Spectral Properties of a Dirac Operator in the Chiral Quark Soliton Model

Asao Arai*, Kunimitsu Hayashi and Itaru Sasaki
Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan

December 16, 2004

Abstract

We consider a Dirac operator H acting in the Hilbert space $L^2(\mathbb{R}^3;\mathbb{C}^4)\otimes\mathbb{C}^2$, which describes a Hamiltonian of the chiral quark soliton model in nuclear physics. The mass term of H is a matrix-valued function formed out of a function $F:\mathbb{R}^3\to\mathbb{R}$, called a profile function, and a vector field $\mathbf{n}$ on $\mathbb{R}^3$, which fixes pointwise a direction in the iso-spin space of the pion. We first show that, under suitable conditions, H may be regarded as a generator of a supersymmetry. In this case, the spetra of H are symmetric with respect to the origin of $\mathbb{R}$. We then identify the essential spectrum of H under some condition for F. For a class of profile functions F, we derive an upper bound for the number of discrete eigenvalues of H. Under suitable conditions, we show the existence of a positive energy ground state or a negative energy ground state for a family of scaled deformations of H. A symmetry reduction of H is also discussed. Finally a unitary transformation of H is given, which may have a physical interpretation.

Keywords: Dirac operator, chiral quark soliton model, supersymmetry, spectrum, ground state

1 Introduction

Let σ_j $(j=1,2,3)$ be the Pauli matrices:

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.1}$$

and

$$\alpha_j := \begin{pmatrix} \sigma_j & 0_2 \\ 0_2 & -\sigma_j \end{pmatrix} \quad (j=1,2,3), \quad \beta := \begin{pmatrix} 0_2 & 1_2 \\ 1_2 & 0_2 \end{pmatrix}, \tag{1.2}$$

*Corresponding author: arai@math.sci.hokudai.ac.jp

where 0_2 and 1_2 are the 2×2 zero matrix and the 2×2 identity matrix respectively. The matrix

$$\gamma_5 := -i\alpha_1\alpha_2\alpha_3 \tag{1.3}$$

is Hermitian with $\gamma_5^2 = 1_4$ (the 4×4 identity matrix) satisfying the following relations:

$$[\alpha_j, \gamma_5] = 0 \ (j = 1, 2, 3), \quad \{\beta, \gamma_5\} = 0, \tag{1.4}$$

where $[A, B] := AB - BA$ and $\{A, B\} := AB + BA$. We set

$$\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \sigma_3), \quad \boldsymbol{\alpha} := (\alpha_1, \alpha_2, \alpha_3). \tag{1.5}$$

For objects $\mathbf{A} = (A_1, A_2, A_3)$ and $\mathbf{B} = (B_1, B_2, B_3)$ such that the products A_jB_j $(j = 1, 2, 3)$ and their sum are defined, we write $\mathbf{A}\cdot\mathbf{B} := \sum_{j=1}^3 A_jB_j$.

We consider a Dirac operator acting in the Hilbert space

$$\mathcal{H} := L^2(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathbb{C}^2, \tag{1.6}$$

where $L^2(\mathbb{R}^3; \mathbb{C}^4)$ is the Hilbert space of $\mathbb{C}^4$-valued square integrable functions on $\mathbb{R}^3$. Let $\nabla := (D_1, D_2, D_3)$ with D_j the generalized partial differential operator in the variable x_j, the j-th component of $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. Then the free Dirac operator with mass zero is defined by

$$H_0 := -i\boldsymbol{\alpha}\cdot\nabla \otimes 1_2 \tag{1.7}$$

acting in $\mathcal{H}$. To introduce a perturbation to H_0, let $F : \mathbb{R}^3 \to \mathbb{R}$ be Borel measurable and finite almost everywhere (a.e.) in $\mathbb{R}^3$ and set

$$U_F := \cos F + i\gamma_5 \otimes \boldsymbol{\tau}\cdot\mathbf{n}\sin F \tag{1.8}$$

where $\boldsymbol{\tau} := (\tau_1, \tau_2, \tau_3)$ with $\tau_j := \sigma_j$ $(j = 1, 2, 3)$, $\mathbf{n} := (n_1, n_2, n_3)$ with n_j a real-valued measurable function on $\mathbb{R}^3$ such that

$$|\mathbf{n}(\mathbf{x})|^2 = 1, \quad \text{a.e.}\mathbf{x} \in \mathbb{R}^3. \tag{1.9}$$

Let $M > 0$ be a constant. Then, by the second relation in (1.4), $M(\beta\otimes 1_2)U_F$ is a bounded self-adjoint operator on $\mathcal{H}$. Hence, by the Kato-Rellich theorem, the operator

$$H := H_0 + M(\beta \otimes 1_2)U_F \tag{1.10}$$

is self-adjoint with domain $D(H) = D(H_0)$. This is the Dirac operator we consider in this paper. The operator H appears as the Hamiltonian of the so-called the chiral quark soliton model in nuclear physics (e.g., [5] and references therein). In this context, M and

$$\Phi_F := \cos F + i\sin F \otimes \boldsymbol{\tau}\cdot\mathbf{n} \tag{1.11}$$

(U_F with γ_5 replaced by 1_4) denote the mass of a quark and the pion field respectively, and F is called a profile function. The Dirac operator H is not only physically important, but also may have interests from purely mathematical points of view. As far as we know, no mathematically rigorous analysis has been made on the Dirac operator H (a study of a Dirac operator with a variable mass is given in [1], but, in that paper, the mass is a scalar function).

The present paper is organized as follows. In Section 2, we show that the Dirac operator H can be regarded as a generator of a supersymmetry, and describe its implications on the spectra of H. In Section 3 we idetify the essential spectrum of H. We also derive an upper bound for the number of discrete eigenvalues of H. In particular, for a class of F and $\mathbf{n}$, the absence of discrete eigenvalues of H is proven. Sections 4 and 5 are concerned with existence of discrete eigenvalues of H. In Section 4 we introduce a concept of a positive energy ground state and that of a negative energy ground state of H and show, under some condition for F, that a scaled deformation of H has a positive energy ground state or a negative ground state. In Section 5 we discuss a symmetry reduction of H to smaller mutually orthogonal closed subspaces which are indexed by triples $(\ell, s, t) \in \mathbb{Z} \times \{\pm 1\} \times \{\pm 1\}$, where ℓ denote an eigenvalue of the third component of the angular momentum operator, $s/2$ the spin of the quark and $t/2$ the iso-spin of the pion. We prove that, under suitable conditions, each reduced part of H or its scaled version has a discrete positive ground state or a discrete negative ground state. In the last section we present a unitary transformation which brings H to a Dirac operator with a magnetic moment.

