REMARK ON THE WEIGHT ENUMERATORS AND SIEGEL MODULAR FORMS

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Abstract. The purpose of this note is to study the coefficients of the polynomials if we express the weight enumerator as the polynomial of the fixed generators.

1. Introduction

It is well known that the graded ring $\mathbf{C}[W_C^{(1)}]$ generated by the weight enumerators of all self-dual doubly-even codes over the complex field $\mathbf{C}$ can be generated by two elements[1];

$$\mathbf{C}[W_C^{(1)}] = \mathbf{C}[W_{e_8}^{(1)}, W_{g_{24}}^{(1)}].$$

From this, the weight enumerator of every self-dual doubly-even code can be expressed as the polynomial of $W_{e_8}^{(1)}$ and $W_{g_{24}}^{(1)}$ over $\mathbf{C}$. However, the coefficients of the said polynomial may be in the ring smaller than $\mathbf{C}$. Actually, we can replace $\mathbf{C}$ in the equality above by the smaller ring $\mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{7}]$.

We start with the definitions, the notation and the known facts which are needed in this note. Let $\mathfrak{S}_n$ be the Siegel upper-half space of degree n and denote by $A(\Gamma_n)_k$ the ring of modular forms of weight k on $\Gamma_n = Sp_{2n}(\mathbf{Z})$ over $\mathbf{C}$. If f is an element of $A(\Gamma_n)_k$, then $f(\tau)$ can be expanded into a Fourier series of the following form:

$$f(\tau) = \sum_{s \geqq 0} a_f(s) \exp(2\pi\sqrt{-1}\mathrm{trace}(s\tau)) = \sum_{s_{ii} \geqq 0} \left(\sum a_f(s) \cdot \prod_{i<j} q_{ij}^{2s_{ij}} \right) \prod_{i=1}^{n} q_{ii}^{s_{ii}},$$

in which $q_{ij} = \exp(2\pi\sqrt{-1}\tau_{ij})$ and s runs over the set of half-integral positive (semi-definite) matrices of degree n. For any subring R of $\mathbf{C}$ we denote by

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$A_R(\Gamma_n)_k$ the R-module consisting of those $f \in A(\Gamma_n)_k$ such that $a_f(s)$ is in R for every s and by $A_R(\Gamma_n) := \oplus_{k\geq 0} A_R(\Gamma_n)_k$ taken in $A(\Gamma_n) := \oplus_{k\geq 0} A(\Gamma_n)_k$; then $A_R(\Gamma_n)$ forms a graded integral ring over R. The explicit structure of the ring $A_{\mathbf{Z}}(\Gamma_n)$ is known only for $n = 1, 2$ and we shall use them later.

Let m', m'' denote elements of $\mathbf{F}_2^n$ and put $\mathfrak{m} = (m' \ m'')$; then the theta constants with characteristic $\mathfrak{m}$ is defined as

$$\theta_{\mathfrak{m}}(\tau) = \sum_{p\in\mathbf{Z}^n} \exp 2\pi\sqrt{-1}\left\{\frac{1}{2}(p+\frac{1}{2}m')\tau\ {}^t(p+\frac{1}{2}m') + \frac{1}{2}(p+\frac{1}{2}m')m''\right\}.$$

Let C be a (linear) code of length k over $\mathbf{F}_2$. The weight enumerator $W_C^{(n)}(x_a : a \in \mathbf{F}_2^n)$ of degree n is defined as

$$W_C^{(n)} = W_C^{(n)}(x_a : a \in \mathbf{F}_2^n) = \sum_{v_1,\ldots,v_k\in C} \prod_{a\in\mathbf{F}_2^n} x_a^{n_a(v_1,\ldots,v_k)},$$

