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Stability of facets of crystals growing from vapor

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Abstract

Consider a Stefan-like problem with Gibbs-Thomson and kinetic effects as a model of crystal growth from vapor. The equilibrium shape is assumed to be a regular circular cylinder. Our main concern is a problem whether or not a surface of cylindrical crystals (called a facet) is stable under evolution in the sense that its normal velocity is constant over the facet. If a facet is unstable, then it breaks or bends. A typical result we establish is that all facets are stable if the evolving crystal is near the equilibrium. The stability criterion we use is a variational principle for selecting the correct Cahn-Hoffman vector. The analysis of the phase plane of an evolving cylinder (identified with points in the plane) near the unique equilibrium provides a bound for ratio of velocities of top and lateral facets of the cylinders. [1]

1 Introduction

This is a continuation of our series of papers, [GR1], [GR2], [GR3], [GR4]. We studied there a one phase Stefan-like problem with Gibbs-Thomson and kinetic effect where the interfacial energy is so singular that its Wulff shape-equilibrium shape is a regular circular cylinder. This is a model describing a single crystal grown from vapor. We assume that the initial shape of crystal is a cylinder which may have not have the aspect ratio of the Wulff shape. The main issue we are concerned with is to find conditions for stability of a facet (a surface of a cylinder). We shall say that a facet is stable if its normal velocity is a constant on the facet. In the opposite case the facet breaks or bends. Such a problem is important in pattern formation and of course in the studies of habit of growing shape of crystals (see e.g. [YSF]). We shall prove, among other results, that if the evolving crystal is near the unique equilibrium shape, then its facets are stable (Corollary 4.9). To achieve it we perform a phase plane analysis of our evolution system near the unique steady state. In particular, we shall prove that there exists a one-dimensional stable manifold and a one-dimensional unstable one. Such an observation is useful to control ratio of speeds of lateral and top surfaces and it is a key to derive stability of facets.

The set of equations we are going to study has been justified in [GR1], [GR3], [GR4]. We also presented there its origin. The main object of interest is a crystal $\Omega(t)$, *i.e.* an evolving region in $\mathbb{R}^3$. It evolves at the expense of vapor (or other matter) whose supersaturation (see [GR3] for explanation) is denoted by σ. We are going now to present the evolution equations. The supersaturation σ satisfies

$$\Delta\sigma = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \Omega(t). \tag{1.1}$$

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It is also physically reasonable to assume that σ has a specific value at infinity, *i.e.*

$$\lim_{|x|\to\infty} \sigma(t,x) = \sigma^\infty. \tag{1.2}$$

The velocity of the growing crystal is determined by the normal derivative of σ at the surface,

$$\frac{\partial\sigma}{\partial \mathbf{n}} = V \quad \text{on} \quad \partial\Omega(t) \tag{1.3}$$

where $\mathbf{n}$ is the outer normal. This equation expresses the properly rescaled mass conservation, see [GR3].

The value of σ at the surface is coupled to the surface velocity and its curvature through the Gibbs-Thomson relation

$$-\sigma = -\mathrm{div}\,\xi - \beta V, \tag{1.4}$$

where β is the kinetic coefficient and ξ is a Cahn-Hoffman vector. A relation of this sort is quite natural, see Gurtin [G, Chapter 8]. The Cahn-Hoffman vector ξ appearing in (1.4) requires some extra comments. For a smooth surface S and a smooth energy density function γ we have

$$\xi(x) = (\nabla\gamma)(\mathbf{n}(x)),$$

which is a well-defined quantity. However, it is our intention to consider energy density functions γ which are only Lipschitz continuous and (positively) 1-homogeneous, and surfaces S with edges. It follows that γ is differentiable a.e., but the normals to our interface S fall precisely into the set of points on non-differentiability of γ. We must use a different notion of differentiability. For a convex function γ its subdifferential $\partial\gamma$ is a well-defined nonempty convex set. But in general $\partial\gamma$ is not a singleton, thus we have only that

$$\xi(x) \in (\partial\gamma)(\mathbf{n}(x)) \tag{1.5}$$

which should be considered as an additional constraint for (1.1)–(1.4).

We are interested in evolution of simplified shapes, namely straight circular cylinders. In our earlier papers we showed existence of solutions to "averaged" system (1.1)–(1.4), where equation (1.4) is replaced by its average over each facet of $\partial\Omega(t)$, see [GR1]. We also established existence of a special class of solutions, namely self-similar evolution, *i.e.* $\Omega(t) = a(t)\Omega(0)$, [GR3]. Simply the existence of such solutions requires special condition of the surface energy density γ and kinetic coefficients β, (see [GR3]).

We are concerned with the question: are the velocities of facets in the normal direction really constant over each facet? A facet loses its stability if its velocity is not a constant, hence it must break either bend. In [GR4] we addressed this issue for self-similar solutions to (1.1)–(1.4) augmented with (1.5). In this paper we address the question of facet stability of circular straight cylinders evolving by (1.1)–(1.5) for surface energy density γ given by (2.1) below without imposing any additional restrictions on γ nor β.

We used a variational principle in [GR4] to formulate general sufficient and necessary conditions for facet stability valid not only for self-similar solution. However, the characterization of region of stability in term of the scale factor $a(t)$ ([GR4, Theorem 4.8] and [GR4, Theorem 4.14]) depended heavily on the transparent structure of solutions and easily available various estimates especially on velocities of facets. They were obtained with the help of another general result namely so-called Berg's effect, see [GR2]. Here, we obtain similar estimates but they are no longer that clean. Interestingly, their derivation depends on the phase portrait analysis of the flow of (1.1)–(1.5), which is performed in Section 3. The results are collected in Theorem 3.1. Our main focus in Section 4 is on characterization of the regions of facet stability near the unique equilibrium point of the flow in the phase plane. The starting point is the general [GR4, Theorem 4.6]. However, we are not able to give a clean description of this set, but we may claim that it has an open interior. This is the content of Theorem 4.7. Nonetheless, all facets are stable near the unique equilibrium point of the system, see Corollary 4.9. In Section 4 we also comment on the origin of the variational principle we use.

Section 2 is devoted to the presentation of some preliminary material.

2 Setting up the problem

In order to make our presentation self-contained we collect here the basic definitions and recall from [GR1], [GR3], [GR4] the necessary notions. Our main point is that system (1.1)–(1.5) is not convenient for our analysis, thus we will replace it with more tractable one. An important role in this process will be played by the Cahn-Hoffman vector.

2.1 Preliminaries

Our evolving crystal $\Omega(t)$ is assumed to be, as it is done in the physics literature, see [Ne], [YSF], a straight circular cylinder,

$$\Omega(t) = \{(x, x_2, x_3) \in \mathbb{R}^3 : \ x_1^2 + x_2^2 \le R^2(t), \quad |x_3| \le L(t)\}.$$

In other words, we only have to know $R(t)$ and $L(t)$ to describe its evolution. We shall call by *facets* the following subsets of $\partial\Omega(t)$

$$\begin{array}{l} S_\Lambda = \{x \in \partial\Omega(t) : \quad x_1^2 + x_2^2 = R^2(t)\}, \\ S_T = \{x \in \partial\Omega(t) : \quad x_3 = L(t)\}, \quad S_B = \{x \in \partial\Omega(t) : x_3 = -L(t)\} \end{array}$$

and we shall call them the lateral side, top and bottom. We also define the set of indices $I = \{\Lambda, T, B\}$. By V_i we denote the velocity of facet S_i, $i \in I$, in the direction of $\mathbf{n}$, the outer normal to $\partial\Omega(t)$.

