The Cauchy problem for the Navier-Stokes equations with spatially almost periodic initial data

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Abstract

A unique classical solution of the Cauchy problem for the Navier-Stokes equations is considered when the initial velocity is spatially almost periodic. It is shown that the solution is always spatially almost periodic at any time provided that the solution exists. No restriction on the space dimension is imposed. This fact follows from continuous dependence of the solution with respect to initial data in uniform topology. Similar result is also established for Cauchy problem of the three-dimensional Navier-Stokes equations in a rotating frame.

Key words: Navier-Stokes equations, spatially almost periodic solutions, the Cauchy problem

1 Introduction

We consider the Cauchy problem for the Navier-Stokes equations $(n \geq 2)$:

$$\partial_t u - \nu\Delta u + (u, \nabla)u + \nabla p = 0, \quad \text{div } u = 0 \quad \text{in} \quad \mathbf{R}^n \times (0, T), \tag{1.1}$$

$$u|_{t=0} = u_0 \quad (\text{div } u_0 = 0) \quad \text{in} \quad \mathbf{R}^n, \tag{1.2}$$

when the initial data u_0 is spatially nondecreasing, in particular almost periodic. We use a standard convention of notation; $u(x,t) = (u^1(x,t), \cdots, u^n(x,t))$ represents the unknown velocity field while $p(x,t)$ represents the unknown pressure field; $\nu > 0$ denotes the kinematic viscosity and $\partial_t u = \partial u/\partial t$, $(u, \nabla) = \sum_{i=1}^n u^i \partial_{x_i}$, $\partial_{x_i} = \partial/\partial x_i$, $\nabla p = (\partial_{x_1} p, \cdots, \partial_{x_n} p)$ with $x = (x_1, \cdots, x_n)$.

It is by now well-known ([CK],[Ca],[CaM],[GIM],[KT]) that the problem (1.1)-(1.2) admits a local in time classical solution for any bounded initial data. It is unique under

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an extra assumption for pressure. It is also well-known [GMS] that the solution is global-in-time if the space dimension $n = 2$. By the translation invariance in space variables for (1.1)-(1.2) and the uniqueness of the solution it is clear that if the initial data u_0 is spatially periodic, so is the solution.

Our goal in this note is to show that if u_0 is spatially almost periodic in the sense of Bohr [AG], [Co], so is the solution. This fact follows from continuity of the solution with respect to initial data in uniform topology. Fortunately, such a result follows from analysis developed in [GIM]. However, this persistency of almost periodicity is not noted elsewhere so we shall give its statement as well as its proof.

It turns out that this persistency property also holds for the three-dimensional Navier-Stokes equation in a rotating frame with almost periodic initial data (1.2):

$$u_t - \nu\Delta u + (u, \nabla)u + \Omega e_3 \times u + \nabla p = 0, \quad \mathrm{div}\ u = 0 \text{ in } \quad \mathbf{R}^3 \times (0, T) \qquad (1.3)$$

with $\Omega \in \mathbf{R}$, where $e_3 = (0, 0, 1)$ is the direction of the axis of rotation. In [GIMM] we have proved its locally solvability for a some class of bounded initial data not necessarily decaying at space infinity. Since we are forced to use a homogeneous Besov space, a more specific argument is necessary to prove the persistence of almost periodicity. We shall prove this result in Section 3.

An almost periodic function always has its mean value [AG], [Co]. However, the converse does not hold. Nevertheless we also prove that existence of mean value is preserved for the Navier-Stokes flow even if the initial velocity is not necessarily almost periodic.

2 Persistence of almost periodicity

We first recall a well-known existence result. Let $L^\infty_\sigma(\mathbf{R}^n)$ be the space of all divergence-free essentially bounded vector fields on $\mathbf{R}^n$ equipped with essential supremum norm $\|\cdot\|_\infty$. Let $BUC(\mathbf{R}^n)$ be the space of all bounded uniformly continuous functions on $\mathbf{R}^n$. The space $BUC_\sigma(\mathbf{R}^n)(= BUC(\mathbf{R}^n) \cap L^\infty_\sigma)$ is evidently a closed subspace of $L^\infty_\sigma(\mathbf{R}^n)$. The space $C(I, X)$ denotes the space of all continuous functions from I to X, where X is a Banach space. We do not distinguish the space of vector-valued and scalar functions.

