EXISTENCE RESULTS FOR VISCOUS POLYTROPIC FLUIDS WITH VACUUM

YONGGEUN CHO AND HYUNSEOK KIM

Abstract. We consider the full Navier-Stokes equations for viscous polytropic fluids with nonnegative thermal conductivity. We prove the existence of unique local strong solutions for all initial data satisfying some compatibility condition. The initial density need not be positive and may vanish in an open set. Moreover our results hold for both bounded and unbounded domains.

1. Introduction

By ideal *vacuums* in a compressible or incompressible fluid, we mean spatial domains occupied by the fluid where the (mass) density vanishes. In the absence of vacuum, lots of results have been obtained for viscous heat-conducting compressible fluids since the uniqueness result by J. Serrin [23] and the local existence results by J. Nash [21] and N. Itaya [13]. We refer the readers to the papers [5, 6, 12, 18, 19, 20, 28, 29, 32] for some local or global results. The crucial observation for these results is that the energy and momentum conservation equations are parabolic when the density is assumed to be a known positive scalar field. But they lose the parabolicity at the presence of vacuum.

This difficulty was overcome only for simpler fluid models without considering the energy conservation equation. For nonhomogeneous incompressible fluids, the existence of global weak solutions and local strong solutions was proved first by J. Simon [24, 25] and by H.J. Choe and the second author [4], respectively. See also the works [2, 8, 14, 16, 27] for some related results and extensions. Then similar results have been proved for isentropic or barotropic compressible fluids. For details, refer to [1, 3, 7, 10, 15, 17, 22]. But straightforward adaptations of the previous methods to heat-conducting fluids have failed because the energy, velocity and pressure fields are strongly coupled with each other especially when the density is not bounded away from zero.

2000 *Mathematics Subject Classification.* 35Q30, 76N10.

Key words and phrases. viscous polytropic fluids, compressible Navier-Stokes equations, vacuum.

The authors were supported by Japan Society for the Promotion of Science under JSPS Postdoctoral Fellowship For Foreign Researchers.

In this paper, we develop a local existence theory for viscous polytropic fluids with vacuum while the incompressible one will be discussed in a forthcoming paper.

A *viscous polytropic fluid* is a viscous compressible fluid obeying both Joule's and Boyle's laws, and governed by the following system of equations

$$\rho_t + \operatorname{div}(\rho u) = 0, \qquad p = (\gamma - 1)\rho e, \tag{1.1}$$

$$(\rho e)_t + \operatorname{div}(\rho e u) - \kappa \Delta e + p \operatorname{div} u = \frac{\mu}{2}|\nabla u + \nabla u^T|^2 + \lambda (\operatorname{div} u)^2 + \rho h, \tag{1.2}$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu)\nabla \operatorname{div} u + \nabla p = \rho f \tag{1.3}$$

in $(0,T) \times \Omega$. Here we denote by ρ, e, p and u the unknown density, specific internal energy, pressure and velocity fields for the fluid, respectively. The known constants μ, λ are viscosity coefficients, γ is the ratio of specific heats and κ is the thermal conductivity coefficient divided by the specific heat at constant volume. In view of viscosity and classical thermodynamics (see the book [16] by P.L. Lions), these constants are required to satisfy the natural restriction

$$\mu > 0, \qquad 3\lambda + 2\mu \geq 0, \qquad \kappa \geq 0 \quad \text{and} \quad \gamma > 1.$$

The known fields h and f denote a heat source and external force per unit mass. Finally, $(0,T) \times \Omega$ is the time-space domain for the evolution of the fluid, where T is a finite positive number and Ω is either a bounded domain in $\mathbf{R}^3$ with smooth boundary or a usual unbounded domain such as the whole space $\mathbf{R}^3$, the half space $\mathbf{R}^3_+$ and an exterior domain with smooth boundary.

As was already suggested in [1, 3], the standard homogeneous and inhomogeneous Sobolev spaces are natural function spaces for our local theory in both bounded and unbounded domains. Throughout this paper, we adopt the following simplified notations for Sobolev spaces

$$\begin{gathered} L^r = L^r(\Omega), \quad W^{k,r} = W^{k,r}(\Omega), \quad H^k = W^{k,2}, \\ D^{k,r} = \{v \in L^1_{loc}(\Omega) : |v|_{D^{k,r}} < \infty\}, \quad D^k = D^{k,2}, \\ D^1_0 = \{v \in L^6(\Omega) : |v|_{D^1_0} < \infty \ \text{ and } \ v = 0 \ \text{ on } \ \partial\Omega\}, \\ H^1_0 = D^1_0 \cap L^2, \quad |v|_{D^{k,r}} = |\nabla^k v|_{L^r} \quad \text{and} \quad |v|_{D^1_0} = |\nabla v|_{L^2}. \end{gathered}$$

Then it follows from the classical Sobolev embedding results that

$$|v|_{L^6} \leq C|v|_{D^1_0}, \quad |v|_{L^\infty} \leq C|v|_{W^{1,q}} \quad \text{and} \quad |v|_{L^\infty} \leq C|v|_{D^1_0 \cap D^2},$$

provided that $q > 3$. Hereafter we denote by C a generic positive constant depending only on the fixed constants μ, λ, κ, γ, q, T and the norms of h and f. We also adopt the obvious notation

$$|\cdot|_{X \cap Y} = |\cdot|_X + |\cdot|_Y \quad \text{for (semi-)normed spaces} \quad X, Y.$$

Moreover, we denote by H^{-1} the dual space of H_0^1 with $< \cdot, \cdot >$ being the dual paring of H^{-1} and H_0^1. A detailed study of homogeneous Sobolev spaces may be found in the book [11] by G. Galdi.

The main result in this paper is Theorem 3.1, an existence result on local strong solutions to the initial boundary value problem for a heat-conducting viscous polytropic fluid (the case that $\kappa > 0$). The most striking feature of the theorem is that the existence and uniqueness are proved under a minimal assumption on the initial density ρ_0:

$$\rho_0 \geq 0 \quad \text{in} \quad \Omega \qquad \text{and} \qquad \rho_0 - \rho^\infty \in H^1 \cap W^{1,q}$$

for some constants $\rho^\infty \geq 0$ and $q > 3$. The $W^{1,q}$-regularity of $\rho_0 - \rho^\infty$ seems inevitable to prove the local well-posedness in the framework of Sobolev spaces for any compressible fluid model in three dimensions simply because the Sobolev embedding $W^{1,q} \hookrightarrow L^\infty$ holds only for $q > 3$. The H^1-regularity is necessary to prove the theorem for unbounded domains and can be replaced by $W^{1,3}$ in case that $\rho^\infty > 0$. Moreover, we allow ρ_0 and/or ρ^∞ to vanish and so we may consider both interior vacuum and vacuum at infinity. A similar result was obtained by H.J. Choe and the authors [1] for barotropic fluids with $\rho^\infty = 0$. But even in this case, the strong coupling of the energy and velocity fields prevents us from adapting the arguments in [1] to prove Theorem 3.1.

As has been observed in [1, 3, 22], the lack of a positive lower bound of ρ_0 should be compensated with some condition on the initial data (ρ_0, e_0, u_0). If (ρ, e, u) is a sufficiently smooth solution of (1.1)-(1.3), then letting $t \to 0$ in the equations (1.2) and (1.3), we readily derive a natural condition: there exists a pair (g_1, g_2) of scalar and vector fields such that

$$-\kappa \Delta e_0 - Q(\nabla u_0) = \rho_0 g_1 \quad \text{and} \quad L u_0 + \nabla p_0 = \rho_0 g_2 \quad \text{in} \quad \Omega, \tag{1.4}$$

where we adopt the following notations

$$Q(\nabla u_0) = \frac{\mu}{2} \left| \nabla u_0 + \nabla u_0^T \right|^2 + \lambda (\operatorname{div} u_0)^2,$$
$$L u_0 = -\mu \Delta u_0 - (\lambda + \mu) \nabla \operatorname{div} u_0 \quad \text{and} \quad p_0 = (\gamma - 1) \rho_0 e_0.$$

But it turns out that a weaker condition than (1.4) is sufficient to prove the existence and uniqueness of local strong solutions. Indeed our main existence result is proved under the assumption that the initial data (ρ_0, e_0, u_0) satisfies the regularity

$$\rho_0 - \rho^\infty \in H^1 \cap W^{1,q} \quad \text{and} \quad (e_0,\, u_0) \in D_0^1 \cap D^2 \tag{1.5}$$

and the compatibility condition

$$-\kappa \Delta e_0 - Q(\nabla u_0) = \rho_0^{\frac{1}{2}} g_1 \quad \text{and} \quad L u_0 + \nabla p_0 = \rho_0^{\frac{1}{2}} g_2 \quad \text{in} \quad \Omega \tag{1.6}$$

for some $(g_1, g_2) \in L^2$. Roughly speaking, (1.6) is equivalent to the L^2-integrability of $\sqrt{\rho} e_t$ and $\sqrt{\rho} u_t$ at $t = 0$, as can be shown formally by letting

$t \to 0$ in (1.2) and (1.3). Hence the condition (1.6) plays a key role in deducing that $(e_t,\, u_t) \in L^2(0, T_*; D_0^1)$ as well as $(\sqrt{\rho}e_t,\, \sqrt{\rho}u_t) \in L^\infty(0, T_*; L^2)$ for some small time $T_* > 0$. This was observed and justified rigorously first by R. Salvi and I. Straškraba [22] and then by H.J. Choe and the second author [3], independently, for barotropic fluids. Note that the compatibility condition (1.6) is satisfied automatically for all initial data (ρ_0, e_0, u_0) with the regularity (1.5) whenever ρ_0 is bounded away from zero.

We also consider viscous polytropic fluids without heat-conduction (the case that $\kappa = 0$). Compared with the isentropic fluid models, one major difficulty is the presence of the quadratic nonlinear term $Q(\nabla u)$. However, from the viewpoint of a local existence theory with vacuum, this case is much easier than the previous heat-conducting one because the energy equation (1.2) with $\kappa = 0$ can be rewritten equivalently as a hyperbolic equation for the pressure p

$$p_t + u \cdot \nabla p + \gamma p \operatorname{div} u = (\gamma - 1)\left(Q(\nabla u) + \rho h\right). \tag{1.7}$$

Assume for the sake of simplicity that $h = 0$ and $p(0) = p_0 \in H^1 \cap W^{1,q}$ with $3 < q \leq 6$. Then from the standard estimate

$$\begin{aligned} |p(t)|_{H^1 \cap W^{1,q}} \leq & \left(|p_0|_{H^1 \cap W^{1,q}} + C \int_0^t |Q(\nabla u)|_{H^1 \cap W^{1,q}}\, ds \right) \\ & \times \exp\left(C \int_0^t |\nabla u|_{H^1 \cap W^{1,q}}\, ds \right) \end{aligned}$$

based on energy methods, we can deduce that

$$\sup_{0 \leq t \leq T_*} |p(t)|_{H^1 \cap W^{1,q}} \leq C\left(|p_0|_{H^1 \cap W^{1,q}} + 1\right) \tag{1.8}$$

for some small $T_* > 0$, provided that the velocity u is sufficiently regular. Moreover assume that $\rho_0 \in H^1 \cap W^{1,q}$. Then it also follows that

$$\sup_{0 \leq t \leq T_*} |\rho(t)|_{H^1 \cap W^{1,q}} \leq C|\rho_0|_{H^1 \cap W^{1,q}}. \tag{1.9}$$

In view of the local estimates (1.8) and (1.9), we may regard the equations (1.1), (1.3) and (1.7) as a weakly coupled system at least for some small time interval $[0, T_*]$. This is the reason why the arguments in [1, 3] can be adapted to prove the local well-posedness of the initial boundary value problem for the equations (1.1), (1.3) and (1.7) in case that either Ω is a bounded domain or $\rho^\infty = 0$.

A more general result, Theorem 4.1, is proved in the final section. We prove the local existence of a unique strong solution under a minimal regularity assumption on the initial data: (ρ_0, p_0, u_0) is required to satisfy the regularity condition

$$\rho_0 \geq 0, \qquad \rho_0 - \rho^\infty \in C_0 \cap H^1 \cap W^{1,3},$$
$$p_0 - p^\infty \in H^1 \cap W^{1,q}, \qquad u_0 \in D_0^1 \cap D^2$$

for some constants ρ^∞, p^∞ and $q > 3$, and the compatibility condition

$$Lu_0 + \nabla p_0 \;=\; \rho_0^{\frac{1}{2}}\, g \quad \text{in} \quad \Omega \quad \text{for some} \quad g \in L^2.$$

We remark that the $H^1 \cap W^{1,q}$-regularity of the initial density is replaced by a slightly more general one, $C_0 \cap H^1 \cap W^{1,3}$. Being the completion of $W^{1,q}$ in L^∞, the space C_0 consists of continuous functions on $\overline{\Omega}$ vanishing at infinity. Moreover H^1-regularity can be removed in case that $\rho^\infty > 0$. We also remark that the proof of Theorem 4.1 can be easily adapted to prove a similar result for barotropic fluids, which generalizes the results in [1, 3, 22].

As indicated above, the local estimates (1.8) and (1.9) are crucial ingredients to provide a rather simple proof of our existence result for the case that $\kappa = 0$. But in case that

$$\kappa > 0 \qquad \text{and} \qquad \inf_{\Omega} \rho_0 = 0, \tag{1.10}$$

it seems impossible to derive an estimate analogous to (1.8) since p, e and u are strongly coupled. To investigate this in some detail, we try to estimate $|\nabla p|_{L^2}$ by means of a standard procedure. First, we observe that an estimate for $|\nabla p|_{L^2}$ requires some estimate for $|\nabla e|_{L^2}$ because of the equation of state $p = (\gamma - 1)\rho e$. Then in order to estimate $|\nabla e|_{L^2}$, we make use of standard energy methods: multiplying (1.2) by e_t and integrating over Ω, we obtain

$$\int \rho e_t^2 \, dx + \frac{\kappa}{2}\frac{d}{dt}\int |\nabla e|^2 \, dx \leq C \int |\nabla u|^2 |e_t| \, dx + (\text{good terms}). \tag{1.11}$$

The possibility of vanishing density forces us to estimate the first term of the right hand side as follows

$$C \int |\nabla u|^2 |e_t| \, dx \leq C |\nabla u|^2_{L^{\frac{12}{5}}} |e_t|_{L^6} \leq C |\nabla u|^2_{L^{\frac{12}{5}}} |e_t|_{D_0^1}.$$

Hence we need to estimate $|e_t|_{D_0^1}$. Following the ideas in [1, 3], we differentiate (1.2) with respect to t, multiply by e_t and finally integrate over $(0,t)\times\Omega$. Then from the compatibility condition (1.6), we formally derive

$$\begin{aligned} &\frac{1}{2}\int \rho e_t^2(t) \, dx + \kappa \int_0^t \int |\nabla e_t|^2 \, dxds \\ &\qquad \leq C \int_0^t \int |\nabla u||\nabla u_t||e_t| \, dxds + (\text{some terms}). \end{aligned} \tag{1.12}$$

Taking a similar estimate for u_t into account, we find that the only way to estimate the worst term of the right hand side is

$$\begin{aligned} C\int_0^t\int |\nabla u||\nabla u_t||e_t|\,dxds &\le C\int_0^t |\nabla u|_{L^3}|u_t|_{D_0^1}|e_t|_{D_0^1}\,ds \\ &\le C\left(\sup_{0\le s\le t}|\nabla u|_{L^3}^2\right)\int_0^t |u_t|_{D_0^1}^2\,ds + \frac{\kappa}{2}\int_0^t |e_t|_{D_0^1}^2\,ds. \end{aligned}$$

Substituting this into (1.12) and then (1.11), we deduce that

$$\begin{aligned} &|\nabla e(t)|_{L^2}^2 + \int_0^t |e_t|_{D_0^1}^2\,ds \\ &\quad\le C\left(\sup_{0\le s\le t}|\nabla u|_{L^3}^2\right)\left(\int_0^t |u_t|_{D_0^1}^2\,ds\right) + (\text{some terms}). \end{aligned} \tag{1.13}$$

On the other hand, since $L = -\mu\Delta - (\lambda+\mu)\nabla\mathrm{div}$ is an elliptic operator (see (2.33) below), it follows from the momentum equation (1.3) that

$$|\nabla u|_{H^1} \le C|\nabla p|_{L^2} + (\text{some terms}). \tag{1.14}$$

Combining (1.13) and (1.14), we may conclude that in case of a heat-conducting fluid with vacuum (the case (1.10)), p and u are strongly coupled with each other for any small time interval, that is, $\sup_{0\le s\le t}|\nabla p|_{L^2}$ depends on $\sup_{0\le s\le t}|\nabla u|_{H^1}$ for any $t>0$ and vice versa. This is the main difficulty in proving Theorem 3.1.

