THE CENTER MAP OF AN AFFINE IMMERSION

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ABSTRACT. We study the center map of an equiaffine immersion which is introduced using the equiaffine support function. The center map is a constant map if and only if the hypersurface is an equiaffine sphere. We investigate those immersions for which the center map is affine congruent with the original hypersurface. In terms of centroaffine geometry, we show that such hypersurfaces provide examples of hypersurfaces with vanishing centroaffine Tchebychev operator. We also characterize them in equiaffine differential geometry using a curvature condition involving the covariant derivative of the shape operator. From both approaches, assuming the dimension is 2 and the surface is definite, a complete classification follows.

1. INTRODUCTION

A proper affine sphere is one of the most important objects in affine differential geometry. It was introduced by Tzitzéica in the early 20th century, and attracts the attention of many mathematicians. Such a hypersurface is characterized by the condition that all its affine normals pass through a fixed point called the center. In this paper, we generalize the center to a map for an affine hypersurface, not only for a proper affine sphere, by an elementary idea, and call it the center map (Definition 2.1). By definition, we can state that proper affine spheres are affine hypersurfaces whose center map is constant. The main purpose

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of this paper is then to investigate affine hypersurfaces whose center map is centroaffine congruent with the original immersion. The above question is treated both from the centroaffine viewpoint and from the equiaffine viewpoint.

First, we treat the problem using centroaffine techniques. We show that such a hypersurface must have vanishing Tchebychev operator. It is well known that the Tchebychev operator plays a very important role in centroaffine differential geometry. It was first introduced and studied by Wang, Liu, Simon et al (see [3, 6, 7]).

Next we restrict to dimension two and calculate the center maps of some special surfaces with vanishing Tchebychev operator, introduced in [4]. We find that for that class the converse is also true, i.e. the center map is centroaffinely congruent to the original surface. In particular, we show that the Tchebychev operator of such a hypersurface vanishes (Theorem 3.1), and that under some condition the converse is true (Theorem 2.10).

From the view point of equiaffine differential geometry, the property that the center map is centroaffinely congruent to the original hypersurface is also studied. It characterizes the elliptic paraboloid among the affine spheres (Theorem 5.2). Considering non-affine-sphere case, we show that locally strongly convex surfaces with the above property have projectively flat equiaffinely-induced connections and flat equiaffine metrics, from which we can classify such surfaces (Theorem 5.5).

2. Preliminaries and Examples

We briefly recall the basic theory of affine hypersurfaces and fix the notations. For more details, we refer to [5].

Let $f : M \to \mathbb{R}^{n+1}$ be an immersion of an n-dimensional oriented manifold M into $\mathbb{R}^{n+1}$. We denote by D the standard flat affine connection of $\mathbb{R}^{n+1}$, and by Det the standard parallel volume form of $\mathbb{R}^{n+1}$. Throughout this paper, we assume that f is nondegenerate. Let ξ be the Blaschke normal vector field of f, which is, by definition, characterized by the following conditions:

(1) At each point u of M the tangent space $T_{f(u)}\mathbb{R}^{n+1}$ is decomposed as $T_{f(u)}\mathbb{R}^{n+1} = f_*T_uM \oplus \mathbb{R}\xi_u$.

(2) The volume form θ defined by

$$\theta(X_1, \ldots, X_n) := \mathrm{Det}(f_*X_1, \ldots, f_*X_n, \xi) \tag{2.1}$$

for $X_1, \ldots, X_n \in \Gamma(TM)$, is compatible with the orientation of M.

(3) The 1-form τ defined by

$$D_X\xi = -f_*SX + \tau(X)\xi \tag{2.2}$$

for $X \in \Gamma(TM)$, vanishes identically.

(4) The symmetric $(0,2)$-tensor field h defined by

$$D_X f_*Y = f_*\nabla_X Y + h(X,Y)\xi \tag{2.3}$$

for $X, Y \in \Gamma(TM)$, is nondegenerate, and moreover,

(5) the volume form with respect to the pseudo-Riemannian metric h coincides with θ.

It is well known that such a ξ is uniquely determined. We then call ∇, h and S, respectively, the equiaffinely-induced connection, the equiaffine metric, and the equiaffine shape operator of f.

Let ρ be the equiaffine support function of f from the origin $o \in \mathbb{R}^{n+1}$. By definition, it is a function of M written as

$$f(u) = f_*Z_u + \rho(u)\xi_u, \tag{2.4}$$

where $f(u) = \overrightarrow{of(u)}$ is regarded as the element of $T_{f(u)}\mathbb{R}^{n+1}$, and Z is a vector field on M.

An immersion $f : M \to \mathbb{R}^{n+1}$ is called a proper affine hypersphere with radius r if $\rho(u) = r$ for all $u \in M$. Then the equiaffine shape operator of f is given as $S = -r^{-1}\mathrm{id}$. In fact, comparing the ξ components of the derivative of (2.4): $f_*\nabla_X Z + h(X,Z)\xi = D_X f_* Z = D_X(f - r\xi) = f_*X - r(-f_*SX) = f_*(\mathrm{id} + rS)X$, we have $Z = 0$, and then $S = -r^{-1}\mathrm{id}$. Accordingly, if f is a proper affine hypersphere, the map $f - \rho\xi : M \to \mathbb{R}^{n+1}$ is constant. In general, we put the following.

Definition 2.1. For an immersion $f : M \to \mathbb{R}^{n+1}$, we set $c : M \to \mathbb{R}^{n+1}$ by

$$c(u) := c_f(u) := f(u) - \rho(u)\xi_u, \quad \text{for } u \in M,$$

and call it the *center map* of f.

