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# REMARK ON THE WEIGHT ENUMERATORS AND SIEGEL MODULAR FORMS

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ABSTRACT. The purpose of this note is to study the coefficients of the polynomials if we express the weight enumerator as the polynomial of the fixed generators.

## 1. Introduction

It is well known that the graded ring  $\mathbf{C}[W_C^{(1)}]$  generated by the weight enumerators of all self-dual doubly-even codes over the complex field  $\mathbf{C}$  can be generated by two elements[1];

$$\mathbf{C}[W_C^{(1)}] = \mathbf{C}[W_{e_8}^{(1)}, W_{g_{24}}^{(1)}].$$

From this, the weight enumerator of every self-dual doubly-even code can be expressed as the polynomial of  $W_{e_8}^{(1)}$  and  $W_{g_{24}}^{(1)}$  over  $\mathbf{C}$ . However, the coefficients of the said polynomial may be in the ring smaller than  $\mathbf{C}$ . Actually, we can replace  $\mathbf{C}$  in the equality above by the smaller ring  $\mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{7}]$ .

We start with the definitions, the notation and the known facts which are needed in this note. Let  $\mathfrak{S}_n$  be the Siegel upper-half space of degree  $n$  and denote by  $A(\Gamma_n)_k$  the ring of modular forms of weight  $k$  on  $\Gamma_n = Sp_{2n}(\mathbf{Z})$  over  $\mathbf{C}$ . If  $f$  is an element of  $A(\Gamma_n)_k$ , then  $f(\tau)$  can be expanded into a Fourier series of the following form:

$$f(\tau) = \sum_{s \geq 0} a_f(s) \exp(2\pi\sqrt{-1}\text{trace}(s\tau)) = \sum_{s_{ii} \geq 0} \left( \sum a_f(s) \cdot \prod_{i < j} q_{ij}^{2s_{ij}} \right) \prod_{i=1}^n q_{ii}^{s_{ii}},$$

in which  $q_{ij} = \exp(2\pi\sqrt{-1}\tau_{ij})$  and  $s$  runs over the set of half-integral positive (semi-definite) matrices of degree  $n$ . For any subring  $R$  of  $\mathbf{C}$  we denote by

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$A_R(\Gamma_n)_k$  the  $R$ -module consisting of those  $f \in A(\Gamma_n)_k$  such that  $a_f(s)$  is in  $R$  for every  $s$  and by  $A_R(\Gamma_n) := \bigoplus_{k \geq 0} A_R(\Gamma_n)_k$  taken in  $A(\Gamma_n) := \bigoplus_{k \geq 0} A(\Gamma_n)_k$ ; then  $A_R(\Gamma_n)$  forms a graded integral ring over  $R$ . The explicit structure of the ring  $A_{\mathbf{Z}}(\Gamma_n)$  is known only for  $n = 1, 2$  and we shall use them later.

Let  $m', m''$  denote elements of  $\mathbf{F}_2^n$  and put  $\mathbf{m} = (m' \ m'')$ ; then the theta constants with characteristic  $\mathbf{m}$  is defined as

$$\theta_{\mathbf{m}}(\tau) = \sum_{p \in \mathbf{Z}^n} \exp 2\pi \sqrt{-1} \left\{ \frac{1}{2}(p + \frac{1}{2}m')\tau \ ^t(p + \frac{1}{2}m') + \frac{1}{2}(p + \frac{1}{2}m')m'' \right\}.$$

Let  $C$  be a (linear) code of length  $k$  over  $\mathbf{F}_2$ . The weight enumerator  $W_C^{(n)}(x_a : a \in \mathbf{F}_2^n)$  of degree  $n$  is defined as

$$W_C^{(n)} = W_C^{(n)}(x_a : a \in \mathbf{F}_2^n) = \sum_{v_1, \dots, v_k \in C} \prod_{a \in \mathbf{F}_2^n} x_a^{n_a(v_1, \dots, v_k)},$$