Our proof of Theorem 3.1 is based on the following two key observations that (i) the local estimates for $\sup_{0\le s\le t}|\nabla u|_{L^2}$ and $\int_0^t |u_t|_{D_0^1}^2\,ds$ can be derived without resort to the estimates for p and e, and (ii) $|\nabla p|_{L^2}$ depends on $|\nabla^2 u|_{L^2}$ via *a half power*. The latter fact follows immediately from (1.13) because $|\nabla u|_{L^3}^2 \le C|\nabla u|_{L^2}|\nabla u|_{H^1}$. See also Remark 2.3 in the next section. Then using the elliptic regularity result, we obtain the desired local estimates for p and u.

The detailed proof of Theorem 3.1 is given in the next two sections. In Section 2, we consider a linearized problem and derive some local estimates for the solutions independent of the lower bound of the initial density and in Section 3, we prove the theorem by applying a classical iteration argument based on the uniform estimates. The final section, Section 4, is devoted to proving Theorem 4.1.

2. A priori estimates for a linearized problem with $\kappa > 0$

In this section, we consider the following linearized problem with $\kappa > 0$:

$$\rho_t + \operatorname{div}(\rho v) = 0 \tag{2.1}$$

$$(\rho e)_t + \operatorname{div}(\rho e v) - \kappa \Delta e + p \operatorname{div} v = Q(\nabla v) + \rho h \quad \text{in } (0,T) \times \Omega, \tag{2.2}$$

$$(\rho u)_t + \operatorname{div}(\rho v \otimes u) + Lu + \nabla p = \rho f \tag{2.3}$$

$$(\rho,\, \rho e,\, \rho u)|_{t=0} = (\rho_0,\, \rho_0 e_0,\, \rho_0 u_0) \quad \text{in} \quad \Omega, \tag{2.4}$$

$$(e,\, u) = (0,\, 0) \quad \text{on} \quad (0,T) \times \partial\Omega, \tag{2.5}$$

$$(\rho,\, e,\, u)(t,x) \to (\rho^\infty,\ 0,\ 0) \quad \text{as} \quad |x| \to \infty,\ (t,x) \in (0,T) \times \Omega, \tag{2.6}$$

where v is a known vector field on $(0,T) \times \Omega$. Recall that $p = (\gamma - 1)\rho e$, $Q(\nabla v) = \frac{\mu}{2} \left|\nabla v + \nabla v^T\right|^2 + \lambda(\operatorname{div} v)^2$ and $Lu = -\mu \Delta u - (\lambda + \mu)\nabla \operatorname{div} u$.

We first solve the linear transport equation (2.1)

Lemma 2.1. *Assume that ρ_0 and v satisfy the regularity*

$$\rho_0 \geq 0, \quad \rho_0 - \rho^\infty \in C_0 \quad \textit{and} \quad v \in L^\infty(0,T; D_0^1 \cap D^2) \cap L^2(0,T; D^{2,q})$$

for some constants $\rho^\infty \in [0,\infty)$ and $q \in (3,6]$. Then there exists a unique weak solution ρ in $\rho^\infty + C([0,T]; C_0)$ to the linear hyperbolic problem (2.1), (2.4) *and* (2.6)*. Moreover the solution ρ can be represented by*

$$\rho(t,x) = \rho_0(\, U(0,t,x)\,) \exp\left[- \int_0^t \operatorname{div} v(s, U(s,t,x)\,)\, ds \right], \tag{2.7}$$

where $U \in C([0,T] \times [0,T] \times \overline{\Omega})$ is the solution to the initial value problem

$$\begin{cases} \dfrac{\partial}{\partial t} U(t,s,x) = v(t, U(t,s,x)\,), & 0 \leq t \leq T \\ U(s,s,x) = x, \qquad 0 \leq s \leq T, & x \in \overline{\Omega}. \end{cases} \tag{2.8}$$

Assume in addition that $\rho_0 - \rho^\infty \in W^{1,r}$ for some r with $2 \leq r \leq q$. Then we also have

$$\rho - \rho^\infty \in C([0,T]; W^{1,r}) \quad \textit{and} \quad \rho_t \in L^\infty(0,T; L^r). \tag{2.9}$$

Proof. We provide an elementary and self-contained proof based on the classical method of characteristics; note that the existence of a unique solution ρ in $L^\infty(0,T; L^\infty)$ was already proved by R. J. DiPerna and P. L. Lions [9].

To begin with, we construct sequences $\{\rho_0^k\}$ and $\{v^k\}$ of smooth scalar and vector fields such that

$$\rho_0^k - \rho^\infty \in H^2 \cap C^2(\overline{\Omega}), \quad v^k \in L^2(0,T; D_0^1 \cap D^3) \cap C^2([0,T] \times \overline{\Omega})$$

$$\text{and} \qquad |\rho_0^k - \rho_0|_{L^\infty} + \int_0^T |\nabla(v^k - v)(t)|^2_{H^1 \cap W^{1,q}}\, dt \to 0 \quad \text{as} \quad k \to \infty.$$

For this purpose, we first recall that H^4 and $L^2(0,T; H^3)$ are dense in C_0 and $L^2(0,T; H^1 \cap W^{1,q})$, respectively. Then since $\rho_0 - \rho^\infty \in C_0$ and $g =$

$\nabla v \in L^2(0,T;H^1\cap W^{1,q})$, there exist sequences $\{\rho_0^k\}$ in $\rho^\infty + H^4$ and $\{g^k\}$ in $L^2(0,T;H^3)$ such that $\rho_0^k - \rho^\infty \to \rho_0 - \rho^\infty$ in L^∞ (or C_0) and $g^k \to g$ in $L^2(0,T;H^1\cap W^{1,q})$ as $k\to\infty$. For a.e $t\in(0,T)$, let $w^k = w^k(t) \in D_0^1$ be the unique weak solution to the elliptic boundary value problem

$$\Delta w^k = \operatorname{div} g^k \quad \text{in} \quad \Omega \qquad \text{and} \qquad w^k = 0 \quad \text{on} \quad \partial\Omega.$$

Then since $|\nabla w^k(t)|_{L^2} \le |g^k(t)|_{L^2}$ for a.e $t\in(0,T)$ and $g^k\in L^2(0,T;H^3)$, it follows from the standard elliptic regularity result (see [1] for instance) that

$$|w^k(t)|_{D_0^1\cap D^4} \le C\left(|\operatorname{div} g^k(t)|_{H^2} + |w^k(t)|_{D_0^1}\right) \le C|g^k(t)|_{H^3}$$

for a.e $t\in(0,T)$ and $w^k\in L^2(0,T;D_0^1\cap D^4)$. It is easy to show that

$$|w^k(t) - v(t)|_{D_0^1} \le |g^k(t) - g(t)|_{L^2} \qquad \text{for a.e.} \quad t\in(0,T).$$

Hence by virtue of the elliptic regularity result in [1], we deduce that

$$\begin{aligned}|\nabla(w^k - v)(t)|_{H^1} &\le C\left(|\operatorname{div} g^k(t) - \operatorname{div} g(t)|_{L^2} + |w^k(t) - v(t)|_{D_0^1}\right)\\ &\le C|g^k(t) - g(t)|_{H^1}\end{aligned}$$

and

$$\begin{aligned}|\nabla(w^k - v)(t)|_{W^{1,q}} &\le C\left(|\operatorname{div} g^k(t) - \operatorname{div} g(t)|_{L^q} + |\nabla(w^k - v)(t)|_{L^q}\right)\\ &\le C\left(|g^k(t) - g(t)|_{W^{1,q}} + |\nabla(w^k - v)(t)|_{H^1}\right)\end{aligned}$$

for a.e. $t\in(0,T)$, which implies that $\nabla w^k \to \nabla v$ in $L^2(0,T;H^1\cap W^{1,q})$ as $k\to\infty$. Therefore, recalling that $C^\infty([0,T];D_0^1\cap D^4)$ is dense in $L^2(0,T;D_0^1\cap D^4)$, we conclude that there exists a sequence $\{v^k\}$ in $C^\infty([0,T];D_0^1\cap D^4)$ such that $\nabla v^k \to \nabla v$ in $L^2(0,T;H^1\cap W^{1,q})$ as $k\to\infty$. In view of the Sobolev embedding results

$$H^4 \hookrightarrow C^2(\overline{\Omega}) \quad \text{and} \quad D_0^1\cap D^4 \hookrightarrow C^2(\overline{\Omega}),$$

we have proved the existence of sequences $\{\rho_0^k\}$ and $\{v^k\}$ with the desired properties. To treat the case of unbounded domains, we also need a cut-off procedure. Assuming that Ω is an unbounded domain such as the whole space, the half space and an exterior domain, we choose a sufficiently large integer $R_0 > 1$ so that

$$\mathbf{R}^3\setminus\Omega \subset B_{R_0/2} \quad \text{if} \quad \mathbf{R}^3\setminus\Omega \subset\subset \mathbf{R}^3,$$

where for each $R>0$, B_R denotes the open ball of radius R centered at the origin: $B_R = \{x\in\mathbf{R}^3 : |x| < R\}$. Then taking a cut-off function $\varphi\in C_c^\infty(B_1)$ such that $\varphi = 1$ in $B_{1/2}$, we define

$$\begin{gathered}\varphi^R(x) = \varphi\left(x/R\right), \quad \rho_0^R(x) = \rho^\infty + \varphi^R(x)\left(\rho_0(x) - \rho^\infty\right)\\ \text{and} \quad v^R(t,x) = \varphi^R(x)v(t,x) \qquad \text{for} \quad (t,x)\in[0,T]\times\Omega \quad \text{and} \quad R > R_0.\end{gathered}$$

Note that $\rho_0^R = \rho^\infty$ and $v^R = 0$ in $(0,T)\times(\Omega\setminus\Omega_R)$, where $\Omega_R = \Omega\cap B_R$. Moreover, it is easy to show that

$$|\rho_0^R - \rho_0|_{L^\infty} + \int_0^T |\nabla(v^R - v)(t)|^2_{H^1\cap W^{1,q}}\,dt \to 0 \quad \text{as} \quad R\to\infty.$$

Hence applying this cut-off technique to ρ_0^k and v^k for each $k\geq 1$, we may assume without loss of generality that if Ω is an unbounded domain, then

$$\rho_0^k(x) = \rho^\infty \quad \text{and} \quad v^k(t,x) = 0 \quad \text{for} \quad t\in[0,T],\ \ x\in\Omega\setminus\Omega_{R_k}, \tag{2.10}$$

where $\{R_k\}$ is a sequence such that $R_0 < R_1 < R_2 < \cdots$ and $R_k\to\infty$.

Now we consider the following regularized problem

$$\rho_t + \operatorname{div}(\rho v^k) = 0 \quad \text{in} \quad (0,T)\times\Omega \qquad \text{and} \qquad \rho(0) = \rho_0^k \quad \text{in} \quad \Omega \tag{2.11}$$

for $k\geq 1$. Then since $\rho_0^k\in C^2(\overline{\Omega})$, $v^k\in C^2([0,T]\times\overline{\Omega})$ and $v^k = 0$ on $[0,T]\times\partial\Omega$, it follows from the classical linear hyperbolic theory that there exists a unique solution $\rho^k\in C^2([0,T]\times\overline{\Omega})$ to the problem (2.11) and the solution ρ^k can be represented by

$$\rho^k(t,x) = \rho_0^k(U^k(0,t,x))\exp\left[-\int_0^t \operatorname{div} v^k(s,U^k(s,t,x))\,ds\right], \tag{2.12}$$

where $U^k\in C^2([0,T]\times[0,T]\times\overline{\Omega})$ is the solution to the initial value problem

$$\begin{cases} \dfrac{\partial}{\partial t}U^k(t,s,x) = v^k(t,U^k(t,s,x)\,), & 0\leq t\leq T \\ \quad U^k(s,s,x) = x, \qquad 0\leq s\leq T, & x\in\overline{\Omega}. \end{cases} \tag{2.13}$$

It should be noted from (2.10) that if Ω is an unbounded domain, then

$$U^k(t,s,x) = x \quad \text{and} \quad \rho^k(t,x) = \rho^\infty \quad \text{for} \quad t,s\in[0,T],\ \ x\in\Omega\setminus\Omega_{R_k}.$$

We will prove that the sequence $\{\rho^k\}$ converges to a solution of the original problem. To show this, we first observe that

$$\begin{aligned} &|U^k(t,s,x) - U^l(t,s,x)| \leq \int_s^t \left|v^k(\tau,U^k(\tau,s,x)\,) - v^l(\tau,U^l(\tau,s,x)\,)\right|\,d\tau \\ &\leq \int_s^t |v^k(\tau) - v^l(\tau)|_{L^\infty}\,d\tau + \int_s^t |\nabla v^l(\tau)|_{L^\infty}|U^k(\tau,s,x) - U^l(\tau,s,x)|\,d\tau. \end{aligned}$$

Then Gronwall's inequality implies that

$$\begin{aligned} &|U^k(t,s,x) - U^l(t,s,x)| \\ &\leq \left(\int_0^T |v^k(\tau) - v^l(\tau)|_{L^\infty}\,d\tau\right)\exp\left(\int_0^T |\nabla v^l(\tau)|_{L^\infty}\,d\tau\right) \\ &\leq C\left(\int_0^T |v^k(\tau) - v^l(\tau)|_{D_0^1\cap D^2}\,d\tau\right)\exp\left(C\int_0^T |\nabla v^l(\tau)|_{W^{1,q}}\,d\tau\right) \end{aligned}$$

for each $s, t \in [0, T]$ and $x \in \overline{\Omega}$, and thus

$$\sup_{s,\,t\in[0,T]} |U^k(t,s) - U^l(t,s)|_{L^\infty} \to 0 \qquad \text{as} \quad k,l \to \infty. \tag{2.14}$$

Hence it follows from the well-known embedding result $W^{1,\,q} \hookrightarrow C^{0,\,\theta}$ with $\theta = 1 - \frac{3}{q}$ that as $k, l \to \infty$,

$$\begin{aligned} &\int_0^T \left|\operatorname{div} v(s, U^k(s,t,x)) - \operatorname{div} v(s, U^l(s,t,x))\right| \, ds \\ &\leq C \int_0^T |\nabla v(s)|_{W^{1,\,q}} \left|U^k(s,t,x) - U^l(s,t,x)\right|^\theta \, ds \to 0 \end{aligned}$$

uniformly in $(t,\, x) \in [0,\, T] \times \overline{\Omega}$. Therefore, observing that

$$\begin{aligned} &\int_0^t \left|\operatorname{div} v^k(s, U^k(s,t,x)) - \operatorname{div} v^l(s, U^l(s,t,x))\right| \, ds \\ &\leq \int_0^T \Big(|\operatorname{div} v^k(s) - \operatorname{div} v(s)|_{L^\infty} + |\operatorname{div} v^l(s) - \operatorname{div} v(s)|_{L^\infty}\Big) \, ds \\ &\quad + \int_0^T \left|\operatorname{div} v(s, U^k(s,t,x)) - \operatorname{div} v(s, U^l(s,t,x))\right| \, ds, \end{aligned}$$

we easily deduce from (2.12) that

$$\sup_{t\in[0,T]} |\rho^k(t) - \rho^l(t)|_{C_0} \to 0 \quad \text{as} \quad k,l \to \infty.$$

This proves the existence of a limit ρ in $\rho^\infty + C([0,T]; C_0)$ such that

$$\rho^k - \rho^\infty \to \rho - \rho^\infty \quad \text{in} \quad C([0,T]; C_0). \tag{2.15}$$

It is easy to show that ρ is a weak solution to the original problem (2.1), (2.4) and (2.6). Then the uniqueness of solutions in the class $\rho^\infty + C([0,T]; C_0)$ follows immediately from a result by R. J. DiPerna and P. L. Lions in [9].