Proposition 2.1([GIM]), (Existence and uniqueness). *Assume that* $u_0 \in BUC_\sigma(\mathbf{R}^n)$. *There exists* $T_0 > 0$ *and a unique classical solution* $(u, \nabla p)$ *of (1.1)-(1.2) such that*
(i) $u \in C([0, T_0], BUC_\sigma(\mathbf{R}^n))$ *and* u *is smooth in* $\mathbf{R}^n \times (0, T_0)$
(ii) $\sup_{0<t<T_0} \|u\|_\infty(t) < \infty$
(iii) $\nabla p = \nabla \sum_{i,j=1}^n R_i R_j u^i u^i$, *where* $R_i = \partial_{x_i}(-\Delta)^{-1/2}$.
Moreover, $t^{1/2}\nabla u \in C([0, T_0], L^\infty_\sigma(\mathbf{R}^n))$ *and* $T_0\|u_0\|^2_\infty \geq C_0$ *with a constant depending only on* n.

Remark 2.2.

(i) One is able to take T_0 arbitrary large number for $n = 2$. In particular, there is a global in time unique smooth solution u for any $u_0 \in BUC_\sigma(\mathbf{R}^n)$ (and more generally for $u_0 \in L^\infty_\sigma(\mathbf{R}^n)$). [GMS].

(ii) As observed in [GIM] we need some restriction of the form of ∇p to have the uniqueness.

Lemma 2.3 (Continuity with respect to initial data). *Let u_k be the bounded (smooth) solution $\mathbf{R}^n \times (0,T)$ of the Navier-Stokes equations for initial data $u_{0k} \in BUC_\sigma(\mathbf{R}^n)$ for $k = 1, 2, \ldots$ Assume that $\sup_{0\le t\le T}\|u_k\|_\infty(t) \le M$. Then the following two properties hold.*

(i) $\|u_1(\cdot,t) - u_2(\cdot,t)\|_\infty \le C\|u_{01} - u_{02}\|_\infty, t \in [0,T]$ with C depending only on n, T and M.

(ii) If u_{0k} converges to u_0 in $L^\infty_\sigma(\mathbf{R}^n)$, then the solution u with initial data u_0 exists for $\mathbf{R}^n \times (0,T)$ and it is the uniform limit of u_k in $\mathbf{R}^n \times (0,T)$.

Proof. This is easy to prove by applying arguments in [GIM]. The solution in [GIM] is the mild solution of an integral equation so u_k satisfies

$$u_k(t) = e^{t\Delta}u_{0k} - \int_0^t \text{div } e^{(t-s)\Delta}\mathbf{P}(u_k \otimes u_k)ds, \tag{2.1}$$

where $\mathbf{P} = (\delta_{ij} + R_iR_j)$ and $e^{t\Delta}f$ is the solution of the heat equation with initial data f i.e., $(e^{t\Delta}f)(x) = (G_t * f)(x)$ with $G_t(x) = (4\pi t)^{-n/2}\exp(-|x|^2/4t)$. By (2.1) it is clear that the difference $u_1 - u_2$ fulfills

$$u_1(t) - u_2(t) = e^{t\Delta}(u_{01} - u_{02}) - \int_0^t \text{div } e^{(t-s)\Delta}\mathbf{P}((u_1 \otimes u_1) - (u_2 \otimes u_2))ds. \tag{2.2}$$

Since

$$\|\text{div } e^{t\Delta}\mathbf{P}f\|_\infty \le Ct^{-1/2}\|f\|_\infty \tag{2.3}$$

by [GIM], estimating (2.2) yields

$$\|u_1 - u_2\|_\infty(t) \le \|u_{01} - u_{02}\|_\infty + 2C\int_0^t (t-s)^{-1/2}(\|u_1\|_\infty + \|u_2\|_\infty)(\|u_1 - u_2\|_\infty(s))ds$$

so that

$$\sup_{0<\tau<t}\|u_1 - u_2\|_\infty(\tau) \le \|u_{01} - u_{02}\|_\infty + 4C2Mt^{1/2}\sup_{0<\tau<t}\|u_1 - u_2\|_\infty(\tau).$$

If $8CMt^{1/2} \le 1/2$, then we have

$$\sup_{0<\tau<t}\|u_1 - u_2\|_\infty(\tau) \le 2\|u_{01} - u_{02}\|_\infty$$

We divide $[0,T]$ into $[0,T_1]$, $[T_1,T_2]$, $\cdots$, $[T_N,T_0]$ so that the length of each time interval is less than $(16CM)^{-2}$ and repeat this argument on each interval to get (i).

