DOUBLY NONLINEAR EVOLUTION EQUATION ASSOCIATED WITH ELLIPTIC-PARABOLIC FREE BOUNDARY PROBLEMS

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Abstract. We study an abstract doubly nonlinear evolution equations associated with elliptic-parabolic free boundary problems. In this paper we show the existence and uniqueness of solution for the doubly nonlinear evolution equation. Moreover we apply our abstract results to an elliptic-parabolic free boundary problem.

1. Introduction

We study an abstract doubly nonlinear evolution equation in a real Hilbert space $H$ of the form

$$(Bu)'(t) + \partial \varphi^t(Bu(t); u(t)) \ni f(t) \quad \text{for a.e. } t \in (0, T),$$

where $B$ is a monotone operator in $H$, $(Bu)'(t) := \frac{d}{dt} Bu(t)$ and $f$ is a given $H$-valued function. For each $t \in [0, T]$, a function $\varphi^t(\cdot; \cdot) : H \times H \to \mathbb{R} \cup \{\infty\}$ is given such that for all $w \in H$, $\varphi^t(w; \cdot) : H \to \mathbb{R} \cup \{\infty\}$ is a proper, l.s.c. (lower semi-continuous) and convex function, and $\partial \varphi^t(w; \cdot)$ is its subdifferential operator.

For a proper, l.s.c. and convex function $\psi^t(\cdot) : H \to \mathbb{R} \cup \{\infty\}$, many mathematicians studied the doubly nonlinear evolution equation of the form

$$(Bu)'(t) + \partial \psi^t(u(t)) \ni f(t) \quad \text{for a.e. } t \in (0, T).$$

For instance, Kenmochi [9] showed the existence-uniqueness, stability and convergence of solutions to (1.2) in the case when $B$ is bi-Lipschitz.

For the Lipschitz operator $B$, Kenmochi-Pawlow [12, 13] have already established the results on existence-uniqueness and asymptotic behavior of solutions to (1.2). Kenmochi-Kubo [10] proved the existence of periodic solutions to (1.2), when $\psi^t$ and $f(t)$ are periodic functions in $t$ with same period. The author [16] considered the almost periodic problem to (1.2), when $\psi^t$ and $f(t)$ are almost periodic in $t$.

The main object of this paper is to establish an abstract result on existence-uniqueness of solution to (1.1). Since the function $\varphi^t(Bu; u)$ is not convex in $u$, we can not apply to the results of Kenmochi-Pawlow [12]. So we shall refine on the abstract theory of Kenmochi-Pawlow [12] in this paper. Using the idea of Kenmochi-Kubo [11] and Kubo-Yamazaki [14], we shall show the existence of solution to (1.1). In fact, for the given

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function \( w : [0, T] \to H \), let us consider the problem
\[
(Bu)'(t) + \partial \varphi^t(w(t); u(t)) \ni f(t) \quad \text{for a.e. } t \in (0, T).
\]
Assuming some appropriate conditions on the \( t \)- and \( w \)-dependence of the function \( \varphi^t(w; z) \), we can apply the result of Kenmochi-Pawlow [12]. Then we see that the equation (1.3) has a solution \( u \) for each \( w \), and that the mapping \( w \mapsto Bu \) has some compactness property. Hence, using a fixed point argument, we can get the existence of solution to (1.1).

In Section 2 we present our main results on existence and uniqueness of solution to (1.1), and then the uniqueness is proved. In Section 3 we prove the main existence result. In Section 4 we apply our abstract results to an elliptic-parabolic free boundary problem.

**Notation.** Throughout this paper, let \( H \) be a real Hilbert space with norm \( | \cdot |_H \) and inner product \((\cdot, \cdot)\). For a proper l.s.c. convex function \( \psi \) on \( H \) we use the notation \( D(\psi) \), \( \partial \psi \) and \( D(\partial \psi) \) to indicate the effective domain, subdifferential and its domain of \( \partial \psi \), respectively. For their precise definitions and basic properties, see a monograph by Brézis [5].

### 2. Assumptions and Main Result

We consider a Cauchy problem \( CP(u_0) \) for (1.1) of the following form:
\[
CP(u_0) \begin{cases}
(Bu)'(t) + \partial \varphi^t(Bu(t); u(t)) \ni f(t) & \text{in } H \text{ a.e. } t \in (0, T), \\
Bu(0) = Bu_0,
\end{cases}
\]
where \( T \) is a given positive number, \( u_0 \in H \), \( f \in L^2(0, T; H) \), \( B \) is a monotone operator and a function \( \varphi^t(Bu(t); u(t)) \) is introduced in section 1.

**Definition 2.1.** Given \( u_0 \in H \) and \( f \in L^2(0, T; H) \), the function \( u : [0, T] \to H \) will be called a solution to \( CP(u_0) \), if \( Bu \in W^{1,2}(0, T; H) \), \( u \in L^2(0, T; H) \), \( Bu(0) = Bu_0 \), \( u(t) \in D(\partial \varphi^t(Bu(t); \cdot)) \) and \( f(t) - (Bu)'(t) \in \partial \varphi^t(Bu(t); u(t)) \) for a.e. \( t \in [0, T] \), namely
\[
(f(t) - (Bu)'(t), y - u(t)) \leq \varphi^t(Bu(t); y) - \varphi^t(Bu(t); u(t))
\]
for any \( y \in H \), a.e. \( t \in [0, T] \).