To prove a higher regularity result, we will derive an uniform estimate for $\sigma^k = \rho^k - \rho^\infty$ in $W^{1,\,r}$ assuming that $\rho_0 - \rho^\infty \in W^{1,\,r}$ for some $r \in [2, q]$. From (2.11), it follows that

$$\sigma^k_t + v^k \cdot \nabla\sigma^k + \sigma^k \operatorname{div} v^k = -\rho^\infty \operatorname{div} v^k \qquad \text{in} \quad \Omega. \tag{2.16}$$

Then multiplying (2.16) by $|\sigma^k|^{r-2}\sigma^k$ and integrating (by parts) over Ω, we obtain

$$\frac{d}{dt}\int |\sigma^k|^r \, dx \leq C \int \Big(|\nabla v^k||\sigma^k|^r + \rho^\infty |\nabla v^k||\sigma^k|^{r-1}\Big) \, dx$$

and

$$\frac{d}{dt}|\sigma^k|^r_{L^r} \leq C|\nabla v^k|_{W^{1,\,q}}|\sigma^k|^r_{L^r} + C\rho^\infty |\nabla v^k|_{L^r}|\sigma|^{r-1}_{L^r}. \tag{2.17}$$

Similarly, taking the operator ∇ to (2.16), multiplying by $|\nabla\sigma^k|^{r-2}\nabla\sigma^k$ and integrating over Ω, we obtain

$$\begin{aligned}\frac{d}{dt}|\nabla\sigma^k|^r_{L^r} \le C|\nabla v^k|_{W^{1,q}}|\nabla\sigma^k|^r_{L^r} + C|\nabla^2 v^k|_{L^q}|\sigma^k|_{L^{\frac{qr}{q-r}}}|\nabla\sigma^k|^{r-1}_{L^r} \\ + C\rho^\infty|\nabla^2 v^k|_{L^r}|\nabla\sigma^k|^{r-1}_{L^r}.\end{aligned} \tag{2.18}$$

But since $r < \frac{qr}{q-r} < \frac{3r}{3-r}$ if $2 \le r \le 3$ and $r < \frac{qr}{q-r} \le \infty$ if $3 < r \le q$, it follows from the well-known embedding results (see Chapter II in [11] for instance) that $|\sigma^k|_{L^{\frac{qr}{q-r}}} \le C|\sigma^k|_{W^{1,r}}$. Hence combining (2.15), (2.17) and (2.18), we have

$$\frac{d}{dt}|\sigma^k|^r_{W^{1,r}} \le C|\nabla v^k|_{W^{1,q}}|\sigma^k|^r_{W^{1,r}} + C\rho^\infty|\nabla v^k|_{W^{1,r}}|\sigma^k|^{r-1}_{W^{1,r}}$$

and in view of Gronwall's inequality, we thus obtain

$$\begin{aligned}|\rho^k(t)-\rho^\infty|_{W^{1,r}} \le \left(|\rho^k_0-\rho^\infty|_{W^{1,r}} + C\rho^\infty\int_0^t|\nabla v^k(s)|_{W^{1,r}}\,ds\right) \\ \times\exp\left(C\int_0^t|\nabla v^k(s)|_{W^{1,q}}\,ds\right)\end{aligned} \tag{2.19}$$

for each $t \in [0,T]$. As a consequence of (2.15) and (2.19), we deduce that

$$\rho^k - \rho^\infty \overset{*}{\rightharpoonup} \rho - \rho^\infty \quad \text{in} \quad L^\infty(0,T;W^{1,r}).$$

Moreover since $\rho_t = -\mathrm{div}\,(\rho v) \in L^\infty(0,T;L^r)$, it follows from the classical embedding result (see [30] for instance) that $\rho-\rho^\infty \in C([0,T];L^r) \cap C([0,T];W^{1,r}-weak)$. To prove the strong time-continuity of $\rho-\rho^\infty$ in $W^{1,r}$, we observe that for each fixed $t \in [0,T]$, $\rho^k(t)-\rho^\infty \to \rho(t)-\rho^\infty$ weakly in $W^{1,r}$. Hence from (2.19), it follows immediately that

$$\begin{aligned}|\rho(t)-\rho^\infty|_{W^{1,r}} \le \left(|\rho_0-\rho^\infty|_{W^{1,r}} + C\rho^\infty\int_0^t|\nabla v(s)|_{W^{1,r}}\,ds\right) \\ \times\exp\left(C\int_0^t|\nabla v(s)|_{W^{1,q}}\,ds\right)\end{aligned} \tag{2.20}$$

for each $t \in [0,T]$. In particular, we have

$$\limsup_{t\to+0}|\rho(t)-\rho^\infty|_{W^{1,r}} \le |\rho_0-\rho^\infty|_{W^{1,r}},$$

which implies that $\rho-\rho^\infty$ is right-continuous in $W^{1,r}$ at $t=0$. Since the equation (2.1) is invariant under the reflections and translations in time, we conclude that $\rho-\rho^\infty \in C([0,T];W^{1,r})$.

Hence in order to complete the proof of the lemma, it remains to prove the assertions about the representation formula (2.7) for the solution ρ. First, recalling that $\nabla v \in L^2(0,T;L^\infty)$, we can easily prove the uniqueness of solutions to the problem (2.8). Then the existence of a solution $U \in C([0,T]\times[0,T]\times\overline{\Omega})$ follows immediately from (2.13) and (2.14). Finally, (2.7) follows from (2.12), (2.14) and (2.15). We have completed the proof of Lemma 2.1. $\square$

Throughout the section, we assume that the known data satisfy the following regularity

$$\begin{gathered}\rho_0 \geq 0, \quad \rho_0 - \rho^\infty \in W^{1,r} \cap W^{1,q}, \quad (e_0, u_0) \in D_0^1 \cap D^2, \\ (h, f) \in C([0,T]; L^2) \cap L^2(0,T; L^q), \quad (h_t, f_t) \in L^2(0,T; H^{-1}), \\ v \in C([0,T]; D_0^1 \cap D^2) \cap L^2(0,T; D^{2,q}) \quad \text{and} \quad v_t \in L^2(0,T; D_0^1)\end{gathered} \tag{2.21}$$

for some constants ρ^∞, q and r such that $\rho^\infty \geq 0$ and $2 \leq r \leq 3 < q \leq 6$. Then the global existence of a unique strong solution (ρ, e, u) to the linearized problem (2.1)–(2.6) can be proved by standard methods at least for the case that ρ_0 is bounded away from zero.

Lemma 2.2. *Assume in addition to* (2.21) *that* $\rho_0 \geq \delta$ *in* Ω *for some constant* $\delta > 0$. *Then there exists a unique strong solution* (ρ, e, u) *to the initial boundary value problem* (2.1)–(2.6) *such that*

$$\begin{gathered}\rho - \rho^\infty \in C([0,T]; W^{1,r} \cap W^{1,q}), \quad \rho_t \in C([0,T]; L^r \cap L^q), \\ (e, u) \in C([0,T]; D_0^1 \cap D^2) \cap L^2(0,T; D^{2,q}), \\ (e_t, u_t) \in C([0,T]; L^2) \cap L^2(0,T; H_0^1), \quad (e_{tt}, u_{tt}) \in L^2(0,T; H^{-1}) \\ \textit{and} \qquad \rho \geq \underline{\delta} \quad \textit{on} \quad [0,T] \times \overline{\Omega} \quad \textit{for some constant} \quad \underline{\delta} > 0.\end{gathered} \tag{2.22}$$

Proof. The existence and regularity of a unique solution ρ of the linear hyperbolic problem (2.1), (2.4) and (2.6) was proved in Lemma 2.1. Moreover, it follows from the representation formula (2.7) that

$$\rho(t,x) \geq \left(\inf_\Omega \rho_0\right) \exp\left(-C \int_0^t |\nabla v(s)|_{W^{1,q}}\, ds\right) \geq \underline{\delta} > 0 \tag{2.23}$$

for $(t,x) \in [0,T] \times \overline{\Omega}$. Hence we can rewrite the equations (2.2) and (2.3) as a linear parabolic equation

$$e_t + v \cdot \nabla e + (\gamma - 1) e \operatorname{div} v - \kappa \rho^{-1} \Delta e = \rho^{-1} Q(\nabla v) + h$$

and a linear parabolic system

$$u_t + v \cdot \nabla u + \rho^{-1} L u = f - (\gamma - 1) \rho^{-1} \nabla(\rho e).$$

The existence and regularity of solutions e and then u to the corresponding linear parabolic problems have been well-known and in fact can be easily proved by classical methods. For instance, in case of bounded domains, we may apply a semi-discrete Galerkin method (as in [2]) or the method of continuity (as in [31]). Then the case of unbounded domains can be easily treated by means of the usual domain expansion technique (see [1] or [4]). Finally, for a proof of the uniqueness, see the proof of Lemma 2.4 below. $\square$

The purpose of the section is to derive *local* (*in time*) a priori estimates for strong solutions to the linearized problem (2.1)–(2.6), which are independent of the lower bound δ of the initial density ρ_0. Let (ρ_0, e_0, u_0) be a given initial data satisfying the hypotheses of Lemma 2.2, and let us choose any fixed c_0 so that

$$c_0 \geq 1 + \rho^\infty + |\rho_0 - \rho^\infty|_{W^{1,r}\cap W^{1,q}} + |(e_0, u_0)|_{D_0^1\cap D^2} + |(g_1, g_2)|_{L^2}^2,$$

where $2 \leq r \leq 3 < q \leq 6$, $g_1 = \rho_0^{-\frac{1}{2}}(-\kappa\Delta e_0 - Q(\nabla u_0))$, $g_2 = \rho_0^{-\frac{1}{2}}(Lu_0 + \nabla p_0)$ and $p_0 = (\gamma-1)\rho_0\, e_0$. Moreover let v be a vector field satisfying the regularity stated in Lemma 2.2, and assume that $v(0) = u_0$ and

$$\sup_{0\leq t\leq T_*}\left(|v(t)|_{D_0^1} + \beta^{-1}|v(t)|_{D^2} \right) + \int_0^{T_*} \left(|v_t(t)|_{D_0^1}^2 + |v(t)|_{D^{2,q}}^2 \right) dt \leq c_1 \tag{2.24}$$

for some fixed constants c_1, β and time T_* such that

$$1 < c_0 < c_1 < c_2 = \beta c_1 \qquad \text{and} \qquad 0 < T_* \leq T.$$

Then we will derive some a priori estimates for the solution (ρ, e, u) which are independent of δ. It should be emphasized again that throughout the paper, we denote by C a generic positive constant depending only on the fixed constants μ, λ, κ, γ, q, T and the norms of h and f.

2.1. **Estimates for the density.** To estimate the density ρ, we first recall from (2.20) that

$$|\rho(t) - \rho^\infty|_{W^{1,r}\cap W^{1,q}} \leq Cc_0 \exp\left(C\int_0^t |\nabla v|_{H^1\cap W^{1,q}}\, ds \right)$$

for $0 \leq t \leq T_*$. Then using the estimate

$$\int_0^t |\nabla v|_{H^1\cap W^{1,q}}\, ds \leq t^{\frac{1}{2}} \left[\int_0^t |\nabla v|_{H^1\cap W^{1,q}}^2\, ds \right]^{\frac{1}{2}} \leq C\left(c_2 t + (c_2 t)^{\frac{1}{2}} \right)$$

together with the equation (2.1), we conclude that

$$|\rho(t) - \rho^\infty|_{W^{1,r}\cap W^{1,q}} \leq Cc_0 \quad \text{and} \quad |\rho_t(t)|_{L^r\cap L^q} \leq Cc_2^2 \tag{2.25}$$

for $0 \leq t \leq \min(T_*, T_1)$, where $T_1 = c_2^{-1} < 1$. Moreover it follows from (2.23) and (2.25) that

$$C^{-1}\delta \leq \rho(t,x) \leq Cc_0 \quad \text{for} \quad 0 \leq t \leq \min(T_*, T_1),\ \ x \in \overline{\Omega}. \tag{2.26}$$

2.2. **Estimates for the internal energy and pressure.** We first derive estimates for the internal energy e. Then estimates for the pressure p follow immediately from the equation of state $p = (\gamma - 1)\rho e$. To derive estimates for e, we differentiate the equation (2.2) with respect to t and obtain

$$\begin{aligned}&\rho e_{tt} + \rho v \cdot \nabla e_t - \kappa \Delta e_t + (p \operatorname{div} v)_t \\ &\qquad = Q(\nabla v)_t + (\rho h)_t - \rho_t v \cdot \nabla e - \rho v_t \cdot \nabla e - \rho_t e_t.\end{aligned}$$

Then multiplying this by e_t, integrating over Ω and using (2.1), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int \rho e_t^2\,dx + \kappa \int |\nabla e_t|^2\,dx + \int (p\mathrm{div}v)_t e_t\,dx \\ &= \int \left(-\rho_t v \cdot \nabla e - \rho v_t \cdot \nabla e + \operatorname{div}(\rho v) e_t + Q(\nabla v)_t + (\rho h)_t\right)\, e_t\,dx\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int \rho e_t^2\,dx + \kappa \int |\nabla e_t|^2\,dx \\ &\le C \int \big(\, |\rho_t||v||\nabla e||e_t| + \rho|v_t||\nabla e||e_t| + \rho|v||\nabla e_t||e_t| \\ &\qquad + |p_t||\nabla v||e_t| + \rho|e||\nabla v_t||e_t| + |\nabla v||\nabla v_t||e_t| + |\rho_t||h||e_t|\,\big)\,dx \\ &\qquad + < h_t, \rho e_t > \equiv \sum_{j=1}^{8} I_j.\end{aligned} \tag{2.28}$$

Making use of (2.24), (2.25) and (2.26), we can estimate each term $I_j = I_j(t)$ for $0 \le t \le \min(T_*, T_1)$ as follows:

$$I_1 \le C|\rho_t|_{L^3}|v|_{L^\infty}|\nabla e|_{L^2}|\nabla e_t|_{L^2} \le Cc_2^6|\nabla e|_{L^2}^2 + \frac{\kappa}{14}|\nabla e_t|_{L^2}^2,$$

$$\begin{aligned}I_2, I_5 &\le C|\rho|_{L^\infty}^{\frac{1}{2}}|\nabla v_t|_{L^2}|\nabla e|_{L^2}|\sqrt{\rho}e_t|_{L^3} \le C|\rho|_{L^\infty}^{\frac{3}{4}}|\nabla v_t|_{L^2}|\nabla e|_{L^2}|\sqrt{\rho}e_t|_{L^2}^{\frac{1}{2}}|\nabla e_t|_{L^2}^{\frac{1}{2}} \\ &\le Cc_0^3|\nabla e|_{L^2}^4|\sqrt{\rho}e_t|_{L^2}^2 + \frac{\kappa}{14}|\nabla e_t|_{L^2}^2 + |\nabla v_t|_{L^2}^2,\end{aligned}$$

$$I_3 \le C|\rho|_{L^\infty}^{\frac{1}{2}}|v|_{L^\infty}|\nabla e_t|_{L^2}|\sqrt{\rho}e_t|_{L^2} \le Cc_2^3|\sqrt{\rho}e_t|_{L^2}^2 + \frac{\kappa}{14}|\nabla e_t|_{L^2}^2,$$

$$\begin{aligned}I_4 &\le |p_t|_{L^2}|\nabla v|_{L^3}|e_t|_{L^6} \le C(|\rho|_{L^\infty}^{\frac{1}{2}}|\sqrt{\rho}e_t|_{L^2} + |\rho_t|_{L^3}|\nabla e|_{L^2})|\nabla v|_{L^3}|\nabla e_t|_{L^2} \\ &\le Cc_2^3|\sqrt{\rho}e_t|_{L^2}^2 + Cc_2^6|\nabla e|_{L^2}^2 + \frac{\kappa}{14}|\nabla e_t|_{L^2}^2,\end{aligned}$$

$$I_6 \le C|\nabla v|_{L^2}^{\frac{1}{2}}|\nabla v|_{H^1}^{\frac{1}{2}}|\nabla v_t|_{L^2}|\nabla e_t|_{L^2} \le Cc_1c_2|\nabla v_t|_{L^2}^2 + \frac{\kappa}{14}|\nabla e_t|_{L^2}^2,$$

and

$$I_7 + I_8 \le Cc_0^2|h_t|_{H^{-1}}^2 + Cc_2^4|h|_{L^2}^2 + |\sqrt{\rho}e_t|_{L^2}^2 + \frac{\kappa}{14}|\nabla e_t|_{L^2}^2.$$