where  $n_a(v_1, \dots, v_k)$  denotes the number of  $i$  such that  $a = (v_{1i}, \dots, v_{ki})$ . We note that  $W_C^{(n)}$  is a homogeneous polynomial of degree  $k$  with non-negative integers as its coefficients. For any subring  $R$  of  $\mathbf{C}$  we denote by  $R[W_C^{(n)}]$  the graded ring generated by the weight enumerators of degree  $n$  of all self-dual doubly-even codes of any length over  $R$ . It is known that the Broué-Enguehard map  $Th : x_a \mapsto \theta_{a0}(2\tau), a \in \mathbf{F}_2^n$ , gives the  $\mathbf{C}$ -algebra homomorphism from  $\mathbf{C}[W_C^{(n)}]$  to  $A(\Gamma_n)^{(4)} = \bigoplus_{k \geq 0, k \equiv 0 \pmod{4}} A(\Gamma_n)_k$ . In particular, it gives the isomorphisms  $\mathbf{C}[W_C^{(n)}] \cong A(\Gamma_n)^{(4)}$  when  $n = 1, 2$  (see [6]). In the next section we explain our problem dealing with the case when  $n = 1$ . The main theme of this note is to investigate this in the case when  $n = 2$ .

## 2. The case when $n = 1$

In this section, we discuss the case when  $n = 1$  (and may omit  $n = 1$  in the notation of the weight enumerator for the sake of simplicity). Before proving the assertion described in the introduction, we modify our setting. We started from the fact (see [1]), called *Gleason Theorem*, that  $\mathbf{C}[W_C]$  is generated by  $W_{e_8}$  and  $W_{g_{24}}$  over  $\mathbf{C}$ , where

$$\begin{aligned} W_{e_8} &= x_0^8 + 14x_0^4x_1^4 + x_1^8, \\ W_{g_{24}} &= x_0^{24} + 759x_0^{16}x_1^8 + 2576x_0^{12}x_1^{12} + 759x_0^8x_1^{16} + x_1^{24}. \end{aligned}$$

The self-dual doubly-even code of length 8 is unique (up to isomorphism), however, we may take another self-dual doubly-even code of length 24 instead of  $g_{24}$ . There exist 7 indecomposable self-dual doubly-even codes of length 24(see [7]):

$$d_{12}^2, d_{10}e_7^2, d_8^3, d_6^4, d_{24}, d_4^6, g_{24}.$$

We call them  $C_{24,1}, \dots, C_{24,7}$ . The following table gives the values  $a, b$ , if we write

$$W_{C_{24,i}} = aW_{e_8}^3 + bW_{g_{24}}, i = 1, 2, \dots, 7.$$

	$a$	$b$
$C_{24,1}$	$5/7$	$2/7$
$C_{24,2}$	$4/7$	$3/7$
$C_{24,3}$	$3/7$	$4/7$
$C_{24,4}$	$2/7$	$5/7$
$C_{24,5}$	$11/7$	$-4/7$
$C_{24,6}$	$1/7$	$6/7$
$C_{24,7}$	$1$	$0$

We state the following proposition.

**Proposition 2.1.** *Let  $\mathcal{R}$  be a ring such that  $\mathbf{Z} \subseteq \mathcal{R} \subseteq \mathbf{C}$ . Then we have*

$$\mathcal{R}[W_C] = \mathcal{R}[W_{e_8}, W_{C_{24,4}}] \text{ if and only if } \mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}] \subseteq \mathcal{R},$$

$$\mathcal{R}[W_C] = \mathcal{R}[W_{e_8}, W_{C_{24,7}}] \text{ if and only if } \mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{7}] \subseteq \mathcal{R},$$

and for  $i = 1, 2, 3, 5, 6$ ,

$$\mathcal{R}[W_C] = \mathcal{R}[W_{e_8}, W_{C_{24,i}}] \text{ if and only if } \mathbf{Z}[\frac{1}{2}, \frac{1}{3}] \subseteq \mathcal{R}.$$

Before proceeding to the proof, we recall the modular forms for  $\Gamma_1$  over  $\mathbf{Z}$ . If we denote by  $E_k$  the Eisenstein series of even weight  $k$  normalized as  $E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n$ ,  $q = e^{2\pi i\tau}$ , and if we put  $\Delta = 2^{-6}3^{-3}(E_4^3 - E_6^2)$ , then it is well known that