It is an interesting problem to characterize already-known classes, or to find new classes in terms of center maps. First of all, we should remark the following again:

Proposition 2.2. *An immersion $f : M \to \mathbb{R}^{n+1}$ is a proper affine hypersphere if and only if the center map c of f is constant.*

Proof. By definition, we have that

$$c_*X = X(f - \rho\xi) = f_*(\mathrm{id} + \rho S)X - (X\rho)\xi \tag{2.5}$$

for $X \in \Gamma(TM)$. If c is a constant map, then ρ is a constant function, that is, f is a proper affine hypersphere. □

Proposition 2.3. *(1) The center map c of an immersion $f : M \to \mathbb{R}^{n+1}$ is an immersion if and only if $\mathcal{D} := \ker(\mathrm{id} + \rho S) \cap \ker d\rho = \{0\}$.*
(2) The center map of an improper affine hypersphere is an immersion.

Proof. The equation (2.5) implies that $c_*X = 0$ if and only if $X \in \ker(\mathrm{id} + \rho S) \cap \ker d\rho$.

To show (2), you should only remember that the equiaffine shape operator of an improper affine hypersphere vanishes identically. □

We now calculate center maps for some examples.

Example 2.4. We consider the elliptic paraboloid given by

$$f(u,v) = {}^t(u, v, (u^2 + v^2)/2).$$

As $uf_u + vf_v = {}^t(u, v, u^2 + v^2) = {}^t(u, v, (u^2 + v^2)/2) - \rho\, {}^t(0, 0, 1)$, where $\rho = -(u^2 + v^2)/2$, we see that

$$c(u, v) = {}^t(u, v, u^2 + v^2).$$

As a consequence, the center map of c is written as

$$c(u, v) = Af(u, v), \text{ where } A := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \in GL(3; \mathbb{R}),$$

that is, c is centroaffinely congruent with f.

Moreover, for $b = {}^t(b^1, b^2, b^3) \in \mathbb{R}^3$, the center map of $f + b$ is given by

$$c_{f+b}(u, v) = {}^t(u + b^1, v + b^2, u^2 + v^2 + ub^1 + vb^2).$$

Therefore, c_{f+b} is centroaffinely congruent with $f + b$ if and only if $b^3 = -\{(b^1)^2 + (b^2)^2\}/2$.

Example 2.5. For $a, b \in \mathbb{R}$ such that $ab(a + b - 1) \neq 0$, we set

$$f_{ab}(u, v) := {}^t(u, v, u^{-a}v^{-b}).$$

Wang [7] obtained these immersions as examples of centroaffine extremal surfaces, which are critical points in the variational problem for the area functional with respect to centroaffine metrics. The center map of f_{ab} is given as

$$c(u, v) = Af_{ab}(u, v), \text{ where}$$
$$A := \begin{bmatrix} -a^{-1}(1 - 2a + b)/2 & 0 & 0 \\ 0 & -b^{-1}(1 + a - 2b)/2 & 0 \\ 0 & 0 & -(a + b - 2)/2 \end{bmatrix}.$$

If $\det A \neq 0$, the center map is again centroaffinely congruent with the original immersion. In particular, when $a = b = 1$, $f_{11}$is a proper affine sphere and c is a constant map 0.

We assume that $f : M \to \mathbb{R}^{n+1}$ is a centroaffine immersion as well. By definition, for each point the position vector is transversal to the tangent space, and the symmetric $(0, 2)$-tensor field h^c defined by

$$D_X f_* Y = f_* \nabla^c_X Y + h^c(X, Y) f \tag{2.6}$$

is nondegenerate. We call ∇^c, h^c, respectively, the centroaffinely-induced connection, centroaffine metric, respectively, of f. The hypersurface is called elliptic if and only if the centroaffine metric is negative definite. We denote the difference tensor of the the centroaffinely-induced connection and the Levi-Civita connection ∇^{h^c} of the centroaffine metric by $K^c = \nabla^c - \nabla^{h^c} \in \Gamma(TM^{(1,2)})$. By definition(for example, see [2]), the Tchebychev vector field T and the Tchebychev operator $\mathcal{T}$ are given by

$$T := \mathrm{tr}_{h^c} K^c = \frac{n+2}{2} \operatorname{grad}_{h^c} \log \rho \ \in \Gamma(TM), \tag{2.7}$$

$$\mathcal{T} := \nabla^{h^c} T \ \in \Gamma(TM^{(1,1)}). \tag{2.8}$$

Example 2.6. For $a, b \in \mathbb{R}$ such that $(2b-1)(a^2+b^2) \neq 0$, we set

$$f_{ab}(u,v) := {}^t \left(u, v, (u^2+v^2)^b \exp\left(-a \arctan(u/v)\right)\right).$$

The center map of f_{ab} is given as

$$c(u,v) = A f_{ab}(u,v), \text{ where}$$
$$A := \frac{1}{2}(a^2+4b^2)^{-1} \begin{bmatrix} 3a^2+4b(1+b) & -2a(1-2b) & 0 \\ 2a(1-2b) & 3a^2+4b(1+b) & 0 \\ 0 & 0 & 2(a^2+4b^2)(1+b) \end{bmatrix}.$$

If $b \neq -1$, then $\det A = (1+b)(a^2+4b^2)^{-1}\{9a^2+4(1+b)^2\}/4 \neq 0$, and the center map is centroaffinely congruent with the original immersion.