Now we assume that the single valued operator \( B \) from \( D(B)(\subset H) \) into \( H \) satisfies the following five conditions:

(B1) There is a proper l.s.c. convex function \( j_B \) on \( H \) such that its subdifferential \( \partial j_B \) coincides with \( B \);

(B2) There is a positive constant \( C_1 > 0 \) such that
\[
C_1 |Bz_1 - Bz_2|_H^2 \leq (Bz_1 - Bz_2, z_1 - z_2), \quad \forall z_1, z_2 \in H;
\]

(B3) \( Bz \in D(\varphi^t(0; \cdot)) \) for any \( t \in [0, T] \) and \( z \in D(\varphi^t(0; \cdot)) \);

(B4) There are positive constants \( C_2 > 0 \) and \( C_3 > 0 \) such that
\[
\varphi^t(0; Bz) \leq C_2 \varphi^t(0; z) + C_3, \quad \forall z \in H, \forall t \in [0, T];
\]

(B5) \( B \) is bounded in \( H \), namely, there is a positive constant \( C_B > 0 \) such that \( |Bz|_H \leq C_B \) for any \( z \in H \).

**Definition 2.2.** Given a positive number \( T \) and a function \( \alpha \in W^{1,2}(0, T) \), we denote by \( \{ \varphi^t \} \in \Phi(\{\alpha\}) \) the set of all time-dependent functions \( \varphi^t(\cdot, \cdot) \) from \( H \times H \) into \( \mathbb{R} \cup \{\infty\} \) satisfying the following six conditions:
For each $w \in H$ and $t \in [0, T]$, $\varphi^t(w; \cdot) : H \to \mathbb{R} \cup \{\infty\}$ is a proper l.s.c. convex function;

(\Phi 2) There exists a positive constant $C_4 > 0$ such that
\[ \varphi^t(w; z) \geq C_4|z|^2_H, \quad \forall w \in H, \forall t \in [0, T], \forall z \in D(\varphi^t(w; \cdot)). \]

(\Phi 3) For each $t \in [0, T]$, $w \in H$ and $r > 0$, the level set $\{z \in H; \varphi^t(w; z) \leq r\}$ is compact in $H$;

(\Phi 4) $D(\varphi^t(w; \cdot))$ is independent of $w \in H$ for any $t \in [0, T]$;

(\Phi 5) For any $s, t \in [0, T]$ with $s \leq t$, $w \in D(\varphi^s(0; \cdot))$ with $|w|_H \leq C_B$ and $z \in D(\varphi^s(w; \cdot))$, there exists an element $\tilde{z} \in D(\varphi^t(w; \cdot))$ such that
\[ |\tilde{z} - z|_H \leq |\alpha(t) - \alpha(s)| \left( 1 + \varphi^s(0; z)^{\frac{1}{2}} \right), \]
\[ \varphi^t(w; \tilde{z}) - \varphi^s(w; z) \leq |\alpha(t) - \alpha(s)| \left( 1 + \varphi^s(0; z) + \varphi^s(0; w)^{\frac{1}{2}} \varphi^s(0; z)^{\frac{1}{2}} \right); \]

(\Phi 6) There is a positive constants $C_5 > 0$ such that
\[ |\varphi^t(\tilde{w}; z) - \varphi^t(w; z)| \leq C_5|\tilde{w} - w|_H \varphi^t(0; z)^{\frac{1}{2}} \]
\[ \forall t \in [0, T], w \in H \text{ with } |w|_H \leq C_B, \tilde{w} \in H \text{ with } |\tilde{w}|_H \leq C_B \text{ and } z \in D(\varphi^t(0, \cdot)). \]

Let us begin with the uniqueness of solution to CP$(u_0)$. To do so, we shall introduce a subclass of $\Phi(\{\alpha\})$.

Definition 2.3. Let $\gamma$ be a non-negative continuous and convex function on $H$ such that $\gamma(z) + \gamma(-z) = 0$ if and only if $z = 0$. Then $\{\varphi^t\} \in \Phi_{B, \gamma}(\{\alpha\})$ if and only if $\{\varphi^t\} \in \Phi(\{\alpha\})$ satisfies the $\gamma$-accretiveness $(\ast)$ for $\varphi^t$ and $B$ as follows:

$(\ast)$ For any $w_i \in H$, $z_i \in D(\partial \varphi^t(w_i; \cdot))$ and $z^*_i \in \partial \varphi^t(w_i; z_i)$ $(i = 1, 2)$, there is an element $w_0 \in \partial \gamma(Bz_1 - Bz_2)$ such that $(z^*_1 - z^*_2, w_0) \geq 0$, where $\partial \gamma$ is the subdifferential of $\gamma$ in $H$.

Now let us mention our abstract uniqueness result in this paper.

Theorem 2.4. Let $T$ be any positive number. Assume $\{\varphi^t\} \in \Phi_{B, \gamma}(\{\alpha\})$, $f \in L^2(0, T; H)$ and $B$ satisfies the conditions (B1)-(B5).

(i) Let $u$ and $v$ be solutions to CP$(u_0)$ and CP$(v_0)$, respectively. Then, we have
\[ \gamma(Bu(t) - Bv(t)) \leq \gamma(Bu(s) - Bv(s)) \quad \text{for any } 0 \leq s \leq t \leq T. \]

(ii) For each $u_0 \in H$, the function $Bu$ is uniquely determined, where $u$ is the solution to CP$(u_0)$.

(iii) Furthermore, we assume that for each $w \in H$ $\varphi^t(w; \cdot)$ is strictly convex on $D(\varphi^t(w; \cdot))$. Then the solution $u$ to CP$(u_0)$ is unique.