$$A_{\mathbf{Z}}(\Gamma_1) = \mathbf{Z}[E_4, E_6, \Delta].$$

Moreover we have

$$A_{\mathbf{Z}}(\Gamma_1)^{(4)} = \mathbf{Z}[E_4, \Delta].$$

**Proof of Proposition 2.1** First we consider the case when  $C_{24,7} \cong g_{24}$ . Suppose that we have the equality  $\mathcal{R}[W_C] = \mathcal{R}[W_{e_8}, W_{g_{24}}]$ . We pick the self-dual doubly-even code  $C_{32,50}$  of length 32, which is No.50 in the list taken from Sloane's homepage (<http://www.research.att.com/njas/>). Direct computation gives

$$W_{C_{32,50}} = \frac{1}{42}W_{e_8}^4 + \frac{41}{42}W_{e_8}W_{g_{24}}.$$

Therefore  $\mathcal{R}$  must contain  $\frac{1}{2}, \frac{1}{3}, \frac{1}{7}$ .

Conversely, suppose that  $\mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{7}] \subseteq \mathcal{R}$ . The inclusion  $\mathcal{R}[W_C] \supseteq \mathcal{R}[W_{e_8}, W_{g_{24}}]$  is trivial and we show the converse. Let  $C$  be any self-dual doubly-even code of any length  $k$ . Then  $Th(W_C)$  is in  $A(\Gamma_1)_{\frac{k}{2}}$ . As we noted above,  $Th(W_C)$  can be expressed as in the form

$$Th(W_C) = W_C(\theta_{00}(2\tau), \theta_{10}(2\tau)) = \sum c_{ab}E_4^a\Delta^b, \text{ for some } c_{ab} \in \mathbf{Z}.$$

Since

$$E_4(\tau) = Th(W_{e_8}), \quad \Delta(\tau) = \frac{1}{2^5 3 \cdot 7} (Th(W_{e_8})^3 - Th(W_{g_{24}})),$$

we get

$$\begin{aligned} Th(W_C) &= \sum c_{ab}E_4^a\Delta^b \\ &= \sum c_{ab}Th(W_{e_8})^a \left\{ \frac{1}{2^5 3 \cdot 7} (Th(W_{e_8})^3 - Th(W_{g_{24}})) \right\}^b \\ &= \sum \widetilde{c_{a'b'}} Th(W_{e_8})^{a'} Th(W_{g_{24}})^{b'}, \end{aligned}$$

in which  $\widetilde{c_{a'b'}}$ 's are elements of  $\mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{7}]$ . Therefore  $W_C$  is contained in  $\mathcal{R}[W_{e_8}, W_{g_{24}}]$ . This completes a proof of the case when  $C_{24,7} \cong g_{24}$ .

For other cases in Proposition, the similar method can be applied and so we omit the detailed proof.  $\square$

### 3. The case when $n = 2$

In this section, we shall discuss the case when  $n = 2$  (and may omit  $n = 2$  in the notation of the weight enumerator). Our starting point is the following equality given in [2]:

$$\mathbf{C}[W_C] = \mathbf{C}[W_{e_8}, W_{g_{24}}, W_{d_{24}^+}, W_{d_{32}^+}, W_{d_{40}^+}],$$

where

$$\begin{aligned} W_{g_{24}} &= (24) + 759(16, 8) + 2576(12, 12) + 212520(12, 4, 4, 4) \\ &\quad + 340032(10, 6, 6, 2) + 22770(8, 8, 8) + 1275120(8, 8, 4, 4) \\ &\quad + 4080384(6, 6, 6, 6), \end{aligned}$$

$$W_{d_k^+} = \frac{1}{2^2} \sum_{\beta, \gamma \in \mathbf{F}_2^2} \left( \sum_{\alpha \in \mathbf{F}_2^2} (-1)^{\alpha \cdot \beta} x_{\alpha + \gamma} x_\alpha \right)^{\frac{k}{2}}, \quad k = 8, 24, 32, 40$$

with the usual inner product  $\cdot$  of  $\mathbf{F}_2^2$ . Here we write  $e_8$  instead of  $d_8^+$  and use the convention  $(*, *, \dots)$  to express the symmetric polynomials, such as  $(24) = x_{00}^{24} + x_{01}^{24} + x_{10}^{24} + x_{11}^{24}$ ,  $(8, 8, 8) = x_{00}^8 x_{01}^8 x_{10}^8 + x_{00}^8 x_{01}^8 x_{11}^8 + x_{00}^8 x_{10}^8 x_{11}^8 + x_{01}^8 x_{10}^8 x_{11}^8$ , etc. In [2] it was shown that  $W_{e_8}, W_{g_{24}}, W_{d_{24}^+}, W_{d_{40}^+}$  are algebraically independent over  $\mathbf{C}$  and there exists a unique relation, which is explicitly given in [8]:

$$\begin{aligned} (3.1) \quad W_{d_{32}^+}^2 &= -113 \cdot 32621 \cdot 3^{-4} 5^{-1} 7^{-2} 41^{-1} W_{e_8}^8 \\ &\quad - 2^8 60289 \cdot 3^{-4} 5^{-1} 7^{-2} 11^{-1} 41^{-1} W_{e_8}^5 W_{g_{24}} \\ &\quad + 2^4 821477 \cdot 3^{-4} 5^{-1} 7^{-1} 11^{-1} 41^{-1} W_{e_8}^5 W_{d_{24}^+} \\ &\quad + 2 \cdot 751 \cdot 3^{-2} 7^{-1} 41^{-1} W_{e_8}^4 W_{d_{32}^+} \\ &\quad - 2^9 11^2 \cdot 3^{-3} 5^{-1} 7^{-1} 41^{-1} W_{e_8}^3 W_{d_{40}^+} \\ &\quad + 2^{14} 163 \cdot 3^{-4} 7^{-2} 11^{-2} 41^{-1} W_{e_8}^2 W_{g_{24}}^2 \\ &\quad + 2^{11} 73 \cdot 79 \cdot 3^{-4} 7^{-1} 11^{-2} 41^{-1} W_{e_8}^2 W_{g_{24}} W_{d_{24}^+} \\ &\quad - 2^6 107 \cdot 499 \cdot 3^{-4} 11^{-2} 41^{-1} W_{e_8}^2 W_{d_{24}^+}^2 \\ &\quad - 2^8 389 \cdot 3^{-2} 7^{-1} 11^{-1} 41^{-1} W_{e_8} W_{g_{24}} W_{d_{32}^+} \\ &\quad + 2^4 5 \cdot 197 \cdot 3^{-2} 11^{-1} 41^{-1} W_{e_8} W_{d_{24}^+} W_{d_{32}^+} \\ &\quad + 2^{12} 3^{-1} 5^{-1} 7^{-1} 41^{-1} W_{g_{24}} W_{d_{40}^+} \\ &\quad + 2^9 3^{-1} 5^{-1} 41^{-1} W_{d_{24}^+} W_{d_{40}^+}. \end{aligned}$$

So, finally we state the main result;

**Theorem 3.1.** *Let  $\mathcal{R}$  be a ring such that  $\mathbf{Z} \subseteq \mathcal{R} \subseteq \mathbf{C}$ . Then we have*

$$\mathcal{R}[W_{\mathcal{C}}] = \mathcal{R}[W_{e_8}, W_{g_{24}}, W_{d_{24}^+}, W_{d_{32}^+}, W_{d_{40}^+}]$$

if and only if  $\mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{41}] \subseteq \mathcal{R}$ .

The proof of this theorem is carried out by the similar method to that of Proposition 2.1. We recall that  $A(\Gamma_2)$  is generated by homogeneous elements  $\psi_4, \psi_6, \chi_{10}, \chi_{12}, \chi_{35}$  over  $\mathbf{C}$ , each with the subscript as its weight. The normalization is made as follows (we follow the notation in [4]):

$$\begin{aligned}\psi_4(\tau) &= 1 + \cdots, \\ \psi_6(\tau) &= 1 + \cdots, \\ \chi_{10}(\tau) &= (q_{11}q_{22} + \cdots)(\pi\tau_{12})^2 + \cdots, \\ \chi_{12}(\tau) &= (q_{11}q_{22} + \cdots) + \cdots, \\ \chi_{35}(\tau) &= (q_{11}^2q_{22}^2(q_{11} - q_{22}) + \cdots)(\pi\tau_{12}) + \cdots.\end{aligned}$$

