$

$$= \chi(x, \ell)\sqrt{n}S_n(\chi_{\ell, V} \otimes \Phi^{(n-1)}).$$

Hence, we have

$$\hat{a}^*(v(x, \ell)\chi_{\ell, V}(k))\Phi = v(x, \ell) \otimes a^*(\chi_{\ell, V})\Phi.$$ 

Therefore, we get

$$\hat{a}^*(v_{K, V}) = \sum_{\ell \in \Gamma_V, |\ell| < K} v(\cdot, \ell) \otimes a^*(\chi_{\ell, V}). \quad (23)$$

Hence, its adjoint is

$$\hat{a}(v_{K, V}) = \sum_{\ell \in \Gamma_V, |\ell| < K} v(\cdot, \ell)^* \otimes a(\chi_{\ell, V}). \quad (24)$$

This means that the operator $H_{DG}(K, V)$ is a special case of the GSB Hamiltonian (see [2]). Hence, by [2, Lemma 3.7] $H_{DG}(K, V)$ is reduced by $F_V$. 

Lemma 4.15. $H_{DG}(K, V)[F_V^\perp \geq E_0(H_{DG}(K, V)) + m$

Proof. Similar to the proof of [2, Lemma 3.10].

Lemma 4.16. For all $\Phi \in D(I \otimes H_0^{1/2})$, and for all $\epsilon' > 0$,

$$|\langle \Phi, \tilde{\phi}(v)\Phi \rangle| \leq \epsilon' \frac{\|v\|}{\sqrt{\omega}} \left\| I \otimes H_0^{1/2} \right\|^2 + \frac{\|v\|^2}{2} \left( \epsilon' + \frac{1}{\epsilon'} \right) \|\Phi\|^2.$$

26
Proof. For all $\Phi \in D(I \otimes H_b^{1/2})$, $\epsilon' > 0$,
\[
|\langle \Phi, \tilde{\phi}(v)\Phi \rangle| \leq \frac{1}{\sqrt{2}} \left( \epsilon \|a(v)\Phi\|^2 + \frac{1}{4\epsilon} \|\Phi\|^2 + \epsilon \|a^*(v)\Phi\|^2 + \frac{1}{4\epsilon} \|\Phi\|^2 \right)
\leq \frac{1}{\sqrt{2}} \left( 2\epsilon \left\| \frac{v}{\sqrt{\omega}} \right\|^2 \|I \otimes H_b^{1/2}\Phi\|^2 + \epsilon \|v\| \|\Phi\|^2 + \frac{1}{2\epsilon} \|\Phi\|^2 \right)
= \sqrt{2\epsilon} \left\| \frac{v}{\sqrt{\omega}} \right\|^2 \|I \otimes H_b^{1/2}\Phi\|^2 + \frac{\|v\|}{2} \left( \sqrt{2\epsilon} \|v\| + \frac{1}{\sqrt{2\epsilon} \|v\|} \right) \|\Phi\|^2,
\]
where we have used Lemma 4.8 and 4.10. Let $\sqrt{2\epsilon} \|v\| =: \epsilon'$. Then, for all $\epsilon' > 0$, we have
\[
|\langle \Phi, \tilde{\phi}(v)\Phi \rangle| \leq \frac{\epsilon'}{|v|} \left\| \frac{v}{\sqrt{\omega}} \right\|^2 \|I \otimes H_b^{1/2}\Phi\|^2 + \frac{\|v\|}{2} \left( \epsilon' + \frac{1}{\epsilon'} \right) \|\Phi\|^2.
\]

Proof of Theorem 4.4. From (23) and (24), $H_{DG}(K, V)$ is equal to the special case of the GSB model. Therefore, $H_{DG}(K, V)[\mathcal{F}_V$ has the same form with $H_{DG}(K, V)$. Using Lemma 4.16 we have on $D(H_0) \cap \mathcal{F}_V$
\[
H_{DG}(K, V)
= A \otimes I + I \otimes H_{b,V} + \tilde{\phi}(v_{K,V})
\geq A \otimes I + I \otimes H_{b,V} - \frac{\epsilon'}{|v_{K,V}|} \left\| \frac{v_{K,V}}{\sqrt{\omega_V}} \right\|^2 \|I \otimes H_{b,V} - \frac{\|v_{K,V}\|}{2} \left( \epsilon' + \frac{1}{\epsilon'} \right)
= A \otimes I + \left(1 - \frac{\epsilon'}{|v_{K,V}|} \left\| \frac{v_{K,V}}{\sqrt{\omega_V}} \right\|^2 \right) I \otimes H_{b,V} - \frac{\|v_{K,V}\|}{2} \left( \epsilon' + \frac{1}{\epsilon'} \right),
\]
where $\epsilon' > 0$ is an arbitrary constant. By Lemma 3.10, $H_{b,V}[\mathcal{F}_{b,V}$ has compact resolvent. Thus, for $\epsilon' > 0$ satisfying
\[
1 - \frac{\epsilon'}{|v_{K,V}|} \left\| \frac{v_{K,V}}{\sqrt{\omega_V}} \right\|^2 > 0,
\]
the bottom of the essential spectrum of (25) is equal to
\[
\Sigma(A) - \frac{\|v_{K,V}\|}{2} \left( \epsilon' + \frac{1}{\epsilon'} \right).
\]
Let, $D_K$ and $D_{K,V}$ be $D$ with $v$ replaced by $v_K$, $v_{K,V}$, respectively. It is easy to see that
\[
\lim_{K \to \infty} D_K = D, \quad \lim_{V \to \infty} D_{K,V} = D_K.
\]

By Lemma 4.13, one has
\[
\lim_{K \to \infty} E_0(H_{DG}(K)) = E_0(H_{DG}), \quad \lim_{V \to \infty} E_0(H_{DG}(K,V)) = E_0(D_{DG}(K)).
\]

From the assumption of Theorem 4.4, for all $K > 0$, there exists a constant $V_0$ such that for $V > V_0$,
\[
\Sigma(A) - \|v_{K,V}\| D_{K,V} - E_0(H_{DG}(K,V)) > 0.
\]

By the definition of $D_{K,V}$, for all $K > 0$ and $V > V_0$, and for all $\epsilon'$ which satisfies (26), we have
\[
\Sigma(A) - \frac{\|v_{K,V}\|}{2} D_{K,V} - E_0(H_{DG}(K,V)) > 0.
\]

Therefore, by Theorem 2.1, we have that $H_{DG}(K,V)[\mathcal{F}_V$ has purely discrete spectrum in
\[
[E_0(H_{DG}(K,V)), \Sigma(A) - \|v_{K,V}\| D_{K,V}).
\]

This fact and Lemma 4.15 mean that $H_{DG}(K,V)$ has purely discrete spectrum in
\[
[E_0(H_{DG}(K,V)), \Sigma(A) - \|v_{K,V}\| D_{K,V} + \min\{E_0(H_{DG}(K,V)) + \Sigma(A) - \|v_{K,V}\| D_{K,V}\}].
\]

Finally, we use Lemma 3.13 and Lemma 4.13, to conclude that $H_{DG}$ has purely discrete spectrum in the interval
\[
[E_0(H_{DG}), \min\{E_0(H_{DG}) + \Sigma(A) - \|v\| D\}).
\]

Acknowledgements

We would like to thank Professor A. Arai of Hokkaido University for proposing a problem, discussions and helpful comments.
References