2 Supersymmetric Aspects

In this section we assume the following:

Hypothesis (I) Each n_j $(j = 1, 2, 3)$ is continuously differentiable on $\mathbb{R}^3$ and

$$(n_1(\mathbf{x}), n_2(\mathbf{x})) \neq (0, 0), \quad \mathbf{x} \in \mathbb{R}^3. \tag{2.1}$$

Let

$$\xi(\mathbf{x}) := \frac{(\tau_1 n_2(\mathbf{x}) - \tau_2 n_1(\mathbf{x}))}{\sqrt{n_1(\mathbf{x})^2 + n_2(\mathbf{x})^2}}, \quad \mathbf{x} \in \mathbb{R}^3. \tag{2.2}$$

Then $\xi(\mathbf{x})^2 = 1$, $\mathbf{x} \in \mathbb{R}^3$. For all $\mathbf{x} \in \mathbb{R}^3$, we can define a matrix tensor

$$\Gamma(\mathbf{x}) := \alpha_1 \alpha_2 \alpha_3 \beta \otimes \xi(\mathbf{x}) \tag{2.3}$$

acting on $\mathbb{C}^4 \otimes \mathbb{C}^2$. It is easy to see that $\Gamma(\mathbf{x})$ is self-adjoint with $\Gamma(\mathbf{x})^2 = I$ (I denotes identity). By the natural identification $\mathcal{H} = L^2(\mathbb{R}^3; \mathbb{C}^4 \otimes \mathbb{C}^2)$, we denote the multiplication operator by the matrix-tensor valued function $\Gamma(\cdot)$ by the same symbol Γ. Then Γ is self-adjoint and unitary on $\mathcal{H}$.

Proposition 2. 1 *Suppose that Hypothesis (I) holds and $\xi(\mathbf{x})$ is a constant matrix. Then, for all $\psi \in D(H)$, $\Gamma\psi \in D(H)$ and*

$$\{\Gamma, H\}\psi = 0, \quad \psi \in D(H). \tag{2.4}$$

Proof. By direct computations, we have

$$\{\alpha_1\alpha_2\alpha_3\beta, \alpha_j\} = 0\,(j=1,2,3), \quad \{\xi(\mathbf{x}), \boldsymbol{\tau}\cdot\mathbf{n}(\mathbf{x})\} = 0. \tag{2.5}$$

Using these relations and the constancy of $\xi(\cdot)$, we see that, for all $\psi \in D(H) = D(H_0)$, $\Gamma\psi \in D(H_0)$ and $H_0\Gamma\psi = -\Gamma H_0\psi$. Similarly, using (2.5) and $[\alpha_1\alpha_2\alpha_3\beta, \beta\gamma_5] = 0$, we see that $\{M(\beta\otimes 1_2)U_F, \Gamma\}\psi = 0$. Thus (2.4) follows. ■

Proposition 2. 1 shows that the Dirac operator H may be regarded as a generator of a supersymmetry, i.e., a supercharge with respect to Γ (e.g., [6, p.140]).

For a self-adjoint operator T, we denote by $\sigma(T)$ (resp. $\sigma_{\rm p}(T)$) the spectrum of T (resp. the point spectrum of T). The discrete spectrum of T (the set of isolated eigenvalues of T with finite multiplicity) is denoted $\sigma_{\rm d}(T)$.

Theorem 2. 2 *Suppose that Hypothesis (I) holds and $\xi(\mathbf{x})$ is a constant matrix. Then:*

(i) $\sigma(H)$ is symmetric with respect to the origin of $\mathbb{R}$, i.e., if $\lambda \in \sigma(H)$, then $-\lambda \in \sigma(H)$.

(ii) $\sigma_\#(H)$ ($\# = {\rm p}, {\rm d}$) is symmetric with respect to the origin of $\mathbb{R}$. The multiplicity $\lambda \in \sigma_\#(H)$ coincides with that of $-\lambda \in \sigma_\#(H)$.

Proof. By Proposition 2. 1 we have $\Gamma H\Gamma^{-1} = -H$ (the unitary equivalence of H and $-H$). This implies the desired results. ■

Remark 2. 1 The properties stated in Theorem 2. 2 may differ from spectral properties of the usual Dirac operator $H_0 + M\beta + V$, where V is a scalar potential.

3 The Essential Spectrum and Finiteness of the Discret Spectrum of H

3.1 Structure of the spectrum of H

For a self-adjoint operator T, we denote by $\sigma_{\rm ess}(T)$ the essential spectrum of T.

Theorem 3. 1 *Suppose that*

$$\lim_{|\mathbf{x}|\to\infty} F(\mathbf{x}) = 0. \tag{3.1}$$

Then

$$\sigma_{\rm ess}(H) = (-\infty, -M] \cup [M, \infty), \tag{3.2}$$

$$\sigma_{\rm d}(H) \subset (-M, M). \tag{3.3}$$

Proof. We write $H = H_0 + M(\beta\otimes I_2) + V$ with $V := M(\beta\otimes I_2)(U_F - I)$. We have $\|V(x)\| \le M(|1-\cos F(x)| + |\sin F(x)|) \to 0\,(|x|\to\infty)$. Hence we can apply [6, Theorem 4.7, Remark 2 on p.117] to H to obtain (3.2). This implies (3.3) ■

3.2 Bound for the number of discrete eigenvalues of H

Assume (3.1). Then, by Theorem 3. 1 , we can define the number of discrete eigenvalues of H counting multiplicities:

$$N_H := \dim \mathrm{Ran} E_H((-M, M)), \tag{3.4}$$

where E_H is the spectral measure of H and $\mathrm{Ran}E_H((-M,M))$ means the range of $E_H((-M,M))$. To estimate an upper bound for N_H, we introduce a hypothesis for F and $\mathbf{n}$:

Hypothesis (II)

(i) The functions F and n_j $(j = 1, 2, 3)$ are continuously differentiable on $\mathbb{R}^3$.