where $n_a(v_1, \ldots, v_k)$ denotes the number of i such that $a = (v_{1i}, \ldots, v_{ki})$. We note that $W_C^{(n)}$ is a homogeneous polynomial of degree k with non-negative integers as its coefficients. For any subring R of $\mathbf{C}$ we denote by $R[W_C^{(n)}]$ the graded ring generated by the weight enumerators of degree n of all self-dual doubly-even codes of any length over R. It is known that the Broué-Enguehard map $Th : x_a \mapsto \theta_{a0}(2\tau), a \in \mathbf{F}_2^n$, gives the $\mathbf{C}$–algebra homomorphism from $\mathbf{C}[W_C^{(n)}]$ to $A(\Gamma_n)^{(4)} = \oplus_{k\geqq 0, k\equiv 0 \pmod 4} A(\Gamma_n)_k$. In particular, it gives the isomorphisms $\mathbf{C}[W_C^{(n)}] \cong A(\Gamma_n)^{(4)}$ when $n = 1, 2$(see [6]). In the next section we explain our problem dealing with the case when $n = 1$. The main theme of this note is to investigate this in the case when $n = 2$.

2. The case when $n = 1$

In this section, we discuss the case when $n = 1$ (and may omit $n = 1$ in the notation of the weight enumerator for the sake of simplicity). Before proving the assertion described in the introduction, we modify our setting. We started from the fact(see [1]) , called *Gleason Theorem*, that $\mathbf{C}[W_C]$ is generated by W_{e_8} and $W_{g_{24}}$ over $\mathbf{C}$, where

$$W_{e_8} = x_0^8 + 14x_0^4x_1^4 + x_1^8,$$
$$W_{g_{24}} = x_0^{24} + 759x_0^{16}x_1^8 + 2576x_0^{12}x_1^{12} + 759x_0^8x_1^{16} + x_1^{24}.$$

The self-dual doubly-even code of length 8 is unique (up to isomorphism), however, we may take another self-dual doubly-even code of length 24 instead of g_{24}. There exist 7 indecomposable self-dual doubly-even codes of length 24(see [7]):

$$d_{12}^2, d_{10}e_7^2, d_8^3, d_6^4, d_{24}, d_4^6, g_{24}.$$

We call them $C_{24,1}, \ldots, C_{24,7}$. The following table gives the values a, b, if we write

$$W_{C_{24,i}} = aW_{e_8}^3 + bW_{g_{24}}, i = 1, 2, \cdots, 7.$$

	a	b
$C_{24,1}$	5/7	2/7
$C_{24,2}$	4/7	3/7
$C_{24,3}$	3/7	4/7
$C_{24,4}$	2/7	5/7
$C_{24,5}$	11/7	−4/7
$C_{24,6}$	1/7	6/7
$C_{24,7}$	1	0

We state the following proposition.

Proposition 2.1. *Let* $\mathcal{R}$ *be a ring such that* $\mathbf{Z} \subseteq \mathcal{R} \subseteq \mathbf{C}$. *Then we have*

$$\mathcal{R}[W_C] = \mathcal{R}[W_{e_8}, W_{C_{24,4}}] \textit{ if and only if } \mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}] \subseteq \mathcal{R},$$

$$\mathcal{R}[W_C] = \mathcal{R}[W_{e_8}, W_{C_{24,7}}] \textit{ if and only if } \mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{7}] \subseteq \mathcal{R},$$

and for $i = 1, 2, 3, 5, 6$,

$$\mathcal{R}[W_C] = \mathcal{R}[W_{e_8}, W_{C_{24,i}}] \textit{ if and only if } \mathbf{Z}[\frac{1}{2}, \frac{1}{3}] \subseteq \mathcal{R}.$$

Before proceeding to the proof, we recall the modular forms for Γ_1 over $\mathbf{Z}$. If we denote by E_k the Eisenstein series of even weight k normalized as $E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$, $q = e^{2\pi i \tau}$, and if we put $\Delta = 2^{-6}3^{-3}(E_4^3 - E_6^2)$, then it is well known that

$$A_{\mathbf{Z}}(\Gamma_1) = \mathbf{Z}[E_4, E_6, \Delta].$$

Moreover we have

$$A_{\mathbf{Z}}(\Gamma_1)^{(4)} = \mathbf{Z}[E_4, \Delta].$$

Proof of **Proposition2.1** First we consider the case when $C_{24,7} \cong g_{24}$. Suppose that we have the equality $\mathcal{R}[W_C] = \mathcal{R}[W_{e_8}, W_{g_{24}}]$. We pick the self-dual doubly-even code $C_{32,50}$ of length 32, which is No.50 in the list taken from Sloane's homepage(http://www.research.att.com/ njas/). Direct computation gives

$$W_{C_{32,50}} = \frac{1}{42} W_{e_8}^4 + \frac{41}{42} W_{e_8} W_{g_{24}}.$$

Therefore $\mathcal{R}$ must contain $\frac{1}{2}, \frac{1}{3}, \frac{1}{7}$.