We put

$$X_4 = \psi_4, \quad X_6 = \psi_6, \quad X_{10} = -2^2\chi_{10}, \quad X_{12} = 2^2\chi_{12}, \quad X_{35} = 2^2i\chi_{35},$$

and

$$\begin{aligned}Y_{12} &= 2^{-6}3^{-3}(X_4^3 - X_6^2) + 2^43^2X_{12}, \\ X_{16} &= 2^{-2}3^{-1}(X_4X_{12} - X_6X_{10}), \\ X_{18} &= 2^{-2}3^{-1}(X_6X_{12} - X_4^2X_{10}), \\ X_{24} &= 2^{-3}3^{-1}(X_{12}^2 - X_4X_{10}^2), \\ X_{28} &= 2^{-1}3^{-1}(X_4X_{24} - X_{10}X_{18}), \\ X_{30} &= 2^{-1}3^{-1}(X_6X_{24} - X_4X_{10}X_{16}), \\ X_{36} &= 2^{-1}3^{-2}(X_{12}X_{24} - X_{10}^2X_{16}), \\ X_{40} &= 2^{-2}(X_4X_{36} - X_{10}X_{30}), \\ X_{42} &= 2^{-2}3^{-1}(X_{12}X_{30} - X_4X_{10}X_{28}), \\ X_{48} &= 2^{-2}(X_{12}X_{36} - X_{24}^2).\end{aligned}$$

Igusa [4] showed that *the fifteen elements*

$$X_4, X_6, X_{10}, X_{12}, Y_{12}, X_{16}, X_{18}, X_{24}, X_{28}, X_{30}, X_{35}, X_{36}, X_{40}, X_{42}, X_{48}$$

*form a minimal set of generators of  $A_{\mathbf{Z}}(\Gamma_2)$  over  $\mathbf{Z}$ .* For our purpose, we deduce the following lemma.

**Lemma 3.2.** *The ring  $A_{\mathbf{Z}}(\Gamma_2)^{(4)}$  can be generated over  $\mathbf{Z}$  by the following thirty elements:*

$$X_4, X_{12}, Y_{12}, X_{16}, X_{24}, X_{28}, X_{36}, X_{40}, X_{48},$$

and

$$\begin{array}{cccccc} X_6^2, & X_6 X_{10}, & X_6 X_{18}, & X_6 X_{30}, & X_6 X_{42}, & X_6 X_{35}^2, \\ & X_{10}^2, & X_{10} X_{18}, & X_{10} X_{30}, & X_{10} X_{42}, & X_{10} X_{35}^2, \\ & & X_{18}^2, & X_{18} X_{30}, & X_{18} X_{42}, & X_{18} X_{35}^2, \\ & & & X_{30}^2, & X_{30} X_{42}, & X_{30} X_{35}^2, \\ & & & & X_{42}^2, & X_{42} X_{35}^2, \\ & & & & & X_{35}^4. \end{array}$$

**Proof.** This is derived from the usual argument on the graded ring. See Chapter III in [3].  $\square$

We notice that the thirty elements in Lemma 3.2 do *not* form a minimal set of generators of  $A_{\mathbf{Z}}(\Gamma_2)^{(4)}$ , however, it is enough for our purpose. We put

$$\mathcal{Z} = \mathbf{Z}\left[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{41}\right][Th(W_{e_8}), Th(W_{g_{24}}), Th(W_{d_{24}^+}), Th(W_{d_{32}^+}), Th(W_{d_{40}^+})].$$

By the following two lemmas, we shall show that the thirty elements in Lemma 3.2 are in  $\mathcal{Z}$ .

**Lemma 3.3.** *If the elements  $X_4, X_{12}, X_6^2, X_6 X_{10}, X_{10}^2$  are in  $\mathcal{Z}$ , then the remaining twenty five elements in Lemma 3.2 are also in  $\mathcal{Z}$ .*

**Proof.** This is derived from the definition of each element and the formula

$$\begin{aligned} X_{35}^2 &= (-2^2 X_4^2 X_{16} + Y_{12}^2) X_{36} X_{10} + (-2^6 X_4 X_{12}^3 + 2^3 Y_{12} X_{28}) X_{10}^3 \\ &\quad + (2 Y_{12} X_{18} + 2^{10} X_{30}) X_{10}^4 + 3 \cdot 61 X_4^2 X_{12} X_{10}^5 - 2 \cdot 73 X_4 X_6 X_{10}^6 + 2^{10} 5^5 X_{10}^7, \end{aligned}$$

which was given in [4]. For example, by the assumption that  $X_4, X_6^2, X_{12}$  are in  $\mathcal{Z}$ , we conclude that  $Y_{12} = 2^{-6} 3^{-3} (X_4^3 - X_6^2) + 2^4 3^2 X_{12}$  is in  $\mathcal{Z}$ . Since the assertion can be checked directly, we omit the detailed proof.  $\square$