(ii) The functions D_jF and D_jn_k $(j, k = 1, 2, 3)$ are bounded on $\mathbb{R}^3$.

Under this assumption, we can define

$$V_F(\mathbf{x}) := \sqrt{|\nabla F(\mathbf{x})|^2 + \sum_{k=1}^{3} |\nabla n_k(\mathbf{x})|^2 \sin^2 F(\mathbf{x})}, \quad \mathbf{x} \in \mathbb{R}^3. \tag{3.5}$$

Theorem 3. 2 *Assume (3.1) and Hypothesis (II). Suppose that*

$$C_F := \int_{\mathbb{R}^6} \frac{V_F(\mathbf{x})V_F(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2} d\mathbf{x}d\mathbf{y} < \infty. \tag{3.6}$$

Then N_H is finite with

$$N_H \le \frac{M^2 C_F}{2\pi^2}. \tag{3.7}$$

To prove this theorem we present a general lemma. Let $\mathcal{K}$ be a complex Hilbert space and $\mathsf{B}(\mathcal{H})$ be the Banach space of bounded linear operators on $\mathcal{K}$. Let $V : \mathbb{R}^d \to \mathsf{B}(\mathcal{K})$ $(d \in \mathbb{N})$ be a measurable function. The function V defines a unique multiplication operator acting in the Hilbert space $L^2(\mathbb{R}^d; \mathcal{K})$ of $\mathcal{K}$-valued square integrable functions on $\mathbb{R}^d$. We denote it by the same symbol V. We assume the following (Δ is the d-dimensional generalized Laplacian) :

(V.1) $D((-\Delta)^{1/2}) \subset D(|V|^{1/2}) \cap D(|V^*|^{1/2})$ and the form sum

$$L_0 := -\Delta \dot{+} \begin{pmatrix} -|V| & 0 \\ 0 & -|V^*| \end{pmatrix}$$

acting in $\oplus^2 L^2(\mathbb{R}^d; \mathcal{K})$ with form domain $D((-\Delta)^{1/2})$ defines a unique self-adjoint operator bounded from below. Moreover, $\sigma_{\mathrm{ess}}(L_0) \subset [0, \infty)$.

(V.2) The operator

$$L := -\Delta + \begin{pmatrix} 0 & V^* \\ V & 0 \end{pmatrix}$$

acting in $\oplus^2 L^2(\mathbb{R}^d; \mathcal{K})$ is self-adjoint on $D(\Delta)$, bounded from below, and $\sigma_{\mathrm{ess}}(L) \subset [0, \infty)$.

For a self-adjoint operator A, we denote by $N_-(A)$ the number of negative eigenvalues of A counting multiplicities.

Lemma 3. 3 *Assume (V.1) and (V.2). Then $N_-(L) \le N_-(L_0)$.*

Proof. Let

$$Q := \begin{pmatrix} 0 & V^* \\ V & 0 \end{pmatrix}.$$

Then Q is self-adjoint and

$$Q^2 = \begin{pmatrix} |V|^2 & 0 \\ 0 & |V^*|^2 \end{pmatrix},$$

which imlies that

$$|Q| = \begin{pmatrix} |V| & 0 \\ 0 & |V^*| \end{pmatrix}.$$

It is obvious that $Q \ge -|Q|$. Hence $L \ge L_0$. This inequality and the min-max principle (e.g., [4, Theorem XIII.1, Problem 1]) imply the inequality $N_-(L) \le N_-(L_0)$. ■

Proof of Theorem 3. 2

We note that H has the operator matrix representation

$$H = H_0 + M \begin{pmatrix} 0 & \Phi_F^* \\ \Phi_F & 0 \end{pmatrix}, \tag{3.8}$$

where Φ_F is defined by (1.11). Hence

$$H^2 = L(F) + M^2 \tag{3.9}$$

with

$$L(F) := -\Delta + M \begin{pmatrix} 0 & W_F^* \\ W_F & 0 \end{pmatrix}, \tag{3.10}$$

where $W_F := i\boldsymbol{\sigma} \cdot (\nabla \Phi_F)$. Note that, by Hypothesis (II)-(ii), the second term on the right hand side of (3.10) is a bounded self-adjoint operator and hence $L(F)$ is self-adjoint with $D(L(F)) = D(\Delta)$. By direct computations, we have

$$W_F(\mathbf{x})^* W_F(\mathbf{x}) = W_F(\mathbf{x}) W_F(\mathbf{x})^* = |\nabla F(\mathbf{x})|^2 + \sum_{j=1}^{3} |\nabla n_j(\mathbf{x})|^2 \sin^2 F(\mathbf{x}),$$

where we have used (1.9). Hence $|W_F| = |W_F^*| = V_F$. Let $L_0(F) := -\Delta - MV_F$. By Theorem 3. 1 , $\sigma_{\mathrm{ess}}(L(F)) = [0, \infty)$. Condition (3.6) implies that V_F is a potential in the Rollnik class [3, p.170]. Hence it follows from [4, p.118, Example 7] and Weyl's essential spectrum theorem [4, p.112, Theorem XIII.14] that $\sigma_{\mathrm{ess}}(L_0(F)) = \sigma_{\mathrm{ess}}(-\Delta) = [0, \infty)$. Therefore the assumption of Lemma 3. 3 with $L = L(F)$ and $L_0 = L_0(F)$ is satisfied. Hence $N_-(L(F)) \le N_-(L_0(F))$. It is well-known that $N_-(L_0(F)) \le 8M^2 C_F/(4\pi)^2$ ([4, p.98,Theorem XIII.10]), where the factor $8 = \dim \mathbb{C}^4 \otimes \mathbb{C}^2$. On the other hand, by the spectral theorem, $N_H \le N_-(L_F)$. Thus (3.7) follows. ■

Theorem 3. 2 impilies the absence of discrete eigenvalues of H for F's such that the Rollnik norm of MV_F is sufficiently small:

Corollary 3. 4 *Assume (3.1) and Hypothesis (II). Let $M^2 C_F < 2\pi^2$. Then $\sigma_{\mathrm{d}}(H) = \emptyset$.*

4 Existence of Discrete Ground States

For a self-adjoint operator A bounded from below, we set

$$E_0(A) := \inf \sigma(A).$$

If $E_0(A) \in \sigma_{\mathrm{p}}(A)$, then we say that A has a ground state and we call a non-zero vector in $\ker(A - E_0(A))$ a ground state of A. If $E_0(A) \in \sigma_{\mathrm{d}}(A)$, then we say that A has a discrete ground state.