Conversely, suppose that $\mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{7}] \subseteq \mathcal{R}$. The inclusion $\mathcal{R}[W_C] \supseteq \mathcal{R}[W_{e_8}, W_{g_{24}}]$ is trivial and we show the converse. Let C be any self-dual doubly-even code of any length k. Then $Th(W_C)$ is in $A(\Gamma_1)_{\frac{k}{2}}$. As we noted above, $Th(W_C)$ can be expressed as in the form

$$Th(W_C) = W_C(\theta_{00}(2\tau), \theta_{10}(2\tau)) = \sum c_{ab} E_4^a \Delta^b, \quad \text{for some} \quad c_{ab} \in \mathbf{Z}.$$

Since

$$E_4(\tau) = Th(W_{e_8}), \qquad \Delta(\tau) = \frac{1}{2^5 3 \cdot 7} \left(Th(W_{e_8})^3 - Th(W_{g_{24}}) \right),$$

we get

$$\begin{aligned} Th(W_C) &= \sum c_{ab} E_4^a \Delta^b \\ &= \sum c_{ab} Th(W_{e_8})^a \left\{ \frac{1}{2^5 3 \cdot 7} \left(Th(W_{e_8})^3 - Th(W_{g_{24}}) \right) \right\}^b \\ &= \sum \widetilde{c_{a'b'}} Th(W_{e_8})^{a'} Th(W_{g_{24}})^{b'}, \end{aligned}$$

in which $\widetilde{c_{a'b'}}$'s are elements of $\mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{7}]$. Therefore W_C is contained in $\mathcal{R}[W_{e_8}, W_{g_{24}}]$. This completes a proof of the case when $C_{24,7} \cong g_{24}$.

For other cases in Proposition, the similar method can be applied and so we omit the detailed proof. □

3. The case when $n = 2$

In this section, we shall discuss the case when $n = 2$ (and may omit $n = 2$ in the notation of the weight enumerator). Our starting point is the following equality given in [2]:

$$\mathbf{C}[W_C] = \mathbf{C}[W_{e_8}, W_{g_{24}}, W_{d_{24}^+}, W_{d_{32}^+}, W_{d_{40}^+}],$$

where

$$
\begin{aligned}
W_{g_{24}} &= (24) + 759(16,8) + 2576(12,12) + 212520(12,4,4,4) \\
&\quad + 340032(10,6,6,2) + 22770(8,8,8) + 1275120(8,8,4,4) \\
&\quad + 4080384(6,6,6,6),
\end{aligned}
$$

$$
W_{d_k^+} = \frac{1}{2^2} \sum_{\beta,\gamma\in\mathbf{F}_2^2} \left(\sum_{\alpha\in\mathbf{F}_2^2} (-1)^{\alpha\cdot\beta} x_{\alpha+\gamma} x_\alpha \right)^{\frac{k}{2}}, \ k = 8, 24, 32, 40
$$

with the usual inner product $\cdot$ of $\mathbf{F}_2^2$. Here we write e_8 instead of d_8^+ and use the convention $(*,*,\ldots)$ to express the symmetric polynomials, such as $(24) = x_{00}^{24}+x_{01}^{24}+x_{10}^{24}+x_{11}^{24}$, $(8,8,8) = x_{00}^8x_{01}^8x_{10}^8+x_{00}^8x_{01}^8x_{11}^8+x_{00}^8x_{10}^8x_{11}^8+x_{01}^8x_{10}^8x_{11}^8$, etc. In [2] it was shown that $W_{e_8}, W_{g_{24}}, W_{d_{24}^+}, W_{d_{40}^+}$ are algebraically independent over $\mathbf{C}$ and there exists a unique relation, which is explicitly given in [8]:

$$
\begin{aligned}
W_{d_{32}^+}^2 &= -113\cdot 32621\cdot 3^{-4}5^{-1}7^{-2}41^{-1}W_{e_8}^8 \\
&\quad - 2^8 60289\cdot 3^{-4}5^{-1}7^{-2}11^{-1}41^{-1}W_{e_8}^5W_{g_{24}} \\
&\quad + 2^4 821477\cdot 3^{-4}5^{-1}7^{-1}11^{-1}41^{-1}W_{e_8}^5W_{d_{24}^+} \\
&\quad + 2\cdot 751\cdot 3^{-2}7^{-1}41^{-1}W_{e_8}^4W_{d_{32}^+} \\
&\quad - 2^9 11^2\cdot 3^{-3}5^{-1}7^{-1}41^{-1}W_{e_8}^3W_{d_{40}^+} \\
&\quad + 2^{14}163\cdot 3^{-4}7^{-2}11^{-2}41^{-1}W_{e_8}^2W_{g_{24}}^2 \\
&\quad + 2^{11}73\cdot 79\cdot 3^{-4}7^{-1}11^{-2}41^{-1}W_{e_8}^2W_{g_{24}}W_{d_{24}^+} \\
&\quad - 2^6 107\cdot 499\cdot 3^{-4}11^{-2}41^{-1}W_{e_8}^2W_{d_{24}^+}^2 \\
&\quad - 2^8 389\cdot 3^{-2}7^{-1}11^{-1}41^{-1}W_{e_8}W_{g_{24}}W_{d_{32}^+} \\
&\quad + 2^4 5\cdot 197\cdot 3^{-2}11^{-1}41^{-1}W_{e_8}W_{d_{24}^+}W_{d_{32}^+} \\
&\quad + 2^{12}3^{-1}5^{-1}7^{-1}41^{-1}W_{g_{24}}W_{d_{40}^+} \\
&\quad + 2^9 3^{-1}5^{-1}41^{-1}W_{d_{24}^+}W_{d_{40}^+}.
\end{aligned} \tag{3.1}
$$

So, finally we state the main result;

Theorem 3.1. *Let $\mathcal{R}$ be a ring such that $\mathbf{Z}\subseteq\mathcal{R}\subseteq\mathbf{C}$. Then we have*

$$
\mathcal{R}[W_C] = \mathcal{R}[W_{e_8}, W_{g_{24}}, W_{d_{24}^+}, W_{d_{32}^+}, W_{d_{40}^+}]
$$

if and only if $\mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{41}] \subseteq \mathcal{R}$.

The proof of this theorem is carried out by the similar method to that of Proposition 2.1. We recall that $A(\Gamma_2)$ is generated by homogeneous elements $\psi_4, \psi_6, \chi_{10}, \chi_{12}, \chi_{35}$ over $\mathbf{C}$, each with the subscript as its weight. The normalization is made as follows(we follow the notation in [4]):

$$\begin{aligned}
\psi_4(\tau) &= 1 + \cdots, \\
\psi_6(\tau) &= 1 + \cdots, \\
\chi_{10}(\tau) &= (q_{11}q_{22} + \cdots)(\pi\tau_{12})^2 + \cdots, \\
\chi_{12}(\tau) &= (q_{11}q_{22} + \cdots) + \cdots, \\
\chi_{35}(\tau) &= (q_{11}^2 q_{22}^2 (q_{11} - q_{22}) + \cdots)(\pi\tau_{12}) + \cdots .
\end{aligned}$$