Example 2.7. For $a, b \in \mathbb{R}$ such that $b(a+b) \neq 0$, we set

$$f_{ab}(u,v) := {}^t \left(u, v, -u(a \log u + b \log v)\right).$$

The center map of f_{ab} is given as

$$c(u,v) = A f_{ab}(u,v), \quad \text{where } A := \begin{bmatrix} 3/2 & 0 & 0 \\ 0 & 1-(1/2)ab^{-1} & 0 \\ -(a+b) & 0 & 3/2 \end{bmatrix}.$$

If $a - 2b \neq 0$, then $\det A = -(9/8)(a-2b)b^{-1} \neq 0$, and the center map is centroaffinely congruent with the original immersion.

Example 2.8. Let $a = a(v), b = b(v)$ be the fundamental solutions of the ordinary differential equation

$$y''(v) - y'(v) - \vartheta(v) y(v) = 0 \tag{2.9}$$

for some function ϑ. we set

$$f_{ab}(u,v) := {}^t\left(e^v, a(v)e^u, b(v)e^u\right).$$

The center map of f_{ab} is given as

$$c(u,v) = {}^t\left(e^v, (a(v)+a'(v))\,e^u, (b(v)+b'(v))\,e^u\right).$$

The triplet of functions $a(v) = e^{v/2}$, $b(v) = ve^{v/2}$, $\vartheta(v) = -1/4$, satisfy the equation (2.9). In this case, we have

$$\begin{aligned} f_{ab}(u,v) &= {}^t\left(e^v, e^{u+v/2}, ve^{u+v/2}\right), \\ c(u,v) &= {}^t\left(e^v, (3/2)e^{u+v/2}, (v^{-1}+3/2)ve^{u+v/2}\right). \end{aligned}$$

Thus the center map is not centroaffinely congruent with the original immersion.

The centroaffine metric of a surface in Example 2.8 is indefinite.

Example 2.9. We consider the surface parameterized by

$$f(u,v) = {}^t\left(e^{2v}, \sqrt{2}ue^{2v}, 2(u^2+v)e^{2v}\right).$$

It follows that

$$\begin{aligned} f_{uu} &= 2f_v - 4f, \\ f_{uv} &= 2f_u, \\ f_{vv} &= 4f_v - 4f, \end{aligned}$$

from which it follows that f has flat centroaffine metric. It is clear that this metric is elliptic and that u and v are flat coordinates for this metric. Moreover, as the Tchebychev vector field is a constant multiple of f_v it follows immediately that the Tchebychev operator vanishes.

A straightforward calculation shows that the center map of f is given as

$$c(u,v) = Af(u,v), \text{ where } A := \frac{3}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

from which we see that the center map is centroaffinely congruent with the original immersion.

Liu and Wang [3] studied the nondegenerate centroaffine surfaces in $\mathbb{R}^3$ with $\mathcal{T} = 0$. However the classification result obtained therein is incomplete. Its proof uses Theorem 1.3 of [4] which for surfaces is only valid if the metric is not elliptic. Hence the correct formulation of Theorem 4.1 of [3] should be that a nondegenerate centroaffine surface in $\mathbb{R}^3$, which is not elliptic but with vanishing Tchebychev operator $\mathcal{T}$ belongs to one of the following five types:

(i) proper affine spheres with center 0 (The center maps are given in Proposition 2.2).

(ii)-(v) surfaces in Examples 2.5-2.8, respectively.

If the centroaffine metric of a surface f with vanishing Tchebychev operator is positive definite (not elliptic), f is centroaffinely equivalent to an open set of one of (i) - (iv). Accordingly, we obtain the following.

Theorem 2.10. *Let $f : M \to \mathbb{R}^3$ be a surface with positive definite centroaffine metric. Suppose that the center map c is an immersion. If the Tchebychev operator vanishes identically, then c is centroaffinely congruent to f.*

Note that Example 2.9 shows that the theorem of Liu and Wang is not correct without the additional assumption elliptic. However, the authors believe it is the only surface missing in the classification. Some evidence of this is given in Theorem 5.5.

3. Centroaffine Properties

In this section, we consider the converse of Theorem 2.10. In fact, we prove the following.

Theorem 3.1. *Let $f : M \to \mathbb{R}^{n+1}$ be a centroaffine immersion with center map c. If c is centroaffinely congruent to f, then the Tchebychev operator of f vanishes identically. In particular, f is a centroaffine extremal immersion.*

We denote by $\widetilde{\nabla}^c$ the centroaffinely-induced connection of c, and by $\widetilde{h}^c$ the centroaffine metric:

$$D_X c_* Y = c_* \widetilde{\nabla}^c_X Y + \widetilde{h}^c(X, Y)c \tag{3.1}$$

for $X, Y \in \Gamma(TM)$.

Lemma 3.2. *The following formula holds for any $X, Y \in \Gamma(TM)$:*

$$h^c(X, \mathcal{T}Y) = -\frac{1}{2} h^c(\nabla^c_X Y - \widetilde{\nabla}^c_X Y, T).$$

This lemma implies Theorem 3.1 immediately. In fact, if f and c are centroaffinely congruent, then $\nabla^c = \widetilde{\nabla}^c$, and hence $\mathcal{T}$ vanishes identically. To prove Lemma 3.2, we need some calculation as follows.

Lemma 3.3. *Let c be the center map of an immersion $f : M \to \mathbb{R}^{n+1}$. Then the following formula holds:*

$$c = -\frac{2}{n+2} f_* T,$$

where T is the Tchebychev vector field of f.