If $\{u_{0k}\}$ is the Cauchy sequence in $L^\infty_\sigma(\mathbf{R}^n)$, $\{u_k\}$ is the Cauchy sequence in $L^\infty(\mathbf{R}^n \times (0,T))$ (= the space of all essentially bounded functions in $\mathbf{R}^n \times (0,T)$) by (i). Thus sending k to infinity in (2.1) yields (ii) if we note (2.3). □

We now recall the notion of almost periodicity in the sense of Bohr [AG], [Co]. Let f be in $BUC(\mathbf{R}^n)$. We say that f is almost periodic if the set

$$\Sigma_f = \{f(\cdot+\xi)|\xi \in \mathbf{R}^n\} \subset L^\infty(\mathbf{R}^n)$$

is relatively compact in $L^\infty(\mathbf{R}^n)$, i.e. any sequence in Σ_f has an convergent subsequence in $L^\infty(\mathbf{R}^n)$. (Actually, uniformly continuity assumption is redundant. In fact, if f is bounded continuous and Σ_f is relatively compact in $L^\infty(\mathbf{R}^n)$, then f must be uniformly continuous [AG], [Co]). If f is periodic in $(x_1,\cdots,x_n)$ i.e. f

$$f(x+\eta) = f(x) \text{ for all } x \in \mathbf{R}^n$$

for some $\eta = (\eta_1,\cdots,\eta_n), \eta_i > 0, i = 1,\cdots,n$, then $\Sigma_f = \prod_{i=1}^n (\mathbf{R}/\eta_i\mathbf{Z})$ is a torus so any periodic function is almost periodic. A finite sum of periodic continuous functions in L^∞ is almost periodic. If an infinite sum of periodic continuous functions converges in $L^\infty(\mathbf{R}^n)$, then it is almost periodic.

We are now in position to state our main result.

Theorem 2.4. *Assume that $u_0 \in BUC_\sigma(\mathbf{R}^n)$ is almost periodic. Let u be the bounded (smooth) solution of the Navier-Stokes equation in $\mathbf{R}^n \times (0,T)$ with initial data u_0 in Proposition 2.1. Then $u(\cdot,t)$ is almost periodic as a function of $\mathbf{R}^n$ for all $t \in (0,T)$.*

Proof. Let $u(t) = u(\cdot,t)$ be the solution (in Proposition 2.1) of the Navier-Stokes equation with initial data u_0. We denote the mapping $u_0 \longmapsto u(t)$ by $S(t)$.

Since the Navier-Stokes equations are invariant under translation in the space variable, the solution u_η with initial data $u_{0\eta}(x) = u_0(x+\eta)$, $\eta \in \mathbf{R}^n$ fulfills $u_\eta(x,t) = u(x+\eta,t)$. Thus $S(t)$ maps Σ_{u_0} on to $\Sigma_{u(t)}$. Since

$$\sup_{0<t<T} \|u_\eta\|_\infty(t) = \sup_{0<t<T} \|u\|_\infty(t)$$

is independent of η, Lemma 2.3 implies that $S(t)$ is a well-defined continuous mapping from the closure $\bar{\Sigma}_{u_0}$ to $\bar{\Sigma}_{u(\cdot,t)}$. Thus if $\bar{\Sigma}_{u_0}$ is relatively compact, so is $\bar{\Sigma}_{u(\cdot,t)}$. We now conclude that $u(\cdot,t)$ is almost periodic. □