We explicitly assume that, at each time instant t, $\sigma(t)$ enjoys the symmetry of $\Omega(t)$, *i.e.* σ is axisymmetric and symmetric with respect to the plane $x_3 = 0$:

$$\sigma(t) = \bar{\sigma}(t, \sqrt{x_1^2 + x_2^2}, |x_3|).$$

The choice of the surface energy density γ determines evolution of $\Omega(t)$. We take such γ that it will be consistent with the cylindrical shape of $\Omega(t)$. Specifically, we take

$$\gamma(x_1, x_2, x_3) = r\gamma_\Lambda + |x_3|\gamma_{TB}, \quad \gamma_\Lambda, \gamma_{TB} > 0, \tag{2.1}$$

where $r^2 = x_1^2 + x_2^2$ and γ_Λ, γ_{TB} are positive constants. Hence, its Frank diagram F_γ defined as

$$F_\gamma = \{p \in \mathbb{R}^3 : \ \gamma(p) \le 1\}$$

consists of two straight cones with common base, which is the disk $\{(x_1, x_2, 0) : \ x_1^2 + x_2^2 \le 1/\gamma_\Lambda\}$, same height and the vertices at

$$(0, 0, \pm 1/\gamma_{TB}).$$

Now, the Wulff shape of γ is defined by

$$W_\gamma = \{x \in \mathbb{R}^3 : \ \forall \mathbf{n} \in \mathbb{R}^3, \ |\mathbf{n}| = 1, \quad x \cdot \mathbf{n} \le \gamma(\mathbf{n})\}.$$

In our setting W_γ is a cylinder of radius R_0 equal to γ_Λ and half-height L_0 equal to γ_{TB}. Hence, all cylinders like $\Omega(t)$ above are *admissible*, in the sense that normal $\mathbf{n}$ to the top facet of $\Omega(t)$ (respectively: bottom, lateral surface of $\Omega(t)$) is the normal to top facet of W_γ (respectively: bottom, lateral surface of W_γ).

Now, we are in a position to explain the additional restrictions on γ which were sufficient to guarantee existence of self-similar solutions to (1.1)–(1.5). Namely, in addition to the necessary condition

$$\gamma_i \cdot \beta_i = \text{const.} \qquad i = T, B, \Lambda,$$

the aspect ratio $\varrho = L_0/R_0$ of the Wulff shape had to be a root of a transcendental equation (see [GR3, equation (4.8)]).

Our choice of γ made in (2.1) is natural from the view point of modeling the crystal growth, but since γ is Lipschitz continuous it is differentiable just almost everywhere. We have already explained why this is not sufficient for us. However, this function is convex, hence its subdifferential $\partial\gamma(\mathbf{n})$ is well-defined for each vector $\mathbf{n} \in \mathbb{R}^3$. Thus, a Cahn-Hoffman vector $x \to \xi(x)$ is just a section of the subdifferential $\partial\gamma(\mathbf{n}(x))$, where $\mathbf{n}(x)$ is the outer normal to $\partial\Omega(t)$, which is well-defined $\mathcal{H}^2$-a.e. We should be a bit careful about the meaning of surface divergence $\mathrm{div}_S\xi$. At this point we recall that for ξ defined in U a neighborhood of S_i

$$\mathrm{div}_S\xi = \text{trace}\,(Id - \mathbf{n}\otimes\mathbf{n})\nabla\xi, \quad \text{for } x \in S_i,$$

where $\mathbf{n}$ is an outer normal to the surface. This definition is independent of the extension of ξ to U (see [Si], [GPR]).

2.2 Crystalline curvature and div$_S\xi$

Here, we will recall the assumptions on ξ, which enable us to proceed, (see [GR4]). In order to make the quantities appearing in (1.1)–(1.5) well-defined, we require that

$$\xi(t,\cdot)|_{S_i(t)} \in L^2(S_i(t)) \quad \text{and} \quad \mathrm{div}_S(\xi(t,\cdot)|_{S_i}(t)) \in L^2(S_i(t)), \qquad i \in \{T, B, \Lambda\}, \tag{2.2}$$

and in addition

$$\xi(t,\cdot) \in \partial\gamma(\mathbf{n}), \quad \mathcal{H}^2\text{-a.e.} \tag{2.3}$$

It is worth noting that the authors of [BNP2] and [BNP3] considering motion by mean crystalline curvature were able to prove there that $\mathrm{div}_S(\xi(t,\cdot) \in L^\infty(S(t))$. At the end however, our choice of ξ will turn out to be quite smooth, see Remark (1) after Proposition 4.1.

The present smoothness assumptions are sufficient to make sure the traces are well-defined, (see [GR4, §2]). If we combine it with an observation that

$$\partial\gamma(\mathbf{n}_\Lambda(x)) \cap \partial\gamma(\mathbf{n}_i(x)) = \{\gamma_{BT}\mathbf{n}_i + \gamma_\Lambda\mathbf{n}_\Lambda\}, \qquad i = T, B, \tag{2.4}$$

then we conclude that

$$\xi\cdot\nu_i = \gamma_\Lambda \quad \text{in } W^{-1/2,2}(\partial S_i), \quad i = T, B, \qquad \xi\cdot\nu_\Lambda = \gamma_{TB} \quad \text{in } L^2(\partial S_\Lambda). \tag{2.5}$$

By definition a solution to (1.1)–(1.5) is a triple $(\Omega(t), \sigma(t), \xi(t))$. A separate issue is its proper definition. At the moment we abstain from it apart from the requirement that (1.1)–(1.5) hold for all $t \geq 0$.

We are interested in such kinds of evolution that initial cylinder $\Omega(0)$ retains its form all time instances, that is we want that $\Omega(t)$ be another cylinder, possibly of different aspect ratio $L(t)/R(t)$. This is possible only if

$$V_i(t) \text{ does not depend upon the point } x \in S_i,\ i \in I. \tag{2.6}$$

But of course V_Λ may be different from V_T. However, $V_T = V_B$ due to assumed symmetry with respect to the plane $\{x_3 = 0\}$.

In [GR1], [GR4] we reduced the number of unknown in system (1.1)–(1.5). This is in fact possible for evolution conforming to (2.6) due to the following fact.

Proposition 2.1. ([GR4, Proposition 2.1]). *Let us suppose that γ is defined by (2.1), Ω is an admissible cylinder. Then, if ξ satisfies (2.2), (2.3) and (2.5), then*

$$\int_{S_i} \mathrm{div}_S\xi\, d\mathcal{H}^2 = -\kappa_i|S_i|,$$

where the numbers $\kappa_T = \kappa_B, \kappa_\Lambda$ are called crystalline curvatures *of the top, bottom, and the lateral surfaces, and*

$$\kappa_\Lambda = -\frac{\gamma_\Lambda}{R} - \frac{\gamma_{TB}}{L}, \qquad \kappa_T = -2\frac{\gamma_\Lambda}{R}. \tag{2.7}$$

Remark. We use the symbol $\mathcal{H}^k$ to denote the k-dimensional Hausdorff measure, $k = 1, 2$. However, for the sake of simplicity we shall write $|S_i|$ instead of $\mathcal{H}^2(S_i)$.

This fact allows us to simplify equations (1.1-1.5) if the following condition holds

$$(\sigma - \mathrm{div}_S \xi)|_{S_i} = \text{ const.}_i. \tag{2.8}$$

Namely, integrating (1.4) over S_i yields

$$-\int_{S_i(t)} \sigma(t,x)\, d\mathcal{H}^2(x) = (\kappa_i(t) - \beta_i V_i(t))|S_i(t)|. \tag{2.9}$$

In fact the system (1.1)–(1.3), (2.9) augmented with initial data has a unique local in time solution $(R(t), L(t), \sigma(t))$:

Proposition 2.2. ([GR1, Theorem 1]). *There exists $(R(t), L(t), \sigma(t))$ a unique weak solution to*

$$\Delta\sigma = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \Omega(t), \qquad \lim_{|x|\to+\infty} \sigma(x) = \sigma^\infty \tag{2.10}$$

$$\frac{\partial \sigma}{\partial \mathbf{n}} = V_i \quad \text{on} \quad S_i, \quad i = \Lambda, T, B \tag{2.11}$$

$$-\int_{S_i} \sigma\, d\mathcal{H}^2 = (\kappa_i - \beta_i V_i)|S_i|, \quad i = \Lambda, T, B \tag{2.12}$$

augmented with an initial condition $\Omega(0) = \Omega_0$, where Ω_0 is an admissible cylinder. Moreover, $V_\Lambda = \dot{R}$, $V_T = \dot{L}$ and

$$R, L \in C^{1,1}([0,T))$$
$$\nabla\sigma \in C^{0,1}([0,T); L^2(\mathbb{R}^3 \setminus \Omega(t))). \qquad \square$$

In order to make the notation more concise we shall write $(\Omega(t), \sigma(t))$ in place of $(R(t), L(t), \sigma(t))$, (but sometimes we suppress the argument t).