Proof. By (2.4), we have that $c = f_* Z$, and

$$c_* X = D_X f_* Z = f_* \nabla_X Z + h(X, Z)\xi.$$

Comparing it with (2.5), we get

$$\nabla_X Z = (\mathrm{id} + \rho S)\, X, \tag{3.2}$$

$$h(X, Z) = -X\rho. \tag{3.3}$$

By $f_* \nabla_X Y + h(X, Y)\xi = D_X f_* Z = f_* \nabla^c_X Y + h^c(X, Y)\,(\rho\xi + f_* Z)$, we should remark that the centroaffine invariants ∇^c, h^c are written by equiaffine invariants as follows:

$$h^c = \rho^{-1} h, \tag{3.4}$$

$$\nabla^c_X Y = \nabla_X Y - \rho^{-1} h(X, Y) Z. \tag{3.5}$$

By (3.3) and (3.4), we get $h^c(X, Z) = -\rho^{-1} X\rho$, which implies

$$Z = -\operatorname{grad}_{h^c} \log \rho = -\frac{2}{n+2} T.$$

$\square$

Proof of Lemma 3.2. First, we prove the following formula:

$$h^c(X, \nabla^c_Y T) + X h^c(Y, T) = h^c(\widetilde{\nabla}^c_X Y, T). \tag{3.6}$$

To get it, we consider the decomposition of $D_X c_* Y$ centroaffinely. By Lemma 3.3, we have that

$$D_X c_* Y = D_X(Yc) = -\frac{2}{n+2} D_X D_Y f_* T,$$

from which,

$$\begin{aligned}
&-\frac{n+2}{2} D_X c_* Y \\
=\ & D_X \left\{ f_* \nabla^c_Y T + h^c(Y, T) f \right\} \\
=\ & f_* \left\{ \nabla^c_X \nabla^c_Y T + h^c(Y, T) X \right\} + \left\{ h^c(X, \nabla^c_Y T) + X h^c(Y, T) \right\} f.
\end{aligned}$$

On the other hand, by Lemma 3.3 again we have that

$$\begin{aligned}
&-\frac{n+2}{2} \left\{ c_* \widetilde{\nabla}^c_X Y + \widetilde{h}^c\ (X, Y) c \right\} \\
=\ & D_{\widetilde{\nabla}^c_X Y} f_* T + \widetilde{h}^c(X, Y) f_* T \\
=\ & f_* \left\{ \nabla^c_{\widetilde{\nabla}^c_X Y} T + \widetilde{h}^c\ (X, Y) T \right\} + h^c(\widetilde{\nabla}^c_X Y, T) f.
\end{aligned}$$

Comparing the f components, we obtain the formula (3.6). Using well-known formulas

$$h^c(X, \mathcal{T} Y) = h^c(\mathcal{T} X, Y),$$
$$h^c(X, K^c_Y Z) = h^c(K^c_X Y, Z),$$

by (3.6) we calculate that

$$\begin{aligned}
0 \ =\ & h^c(X, \nabla^c_Y T) + X h^c(Y, T) - h^c(\widetilde{\nabla}^c_X Y, T) \\
=\ & \left\{ h^c(X, \nabla^{h^c}_Y T) + h^c(X, K^c_Y T) \right\} \\
& + \left\{ h^c(\nabla^c_X Y - K^c_X Y, T) + h^c(Y, \nabla^{h^c}_X T) \right\} - h^c(\widetilde{\nabla}^c_X Y, T) \\
=\ & h^c(X, \mathcal{T} Y) + h^c(Y, \mathcal{T} X) + h^c(\nabla^c_X Y - \widetilde{\nabla}^c_X Y, T) \\
& + h^c(X, K^c_Y T) - h^c(K^c_X Y, T) \\
=\ & 2h^c(X, \mathcal{T} Y) + h^c(\nabla^c_X Y - \widetilde{\nabla}^c_X Y, T),
\end{aligned}$$

which completes the proof of Lemma 3.2 and hence Theorem 3.1. □

4. Equiaffine Properties

We study hypersurfaces whose center map is centroaffinely congruent to the original hypersurface from the view point of equiaffine differential geometry. Throughout this section, we assume that the center map c of an immersion $f : M \to \mathbb{R}^{n+1}$ is centroaffinely congruent to f.

For f we define a vector field on M by

$$Z^* := \rho^{-1} Z, \tag{4.1}$$

where ρ, Z are defined in (2.4). The formulas (3.3) and (3.2) are written as

$$X\rho = -\rho h(X, Z^*), \tag{4.2}$$

$$\nabla_X Z^* = h(X, Z^*)Z^* + \rho^{-1}(\mathrm{id} + \rho S)X. \tag{4.3}$$

We consider the decomposition of $D_X c_* Y$ equiaffinely. By (2.5) we have

$$\begin{aligned} D_X c_* Y &= D_X \{f_*(\mathrm{id} + \rho S)Y - (Y\rho)\xi\} \\ &= f_* [\nabla_X \{(\mathrm{id} + \rho S)Y\} + (Y\rho)SX] \\ &\quad + [h(X, (\mathrm{id} + \rho S)Y) - X(Y\rho)]\,\xi. \end{aligned}$$

On the other hand, by (3.4) and (3.5) we get

$$\begin{aligned} c_* \widetilde{\nabla}^c_X Y + \widetilde{h}^c(X, Y)c &= c_* \nabla^c_X Y + h^c(X, Y)c \\ &= c_* \left\{\nabla_X Y - \rho^{-1} h(X, Y)Z\right\} + \rho^{-1} h(X, Y) f_* Z \\ &= f_* [(\mathrm{id} + \rho S)\nabla_X Y - h(X, Y)SZ] \\ &\quad - \left[\left\{\nabla_X Y - \rho^{-1} h(X, Y)Z\right\}\rho\right]\xi. \end{aligned}$$

Comparing the tangential components, we get

$$\rho(\nabla_X S)Y = -(X\rho)SY - (Y\rho)SX - h(X, Y)SZ,$$

and hence,

$$(\nabla_X S)Y = h(X, Z^*)SY + h(Y, Z^*)SX - h(X, Y)SZ^*. \tag{4.4}$$

Comparing the ξ components, we have

$$h(X,Y)+\rho h(X,SY)-X(Y\rho)=-\rho^{-1}h(X,Y)h(Z,Z)+h(\nabla_X Y,Z),$$

and hence,

$$(\nabla_X h)(Y,Z^*) = -2\rho^{-1}h(X,Y)-2h(X,SY)-h(X,Y)h(Z^*,Z^*). \tag{4.5}$$

We should remark that c is a centroaffine immersion if and only if

$$\dim\{f_*Z^*_u, f_*(\mathrm{id}+\rho S)X-(X\rho)\xi; X\in T_uM\}=n+1 \tag{4.6}$$

for any $u\in M$. To sum up, we have the following.

