DOUBLY NONLINEAR EVOLUTION EQUATION ASSOCIATED WITH ELLIPTIC-PARABOLIC FREE BOUNDARY PROBLEMS

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Abstract. We study an abstract doubly nonlinear evolution equations associated with elliptic-parabolic free boundary problems. In this paper we show the existence and uniqueness of solution for the doubly nonlinear evolution equation. Moreover we apply our abstract results to an elliptic-parabolic free boundary problem.

1. Introduction

We study an abstract doubly nonlinear evolution equation in a real Hilbert space H of the form

$$(1.1) \qquad (Bu)'(t) + \partial\varphi^t(Bu(t); u(t)) \ni f(t) \qquad \text{for a.e. } t \in (0,T),$$

where B is a monotone operator in H, $(Bu)'(t) := \frac{d}{dt}Bu(t)$ and f is a given H-valued function. For each $t \in [0,T]$, a function $\varphi^t(\cdot;\cdot) : H \times H \to \mathbf{R} \cup \{\infty\}$ is given such that for all $w \in H$, $\varphi^t(w;\cdot) : H \to \mathbf{R} \cup \{\infty\}$ is a proper, l.s.c. (lower semi-continuous) and convex function, and $\partial\varphi^t(w;\cdot)$ is its subdifferential operator.

For a proper, l.s.c. and convex function $\psi^t(\cdot) : H \to \mathbf{R} \cup \{\infty\}$, many mathematicians studied the doubly nonlinear evolution equation of the form

$$(1.2) \qquad (Bu)'(t) + \partial\psi^t(u(t)) \ni f(t) \qquad \text{for a.e. } t \in (0,T).$$

For instance, Kenmochi [9] showed the existence-uniqueness, stability and convergence of solutions to (1.2) in the case when B is bi-Lipschitz.

For the Lipschitz operator B, Kenmochi-Pawłow [12, 13] have already established the results on existence-uniqueness and asymptotic behavior of solutions to (1.2). Kenmochi-Kubo [10] proved the existence of periodic solutions to (1.2), when ψ^t and $f(t)$ are periodic functions in t with same period. The author [16] considered the almost periodic problem to (1.2), when ψ^t and $f(t)$ are almost periodic in t.

The main object of this paper is to establish an abstract result on existence-uniqueness of solution to (1.1). Since the function $\varphi^t(Bu; u)$ is not convex in u, we can not apply to the results of Kenmochi-Pawłow [12]. So we shall refine on the abstract theory of Kenmochi-Pawłow [12] in this paper. Using the idea of Kenmochi-Kubo [11] and Kubo-Yamazaki [14], we shall show the existence of solution to (1.1). In fact, for the given

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function $w : [0,T] \to H$, let us consider the problem

$$(Bu)'(t) + \partial\varphi^t(w(t); u(t)) \ni f(t) \qquad \text{for a.e. } t \in (0,T). \tag{1.3}$$

Assuming some appropriate conditions on the t- and w-dependence of the function $\varphi^t(w; z)$, we can apply the result of Kenmochi-Pawłow [12]. Then we see that the equation (1.3) has a solution u for each w, and that the mapping $w \mapsto Bu$ has some compactness property. Hence, using a fixed point argument, we can get the existence of solution to (1.1).

In Section 2 we present our main results on existence and uniqueness of solution to (1.1), and then the uniqueness is proved. In Section 3 we prove the main existence result. In Section 4 we apply our abstract results to an elliptic-parabolic free boundary problem.

Notation. Throughout this paper, let H be a real Hilbert space with norm $|\cdot|_H$ and inner product $(\cdot,\cdot)$. For a proper l.s.c. convex function ψ on H we use the notation $D(\psi)$, $\partial\psi$ and $D(\partial\psi)$ to indicate the effective domain, subdifferential and its domain of $\partial\psi$, respectively. For their precise definitions and basic properties, see a monograph by Brézis [5].

2. Assumptions and main result

We consider a Cauchy problem CP(u_0) for (1.1) of the following form:

$$\mathrm{CP}(u_0) \begin{cases} (Bu)'(t) + \partial\varphi^t(Bu(t); u(t)) \ni f(t) & \text{in } H \text{ a.e. } t \in (0,T), \\ Bu(0) = Bu_0, \end{cases}$$

where T is a given positive number, $u_0 \in H$, $f \in L^2(0,T;H)$, B is a monotone operator and a function $\varphi^t(Bu(t); u(t))$ is introduced in section 1.

Definition 2.1. *Given $u_0 \in H$ and $f \in L^2(0,T;H)$, the function $u : [0,T] \to H$ will be called a solution to CP(u_0), if $Bu \in W^{1,2}(0,T;H)$, $u \in L^2(0,T;H)$, $Bu(0) = Bu_0$, $u(t) \in D(\partial\varphi^t(Bu(t);\cdot)$ and $f(t) - (Bu)'(t) \in \partial\varphi^t(Bu(t); u(t))$ for a.e. $t \in [0,T]$, namely*

$$(f(t) - (Bu)'(t),\ y - u(t)) \le \varphi^t(Bu(t); y) - \varphi^t(Bu(t); u(t))$$
$$\text{for any } y \in H, \text{ a.e. } t \in [0,T].$$

Now we assume that the single valued operator B from $D(B)(\subset H)$ into H satisfies the following five conditions:

(B1) There is a proper l.s.c. convex function j_B on H such that its subdifferential ∂j_B coincides with B;

(B2) There is a positive constant $C_1 > 0$ such that

$$C_1|Bz_1 - Bz_2|_H^2 \le (Bz_1 - Bz_2, z_1 - z_2), \quad \forall z_1,\ z_2 \in H;$$

(B3) $Bz \in D(\varphi^t(0;\cdot))$ for any $t \in [0,T]$ and $z \in D(\varphi^t(0;\cdot))$;

(B4) There are positive constants $C_2 > 0$ and $C_3 > 0$ such that

$$\varphi^t(0; Bz) \le C_2\varphi^t(0; z) + C_3, \quad \forall z \in H,\ \forall t \in [0,T];$$

(B5) B is bounded in H, namely, there is a positive constant $C_B > 0$ such that $|Bz|_H \le C_B$ for any $z \in H$.