Definition 4. 1 Let

$$E_0^+(H) := \inf\left(\sigma(H) \cap [0, \infty)\right), \quad E_0^-(H) := \sup\left(\sigma(H) \cap (-\infty, 0]\right). \tag{4.1}$$

If $E_0^+(H)$ (resp. $E_0^-(H)$) is an eigenvalue of H, then we say that H has a positive (resp. negative) energy ground state and we call a non-zero vector in $\ker(H - E_0^+(H))$ (resp. $\ker(H - E_0^-(H))$ a positive (resp. negative) energy ground state of H. If $E_0^+(H)$ (resp. $E_0^-(H)$) is a discrete eigenvalue of H, then we say that H has a discrete positive (resp. negative) energy ground state.

Remark 4. 1 If the spectrum of H is symmetric with respect to the origin of $\mathbb{R}$ as in Theorem 2. 2 , then $E_0^+(H) = -E_0^-(H)$, and H has a positive energy ground state if and only if it has a negative energy ground state.

We assume Hypothesis (II). Then the operators

$$S_\pm(F) := -\Delta \pm M(D_3 \cos F) = -\Delta \mp M(D_3 F) \sin F. \tag{4.2}$$

are self-adjoint with $D(S_\pm(F)) = D(\Delta)$ and bounded from below.

Theorem 4. 2 *Assume Hypothesis (II) and (3.1). Suppose that $E_0(S_+(F)) < 0$ or $E_0(S_-(F)) < 0$. Then H has a discrete positive energy ground state or a discrete negative ground state.*

Proof. For each $f \in D(\Delta)$ and $u \in \mathbb{C}^2$ with $\|u\| = 1$, we define

$$\psi_f^+ := (f \otimes u, 0, if \otimes u, 0) \in \mathcal{H}, \quad \psi_f^- := (0, f \otimes u, 0, if \otimes u) \in \mathcal{H}.$$

Then we have

$$\left\langle \psi_f^\pm, L(F)\psi_f^\pm \right\rangle = 2\left\langle f, S_\pm(F) f\right\rangle.$$

In the case where $E_0(S_+(F)) < 0$, there exists a unit vector $f \in D(\Delta)$ such that $\langle f, S_+(F) f\rangle < 0$. Hence $\left\langle \psi_f^+, L(F)\psi_f^+ \right\rangle < 0$. By Theorem 3. 1 and the spectral theorem, we have

$$\sigma_{\mathrm{ess}}(L(F)) = [0, \infty). \tag{4.3}$$

Thus, by the min-max principle, $L(F)$ has a discrete ground state. Similarly, in the case where $E_0(S_-(F)) < 0$ too, $L(F)$ has a discrete ground state. This implies that H has a discrete positive energy ground state or a discrete negative ground state. ■

To construct examples of F satisfying the conditions as stated in Theorem 4. 2 , we consider a scaling. For a constant $\varepsilon > 0$ and a function f on $\mathbb{R}^d$, we define a function f_ε on $\mathbb{R}^d$ by

$$f_\varepsilon(x) := f(\varepsilon x), \quad x \in \mathbb{R}^d.$$

Lemma 4. 3 *Let $V : \mathbb{R}^d \to \mathbb{R}$ be in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and, for a constant $\varepsilon > 0$,*

$$S_\varepsilon := -\Delta + V_\varepsilon.$$

Suppose that the following conditions are satisfied:

(i) For all $\varepsilon > 0$, S_ε is self-adjoint and bounded from below and $\sigma_{\mathrm{ess}}(S_\varepsilon) \subset [0, \infty)$.

(ii) There exists a non-empty open set $\Omega \subset \{x \in \mathbb{R}^d | V(x) < 0\}$.

Then there exists a constant $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, S_ε has a discrete ground state.

Proof. By condition (ii), we can take a non-zero vector $f \in C_0^\infty(\Omega)$ (the set of infinitely differentiable functions on $\mathbb{R}^d$ with compact support in Ω). Then it is easy to see that $\langle f_\varepsilon, S_\varepsilon f_\varepsilon\rangle = \varepsilon^{-d}(a_f\varepsilon^2 - |b_f|)$, where $a_f := \|\nabla f\|^2, b_f = \langle f, Vf\rangle < 0$. Hence, taking $\varepsilon_0 := \sqrt{|b_f|/a_f}$ (note that $a_f \neq 0$), we have $\langle f_\varepsilon, S_\varepsilon f_\varepsilon\rangle < 0$ for all $\varepsilon \in (0, \varepsilon_0)$. Hence, by the min-max principle and condition (i), $E_0(S_\varepsilon) \in \sigma_{\mathrm{d}}(S_\varepsilon)$. ■

Lemma 4. 4 *Let $V : \mathbb{R}^d \to \mathbb{R}$ be continuous on $\mathbb{R}^d$ with $\lim_{|\mathbf{x}|\to\infty} V(\mathbf{x}) = 0$. Suppose that $\Omega_- := \{x \in \mathbb{R}^d | V(x) < 0\} \neq \emptyset$. Then the following hold:*

(i) $-\Delta + V$ acting in $L^2(\mathbb{R}^d)$ is self-adjoint and bounded from below.

(ii) $\sigma_{\mathrm{ess}}(-\Delta + V) = [0, \infty)$.