Substituting these estimates into (2.28), we have

$$\begin{aligned}&\frac{d}{dt}|\sqrt{\rho}e_t|_{L^2}^2+\kappa|\nabla e_t|_{L^2}^2\\&\le Cc_2^6\left(1+|\nabla e|_{L^2}^4\right)\left(1+|\sqrt{\rho}e_t|_{L^2}^2\right)+C(c_1c_2|\nabla v_t|_{L^2}^2+c_0^2|h_t|_{H^{-1}}^2+c_2^4|h|_{L^2}^2)\end{aligned}$$

and then integrating this over (τ,t), we also have

$$\begin{aligned}|\sqrt{\rho}e_t(t)|_{L^2}^2+\kappa\int_\tau^t|\nabla e_t|_{L^2}^2\,ds&\le|\sqrt{\rho}e_t(\tau)|_{L^2}^2+Cc_1^2c_2+Cc_2^4t\\&\quad+Cc_2^6\int_\tau^t\left(1+|\nabla e|_{L^2}^4\right)\left(1+|\sqrt{\rho}e_t|_{L^2}^2\right)\,ds\end{aligned}\tag{2.29}$$

for $0<\tau<t\le\min(T_*,T_1)$. To estimate $\limsup_{\tau\to0}|\sqrt{\rho}e_t(\tau)|_{L^2}^2$, we observe from the equation (2.2) that

$$\int\rho e_t^2\,dx\le C\int\left(\rho|h|^2+\rho|v|^2|\nabla e|^2+\rho|e|^2|\nabla v|^2+\rho^{-1}\left|\kappa\Delta e+Q(\nabla v)\right|^2\right)\,dx$$

and thus

$$\begin{aligned}\limsup_{\tau\to0}|\sqrt{\rho}e_t(\tau)|_{L^2}^2&\le C\left(|\rho_0|_{L^\infty}|h(0)|_{L^2}^2+|\rho_0|_{L^\infty}|\nabla u_0|_{H^1}^2|\nabla e_0|_{L^2}^2+|g_1|_{L^2}^2\right)\\&\le Cc_0^5.\end{aligned}$$

Hence, letting $\tau\to0$ in (2.29), we deduce that

$$\begin{aligned}&|\sqrt{\rho}e_t(t)|_{L^2}^2+\int_0^t|\nabla e_t|_{L^2}^2\,ds\\&\quad\le Cc_1^4c_2+Cc_2^6\int_0^t\left(1+|\nabla e|_{L^2}^4\right)\left(1+|\sqrt{\rho}e_t|_{L^2}^2\right)\,ds\end{aligned}\tag{2.30}$$

for $0\le t\le\min(T_*,T_2)$, where $T_2=c_2^{-4}<T_1$. On the other hand, since

$$\frac{d}{dt}|\nabla e|_{L^2}^2=2\int\nabla e\cdot\nabla e_t\,dx\le2|\nabla e|_{L^2}|\nabla e_t|_{L^2},$$

it follows that

$$|\nabla e(t)|_{L^2}^2\le C|\nabla e_0|_{L^2}^2+C\int_0^t|\nabla e_t|_{L^2}^2\,ds\quad\text{for}\quad0\le t\le\min(T_*,T_2).$$

Combining this and (2.30), we deduce that

$$\begin{aligned}&|\nabla e(t)|_{L^2}^2+|\sqrt{\rho}e_t(t)|_{L^2}^2+\int_0^t|\nabla e_t|_{L^2}^2\,ds\\&\quad\le Cc_1^4c_2+Cc_2^6\int_0^t\left(1+|\nabla e|_{L^2}^4\right)\left(1+|\sqrt{\rho}e_t|_{L^2}^2\right)\,ds\end{aligned}\tag{2.31}$$

for $0\le t\le\min(T_*,T_2)$. Now we define a function $\Gamma(t)$ by

$$\Gamma(t)=1+|e(t)|_{D_0^1}^2+|\sqrt{\rho}e_t(t)|_{L^2}^2.$$

Then it follows from (2.31) that

$$\Gamma(t) \le Cc_1^4 c_2 + Cc_2^6 \int_0^t \Gamma(s)^3\, ds \quad \text{for} \quad 0 \le t \le \min(T_*, T_2).$$

Solving this integral inequality, we easily derive

$$\Gamma(t) \le Cc_1^4 c_2 (1 - Cc_2^{16} t)^{-\frac{1}{2}} \quad \text{for all small} \quad t \ge 0.$$

Therefore, taking $T_3 = (2Cc_2^{16})^{-1}$ with a large $C > 1$, we conclude that

$$|e(t)|_{D_0^1}^2 + |\sqrt{\rho} e_t(t)|_{L^2}^2 + \int_0^t |e_t(s)|_{D_0^1}^2\, ds \le Cc_1^4 c_2 \tag{2.32}$$

for $0 \le t \le \min(T_*, T_3)$.

To obtain further estimates, we recall the following elliptic regularity result: if $(c, w) \in D_0^1 \cap D^{1,r}$ is a solution of the elliptic system

$$-\kappa \Delta c = F \quad \text{and} \quad Lw = G \quad \text{in} \quad \Omega$$

with $(F, G) \in L^r$ for some $r \in (1, \infty)$, then $(c, w) \in D^{2,r}$,

$$|c|_{D^{2,r}} \le C(|F|_{L^r} + |c|_{D^{1,r}}) \quad \text{and} \quad |w|_{D^{2,r}} \le C(|G|_{L^r} + |w|_{D^{1,r}}). \tag{2.33}$$

For a detailed proof, one may refer to [1]. It should be noted that the estimate (2.33) holds for both bounded and unbounded domains. Applying the elliptic regularity result (2.33) to the equation $-\kappa\Delta e = F$ in Ω, where $F = \rho(h - e_t - v \cdot \nabla e) - p \operatorname{div} v + Q(\nabla v)$, we have

$$\begin{aligned} &|\nabla e|_{H^1} \\ &\le C(|\rho h|_{L^2} + |\rho e_t|_{L^2} + |\rho v \cdot \nabla e|_{L^2} + |\rho e \operatorname{div} v|_{L^2} + |Q(\nabla v)|_{L^2} + |\nabla e|_{L^2}) \\ &\le Cc_2^2 (1 + |\sqrt{\rho} e_t|_{L^2} + |\nabla e|_{L^2}) \end{aligned}$$

and

$$\begin{aligned} &C^{-1} \int_0^t |\nabla e|_{W^{1,q}}^2\, ds \\ &\le \int_0^t (|\rho h|_{L^q}^2 + |\rho e_t|_{L^q}^2 + |\rho v \cdot \nabla e|_{L^q}^2 + |\rho e \operatorname{div} v|_{L^q}^2 + |Q(\nabla v)|_{L^q}^2 + |\nabla e|_{L^q}^2) ds \\ &\le \int_0^t (c_0^2 |h|_{L^q}^2 + c_0^2(|\sqrt{\rho} e_t|_{L^2}^2 + |\nabla e_t|_{L^2}^2) + c_0^2 c_2^2 |\nabla e|_{H^1}^2 + c_2^2 |\nabla v|_{W^{1,q}}^2) ds \end{aligned}$$

for $0 \le t \le \min(T_*, T_1)$. Hence by virtue of (2.32), we deduce that

$$|e(t)|_{D^2} \le Cc_2^5 \quad \text{and} \quad \int_0^t |e(s)|_{D^{2,q}}^2\, ds \le Cc_2^7 \tag{2.34}$$

for $0 \le t \le \min(T_*, T_3)$. Finally, recalling that $p = (\gamma - 1)\rho e$, we also deduce from (2.32) and (2.34) that

$$|\nabla p(t)|_{L^2} \le Cc_1^3 c_2^{\frac{1}{2}}, \quad |\nabla p(t)|_{L^q} \le Cc_2^6 \quad \text{and} \quad |p_t(t)|_{L^2} \le Cc_2^5 \tag{2.35}$$

for $0 \le t \le \min(T_*, T_3)$.

Remark 2.3. *We emphasize that* $|\nabla p(t)|_{L^2}$ *depends on a half power of* c_2 *for all sufficiently small time* t*. This is the key observation to proving Theorem* 3.1*, the main result in this paper.*

2.3. **Estimates for the velocity.** To derive estimates for the velocity u, we differentiate the equation (2.3) with respect to t, multiply by u_t and integrate over Ω. Then using the equation (2.1), we derive

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2\,dx + \int \left(\mu|\nabla u_t|^2 + (\lambda+\mu)(\mathrm{div}u_t)^2\right)\,dx \\ &= \int \left(-\nabla p_t + (\rho f)_t - \rho_t v\cdot\nabla u - \rho(2v\cdot\nabla u_t + v_t\cdot\nabla u)\right)\cdot u_t\,dx.\end{aligned} \tag{2.36}$$

Making repeated use of Hölder, Young's and Sobolev inequalities, we easily deduce that

$$\begin{aligned}&\frac{d}{dt}|\sqrt{\rho}u_t|_{L^2}^2 + \mu|\nabla u_t|_{L^2}^2 \\ &\le C\left(|p_t|_{L^2}^2 + |\rho|_{L^\infty}|\sqrt{\rho}u_t|_{L^2}^2 + \left(|\rho|_{L^\infty}^2 + |\nabla\rho|_{L^3}^2\right)|f_t|_{H^{-1}}^2\right.\\ &\qquad \left.+\,|\rho_t|_{L^3}^2|f|_{L^2}^2 + |\rho_t|_{L^3}^2|v|_{L^\infty}^2|\nabla u|_{L^2}^2 + |\rho|_{L^\infty}|v|_{L^\infty}^2|\sqrt{\rho}u_t|_{L^2}^2\right)\\ &\qquad +\,\eta^{-1}C|\rho|_{L^\infty}|\nabla u|_{L^2}|\nabla u|_{H^1} + \eta|\nabla v_t|_{L^2}^2|\sqrt{\rho}u_t|_{L^2}^2.\end{aligned}$$

Then it follows from the estimates (2.25), (2.26) and (2.35) that

$$\begin{aligned}\frac{d}{dt}|\sqrt{\rho}u_t|_{L^2}^2 + \mu|\nabla u_t|_{L^2}^2 &\le Cc_2^{10} + Cc_0^2|f_t|_{H^{-1}}^2 + C\left(c_2^3 + \eta|\nabla v_t|_{L^2}^2\right)|\sqrt{\rho}u_t|_{L^2}^2\\ &\quad + C\left(c_2^6 + \eta^{-2}c_0^2\right)|\nabla u|_{L^2}^2 + |\nabla u|_{H^1}^2\end{aligned}$$

for $0 \le t \le \min(T_*, T_3)$. Hence taking $\eta = c_1^{-1}$ and using the facts that

$$\rho(\tau)^{-\frac{1}{2}}\left(Lu(\tau) + \nabla p\,(\tau)\right) \to g_2 \quad \text{in} \quad L^2 \qquad \text{as} \quad \tau \to 0$$

and

$$|\nabla u(t)|_{L^2}^2 \le C|\nabla u_0|_{L^2}^2 + C\int_0^t |\nabla u_t|_{L^2}^2\,ds \quad \text{for} \quad 0 \le t \le \min(T_*, T_3),$$

we easily obtain

$$\begin{aligned}&|\nabla u(t)|_{L^2}^2 + |\sqrt{\rho}u_t(t)|_{L^2}^2 + \int_0^t |\nabla u_t|_{L^2}^2\,ds\\ &\le Cc_0^5 + C\int_0^t \left(c_2^6 + c_1^{-1}|\nabla v_t|_{L^2}^2\right)\left(|\nabla u|_{L^2}^2 + |\sqrt{\rho}u_t|_{L^2}^2\right)\,ds + C\int_0^t |\nabla u|_{H^1}^2\,ds\end{aligned}$$

for $0 \le t \le \min(T_*, T_3)$. In view of Gronwall's inequality, we thus have

$$|\nabla u(t)|_{L^2}^2 + |\sqrt{\rho}u_t(t)|_{L^2}^2 + \int_0^t |\nabla u_t|_{L^2}^2\,ds \le Cc_0^5 + C\int_0^t |\nabla u|_{H^1}^2\,ds \tag{2.37}$$

for $0 \le t \le \min(T_*, T_3)$.

To estimate $|\nabla u|_{H^1}$, we apply the elliptic regularity result (2.33) to the equation (2.3). Then it follows from (2.25), (2.26) and (2.35) that

$$
\begin{aligned}
|\nabla u|_{H^1} &\leq C\,(\,|\rho f|_{L^2} + |\rho u_t|_{L^2} + |\rho v \cdot \nabla u|_{L^2} + |\nabla p|_{L^2} + |\nabla u|_{L^2}\,) \\
&\leq C(c_1 + c_1^{\frac{1}{2}}|\sqrt{\rho} u_t|_{L^2} + c_1^2 |\nabla u|_{L^2}^{\frac{1}{2}} |\nabla u|_{H^1}^{\frac{1}{2}} + c_1^3 c_2^{\frac{1}{2}} + |\nabla u|_{L^2}) \\
&\leq C c_1^3 c_2^{\frac{1}{2}} + C c_1^4\,(|\sqrt{\rho} u_t|_{L^2} + |\nabla u|_{L^2}) + \frac{1}{2}|\nabla u(t)|_{H^1}
\end{aligned}
$$

and thus

$$
|\nabla u(t)|_{H^1} \leq C c_1^3 c_2^{\frac{1}{2}} + C c_1^4\,(|\nabla u|_{L^2} + |\sqrt{\rho} u_t|_{L^2})
$$

for $0 \leq t \leq \min(T_*, T_3)$. Therefore, substituting this into (2.37) and using Gronwall's inequality again, we conclude that

$$
\begin{aligned}
&|u(t)|_{D_0^1}^2 + |\sqrt{\rho} u_t(t)|_{L^2}^2 + \int_0^t |u_t(s)|_{D_0^1}^2\,ds \leq C c_0^5 \\
&\text{and} \qquad |u(t)|_{D^2} \leq C c_1^6 c_2^{\frac{1}{2}} \qquad \text{for} \quad 0 \leq t \leq \min(T_*, T_3).
\end{aligned}
\tag{2.38}
$$

Moreover, it follows from (2.33) with $r = q$ that if $0 \leq t \leq \min(T_*, T_3)$, then

$$
|\nabla u|_{W^{1,q}} \leq C\left(c_0 |f|_{L^q} + c_0(|\sqrt{\rho} u_t|_{L^2} + |\nabla u_t|_{L^2}) + c_0 c_2 |\nabla u|_{H^1} + c_2^7\right)
$$

and thus

$$
\int_0^t |\nabla u(s)|_{W^{1,q}}^2\,ds \leq C c_0^7 \quad \text{for} \quad 0 \leq t \leq \min(T_*, T_3). \tag{2.39}
$$

Combining (2.38) and (2.39), we finally conclude that

$$
\begin{aligned}
&|u(t)|_{D_0^1} + c_0^7 c_1^{-6} c_2^{-\frac{1}{2}}\,|u(t)|_{D^2} + |\sqrt{\rho} u_t(t)|_{L^2} \\
&\qquad + \int_0^t \left(|u_t(s)|_{D_0^1}^2 + |u(s)|_{D^{2,q}}^2\right) ds \leq C c_0^7
\end{aligned}
\tag{2.40}
$$

for $0 \leq t \leq \min(T_*, T_3)$. Note that the constant $C > 1$ in (2.40) is independent of any one of c_0, c_1 and c_2.