Definition 2.2. *Given a positive number T and a function $\alpha \in W^{1,2}(0,T)$, we denote by $\{\varphi^t\} \in \Phi(\{\alpha\})$ the set of all time-dependent functions $\varphi^t(\cdot,\cdot)$ from $H \times H$ into $\mathbf{R} \cup \{\infty\}$ satisfying the following six conditions:*

(Φ1) *For each* $w \in H$ *and* $t \in [0, T]$, $\varphi^t(w; \cdot) : H \to \mathbf{R} \cup \{\infty\}$ *is a proper l.s.c. convex function;*
(Φ2) *There exists a positive constant* $C_4 > 0$ *such that*

$$\varphi^t(w; z) \geq C_4 |z|_H^2, \quad \forall w \in H, \ \forall t \in [0, T], \ \forall z \in D(\varphi^t(w; \cdot));$$

(Φ3) *For each* $t \in [0, T]$, $w \in H$ *and* $r > 0$, *the level set* $\{z \in H; \varphi^t(w; z) \leq r\}$ *is compact in* H*;*
(Φ4) $D(\varphi^t(w; \cdot))$ *is independent of* $w \in H$ *for any* $t \in [0, T]$*;*
(Φ5) *For any* $s, \ t \in [0, T]$ *with* $s \leq t$, $w \in D(\varphi^s(0; \cdot))$ *with* $|w|_H \leq C_B$ *and* $z \in D(\varphi^s(w; \cdot))$, *there exists an element* $\tilde{z} \in D(\varphi^t(w; \cdot))$ *such that*

$$|\tilde{z} - z|_H \leq |\alpha(t) - \alpha(s)| \left(1 + \varphi^s(0; z)^{\frac{1}{2}}\right),$$

$$\varphi^t(w; \tilde{z}) - \varphi^s(w; z) \leq |\alpha(t) - \alpha(s)| \left(1 + \varphi^s(0; z) + \varphi^s(0; w)^{\frac{1}{2}} \varphi^s(0; z)^{\frac{1}{2}}\right);$$

(Φ6) *There is a positive constants* $C_5 > 0$ *such that*

$$|\varphi^t(\tilde{w}; z) - \varphi^t(w; z)| \leq C_5 |\tilde{w} - w|_H \varphi^t(0; z)^{\frac{1}{2}}$$

$$\forall t \in [0, T], w \in H \text{ with } |w|_H \leq C_B, \tilde{w} \in H \text{ with } |\tilde{w}|_H \leq C_B \text{ and } z \in D(\varphi^t(0, \cdot)).$$

Let us begin with the uniqueness of solution to CP(u_0). To do so, we shall introduce a subclass of $\Phi(\{\alpha\})$.

Definition 2.3. *Let* γ *be a non-negative continuous and convex function on* H *such that* $\gamma(z) + \gamma(-z) = 0$ *if and only if* $z = 0$. *Then* $\{\varphi^t\} \in \Phi_{B,\gamma}(\{\alpha\})$ *if and only if* $\{\varphi^t\} \in \Phi(\{\alpha\})$ *satisfies the* γ*-accretiveness* ($\star$) *for* φ^t *and* B *as follows:*

($\star$) *For any* $w_i \in H$, $z_i \in D(\partial\varphi^t(w_i; \cdot))$ *and* $z_i^* \in \partial\varphi^t(w_i; z_i)$ $(i = 1, \ 2)$, *there is an element* $w_0 \in \partial\gamma(Bz_1 - Bz_2)$ *such that* $(z_1^* - z_2^*, w_0) \geq 0$, *where* $\partial\gamma$ *is the subdifferential of* γ *in* H*.*

Now let us mention our abstract uniqueness result in this paper.

Theorem 2.4. *Let* T *be any positive number. Assume* $\{\varphi^t\} \in \Phi_{B,\gamma}(\{\alpha\})$, $f \in L^2(0, T; H)$ *and* B *satisfies the conditions (B1)-(B5).*
(i) Let u *and* v *be solutions to CP(*u_0*) and CP(*v_0*), respectively. Then, we have*

$$\gamma(Bu(t) - Bv(t)) \leq \gamma(Bu(s) - Bv(s)) \quad \text{for any } 0 \leq s \leq t \leq T.$$

(ii) For each $u_0 \in H$, *the function* Bu *is uniquely determined, where* u *is the solution to CP(*u_0*).*
(iii) Furthermore, we assume that for each $w \in H$ $\varphi^t(w; \cdot)$ *is strictly convex on* $D(\varphi^t(w; \cdot))$*. Then the solution* u *to CP(*u_0*) is unique.*

Proof. (i) Let u and v be solutions to CP(u_0) and CP(v_0), respectively. By the γ-accretiveness of φ^t and B, for a.e. $\tau \in [0, T]$ there exists $z^*(\tau) \in \partial\gamma(Bu(\tau) - Bv(\tau))$ such that

$$\begin{aligned} 0 &\leq \left([f(\tau) - \frac{d}{d\tau}Bu(\tau)] - [f(\tau) - \frac{d}{d\tau}Bv(\tau))], z^*(\tau)\right) \\ &= \left(-\frac{d}{d\tau}Bu(\tau) + \frac{d}{d\tau}Bv(\tau), z^*(\tau)\right) = -\frac{d}{d\tau}\gamma(Bu(\tau) - Bv(\tau)). \end{aligned} \tag{2.1}$$

Hence, integrating (2.1) over (s,t), we get

$$\gamma(Bu(t)-Bv(t)) \le \gamma(Bu(s)-Bv(s)) \quad \text{for any } 0 \le s \le t \le T. \tag{2.2}$$

Thus the assertion (i) holds.