Proof. (i) Let $u$ and $v$ be solutions to CP$(u_0)$ and CP$(v_0)$, respectively. By the $\gamma$-accretiveness of $\varphi^t$ and $B$, for a.e. $\tau \in [0, T]$ there exists $z^*(\tau) \in \partial \gamma(Bu(\tau) - Bv(\tau))$ such that
\[ 0 \leq \left( [f(\tau) - \frac{d}{d\tau}Bu(\tau)] - [f(\tau) - \frac{d}{d\tau}Bv(\tau)], z^*(\tau) \right) \]
\[ = \left( -\frac{d}{d\tau}Bu(\tau) + \frac{d}{d\tau}Bv(\tau), z^*(\tau) \right) = -\frac{d}{d\tau}\gamma(Bu(\tau) - Bv(\tau)). \]
Hence, integrating (2.1) over \((s, t)\), we get
\[
\gamma(Bu(t) - Bv(t)) \leq \gamma(Bu(s) - Bv(s)) \quad \text{for any } 0 \leq s \leq t \leq T.
\]
Thus the assertion (i) holds.

Here if \(u_0 = v_0\), then (2.2) implies that \(\gamma(Bu(t) - Bv(t)) = 0\) for all \(t \in [0, T]\).

Similarly we can get \(\gamma(Bv(t) - Bu(t)) = 0\) for all \(t \in [0, T]\). Hence we have \(Bu = Bv\).
Thus the assertion (ii) has been shown.

From the assertion of (ii) it follows that
\[
\partial \varphi^t(Bu(t); u(t)) \cap \partial \varphi^t(Bv(t); v(t)) \neq \emptyset, \quad \text{a.e. } t \in [0, T].
\]
Furthermore, if \(\varphi^t(w; \cdot)\) is strictly convex on \(D(\varphi^t(w; \cdot))\) for any \(w \in H\), then \(\partial \varphi^t(w; \cdot)\) is strictly monotone for a.e. \(t \in [0, T]\). Hence from (2.3), the assertion (ii) and the strictly monotonicity of \(\partial \varphi^t(Bu(t); \cdot)\) we have \(u(t) = v(t)\) for a.e. \(t \in [0, T]\). Thus, the assertion (iii) has been proved.

Next main result is concerned with the existence of solutions to \(CP(u_0)\).

**Theorem 2.5.** Let \(T\) be any positive number. Assume \(\{\varphi^t\} \in \Phi_{B, \gamma}(\{\alpha\})\), \(f \in W^{1,1}(0, T; H)\) and \(B\) satisfies the conditions (B1)-(B5). Then, for each \(u_0 \in D(\varphi^0(0; \cdot))\) there exists at least one solution \(u\) of \(CP(u_0)\) on \([0, T]\).

### 3. Proof of Theorem 2.5

In this section we shall show Theorem 2.5 by the fixed point argument and a regularization method. Let us begin with the existence of local solution to \(CP(u_0)\).

3.1. **Local solution.** Using the fixed point argument we shall prove the existence of local solution to \(CP(u_0)\). To do so, for given positive numbers \(T > 0\) and \(M > 0\), let us consider a Banach space
\[
E_M(T) \equiv \left\{ w \in W^{1,2}(0, T; H) ; \begin{array}{l}
\sup_{t \in [0, T]} \varphi^t(0; w(t)) \leq M, \\
|w'|_{L^2(0, T; H)} \leq M, \\
w(0) = Bu_0, \\
\sup_{t \in [0, T]} |w(t)|_H \leq C_B.
\end{array} \right\}.
\]

Now, for each \(w \in E_M(T)\) let us consider a following Cauchy problem \(CP(w; u_0)\):
\[
CP(w; u_0) \left\{ \begin{array}{l}
(Bu)'(t) + \partial \varphi^t(w(t); u(t)) \ni f(t) \quad \text{in } H \text{ a.e. } t \in (0, T), \\
Bu(0) = Bu_0.
\end{array} \right.
\]

**Lemma 3.1.** For each \(w \in E_M(T)\) we put \(\psi^t_w(z) := \varphi^t(w(t); z)\) for \(z \in H\). Then, there is a positive constant \(N_1 > 0\) independent of \(w\) satisfying the following:
for any \(s, t \in [0, T]\) with \(s \leq t\) and \(z \in D(\psi^s_w)\), there exists \(\tilde{z} \in D(\psi^t_w)\) such that
\[
|\tilde{z} - z|_H \leq N_1 |\alpha(t) - \alpha(s)| \left(1 + \psi^s_w(z)^{\frac{1}{2}}\right)
\]
and
\[
\psi^t_w(\tilde{z}) - \psi^t_w(z) \leq N_1 \left\{ |\alpha(t) - \alpha(s)|(1 + \psi^s_w(z)^{\frac{1}{2}}) + |w(t) - w(s)|_H (1 + \psi^s_w(z)^{\frac{1}{2}})\right\}.
\]

(3.1)
Proof. Taking $w = w(s)$ in (Φ5), then for any $s, t \in [0, T]$ with $s \leq t$ and $z \in D(\varphi^s(w(s); \cdot))$ there exists $\tilde{z} \in D(\varphi^s(w(s); \cdot))$ such that

$$
(3.2) \quad |\tilde{z} - z|_H \leq |\alpha(t) - \alpha(s)| \left(1 + \varphi^s(0; z)^{\frac{1}{2}}\right),
$$

$$
\varphi^t(w(s); \tilde{z}) - \varphi^s(w(s); z)
$$

$$
(3.3) \quad \leq |\alpha(t) - \alpha(s)| \left(1 + \varphi^s(0; z) + \varphi^s(0; w(s))^{\frac{1}{2}}\varphi^s(0; z)^{\frac{1}{2}}\right).
$$

It follows from (Φ4) that

$$
(3.4) \quad \varphi^s(0; z) \leq 2\varphi^s(w(s); z) + C_5^2|w(s)|^2_H \leq 2\psi^s_w(z) + C_5^2C_B^2.
$$