3 Three-dimensional Navier-Stokes equations in a rotating frame

We shall show persistence of almost periodicity for the Cauchy problem (1.2)-(1.3). For this problem it seems impossible to establish well-posedness for L^∞- initial data. We recall a result of existence of a unique local solution for (1.2)-(1.3) [GIMM](see also [S]). Let $\mathcal{W}$ be

$$\mathcal{W} = \{u \in L^\infty_\sigma(\mathbf{R}^3) | \partial u/\partial x_3 = 0 \text{ in } \mathbf{R}^n \text{ in the sense of distribution } \}.$$

An element of this space is often called a two-dimensional three component divergence free vector field. We need to recall a Besov space

$$\dot{B}^0_{\infty,1}(\mathbf{R}^3) = \{f \in \mathcal{S}'(\mathbf{R}^3) | f = \sum_{j=-\infty}^{\infty} \varphi_j * f \text{ in } \mathcal{S}' (\mathbf{R}^3),$$

$$\|f\|_{\dot{B}^0_{\infty,1}} := \sum_{j=-\infty}^{\infty} \|\varphi_j * f\|_\infty < \infty\},$$

where $\{\varphi_j\}_{j=-\infty}^{\infty}$ is the Littlewood-Paley decomposition satisfying

$$\hat{\varphi}_j(\xi) = \hat{\varphi}_0(2^{-j}\xi) \in C_0^\infty(\mathbf{R}^3), \text{ supp } \hat{\varphi}_0 \subset \{1/2 \le |\xi| \le 2\}, \sum_{j=-\infty}^{\infty} \hat{\varphi}_j(\xi) = 1(\xi \ne 0);$$

here $\hat{\varphi}$ denotes the Fourier transform of $\hat{\varphi}$ and $C_0^\infty(\mathbf{R}^n)$ is the space of all smooth functions with compact support in $\mathbf{R}^n$.

We need to prepare several function spaces. We say that $u \in L^\infty_\sigma(\mathbf{R}^3)$ admits vertical averaging if

$$\lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} u(x_1, x_2, x_3) dx_3 = \bar{u}(x_1, x_2)$$

exists almost everywhere (a.e). The vector field $\bar{u}$ is called vertical average of $u(x_1, x_2, x_3)$. Let $L^\infty_{\sigma,a}$ be the topological direct sum of the form

$$L^\infty_{\sigma,a} = \mathcal{W} \oplus \mathcal{B}^0$$

with

$$\mathcal{B}^0 = \{u \in \dot{B}^0_{\infty,1} \cap L^\infty_\sigma | \bar{u}(x_1, x_2) = 0 \text{ a.e.}(x_1, x_2) \in \mathbf{R}^2\}.$$

We often consider a smaller space

$$X = BUC_{\sigma,a} := \{u \in L^\infty_{\sigma,a}(\mathbf{R}^3) | u \in BUC(\mathbf{R}^3)\}$$

$$= \{u = u_1 + u_2 | u_1 \in \mathcal{W} \cap BUC, u_2 \in \mathcal{B}^0\}.$$

The second identity follows from the fact that $\mathcal{B}^0 \subset BUC$. The space X is equipped with the norm

$$\|u\|_X = \|u_1\|_{L^\infty} + \|u_2\|_{\dot{B}^0_{\infty,1}}$$

and it is a Banach space. We shall fix $\Omega \in \mathbf{R}$ below.

Proposition 3.1 ([GIMM]). *Assume that $u_0 \in X$. There exists $T_0 > 0$ and a unique classical solution $(u, \nabla p)$ of (1.2)-(1.3) such that*
(i) $u \in C([0,T_0), X)$ for any $\delta > 0$ and u is smooth in $\mathbf{R}^3 \times (0,T_0)$
(ii)

$$\frac{\partial p}{\partial x_\ell} = \frac{\partial}{\partial x_\ell} \sum_{i,j=1}^{3} R_i R_j u^i u^j - \Omega R_\ell (R_2 u^1 - R_1 u^2), \ell = 1,2,3.$$

Moreover, $t^{1/2}\nabla u \in C([0,T_0]; L^\infty_\sigma(\mathbf{R}^3))$ and $T_0 \|u_0\|_X^2 \geq c_0$ with a positive constant c_0.