Here if $u_0 = v_0$, then (2.2) implies that $\gamma(Bu(t)-Bv(t)) = 0$ for all $t \in [0,T]$.

Similarly we can get $\gamma(Bv(t)-Bu(t)) = 0$ for all $t \in [0,T]$. Hence we have $Bu = Bv$. Thus the assertion (ii) has been shown.

From the assertion of (ii) it follows that

$$\partial\varphi^t(Bu(t);u(t)) \cap \partial\varphi^t(Bv(t);v(t)) \neq \emptyset, \qquad \text{a.e. } t \in [0,T]. \tag{2.3}$$

Furthermore, if $\varphi^t(w;\cdot)$ is strictly convex on $D(\varphi^t(w;\cdot))$ for any $w \in H$, then $\partial\varphi^t(w;\cdot)$ is strictly monotone for a.e. $t \in [0,T]$. Hence from (2.3), the assertion (ii) and the strictly monotonicity of $\partial\varphi^t(Bu(t);\cdot)$ we have $u(t) = v(t)$ for a.e. $t \in [0,T]$. Thus, the assertion (iii) has been proved. ☐

Next main result is concerned with the existence of solutions to CP(u_0).

Theorem 2.5. *Let T be any positive number. Assume $\{\varphi^t\} \in \Phi_{B,\gamma}(\{\alpha\})$, $f \in W^{1,1}(0,T;H)$ and B satisfies the conditions (B1)-(B5). Then, for each $u_0 \in D(\varphi^0(0;\cdot))$ there exists at least one solution u of CP(u_0) on $[0,T]$.*

3. Proof of Theorem 2.5

In this section we shall show Theorem 2.5 by the fixed point argument and a regularization method. Let us begin with the existence of local solution to CP(u_0).

3.1. **Local solution.** Using the fixed point argument we shall prove the existence of local solution to CP(u_0). To do so, for given positive numbers $T > 0$ and $M > 0$, let us consider a Banach space

$$E_M(T) \equiv \left\{ w \in W^{1,2}(0,T;H)\ ;\ \begin{array}{l} \sup\limits_{t\in[0,T]} \varphi^t(0;w(t)) \le M, \quad |w'|^2_{L^2(0,T;H)} \le M, \\ w(0) = Bu_0, \quad \sup\limits_{t\in[0,T]} |w(t)|_H \le C_B. \end{array} \right\}.$$

Now, for each $w \in E_M(T)$ let us consider a following Cauchy problem CP$(w;u_0)$:

$$\text{CP}(w;u_0) \begin{cases} (Bu)'(t) + \partial\varphi^t(w(t);u(t)) \ni f(t) & \text{in } H \text{ a.e. } t \in (0,T), \\ Bu(0) = Bu_0. \end{cases}$$

Lemma 3.1. *For each $w \in E_M(T)$ we put $\psi^t_w(z) := \varphi^t(w(t);z)$ for $z \in H$. Then, there is a positive constant $N_1 > 0$ independent of w satisfying the following :*
for any $s,\ t \in [0,T]$ with $s \le t$ and $z \in D(\psi^s_w)$, there exists $\tilde{z} \in D(\psi^t_w)$ such that

$$|\tilde{z} - z|_H \le N_1|\alpha(t)-\alpha(s)|\left(1 + \psi^s_w(z)^{\frac12}\right)$$

and

$$\begin{aligned} \psi^t_w(\tilde{z}) - \psi^s_w(z) \ \le\ & N_1 \Big\{ |\alpha(t)-\alpha(s)|(1+\psi^s_w(z)) + |w(t)-w(s)|_H(1+\psi^s_w(z))^{\frac12} \\ & + |\alpha(t)-\alpha(s)|\varphi^s(0;w(s))^{\frac12}(1+\psi^s_w(z))^{\frac12} \Big\}. \end{aligned} \tag{3.1}$$

Proof. Taking $w = w(s)$ in (Φ5), then for any $s,\ t \in [0,T]$ with $s \le t$ and $z \in D(\varphi^s(w(s);\cdot)$ there exists $\tilde z \in D(\varphi^t(w(s);\cdot))$ such that

$$|\tilde z - z|_H \le |\alpha(t)-\alpha(s)|\left(1+\varphi^s(0;z)^{\frac12}\right), \tag{3.2}$$

$$\begin{aligned}&\varphi^t(w(s);\tilde z)-\varphi^s(w(s);z)\\ \le\ & |\alpha(t)-\alpha(s)|\left(1+\varphi^s(0;z)+\varphi^s(0;w(s))^{\frac12}\varphi^s(0;z)^{\frac12}\right).\end{aligned} \tag{3.3}$$

It follows from (Φ4) that

$$z \in D(\varphi^s(w(s);\cdot)=D(\psi^s_w),\quad \tilde z \in D(\varphi^t(w(s);\cdot))=D(\psi^t_w). \tag{3.4}$$