Proposition 4.1. *Suppose that the center map c of an immersion $f : M \to \mathbb{R}^{n+1}$ is a centroaffine immersion which is centroaffinely congruent with f. Then there exist a non vanishing function ρ and a vector field Z^* satisfying (4.2) – (4.6).*

Conversely, we prove

Proposition 4.2. *Given an arbitrary equiaffine hypersurface f which admits a positive function ρ and a vector field Z^* satisfying (4.2) – (4.6), there exists a constant vector v such that $f+v$ is centroaffinely congruent to the center map c_{f+v}.*

Proof. We define a vector field η along the immersion by

$$\eta=f-\rho\xi-f_*(\rho Z^*).$$

It then follows in a straightforward way that $D\eta=0$. Hence η is a constant vector field. It is now straightforward to verify that the center map of $f-\eta$ is centroaffinely congruent to $f-\eta$. □

Proposition 4.3. *Suppose that the center map c of an immersion $f : M \to \mathbb{R}^{n+1}$ is a centroaffine immersion which is centroaffinely congruent with f. Assume that the equiaffine metric h is positive definite. Denote by λ_i the eigenvalues of the equiaffine shape operator S. Then there exist constants ν_j such that*

$$\lambda_j=\nu_j\rho^{-1},$$

where ρ is the equiaffine support function of f.

Proof. As h is positive definite, we know that there exist local orthonormal vector fields X_i such that $SX_i = \lambda_i X_i$. In terms of this basis, (4.4) reduces to

$$\begin{aligned} & (X_i\lambda_j)X_j - S\nabla_{X_i}X_j + \lambda_j\nabla_{X_i}X_j \\ = \;& h(X_i, Z^*)\lambda_j X_j + h(X_j, Z^*)\lambda_i X_i - \delta_{ij}SZ^*, \end{aligned} \tag{4.7}$$

from which when $i \neq j$ we have

$$\begin{aligned} 0 \;=\; & \{X_i\lambda_j - h(X_i, Z^*)\lambda_j\}\, X_j \\ & - \{h(X_j, Z^*)\lambda_i + (\lambda_i - \lambda_j)h(\nabla_{X_i}X_j, X_i)\}\, X_i \\ & - \sum_{k\neq i,j}(\lambda_k - \lambda_j)h(\nabla_{X_i}X_j, X_k)X_k, \end{aligned} \tag{4.8}$$

and when $i = j$ we have

$$\begin{aligned} 0 \;=\; & \{X_j\lambda_j - h(X_j, Z^*)\lambda_j\}\, X_j \\ & - \sum_{k\neq j}\big\{(\lambda_k - \lambda_j)h(\nabla_{X_j}X_j, X_k) - h(X_k, Z^*)\lambda_k\big\}\, X_k. \end{aligned} \tag{4.9}$$

By looking at the X_j components of the above equations, we deduce (both in the case that $i = j$ and $i \neq j$) that $X_i\lambda_j = h(X_i, Z^*)\lambda_j$. As the above is valid for all i, it follows that for an arbitrary vector field X, we have that

$$X\lambda_j = h(X, Z^*)\lambda_j. \tag{4.10}$$

Using now (4.2), it follows that

$$X(\lambda_j\rho) = \rho h(X, Z^*)\lambda_j + \lambda_j(-1)\rho h(X, Z^*) = 0.$$

Hence for all indices j, the functions $\lambda_j\rho$ are constant. □

5. Equiaffine Surfaces

In this section, we will now further restrict ourselves to the case that M is 2-dimensional. First we assume that $f : M \to \mathbb{R}^3$ is an affine sphere. In this case, we have the following:

Lemma 5.1. *Let $f : M^2 \to \mathbb{R}^3$ be an affine sphere with positive definite equiaffine metric. If the non-zero vector field Z^* satisfies* (4.4)*, then f is an improper affine sphere. If moreover f also satisfies* (4.5)*, then f is equiaffinely congruent with the elliptic paraboloid.*

Proof. As an affine sphere has constant mean curvature, we have that $S = H\mathrm{id}$, where H is a constant. Then (4.4) reduces to

$$H(h(X, Z^*)Y + h(Y, Z^*)X - h(X, Y)Z^*) = 0.$$

So either $H = 0$, or by taking X and Y orthogonal it follows that $Z^* = 0$. The latter is a contradiction.