**Lemma 3.4.** *The elements  $X_4, X_{12}, X_6^2, X_6 X_{10}, X_{10}^2$  are in  $\mathcal{Z}$ .*



**Proof.** It is known that Broué-Enguehard map gives rise to the isomorphism  $\mathbf{C}[W_{e_8}, h_{12}, F_{20}, W_{g_{24}}, W_{d_{40}^+}] \cong A(\Gamma_2)^{(2)}$ , where

$$h_{12} = (12) - 33(8, 4) + 330(4, 4, 4) + 792(6, 2, 2, 2),$$

$$F_{20} = (20) - 19(16, 4) - 336(14, 2, 2, 2) - 494(12, 8) + 716(12, 4, 4)$$

$$+ 1038(8, 8, 4) + 7632(10, 6, 2, 2) + 106848(6, 6, 6, 2) + 129012(8, 4, 4, 4).$$

The relations among the polynomials and Siegel modular forms hold as follows (*cf.* [5]);

$$W_{d_{24}^+} = 11^2 3^{-2} 7^{-1} W_{e_8}^3 + 2 \cdot 3^{-2} h_{12}^2 - 2^3 7^{-1} W_{g_{24}},$$

$$W_{d_{32}^+} = 43 \cdot 53 \cdot 3^{-4} 7^{-1} W_{e_8}^4 + 2^4 5 \cdot 23 \cdot 3^{-5} 11^{-1} W_{e_8} h_{12}^2 \\ - 2^6 43 \cdot 3^{-2} 7^{-1} 11^{-1} W_{e_8} W_{g_{24}} + 2^6 3^{-5} h_{12} F_{20},$$

$$W_{d_{40}^+} = 3 \cdot 19 \cdot 7^{-1} W_{e_8}^5 + 2 \cdot 5 \cdot 7 \cdot 557 \cdot 3^{-7} 11^{-1} W_{e_8}^2 h_{12}^2 \\ - 2^3 5 \cdot 19 \cdot 7^{-1} 11^{-1} W_{e_8}^2 W_{g_{24}} + 2^6 5^2 3^{-7} W_{e_8} h_{12} F_{20} + 2^2 5 \cdot 41 \cdot 3^{-7} F_{20}^2,$$

and

$$Th(W_{e_8}) = \psi_4,$$

$$Th(h_{12}) = \psi_6,$$

$$Th(F_{20}) = \psi_4 \psi_6 + 2^{12} 3^4 \chi_{10},$$

$$Th(W_{g_{24}}) = 11 \cdot 2^{-1} 3^{-2} \psi_4^3 + 7 \cdot 2^{-1} 3^{-2} \psi_6^2 - 2^{10} 3^2 7 \cdot 11 \chi_{12}.$$

So, we have

$$X_4 = Th(W_{e_8}),$$

$$X_{12} = Th(-2^{-10} 3^{-1} 7^{-1} W_{e_8}^3 + 2^{-8} 3^{-1} 7^{-1} 11^{-1} W_{g_{24}} + 2^{-10} 3^{-1} 11^{-1} W_{d_{24}^+}),$$

$$X_6^2 = Th\left(-11^2 2^{-1} 7^{-1} W_{e_8}^3 + 2^2 3^2 7^{-1} W_{g_{24}} + 3^2 2^{-1} W_{d_{24}^+}\right),$$

$$X_6 X_{10} = Th(-5 \cdot 53 \cdot 2^{-16} 3^{-1} 7^{-1} W_{e_8}^4 + 5 \cdot 2^{-9} 3^{-1} 7^{-1} 11^{-1} W_{e_8} W_{g_{24}} \\ + 53 \cdot 2^{-13} 3^{-1} 11^{-1} W_{e_8} W_{d_{24}^+} - 3 \cdot 2^{-16} W_{d_{32}^+}),$$

$$X_{10}^2 = Th(-461 \cdot 2^{-25} 3^{-1} 5^{-1} 7^{-1} 41^{-1} W_{e_8}^5 + 2^{-18} 3^{-1} 7^{-1} 11^{-1} 41^{-1} W_{e_8}^2 W_{g_{24}} \\ + 13 \cdot 2^{-21} 3^{-1} 11^{-1} 41^{-1} W_{e_8}^2 W_{d_{24}^+} - 3 \cdot 2^{-25} 41^{-1} W_{e_8} W_{d_{32}^+} \\ + 2^{-22} 3^{-1} 5^{-1} 41^{-1} W_{d_{40}^+}).$$