Note that by (Φ6) and $w \in E_M(T)$ we have

$$\varphi^s(0;z) \le 2\varphi^s(w(s);z)+C_5^2|w(s)|_H^2 \le 2\psi^s_w(z)+C_5^2C_B^2. \tag{3.5}$$

Then, by (3.2) and (3.5) there is a positive number $N_2>0$ independent of w satisfying

$$\begin{aligned}|\tilde z - z|_H &\le |\alpha(t)-\alpha(s)|\left(1+\sqrt2\psi^s_w(z)^{\frac12}+C_5C_B\right)\\ &\le N_2|\alpha(t)-\alpha(s)|\left(1+\psi^s_w(z)^{\frac12}\right).\end{aligned} \tag{3.6}$$

Moreover, we observe that by (3.3), (3.5), (Φ6) there is a positive number $N_3>0$ independent of w satisfying the following :

$$\begin{aligned}&\psi^t_w(\tilde z)-\psi^s_w(z)\ \left(=\varphi^t(w(t);\tilde z)-\varphi^t(w(s);\tilde z)+\varphi^t(w(s);\tilde z)-\varphi^s(w(s);z)\right)\\ \le\ & N_3\Big\{|w(t)-w(s)|_H\psi^t_w(\tilde z)^{\frac12}+|w(t)-w(s)|_H\\ &+|\alpha(t)-\alpha(s)|(1+\psi^s_w(z))+|\alpha(t)-\alpha(s)|\varphi^s(0;w(s))^{\frac12}(1+\psi^s_w(z))^{\frac12}\Big\}.\end{aligned} \tag{3.7}$$

From $\alpha \in W^{1,2}(0,T)$, $w \in E_M(T)$ and (3.7) it follows that

$$\psi^t_w(\tilde z) \le N_4\left(1+\psi^s_w(z)+|\alpha(t)-\alpha(s)|^2\varphi^s(0;w(s))\right) \tag{3.8}$$

for some constant $N_4>0$. Therefore, using (3.8) in the right hand side of (3.7), and by (3.4)-(3.6), we get this Lemma 3.1 for some constant $N_1>0$ independent of w. □

Proposition 3.2. *For each $w \in E_M(T)$, there exists a solution u to CP($w;u_0$) such that the function Bu is uniquely determined.*

Proof. We note that CP($w;u_0$) can be regarded as the Cauchy problem for the doubly nonlinear evolution equation of the form:

$$\begin{cases}(Bu)'(t)+\partial\psi^t_w(u(t)) \ni f(t) & \text{in } H \text{ a.e. } t\in(0,T),\\ Bu(0)=Bu_0.\end{cases}$$

By Lemma 3.1 we get the time-dependence of ψ^t_w. Therefore it follows from the assumptions (Φ1)-(Φ3), (B1)-(B2) that we can apply the abstract theory established by Kenmochi-Pawłow [12]. Thus we can get the existence of solution u for CP($w;u_0$). For detail proof, see [12, Theorem 1.1].

Moreover by the same argument of Kenmochi-Pawłow [12, Theorem 1.2] or Theorem 2.4 (ii), we can get the uniqueness of the function Bu. □

By Proposition 3.2, we can define a mapping $Q : E_M(T) \longrightarrow L^2(0,T;H)$ by $Qw = Bu$ for each $w \in E_M(T)$, where u is a solution for CP($w;u_0$).

Lemma 3.3. *There are positive constants T_0 and M_0 such that Q is a self-mapping on $E_{M_0}(T_0)$, i.e., $Qw(=Bu) \in E_{M_0}(T_0)$ for any $w \in E_{M_0}(T_0)$.*

Proof. We consider the approximate problem $\mathrm{CP}(w;u_0)_{\varepsilon,\lambda}$ $(0<\varepsilon,\lambda \le 1)$ of $\mathrm{CP}(w;u_0)$:

$$\mathrm{CP}(w;u_0)_{\varepsilon,\lambda} \begin{cases} (B_\varepsilon u_{\varepsilon,\lambda})'(t) + \partial\varphi^t_\lambda(w(t);u_{\varepsilon,\lambda}(t)) = f(t), \qquad 0<t<T, \\ B_\varepsilon u_{\varepsilon,\lambda}(0) = B_{0,\varepsilon}(:= B_\varepsilon u_0). \end{cases}$$

Here we put $B_\varepsilon := B+\varepsilon I$ and $\varphi^t_\lambda(w(t);\cdot)$ is the Moreau-Yosida approximation of $\varphi^t(w(t);\cdot)$ defined by $\varphi^t_\lambda(w(t);z) := \inf_{y\in H}\left\{\frac{1}{2\lambda}|z-y|^2_H + \varphi^t(w(t);y)\right\}$ for $z \in H$.

By the same argument in [12, Lemma 2.2] we see that there is a positive constant $C_4' > 0$ independent of t, w, z and $0<\lambda\le 1$ so that

$$\varphi^t_\lambda(w;z) \ge C_4'|z|^2_H. \tag{3.9}$$

Moreover, we observe that the problem $\mathrm{CP}(w;u_0)_{\varepsilon,\lambda}$ has a unique solution $u_{\varepsilon,\lambda}$ which converges to the solution of $\mathrm{CP}(w;u_0)$ in some sense. For detail proof, see [12]. By the slight modification of [12, Lemma 2.3], we see that $\Psi_{\varepsilon,\lambda}(t) := \varphi^t_\lambda(w(t);u_{\varepsilon,\lambda}(t))$ is of bounded variation on $[0,T]$ and satisfies the following inequality :