Let us suppose now that $(\Omega(t), \sigma(t))$ is a solution to (2.10)–(2.12). In order to obtain a solution to (1.1)–(1.5) we have to prove existence of sufficiently regular section

$$\mathbb{R} \times \Omega \ni (t,x) \to \xi(t,x) \in \partial\gamma(\mathbf{n}(x))$$

such that (2.8) holds. Indeed, we can do this.

Proposition 2.3. ([GR4, Theorem 2.3]).

(1) Let us suppose that $(\Omega(t), \sigma(t), \xi(t))$ is such that equations (1.1)–(1.5) are satisfied for each $t \geq 0$ and $\nabla\sigma(t) \in L^2(\mathbb{R}^3 \setminus \Omega(t))$, $\Omega(t)$ is an admissible cylinder, $\xi|_{S_i}(t,\cdot) \in L^2(S_i)$, $\mathrm{div}_S \xi|_{S_i}(t,\cdot) \in L^2(S_i)$, $i = B, \Lambda, T$ and the initial position Ω_0 of $\Omega(t)$ is given. Finally, we assume that (2.8) holds. Then $(\Omega(t), \sigma(t))$ satisfies (2.10)–(2.12) with crystalline curvatures κ_i given by (2.7).

(2) Let us suppose that $(\Omega(t), \sigma(t))$ is a solution to (2.10)–(2.12) given in Proposition 2.2 above. If there exists ξ such that $\xi|_{S_i} \in L^2(S_i)$, $\mathrm{div}_S \xi|_{S_i}(t,\cdot) \in L^2(S_i)$, $i = T, \Lambda, B$, $\xi(x) \in \partial\gamma(\mathbf{n}(x))$, $\mathcal{H}^2$-a.e., and satisfying (2.8), then the triple $(\Omega(t), \sigma(t), \xi(t))$ is a solution to (1.1)–(1.5). $\square$

In [GR4] we constructed the proper ξ for self-similar solutions. Here, we will drop this restriction. We will perform the construction in Section 4.

2.3 The structure of σ and the scalings

In order to proceed we need a detailed knowledge of the structure of σ. For this purpose we have to introduce some additional objects. Namely, for an admissible cylinder Ω we need f_i which is a unique solution to (see [GR1])

$$-\Delta f_i = 0, \qquad \text{in } \mathbb{R}^3 \setminus \Omega, \tag{2.13}$$

$$\frac{\partial f_i}{\partial \nu} = \delta_{ij}, \qquad \text{on } S_j, \quad j \in I, \tag{2.14}$$

such that $\nabla f_i \in L^2(\mathbb{R}^3 \setminus \Omega)$ and $\lim_{x\to\infty} f_i(x) = 0$. Here, δ_{ij} is the Kronecker delta and ν denotes the inner normal to $\partial\Omega$. For functions f, g such that ∇f, $\nabla g \in L^2(\mathbb{R}^3 \setminus \Omega)$ we also define the quantity

$$(f, g) := \int_{\mathbb{R}^3\setminus\Omega} \nabla f(x) \cdot \nabla g(x)\, dx.$$

Let us mention that the equation above takes the following weak form

$$\int_{\mathbb{R}^3\setminus\Omega} \nabla f_i(x) \cdot \nabla h(x)\, dx = \int_{S_i} h(x)\, d\mathcal{H}^2(x) \tag{2.15}$$

for all h such that $\nabla h \in L^2(\mathbb{R}^3 \setminus \Omega)$.

We showed in [GR1] that (2.10)–(2.12) can be reduced to a system of ODE's, (see [GR1, equation (3.4)]). After taking into account that $V_T = V_B$ that system can be rewritten as

$$(\mathcal{A} + \mathcal{D})\mathbf{V} = \mathbf{B} \tag{2.16}$$

where

$$\mathcal{A} = \begin{bmatrix} (f_\Lambda, f_\Lambda) & (f_\Lambda, f_T + f_B) \\ (f_\Lambda, f_T + f_B) & (f_T + f_B, f_T + f_B) \end{bmatrix},$$

$$\mathbf{V} = (V_\Lambda, V_T), \quad \mathbf{B} = (|S_\Lambda|(\sigma^\infty + \kappa_\Lambda), |S_T|(\sigma^\infty + \kappa_T)), \quad \mathcal{D} = \text{diag}\,\{\beta_\Lambda|S_\Lambda|, \beta_T|S_T|\} \tag{2.17}$$

Moreover, it was shown in the course of proof of [GR1, Theorem 1] that we have a representation formula for σ. It is given by

$$\sigma(t) = -\sum_{i\in I} V_i(t) f_i(t) + \sigma^\infty. \tag{2.18}$$

The studies of non-breaking of facet require clarifying the behavior of our system under scaling of domains. Suppose we define a new variable y by formula

$$y = ax$$

where $a > 0$, thus Ω is transformed to $a\Omega = \tilde{\Omega}$, S_i goes to $aS_i = \tilde{S}_i$. If h is defined on Ω, then we transform it to $\tilde{h} : \tilde{\Omega} \to \mathbb{R}$, by setting

$$\tilde{h}(y) = h(\frac{y}{a}).$$

We also define

$$f^a(y) = af(\frac{y}{a}). \tag{2.19}$$

We recall from [GR3] the fact below clarifying the role of definition of f_i^a.

Proposition 2.4. ([GR3, Proposition 2.2]). *Let us suppose that f_i satisfies (2.15) on $\mathbb{R}^3 \setminus \Omega$, then*

$$\int_{\mathbb{R}^3\setminus a\Omega} \nabla_y f_i^a(y) \nabla_y \tilde{h}(y)\, dy = \int_{aS_i} \tilde{h}(y)\, d\mathcal{H}^2(y)$$

for all $\tilde{h}$ with $\nabla\tilde{h} \in L^2(\mathbb{R}^3 \setminus a\Omega)$. □

Scaling will play an important role in the last section. However, we have to keep tabs on the aspect ratio $\varrho = L/R$ of Ω. We will denote by

$$f_i^{1,\varrho}, \quad i = \Lambda, T, B$$

a unique solution to (2.13), where Ω has aspect ratio equal to ϱ and $R = R_0$. Consequently, $f_i^{a,\varrho}$ is the result of scaling $f_i^{1,\varrho}$ by a, *i.e.* it is given by (2.19).

3 Phase portrait

The existence result, Proposition 2.2, gives us solutions to (2.10)–(2.12) in terms of R, L and σ. However, the representation formula (2.18) for σ makes it plain that the dynamics of system (2.10)–(2.12) is governed solely by R and L. They are solution to an ODE system (2.16), *i.e.*

$$\begin{bmatrix} \dot{R} \\ \dot{L} \end{bmatrix} = (\mathcal{A} + \mathcal{D})^{-1}\mathbf{B} \tag{3.1}$$

where $R(0)$ and $L(0)$ are given. Due to the physics of the problem the phase plane $\mathcal{P}$ consists of the first quarter of $\mathbb{R}^2$,

$$\mathcal{P} = \{(R, L) \in \mathbb{R}^2 : \ R > 0, \ L > 0\}.$$

The first step in drawing our phase portrait is locating the stationary points. We have seen in [GR4, §2.4] that there is a unique steady state of (2.10)–(2.12), as well as of (3.1). It is the Wulff shape scaled by $\frac{2}{\sigma^\infty}$. This corresponds to point $z_0 = (R_0\frac{2}{\sigma^\infty}, L_0\frac{2}{\sigma^\infty})$ of the phase plane $\mathcal{P}$, where R_0 and L_0 are, respectively, the radius and half height of W_γ.

We also know that for a special value ϱ_0 of the aspect ratio $\frac{L_0}{R_0}$ there exists a self-similar solution being in fact a scaling of the Wulff shape *i.e.* $\Omega(t) = a(t)\frac{2}{\sigma^\infty}W_\gamma$, (see [GR3]). Moreover, if $a(t) > 1$, then $\dot{a}(t) > 0$ (respectively, if $a(t) < 1$, then $\dot{a}(t) < 0$). The case $a(t) \equiv 1$ corresponds to the equilibrium. This shows that the steady state of the flow is dynamically unstable. Moreover the unstable manifold of $\frac{2}{\sigma^\infty}W_\gamma$ is given explicitely, $W^U(z_0) = \{z \in \mathcal{P} : \ z = sz_0, \ s \in \mathbb{R}_+\}$. We can ask about existence of a stable (or rather local stable) manifold. Fortunately, the answer is at hand for any value of the aspect ratio of the Wulff shape W_γ. In the following the value of ϱ is immaterial.