This shows Lemma 3.4.  $\square$

**Proof of Theorem 3.1.** Suppose that  $\mathcal{R}[W_C] = \mathcal{R}[W_{e_8}, W_{g_{24}}, W_{d_{24}^+}, W_{d_{32}^+}, W_{d_{40}^+}]$ . Since the weight enumerator  $W_{d_{48}^+}$  is uniquely expressed with our fixed generators as

$$\begin{aligned} W_{d_{48}^+} &= 23 \cdot 22229 \cdot 2^{-2} 3^{-2} 5^{-1} 7^{-2} 41^{-1} W_{e_8}^6 - 2^5 13 \cdot 23 \cdot 3^{-2} 7^{-2} 11^{-1} 41^{-1} W_{e_8}^3 W_{g_{24}}^3 \\ &\quad + 2 \cdot 23 \cdot 113 \cdot 3^{-2} 7^{-1} 11^{-1} 41^{-1} W_{e_8}^3 W_{d_{24}^+} - 3^2 5 \cdot 23 \cdot 2^{-2} 41^{-1} W_{e_8}^2 W_{d_{32}^+} \\ &\quad + 2^4 7 \cdot 23 \cdot 3^{-1} 5^{-1} 41^{-1} W_{e_8} W_{d_{40}^+} - 2^9 19 \cdot 3^{-2} 7^{-2} 11^{-2} W_{g_{24}}^2 \\ &\quad + 2^6 23 \cdot 3^{-2} 7^{-1} 11^{-2} W_{g_{24}} W_{d_{24}^+} + 2 \cdot 23 \cdot 37 \cdot 3^{-2} 11^{-2} W_{d_{24}^+}^2, \end{aligned}$$

we see that  $\mathcal{R}$  must contain  $\mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{41}]$ .

Conversely, suppose that  $\mathcal{R}$  contains  $\mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{41}]$ . We have only to show that  $W_C$  is in  $\mathcal{R}[W_{e_8}, W_{g_{24}}, W_{d_{24}^+}, W_{d_{32}^+}, W_{d_{40}^+}]$  for *any* self-dual doubly-even code  $C$ . Take any self-dual doubly-even code  $C$  of length  $k$ . Then  $Th(W_C)$  is in  $A_{\mathbf{Z}}(\Gamma_2)_k$ , with weight  $\frac{k}{2}$  and  $k \equiv 0 \pmod{8}$ . By Lemma 3.2,  $Th(W_C)$  is expressed as the polynomial of the thirty elements  $X_4, \dots, X_6^2, \dots$  over  $\mathbf{Z}$ , say

$$Th(W_C) = \sum_{a, \dots, b, \dots} c_{a \dots b \dots} X_4^a \cdots X_6^{2b} \cdots,$$

in which  $c_{a \dots b \dots}$ 's are integers. By Lemmas 3.3 and 3.4, all thirty elements are in  $\mathcal{Z}$  and we have

$$Th(W_C) = Th\left( \sum_{a', b', c', d', e' \in \mathbf{Z}} \widetilde{c_{a' b' c' d' e'}} W_{e_8}^{a'} W_{g_{24}}^{b'} W_{d_{24}^+}^{c'} W_{d_{32}^+}^{d'} W_{d_{40}^+}^{e'} \right),$$

in which the coefficients  $\widetilde{c_{a' b' c' d' e'}}$  are in  $\mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{41}]$ . Here there is an ambiguity whether or not we replace  $W_{d_{32}^+}^2$  by the relation given in (3.1). This causes, however, nothing in our argument since the coefficients of the right-hand side of (3.1) is contained in  $\mathcal{R}$ . At any rate,  $W_C$  is in  $\mathcal{R}[W_{e_8}, W_{g_{24}}, W_{d_{24}^+}, W_{d_{32}^+}, W_{d_{40}^+}]$ . This completes a proof.  $\square$

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