$$\begin{aligned} &\Psi_{\varepsilon,\lambda}(t) - \Psi_{\varepsilon,\lambda}(s) + \int_s^t ((B_\varepsilon u_{\varepsilon,\lambda})'(\tau) - f(\tau), u'_{\varepsilon,\lambda}(\tau))d\tau \\ \le\ & N_1\int_s^t \Big[|\alpha'(\tau)||(B_\varepsilon u_{\varepsilon,\lambda})'(\tau) - f(\tau)|\{1+\Psi_{\varepsilon,\lambda}(\tau)^{\frac12}\} + |\alpha'(\tau)|(1+\Psi_{\varepsilon,\lambda}(\tau)) \\ &\qquad + (|w'(\tau)|_H + |\alpha'(\tau)|\varphi^\tau(0;w(\tau))^{\frac12})(1+\Psi_{\varepsilon,\lambda}(\tau))^{\frac12}\Big]\,d\tau \end{aligned} \tag{3.10}$$

for $0\le s\le t\le T$. Note that the following inequalities hold (cf. [12, Section 3]) :

$$\big((B_\varepsilon u_{\varepsilon,\lambda})'(\tau), u'_{\varepsilon,\lambda}(\tau)\big) \ge \frac{C_1}{1+\varepsilon C_1}\left|(B_\varepsilon u_{\varepsilon,\lambda})'(\tau)\right|^2_H, \tag{3.11}$$

$$\begin{aligned} \int_s^t (f(\tau), u'_{\varepsilon,\lambda}(\tau))d\tau &= -\int_s^t (f'(\tau), u_{\varepsilon,\lambda}(\tau))d\tau + (f(t),u_{\varepsilon,\lambda}(t)) - (f(s),u_{\varepsilon,\lambda}(s)) \\ &\le \int_s^t \{|f'(\tau)|^2_H + N_5\Psi_{\varepsilon,\lambda}(\tau)\}d\tau + (f(t),u_{\varepsilon,\lambda}(t)) - (f(s),u_{\varepsilon,\lambda}(s)), \end{aligned} \tag{3.12}$$

$$\begin{aligned} &|\alpha'(\tau)||(B_\varepsilon u_{\varepsilon,\lambda})'(\tau) - f(\tau)|_H\{1+\Psi_{\varepsilon,\lambda}(\tau)^{\frac12}\} \\ \le\ & 2\delta|(B_\varepsilon u_{\varepsilon,\lambda})'(\tau)|^2_H + 2\delta|f(\tau)|^2_H + \delta^{-1}|\alpha'(\tau)|^2(1+\Psi_{\varepsilon,\lambda}(\tau)), \end{aligned} \tag{3.13}$$

where the positive constant $N_5 > 0$ in (3.12) depends on C_4' and the constant $\delta > 0$ will be defined below. Using (3.10)-(3.13), we have

$$\begin{aligned} &X_{\varepsilon,\lambda}(t) - X_{\varepsilon,\lambda}(s) + N_6\int_s^t |(B_\varepsilon u_{\varepsilon,\lambda})'(\tau)|^2_H d\tau \\ \le\ & N_7\int_s^t \left\{G(\tau)(1+\Psi_{\varepsilon,\lambda}(\tau)) + W(\tau)(1+\Psi_{\varepsilon,\lambda}(\tau))^{\frac12}\right\}d\tau \end{aligned} \tag{3.14}$$

for $0\le s\le t\le T$ and $0<\varepsilon,\lambda\le 1$,

where we put $X_{\varepsilon,\lambda}(t) := \Psi_{\varepsilon,\lambda}(t) - (f(t),u_{\varepsilon,\lambda}(t))$, $N_6 := \frac{C_1}{1+\varepsilon C_1} - 2\delta N_1$,

$$G(t) := |f(t)|^2_H + |f'(t)|^2_H + |\alpha'(t)|^2 + 1, \quad W(t) := |w'(t)|_H + |\alpha'(t)|\varphi^t(0;w(t))^{\frac12}.$$

Now we take $\delta > 0$ so that $N_6 > 0$. Then, note that $N_7 > 0$ is dependent only on C_1, N_1, N_5. Here by (3.9) we get

$$X_{\varepsilon,\lambda}(t) = \Psi_{\varepsilon,\lambda}(t) - (f(t), u_{\varepsilon,\lambda}(t)) \geq N_8 \Psi_{\varepsilon,\lambda}(t) - \delta_1^{-1}|f(t)|_H^2, \tag{3.15}$$

where $\delta_1 > 0$ is the constant so that $N_8 > 0$. Using (3.15) in (3.14), applying Gronwall's inequality and letting $\varepsilon \to 0, \lambda \to 0$, we obtain

$$\begin{aligned} &\sup_{0\leq t\leq T} \varphi^t(w(t); u(t)) + \int_0^T |(Bu)'(t)|_H^2 dt \\ \leq\ & N_{10} e^{N_9(|G|_{L^1(0,T)}+|W|_{L^1(0,T)})}\{1 + N_9(|G|_{L^1(0,T)} + |W|_{L^1(0,T)})\}, \end{aligned} \tag{3.16}$$

where N_9 and N_{10} are positive constants dependent on the given data.