2.4. **Conclusion.** Let us define c_1, β and c_2 by

$$
c_1 = C c_0^7, \quad \beta = c_0^{-14} c_1^{13} \quad \text{and} \qquad c_2 = \beta c_1 = (c_0^{-1} c_1)^{14}, \tag{2.41}
$$

where $C > 1$ is the constant in the estimate (2.40). Then choosing any T_* such that $0 < T_* \leq T_{**} \equiv \min(T, T_3(c_2))$, we conclude from (2.25), (2.32), (2.34) and (2.40) that

$$
\begin{aligned}
&\sup_{0\le t\le T_*}\Big(|\rho(t)-\rho^\infty|_{W^{1,r}\cap W^{1,q}}+|\rho_t(t)|_{L^r\cap L^q}+|e(t)|_{D_0^1\cap D^2}\Big)\le Cc_2^7,\\
&\operatorname*{ess\,sup}_{0\le t\le T_*}|(\sqrt{\rho}e_t,\ \sqrt{\rho}u_t)(t)|_{L^2}+\int_0^{T_*}(|e_t(t)|^2_{D_0^1}+|e(t)|^2_{D^{2,q}})\,dt\le Cc_2^7,\\
&\sup_{0\le t\le T_*}\Big(\,|u(t)|_{D_0^1}+\beta^{-1}\,|u(t)|_{D^2}\,\Big)\\
&\qquad\qquad\qquad+\int_0^{T_*}\Big(|u_t(t)|^2_{D_0^1}+|u(t)|^2_{D^{2,q}}\Big)\,dt\le c_1.
\end{aligned}
\tag{2.42}
$$

Here it deserves to emphasize that the constants c_1, β, c_2 and T_* (or T_{**}) depend only on c_0 and the parameters of C, but not on the lower bound δ of the initial density ρ_0.

Now we can prove the key lemma to prove our main result.

Lemma 2.4. *Assume in addition to* (2.21) *that the initial data* (ρ_0,e_0,u_0) *satisfies the compatibility condition*

$$
-\kappa\Delta e_0-Q(\nabla u_0)=\rho_0^{\frac12}g_1 \quad \textit{and} \quad Lu_0+\nabla p_0=\rho_0^{\frac12}g_2 \quad \textit{in} \quad \Omega \tag{2.43}
$$

for some $(g_1,g_2)\in L^2$, *where* $p_0=(\gamma-1)\rho_0e_0$. *Assume further that*

$$
\sup_{0\le t\le T_*}\Big(\,|v(t)|_{D_0^1}+\beta^{-1}|v(t)|_{D^2}\Big)+\int_0^{T_*}\Big(|v_t(t)|^2_{D_0^1}+|v(t)|^2_{D^{2,q}}\Big)\,dt\le c_1
$$

for the positive constants c_1, β, $c_2=\beta c_1$ *and* T_*, *chosen as before and dependent only on* c_0 *(and of course on the parameters of* C*), where*

$$
c_0=2+\rho^\infty+|\rho_0-\rho^\infty|_{W^{1,r}\cap W^{1,q}}+|(e_0,u_0)|_{D_0^1\cap D^2}+|(g_1,g_2)|^2_{L^2}.
$$

Then there exists a unique strong solution (ρ,e,u) *to the linearized problem* (2.1)–(2.6) *in* $[0,T_*]$ *satisfying the estimate* (2.42) *as well as the regularity*

$$
\begin{gathered}
\rho-\rho^\infty\in C([0,T_*];W^{1,r}\cap W^{1,q}),\quad \rho_t\in C([0,T_*];L^r\cap L^q),\\
(e,u)\in C([0,T_*];D_0^1\cap D^2)\cap L^2(0,T_*;D^{2,q}),\\
(e_t,u_t)\in L^2(0,T_*;D_0^1)\quad \textit{and}\quad (\sqrt{\rho}e_t,\sqrt{\rho}u_t)\in L^\infty(0,T_*;L^2).
\end{gathered}
$$

Remark 2.5. *It follows from Lemma* 2.1 *that the solution* ρ *of the linear transport equation* (2.1) *indeed exists for the whole interval* $[0,T]$. *Then following the arguments in* [1], *we can also prove the global existence of the solution* (e,u).

Proof. We define $\rho_0^\delta=\rho_0+\delta$ for each $\delta\in(0,1)$. Then from the compatibility condition (2.43), we derive

$$
-\kappa\Delta e_0-Q(\nabla u_0)=(\rho_0^\delta)^{\frac12}g_1^\delta \quad \text{and} \quad Lu_0+(\gamma-1)\nabla\,(\rho_0^\delta e_0)=(\rho_0^\delta)^{\frac12}g_2^\delta,
$$

where

$$g_1^\delta = \left(\frac{\rho_0}{\rho_0^\delta}\right)^{\frac{1}{2}} g_1 \quad \text{and} \quad g_2^\delta = \left(\frac{\rho_0}{\rho_0^\delta}\right)^{\frac{1}{2}} g_2 + (\gamma-1)\frac{\delta}{(\rho_0^\delta)^{\frac{1}{2}}}\nabla e_0.$$

Moreover we observe that for all small $\delta > 0$,

$$c_0 \geq 1 + (\rho^\infty + \delta) + |\rho_0^\delta - (\rho^\infty + \delta)|_{W^{1,r}\cap W^{1,q}} + |(e_0, u_0)|_{D_0^1 \cap D^2} + |(g_1^\delta, g_2^\delta)|_{L^2}^2.$$

Hence from the previous results for positive initial densities, we deduce that corresponding to the initial data $(\rho_0^\delta, e_0, u_0)$ with small $\delta > 0$, there exists a unique strong solution $(\rho^\delta, e^\delta, u^\delta)$ of the linearized equations (2.1)–(2.3) satisfying the local estimate (2.42). From this uniform estimate on δ, we conclude that a subsequence of solutions $(\rho^\delta, e^\delta, u^\delta)$ converges to a limit (ρ, e, u) in an obvious weak or weak-* sense. It is then easy to show that (ρ, e, u) is a weak solution to the linearized problem (2.1)–(2.6) in $[0, T_*]$. Finally, thanks to the lower semi-continuity of various norms, we find that (ρ, e, u) also satisfies the estimate (2.42). This proves the existence of a strong solution (ρ, e, u) with the regularity

$$\begin{aligned} &\rho - \rho^\infty \in L^\infty(0, T_*; W^{1,r} \cap W^{1,q}), \quad \rho_t \in L^\infty(0, T_*; L^r \cap L^q), \\ &(e, u) \in L^\infty(0, T_*; D_0^1 \cap D^2) \cap L^2(0, T_*; D^{2,q}), \\ &(e_t, u_t) \in L^2(0, T_*; D_0^1) \quad \text{and} \quad (\sqrt{\rho} e_t, \sqrt{\rho} u_t) \in L^\infty(0, T_*; L^2). \end{aligned} \tag{2.44}$$

Now we prove the uniqueness of solutions in this regularity class. Let (ρ_1, e_1, u_1) and (ρ_2, e_2, u_2) be two solutions to the problem (2.1)–(2.6) satisfying the regularity (2.44), and we denote

$$\overline{\rho} = \rho_1 - \rho_2, \quad \overline{e} = e_1 - e_2 \quad \text{and} \quad \overline{u} = u_1 - u_2.$$

First, since $\overline{\rho} \in L^\infty(0, T_*; L^q)$ is a solution of the linear transport equation $\overline{\rho}_t + \operatorname{div}(\overline{\rho} v) = 0$, it follows from a uniqueness result by R. J. DiPerna and P. L. Lions in [9] that $\overline{\rho} = 0$, that is, $\rho_1 = \rho_2$ in $[0, T_*] \times \Omega$. Next, to show that $\overline{e} = 0$ in $[0, T_*] \times \Omega$, we multiply the both sides of

$$\rho_1 \overline{e}_t + \rho_1 v \cdot \nabla \overline{e} - \kappa \Delta \overline{e} + (\gamma - 1)\rho_1 \overline{e} \operatorname{div} v = 0. \tag{2.45}$$

by $\overline{e}$ and integrate over $(0, t) \times \Omega$. Then since $(\rho_1)_t + \operatorname{div}(\rho_1 v) = 0$ and $\overline{e}(0) = 0$ in Ω, we deduce at least *formally* that

$$\begin{aligned} \frac{1}{2}\int \rho_1 |\overline{e}|^2(t)\, dx + \kappa \int_0^t\!\!\int |\nabla \overline{e}|^2\, dxds \\ = -(\gamma - 1)\int_0^t\!\!\int \rho_1 |\overline{e}|^2 |\operatorname{div} v|\, dxds \end{aligned} \tag{2.46}$$

and applying Gronwall's inequality, we also conclude that $\overline{e} = 0$ in $(0, T) \times \Omega$. But this argument is somewhat formal since it is not obvious that $\sqrt{\rho_1}\overline{e} \in L^\infty(0, T_*; L^2)$ and $\overline{e} \in L^2(0, T_*; H_0^1)$ for the case of unbounded domains. Hence we have to justify this formal argument by deriving the identity (2.46) rigorously. For this purpose, we assume that Ω is an unbounded domain and define $\overline{e}_R \in L^\infty(0,T_*;H_0^1(\Omega_R))$ by

$$\overline{e}_R(t,x) = \overline{e}(t,x)\,\varphi_R(x) \qquad \text{for} \quad (t,x)\in[0,T_*]\times\Omega,$$

where φ_R is the same cut-off function as in the proof of Lemma 2.1. Then from (2.45), we derive

$$\rho_1\,(\overline{e}_R)_t + \rho_1 v\cdot\nabla\overline{e}_R - \kappa\varphi_R\Delta\overline{e} + (\gamma-1)\rho_1\overline{e}_R\,\mathrm{div}\,v = \rho_1\overline{e}\,v\cdot\nabla\varphi_R.$$

Hence multiplying this by $\overline{e}_R$ and integrating over $[0,t]\times\Omega$, we deduce that

$$\begin{aligned}\frac{1}{2}\int\rho_1|\overline{e}_R|^2(t)\,dx + \kappa\int_0^t\int\varphi_R^2|\nabla\overline{e}|^2\,dxds& \\ = -(\gamma-1)\int_0^t\int\rho_1|\overline{e}_R|^2\mathrm{div}\,v\,dxds + I_R(t)&,\end{aligned}\tag{2.47}$$

where the remainder term $I_R(t)$ satisfies

$$|I_R(t)| \le C\int_0^t\int(\,|\overline{e}_R||\nabla\overline{e}| + \rho_1|\overline{e}||\overline{e}_R||v|\,)\,|\nabla\varphi_R|\,dxds.$$

Since $\rho_1\in L^\infty(0,T_*;L^\infty)$ and $(\overline{e},v)\in L^\infty(0,T_*;D_0^1)$, it follows that

$$\begin{aligned}|I_R(t)| &\le C\int_0^t\int\rho_1|\overline{e}_R|^2\,dxds + \frac{C}{R}\int_0^t\int_{\Omega_R}\left(|\overline{e}||\nabla\overline{e}| + \rho_1|\overline{e}|^2|v|^2\right)\,dx \\ &\le C\int_0^t\int\rho_1|\overline{e}_R|^2\,dxds + \tilde{C}\end{aligned}$$

for some constant $\tilde{C}$ independent of R. Therefore, substituting this estimate into (2.47) and applying Gronwall's inequality, we conclude that

$$\sup_{0\le t\le T_*}|\sqrt{\rho_1}\overline{e}(t)|^2_{L^2(\Omega_{R/2})} + \int_0^{T_*}|\nabla\overline{e}(t)|^2_{L^2(\Omega_{R/2})}\,dt \le \tilde{C}.$$

It follows that $\sqrt{\rho_1}\overline{e}\in L^\infty(0,T_*;L^2)$. As a consequence, we can estimate $I_R(t)$ again to deduce that

$$\begin{aligned}|I_R(t)| &\le C\int_0^{T_*}\left(|\nabla\overline{e}|_{L^2} + |\rho_1|^{\frac{1}{2}}_{L^\infty}|\sqrt{\rho_1}\overline{e}|_{L^2}|v|_{L^\infty}\right)|\nabla\overline{e}|_{L^2(\Omega\setminus B_{R/2})}\,ds \\ &\le \tilde{C}\left(\int_0^{T_*}|\nabla\overline{e}|^2_{L^2(\Omega\setminus B_{R/2})}\,ds\right)^{\frac{1}{2}} \quad\to 0 \qquad \text{as} \quad R\to\infty.\end{aligned}$$

Hence letting $R\to\infty$ in (2.47), we derive the identity (2.46). From this identity, it follows that $\overline{e}=0$ in $(0,T)\times\Omega$. A similar argument also shows that $\overline{u}=0$. This completes the proof of the uniqueness.

Finally, we prove the time-continuity of the solution (ρ,e,u) with the regularity (2.44). The continuity of ρ follows immediately from Lemma 2.1 since the solutions in $L^\infty(0,T_*;L^q)$ of the linear transport equation (2.1) are unique. To show the time-continuity of (e,u), we first observe that $(e,u)\in$

$C([0,T_*];D_0^1)\cap C([0,T_*];D^2-weak)$. From the equations (2.2) and (2.3), we also observe that $(\,(\rho e_t)_t,(\rho u_t)_t\,)\in L^2(0,T_*;H^{-1})$. Then since $(\rho e_t,\rho u_t)\in L^2(0,T_*;H_0^1)$, it follows immediately that $(\rho e_t,\rho u_t)\in C([0,T_*];L^2)$. Hence we conclude that for each $t\in[0,T_*]$, $(e,u)=(e(t),u(t))\in D_0^1\cap D^2$ is a solution of the elliptic system $-\kappa\Delta e=F$ and $Lu=G$ in Ω for some $(F,G)\in C([0,T_*];L^2)$. Using the elliptic regularity estimate (2.33), we easily show that $(e,u)\in C([0,T_*];D^2)$. We have completed the proof of Lemma 2.4. $\square$

3. An existence result for polytropic fluids with $\kappa>0$

In this section, we consider the following initial boundary value problem for a viscous polytropic fluid with $\kappa>0$:

$$\rho_t+\mathrm{div}\,(\rho u)=0 \tag{3.1}$$
$$(\rho e)_t+\mathrm{div}\,(\rho e u)-\kappa\Delta e+p\,\mathrm{div}\,u=Q(\nabla u)+\rho h\quad\text{in }\ (0,T)\times\Omega, \tag{3.2}$$
$$(\rho u)_t+\mathrm{div}\,(\rho u\otimes u)+Lu+\nabla p=\rho f \tag{3.3}$$
$$(\rho,\,\rho e,\,\rho u)|_{t=0}=(\rho_0,\,\rho_0e_0,\,\rho_0u_0)\quad\text{in}\quad\Omega, \tag{3.4}$$
$$(e,\,u)=(0,\,0)\quad\text{on}\quad(0,T)\times\partial\Omega, \tag{3.5}$$
$$(\rho,\,e,\,u)(t,x)\to(\rho^\infty,\ 0,\ 0)\quad\text{as}\quad|x|\to\infty,\ (t,x)\in(0,T)\times\Omega. \tag{3.6}$$

Here we used the familiar notations: $Q(\nabla u)=\frac{\mu}{2}\left|\nabla u+\nabla u^T\right|^2+\lambda(\mathrm{div}\,u)^2$, $Lu=-\mu\Delta u-(\lambda+\mu)\nabla\mathrm{div}\,u$ and $p=(\gamma-1)\rho e$.