We put

$$X_4 = \psi_4, \quad X_6 = \psi_6, \quad X_{10} = -2^2\chi_{10}, \quad X_{12} = 2^2 3\chi_{12}, \quad X_{35} = 2^2 i\chi_{35},$$

and

$$\begin{aligned}
Y_{12} &= 2^{-6}3^{-3}(X_4^3 - X_6^2) + 2^4 3^2 X_{12}, \\
X_{16} &= 2^{-2}3^{-1}(X_4X_{12} - X_6X_{10}), \\
X_{18} &= 2^{-2}3^{-1}(X_6X_{12} - X_4^2X_{10}), \\
X_{24} &= 2^{-3}3^{-1}(X_{12}^2 - X_4X_{10}^2), \\
X_{28} &= 2^{-1}3^{-1}(X_4X_{24} - X_{10}X_{18}), \\
X_{30} &= 2^{-1}3^{-1}(X_6X_{24} - X_4X_{10}X_{16}), \\
X_{36} &= 2^{-1}3^{-2}(X_{12}X_{24} - X_{10}^2X_{16}), \\
X_{40} &= 2^{-2}(X_4X_{36} - X_{10}X_{30}), \\
X_{42} &= 2^{-2}3^{-1}(X_{12}X_{30} - X_4X_{10}X_{28}), \\
X_{48} &= 2^{-2}(X_{12}X_{36} - X_{24}^2).
\end{aligned}$$

Igusa [4] showed that *the fifteen elements*

$$X_4, X_6, X_{10}, X_{12}, Y_{12}, X_{16}, X_{18}, X_{24}, X_{28}, X_{30}, X_{35}, X_{36}, X_{40}, X_{42}, X_{48}$$

form a minimal set of generators of $A_{\mathbf{Z}}(\Gamma_2)$ *over* $\mathbf{Z}$. For our purpose, we deduce the following lemma.

Lemma 3.2. *The ring $A_{\mathbf{Z}}(\Gamma_2)^{(4)}$ can be generated over* $\mathbf{Z}$ *by the following thirty elements:*

$$X_4, X_{12}, Y_{12}, X_{16}, X_{24}, X_{28}, X_{36}, X_{40}, X_{48},$$

and

$$\begin{array}{cccccc} X_6^2\ , & X_6X_{10}, & X_6X_{18}\ , & X_6X_{30}\ , & X_6X_{42}\ , & X_6X_{35}^2, \\ & X_{10}^2\ , & X_{10}X_{18}\ , & X_{10}X_{30}\ , & X_{10}X_{42}\ , & X_{10}X_{35}^2, \\ & & X_{18}^2\ , & X_{18}X_{30}\ , & X_{18}X_{42}\ , & X_{18}X_{35}^2 \\ & & & X_{30}^2, & X_{30}X_{42}, & X_{30}X_{35}^2, \\ & & & & X_{42}^2, & X_{42}X_{35}^2, \\ & & & & & X_{35}^4. \end{array}$$

Proof. This is derived from the usual argument on the graded ring. See Chapter III in [3]. □

We notice that the thirty elements in Lemma 3.2 do *not* form a minimal set of generators of $A_{\mathbf{Z}}(\Gamma_2)^{(4)}$, however, it is enough for our purpose. We put

$$\mathcal{Z} = \mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{41}][Th(W_{e_8}), Th(W_{g_{24}}), Th(W_{d_{24}^+}), Th(W_{d_{32}^+}), Th(W_{d_{40}^+})].$$

By the following two lemmas, we shall show that the thirty elements in Lemma 3.2 are in $\mathcal{Z}$.

Lemma 3.3. *If the elements $X_4, X_{12}, X_6^2, X_6X_{10}, X_{10}^2$ are in $\mathcal{Z}$, then the remaining twenty five elements in Lemma 3.2 are also in $\mathcal{Z}$.*

Proof. This is derived from the definition of each element and the formula

$$\begin{aligned} X_{35}^2 &= (-2^2X_4^2X_{16} + Y_{12}^2)X_{36}X_{10} + (-2^6X_4X_{12}^3 + 2^3Y_{12}X_{28})X_{10}^3 \\ &\quad + (2Y_{12}X_{18} + 2^{10}X_{30})X_{10}^4 + 3\cdot 61X_4^2X_{12}X_{10}^5 - 2\cdot 73X_4X_6X_{10}^6 + 2^{10}5^5X_{10}^7, \end{aligned}$$

which was given in [4]. For example, by the assumption that X_4, X_6^2, X_{12} are in $\mathcal{Z}$, we conclude that $Y_{12} = 2^{-6}3^{-3}(X_4^3 - X_6^2) + 2^43^2X_{12}$ is in $\mathcal{Z}$. Since the assertion can be checked directly, we omit the detailed proof. □