By Lemma 3.1 we get the time-dependence of $N$ such that

$$
|\tilde{z} - z|_H \leq N_2|\alpha(t) - \alpha(s)| \left(1 + \psi^s_w(z)^{\frac{1}{2}}\right).
$$

Moreover, we observe that by (3.3), (3.5), (Φ6) there is a positive number $N_3 > 0$ independent of $w$ satisfying the following:

$$
\psi^t_w(\tilde{z}) - \psi^s_w(z) = \varphi^t(w(t); \tilde{z}) - \varphi^t(w(s); \tilde{z}) + \varphi^t(w(s); \tilde{z}) - \varphi^s(w(s); z)
$$

$$
\leq N_3 \left\{|w(t) - w(s)|_H \psi^t_w(\tilde{z})^{\frac{1}{2}} + |w(t) - w(s)|_H\right\}
$$

$$
(3.7) \quad +|\alpha(t) - \alpha(s)|\left(1 + \psi^s_w(z)^{\frac{1}{2}}\right) + |\alpha(t) - \alpha(s)|\varphi^s(0; w(s))^{\frac{1}{2}}(1 + \psi^s_w(z)^{\frac{1}{2}}).
$$

From $\alpha \in W^{1,2}(0, T), w \in E_M(T)$ and (3.7) it follows that

$$
(3.8) \quad \psi^t_w(\tilde{z}) \leq N_4 \left(1 + \psi^s_w(z) + |\alpha(t) - \alpha(s)|^2\varphi^s(0; w(s))\right)
$$

for some constant $N_4 > 0$. Therefore, using (3.8) in the right hand side of (3.7), and by (3.4)-(3.6), we get this Lemma 3.1 for some constant $N_1 > 0$ independent of $w$. □

**Proposition 3.2.** For each $w \in E_M(T)$, there exists a solution $u$ to $CP(w; u_0)$ such that the function $Bu$ is uniquely determined.

**Proof.** We note that $CP(w; u_0)$ can be regarded as the Cauchy problem for the doubly nonlinear evolution equation of the form:

$$
\left\{
\begin{array}{ll}
(Bu)^t(t) + \partial\psi^t_w(u(t)) & \ni f(t) & \text{in } H \text{ a.e. } t \in (0, T), \\
Bu(0) = Bu_0.
\end{array}
\right.
$$

By Lemma 3.1 we get the time-dependence of $\psi^t_w$. Therefore it follows from the assumptions (Φ1)-(Φ3), (B1)-(B2) that we can apply the abstract theory established by Kenmochi-Pawłow [12]. Thus we can get the existence of solution $u$ for $CP(w; u_0)$. For detail proof, see [12, Theorem 1.1].

Moreover by the same argument of Kenmochi-Pawłow [12, Theorem 1.2] or Theorem 2.4 (ii), we can get the uniqueness of the function $Bu$. □

By Proposition 3.2, we can define a mapping $Q : E_M(T) \rightarrow L^2(0, T; H)$ by $Qw = Bu$ for each $w \in E_M(T)$, where $u$ is a solution for $CP(w; u_0)$. 
Lemma 3.3. There are positive constants $T_0$ and $M_0$ such that $Q$ is a self-mapping on $E_{M_0}(T_0)$, i.e., $Qw(= Bu) \in E_{M_0}(T_0)$ for any $w \in E_{M_0}(T_0)$.

Proof. We consider the approximate problem $\text{CP}(w; u_0)_{\varepsilon, \lambda}$ ($0 < \varepsilon, \lambda \leq 1$) of $\text{CP}(w; u_0)$:

$$\text{CP}(w; u_0)_{\varepsilon, \lambda} \begin{cases} (B_{\varepsilon}u_{\varepsilon, \lambda})'(t) + \partial \varphi_{\lambda}(w(t); u_{\varepsilon, \lambda}(t)) = f(t), & 0 < t < T, \\ B_{\varepsilon}u_{\varepsilon, \lambda}(0) = B_{\varepsilon}u_0. \end{cases}$$

Here we put $B_{\varepsilon} := B + \varepsilon I$ and $\varphi^\varepsilon_\lambda(w(t); \cdot)$ is the Moreau-Yosida approximation of $\varphi^\varepsilon(w(t); \cdot)$, defined by $\varphi^\varepsilon_\lambda(w(t); z) := \inf_{y \in H} \left\{ \frac{1}{2\varepsilon}||z - y||_H^2 + \varphi^\varepsilon(w(t); y) \right\}$ for $z \in H$.

By the same argument in [12, Lemma 2.2] we see that there is a positive constant $C'_4 > 0$ independent of $t, w, z$ and $0 < \lambda \leq 1$ so that

$$\varphi^\varepsilon_\lambda(w; z) \geq C'_4 ||z||_H^2.$$  \hfill (3.9)

Moreover, we observe that the problem $\text{CP}(w; u_0)_{\varepsilon, \lambda}$ has a unique solution $u_{\varepsilon, \lambda}$ which converges to the solution of $\text{CP}(w; u_0)$ in some sense. For detail proof, see [12]. By the slight modification of [12, Lemma 2.3], we see that $\Psi_{\varepsilon, \lambda}(t) := \varphi^\varepsilon_\lambda(w(t); u_{\varepsilon, \lambda}(t))$ is of bounded variation on $[0, T]$ and satisfies the following inequality:

$$\Psi_{\varepsilon, \lambda}(t) - \Psi_{\varepsilon, \lambda}(s) + \int_s^t ((B_{\varepsilon}u_{\varepsilon, \lambda})'(\tau) - f(\tau), u_{\varepsilon, \lambda}'(\tau))d\tau \leq N_1 \int_s^t \left[ |\alpha'(\tau)||((B_{\varepsilon}u_{\varepsilon, \lambda})'(\tau) - f(\tau)||1 + \Psi_{\varepsilon, \lambda}(\tau)\right]^{\frac{1}{2}} + |\alpha'(\tau)|1 + \Psi_{\varepsilon, \lambda}(\tau)\right]^{\frac{1}{2}} + (|w'(\tau)||H + |\alpha'(\tau)||\varphi^\varepsilon(0; w(\tau))\right]^{\frac{1}{2}} (1 + \Psi_{\varepsilon, \lambda}(\tau))^{\frac{1}{2}} d\tau \leq N_1 \int_s^t (f(\tau), u_{\varepsilon, \lambda}'(\tau))d\tau - \int_s^t (f'(\tau), u_{\varepsilon, \lambda}(\tau))d\tau + (f(t), u_{\varepsilon, \lambda}(t)) - (f(s), u_{\varepsilon, \lambda}(s)) \leq \int_s^t \left\{ |f'(\tau)|H + N_5 \Psi_{\varepsilon, \lambda}(\tau) \right\} d\tau + (f(t), u_{\varepsilon, \lambda}(t)) - (f(s), u_{\varepsilon, \lambda}(s)) \leq 2\delta((B_{\varepsilon}u_{\varepsilon, \lambda})'(\tau)||H^2 + 2\delta|f(\tau)||H^2 + \delta^{-1}|\alpha'(\tau)||1 + \Psi_{\varepsilon, \lambda}(\tau)||^2) \leq 2\delta((B_{\varepsilon}u_{\varepsilon, \lambda})'(\tau)||H^2 + 2\delta|f(\tau)||H^2 + \delta^{-1}|\alpha'(\tau)||1 + \Psi_{\varepsilon, \lambda}(\tau)||^2) \leq 2\delta((B_{\varepsilon}u_{\varepsilon, \lambda})'(\tau)||H^2 + 2\delta|f(\tau)||H^2 + \delta^{-1}|\alpha'(\tau)||1 + \Psi_{\varepsilon, \lambda}(\tau)||^2),$$

where the positive constant $N_5 > 0$ in (3.12) depends on $C'_4$ and the constant $\delta > 0$ will be defined below. Using (3.10)-(3.13), we have

$$X_{\varepsilon, \lambda}(t) - X_{\varepsilon, \lambda}(s) + N_6 \int_s^t ((B_{\varepsilon}u_{\varepsilon, \lambda})'(\tau)||Hd\tau \leq N_7 \int_s^t \left\{ G(\tau)(1 + \Psi_{\varepsilon, \lambda}(\tau)) + W(\tau)(1 + \Psi_{\varepsilon, \lambda}(\tau))^{\frac{1}{2}} \right\} d\tau$$

for $0 \leq s \leq t \leq T$ and $0 < \varepsilon, \lambda \leq 1$, where we put $X_{\varepsilon, \lambda}(t) := \Psi_{\varepsilon, \lambda}(t) - (f(t), u_{\varepsilon, \lambda}(t))$, $N_6 := \frac{C_1}{1 + \varepsilon C_1} - 2\delta N_1$, $G(t) := |f(t)||H + |f'(t)||H + |\alpha'(t)||1 + \Psi_{\varepsilon, \lambda}(\tau)||^2$, $W(t) := |w'(t)||H + |\alpha'(t)||\varphi^\varepsilon(0; w(t))||$.
Now we take $\delta > 0$ so that $N_6 > 0$. Then, note that $N_7 > 0$ is dependent only on $C_1, N_1, N_5$. Here by (3.9) we get
\begin{equation}
X_{\varepsilon, \lambda}(t) = \Psi_{\varepsilon, \lambda}(t) - (f(t), u_{\varepsilon, \lambda}(t)) \geq N_8 \Psi_{\varepsilon, \lambda}(t) - \delta_1 |f(t)|_H^2,
\end{equation}
where $\delta_1 > 0$ is the constant so that $N_8 > 0$. Using (3.15) in (3.14), applying Gronwall’s inequality and letting $\varepsilon \to 0$, $\lambda \to 0$, we obtain
\begin{equation}
\sup_{0 \leq t \leq T} \varphi'(w(t); u(t)) + \int_0^T |(Bu)'(t)|_H^2 dt \leq N_{10} e^{N_9 (|G|_{L^1(0,T)} + |W|_{L^1(0,T)})} \{1 + N_9 (|G|_{L^1(0,T)} + |W|_{L^1(0,T)})\},
\end{equation}
where $N_9$ and $N_{10}$ are positive constants dependent on the given data.

Now we show that $Q$ is the self-mapping on $E_{M_0}(T_0)$ for some chosen $T_0 > 0$ and $M_0 > 0$. Taking account of (3.5) and (B4), for any $w \in E_M(T)$ we have
\begin{equation}
\varphi'(0; Bu(t)) \leq C_2 \varphi'(0; u(t)) + C_3 \leq 2C_2 \varphi'(w(t); u(t)) + C_2 C_5^{-2} C_B + C_3.
\end{equation}
Here we put the constant $M_0 > 0$ so that
\begin{equation}
(1 + 2C_2) N_{10} e^{2N_9 (1 + 2N_9)} + C_2 C_5^{-2} C_B + C_3 \leq M_0,
\end{equation}
and then $T_0 > 0$ such that
\begin{equation}
|G|_{L^1(0,T_0)} \leq 1, \quad |W|_{L^1(0,T)} \leq T_0^{\frac{1}{2}} M_0^{\frac{1}{2}} + T_0^{\frac{1}{2}} M_0^{\frac{1}{2}} |\alpha'|_{L^2(0,T_0)} \leq 1.
\end{equation}
Then from (3.16) and (3.17) it follows that $Qw(= Bu) \in E_{M_0}(T_0)$ for any $w \in E_{M_0}(T_0)$, hence, $Q$ is the self-mapping on $E_{M_0}(T_0)$.