This section is devoted to proving the existence of a unique local solution with minimal regularity, which is the main result in the paper.

Theorem 3.1. *Let $\rho^\infty\ge 0$ and $q>3$ be fixed constants, and let us define r by*

$$r=2\quad\text{if}\quad\rho^\infty=0\qquad\text{and}\qquad r=2\ \text{ or }3\quad\text{if}\quad\rho^\infty>0.$$

Assume that the data (ρ_0,e_0,u_0,h,f) satisfies the regularity condition

$$\rho_0\ge 0,\quad \rho_0-\rho^\infty\in W^{1,r}\cap W^{1,q},\quad (e_0,u_0)\in D_0^1\cap D^2,$$
$$(h,f)\in C([0,T];L^2)\cap L^2(0,T;L^q)\quad\text{and}\quad(h_t,f_t)\in L^2(0,T;H^{-1})$$

and the compatibility condition

$$-\kappa\Delta e_0-Q(\nabla u_0)=\rho_0^{\frac12}g_1\quad\text{and}\quad Lu_0+\nabla p_0=\rho_0^{\frac12}g_2\quad\text{in}\quad\Omega \tag{3.7}$$

for some $(g_1,g_2)\in L^2$, where $p_0=(\gamma-1)\rho_0e_0$. Then there exist a small time $T_*>0$ and a unique strong solution (ρ,e,u) to the initial boundary value problem (3.1)–(3.6) *such that*

$$\begin{gathered}\rho-\rho^\infty\in C([0,T_*];W^{1,r}\cap W^{1,q_0}),\quad \rho_t\in C([0,T_*];L^r\cap L^{q_0}),\\ (e,u)\in C([0,T_*];D_0^1\cap D^2)\cap L^2(0,T_*;D^{2,q_0}),\\ (e_t,u_t)\in L^2(0,T_*;D_0^1)\quad\text{and}\quad(\sqrt{\rho}e_t,\sqrt{\rho}u_t)\in L^\infty(0,T_*;L^2),\end{gathered} \tag{3.8}$$

where $q_0 = \min(6, q)$.

Before providing a proof, we make a few remarks on this theorem.

Remark 3.2. *The proof of the uniqueness part of Theorem* 3.1 *also shows that the solution depends continuously in some weaker norms on the initial data. We refer the readers to* [3] *for a detailed result for the isentropic fluids.*

Remark 3.3. *Using a standard technique (see Section* 7.6 *in* [17] *for instance), we can obtain a similar result to Theorem* 3.1 *even though we impose an inhomogeneous boundary condition on the internal energy* e.

Proof of Theorem 3.1. It suffices to consider the case $3 < q \le 6$. Our proof will be based on the usual iteration argument and on the results (in particular, Lemma 2.4) in the last section.

Let us denote

$$c_0 = 2 + \rho^\infty + |\rho_0 - \rho^\infty|_{W^{1,r}\cap W^{1,q}} + |(e_0, u_0)|_{D_0^1\cap D^2} + |(g_1, g_2)|_{L^2}^2,$$

and we choose the positive constants c_1, β, c_2 and T_{**} as in Section 2.4, dependently only on c_0. Next, let $u^0 \in C([0,\infty); D_0^1 \cap D^2) \cap L^2(0,\infty; D^3)$ be the solution to the linear parabolic problem

$$w_t - \Delta w = 0 \quad \text{in} \quad (0,\infty)\times\Omega \qquad \text{and} \qquad w(0) = u_0 \quad \text{in} \quad \Omega.$$

Then taking a small time $T_1 \in (0, T_{**}]$, we have

$$\sup_{0\le t\le T_1}\Big(|u^0(t)|_{D_0^1} + \beta^{-1}|u^0(t)|_{D^2}\Big) + \int_0^{T_1}\Big(|u_t^0(t)|_{D_0^1}^2 + |u^0(t)|_{D^{2,q}}^2\Big)\,dt \le c_1.$$

Hence it follows from Lemma 2.4 that there exists a unique strong solution (ρ^1, e^1, u^1) to the linearized problem (2.1)–(2.6) with v replaced by u^0, which satisfies the regularity estimate (2.42) with T_* replaced by T_1. Similarly, we construct approximate solutions (ρ^k, e^k, u^k), inductively, as follows: assuming that u^{k-1} was defined for $k \ge 1$, let (ρ^k, e^k, u^k) be the unique solution to the problem (2.1)–(2.6) with v replaced by u^{k-1}. Then it also follows from Lemma 2.4 that there exists a constant $\tilde{C} > 1$ such that

$$\begin{aligned}
&\sup_{0\le t\le T_1}\Big(|\rho^k(t) - \rho^\infty|_{W^{1,r}\cap W^{1,q}} + |\rho_t^k(t)|_{L^r\cap L^q}\Big)\\
&+ \sup_{0\le t\le T_1}|(e^k, u^k)(t)|_{D_0^1\cap D^2} + \operatorname*{ess\,sup}_{0\le t\le T_1}|(\sqrt{\rho^k}e_t^k, \sqrt{\rho^k}u_t^k)(t)|_{L^2}\\
&+ \int_0^{T_1}\Big(|(e_t^k, u_t^k)(t)|_{D_0^1}^2 + |(e^k, u^k)(t)|_{D^{2,q}}^2\Big)\,dt \le \tilde{C}
\end{aligned} \tag{3.9}$$

for all $k \ge 1$. Throughout the proof, we denote by $\tilde{C}$ a generic constant depending only on c_0 and the parameters of the constant C, but independent of k.

From now on, we show that the full sequence (ρ^k, e^k, u^k) converges to a solution to the original nonlinear problem (3.1)–(3.6) in a strong sense. Let us define

$$\overline{\rho}^{k+1} = \rho^{k+1} - \rho^k, \quad \overline{e}^{k+1} = e^{k+1} - e^k, \quad \overline{u}^{k+1} = u^{k+1} - u^k.$$

Then from (3.1)–(3.3), we derive the equations for the differences

$$\overline{\rho}_t^{k+1} + \operatorname{div}(\overline{\rho}^{k+1}u^k) + \operatorname{div}(\rho^k\overline{u}^k) = 0, \tag{3.10}$$

$$\begin{aligned}
&\rho^{k+1}\overline{e}_t^{k+1} + \rho^{k+1}u^k\cdot\nabla\overline{e}^{k+1} - \kappa\Delta\overline{e}^{k+1}\\
&= Q(\nabla u^k) - Q(\nabla u^{k-1}) - \overline{\rho}^{k+1}e_t^k\\
&\quad + \overline{\rho}^{k+1}\left(h - u^{k-1}\cdot\nabla e^k - (\gamma-1)e^k\operatorname{div}u^{k-1}\right)\\
&\quad - \rho^{k+1}\left(\overline{u}^k\cdot\nabla e^k + (\gamma-1)\overline{e}^{k+1}\operatorname{div}u^k + (\gamma-1)e^k\operatorname{div}\overline{u}^k\right),
\end{aligned}\tag{3.11}$$

$$\begin{aligned}
&\rho^{k+1}\overline{u}_t^{k+1} + \rho^{k+1}u^k\cdot\nabla\overline{u}^{k+1} + L\overline{u}^{k+1}\\
&= \overline{\rho}^{k+1}\left(f - u_t^k - u^{k-1}\cdot\nabla u^k\right) - \rho^{k+1}\overline{u}^k\cdot\nabla u^k\\
&\quad - (\gamma-1)\nabla(\rho^{k+1}\overline{e}^{k+1} - \overline{\rho}^{k+1}e^k).
\end{aligned}\tag{3.12}$$

First, we consider the case that $\rho^\infty > 0$. Multiplying (3.10) by $\overline{\rho}^{k+1}$ and integrating over Ω, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int|\overline{\rho}^{k+1}|^2\,dx\\
&\le C\int\left(|\nabla u^k||\overline{\rho}^{k+1}|^2 + |\nabla\rho^k||\overline{u}^k||\overline{\rho}^{k+1}| + \rho^k|\nabla\overline{u}^k||\overline{\rho}^{k+1}|\right)dx\\
&\le C\left(|\nabla u^k|_{W^{1,q}}|\overline{\rho}^{k+1}|_{L^2}^2 + (|\nabla\rho^k|_{L^3} + |\rho^k|_{L^\infty})|\nabla\overline{u}^k|_{L^2}|\overline{\rho}^{k+1}|_{L^2}\right).
\end{aligned}$$

Hence, by virtue of Young's inequality, we have

$$\frac{d}{dt}|\overline{\rho}^{k+1}|_{L^2}^2 \le A_\eta^k(t)|\overline{\rho}^{k+1}|_{L^2}^2 + \eta|\nabla\overline{u}^k|_{L^2}^2, \tag{3.13}$$

where $A_\eta^k(t) = C|\nabla u^k(t)|_{W^{1,q}} + \eta^{-1}C\left(|\nabla\rho^k(t)|_{L^3}^2 + |\rho^k(t)|_{L^\infty}^2\right)$. Notice from the estimate (3.9) that $A_\eta^k(t) \in L^1(0,T_1)$ and $\int_0^t A_\eta^k(s)\,dt \le \tilde{C} + \tilde{C}_\eta t$ for all $k \ge 1$ and $t \in [0,T_1]$. Here we denote by $\tilde{C}_\eta$ a generic positive constant depending only on η^{-1} and the parameters of $\tilde{C}$, where $\eta \in (0,1)$ is a small number.

Next, multiplying (3.11) by $\overline{e}^{k+1}$, integrating over Ω and recalling that

$$(\rho^{k+1})_t + \operatorname{div}(\rho^{k+1}u^k) = 0 \qquad \text{in} \quad \Omega, \tag{3.14}$$

we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho^{k+1}|\overline{e}^{k+1}|^2\,dx + \kappa\int |\nabla\overline{e}^{k+1}|^2\,dx\\
&\leq C\int \Big[\Big(|\nabla u^k| + |\nabla u^{k-1}|\Big)|\nabla\overline{u}^k||\overline{e}^{k+1}| + |\overline{\rho}^{k+1}||e_t^k||\overline{e}^{k+1}|\\
&\qquad + |\overline{\rho}^{k+1}|\Big(|h| + |u^{k-1}\cdot\nabla e^k| + |e^k\,\mathrm{div}\,u^{k-1}|\Big)|\overline{e}^{k+1}|\\
&\qquad + \rho^{k+1}\Big(|\overline{u}^k||\nabla e^k| + |\overline{e}^{k+1}||\mathrm{div}\,u^k| + |e^k||\mathrm{div}\,\overline{u}^k|\Big)|\overline{e}^{k+1}|\Big]\,dx.
\end{aligned}$$

By virtue of Hölder and Sobolev inequalities, we also have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho^{k+1}|\overline{e}^{k+1}|^2\,dx + \kappa\int |\nabla\overline{e}^{k+1}|^2\,dx\\
&\leq C\Big(|\nabla u^k|_{L^3} + |\nabla u^{k-1}|_{L^3}\Big)|\nabla\overline{u}^k|_{L^2}|\nabla\overline{e}^{k+1}|_{L^2} + C\int |\overline{\rho}^{k+1}||e_t^k||\overline{e}^{k+1}|\,dx\\
&\quad + C|\overline{\rho}^{k+1}|_{L^2}|h|_{L^3}|\nabla\overline{e}^{k+1}|_{L^2} + C|\overline{\rho}^{k+1}|_{L^2}|\nabla u^{k-1}|_{H^1}|\nabla e^k|_{H^1}|\nabla\overline{e}^{k+1}|_{L^2}\\
&\quad + C|\nabla u^k|_{L^\infty}|\sqrt{\rho^{k+1}}\overline{e}^{k+1}|_{L^2}^2 + C|\rho^{k+1}|_{L^\infty}^{\frac{1}{2}}|\nabla\overline{u}^k|_{L^2}|\nabla e^k|_{H^1}|\sqrt{\rho^{k+1}}\overline{e}^{k+1}|_{L^2}.
\end{aligned}$$

Hence it follows from (3.9) that

$$\begin{aligned}
&\frac{d}{dt}|\sqrt{\rho^{k+1}}\overline{e}^{k+1}|_{L^2}^2 + \kappa|\nabla\overline{e}^{k+1}|_{L^2}^2\\
&\qquad\leq B^k(t)\Big(|\overline{\rho}^{k+1}|_{L^2}^2 + |\sqrt{\rho^{k+1}}\overline{e}^{k+1}|_{L^2}^2\Big)\\
&\qquad\quad + \tilde{C}|\nabla\overline{u}^k|_{L^2}^2 + C\int |\overline{\rho}^{k+1}||e_t^k||\overline{e}^{k+1}|\,dx
\end{aligned}\tag{3.15}$$

for some $B^k(t) \in L^1(0,T_1)$ such that $\int_0^t B^k(s)\,ds \leq \tilde{C}$ for $0 \leq t \leq T_1$ and $k \geq 1$.

Finally, multiplying (3.12) by $\overline{u}^{k+1}$ and integrating over Ω, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho^{k+1}|\overline{u}^{k+1}|^2\,dx + \mu\int |\nabla\overline{u}^{k+1}|^2\,dx\\
&\leq C\int \Big[|\overline{\rho}^{k+1}|\Big(|f| + |u_t^k| + |u^{k-1}\cdot\nabla u^k|\Big)|\overline{u}^{k+1}| + |\rho^{k+1}||\overline{u}^k||\nabla u^k||\overline{u}^{k+1}|\\
&\qquad + \Big(|\rho^{k+1}||\overline{e}^{k+1}| + |\overline{\rho}^{k+1}||e^k|\Big)|\nabla\overline{u}^{k+1}|\Big]\,dx.
\end{aligned}$$

Then it also follows from (3.9) that

$$\begin{aligned}
&\frac{d}{dt}|\sqrt{\rho^{k+1}}\overline{u}^{k+1}|_{L^2}^2 + \mu|\nabla\overline{u}^{k+1}|_{L^2}^2\\
&\leq D_\eta^k(t)\Big(|\overline{\rho}^{k+1}|_{L^2}^2 + |\sqrt{\rho^{k+1}}\overline{u}^{k+1}|_{L^2}^2\Big) + \tilde{C}|\sqrt{\rho^{k+1}}\overline{e}^{k+1}|_{L^2}^2\\
&\quad + \eta|\nabla\overline{u}^k|_{L^2}^2 + C\int |\overline{\rho}^{k+1}||u_t^k||\overline{u}^{k+1}|\,dx
\end{aligned}\tag{3.16}$$

for some $D_\eta^k(t) \in L^1(0,T_1)$ such that $\int_0^t D_\eta^k(s)\,ds \le \tilde{C} + \tilde{C}_\eta t$ for $0 \le t \le T_1$ and $k \ge 1$.

Therefore, combining (3.13), (3.15) and (3.16) and defining

$$\psi^{k+1}(t) = |\overline{\rho}^{k+1}|_{L^2}^2 + \frac{\eta}{\kappa}|\sqrt{\rho^{k+1}}\overline{e}^{k+1}|_{L^2}^2 + |\sqrt{\rho^{k+1}}\overline{u}^{k+1}|_{L^2}^2,$$

we deduce that

$$\begin{aligned} &\frac{d}{dt}\psi^{k+1} + \eta|\nabla\overline{e}^{k+1}|_{L^2}^2 + \mu|\nabla\overline{u}^{k+1}|_{L^2}^2 \\ &\le E_\eta^k(t)\psi^{k+1} + 3\eta\tilde{C}|\nabla\overline{u}^k|_{L^2}^2 \\ &\qquad + C\int |\overline{\rho}^{k+1}|\left(\eta|e_t^k||\overline{e}^{k+1}| + |u_t^k||\overline{u}^{k+1}|\right)dx \end{aligned} \tag{3.17}$$

for some $E_\eta^k(t) \in L^1(0,T_1)$ such that $\int_0^t E_\eta^k(s)\,ds \le \tilde{C} + \tilde{C}_\eta t$ for $0 \le t \le T_1$ and $k \ge 1$.