Now we show that Q is the self-mapping on $E_{M_0}(T_0)$ for some chosen $T_0 > 0$ and $M_0 > 0$. Taking account of (3.5) and $(B4)$, for any $w \in E_M(T)$ we have

$$\varphi^t(0; Bu(t)) \leq C_2\varphi^t(0; u(t)) + C_3 \leq 2C_2\varphi^t(w(t); u(t)) + C_2C_5^2C_B^2 + C_3. \tag{3.17}$$

Here we put the constant $M_0 > 0$ so that

$$(1 + 2C_2)N_{10}e^{2N_9}(1 + 2N_9) + C_2C_5^2C_B^2 + C_3 \leq M_0,$$

and then take $T_0 > 0$ such that

$$|G|_{L^1(0,T_0)} \leq 1, \quad |W|_{L^1(0,T)} \leq T_0^{\frac{1}{2}} M_0^{\frac{1}{2}} + T_0^{\frac{1}{2}} M_0^{\frac{1}{2}} |\alpha'|_{L^2(0,T_0)} \leq 1.$$

Then from (3.16) and (3.17) it follows that $Qw(= Bu) \in E_{M_0}(T_0)$ for any $w \in E_{M_0}(T_0)$, hence, Q is the self-mapping on $E_{M_0}(T_0)$. □

Lemma 3.4. *Let $M_0 > 0$ and $T_0 > 0$ be constants obtained in Lemma 3.3. Let $\{w_n\} \subset E_{M_0}(T_0)$, $w \in E_{M_0}(T_0)$ and u_n be the solution of CP$(w_n; u_0)$. Suppose $w_n \longrightarrow w$ in $C([0, T_0]; H)$ as $n \to +\infty$. Then, there is a solution u of CP$(w; u_0)$ on $[0, T_0]$ such that $Bu \in E_{M_0}(T_0)$ and $Bu_n \longrightarrow Bu$ in $C([0, T_0]; H)$ as $n \to +\infty$.*

Proof. Since $\{w_n\} \subset E_{M_0}(T_0)$, Lemma 3.3 and (3.17), we have

$$\sup_{t\in[0,T_0]} \varphi^t(0; Bu_n(t)) \leq M_0, \qquad |(Bu_n)'|_{L^2(0,T_0;H)}^2 \leq M_0, \quad \forall n = 1, 2, \cdots, \tag{3.18}$$

$$\sup_{t\in[0,T_0]} \varphi^t(0; u_n(t)) \leq C_2^{-1} M_0, \quad \forall n = 1, 2, \cdots. \tag{3.19}$$

Here we note that the function Bu_n is uniquely determined (cf. Theorem 2.4 (ii)).

By $(\Phi2)$, $(\Phi3)$, (3.18), (3.19) there are a subsequence $\{n_k\}$ of $\{n\}$, a countable dense subset J_D of $[0, T_0]$ and functions $\tilde{u} \in W^{1,2}(0, T_0; H)$, $u \in L^\infty(0, T_0; H)$ such that

$$Bu_{n_k}(t) \rightharpoonup \tilde{u}(t) \quad \text{weakly in } H \quad \text{for all } t \in [0, T_0], \tag{3.20}$$

$$(Bu_{n_k})' \rightharpoonup (\tilde{u})' \quad \text{weakly in } L^2(0, T_0; H), \tag{3.21}$$

$$u_{n_k} \rightharpoonup u \quad \text{weakly-}* \text{ in } L^\infty(0, T_0; H), \tag{3.22}$$

$$u_{n_k}(t) \longrightarrow u(t) \quad \text{strongly in } H, \quad \text{for } t \in J_D \tag{3.23}$$

as $k \to +\infty$.

Since $C_1|Bu_{n_k}(t) - Bu(t)|_H \leq |u_{n_k}(t) - u(t)|_H$, we observe that $Bu_{n_k}(t) \longrightarrow Bu(t)$ strongly in H as k $+\infty$ for all $t \in J_D$. On account of (3.20)-(3.21), we see that $Bu(t) = \tilde{u}(t)$ for all $t \in J_D$. Therefore by (3.18)-(3.23) and the uniqueness of the function Bu_n, we observe that

$Bu \in E_{M_0}(T_0)$ and $Bu_n \longrightarrow Bu$ strongly in $C([0,T_0];H)$ and weakly in $W^{1,2}(0,T_0;H)$ as $n \to +\infty$.

Now, let us show that u is a solution of $\mathrm{CP}(w;u_0)$ on $[0,T_0]$. To do so, we define $\Phi(w;z) = \int_0^{T_0} \varphi^t(w(t);z(t))dt$. Then by the assumption (Φ6) we see that

$$\Phi(w_n;z) \longrightarrow \Phi(w;z) \text{ as } n \to +\infty \tag{3.24}$$

for any $z \in L^2(0,T_0;H)$ with $\varphi^{(\cdot)}(0;z(\cdot)) \in L^1(0,T_0)$. From (3.19), ($\Phi$1), ($\Phi$2), ($\Phi$6) and the Fatou's lemma, it follows that

$$\begin{aligned}\liminf_{k\to+\infty} \Phi(w_{n_k};u_{n_k}) &= \liminf_{k\to+\infty} \{\Phi(w_{n_k};u_{n_k}) - \Phi(w;u_{n_k}) + \Phi(w;u_{n_k})\} \\ &\geq \liminf_{k\to+\infty} \Phi(w;u_{n_k}) \geq \Phi(w;u).\end{aligned} \tag{3.25}$$

Moreover, let j_B^* be a conjugate function of j_B on H. Clearly, j_B^* is a proper l.s.c. convex function on H such that $\partial j_B^* = B^{-1}$. Then, we have

$$\begin{aligned}\liminf_{k\to+\infty} \int_0^{T_0} ((Bu_{n_k})'(t), u_{n_k}(t))\,dt &= \liminf_{k\to+\infty} \{j_B^*(Bu_{n_k}(T_0)) - j_B^*(Bu_0)\} \\ &\geq j_B^*(Bu(T_0)) - j_B^*(Bu_0) = \int_0^{T_0} ((Bu)'(t), u(t))\,dt.\end{aligned} \tag{3.26}$$