(iii) S_ε has a discrete ground state for all $\varepsilon \in (0, \varepsilon_0)$ with some $\varepsilon_0 > 0$.

Proof. Part (i) follows from the Kato-Rellich theorem. Part (ii) is proven by a simple applcation of [4, p.119, Theorem XIII.15-(b)].

Since V is continuous, the set Ω_- is open. Hence Lemma 4. 3 implies the existence of a ground state of S_ε for all $\varepsilon \in (0, \varepsilon_0)$ with some $\varepsilon_0 > 0$. ■

We consider a one-parameter family of Dirac operators:

$$H_\varepsilon := H_0 + \frac{1}{\varepsilon} M(\beta \otimes 1_2) U_{F_\varepsilon}, \tag{4.4}$$

which is a scaled deformation of H.

Theorem 4. 5 *Assume Hypothesis (II) and (3.1). Suppose that $D_3 \cos F$ is not identically zero. Then there exists a constant $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, H_ε has a discrete positive energy ground state or a discrete negative ground state.*

Proof. We write $S_\pm(F, M) := S_\pm(F)$ to make explicit the dependence of $S_\pm(F)$ on M. At least one of the sets $\{\mathbf{x} \in \mathbb{R}^3 | (D_3 \cos F)(\mathbf{x}) > 0\}$ and $\{\mathbf{x} \in \mathbb{R}^3 (D_3 \cos F)(\mathbf{x}) < 0\}$ is not empty. The function $D_3 \cos F = -(D_3 F) \sin F$ is bounded and continuous satisfying $\lim_{|\mathbf{x}|\to\infty}(D_3 F)(\mathbf{x}) = 0$. Hence we can apply Lemma 4. 4 to conclude that $S_+(F_\varepsilon, \varepsilon^{-1}M)$ or $S_-(F_\varepsilon, \varepsilon^{-1}M)$ has a discrete ground state for all $\varepsilon \in (0, \varepsilon_0)$ with some $\varepsilon_0 > 0$. This fact and Theorem 4. 2 yield the desired result. ■

5 Symmetry Reduction of H

In this section, we show that, if F is invariant under the rotations around the x_3-axis, then there exist infinitely many mutually orthogonal closed subspaces of $\mathcal{H}$ that reduce H_ε for all $\varepsilon > 0$ and each reduced part of H_ε may have a positive energy ground state or a negative energy ground state. We use the cylindrical coordinates for points $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$:

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad x_3 = z,$$

where $\theta \in [0, 2\pi)$, $r > 0$. We assume the following:

Hypothesis (III) There exists a continuously differentiable function $G : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ such that (i) $F(\mathbf{x}) = G(r, z), \quad \mathbf{x} \in \mathbb{R}^3 \setminus \{0\}$; (ii) $\lim_{r+|z|\to\infty} G(r, z) = 0$; (iii) $\sup_{r>0,z\in\mathbb{R}}(|\partial G(r, z)/\partial r| + |\partial G(r, z)/\partial z|) < \infty$.

We take the vector field $\mathbf{n}$ to be of the form

$$\mathbf{n}(\mathbf{x}) := (\sin\Theta(r, z)\cos(m\theta), \sin\Theta(r, z)\sin(m\theta), \cos\Theta(r, z)), \tag{5.1}$$

where $\Theta : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and m is a real constant.

Let $L_3 := -ix_1D_2 + ix_2D_1$, the third component of the angular momentum. It is well-known that L_3 is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$. We denote its closure by the same symbol L_3. We set

$$\Sigma_3 := \sigma_3 \oplus \sigma_3$$

acting on $\mathbb{C}^4$ and define

$$K_3 := L_3 \otimes 1_2 + \frac{1}{2}\Sigma_3 \otimes 1_2 + \frac{m}{2} I \otimes \tau_3, \tag{5.2}$$

which is a self-adjoint operator acting in $\mathcal{H}$.

We denote by T_ε ($\varepsilon > 0$) the unitary dilation on $L^2(\mathbb{R}^3)$ with power ε:

$$(T_\varepsilon f)(\mathbf{x}) := \varepsilon^{3/2} f(\varepsilon\mathbf{x}), \quad f \in L^2(\mathbb{R}^3), \quad \text{a.e.}\mathbf{x}. \tag{5.3}$$

Lemma 5. 1 *For all $\varepsilon > 0$, $T_\varepsilon L_3 T_\varepsilon^{-1} = L_3$. Hence $(T_\varepsilon \otimes 1_2)K_3(T_\varepsilon \otimes 1_2)^{-1} = K_3$ for all $\varepsilon > 0$.*

Proof. It is straightforward to see that, for all $f \in C_0^\infty(\mathbb{R}^3)$, $T_\varepsilon L_3 f = L_3 T_\varepsilon f$. Since $C_0^\infty(\mathbb{R}^3)$ is a core of L_3, this equality extends to all $f \in D(L_3)$ showing that $L_3 \subset T_\varepsilon^{-1} L_3 T_\varepsilon$. The both sides are self-adjoint. Hence they conincide. ■

Lemma 5. 2 *Assume that*

$$\Theta(\varepsilon r, \varepsilon z) = \Theta(r, z), \quad (r, z) \in (0, \infty) \times \mathbb{R}, \ \varepsilon > 0. \tag{5.4}$$

Then, for all $t \in \mathbb{R}$ and $\varepsilon > 0$, the operator equality

$$e^{itK_3} H_\varepsilon e^{-itK_3} = H_\varepsilon \tag{5.5}$$

holds.