Theorem 3.1. *If $\mathcal{V}$ is any bounded neighborhood of $z_0 \in \mathcal{P}$, then there exist orthogonal projections P, $Q : \mathbb{R}^2 \to \mathbb{R}^2$, such that* $\dim \operatorname{Im} P = 1$, $\dim \operatorname{Im} Q = 1$. *They are such that $W^U(z_0, \mathcal{V})$ (respectively, $W^S(z_0, \mathcal{V})$) exists and it is a C^1 curve over $P\mathbb{R}^2$ (respectively, $Q\mathbb{R}^2$) which is tangent at z_0 to $P\mathbb{R}^2$ (respectively $Q\mathbb{R}^2$).*

Proof. (a) We shall see that we can apply a standard result to show existence of Lipschitz manifolds $W^S(z_0, \mathcal{V})$, $W^U(z_0, \mathcal{V})$. For this purpose it is sufficient to check that the assumptions of [Ha, Theorem A.2] are satisfied. In other words we have to rewrite our system (3.1) as

$$\dot{z} = Az + F(z) \tag{3.2}$$

where $z = (R, L) - z_0$, A is a constant two by two matrix and $F : \mathcal{V} \to \mathbb{R}^2$ is a Lipschitz continuous vector valued function. We stress that we can take any bounded neighborhood of z_0, this follows from local Lipschitz continuity of matrices $\mathcal{A}$ and $\mathcal{D}$. System (3.1) implies the following definitions for A and $F(z)$:

$$A := (\mathcal{A}(z_0) + \mathcal{D}(z_0))^{-1}DB(z_0),$$

$$\begin{aligned} F(z) &= (\mathcal{A}(z + z_0) + \mathcal{D}(z + z_0))^{-1}(z)B(z + z_0) - (\mathcal{A}(z_0) + \mathcal{D}(z_0))^{-1}DB(z_0)z \\ &= (\mathcal{A}(z + z_0) + \mathcal{D}(z + z_0))^{-1}B(z + z_0) - Az. \end{aligned}$$

We have to show that A and F satisfy the following conditions:

$$\operatorname{Re} \sigma(A) \cap \{0\} = \emptyset \tag{3.3}$$

and

$$\begin{aligned} &F(0) = 0, \\ &|F(x) - F(y)| \le \eta(\sigma)|x - y|, \quad \text{if } |x|, |y| \le \sigma, \end{aligned} \tag{3.4}$$

where $\eta : [0, \infty) \to \mathbb{R}$ is continuous and $\eta(0) = 0$.

Obviously, since $B(z_0) = 0$ it follows that $F(0) = 0$. Subsequently, we shall check (3.4). We have

$$
\begin{aligned}
F(x) \quad - &\quad F(y) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.5)\\
= &\quad (\mathcal{A}(x+z_0)+\mathcal{D}(x+z_0))^{-1}B(x+z_0) - (\mathcal{A}(z_0)+\mathcal{D}(z_0))^{-1}(z_0)DB(z_0)x\\
&\quad -((\mathcal{A}(y+z_0)+\mathcal{D}(y+z_0))^{-1}B(y+z_0) - (\mathcal{A}(z_0)+\mathcal{D}(z_0))^{-1}(z_0)DB(z_0)y\\
= &\quad ((\mathcal{A}(x+z_0)+\mathcal{D}(x+z_0))^{-1} - (\mathcal{A}(y+z_0)+\mathcal{D}(y+z_0))^{-1})(B(x+z_0)-B(z_0))\\
&\quad +((\mathcal{A}(y+z_0)+\mathcal{D}(y+z_0))^{-1} - (\mathcal{A}(z_0)+\mathcal{D}(z_0))^{-1}(z_0))(B(x+z_0)-B(y+z_0))\\
&\quad +(\mathcal{A}(z_0)+\mathcal{D}(z_0))^{-1}(z_0)(B(x+z_0)-B(y+z_0)-DB(z_0)(x-y)\\
= &\quad I + II + III.
\end{aligned}
$$

By [GR1, Lemma 2] and differentiability of B we see that

$$|I| \leq M|x-y||x|$$

for a constant M. By the same token

$$|II| \leq M|y||x-y|.$$

Finally, due to smoothness of B and boundedness of $(\mathcal{A}+\mathcal{D})^{-1}$ we can see

$$|III| \leq M|x-y|^2.$$

Thus we can set $\eta(\sigma) = 4M\sigma$.

Now, we shall check that $\operatorname{Re}\sigma(A) \cap \{0\} = \emptyset$. It is sufficient for this purpose to show that $\det A < 0$. For this will imply that the eigenvalues of A satisfy $\lambda_1 \cdot \lambda_2 < 0$ and that they are real and of different sign. We compute

$$\det A = \det((\mathcal{A}(z_0)+\mathcal{D}(z_0))^{-1}DB(z_0)) = \det(\mathcal{A}(z_0)+\mathcal{D}(z_0))^{-1} \cdot \det DB(z_0).$$

We recall now that $\mathcal{A}(z_0)+\mathcal{D}(z_0)$ is positive definite, thus $\det(\mathcal{A}(z_0)+\mathcal{D}(z_0))^{-1} > 0$. The formula for $B(z)$ (see (2.17)) written explicitely in terms of $\bar{z} = (R, L)$ is

$$B(\bar{z}) = \pi \begin{bmatrix} 4RL\sigma^\infty - 4R\gamma(\mathbf{n}_T) - 4L\gamma(\mathbf{n}_\Lambda) \\ R^2\sigma^\infty - \gamma(\mathbf{n}_\Lambda)R \end{bmatrix}.$$

Hence, simple calculations yield

$$\det DB(\bar{z}) = -4\pi^2(R\sigma^\infty - \gamma(\mathbf{n}_\Lambda))(2R\sigma^\infty - \gamma(\mathbf{n}_\Lambda)).$$

Moreover, we have $\gamma(\mathbf{n}_\Lambda) = R_0$, that is why at $z_0 = \frac{2}{\sigma^\infty}(R_0, L_0)$ we reach

$$\det DB(z_0) = -12\pi^2\gamma(\mathbf{n}_\Lambda)^2 < 0,$$

i.e.

$$\det A < 0. \tag{3.6}$$

(b) We want to show that W^S and W^U are C^1. Away from W_γ, the mapping $t \to (R(t), L(t))$ is a good local parameterization of W^S, because $\frac{d}{dt}(R, L) \neq 0$. So indeed W^S is a manifold. This manifold is tangent to $P\mathbb{R}^2$ at (R_0, L_0), see [Ha, Theorem A.2]. The same argument works for W^U. Hence the claim follows. □

Interestingly, the structure of (3.2) yields estimates on $|V_\Lambda/V_T|$ and $|V_T/V_\Lambda|$ which are independent of behavior of W^S or W^U. This will be done in the next section, see proposition 4.6.

After we drew the phase portrait a number of observations become simpler and easily accessible. The first one is that the equilibrium point is dynamically unstable, and it is a saddle point.

If we are concerned with such geometric characteristics of $\Omega(t)$ like aspect ratio (or the isoperimetric quotient), then we see from the phase portrait that we cannot draw a simple conclusion that is decreases or increases. Its behavior depends very much on position of the corresponding point (R, L) in the phase plane.

The dependence of the system on β_T, β_Λ becomes more transparent. It is now obvious that the tangent line to W^S and W^U at z_0 depends smoothly on β_T and β_Λ, also at $(\beta_\Lambda, \beta_T) = (0,0)$. This is the result of the structure of the matrix A.