Lemma 3.4. *The elements $X_4, X_{12}, X_6^2, X_6X_{10}, X_{10}^2$ are in $\mathcal{Z}$.*

Proof. It is known that Broué-Enguehard map gives rise to the isomorphism $\mathbf{C}[W_{e_8}, h_{12}, F_{20}, W_{g_{24}}, W_{d_{40}^+}] \cong A(\Gamma_2)^{(2)}$, where

$$h_{12} = (12) - 33(8,4) + 330(4,4,4) + 792(6,2,2,2),$$

$$F_{20} = (20) - 19(16,4) - 336(14,2,2,2) - 494(12,8) + 716(12,4,4) + 1038(8,8,4) + 7632(10,6,2,2) + 106848(6,6,6,2) + 129012(8,4,4,4).$$

The relations among the polynomials and Siegel modular forms hold as follows (*cf.* [5]);

$$W_{d_{24}^+} = 11^2 3^{-2} 7^{-1} W_{e_8}^3 + 2 \cdot 3^{-2} h_{12}^2 - 2^3 7^{-1} W_{g_{24}},$$

$$W_{d_{32}^+} = 43 \cdot 53 \cdot 3^{-4} 7^{-1} W_{e_8}^4 + 2^4 5 \cdot 23 \cdot 3^{-5} 11^{-1} W_{e_8} h_{12}^2 - 2^6 43 \cdot 3^{-2} 7^{-1} 11^{-1} W_{e_8} W_{g_{24}} + 2^6 3^{-5} h_{12} F_{20},$$

$$W_{d_{40}^+} = 3 \cdot 19 \cdot 7^{-1} W_{e_8}^5 + 2 \cdot 5 \cdot 7 \cdot 557 \cdot 3^{-7} 11^{-1} W_{e_8}^2 h_{12}^2 - 2^3 5 \cdot 19 \cdot 7^{-1} 11^{-1} W_{e_8}^2 W_{g_{24}} + 2^6 5^2 3^{-7} W_{e_8} h_{12} F_{20} + 2^2 5 \cdot 41 \cdot 3^{-7} F_{20}^2,$$

and

$$Th(W_{e_8}) = \psi_4,$$

$$Th(h_{12}) = \psi_6,$$

$$Th(F_{20}) = \psi_4 \psi_6 + 2^{12} 3^4 \chi_{10},$$

$$Th(W_{g_{24}}) = 11 \cdot 2^{-1} 3^{-2} \psi_4^3 + 7 \cdot 2^{-1} 3^{-2} \psi_6^2 - 2^{10} 3^2 7 \cdot 11 \chi_{12}.$$

So, we have

$$X_4 = Th(W_{e_8}),$$

$$X_{12} = Th(-2^{-10} 3^{-1} 7^{-1} W_{e_8}^3 + 2^{-8} 3^{-1} 7^{-1} 11^{-1} W_{g_{24}} + 2^{-10} 3^{-1} 11^{-1} W_{d_{24}^+}),$$

$$X_6^2 = Th\left(-11^2 2^{-1} 7^{-1} W_{e_8}^3 + 2^2 3^2 7^{-1} W_{g_{24}} + 3^2 2^{-1} W_{d_{24}^+}\right),$$

$$X_6 X_{10} = Th(-5 \cdot 53 \cdot 2^{-16} 3^{-1} 7^{-1} W_{e_8}^4 + 5 \cdot 2^{-9} 3^{-1} 7^{-1} 11^{-1} W_{e_8} W_{g_{24}} + 53 \cdot 2^{-13} 3^{-1} 11^{-1} W_{e_8} W_{d_{24}^+} - 3 \cdot 2^{-16} W_{d_{32}^+}),$$

$$X_{10}^2 = Th(-461 \cdot 2^{-25} 3^{-1} 5^{-1} 7^{-1} 41^{-1} W_{e_8}^5 + 2^{-18} 3^{-1} 7^{-1} 11^{-1} 41^{-1} W_{e_8}^2 W_{g_{24}} + 13 \cdot 2^{-21} 3^{-1} 11^{-1} 41^{-1} W_{e_8}^2 W_{d_{24}^+} - 3 \cdot 2^{-25} 41^{-1} W_{e_8} W_{d_{32}^+} + 2^{-22} 3^{-1} 5^{-1} 41^{-1} W_{d_{40}^+}).$$