Proof. We first prove (5.5) with $\varepsilon = 1$. We have for all $t \in \mathbb{R}$

$$e^{itK_3} = e^{itL_3} e^{it\Sigma_3/2} \otimes e^{itm\tau_3/2}.$$

For all $f \in C_0^\infty(\mathbb{R}^3)$, we have

$$(e^{itL_3} f)(\mathbf{x}) = f(x_1 \cos t - x_2 \sin t, x_1 \sin t + x_2 \cos t, z), \quad \mathbf{x} \in \mathbb{R}^3.$$

Hence e^{itL_3} leaves $C_0^\infty(\mathbb{R}^3)$ invariant. It follows that, for all $f \in C_0^\infty(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathbb{C}^2$, $e^{itK_3} f \in D(H_0) = D(H)$ and

$$H_0 e^{itL_3} f = e^{itL_3} \{(-i\alpha_1 \cos t + i\alpha_2 \sin t) D_1 f + (-i\alpha_1 \sin t - i\alpha_2 \cos t) D_2 f - i\alpha_3 D_3 f\}. \tag{5.6}$$

Using the matrix representation of α_j, one can check that

$$\alpha_j e^{it\Sigma_3/2} = e^{-it\Sigma_3/2} \alpha_j \ (j = 1, 2), \quad [\alpha_3, e^{it\Sigma_3}] = 0.$$

It follows from these relations and (5.6)

$$H_0 e^{itK_3} f = e^{itK_3} H_0 f. \tag{5.7}$$

We have

$$\tau_j e^{itm\tau_3/2} = e^{itm\tau_3/2} \tau_j e^{itm\tau_3} \ (j = 1, 2), \quad \tau_3 e^{itm\tau_3/2} = e^{itm\tau_3/2} \tau_3$$

and

$$e^{-itL_3} \mathbf{n}(\mathbf{x}) e^{itL_3} = (\sin \Theta(r, z) \cos m(\theta - t), \sin \Theta(r, z) \sin m(\theta - t), \cos \Theta(r, z)).$$

It follows from these relations that

$$\beta \otimes 1_2 U_F e^{itK_3} f = e^{itK_3} (\beta \otimes 1_2) U_F f. \tag{5.8}$$

Combining (5.7) together with (5.8), we obtain $He^{itK_3} f = e^{itK_3} Hf$. Since $C_0^\infty(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathbb{C}^2$ is a core of H, this equality extends to all $f \in D(H) = D(H_0)$ showing $H \subset e^{-itK_3} H e^{itK_3}$. The both sides are self-adjoint. Thus (5.5) follows.

We next consider the case where $\varepsilon \neq 1$. We write $U_F = U(F, \mathbf{n})$. By Lemma 5. 1 , (5.8) and the fact that T_ε is a bijection from $C_0^\infty(\mathbb{R}^3)$ onto itself, we have $\beta \otimes 1_2 U(F_\varepsilon, \mathbf{n}_\varepsilon) e^{itK_3} f = e^{itK_3} (\beta \otimes 1_2) U(F_\varepsilon, \mathbf{n}_\varepsilon) f$. By condition (5.4), $\mathbf{n}_\varepsilon = \mathbf{n}$. Hence $\beta \otimes 1_2 U(F_\varepsilon, \mathbf{n}_\varepsilon) e^{itK_3} f = e^{itK_3} (\beta \otimes 1_2) U(F_\varepsilon, \mathbf{n}) f$. Therefore (5.8) holds with F replaced by F_ε. Thus, in the same way as in the preceding paragraph, one can prove (5.5). ■

We say that two self-adjoint operators on a Hilbert space strongly commute if their spectral measures commute.

Lemma 5. 3 *Assume (5.4). Then, for all $\varepsilon > 0$, H_ε and K_3 strongly commute.*

Proof. It follows from Lemma 5. 2 and the functional calculus for self-adjoint operators that $e^{itK_3} e^{isH_\varepsilon} = e^{isH_\varepsilon} e^{itK_3}$ for all $s, t \in \mathbb{R}$ and all $\varepsilon > 0$. This implies the strong commutativity of H_ε and K_3 (see [2, p.271, Theorem VIII.13] for general criteria of the strong commutativity of self-adjoint operators). ■

Let

$$E := (0,\infty) \times [0, 2\pi) \times \mathbb{R} = \{(r, \theta, z) | r > 0, \theta \in [0, 2\pi), z \in \mathbb{R}\}$$

and $d\mu := rdr \otimes d\theta \otimes dz$, a measure on E. Then one can define a unitary operator $Y : L^2(\mathbb{R}^3) \to L^2(E, d\mu)$ by

$$(Yf)(r, \theta, z) := f(r\cos\theta, r\sin\theta, z), \quad f \in L^2(\mathbb{R}^3).$$

For each $\ell \in \mathbb{Z}$, we define $\phi_\ell : [0, 2\pi) \to \mathbb{C}$ by

$$\phi_\ell(\theta) := \frac{1}{\sqrt{2\pi}} e^{i\ell\theta}, \quad \theta \in [0, 2\pi). \tag{5.9}$$

It is well-known that $\{\phi_\ell\}_{\ell\in\mathbb{Z}}$ is a complete orthonormal system of $L^2([0, 2\pi))$. For each $f \in L^2(E, d\mu)$, we define $\hat{f} : (0,\infty) \times \mathbb{Z} \times \mathbb{R}$ by

$$\hat{f}(r, \ell, z) := \int_0^{2\pi} \phi_\ell(\theta)^* f(r, \theta, z) d\theta.$$

We define an operator D_θ on $L^2(E, d\mu)$ as follows:

$$D(D_\theta) := \left\{ f \in L^2(E, d\mu) \middle| \sum_{\ell=-\infty}^{\infty} \ell^2 \int_0^\infty drr \int_{\mathbb{R}} dz |\hat{f}(r, \ell, z)|^2 < \infty \right\},$$
$$(\widehat{D_\theta f})(r, \ell, \theta) = i\ell \hat{f}(r, \ell, \theta), \quad f \in D(D_\theta).$$

Then $-iD_\theta$ is self-adjoint with

$$\sigma(-iD_\theta) = \sigma_{\mathrm{p}}(-iD_\theta) = \{\ell\}_{\ell\in\mathbb{Z}} = \mathbb{Z}, \tag{5.10}$$
$$\ker(-iD_\theta - \ell) = \left\{ g\phi_\ell \middle| g : (0,\infty) \times \mathbb{R} \to \mathbb{C}, \int_0^\infty drr \int_{\mathbb{R}} dz |g(r, z)|^2 < \infty \right\}. \tag{5.11}$$