4 Stability of facets

The starting observation is Proposition 2.3, which tells us that the flows of (1.1)–(1.5) and (2.10)–(2.12) are equivalent as long as we can construct an appropriate Cahn-Hoffman vector ξ. In [GR4] we postulated a variational principle for selecting the right ξ. We defined three functionals

$$\mathcal{E}_i(\xi) = \frac{1}{2}\int_{S_i} |\mathrm{div}_S\xi - \sigma|^2 d\mathcal{H}^2, \qquad i = \Lambda, T, B.$$

In [GR4] we postulated that the correct choice of $\xi(t,\cdot)$ is such that it is a solution to the minimization problems,

$$\mathcal{E}_i(\xi(t,\cdot)) = \min\{\mathcal{E}_i(\zeta) : \ \zeta \in \mathcal{D}_i\}, \qquad i = \Lambda, T, B. \tag{4.1}$$

where

$$\mathcal{D}_i \quad = \quad \{\xi \in L^2(S_i) : \ \mathrm{div}_S\xi \in L^2(S_i), \ (2.3), (2.5) \text{ hold}\}. \tag{4.2}$$

Thus, selecting the right Cahn-Hoffman vector amounts to choosing ξ with minimizing $\mathcal{E}_i$, $i = T, B, \Lambda$. This idea was justified by [FG] for the graph evolution and it was further developed by M.H.Giga-Y.Giga in [GG]. Similar ideas were used by Bellettini, Novaga and Paolini, (see [BNP1]–[BNP3]) as well as in [GPR].

Existence of solutions to these minimization problems is obvious, once we realize that for a minimizer ξ of $\mathcal{E}_i$ it is true that

$$\mathcal{E}_i(\xi) = \frac{1}{2}\mathrm{dist}\,(\sigma, \mathcal{D}_i),$$

and $\mathcal{D}_i$ is a convex, closed subset of $L^2(S_i)$. Thus, $\mathrm{div}_S\xi$ is uniquely defined, while ξ is not unique.

We are interested in establishing a region in the phase plane $\mathcal{P}$ of facet stability in a neighborhood of a unique equilibrium point. Now we may precisely define this notion. We say that a facet S_i is *stable* if there exists a minimizer $\xi \in \mathcal{D}_i$ of (4.1) such that

$$\mathrm{div}_S\xi - \sigma = \text{const.} \qquad \text{on } S_i.$$

We notice that these are precisely the Euler-Lagrange equations of $\mathcal{E}_i$.

Let us stress that the steady states of (1.1)–(1.5) are always stable (see the Remark below Proposition 4.1). In our previous paper [GR4] we were able to find precise sufficient and necessary conditions for stability of facets in the case of self-similar evolution. Now our goal is broader, *i.e.* we make no special assumption on the aspect ratio ϱ of W_γ, nor on the type of solution to (1.1)–(1.5). We will use the same tools as in [GR4].

We saw that these necessary and sufficient conditions were sensitive to the signs of V_T, V_Λ. In [GR4] we had to consider separately growing and shrinking crystals. Here, in fact we consider four basic cases: $\{V_T > 0, V_\Lambda > 0\}$, $\{V_T < 0, V_\Lambda < 0\}$, $\{V_T > 0, V_\Lambda < 0\}$, $\{V_T < 0, V_\Lambda > 0\}$. We shall consider in details only the representative cases, the remaining ones will receive a sketchy treatment.

Our basic observation is that the main tools for stability study ([GR4, Theorem 4.6]) do not depend on the value of the aspect ratio ϱ. That is why we are able to consider here a general situation. In [GR4] we could freely scale our self-similar solutions. Here the situation is more difficult, we have to play with the scale and aspect ratio at the same time to determine stability. For that reason our present result is not as clear as [GR4, Theorem 4.8] and [GR4, Theorem 4.14].

The sets of stability in $\mathcal{P}$ will be determined for S_Λ and S_T, S_B separately, the point is, however, their intersection contains a nonempty open neighborhood of the equilibrium. More precisely, we are not able to give explicit detailed characterization of these sets, but we will show that they have open, non-empty interiors.

Let us recall the basic facts related to our analysis.

Proposition 4.1. ([GR4, Theorem 4.6]). *Let us suppose that σ is given by Proposition 2.2, thus in particular $\sigma|_{S_i} \in L^2(S_i)$.*

(a) If $\bar\xi \in \mathcal{D}_i$ is a solution to (4.1), then there exists $\xi \in \mathcal{D}_i$ another minimizer of $\mathcal{E}_i$, which is of the form

$$\xi(x_1, x_2, x_3) = \nabla(\varphi(r) + \psi(x_3)), \tag{4.3}$$

where φ_r and ψ are explicitely given by the formulas

$$\varphi_r(r) = \frac{1}{r}\int_0^r s\sigma(s, L)\, ds + \frac{r}{R}\left(\gamma(\mathbf{n}_\Lambda) - \frac{1}{R}\int_0^R s\sigma(s, L)\, ds\right) \tag{4.4}$$

$$\psi_z(z) = \int_0^z \sigma(R, s)\, ds - \frac{z}{L}\int_0^L \sigma(R, s)\, ds + \frac{\gamma(\mathbf{n}_T)}{L} z. \tag{4.5}$$

Moreover,

$$\mathrm{div}_S \xi = \mathrm{div}_S \bar\xi.$$

(b) *Let us assume that ξ is the minimizer of $\mathcal{E}_i$ given by (4.3), (4.4) and (4.5). Then,*

(i) *Facet S_T (and S_B) is stable if and only if*

$$\varphi_r(r) \in [-\gamma(\mathbf{n}_\Lambda), \gamma(\mathbf{n}_\Lambda)], \quad \text{for } r \in [0, R], \qquad \varphi_r(0) = 0 \quad \text{and} \quad \varphi_r(R) = \gamma(\mathbf{n}_\Lambda).$$

(ii) *Facet S_Λ is stable if and only if*

$$\psi_{x_3}(x_3) \in [-\gamma(\mathbf{n}_T), \gamma(\mathbf{n}_T)], \quad \text{for } r \in [-L, L], \qquad \psi_{x_3}(0) = 0, \quad \psi_{x_3}(L) = \gamma(\mathbf{n}_T).$$

□

Remarks. (1) In fact $\sigma|_{\partial\Omega(t)} \in C^{1,\alpha}$, see [GR2, Lemma 1]. Thus, our ξ is quite smooth, namely, $\xi \in C(\partial\Omega(t))$ and $\xi \in C^{2,\alpha}(S_i(t))$, $i = T, B, \Lambda$.

(2) A particular conclusion can be drawn for steady states. Namely the facets of $\frac{2}{\sigma^\infty} W_\gamma$ are stable. Since, $V_T = V_\Lambda = 0$, then the supersaturation is constant, $\sigma = \sigma^\infty$. Therefore formulas (4.4) and (4.5) simplify to, respectively,

$$\varphi_r(r) = \frac{r}{R}\gamma(\mathbf{n}_\Lambda), \qquad \psi_{x_3}(x_3) = \frac{x_3}{L}\gamma(\mathbf{n}_T)$$

in support of our claim.

Proposition 4.1 contains a general statement of necessary and sufficient conditions for facet stability. Our task is to study them in more detail for orbits near the stationary point. It turns out that some of these conditions are satisfied automatically depending on the sign of speeds V_T, V_Λ. We collect these observations below. For the purpose of their proof we recall Berg's effect.

Proposition 4.2. (Berg's effect, [GR2, Theorem 1]). *Suppose that σ is a unique solution to*

$$\Delta\sigma = 0 \quad \textit{in} \quad \mathbb{R}^3 \setminus \Omega, \quad \sigma(\infty) = \sigma^\infty,$$
$$\frac{\partial\sigma}{\partial\mathbf{n}} = V_i \quad \textit{on} \quad S_i, \quad i = \Lambda, T, B,$$

where $\sigma = \sigma(\sqrt{x_1^2 + x_2^2}, |x_3|)$, $\mathbf{n}$ is the outer normal to Ω and V_i, $i = \Lambda, T, B$ are constants, moreover $V_T = V_B$.

(a) If $V_T > 0$, then $\frac{\partial\sigma}{\partial x_3} > 0$ for $x_3 > 0$ and $\frac{\partial\sigma}{\partial x_3} < 0$ for $x_3 < 0$ on S_Λ.

(b) If $V_\Lambda > 0$, then $\frac{\partial\sigma}{\partial r} > 0$ on $S_T \cup S_B$. □

Remark. A similar statement holds if we reverse the inequality signs.