**Lemma 3.4.** Let $M_0 > 0$ and $T_0 > 0$ be constants obtained in Lemma 3.3. Let $\{w_n\} \subset E_{M_0}(T_0)$, $w \in E_{M_0}(T_0)$ and $u_n$ be the solution of $CP(w_n; u_0)$. Suppose $w_n \rightharpoonup w$ in $C([0,T_0]; H)$ as $n \to +\infty$. Then, there is a solution $u$ of $CP(w; u_0)$ on $[0,T_0]$ such that $Bu \in E_{M_0}(T_0)$ and $Bu_n \rightharpoonup Bu$ in $C([0,T_0]; H)$ as $n \to +\infty$.

**Proof.** Since $\{w_n\} \subset E_{M_0}(T_0)$, Lemma 3.3 and (3.17), we have
\begin{align}
\sup_{t \in [0,T_0]} \varphi'(0; Bu_n(t)) & \leq M_0, \quad |(Bu_n)'|_{L^2(0,T_0; H)} \leq M_0, \quad \forall n = 1, 2, \cdots, \\
\sup_{t \in [0,T_0]} \varphi'(0; u_n(t)) & \leq C_2^{-1} M_0, \quad \forall n = 1, 2, \cdots.
\end{align}
Here we note that the function $Bu_n$ is uniquely determined (cf. Theorem 2.4 (ii)).

By (Φ2), (Φ3), (3.18), (3.19) there are a subsequence $\{u_{n_k}\}$ of $\{u_n\}$, a countable dense subset $J_D$ of $[0,T_0]$ and functions $\tilde{u} \in W^{1,2}(0,T_0; H)$, $u \in L^\infty(0,T_0; H)$ such that
\begin{align}
Bu_{n_k}(t) & \rightharpoonup \tilde{u}(t) \quad \text{weakly in } H \quad \text{for all } t \in [0,T_0], \\
(Bu_{n_k})' & \rightharpoonup (\tilde{u})' \quad \text{weakly in } L^2(0,T_0; H), \\
\left|u_{n_k}(t) - u(t) \right| & \rightharpoonup 0 \quad \text{weakly-}^* \text{ in } L^\infty(0,T_0; H), \\
u_{n_k}(t) & \rightharpoonup u(t) \quad \text{strongly in } H, \quad \text{for } t \in J_D
\end{align}
as $k \to +\infty$.

Since $C_1 |Bu_{n_k}(t) - Bu(t)|_H \leq |u_{n_k}(t) - u(t)|_H$, we observe that $Bu_{n_k}(t) \longrightarrow Bu(t)$ strongly in $H$ as $k \to +\infty$ for all $t \in J_D$. On account of (3.20)-(3.21), we see that $Bu(t) = \tilde{u}(t)$ for all $t \in J_D$. Therefore by (3.18)-(3.23) and the uniqueness of the function $Bu_n$, we observe that
for $0 \leq (3.28)$

Thus solution of $\text{CP}(w; u_0)$ as $n \to +\infty$.

Now, let us show that $u$ is a solution of $\text{CP}(w; u_0)$ on $[0, T_0]$. To do so, we define

$$
\Phi(w; z) = \int_0^{T_0} \varphi'(w(t); z(t)) dt.
$$

Then by the assumption (Φ6) we see that

$$
\Phi(w_n; z) \to \Phi(w; z) \text{ as } n \to +\infty
$$

for any $z \in L^2(0, T_0; H)$ with $\varphi^{(i)}(0; z(\cdot)) \in L^1(0, T_0)$. From (3.19), (Φ1), (Φ2), (Φ6) and the Fatou’s lemma, it follows that

$$
\liminf_{k \to +\infty} \Phi(w_{n_k}; u_{n_k}) = \liminf_{k \to +\infty} \{\Phi(w_{n_k}; u_{n_k}) - \Phi(w; u_{n_k}) + \Phi(w; u_{n_k})\}
$$

(3.25)

$$
\geq \liminf_{k \to +\infty} \Phi(w; u_{n_k}) \geq \Phi(w; u).
$$

Moreover, let $j_B^*$ be a conjugate function of $j_B$ on $H$. Clearly, $j_B^*$ is a proper l.s.c. convex function on $H$ such that $\partial j_B^* = B^{-1}$. Then, we have

$$
\liminf_{k \to +\infty} \int_0^{T_0} (Bu_{n_k})'(t), u_{n_k}(t) dt = \liminf_{k \to +\infty} \{j_B^*(Bu_{n_k}(T_0)) - j_B^*(Bu_0)\}
$$

(3.26)

$$
\geq j_B^*(Bu(T_0)) - j_B^*(Bu_0) = \int_0^{T_0} (Bu)'(t), u(t) dt.
$$

Now, let $z$ be any function in $L^2(0, T_0; H)$ with $\varphi^{(i)}(0; z(\cdot)) \in L^1(0, T_0)$. Since $u_{n_k}$ is the solution of $\text{CP}(w_{n_k}; u_0)$, then the following inequality holds:

$$
\int_0^{T_0} (f(t) - (Bu_{n_k})'(t), z(t) - u_{n_k}(t)) dt \leq \Phi(w_{n_k}; z) - \Phi(w_{n_k}; u_{n_k}).
$$

Taking account of (3.21), (3.24)-(3.26) and letting $k \to +\infty$ in (3.27), we get

$$
\int_0^{T_0} (f(t) - (Bu)'(t), z(t) - u(t)) dt \leq \Phi(w; z) - \Phi(w; u),
$$

which implies that $f(t) - (Bu)'(t) \in \partial \varphi'(w(t); u(t))$ for a.e. $t \in [0, T_0]$ (cf. [3, Proposition 3.3]). Thus $u$ is the solution of $\text{CP}(w; u_0)$.