If f also satisfies (4.5) it follows from the apolarity condition that

$$-2 - \rho h(Z^*, Z^*) = 0.$$

Therefore (4.5) reduces to $(\nabla h)(X, Y, Z^*) = -2h(K(X, Y), Z^*) = 0$, where K denotes the difference tensor of the equiaffinely-induced connection and the Levi-Civita connection of h. Hence image(K) is less than 2-dimensional. However, as in general, from the symmetries of the difference tensor, we have

$$\begin{aligned} K(X_1, X_1) &= pX_1 + qX_2, \\ K(X_1, X_2) &= qX_1 - pX_2, \\ K(X_2, X_2) &= -pX_1 - qX_2, \end{aligned} \tag{5.1}$$

where $\{X_1, X_2\}$ is an arbitrary orthonormal basis. Hence image(K) is less than 2-dimensional if and only if $p = q = 0$. In this case, K vanishes identically and the Pick-Berwald theorem implies that f is equiaffine congruent with the elliptic paraboloid. □

Combining Example 2.4 with the previous lemma, we have

Theorem 5.2. *Let $f : M^2 \to \mathbb{R}^3$ be an affine sphere with positive definite equiaffine metric. There exists a constant vector v such that $f + v$ is centroaffinely congruent to the center map c_{f+v} if and only if f is equiaffinely congruent to the elliptic paraboloid.*

Next we consider the case that f is not an affine sphere. In view of Proposition 4.3, we therefore have that there exists a constant μ such that $\lambda_2 = \mu\lambda_1$. Looking back at the equation (4.8), we see that the X_i component reduces to

$$(\lambda_j - \lambda_i)h(\nabla_{X_i}X_j, X_i) = \lambda_i h(X_j, Z^*) = X_j\lambda_i,$$

whereas looking at the component X_k $(k \neq i)$ of (4.9) gives

$$(\lambda_i - \lambda_k)h(\nabla_{X_i}X_i, X_k) = -\lambda_k h(X_k, Z^*) = -X_k\lambda_k.$$

Introducing now functions α and β by

$$\alpha = h(\nabla_{X_2}X_2, X_1), \quad \beta = -h(\nabla_{X_1}X_2, X_1), \tag{5.2}$$

we write that

$$\begin{aligned} X_1\lambda_1 &= \alpha(\lambda_1 - \lambda_2) = \alpha\lambda_1(1-\mu), \\ X_2\lambda_1 &= \beta(\lambda_1 - \lambda_2) = \beta\lambda_1(1-\mu). \end{aligned} \tag{5.3}$$

It follows using the apolarity condition that we can write

$$\nabla_{X_1}X_1 = pX_1 - \mu\beta X_2, \ \nabla_{X_1}X_2 = -\beta X_1 - pX_2,$$
$$\nabla_{X_2}X_1 = qX_1 + \mu\alpha X_2, \ \nabla_{X_2}X_2 = \alpha X_1 - qX_2.$$

Using the Codazzi equations for h:

$$(\nabla h)(X_1, X_2, X_2) = (\nabla h)(X_2, X_1, X_2),$$
$$(\nabla h)(X_2, X_1, X_1) = (\nabla h)(X_1, X_2, X_1),$$

we get that

$$2p = -(\mu+1)\alpha, \quad 2q = -(\mu+1)\beta. \tag{5.4}$$

Therefore, we obtain that

$$\begin{aligned}
\nabla_{X_1}X_1 &= -\frac{1+\mu}{2}\alpha \; X_1 - \mu\beta \; X_2,\\
\nabla_{X_1}X_2 &= -\beta \; X_1 + \frac{1+\mu}{2}\alpha \; X_2,\\
\nabla_{X_2}X_1 &= -\frac{1+\mu}{2}\beta \; X_1 + \mu\alpha \; X_2,\\
\nabla_{X_2}X_2 &= \alpha \; X_1 + \frac{1+\mu}{2}\beta \; X_2.
\end{aligned} \tag{5.5}$$

Computing $[X_1, X_2]\lambda_1$ in two different ways, we find on one hand that

$$X_1(X_2\lambda_1) - X_2(X_1\lambda_1) = (1-\mu)\lambda_1(X_1\beta - X_2\alpha).$$

As on the other hand, we have that

$$(\nabla_{X_1}X_2 - \nabla_{X_2}X_1)\,\lambda_1 = -\frac{1}{2}(1-\mu)\,(\beta X_1 - \alpha X_2)\,\lambda_1 = 0,$$

we deduce that

$$X_2\alpha - X_1\beta = 0. \tag{5.6}$$

Finally, we look at the Gauss equation. This yields

$$\begin{aligned}
\lambda_1 X_1 &= R(X_1, X_2)X_2\\
&= \nabla_{X_1}(\alpha X_1 + \frac{1}{2}\beta(1+\mu)X_2) - \nabla_{X_2}(-\beta X_1 + \frac{1}{2}\alpha(1+\mu)X_2)\\
&\qquad + \nabla_{\frac{1}{2}\beta(1-\mu)X_1 - \frac{1}{2}\alpha(1-\mu)X_2}X_2\\
&= \left\{(X_1\alpha + X_2\beta) - \frac{1}{2}(3+\mu)(\alpha^2+\beta^2)\right\}X_1.
\end{aligned}$$

So, we obtain that

$$\lambda_1 = X_1\alpha + X_2\beta - \frac{1}{2}(3+\mu)(\alpha^2+\beta^2). \tag{5.7}$$

Similarly we obtain that

$$\begin{aligned}
\lambda_2 X_2 &= R(X_2, X_1)X_1\\
&= \nabla_{X_2}(-\frac{1}{2}\alpha(1+\mu)X_1 - \mu\beta X_2) - \nabla_{X_1}(-\frac{1}{2}\beta(1+\mu)X_1 + \mu\alpha X_2)\\
&\qquad - \nabla_{\frac{1}{2}\beta(1-\mu)X_1 - \frac{1}{2}\alpha(1-\mu)X_2}X_1\\
&= \left\{-\mu\,(X_1\alpha + X_2\beta) - \frac{1}{2}\mu(1+3\mu)(\alpha^2+\beta^2)\right\}X_2.
\end{aligned}$$

Hence, when $\mu \neq 0$ it follows that

$$\lambda_1 = -X_1\alpha - X_2\beta - \frac{1}{2}(1+3\mu)(\alpha^2+\beta^2). \tag{5.8}$$

We prove that the case $\mu = 0$ does not occur.