Proposition 3.2 (Continuity with respect to initial data). *Let u_k be the bounded (smooth) solution in $\mathbf{R}^3 \times (0,T)$ of the Navier-Stokes equations (1.3) in a rotating frame with initial data $u_{0k} \in X$ for $k = 1,2,\ldots$ Assume that $\sup_{0 \leq t < T} \|u_k\|_\infty(t) \leq M$. Then the following two properties hold*
(i) $\|u_1(\cdot,t) - u_2(\cdot,t)\|_\infty \leq C\|u_{01} - u_{02}\|_X, t \in (0,T)$ with C depending only on T and M.
(ii) If $\{u_{0k}\}_{k=1}^\infty$ converges to u_0 in X, then the solution u with initial data u_0 exists for $\mathbf{R}^3 \times (0,T)$ and it is the uniform limit in $\mathbf{R}^3 \times (0,T)$.

Proof of Lemma 3.2. This is easy to prove by applying arguments in [GIMM]. The solution in [GIMM] is the mild solution of an integral equation so u_k
fulfils

$$u_k(t) = e^{-A(\Omega)t} u_{0k} - \int_0^t \mathrm{div} e^{-A(\Omega)(t-s)\Delta} \mathbf{P}(u_k \otimes u_k) ds$$

with $A(\Omega) = -\Delta + \Omega \mathbf{P} J \mathbf{P}$, where $Ja = e_3 \times a$ for $a \in \mathbf{R}^3$. We estimate the difference $u_1 - u_2$ similarly to [GIMM, (4.8)] and obtain

$$\|u_1 - u_2\|_\infty(t) \leq \|u_{01} - u_{02}\|_X + 2CM \int_0^t (t-s)^{-1/2} \|u_1 - u_2\|_\infty(s) ds,$$

with a constant $C > 0$. As before we apply the Gronwall inequality [GMS] to get

$$\|u_1 - u_2\|_\infty(t) \leq C_1 \|u_{01} - u_{02}\|_X e^{C_2 t}, \quad t \in (0,T)$$

with some positive constants C_1, C_2 depending only on C and M. This yields (i). From (i) it follows (ii) as before. $\square$

We are now in position to state our main result.

Theorem 3.3. *Assume that $u_0 \in X$ is almost periodic. Let u be the bounded (smooth) solution of the Navier-Stokes equations (1.3) in a rotating frame in $\mathbf{R}^3 \times (0,T)$ with initial data u_0 in Proposition 3.1. Then $u(\cdot,t)$ is almost periodic as a function of $\mathbf{R}^n$.*

The proof parallels that of Theorem 2.4 by setting the solution operator $S(t) : u_0 \mapsto u(\cdot, t)$ if we use the next lemma.

Lemma 3.4. *A function $f \in \dot{B}^0_{\infty,1}(\mathbf{R}^n)$ is almost periodic if and only if Σ_f is relatively compact in $f \in \dot{B}^0_{\infty,1}(\mathbf{R}^n)$.*

Proof. It suffices to prove 'only if' part since $\dot{B}^0_{\infty,1} \subset L^\infty$ is continuous. Suppose that $\Sigma_f (\subset \dot{B}^0_{\infty,1})$ is relatively compact in $L^\infty(\mathbf{R}^n)$. Then any sequence $\{f_{\eta_k}\} \subset \Sigma_f$ has a convergent subsequence $\{f_\ell = f(\cdot + \eta_{k(\ell)})\} \to f_0$ in $L^\infty(\mathbf{R}^3)$ as $\ell \to \infty$. We note that

$$\|\varphi_i * f_\ell\|_\infty = \|\varphi_i * f\|_\infty \quad \text{for all} \quad i \in \mathbf{Z}.$$

to get

$$\|f_\ell - f_0\|_{\dot{B}^0_{\infty,1}} \leq \sum_{|i| \geq N} \|\varphi_i * (f_\ell - f_0)\|_\infty + \sum_{|i| \leq N-1} \|(f_\ell - f_0) * \varphi_i\|_\infty \leq 2 \sum_{|i| \geq N} \|\varphi_i * f\|_\infty + C_N \|f_\ell - f\|_\infty$$

with $C_N = 2N\|\varphi_0\|_{L^1}$. Sending $\ell \to \infty$, we observe that

$$\lim_{\ell \to \infty} \sup \|f_\ell - f_0\|_{\dot{B}^0_{\infty,1}} \leq 2 \sum_{|i| \geq N} \|\varphi_i * f\|_\infty.$$

Since $f \in \dot{B}^0_{\infty,1}$, the right hand side tends to zero as $N \to \infty$ so $f_\ell \to f_0$ and $f_0 \in \dot{B}^0_{\infty,1}$. $\square$

Remark 3.5.