**Proof. [Proof of Theorem 2.5; Local existence]** By Lemma 3.3, we can define a self-mapping $Q : E_{M_0}(T_0) \to E_{M_0}(T_0)$ by $Qw = Bu$ for each $w \in E_{M_0}(T_0)$, where $u$ is a solution of $\text{CP}(w; u_0)$. Clearly, $E_{M_0}(T_0)$ is compact in $C([0, T_0]; H)$.

Moreover, it follows from Lemma 3.4 that $Q$ is continuous with respect to the topology of $C([0, T_0]; H)$. Therefore, the Schauder’s fixed point theorem implies that the self-mapping $Q$ has a fixed point $Bu$ in $E_{M_0}(T_0)$, i.e. $QBu = Bu$. Clearly $u$ is the solution of $\text{CP}(u_0)$, thus we can get the local existence of solution $u$ of $\text{CP}(u_0)$.

**3.2. Global solution.** Now let us begin with the inequality (3.14). Applying Schwarz inequality to the term $W(\tau)(1 + \Psi_{\varepsilon, \lambda}(\tau))^{1/2}$ and using (3.15), we get

$$
X_{\varepsilon, \lambda}(t) - X_{\varepsilon, \lambda}(s) + N_6 \int_0^t ||(B_{\varepsilon, \lambda})'(\tau)||_H^2 d\tau
$$

(3.28)

$$
\leq N_{11} \int_0^t G(\tau)(1 + X_{\varepsilon, \lambda}(\tau)) d\tau + \frac{N_8}{2} \int_0^t ||w'(\tau)||_H^2 + |\varphi'(0; w(\tau))| d\tau
$$

for $0 \leq s \leq t \leq T$ and $0 < \varepsilon, \lambda \leq 1$, where $N_{11} > 0$ is dependent on $N_6, N_7, N_8, \delta_1$ and $|f|_{L^\infty(0,T;H)}$. 
Applying Gronwall’s inequality to (3.28), and letting $\varepsilon, \lambda \to 0$, we have
\begin{equation}
\varphi^{t}(w(t); u(t)) + \int_{0}^{t} e^{N_{11} \int_{0}^{\tau} \theta^{(s)} ds} \left\{ N_{6} |(Bu)^{(s)}(\tau)|_{H}^2 - \frac{N_{6}}{2} \{ |w^{(s)}(\tau)|_{H}^2 + \varphi^{\tau}(0; w(\tau)) \} \right\} d\tau \leq N_{12}
\end{equation}
for $0 \leq t \leq T$, where $N_{12} > 0$ is dependent on the given data.

Note that the inequality (3.29) holds for any $w \in \bar{E}_{M_{6}}(T_{0})$. Then by the result in Section 3.1, we can take $w = Bu \in \bar{E}_{M_{6}}(T_{0})$, where $u$ is the solution of $\text{CP}(u_{0})$ on $[0, T_{0}]$. Hence, using (3.17) and (3.29), we have
\begin{equation}
\varphi^{t}(Bu(t); u(t)) + \int_{0}^{t} e^{N_{11} \int_{0}^{\tau} \theta^{(s)} ds} \frac{N_{6}}{2} |(Bu)^{(s)}(\tau)|_{H}^2 d\tau
\end{equation}
(3.30)
\begin{equation}
\leq N_{13} \left( 1 + \int_{0}^{t} \varphi^{\tau}(Bu(\tau); u(\tau)) d\tau \right)
\end{equation}
for $0 \leq t \leq T_{0}$,
where $N_{13}$ depends on $C_{2}, C_{3}, C_{5}, C_{B}, N_{6}, N_{11}, N_{12}, |G|_{L^{1}(0, T)}$.

Applying Gronwall’s inequality to (3.30), we get
\begin{equation}
\sup_{t \in [0, T_{0}]} \varphi^{t}(Bu(t); u(t)) + \int_{0}^{T_{0}} |(Bu)^{(s)}(t)|_{H}^2 dt \leq N_{14},
\end{equation}
where $N_{14}$ depends only on the given data and is independent of $T_{0}( \leq T)$.

Now let us prove the global existence of solution to $\text{CP}(u_{0})$.

**Proof.** [**Proof of Theorem 2.5; Global existence**] Assume that
\begin{equation}
T^{*} := \sup \{ T_{0}; \text{CP}(u_{0}) \text{ has a solution on } [0, T_{0}] \} < +\infty.
\end{equation}
By the local existence results in Section 3.1, we note $T^{*} > 0$. By the definition of $T^{*}$, there is a function $u : [0, T^{*}) \to H$ such that for any $T_{0} (< T^{*})$ $u$ is the solution of $\text{CP}(u_{0})$ on $[0, T_{0}]$. By (B5) and (3.31) we have
\begin{equation}
Bu \in W^{1,2}(0, T^{*}; H), \quad \varphi^{(3)}(Bu(\cdot); u(\cdot)) \in L^{\infty}(0, T^{*}).
\end{equation}
Hence we observe that the limit $B^{*} := \lim_{t \to T^{*}} Bu(t)$ strongly in $H$ exists. By assumptions (B2), (Φ1), (Φ3), (Φ5), (Φ6) we see that
\begin{equation}
B^{*} = Bu^{*} \text{ for some } u^{*} \in D(\varphi^{T^{*}}(0; \cdot)).
\end{equation}
Now, taking $B^{*}$ as the initial value at $t = T^{*}$, we can get the solution $u$ beyond the time interval $[0, T^{*}]$. Thus $T^{*} = +\infty$, so we have the global existence of solution. \qed