Now, let z be any function in $L^2(0,T_0;H)$ with $\varphi^{(\cdot)}(0;z(\cdot)) \in L^1(0,T_0)$. Since u_{n_k} is the solution of $\mathrm{CP}(w_{n_k};u_0)$, then the following inequality holds:

$$\int_0^{T_0} (f(t) - (Bu_{n_k})'(t), z(t) - u_{n_k}(t))\,dt \leq \Phi(w_{n_k};z) - \Phi(w_{n_k};u_{n_k}). \tag{3.27}$$

Taking account of (3.21), (3.24)-(3.26) and letting $k \to +\infty$ in (3.27), we get

$$\int_0^{T_0} (f(t) - (Bu)'(t), z(t) - u(t))\,dt \leq \Phi(w;z) - \Phi(w;u),$$

which implies that $f(t) - (Bu)'(t) \in \partial\varphi^t(w(t);u(t))$ for a.e. $t \in [0,T_0]$ (cf. [3, Proposition 3.3]). Thus u is the solution of $\mathrm{CP}(w;u_0)$. □

Proof. [**Proof of Theorem 2.5; Local existence**] By Lemma 3.3, we can define a self-mapping $Q : E_{M_0}(T_0) \longrightarrow E_{M_0}(T_0)$ by $Qw = Bu$ for each $w \in E_{M_0}(T_0)$, where u is a solution of $\mathrm{CP}(w;u_0)$. Clearly, $E_{M_0}(T_0)$ is compact in $C([0,T_0];H)$.

Moreover, it follows from Lemma 3.4 that Q is continuous with respect to the topology of $C([0,T_0];H)$. Therefore, the Schauder's fixed point theorem implies that the self-mapping Q has a fixed point Bu in $E_{M_0}(T_0)$, i.e. $QBu = Bu$. Clearly u is the solution of $\mathrm{CP}(u_0)$, thus we can get the local existence of solution u of $\mathrm{CP}(u_0)$. □

3.2. **Global solution.** Now let us begin with the inequality (3.14). Applying Schwarz inequality to the term $W(\tau)(1+\Psi_{\varepsilon,\lambda}(\tau))^{1/2}$ and using (3.15), we get

$$\begin{aligned}&X_{\varepsilon,\lambda}(t) - X_{\varepsilon,\lambda}(s) + N_6 \int_s^t |(B_\varepsilon u_{\varepsilon,\lambda})'(\tau)|_H^2 d\tau \\ &\leq N_{11} \int_s^t G(\tau)(1 + X_{\varepsilon,\lambda}(\tau))d\tau + \frac{N_6}{2} \int_s^t (|w'(\tau)|_H^2 + \varphi^\tau(0;w(\tau)))d\tau\end{aligned} \tag{3.28}$$

for $0 \leq s \leq t \leq T$ and $0 < \varepsilon, \lambda \leq 1$, where $N_{11} > 0$ is dependent on N_6, N_7, N_8, δ_1 and $|f|_{L^\infty(0,T;H)}$.

Applying Gronwall's inequality to (3.28), and letting $\varepsilon, \lambda \to 0$, we have

$$\begin{aligned} (3.29) \qquad & \varphi^t(w(t);u(t)) \\ & + \int_0^t e^{N_{11}\int_\tau^t G(s)ds}\{N_6|(Bu)'(\tau)|_H^2 - \frac{N_6}{2}\{|w'(\tau)|_H^2 + \varphi^\tau(0;w(\tau))\}\}d\tau \le N_{12} \end{aligned}$$

for $0 \le t \le T$, where $N_{12} > 0$ is dependent on the given data.

Note that the inequality (3.29) holds for any $w \in E_{M_0}(T_0)$. Then by the result in Section 3.1, we can take $w = Bu \in E_{M_0}(T_0)$, where u is the solution of $\mathrm{CP}(u_0)$ on $[0, T_0]$. Hence, using (3.17) and (3.29), we have

$$\begin{aligned} & \varphi^t(Bu(t);u(t)) + \int_0^t e^{N_{11}\int_\tau^t G(s)ds}\frac{N_6}{2}|(Bu)'(\tau)|_H^2 d\tau \\ (3.30) \qquad \le \ & N_{13}\left(1 + \int_0^t \varphi^\tau(Bu(\tau);u(\tau))d\tau\right) \qquad \text{for } 0 \le t \le T_0, \end{aligned}$$

where N_{13} depends on C_2, C_3, C_5, C_B, N_6, N_{11}, N_{12}, $|G|_{L^1(0,T)}$.

Applying Gronwall's inequality to (3.30), we get

$$(3.31) \qquad \sup_{t\in[0,T_0]} \varphi^t(Bu(t);u(t)) + \int_0^{T_0} |(Bu)'(t)|_H^2 dt \le N_{14},$$

where N_{14} depends only on the given data and is independent of $T_0 (\le T)$.

Now let us prove the global existence of solution to $\mathrm{CP}(u_0)$.