To estimate the last integral term in (3.17), we assume for the time being that Ω is an unbounded domain in $\mathbf{R}^3$. Then since $\rho_0 - \rho^\infty \in W^{1,q}$ and $W^{1,q} \hookrightarrow C_0$, where C_0 consists of continuous functions on $\overline{\Omega}$ vanishing at infinity, we can choose a large radius $R > 1$ (of course, independent of k) so that

$$\frac{3}{4}\rho^\infty \le \rho_0(x) \le \frac{5}{4}\rho^\infty \qquad \text{for} \quad x \in \Omega \setminus B_{R/2}, \tag{3.18}$$

where $B_{R/2}$ is the open ball of radius $R/2$ centered at the origin, and since

$$|\rho^{k+1}(t) - \rho_0|_{L^\infty} \to 0 \qquad \text{as} \quad t \to 0,$$

there exists a small time $T_2 \in (0,T_1)$ such that

$$\frac{3}{8}\rho^\infty \le \rho^{k+1}(t,x) \le \frac{5}{2}\rho^\infty \qquad \text{for} \quad (t,x) \in [0,T_2] \times (\Omega \setminus B_R). \tag{3.19}$$

It is easy to show that T_2 can be chosen independently of k. For this purpose, we first observe that

$$\begin{aligned} &\rho^{k+1}(t,x) \\ &= \rho_0(U^{k+1}(0,t,x))\,\exp\left[-\int_0^t \operatorname{div} u^k(s,U^{k+1}(s,t,x))\,ds\right], \end{aligned} \tag{3.20}$$

where $U^{k+1} = U^{k+1}(t,s,x)$ is the solution to the initial value problem

$$\begin{cases} \dfrac{\partial}{\partial t}U^{k+1}(t,s,x) = u^k(t,U^{k+1}(t,s,x)), & 0 \le t \le T_1 \\ U^{k+1}(s,s,x) = x, \qquad 0 \le s \le T_1, \ x \in \overline{\Omega}. \end{cases}$$

Moreover, in view of (3.9), we deduce that

$$\int_0^t \left|\operatorname{div} u^k(s,U^{k+1}(s,t,x))\right| ds \le \int_0^t |\nabla u^k|_{L^\infty}\,ds \le \tilde{C}t^{\frac{1}{2}} \le \ln 2$$

and

$$\begin{aligned}\left|U^{k+1}(0,t,x)-x\right| &= \left|U^{k+1}(0,t,x)-U^{k+1}(t,t,x)\right| \\ &\le \int_0^t \left|u^k(\tau,U^{k+1}(\tau,t,x))\right| d\tau \le \tilde{C}t \le \frac{R}{2}\end{aligned}$$

for all (t,x) in $[0,T_2]\times\Omega$, where T_2 is a small positive time depending only on T_1 and the parameters of $\tilde{C}$. In particular, it follows that if $0\le t\le T_2$ and $x\in\Omega\setminus B_R$, then $U^{k+1}(0,t,x)\in\Omega\setminus B_{R/2}$. Hence the desired result (3.19) follows immediately from (3.18) and (3.20).

Remark 3.4. *The proof of* (3.19) *requires only that* $\rho_0-\rho^\infty\in C_0$. *This fact will be used later to prove Theorem* 4.1.

We are ready to estimate the integral term in (3.17) as follows: for $0\le t\le T_2$,

$$\begin{aligned}&C\int_{\Omega\cap B_R}|\overline{\rho}^{k+1}|\left(\eta|e_t^k||\overline{e}^{k+1}|+|u_t^k||\overline{u}^{k+1}|\right)dx \\ &\le \tilde{C}\left(|\nabla e_t^k|_{L^2}^2+|\nabla u_t^k|_{L^2}^2\right)|\overline{\rho}^{k+1}|_{L^2}^2+\frac14\left(\eta|\nabla\overline{e}^{k+1}|_{L^2}^2+\mu|\nabla\overline{u}^{k+1}|_{L^2}^2\right)\end{aligned}$$

and

$$\begin{aligned}&C\int_{\Omega\setminus B_R}|\overline{\rho}^{k+1}|\left(\eta|e_t^k||\overline{e}^{k+1}|+|u_t^k||\overline{u}^{k+1}|\right)dx \\ &\le \frac{C}{\sqrt{\rho^\infty}}\int|\overline{\rho}^{k+1}|\left(\eta|e_t^k||\sqrt{\rho^{k+1}}\overline{e}^{k+1}|+|u_t^k||\sqrt{\rho^{k+1}}\overline{u}^{k+1}|\right)dx \\ &\le \tilde{C}(\rho^\infty)\left(|\nabla e_t^k|_{L^2}^2+|\nabla u_t^k|_{L^2}^2+1\right)\psi^{k+1} \\ &\qquad +\frac14\left(\eta|\nabla\overline{e}^{k+1}|_{L^2}^2+\mu|\nabla\overline{u}^{k+1}|_{L^2}^2\right)\end{aligned}$$

for some constant $\tilde{C}(\rho^\infty)$ depending also on ρ^∞. Therefore, substituting these estimates into (3.17), we obtain

$$\frac{d}{dt}\psi^{k+1}+\frac{\eta}{2}|\nabla\overline{e}^{k+1}|_{L^2}^2+\frac{\mu}{2}|\nabla\overline{u}^{k+1}|_{L^2}^2\le F_\eta^k(t)\psi^{k+1}+3\eta\tilde{C}|\nabla\overline{u}^k|_{L^2}^2 \tag{3.21}$$

for some $F_\eta^k(t)\in L^1(0,T_2)$ such that $\int_0^t F_\eta^k(t)\,ds\le\tilde{C}(\rho^\infty)+\tilde{C}_\eta t$ for $k\ge1$ and $0\le t\le T_2$. Note that this estimate holds also for a bounded domain Ω since we can choose a sufficiently large R so that $\Omega\subset B_R$.

Now recalling that $\psi^{k+1}(0)=0$ and using Gronwall's inequality, we deduce from (3.21) that

$$\begin{aligned}&\psi^{k+1}(t)+\int_0^t\left(\eta|\nabla\overline{e}^{k+1}|_{L^2}^2+\mu|\nabla\overline{u}^{k+1}|_{L^2}^2\right)ds \\ &\qquad\le\left(6\eta\tilde{C}\int_0^t|\nabla\overline{u}^k|_{L^2}^2\,ds\right)\exp\left(\tilde{C}(\rho^\infty)+\tilde{C}_\eta t\right).\end{aligned}$$

Hence choosing small constants $\eta > 0$ and $T_3 > 0$ so that

$$6\eta\tilde{C}\exp\left(\tilde{C}(\rho^\infty)\right) = \frac{1}{2} \qquad \text{and} \qquad \exp(\tilde{C}_\eta T_3) = \mu,$$

we easily deduce that

$$\sum_{k=1}^\infty \sup_{0\le t\le T_*} \psi^{k+1}(t) + \sum_{k=1}^\infty \int_0^{T_*} \left(\eta|\nabla \overline{e}^{k+1}|_{L^2}^2 + \mu|\nabla \overline{u}^{k+1}|_{L^2}^2\right) dt \le \tilde{C} < \infty,$$

where $T_* = \min(T_2, T_3)$.

Therefore, we conclude that the full sequence (ρ^k, e^k, u^k) converges to a limit (ρ, e, u) in the following strong sense:

$$\begin{cases} \rho^k - \rho^1 \to \rho - \rho^1 & \text{in} \quad L^\infty(0, T_*; L^2), \\ (e^k, u^k) \to (e, u) & \text{in} \quad L^2(0, T_*; D_0^1). \end{cases} \tag{3.22}$$

It is easy to show that the limit (ρ, e, u) is a weak solution to the original nonlinear problem (3.1)–(3.6). Furthermore, it follows from (3.9) that (ρ, e, u) satisfies the following regularity estimate:

$$\begin{aligned} &\operatorname*{ess\,sup}_{0\le t\le T_*} |(\sqrt{\rho}e_t, \sqrt{\rho}u_t)(t)|_{L^2} + \int_0^{T_*} \left(|(e_t, u_t)(t)|_{D_0^1}^2 + |(e, u)(t)|_{D^{2,q}}^2\right) dt \\ &+ \sup_{0\le t\le T_*} \left(|\rho(t) - \rho^\infty|_{W^{1,r}\cap W^{1,q}} + |\rho_t(t)|_{L^r\cap L^q} + |(e, u)(t)|_{D_0^1\cap D^2}\right) \le \tilde{C}. \end{aligned}$$

This proves the existence of a strong solution. Then adapting the arguments in the proof of Lemma 2.4, we can easily prove the time-continuity of the solution (ρ, e, u). One may also refer to [1] for a detailed proof. Now it remains to prove the uniqueness of the strong solutions. To prove the uniqueness, let (ρ_1, e_1, u_1) and (ρ_2, e_2, u_2) be two strong solutions to the problem (3.1)–(3.6) with the regularity (3.8) and we denote by $(\overline{\rho}, \overline{e}, \overline{u})$ their difference. Then following the same arguments as in the derivations of (3.13), (3.15) and (3.16), we can show that

$$\begin{aligned} &\frac{d}{dt}\left(|\overline{\rho}|_{L^2}^2 + |\sqrt{\rho_1}\overline{e}|_{L^2}^2\right) + \kappa|\nabla\overline{e}|_{L^2}^2 \\ &\le A(t)\left(|\overline{\rho}|_{L^2}^2 + |\sqrt{\rho_1}\overline{e}|_{L^2}^2\right) + \widetilde{C}|\nabla\overline{u}|_{L^2}^2 + C\int |\overline{\rho}||(e_2)_t||\overline{e}|\,dx \end{aligned}$$

and

$$\begin{aligned} &\frac{d}{dt}|\sqrt{\rho_1}\overline{u}|_{L^2}^2 + \mu|\nabla\overline{u}|_{L^2}^2 \\ &\le B(t)\left(|\overline{\rho}|_{L^2}^2 + |\sqrt{\rho_1}\overline{e}|_{L^2}^2 + |\sqrt{\rho_1}\overline{u}|_{L^2}^2\right) + C\int |\overline{\rho}||(u_2)_t||\overline{u}|\,dx \end{aligned}$$

for some $A(t)$, $B(t) \in L^1(0,T_*)$. Then from these results, it follows that

$$\frac{d}{dt}\left(|\overline{\rho}|_{L^2}^2 + |\sqrt{\rho_1}\overline{e}|_{L^2}^2 + \tilde{C}|\sqrt{\rho_1}\overline{u}|_{L^2}^2\right) + \left(\kappa|\overline{e}|_{D_0^1} + \tilde{C}|\overline{u}|_{D_0^1}^2\right)$$
$$\leq D(t)\left(|\overline{\rho}|_{L^2}^2 + |\sqrt{\rho_1}\overline{e}|_{L^2}^2 + \tilde{C}|\sqrt{\rho_1}\overline{u}|_{L^2}^2\right) + \tilde{C}\int |\overline{\rho}|\left(|(e_2)_t||\overline{e}| + |(u_2)_t||\overline{u}|\right)dx$$

for some $D(t) \in L^1(0,T_*)$. We observe that

$$\sup_{0\leq t\leq T_*}|\rho_1(t)-\rho^\infty|_{L^\infty(\Omega\setminus B_R)} \leq C\sup_{0\leq t\leq T_*}|\rho_1(t)-\rho^\infty|_{W^{1,q}(\Omega\setminus B_R)} \to 0$$

as $R\to\infty$. Hence, following the arguments used to derive (3.21), we easily deduce that

$$\frac{d}{dt}\left(|\overline{\rho}|_{L^2}^2 + |\sqrt{\rho_1}\overline{e}|_{L^2}^2 + \tilde{C}|\sqrt{\rho_1}\overline{u}|_{L^2}^2\right) + \frac{\kappa}{2}|\overline{e}|_{D_0^1} + \tilde{C}|\overline{u}|_{D_0^1}^2$$
$$\leq E(t)\left(|\overline{\rho}|_{L^2}^2 + |\sqrt{\rho_1}\overline{e}|_{L^2}^2 + \tilde{C}|\sqrt{\rho_1}\overline{u}|_{L^2}^2\right)$$

for some $E(t) \in L^1(0,T_*)$. Therefore, in view of Gronwall's inequality, we conclude that $\overline{\rho}=\overline{e}=0$ and $\overline{u}=0$ in $(0,T_*)\times\Omega$. This completes the proof of the theorem for the case that $\rho^\infty>0$.

Now we consider the case that $\rho^\infty=0$. To prove the convergence in this case, we need to modify slightly the previous arguments. First, multiplying (3.10) by $sgn(\overline{\rho}^{k+1})|\overline{\rho}^{k+1}|^{\frac{1}{2}}$ and integrating over Ω, we obtain

$$\frac{d}{dt}\int|\overline{\rho}^{k+1}|^{\frac{3}{2}}\,dx \leq C\int|\nabla u^k||\overline{\rho}^{k+1}|^{\frac{3}{2}} + (\,|\nabla\rho^k||\overline{u}^k| + \rho^k|\nabla\overline{u}^k|\,)|\overline{\rho}^{k+1}|^{\frac{1}{2}}\,dx$$
$$\leq C|\nabla u^k|_{W^{1,q}}|\overline{\rho}^{k+1}|_{L^{\frac{3}{2}}}^{\frac{3}{2}} + C|\rho^k|_{H^1}|\nabla\overline{u}^k|_{L^2}|\overline{\rho}^{k+1}|_{L^{\frac{3}{2}}}^{\frac{1}{2}}.$$

Hence, multiplying this by $|\overline{\rho}^{k+1}|_{L^{\frac{3}{2}}}^{\frac{1}{2}}$, we have

$$\frac{d}{dt}|\overline{\rho}^{k+1}|_{L^{\frac{3}{2}}}^2 \leq A_\eta^k(t)|\overline{\rho}^{k+1}|_{L^{\frac{3}{2}}}^2 + \eta|\nabla\overline{u}^k|_{L^2}^2, \tag{3.23}$$

where $A_\eta^k(t) = C|\nabla u^k(t)|_{W^{1,q}} + \eta^{-1}C|\rho^k(t)|_{H^1\cap W^{1,q}}^2$. Notice from the uniform bound (3.9) that $\int_0^t A_\eta^k(s)\,dt \leq \tilde{C}+\tilde{C}_\eta t$ for all $k\geq 1$ and $t\in[0,T_1]$. In a similar manner, we can also show that

$$\frac{d}{dt}|\overline{\rho}^{k+1}|_{L^2}^2 \leq B_\eta^k(t)|\overline{\rho}^{k+1}|_{L^2}^2 + \eta|\nabla\overline{u}^k|_{L^2}^2 \tag{3.24}$$

for some $B_\eta^k(t)\in L^1(0,T_1)$ such that $\int_0^t B_\eta^k(s)\,dt \leq \tilde{C}+\tilde{C}_\eta t$ for $0\leq t\leq T_1$ and $k\geq 1$.

Next, multiplying (3.11) by $\overline{e}^{k+1}$ and integrating over Ω, we also deduce formally that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho^{k+1}|\overline{e}^{k+1}|^2\,dx + \kappa\int|\nabla\overline{e}^{k+1}|^2\,dx \\
&\le C\left(|\nabla u^k|_{L^3}+|\nabla u^{k-1}|_{L^3}\right)|\nabla\overline{u}^k|_{L^2}|\nabla\overline{e}^{k+1}|_{L^2} \\
&\quad + C|\overline{\rho}^{k+1}|_{L^{\frac{3}{2}}\cap L^2}\left(|e_t^k|_{D_0^1}+|h|_{L^3}+|\nabla u^{k-1}|_{H^1}|\nabla e^k|_{H^1}\right)|\nabla\overline{e}^{k+1}|_{L^2} \\
&\quad + C|\nabla u^k|_{L^\infty}|\sqrt{\rho^{k+1}}\overline{e}^{k+1}|_{L^2}^2 + C|\rho^{k+1}|_{L^\infty}^{\frac{1}{2}}|\nabla\overline{u}^k|_{L^2}|\nabla e^k|_{H^1}|\sqrt{\rho^{k+1}}\overline{e}^{k+1}|_{L^2}
\end{aligned}$$

and thus

$$\begin{aligned}
&\frac{d}{dt}|\sqrt{\rho^{k+1}}\overline{e}^{k+1}|_{L^2}^2 + \kappa|\nabla\overline{e}^{k+1}|_{L^2}^2 \\
&\le D^k(t)\left(|\overline{\rho}^{k+1}|^2_{L^{\frac{3}{2}}\cap L^2} + |\sqrt{\rho^{k+1}}\overline{e}^{k+1}|_{L^2}^2\right) + \tilde{C}|\nabla\overline{u}^k|_{L^2}^2
\end{aligned} \tag{3.25}$$

for some $D^k(t)\in L^1(0,T_1)$ such that $\int_0^{T_1} D^k(t)\,dt\le\tilde{C}$ for $k\ge 1$.