It is not so hard to see that

$$YL_3Y^{-1} = -iD_\theta. \tag{5.12}$$

Hence

$$\sigma(L_3) = \sigma_{\mathrm{p}}(L_3) = \mathbb{Z}. \tag{5.13}$$

Let

$$\mathcal{M}_\ell := \ker(L_3 - \ell) = Y^{-1} \ker(-iD_\theta - \ell). \tag{5.14}$$

Then we have the orthogonal decomposition

$$L^2(\mathbb{R}^3) = \oplus_{\ell=-\infty}^{\infty} \mathcal{M}_\ell, \quad L^2(E, d\mu) = \oplus_{\ell=-\infty}^{\infty} Y\mathcal{M}_\ell. \tag{5.15}$$

By (5.13), we have

$$\sigma(K_3) = \sigma_{\mathrm{p}}(K_3) = \left\{ \ell + \frac{s}{2} + \frac{mt}{2} \middle| \ell \in \mathbb{Z}, s = \pm 1, t = \pm 1 \right\}. \tag{5.16}$$

The eigenspace of K_3 with eigenvalue $\ell + (s/2) + (mt/2)$ is given by

$$\mathcal{M}_{\ell,s,t} := \mathcal{M}_\ell \otimes \mathcal{C}_s \otimes \mathcal{T}_t \tag{5.17}$$

under the natural identificaion $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^4 \otimes \mathbb{C}^2$, where $\mathcal{C}_s := \ker(\Sigma_3 - s)$ and $\mathcal{T}_t := \ker(\tau_3 - t)$. Then $\mathcal{H}$ has the orthogonal decomposition

$$\mathcal{H} = \oplus_{\ell \in \mathbb{Z}, s,t \in \{\pm 1\}} \mathcal{M}_{\ell,s,t}. \tag{5.18}$$

Lemma 5. 3 implies the following fact:

Lemma 5. 4 *Assume (5.4). Then, for all $\varepsilon > 0$, H_ε is reduced by each $\mathcal{M}_{\ell,s,t}$.*

We denote by $H_\varepsilon(\ell, s, t)$ by the reduced part of H_ε to $\mathcal{M}_{\ell,s,t}$ and set

$$H(\ell, s, t) := H_1(\ell, s, t), \tag{5.19}$$

the reduced part of H to $\mathcal{M}_{\ell,s,t}$.

For $s = \pm 1$ and $\ell \in \mathbb{Z}$, we define

$$S_s(G, \ell) := -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\ell^2}{r^2} + \frac{\partial^2}{\partial z^2} + sM\frac{\partial \cos G}{\partial z} \tag{5.20}$$

acting in $L^2((0,\infty) \times \mathbb{R}, rdrdz)$ with domain $D(S_s(G,\ell)) := C_0^\infty((0,\infty) \times \mathbb{R})$ and set

$$\mathcal{E}_0(S_s(G,\ell)) := \inf_{f \in C_0^\infty((0,\infty)\times\mathbb{R}), \|f\|_{L^2((0,\infty)\times\mathbb{R}, rdrdz)}=1} \langle f, S_s(G,\ell) f \rangle .$$

Theorem 5. 5 *Assume Hypothesis (III). Fix an $\ell \in \mathbb{Z}$ arbitrarily and $s = \pm 1$. Suppose that $\mathcal{E}_0(S_s(G,\ell)) < 0$. Then, for each $t = \pm 1$, $H(\ell, s, t)$ has a discrete positive energy ground state or a discrete negative ground state.*

Proof. Let

$$\begin{aligned}
& c_\ell := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\theta e^{-i\ell\theta} \cos(m\theta), \quad d_\ell := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\theta e^{-i\ell\theta} \sin(m\theta), \\
& n_{j,\ell}(r,z) := (\sin\Theta(r,z) c_\ell, \sin\Theta(r,z) d_\ell, \cos\Theta(r,z)), \\
& \Phi_{G,\ell,t} := \cos G + i \sum_{j=1} n_{j,\ell} \sin G \otimes \tau_j + itn_{3,\ell} \sin G, \\
& D_{1,\ell} := c_\ell \frac{\partial}{\partial r} - \frac{d_\ell}{r}\frac{\partial}{\partial \theta}, \quad D_{2,\ell} := d_\ell \frac{\partial}{\partial r} + \frac{c_\ell}{r}\frac{\partial}{\partial \theta}
\end{aligned}$$

and

$$W_{G_\varepsilon,\ell,s,t} := i \sum_{j=1}^{2} \sigma_j D_{j,\ell} \Phi_{G_\varepsilon,\ell,t} + isD_z \Phi_{G_\varepsilon,\ell,t}, \quad \varepsilon > 0.$$

Then we have

$$\begin{aligned}
(Y \otimes 1_2) H_\varepsilon(\ell,s,t)^2 (Y \otimes 1_2)^{-1} &= -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\ell^2}{r^2} + \frac{\partial^2}{\partial z^2} \\
&\quad + \varepsilon^{-1} M \begin{pmatrix} 0 & W^*_{G_\varepsilon,\ell,s,t} \\ W_{G_\varepsilon,\ell,s,t} & \end{pmatrix} + M^2 \\
&=: L_\varepsilon(\ell,s,t) + M^2
\end{aligned}$$

on $C_0^\infty((0,\infty)\times \mathbb{R})$.