4. Application

In this section we consider an elliptic-parabolic free boundary problem with Signorini-Dirichlet-Neumann mixed boundary condition:
\begin{align}
\tag{4.1}
b(u)_{t} - \nabla \cdot a(x, b(u), \nabla u) = f(t, x) & \quad \text{in } (0, T) \times \Omega, \\
\tag{4.2}
u \cdot a(x, b(u), \nabla u) \leq 0, \quad (u - p(t)) \nu \cdot a(x, b(u), \nabla u) = 0 & \quad \text{on } \Gamma_{S}, \\
\tag{4.3}
u \cdot a(x, b(u), \nabla u) = 0 & \quad \text{on } \Gamma_{D}, \\
\tag{4.4}
\nu \cdot a(x, b(u), \nabla u) = 0 & \quad \text{on } \Gamma_{N}, \\
\tag{4.5}
b(u(0, \cdot)) = b_{0} & \quad \text{in } \Omega.
\end{align}
Here $\nu$ is an outward normal vector on the boundary, and $p$ is a given function. $\Omega$ is a bounded domain in $\mathbf{R}^N$ ($1 \leq N < +\infty$) with smooth boundary $\Gamma$ consisting of three disjoint parts $\Gamma_\nu$, $\nu = S, D, N$, namely, $\Gamma = \Gamma_S \cup \Gamma_D \cup \Gamma_N$.

The problem (4.1)-(4.5) is a model of partially saturated porous media, in which $\Gamma_S$, $\Gamma_D$ and $\Gamma_N$ refer to the parts of the boundary in contact with the atmosphere, reservoirs and impervious layer, respectively. The function $p$ is the pressure in the reservoirs on $\Gamma_D$, and to the atmospheric pressure on $\Gamma_S$.

Many mathematicians have already studied the various models of porous media. For instance, see [1, 2, 4, 8, 11, 13, 14, 15].

The aim of this section is to consider the problem (4.1)-(4.5) as an application of the abstract evolution equation $CP(u_0)$. To do so, we suppose that

(A1) $a(x, s, p) = \partial_p A(x, s, p)$ for some potential function $A(x, s, p)$. There exist constants $\mu_1 > 0$, $\mu_2 = \mu_2(a) > 0$ and $\mu_3 = \mu_3(a) > 0$ such that
\[
\begin{align*}
[a(x, s, p) - a(x, \hat{s}, \hat{p})] \cdot (p - \hat{p}) & \geq \mu_1 |p - \hat{p}|^2, \\
|a(x, s, p)|^2 + |A(x, s, p)| + |\partial_s A(x, s, p)|^2 & \leq \mu_2 (1 + |s|^2 + |p|^2), \\
|a(x, s, p) - a(x, \hat{s}, \hat{p})| & \leq \mu_3 (1 + |p|)|s - \hat{s}|
\end{align*}
\]

for all $x \in \Omega$, $s, \hat{s} \in \mathbf{R}$, $p, \hat{p} \in \mathbf{R}^N$.

(A2) $b : \mathbf{R} \to \mathbf{R}$ is bounded, nondecreasing and Lipschitz continuous.

Here for each $t \in [0, T]$ let us define the convex set $K(t)$ by

$$K(t) := \{z \in H^1(\Omega); z \leq p(t) \text{ on } \Gamma_S \text{ and } z = p(t) \text{ on } \Gamma_D \}.$$ 

As a direct application of Theorems 2.4 and 2.5, we have:

**Proposition 4.1.** Assume (A1) and (A2). Then, for each $f \in W^{1,1}(0, T; L^2(\Omega))$, $p \in W^{1,2}(0, T; H^1(\Omega))$ and $b_0 = b(u_0)$ for some $u_0 \in K(0)$, the problem (4.1)-(4.5) has a solution $u$ on $[0, T]$.

**Proof.** To apply Theorems 2.4 and 2.5 to the problem (4.1)-(4.5), we choose $L^2(\Omega)$ as a real Hilbert space $H$, and define a function $\varphi^t : L^2(\Omega) \times L^2(\Omega) \to \mathbf{R} \cup \{\infty\}$ by

$$\varphi^t(w; z) := \begin{cases} 
\int_\Omega A(x, w(x), \nabla z(x)) dx, & \text{if } z \in K(t), \\
+\infty, & \text{otherwise.}
\end{cases}$$

Let us define an operator $B : L^2(\Omega) \to L^2(\Omega)$ by $Bz := b(z)$ in $L^2(\Omega)$. Also we define a function $\gamma$ by $\gamma(z) := \int_\Omega z^+(x) dx$ for $z \in L^2(\Omega)$, where $z^+ := \max\{z, 0\}$.

Now we put for any $t \in [0, T]$

$$\alpha(t) := N_{15} \int_0^t |p'(\tau)|_{H^1(\Omega)} d\tau,$$

where $N_{15}$ is a (sufficient large) positive constant. Then, we easily see that $\{\varphi^t\} \in \Phi_{B, \gamma, \{\alpha\}}$ and the operator $B$ satisfies the assumptions (B1)-(B5). Clearly, the problem (4.1)-(4.5) can be reformulated in the abstract evolution equation $CP(u_0)$. Thus, applying Theorems 2.4 and 2.5, we see that the problem (4.1)-(4.5) has a solution $u$ on $[0, T]$ such that the function $b(u)$ is uniquely determined. \hfill \Box
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References


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