Lemma 5.3. *Let $f : M \to \mathbb{R}^3$ be a surface with positive definite equiaffine metric. Suppose that f is not an affine sphere, and that there exists a constant vector v such that $f+v$ is centroaffinely congruent to the center map c_{f+v}. Then* $\det S$ *does not vanish, that is,* $\mu \neq 0$.

Proof. The proof is by contradiction. Suppose that $\mu = 0$. By (4.10) we have that $Z^* = \alpha X_1 + \beta X_2$. From Proposition 4.3 we know that there exists a non-zero constant ν such that $\lambda_1 = (2/\nu)\rho^{-1}$. Taking the trace of (4.5) and using the apolarity condition, we show that

$$\begin{aligned} 0 &= \mathrm{tr}_h\,(\nabla h)\,(\cdot,\cdot,Z^*) \\ &= -4\rho^{-1} - 2\lambda_1 - 2(\alpha^2+\beta^2) \\ &= -2\left\{\alpha^2+\beta^2+\lambda_1(1+\nu)\right\}. \end{aligned}$$

Hence $\nu \neq -1$ and there exists a local function σ such that

$$\alpha = \sqrt{-(1+\nu)\lambda_1}\cos\sigma,$$
$$\beta = \sqrt{-(1+\nu)\lambda_1}\sin\sigma.$$

By (5.6) we have that

$$\begin{aligned} 0 &= X_2\alpha - X_1\beta \\ &= -\sqrt{-(1+\nu)\lambda_1}\,(\sin\sigma X_2\sigma + \cos\sigma X_1\sigma). \end{aligned}$$

Substituting 0 for μ in (5.7) implies

$$\begin{aligned} 0 &= X_1\alpha + X_2\beta - \frac{3}{2}(\alpha^2+\beta^2) - \lambda_1 \\ &= \lambda_1\nu + \sqrt{-(1+\nu)\lambda_1}\,(\cos\sigma X_2\sigma - \sin\sigma X_1\sigma). \end{aligned}$$

Combining these equations, we get that

$$\begin{aligned} X_1\sigma &= \lambda_1\nu\,\{-(1+\nu)\lambda_1\}^{-1/2}\sin\sigma, \\ X_2\sigma &= -\lambda_1\nu\,\{-(1+\nu)\lambda_1\}^{-1/2}\cos\sigma. \end{aligned}$$

Computing now $[X_1, X_2]\sigma$ in two different ways, we obtain on one hand that

$$(\nabla_{X_1} X_2 - \nabla_{X_2} X_1)\, \sigma = -\frac{1}{2}\lambda_1 \nu,$$

and on the other hand that

$$X_1(X_2\sigma) - X_2(X_1\sigma) = -\lambda_1 \nu \left\{ \frac{1}{2} + \frac{\nu}{1+\nu} \right\}.$$

Hence $\nu = 0$ and a contradiction follows. $\square$

Accordingly, combining (5.7) with (5.8) it follows that

$$X_1\alpha + X_2\beta = \frac{1}{2}(1-\mu)(\alpha^2 + \beta^2), \tag{5.9}$$

$$\lambda_1 = -(1+\mu)(\alpha^2 + \beta^2). \tag{5.10}$$

Hence $\mu \neq -1$. It also follows that we can introduce a function σ such that

$$\alpha = -\frac{1}{1+\mu}\sqrt{-(1+\mu)\lambda_1}\cos\sigma,$$
$$\beta = -\frac{1}{1+\mu}\sqrt{-(1+\mu)\lambda_1}\sin\sigma.$$

Expressing (5.6) now gives that

$$\cos\sigma X_1\sigma + \sin\sigma X_2\sigma = 0,$$

whereas using (5.9) and (5.10) gives

$$\sin\sigma X_1\sigma - \cos\sigma X_2\sigma = 0.$$

Solving both equations for $X_1\sigma$ and $X_2\sigma$, we find that $X_1\sigma = X_2\sigma = 0$ and hence σ is a constant. Also all the components of the induced connection can be expressed in terms of the function λ_1 and constants μ, σ as follows:

$$
\begin{aligned}
\nabla_{X_1}X_1 &= \sqrt{-(1+\mu)\lambda_1}\left\{\frac{1}{2}\cos\sigma X_1 + \frac{\mu}{1+\mu}\sin\sigma X_2\right\},\\
\nabla_{X_1}X_2 &= \sqrt{-(1+\mu)\lambda_1}\left\{\frac{1}{1+\mu}\sin\sigma X_1 - \frac{1}{2}\cos\sigma X_2\right\},\\
\nabla_{X_2}X_1 &= \sqrt{-(1+\mu)\lambda_1}\left\{\frac{1}{2}\sin\sigma X_1 - \frac{\mu}{1+\mu}\cos\sigma X_2\right\},\\
\nabla_{X_2}X_2 &= \sqrt{-(1+\mu)\lambda_1}\left\{-\frac{1}{1+\mu}\cos\sigma X_1 - \frac{1}{2}\sin\sigma X_2\right\}.
\end{aligned} \tag{5.11}
$$