Now, we introduce important ingredients of the analysis (see [GR4, §4.3])

$$\bar\sigma_R - \bar\sigma_r \quad \text{and} \quad \bar\sigma_L - \bar\sigma_z,$$

where

$$\bar\sigma_r := \fint_{S_T \cap \{x_1^2+x_2^2 \le r^2\}} \sigma(x)\, d\mathcal{H}^2(x) \quad \text{and} \quad \bar\sigma_z := \fint_{S_\Lambda \cap \{|x_3| \le z\}} \sigma(x)\, d\mathcal{H}^2(x)$$

for given admissible cylinder Ω. Here, we use the notation

$$\fint_G f(x)\, d\mu(x) = \frac{1}{\mu(G)} \int_G f(x)\, d\mu(x),$$

where μ is a measure.

Our observation is stated below.

Lemma 4.3. *Let us suppose that $\bar\sigma_r$ is defined above.*

(a) If $V_\Lambda > 0$, then $\bar\sigma_R - \bar\sigma_r > 0$ for all $r \in (0, R]$.

(b) If $V_\Lambda < 0$, then $\bar\sigma_R - \bar\sigma_r < 0$ for all $r \in (0, R]$.

(c) If $V_\Lambda = 0$, then $\bar\sigma_R \equiv \bar\sigma_r$ for $r \in (0, R]$.

Proof. (a) By Berg's effect we deduce $\frac{\partial\sigma}{\partial r} > 0$, hence $\bar\sigma_R > \bar\sigma_r$ for all $r < R$. Similarly we deduce (b). We shall see that (c) is a direct consequence of limiting cases of (a) and (b). Namely, we consider and auxiliary function defined by formula $\sigma_\epsilon = \sigma + \epsilon f_\Lambda$, where $\epsilon \in \mathbb{R}$. Then,

$$\frac{\partial\sigma_\epsilon}{\partial \mathbf{n}_\Lambda}|_{S_\Lambda} = -\epsilon \quad \text{on } S_\Lambda \qquad \text{and} \qquad \frac{\partial\sigma_\epsilon}{\partial \mathbf{n}_T}|_{S_T} = V_T \quad \text{on } S_T.$$

Moreover, σ_ϵ converges to σ in $C^1(B(0, M))$, where the radius M is sufficiently large. Now, we may conclude

$$\frac{\partial\sigma}{\partial \mathbf{n}_\Lambda}|_{S_T} = \lim_{\epsilon\to 0^-} \frac{\partial\sigma_\epsilon}{\partial \mathbf{n}_\Lambda}|_{S_T} \ge 0 \quad \text{and} \quad \frac{\partial\sigma}{\partial \mathbf{n}_\Lambda}|_{S_T} = \lim_{\epsilon\to 0^+} \frac{\partial\sigma_\epsilon}{\partial \mathbf{n}_\Lambda}|_{S_T} \le 0.$$

Our claim follows. □

A similar statement is valid for $\bar\sigma_z$, whose proof is omitted.

Lemma 4.4. *Let us suppose that $\bar\sigma_z$ is defined above, then*

(a) If $V_T > 0$, then $\bar\sigma_L - \bar\sigma_z > 0$ for all $z \in (0, L]$.

(b) If $V_T < 0$, then $\bar\sigma_L - \bar\sigma_z < 0$ for all $z \in (0, L]$.

(c) If $V_T = 0$, then $\bar\sigma_L \equiv \bar\sigma_z$ for $z \in (0, L]$. □

These lemmas suggest that the sets

$$\{V_\Lambda = 0\} \equiv \{\dot R = 0\}, \qquad \{V_T = 0\} \equiv \{\dot L = 0\}$$

will play a special role in our analysis, see the proof of Theorem 4.7 and the concluding remark.

Now we can pinpoint the cases, where the conditions of Proposition 4.1 are automatically fulfilled. We stress that we do not impose any restrictions on the aspect ratio of the Wulff shape.

Lemma 4.5. *Let us assume that (Ω, σ) is a unique local solution to (2.10)–(2.12) and ξ is a solution to (4.1) in the form postulated by Proposition 4.1, i.e. $\xi = \nabla(\varphi(r) + \psi(x_3))$, where φ_r and $\psi_{x_3}(x_3)$ are given, respectively, by (4.4) and (4.5). Then,* *(a) if $V_\Lambda < 0$, then $\varphi_r(r) > -\gamma(\mathbf{n}_\Lambda)$ for all $r \in (0, R]$;*

(b) if $V_\Lambda > 0$, then $\varphi_r(r) < \gamma(\mathbf{n}_\Lambda)$ for all $r \in (0, R]$;

(c) if $V_T < 0$, then $\psi_z(z) > -\gamma(\mathbf{n}_T)$ for all $z \in (0, L]$;

(d) if $V_T > 0$, then $\psi_z(z) < \gamma(\mathbf{n}_T)$ for all $z \in (0, L]$.

Basically, the proof of this fact is given in [GR4]. However, we assumed there $V_T \cdot V_\Lambda > 0$, which is neither the case here, nor shall use it in the proof. That is why for the sake of clarity we present the argument for (a). The remaining cases are handled in a similar manner and their proofs are omitted.

Proof. Let us write $\varphi_r(r)$ as

$$\varphi_r(r) = \frac{r}{R}\gamma(\mathbf{n}_\Lambda) + rg(r),$$

where

$$g(r) = \frac{1}{r^2}\int_0^r s\sigma(s, L)\, ds - \frac{1}{R^2}\int_0^R s\sigma(s, L)\, ds.$$

Thus

$$g(r) = \frac{1}{2}(\bar\sigma_r - \bar\sigma_R).$$

By Lemma 4.3 (b) we conclude that $g(r) > 0$. Thus indeed

$$\varphi_r(r) > \frac{r}{R}\gamma(\mathbf{n}_\Lambda) > -\gamma(\mathbf{n}_\Lambda). \qquad \square$$

These conclusions are drawn without a detailed knowledge of the behavior of solutions near the steady state. We should not expect that the general statement of Proposition 4.1 will give us an explicit characterization of point in $\mathcal{P}$ for which the facets are stable. We would rather show that this set has a non-empty interior. Namely, we show existence of U_T a neighborhood of z_0 (respectively, U_Λ) such that if $z \in U_T$, then S_T is stable (respectively, if $z \in U_\Lambda$, then S_Λ is stable). Moreover, on the sets $\{V_\Lambda = 0\}$, $\{V_T = 0\}$ facets are also stable. However, they are not invariant under the flow.

We know (see [GR4, §4]) that estimating velocities is another key ingredient. Now, contrary to [GR4] we cannot claim that $\frac{V_\Lambda}{V_T} = \text{const}$. However, we may bound $|\frac{V_\Lambda}{V_T}|$ or $|\frac{V_T}{V_\Lambda}|$ on segments with one end point in z_0. We shall see that these estimates are sufficient for our purposes. They are possible due to the structure of (3.2) and (3.3), (3.4). For the purpose of their proof we introduce a short-hand, for $z \in B(z_0, r) \setminus \{z_0\} \subset \mathcal{P}$ we set

$$\mathbf{e}(z) = \frac{z - z_0}{|z - z_0|},$$

and we denote the i-th component of vector $A\mathbf{e}$ by $(A\mathbf{e})_i$.

Proposition 4.6. *There exist $r_0 > 0$, $K_0 > 0$ and $C_{\Lambda T}(r_0)$, $C_{T\Lambda}(r_0)$ two open sets with the following properties:*

(a) $C_{\Lambda T}(r_0) \cup C_{T\Lambda}(r_0) = B(z_0, r_0) \setminus \{z_0\}$.