Finally, from (3.12), we easily deduce that

$$\begin{aligned}
&\frac{d}{dt}|\sqrt{\rho^{k+1}}\overline{u}^{k+1}|_{L^2}^2 + \mu|\nabla\overline{u}^{k+1}|_{L^2}^2 \\
&\qquad\le E_\eta^k(t)\left(|\overline{\rho}^{k+1}|^2_{L^{\frac{3}{2}}\cap L^2} + |\sqrt{\rho^{k+1}}\overline{u}^{k+1}|_{L^2}^2\right) \\
&\qquad\quad + \tilde{C}|\sqrt{\rho^{k+1}}\overline{e}^{k+1}|_{L^2}^2 + \eta|\nabla\overline{u}^k|_{L^2}^2
\end{aligned} \tag{3.26}$$

for some $E_\eta^k(t)\in L^1(0,T_1)$ such that $\int_0^t E_\eta^k(s)\,ds\le\tilde{C}+\tilde{C}_\eta t$ for $0\le t\le T_1$ and $k\ge 1$.

Using the estimates (3.23)–(3.26) and following the same arguments as in the proof of (3.22), we can show that the full sequence (ρ^k, e^k, u^k) also converges to a limit (ρ, e, u) in the sense of (3.22). Then adapting the previous arguments for the case $\rho^\infty > 0$, we can easily prove that (ρ, e, u) is a unique solution to the problem (3.1)–(3.6) with the regularity (3.8). This completes the proof of Theorem 3.1. □

Remark 3.5. *Our proof is somewhat formal because it was not proved that $\overline{\rho}^{k+1}\in L^\infty(0,T_*;L^{\frac{3}{2}}\cap L^2)$ and $(\overline{e}^{k+1},\overline{u}^{k+1})\in L^2(0,T_*;H_0^1)$ for unbounded domains. But this can be easily remedied by means of the cut-off function $\varphi_R(x)$ defined in the proof of Lemma 2.1.*

Adapting the proof of Theorem 3.1, we can also prove the following existence result for strong solutions with higher regularity. We omit a detailed proof and refer the readers to [1] for the proof of a similar result on the barotropic fluid models.

Theorem 3.6. *Let (ρ_0, e_0, u_0, h, f) be a given data satisfying the hypotheses of Theorem* 3.1*. Assume in addition that*

$$\rho_0 - \rho^\infty \in H^2 \quad \text{and} \quad (h, f) \in L^2(0, T; H^1).$$

Then there exist a small time $T_* > 0$ and a unique strong solution (ρ, e, u) satisfying the regularity

$$\rho - \rho^\infty \in C([0, T_*]; H^2), \ \ \rho_t \in C([0, T_*]; H^1) \ \text{ and } \ (e, u) \in L^2(0, T_*; D^3)$$

as well as (3.8)*.*

4. Results for polytropic fluids with $\kappa = 0$

In this final section, we consider the initial boundary value problem for a viscous polytropic fluid with $\kappa = 0$:

$$\rho_t + \mathrm{div}\,(\rho u) = 0 \tag{4.1}$$

$$p_t + u \cdot \nabla p + \gamma p \,\mathrm{div}\, u = (\gamma - 1)\,(Q(\nabla u) + \rho h) \quad \text{in } (0, T) \times \Omega; \tag{4.2}$$

$$(\rho u)_t + \mathrm{div}\,(\rho u \otimes u) + Lu + \nabla p = \rho f \tag{4.3}$$

$$(\rho,\, p,\, \rho u)|_{t=0} = (\rho_0,\, p_0,\, \rho_0 u_0) \quad \text{in} \quad \Omega; \quad u = 0 \quad \text{on } (0, T) \times \partial\Omega, \tag{4.4}$$

$$(\rho, p, u)(t, x) \to (\rho^\infty, p^\infty, 0) \ \text{ as } \ |x| \to \infty, \ (t, x) \in (0, T) \times \Omega. \tag{4.5}$$

Recall again that $Q(\nabla u) = \frac{\mu}{2} \left|\nabla u + \nabla u^T\right|^2 + \lambda(\mathrm{div}\, u)^2$ and $Lu = -\mu \Delta u - (\lambda + \mu)\nabla \mathrm{div}\, u$.

We prove the following existence result for local strong solutions.

Theorem 4.1. *Assume that the data (ρ_0, p_0, u_0, h, f) satisfies the regularity condition*

$$\rho_0 \geq 0, \quad \rho_0 - \rho^\infty \in C_0 \cap H^1 \cap W^{1,3},$$
$$p_0 - p^\infty \in H^1 \cap W^{1,q}, \quad u_0 \in D_0^1 \cap D^2, \quad h = 0,$$
$$f \in C([0, T]; L^2) \cap L^2(0, T; L^q) \quad \text{and} \quad f_t \in L^2(0, T; H^{-1})$$

for some constants ρ^∞, p^∞ and $q > 3$, and the compatibility condition

$$Lu_0 + \nabla p_0 \ = \ \rho_0^{\frac{1}{2}}\, g \quad \text{in} \quad \Omega \quad \text{for some} \quad g \in L^2.$$

Then there exists a small time $T_* > 0$ and a unique strong solution (ρ, p, u) to the initial boundary value problem (4.1)–(4.5) *such that*

$$\rho - \rho^\infty \in C([0, T_*]; C_0 \cap H^1 \cap W^{1,3}), \quad p - p^\infty \in C([0, T_*]; H^1 \cap W^{1,q_0}),$$
$$\rho_t \in C([0, T_*]; L^2 \cap L^3), \quad p_t \in C([0, T_*]; L^2 \cap L^3) \cap L^2(0, T_*; L^{q_0}),$$
$$u \in C([0, T_*]; D_0^1 \cap D^2) \cap L^2(0, T_*; D^{2,q_0}),$$
$$u_t \in L^2(0, T_*; D_0^1) \quad \text{and} \quad \sqrt{\rho} u_t \in L^\infty(0, T_*; L^2),$$

where $q_0 = \min(6, q)$.

Remark 4.2. *In case that $\rho^\infty > 0$, the regularity $C_0 \cap H^1 \cap W^{1,3}$ of $\rho_0 - \rho^\infty$ can be replaced by $C_0 \cap W^{1,3}$. We can also prove a higher regularity result similar to Theorem* 3.6.

Remark 4.3. *Assuming that $h \in C([0,T]; L^2 \cap L^3) \cap L^2(0,T; H^1 \cap W^{1,q})$ and $\rho_0 - \rho^\infty \in H^1 \cap W^{1,q}$, we can prove a similar result to Theorem* 4.1.

Proof. To establish the existence of local strong solutions, we follow the same strategy as in the previous sections. Hence we consider the following linearized problem:

$$\rho_t + \operatorname{div}(\rho v) = 0 \tag{4.6}$$

$$p_t + v \cdot \nabla p + \gamma p \operatorname{div} v = (\gamma - 1) Q(\nabla v) \qquad \text{in } (0,T) \times \Omega; \tag{4.7}$$

$$(\rho u)_t + \operatorname{div}(\rho v \otimes u) + Lu + \nabla p = \rho f \tag{4.8}$$

$$(\rho,\, p,\, \rho u)|_{t=0} = (\rho_0,\, p_0,\, \rho_0 u_0) \quad \text{in} \quad \Omega; \quad u = 0 \quad \text{on } (0,T) \times \partial\Omega, \tag{4.9}$$

$$(\rho, p, u)(t,x) \to (\rho^\infty, p^\infty, 0) \ \text{ as } \ |x| \to \infty, \ (t,x) \in (0,T) \times \Omega. \tag{4.10}$$

where the known data (ρ_0, p_0, u_0, v) satisfies the following properties

$$\begin{gathered}
c_0 \geq 1 + \rho^\infty + p^\infty + |\rho_0 - \rho^\infty|_{C_0 \cap H^1 \cap W^{1,3}} + |p_0 - p^\infty|_{H^1 \cap W^{1,q}} \\
+ |u_0|_{D_0^1 \cap D^2} + |g|_{L^2}^2 \\
3 < q \leq 6, \quad \rho_0 \geq \delta > 0, \quad g = \rho_0^{-\frac{1}{2}} \left(Lu_0 + \nabla p_0 \right), \quad v(0) = u_0, \\
\sup_{0 \leq t \leq T_*} \Big(|v(t)|_{D_0^1} + \beta^{-1} |v(t)|_{D^2} \Big) + \int_0^{T_*} \Big(|v_t(t)|_{D_0^1}^2 + |v(t)|_{D^{2,q}}^2 \Big)\, dt \leq c_1
\end{gathered}$$

for some fixed constants c_0, c_1, β and time T_* such that

$$1 < c_0 < c_1 < c_2 = \beta c_1 \qquad \text{and} \qquad 0 < T_* \leq T.$$

Then we derive local a priori estimates for the strong solution (ρ, p, u) to the problem (4.6)–(4.10), which are analogous to the estimates (2.25), (2.34) and (2.40) in Section 2.

First, using (2.7) and (2.20), we easily deduce that

$$|\rho(t) - \rho^\infty|_{C_0 \cap H^1 \cap W^{1,3}} \leq C c_0 \exp\left(\int_0^t |\nabla v(s)|_{H^1 \cap W^{1,q}}\, ds \right) \tag{4.11}$$

and thus

$$|\rho(t) - \rho^\infty|_{C_0 \cap H^1 \cap W^{1,3}} \leq C c_0 \quad \text{and} \quad |\rho_t(t)|_{L^2 \cap L^3} \leq C c_2^2 \tag{4.12}$$

for $0 \leq t \leq \min(T_*, T_1)$, where $T_1 = c_2^{-1}$. To estimate the pressure p, we observe that $\pi = p - p^\infty$ is a solution of the linear transport equation

$$\pi_t + v \cdot \nabla \pi + \gamma \pi \operatorname{div} v = (\gamma - 1) Q(\nabla v) - \gamma p^\infty \operatorname{div} v \tag{4.13}$$

in $(0,T)\times\Omega$. Hence following the same arguments as in the derivation of (2.20), we can also show that

$$\tag{4.14} \begin{aligned} |\pi(t)|_{H^1\cap W^{1,q}} \le Cc_0\left(1+\int_0^t |Q(\nabla v)|_{H^1\cap W^{1,q}}\,ds\right) \\ \times\exp\left(C\int_0^t |\nabla v|_{H^1\cap W^{1,q}}\,ds\right) \end{aligned}$$

for $0\le t\le \min(T_*,T_1)$. On the other hand, from the interpolation inequality

$$|\nabla v|_{L^\infty}\le C|\nabla v|_{H^1}^{\theta}|\nabla v|_{W^{1,q}}^{1-\theta}\quad \text{for some}\quad \theta=\theta(q)\in(0,1),$$

it follows easily that

$$\tag{4.15} \begin{aligned} |Q(\nabla v)|_{H^1\cap L^q}\le C|\nabla v|_{H^1}^{1+\theta}|\nabla v|_{W^{1,q}}^{1-\theta}, \\ |Q(\nabla v)|_{L^2\cap L^3}\le C|\nabla v|_{H^1}^2,\quad |Q(\nabla v)|_{W^{1,q}}\le C|\nabla v|_{H^1}^{\theta}|\nabla v|_{W^{1,q}}^{2-\theta}. \end{aligned}$$

Therefore, using (4.14), (4.15) and the estimates

$$\int_0^t |\nabla v|_{H^1\cap W^{1,q}}\,ds\le t^{\frac12}\left[\int_0^t |\nabla v|_{H^1\cap W^{1,q}}^2\,ds\right]^{\frac12}\le 2(c_2t)^{\frac12},$$

$$\int_0^t |Q(\nabla v)|_{H^1}\,ds\le Cc_2t^{\frac{1+\theta}{2}}\quad\text{and}\quad \int_0^t |Q(\nabla v)|_{W^{1q}}\,ds\le Cc_2t^{\frac{\theta}{2}}$$

together with the equation (4.7), we conclude that

$$\tag{4.16} \begin{aligned} |p(t)-p^\infty|_{H^1\cap W^{1,q}}\le Cc_0, \\ |p_t(t)|_{L^2\cap L^3}\le Cc_2^2\quad\text{and}\quad \int_0^t |p_t(s)|_{L^q}^2\,ds\le Cc_2^3 \end{aligned}$$

for $0\le t\le\min(T_*,T_2)$, where $T_2=c_2^{-\frac{2}{\theta}}<T_1$. Note that the estimate for $|\nabla p|_{L^2}$ do not depend on c_2 contrary to (2.35) for the case that $\kappa>0$. To estimate the velocity u, we observe that the estimate (2.40) relies only on the estimates (2.25) and (2.35) for the density ρ and pressure p, not on the internal energy e. Hence adapting the arguments used to derive (2.40), we can easily show that

$$\tag{4.17} \begin{aligned} |u(t)|_{D_0^1}+c_0^7c_1^{-13}\,|u(t)|_{D^2}+|\sqrt{\rho}u_t(t)|_{L^2} \\ +\int_0^t\left(|u_t(s)|_{D_0^1}^2+|u(s)|_{D^{2,q}}^2\right)ds\le Cc_0^7 \end{aligned}$$

for $0\le t\le\min(T_*,T_3)$, where $T_3=\min(c_2^{-8},T_2)$.

Therefore, if we define

$$c_1=Cc_0^7,\quad \beta=c_0^{-7}c_1^{13}\quad\text{and}\quad c_2=\beta c_1=(c_0^{-1}c_1^2)^7$$

and choose any T_* such that $0 < T_* \leq T_{**} = \min(T, T_3(c_2))$, then we conclude from (4.12), (4.16) and (4.17) that

$$\sup_{0\leq t\leq T_*} \big(|\rho(t) - \rho^\infty|_{C_0\cap H^1\cap W^{1,3}} + |p(t) - p^\infty|_{H^1\cap W^{1,q}} + |\rho_t(t)|_{L^2\cap L^3}\big)$$

$$+ \operatorname*{ess\,sup}_{0\leq t\leq T_*} |\sqrt{\rho} u_t(t)|_{L^2} + \sup_{0\leq t\leq T_*} |p_t(t)|_{L^2\cap L^3} + c_2^{-1} \int_0^{T_*} |p_t(t)|_{L^q}^2\, dt \leq C c_2^2$$

and

$$\sup_{0\leq t\leq T_*} \Big(|u(t)|_{D_0^1} + \beta^{-1}|u(t)|_{D^2}\Big) + \int_0^{T_*} (|u_t(t)|_{D_0^1}^2 + |u(t)|_{D^{2,q}}^2)\, dt \leq c_1.$$

Based on these *a priori estimates*, we can prove the existence and regularity of a unique local solution (ρ, p, u) to the original nonlinear problem by following exactly the same arguments as in the proof of Theorem 3.1. This completes the proof of Theorem 4.1. □

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Department of Mathematics, Hokkaido University, Kita 8 Nishi 5, Sapporo 060-0808, Japan

E-mail address: `ygcho@math.sci.hokudai.ac.jp`

Mathematical Institute, Tohoku University, Sendai 980-8578, Japan

E-mail address: `h-kim@math.tohoku.ac.jp`