For each $f\in C_0^\infty((0,\infty)\times\mathbb{R})$ and $u_t\in\mathbb{C}^2$ satisfying $\|f\|=1$, $\|u_t\|=1$ and $\tau_3 u_t = t u_t$ $(t=\pm 1)$, we define

$$\psi_f^{(1)} := (f\otimes u_t, 0, if\otimes u_t, 0)\in \mathcal{M}(\ell,1,t),$$
$$\psi_f^{(-1)} := (0, f\otimes u_t, 0, if\otimes u_t)\in \mathcal{M}(\ell,-1,t).$$

Then we have

$$\left\langle \psi_f^{(s)}, Y L_1(\ell,s,t) Y^{-1}\psi_f^{(s)}\right\rangle = 2\left\langle f, S_s(F,\ell) f\right\rangle .$$

By the present assumption, there exists a unit vector $f\in C_0^\infty((0,\infty)\times\mathbb{R})$ such that $\langle f, S_s(F,\ell)f\rangle < 0$. Note that $\sigma_{\rm ess}(L_1(\ell,s,t))\subset[0,\infty)$. Hence, by the min-max principle, $L_1(\ell,s,t)$ has a discrete ground state. This implies that $H(\ell,s,t)$ has a discrete positive energy ground state or a discrete negative ground state. ∎

Theorem 5. 6 *Assume Hypothesis (III) and (5.4). Suppose that $\partial\cos G/\partial z$ is not identically zero. Then, for each $\ell\in\mathbb{Z}$, there exists a constant $\varepsilon_\ell>0$ such that, for all $\varepsilon\in(0,\varepsilon_\ell)$, each $H_\varepsilon(\ell,s,t)$ has a discrete positive energy ground state or a discrete negative ground state.*

Proof. We write $S_{s,M}(F,\ell) := S_s(F,\ell)$ to make explicit the dependence of $S_s(F,\ell)$ on M. In the same way as in the proof of Theorem 4. 5 , one can take a vector $f_\varepsilon\in C_0^\infty((0,\infty)\times\mathbb{R})$ such that $\langle f_\varepsilon, S_{s,\varepsilon^{-1}M}(F_\varepsilon(\ell)) f_\varepsilon\rangle<0$ for all sufficiently small $\varepsilon>0$, where the smallness depends on ℓ. It follows from the proof of the preceding theorem that $L_\varepsilon(\ell,s,t)$ has a discrete ground state. ∎

Corollary 5. 7 *Assume Hypothesis (III) and (5.4). Suppose that $\partial\cos G/\partial z$ is not identically zero. Let ε_ℓ be as in Theorem 5. 6 and, for each $N\in\mathbb{N}$ and $k>n$ $(k,n\in\mathbb{Z})$, $\nu_{k,n} := \min_{n+1\le\ell\le k}\varepsilon_\ell$. Then, for each $\varepsilon\in(0,\nu_{k,n})$, H_ε has at least $(k-n)$ discrete eigenvalues counting multiplicities.*

Proof. We have $\sigma_{\rm p}(H_\varepsilon)=\cup_{\ell\in\mathbb{Z},s,t=\pm1}\sigma_{\rm p}(H_\varepsilon(\ell,s,t))$. By the preceding theorem, for each $\ell=n+1,\cdots,k$, $H_\varepsilon(\ell,s,t)$ has a discrete eigenvalue. Thus the desired result follows. ∎

Remark 5. 1 This result is consistent with Theorem 3. 2 , because it reads in the present case

$$N_{H_\varepsilon}\le \frac{1}{\varepsilon^4}\frac{M^2 C_F}{2\pi^2}$$

and the right hand side diverges as $\varepsilon\to 0$.

6 A Unitary Transformation

In this section we show that, under Hypothesis (II), the Hamiltonian H is unitarily equivalent to an operator which resembles a Dirac operator with a magnetic moment.

It is easy to see that the operator

$$X_F := \begin{pmatrix} e^{iF\otimes\boldsymbol{\tau}\cdot\mathbf{n}/2} & 0 \\ 0 & e^{-iF\otimes\boldsymbol{\tau}\cdot\mathbf{n}/2} \end{pmatrix} \tag{6.1}$$

is unitary. Under Hypothesis (II), we can define the following functions:

$$B_j(\mathbf{x}) := \frac{1}{2} D_j(F(\mathbf{x}) \otimes \boldsymbol{\tau} \cdot \mathbf{n}(\mathbf{x})), \quad \mathbf{x} \in \mathbb{R}^3,\ j = 1, 2, 3. \tag{6.2}$$

We set

$$\mathbf{B} := (B_1, B_2, B_3) \tag{6.3}$$

and introduce

$$H(\mathbf{B}) := H_0 + M\beta - \boldsymbol{\sigma} \cdot \mathbf{B} \tag{6.4}$$

acting in $\mathcal{H}$. Note that, under Hypothesis (II), the operator $-\boldsymbol{\sigma} \cdot \mathbf{B}$ is a bounded self-adjoint operator. Hence, by a simple application of the Kato-Rellich theorem, $H(\mathbf{B})$ is self-adjoint with $D(H(\mathbf{B})) = D(H_0)$.

Proposition 6. 1 *Assume Hypothesis (II). Then*

$$X_F H X_F^{-1} = H(\mathbf{B}). \tag{6.5}$$

Proof. Noting the fact that $(\boldsymbol{\tau} \cdot \mathbf{n})^2 = 1_2$, we have

$$\Phi_F = e^{iF\otimes\boldsymbol{\tau}\cdot\mathbf{n}}.$$

It follows from this fact and (3.8) that $X_F H X_F^{-1}\psi = H(\mathbf{B})\psi$ for all $\psi \in [\oplus^4 C_0^\infty(\mathbb{R}^3)] \otimes \mathbb{C}^2$. Since $[\oplus^4 C_0^\infty(\mathbb{R}^3)] \otimes \mathbb{C}^2$ is a core of $H(\mathbf{B})$, the opeartor equality (6.5) follows. ■

Acknowledgement

The authors thank Dr. T. Miyao for discussions.

References

[1] H. Kalf and O. Yamada, Essential self-adjointness of n-dimensional Dirac operators with a variable mass term, *J. Math. Phys.* **42** (2001), 2667–2676.

[2] M. Reed and B. Simon, Methods of Modern Mathematical Physics I: Functional Analysis, Academic Press, New York, 1972.

[3] M. Reed and B. Simon, Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness, Academic Press, New York, 1975.

[4] M. Reed and B. Simon, Methods of Modern Mathematical Physics IV: Analysis of Operators, Academic Press, New York, 1978.

[5] N. Sawado, The SU(3) dibaryons in the chiral quark soliton model, Phys. Lett. **B 524** (2002), 289–296.

[6] B. Thaller, The Dirac Equation, Springer, Berlin, Heidelberg, 1992.