(b) *If $z \in C_{\Lambda T}(r_0)$, then for all $s \in (0, r_0]$ the following estimate holds*

$$\left|\frac{V_\Lambda}{V_T}\right|(z_0 + s\mathbf{e}(z)) \le K_0.$$

(c) *If $z \in C_{T\Lambda}(r_0)$, then for all $s \in (0, r_0]$ the following estimate holds*

$$\left|\frac{V_T}{V_\Lambda}\right|(z_0 + s\mathbf{e}(z)) \le K_0.$$

Proof. Let us define

$$\omega_{\Lambda T} = \left\{\mathbf{e} \in S^1(0,1) :\ \left|\frac{(A\mathbf{e})_1}{(A\mathbf{e})_2}\right| < 2\right\}, \quad \omega_{T\Lambda} = \left\{\mathbf{e} \in S^1(0,1) :\ \left|\frac{(A\mathbf{e})_2}{(A\mathbf{e})_1}\right| < 2\right\}.$$

Of course $\omega_{\Lambda T}$ and $\omega_{T\Lambda}$ are open subsets of $S^1(0,1)$ and $\omega_{\Lambda T} \cup \omega_{T\Lambda} = S^1(0,1)$, because for any real number x (here $x = (A\mathbf{e})_1/(A\mathbf{e})_2$) where have $|x| < 2$ or $|x| > 1/2$. Now, let us set

$$\delta_0 = \min\{\min_{\mathbf{e}\in\bar{\omega}_{\Lambda T}} |(A\mathbf{e})_2|, \min_{\mathbf{e}\in\bar{\omega}_{T\Lambda}} |(A\mathbf{e})_1|\}.$$

This number has to be positive, for otherwise for some $\mathbf{e} \neq 0$ we would have

$$(A\mathbf{e})_1 = 0 = (A\mathbf{e})_2,$$

i.e. $A\mathbf{e} = 0$, which is a contradiction since $\det A < 0$, due to (3.6).

We now take $r_0 > 0$ such that

$$\delta_0 > 4\frac{|F(z)|}{|z - z_0|} \tag{4.6}$$

for all $z \in B(z_0, r_0)$, $z \neq z_0$. Existence of such r_0 follows directly from (3.4). Now, we set

$$C_{\Lambda T}(r_0) = \{z \in B(z_0, r_0)\backslash\{z_0\} :\ \mathbf{e}(z) \in \omega_{\Lambda T}\}, \quad C_{T\Lambda}(r_0) = \{z \in B(z_0, r_0)\backslash\{z_0\} :\ \mathbf{e}(z) \in \omega_{T\Lambda}\}.$$

These sets are obviously open and

$$C_{\Lambda T}(r_0) \cup C_{T\Lambda}(r_0) = B(z_0, r_0) \setminus \{z_0\}.$$

We shall estimate $|V_\Lambda/V_T|(z)$. Namely, for $z \in C_{\Lambda T}(r_0)$ we have

$$\left|\frac{V_\Lambda}{V_T}\right|(z) = \left|\frac{(A\mathbf{e}(z))_1 + F(z)/|z - z_0|}{(A\mathbf{e}(z))_2 + F(z)/|z - z_0|}\right| \leq \frac{\max_{\mathbf{e}\in\omega_{\Lambda T}} |(A\mathbf{e})_1| + \frac{1}{4}\delta_0}{\frac{3}{4}\delta_0}.$$

A similar estimate is valid for $z \in C_{T\Lambda}(r_0)$,

$$\left|\frac{V_T}{V_\Lambda}\right|(z) \leq \frac{4}{3\delta_0}(\max_{\mathbf{e}\in\omega_{T\Lambda}} |(A\mathbf{e})_2| + \frac{1}{4}\delta_0).$$

Hence, we set,

$$K_0 = \frac{4}{3\delta_0}\left(\max\{\max_{\mathbf{e}\in\omega_{\Lambda T}} |(A\mathbf{e})_1|, \max_{\mathbf{e}\in\omega_{T\Lambda}} |(A\mathbf{e})_2|\}\right) + \frac{1}{3}.$$

□

At this moment we introduce another piece of notation. Namely, we will write

$$S_i(R, \varrho), \quad i = \Lambda, T, B$$

for i-th facet of Ω, which is an admissible cylinder with aspect ratio equal to ϱ and radius R.

Now we are going to state and prove the main result, basically it says that the subset of $\mathcal{P}$, where the facets are stable has a nonempty interior. A more transparent characterization is rather beyond our reach.

Theorem 4.7. *Let us assume that r_0 is given by Proposition 4.6.*

(a) *There exists r_T in the interval $(0, r_0]$ such that for all points $z \in B(z_0, r_T)$ of the phase plane $\mathcal{P}$ the facets S_T, S_B are stable.*

(b) *Then exists r_Λ in the interval $(0, r_0] > 0$ such that for all points $z \in B(z_0, r_\Lambda)$ the facet S_Λ is stable.*

Proof. Part (a). We now explain the plan of our work. It is divided in several cases corresponding to the following partition of $B(z_0, r_T)$

$$(B(z_0, r_T) \cap \{V_T > 0\}) \cup (B(z_0, r_T) \cap \{V_T < 0\}) \cup (B(z_0, r_T) \cap \{V_T = 0\}).$$

Subsequently, we consider intersections of each of these sets with $\{V_\Lambda > 0\}$, $\{V_\Lambda < 0\}$ and $\{V_\Lambda = 0\}$. Finally, at the lowest level of our analysis, we take into account if $|V_T/V_\Lambda| \leq K_0$ or $|V_\Lambda/V_T| \leq K_0$ on the rays passing through z_0 and $\mathbf{e}(z)$.

(A) We consider $z \in B(z_0, r_0) \cap \{V_T > 0\} =: Q_I$.

(i) We first look at the points in $Q_I \cap \{V_\Lambda > 0\}$. By Lemma 4.5 (b) it is sufficient to show that

$$\varphi_r(r) \geq -\gamma(\mathbf{n}_\Lambda) \quad \text{for all} \quad r \in (0, R].$$

This condition is equivalent to

$$\frac{1}{2} r(\bar{\sigma}_R - \bar{\sigma}_r) < \gamma(\mathbf{n}_\Lambda)(1 + \frac{r}{R}). \tag{4.7}$$

(1) Let us suppose that $z \in Q_I \cap \{V_\Lambda > 0\}$ belongs to $C_{\Lambda T}(r_0)$. We use the formula for $\bar{\sigma}_r$ and after scaling to $R = R_0$ due to Lemma 4.3 (a) we obtain

$$\begin{aligned} &0 < \bar{\sigma}_R - \bar{\sigma}_r \\ &= a V_T \left(\fint_{S_T(R_0, \varrho)} (f_T^{1,\varrho} + f_B^{1,\varrho} + f_\Lambda^{1,\varrho} \frac{V_\Lambda}{V_T}) \, d\mathcal{H}^2 - \fint_{S_T(R_0,\varrho) \cap \{|x| \leq \vartheta R_0\}} (f_T^{1,\varrho} + f_B^{1,\varrho} + f_\Lambda^{1,\varrho} \frac{V_\Lambda}{V_T}) \, d\mathcal{H}^2 \right) \\ &=: a V_T d_{TT\Lambda}^{++}(\varrho, \vartheta, \tau), \end{aligned} \tag{4.8}$$

where $\vartheta = \frac{r}{R_0}$, $\tau = \frac{V_\Lambda}{V_T}$ and $a = \frac{R}{R_0}$. The meaning of the super- and subscripts in d is: the integration is performed over S_T, positive V_T is pulled in front of the integral while V_Λ is positive.

We notice that due to our assumptions τ is in the interval $[0, K_0]$. Moreover, the values of ϱ will be in a bounded set H separated away from zero, because $z \in B(z_0, r_0)$. Thus, we conclude

$$0 < d_{TT\Lambda}^{++}(\varrho, \vartheta, \tau) \leq K_1 \qquad \text{for} \quad (\varrho, \vartheta, \tau) \in H \times [0,1] \times [0, K_0]. \tag{4.9}$$

Thus, (4.7) takes the form

$$a \frac{r}{2} V_T d_{TT\Lambda}^{++}(\varrho, \vartheta, \tau) \leq \gamma(\mathbf{n}_\Lambda)(1 + \frac{r}{R}),$$

and after dividing by R and recalling that $\gamma(\mathbf{n}_\Lambda) = R_0$,

$$\frac{1}{2} a^2 V_T \leq \frac{1 + \vartheta}{\vartheta d_{TT\Lambda}^{++}(\varrho, \vartheta, \tau)}. \tag{4.10}$$

We know that V_T behaves like $O(|z - z_0|)$ while due to (4.9) the right-hand-side of (4.10) is strictly positive and separated from zero. Thus, there is $r_{Ai1} \in (0, r_0]$ such that (4.10) holds for

$$z \in Q_I \cap \{V_\Lambda > 0\} \cap C_{\Lambda T}(r_0) \cap B(z_0, r_{Ai1}).$$

(2) Let us now suppose that $z \in Q_I \cap \{V_\Lambda > 0\}$ belongs to $C_{T\Lambda}(r_0)$. Then, following the previous argument we come to the inequality

$$\frac{1}{2} a^2 V_\Lambda \leq \frac{1 + \vartheta}{\vartheta d_{T\Lambda T}^{++}(\varrho, \vartheta, \tau)}, \tag{4.11}$$

with an analogous definition of $d_{T\Lambda T}^{++}(\varrho, \vartheta, \tau)$ and the convention of super- and subscripts saying that integration is over S_T, positive V_Λ is pulled in front of the integral while V_T is positive as well.