This shows Lemma 3.4. □

Proof of **Theorem 3.1**. Suppose that $\mathcal{R}[W_C] = \mathcal{R}[W_{e_8}, W_{g_{24}}, W_{d_{24}^+}, W_{d_{32}^+}, W_{d_{40}^+}]$. Since the weight enumerator $W_{d_{48}^+}$ is uniquely expressed with our fixed generators as

$$\begin{aligned} W_{d_{48}^+} &= 23 \cdot 22229 \cdot 2^{-2}3^{-2}5^{-1}7^{-2}41^{-1}W_{e_8}^6 - 2^5 13 \cdot 23 \cdot 3^{-2}7^{-2}11^{-1}41^{-1}W_{e_8}^3 W_{g_{24}} \\ &\quad + 2 \cdot 23 \cdot 113 \cdot 3^{-2}7^{-1}11^{-1}41^{-1}W_{e_8}^3 W_{d_{24}^+} - 3^2 5 \cdot 23 \cdot 2^{-2}41^{-1}W_{e_8}^2 W_{d_{32}^+} \\ &\quad + 2^4 7 \cdot 23 \cdot 3^{-1}5^{-1}41^{-1}W_{e_8} W_{d_{40}^+} - 2^9 19 \cdot 3^{-2}7^{-2}11^{-2}W_{g_{24}}^2 \\ &\quad + 2^6 23 \cdot 3^{-2}7^{-1}11^{-2}W_{g_{24}} W_{d_{24}^+} + 2 \cdot 23 \cdot 37 \cdot 3^{-2}11^{-2}W_{d_{24}^+}^2, \end{aligned}$$

we see that $\mathcal{R}$ must contain $\mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{41}]$.

Conversely, suppose that $\mathcal{R}$ contains $\mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{41}]$. We have only to show that W_C is in $\mathcal{R}[W_{e_8}, W_{g_{24}}, W_{d_{24}^+}, W_{d_{32}^+}, W_{d_{40}^+}]$ for *any* self-dual doubly-even code C. Take any self-dual doubly-even code C of length k. Then $Th(W_C)$ is in $A_{\mathbf{Z}}(\Gamma_2)_k$, with weight $\frac{k}{2}$ and $k \equiv 0 \pmod 8$. By Lemma 3.2, $Th(W_C)$ is expressed as the polynomial of the thirty elements $X_4, \ldots, X_6^2, \ldots$ over $\mathbf{Z}$, say

$$Th(W_C) = \sum_{a,\cdots,b,\cdots} c_{a\cdots b\cdots} X_4^a \cdots X_6^{2b} \cdots,$$

in which $c_{a\cdots b\cdots}$'s are integers. By Lemmas 3.3 and 3.4, all thirty elements are in $\mathcal{Z}$ and we have

$$Th(W_C) = Th(\sum_{a',b',c',d',e' \in \mathbf{Z}} \widetilde{c_{a'b'c'd'e'}} W_{e_8}^{a'} W_{g_{24}}^{b'} W_{d_{24}^+}^{c'} W_{d_{32}^+}^{d'} W_{d_{40}^+}^{e'}),$$

in which the coefficients $\widetilde{c_{a'b'c'd'e'}}$ are in $\mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{41}]$. Here there is an ambiguity whether or not we replace $W_{d_{32}^+}^2$ by the relation given in (3.1). This causes, however, nothing in our argument since the coefficients of the right-hand side of (3.1) is contained in $\mathcal{R}$. At any rate, W_C is in $\mathcal{R}[W_{e_8}, W_{g_{24}}, W_{d_{24}^+}, W_{d_{32}^+}, W_{d_{40}^+}]$. This completes a proof. □

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