Proof. [**Proof of Theorem 2.5; Global existence**] Assume that

$$T^* := \sup\{T_0;\ \mathrm{CP}(u_0) \text{ has a solution on } [0, T_0]\} < +\infty.$$

By the local existence results in Section 3.1, we note $T^* > 0$. By the definition of T^*, there is a function $u : [0, T^*) \to H$ such that for any T_0 $(< T^*)$ u is the solution of $\mathrm{CP}(u_0)$ on $[0, T_0]$. By (B5) and (3.31) we have

$$Bu \in W^{1,2}(0, T^*; H), \quad \varphi^{(\cdot)}(Bu(\cdot);u(\cdot)) \in L^\infty(0, T^*).$$

Hence we observe that the limit $B^* := \lim_{t\uparrow T^*} Bu(t)$ strongly in H exists. By assumptions $(B2)$, $(\Phi 1)$, $(\Phi 3)$, $(\Phi 5)$, $(\Phi 6)$ we see that

$$B^* = Bu^* \text{ for some } u^* \in D(\varphi^{T^*}(0;\cdot)).$$

Now, taking B^* as the initial value at $t = T^*$, we can get the solution u beyond the time interval $[0, T^*]$. Thus $T^* = +\infty$, so we have the global existence of solution. □

4. Application

In this section we consider an elliptic-parabolic free boundary problem with Signorini-Dirichlet-Neumann mixed boundary condition :

$$(4.1) \qquad b(u)_t - \nabla\cdot a(x, b(u), \nabla u) = f(t,x) \qquad \text{in } (0,T)\times\Omega,$$

$$(4.2) \qquad u \le p(t),\ \nu\cdot a(x,b(u),\nabla u) \le 0,\ (u - p(t))\nu\cdot a(x,b(u),\nabla u) = 0 \quad \text{on } \Gamma_S,$$

$$(4.3) \qquad u = p(t) \qquad \text{on } \Gamma_D,$$

$$(4.4) \qquad \nu\cdot a(x,b(u),\nabla u) = 0 \qquad \text{on } \Gamma_N,$$

$$(4.5) \qquad b(u(0,\cdot)) = b_0 \qquad \text{in } \Omega.$$

Here ν is an outward normal vector on the boundary, and p is a given function. Ω is a bounded domain in $\mathbf{R}^N$ $(1 \le N < +\infty)$ with smooth boundary Γ consisting of three disjoint parts $\Gamma_\nu,\ \nu = S, D, N$, namely, $\Gamma = \Gamma_S \cup \Gamma_D \cup \Gamma_N$.

The problem (4.1)-(4.5) is a model of partially saturated porous media, in which Γ_S, Γ_D and Γ_N refer to the parts of the boundary in contact with the atmosphere, reservoirs and impervious layer, respectively. The function p is the pressure in the reservoirs on Γ_D, and to the atmospheric pressure on Γ_S.

Many mathematicians have already studied the various models of porous media. For instance, see [1, 2, 4, 8, 11, 13, 14, 15].

The aim of this section is to consider the problem (4.1)-(4.5) as a application of the abstract evolution equation CP(u_0). To do so, we suppose that

(A1) $a(x, s, p) = \partial_p A(x, s, p)$ for some potential function $A(x, s, p)$. There exist constants $\mu_1 > 0$, $\mu_2 = \mu_2(a) > 0$ and $\mu_3 = \mu_3(a) > 0$ such that

$$[a(x, s, p) - a(x, s, \hat{p})] \cdot (p - \hat{p}) \ge \mu_1 |p - \hat{p}|^2,$$

$$|a(x, s, p)|^2 + |A(x, s, p)| + |\partial_s A(x, s, p)|^2 \le \mu_2(1 + |s|^2 + |p|^2),$$

$$|a(x, s, p) - a(x, \hat{s}, p)| \le \mu_3(1 + |p|)|s - \hat{s}|$$

for all $x \in \Omega,\ s, \hat{s} \in \mathbf{R}, p, \hat{p} \in \mathbf{R}^N$.

(A2) $b : \mathbf{R} \to \mathbf{R}$ is bounded, nondecreasing and Lipschitz continuous.

Here for each $t \in [0, T]$ let us define the convex set $K(t)$ by

$$K(t) := \{z \in H^1(\Omega); z \le p(t) \text{ on } \Gamma_S \text{ and } z = p(t) \text{ on } \Gamma_D\}.$$

As a direct application of Theorems 2.4 and 2.5, we have:

Proposition 4.1. *Assume (A1) and (A2). Then, for each $f \in W^{1,1}(0, T; L^2(\Omega))$, $p \in W^{1,2}(0, T; H^1(\Omega))$ and $b_0 = b(u_0)$ for some $u_0 \in K(0)$, the problem* (4.1)-(4.5) *has a solution u on* $[0, T]$.

Proof. To apply Theorems 2.4 and 2.5 to the problem (4.1)-(4.5), we choose $L^2(\Omega)$ as a real Hilbert space H, and define a function $\varphi^t : L^2(\Omega) \times L^2(\Omega) \to \mathbf{R} \cup \{\infty\}$ by

$$\varphi^t(w; z) := \begin{cases} \displaystyle\int_\Omega A(x, w(x), \nabla z(x))dx, & \text{if } z \in K(t), \\ +\infty, & \text{otherwise.} \end{cases}$$

Let us define an operator $B : L^2(\Omega) \to L^2(\Omega)$ by $Bz := b(z)$ in $L^2(\Omega)$. Also we define a function γ by $\gamma(z) := \int_\Omega z^+(x)dx$ for $z \in L^2(\Omega)$, where $z^+ := \max\{z, 0\}$.

Now we put for any $t \in [0, T]$

$$\alpha(t) := N_{15} \int_0^t |p'(\tau)|_{H^1(\Omega)} d\tau,$$

where N_{15} is a (sufficient large) positive constant. Then, we easily see that $\{\varphi^t\} \in \Phi_{B,\gamma}(\{\alpha\})$ and the operator B satisfies the assumptions (B1)-(B5). Clearly, the problem (4.1)-(4.5) can be reformulated in the abstract evolution equation CP(u_0). Thus, applying Theorems 2.4 and 2.5, we see that the problem (4.1)-(4.5) has a solution u on $[0, T]$ such that the function $b(u)$ is uniquely determined. $\square$

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