In the same way to the proof of Lemma 5.3, Z^* and ρ are written in terms of λ_1, μ, σ. In fact, by (4.10), we have that

$$
\begin{aligned}
Z^* &= \left(\lambda_1^{-1}X_1\lambda_1\right)X_1 + \left(\lambda_1^{-1}X_1\lambda_1\right)X_2\\
&= -(1-\mu)(1+\mu)^{-1}\sqrt{-(1+\mu)\lambda_1}\left(\cos\sigma X_1 + \sin\sigma X_2\right),
\end{aligned} \tag{5.12}
$$

and taking the trace of (4.5) again we show that

$$
\begin{aligned}
0 &= -4\rho^{-1} - 2(1+\mu)\lambda_1 - 2h(Z^*, Z^*)\\
&= -4\rho^{-1} - 8\mu(1+\mu)^{-1}\lambda_1,
\end{aligned}
$$

from which

$$
\rho = -\frac{1}{2}\mu^{-1}(1+\mu)\lambda_1^{-1}. \tag{5.13}
$$

Finally we consider the condition that the center map is a centroaffine immersion. From (4.6) we have that

$$
\{f_*Z^*, f_*(S+\rho^{-1}\mathrm{id})X_1 + h(X_1, Z^*)\xi, f_*(S+\rho^{-1}\mathrm{id})X_2 + h(X_2, Z^*)\xi\}
$$

are linearly independent at each point of M. Defining a vector filed

$$
W := h(X_2, Z^*)(S+\rho^{-1}\mathrm{id})X_1 - h(X_1, Z^*)(S+\rho^{-1}\mathrm{id})X_2,
$$

we have

$$
\begin{aligned}
&|f_*Z^*, f_*(S+\rho^{-1}\mathrm{id})X_1 + h(X_1, Z^*)\xi, f_*(S+\rho^{-1}\mathrm{id})X_2 + h(X_2, Z^*)\xi|\\
&\qquad = |f_*Z^*, f_*W, \xi| = \theta(Z^*, W),
\end{aligned}
$$

from which Z^* and W are linearly independent at each point. We calculate that

$$
\theta(Z^*, W) = \theta(X_1, X_2)(1-\mu)^3(1+\mu)^{-2}\lambda_1^2\left(\sin^2\sigma - \mu\cos^2\sigma\right),
$$

and hence $\sin^2\sigma - \mu\cos^2\sigma \neq 0$.

To sum up, we get the following:

Lemma 5.4. *Let $f : M \to \mathbb{R}^3$ be a surface with positive definite equiaffine metric h. Suppose that f is not an affine sphere and* $\det S \neq 0$.

If there exists a constant vector v such that $f + v$ is centroaffinely congruent to the center map c_{f+v}, then there exist local orthonormal vector fields $\{X_1, X_2\}$ with respect to h and constants $\mu(\neq 0, \pm 1)$ and σ such that

$$SX_1 = \lambda_1 X_1, \quad SX_2 = \mu\lambda_1 X_2,$$

$\mu\cos^2\sigma - \sin^2\sigma \neq 0$, *and ∇ is given by* (5.11).

The converse is also true. Namely, if there exist local orthonormal vector fields $\{X_1, X_2\}$ with respect to h and constants $\mu(\neq 0, \pm 1)$ and σ such that (1) $SX_1 = \lambda_1 X_1$, $SX_2 = \mu\lambda_1 X_2$, (2) $\mu\cos^2\sigma - \sin^2\sigma \neq 0$, and (3) ∇ is given by (5.11)*, then there exists a constant vector v such that $f + v$ is centroaffinely congruent to the center map c_{f+v}.*

Proof. The converse part is given due to Proposition 4.2 as follows. We define a function ρ as (5.13), and a vector field Z^* as (5.12). We can check the condition (4.2) - (4.6) by straightforward calculation. □

Comparing the above expressions with the results of [1], or verifying directly, it follows that f is a surface with projectively flat equiaffinely-induced connection and flat equiaffine metric. This shows the following theorem:

Theorem 5.5. *Let $f : M^2 \to \mathbb{R}^3$ be a surface with positive definite equiaffine metric. Suppose that f is not an affine sphere. If there exists a constant vector v such that $f + v$ is centroaffinely congruent to the center map c_{f+v}, then f has flat equiaffine metric and projectively flat equiaffinely-induced connection. Indeed, f is equiaffinely congruent to one of the surfaces in Examples 2.5, 2.6, 2.7, and 2.9*

Proof. Note that all surfaces with flat equiaffine metric and projectively flat equiaffinely-induced connection were classified in [1]. It was shown that such a surface is locally equiaffine equivalent with

(1) a paraboloid,
(2) an affine minimal ruled surface,
(3) $x^\alpha y^\beta z^\gamma = 1$,
(4) $e^{\alpha \arctan(y/x)}(x^2+y^2)^\beta z^\gamma = 1$,
(5) $z = -x(\alpha \log x + \beta \log y)$,
(6) $(x^2+y^2)^\beta e^{\alpha \arctan(x/y)} = e^z$,
(7) $z = \log x + \alpha \log y$,
(8) $z = y^2 \pm \log x$,
(9) $xz = y^2 \pm x^2 \log x$,

where $\alpha, \beta, \gamma \in \mathbb{R}$. Conversely, it is straightforward to show that, by excluding some discrete values of α, β or γ (in which case the surfaces are degenerate), all of the above surfaces have flat equiaffine metrics and projectively flat equiaffinely-induced connections. Of course, the above list contains both positive definite and indefinite examples. Besides the paraboloid, the positive definite case occurs for suitable values for the constants in the families (3), (4), (5), (8) and (9).

However the families (5) and (8) need to be excluded as in those cases, we have that $\det S = 0$ (or equivalently $\mu = 0$).

The example (9) can be parameterized by

$$f(u,v) = {}^t\left(e^{2v}, \sqrt{2}ue^{2v}, 2(u^2+v)e^{2v}\right),$$

which corresponds to Example 2.9. □

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