The argument similar to that used in (A)(i)(1) leads us to conclusion that (4.11) holds for $z \in Q_I \cap \{V_\Lambda > 0\} \cap C_{\Lambda T}(r_0) \cap B(z_0, r_{Ai2})$ for some $r_{Ai2} \leq r_0$.

(ii) We now look at the points in $Q_I \cap \{V_\Lambda < 0\}$.

(1) As in (i1) we first assume that z belongs to $C_{\Lambda T}(r_0)$. In this case, by Lemma 4.3 (b) we have to make sure that

$$\varphi_r \leq \gamma(\mathbf{n}_\Lambda) \quad \text{for} \quad r \in (0, R].$$

This inequality is equivalent to

$$\frac{r}{R}\gamma(\mathbf{n}_\Lambda) + \frac{r}{2}(\bar\sigma_r - \bar\sigma_R) \le \gamma(\mathbf{n}_\Lambda). \tag{4.12}$$

After scaling to $R = R_0$, by Lemma 4.3 and the definition of $f_\Lambda^{1,\varrho}$ we shall see

$$\begin{aligned} 0 &< \bar\sigma_r - \bar\sigma_R \qquad\qquad (4.13)\\ &= aV_T\left(⨍_{S_T(R_0,\varrho)\cap\{|x|\le\vartheta R_0\}} (f_T^{1,\varrho} + f_\Lambda^{1,\varrho}\frac{V_\Lambda}{V_T})\, d\mathcal{H}^2 - ⨍_{S_T(R_0,\varrho)} (f_T^{1,\varrho} + f_\Lambda^{1,\varrho}\frac{V_\Lambda}{V_T})\, d\mathcal{H}^2\right)\\ &=: aV_T d_{TT\Lambda}^{+-}(\varrho,\vartheta,\tau) \end{aligned}$$

where, $\vartheta = \frac{r}{R_0}$, $\tau = \frac{V_\Lambda}{V_T}$. This time $d_{TT\Lambda}^{+-}$ means that integration is over S_T, positive V_T is pulled in front of the integral while V_Λ is negative. We also notice that $0 \ge \tau \ge -K_0$, in particular $d_{TT\Lambda}^{+-}(\varrho,\vartheta,\tau) < 0$ for $\vartheta < 1$ and $d_{TT\Lambda}^{+-}(\varrho,1,\tau) = 0$.

Thus (4.12) takes the form

$$\vartheta + \frac{\vartheta}{2}a^2 V_T d_{TT\Lambda}^{+-}(\varrho,\vartheta,\tau) \le 1.$$

or rather we shall check that

$$\frac{a^2}{2}V_T \le \min_\vartheta \frac{1-\vartheta}{\vartheta(d_{TT\Lambda}^{+-}(\varrho,1,\tau) - d_{TT\Lambda}^{+-}(\varrho,\vartheta,\tau))} =: m_{TT\Lambda}^{+-}$$

holds. We can use the same argument as before once we establish that $m_{TT\Lambda}^{+-} > 0$. For the proof of this inequality we shall use the following simple fact.

Lemma 4.8. *Let us suppose that $w : B(0,R) \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous. Then*

$$\lim_{\vartheta\to 1^-} \frac{1}{1-\vartheta}\left(⨍_{B(0,R)} w(x)\,dx - ⨍_{B(0,\vartheta R)} w(x)\,dx \right) = 2⨍_{S^1(0,R)} w(x)\,d\mathcal{H}^1(x) + 2⨍_{B(0,R)} w(x)\,dx.$$

We leave the direct calculations to the interested reader. □

We will apply this Lemma to $w = f_T^{1,\varrho} + f_B^{1,\varrho} + f_\Lambda^{1,\varrho}\frac{V_\Lambda}{V_T}$. Due to the definition of f_i's function w satisfies the assumptions of Proposition 4.2 with $V_\Lambda > 0$, thus we conclude that $m_{TT\Lambda}^{+-} > 0$ as desired.

(iii) $Q_I \cap \{V_\Lambda = 0\}$. We should check that

$$-\gamma(\mathbf{n}_\Lambda) \le \varphi_r \le \gamma(\mathbf{n}_\Lambda).$$

Lemma 4.4 implies that $\bar\sigma_r - \bar\sigma_R \equiv 0$. Hence, these two inequalities are equivalent to

$$\frac{r}{R}\gamma(\mathbf{n}_\Lambda) \le \gamma(\mathbf{n}_\Lambda) \quad \text{and} \quad 0 < \gamma(\mathbf{n}_\Lambda)(1 + \frac{r}{R}).$$

(B) The case of $Q_{II} := B(z_0, r_0) \cap \{V_T < 0\}$ is handled in the same manner.

(C) The case $\{V_T = 0\}$ is a bit puzzling, because Lemma 4.4 is of no use. We notice that

$$\begin{aligned} 0 &< \bar\sigma_R - \bar\sigma_r\\ &= aV_\Lambda\left(⨍_{S_T(R_0,\varrho)} f_\Lambda^{1,\varrho}\, d\mathcal{H}^2 - ⨍_{S_T(R_0,\varrho)\cap\{|x|\le\vartheta R_0\}} f_\Lambda^{1,\varrho}\, d\mathcal{H}^2\right)\\ &=: aV_\Lambda d_{T\Lambda T}^{0}(\varrho,\vartheta,\tau). \end{aligned}$$

Strictly speaking we have to consider $V_\Lambda > 0$, $V_\Lambda < 0$ ($V_\Lambda = 0$ is excluded from these considerations because $V_\Lambda = 0 = V_T$ corresponds to the equilibrium point). In either case by Lemma 4.5 we have to prove just one of the inequalities $\varphi_r > -\gamma(\mathbf{n}_\Lambda)$ or $\varphi_r < \gamma(\mathbf{n}_\Lambda)$. We will consider only $V_\Lambda > 0$. In this case by Lemma 4.3 we conclude that $\bar\sigma_R - \bar\sigma_r$ is positive, hence $d_{T\Lambda T}^{\,0}(\varrho,\vartheta,\tau) > 0$. Analogously to (Ai1) we have to show that

$$\frac{1}{2}a^2 V_\Lambda \le \frac{1+\vartheta}{\vartheta d_{T\Lambda T}^{\,0}(\varrho,\vartheta,\tau)} \qquad \text{for } z \in \{V_T = 0\} \cap B(z_0, r_C). \tag{4.14}$$

From this point we proceed as in (Ai1) to conclude that (4.14) holds for some $r_C > 0$.

Part (b). The reasoning is the same as in part (a). We have to use Lemma 4.4 in place of Lemma 4.3 and replace Lemma 4.5 (a), (b) with Lemma 4.5 (c), (d). Moreover, an analog of Lemma 4.8 is valid. We leave the details to the interested reader. □

We draw an obvious conclusion.

Corollary 4.9. *There exists a neighborhood of z_0 in the phase plane, where facets S_Λ and S_T are stable.* □

Remark. It may look surprising that for $(R, L) \in \{V_\Lambda = 0\}$, (respectively, $(R, L) \in \{V_T = 0\}$) the top and bottom facets (respectively, S_Λ) are stable since this set has a large diameter, *i.e.* point S_T is stable no matter how far from W_γ we are. On the other hand, for points (R, L) in this set the radius of S_T is stationary, thus its stability is not unexpected. Finally, neither $\{V_\Lambda = 0\}$ nor $\{V_T = 0\}$ is invariant under the flow.

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