We note that $\Omega e_3 \times u$ restricted to divergence free vector fields is a skew-symmetric term in (1.3). The fast singular oscillating limit (large Ω) of the 3D Navier-Stokes equations in a rotating frame (1.3) with almost periodic initial data (1.2) will be considered elsewhere. Global regularity for large Ω of solutions of the three-dimensional Navier-Stokes equations in a rotating frame with initial data on arbitrary periodic lattices and in bounded cylindrical domains in $\mathbb{R}^3$ was proven in [BMN1], [BMN2], [BMN3] and [MN] without any conditional assumptions on the properties of solutions at later times. The method of proving global regularity for large fixed Ω is based on the analysis of fast singular oscillating limits (singular limit $\Omega \to +\infty$), nonlinear averaging and cancellation of oscillations in the nonlinear interactions for the vorticity field. It uses harmonic analysis tools of lemmas on restricted convolutions and Littlewood-Paley dyadic decomposition to prove global regularity of the limit resonant three-dimensional Navier-Stokes equations which holds without any restriction on the size of initial data and strong convergence theorems for large Ω.

4 Averaging property

As proved in [AG], [Co] an almost periodic function has the mean value at least for functions of one variable. We shall study a class of functions having its mean value.

Definition 4.1. Let D be a bounded C^1 domain in $\mathbf{R}^n$ containing the origin. Let χ_D be its characteristic function, i.e., $\chi_D(x) = 1$ if $x \in D$ and $\chi_D(x) = 0$ if $x \notin D$. Let $\chi_D^R(x) = \chi_D(x/R)R^{-n}|D|^{-1}$ for $R > 0$, where $|D|$ denotes the Lebesque measure of D. (By definition $\int_{\mathbf{R}^n} \chi_D^R dx = 1$.) A function $f \in L^\infty(D)$ is said to have its D- mean value if $\chi_D^R * f$ converges to a constant c uniformly in $\mathbf{R}^n$ as $R \to \infty$. The constant c is called *D-mean value* of f.

Example 4.2. For any $\xi \in \mathbf{R}^n$ the function $e^{i\xi x}$ has its mean value for any D. This is trivial if $\xi = 0$ so we may assume $\xi \neq 0$. By rotation of coordinates we may assume $\xi_1 \neq 0$ for $\xi = (\xi_1, \cdots, \xi_n)$. Integrating by parts, we observe that

$$(\chi_D^R * f)(x) = \frac{1}{R^n|D|}\int_{RD} f(x-y)dy = \frac{1}{R^n|D|}\int_{\partial(RD)} n_1 \frac{e^{i\xi x}}{i\xi_1} d\mathcal{H}^{n-1}$$

where $\mathbf{n} = (n_1, \cdots, n_n)$ is the outer unit normal of $\partial(RD) = R(\partial D) = \{Rx | x \in \partial D\}$ and $d\mathcal{H}^{n-1}$ is the area element. Thus $\|\chi_D^R * f\|_\infty \to 0$ as $R \to \infty$. Thus unless $\xi = 0$, the mean value of $e^{i\xi x}$ equals zero. Since one can prove that an almost periodic function is a uniform limit of Bochner-Fejer trigonometric polynomials as proved in [AG], [Co] (for $n = 1$), an almost periodic function has its D-mean value for all D. (Note that a uniform limit of $\{f_m\}$ has always its D-mean value if f_m has its D-mean value.)

Evidently, even if a function has its D-mean value for any D, this does not imply the function is almost periodic. For example $f(u) = e^{iax}e^{-\varepsilon x^2}, x \in \mathbf{R}, a \neq 0, \varepsilon > 0$ has mean value zero but it is not at all almost periodic. However, $f \in \dot{B}^0_{\infty,1}$ if ε is taken sufficiently small so that the support of $\hat{f}$ is away from the origin. This implies that an element of $\dot{B}^0_{\infty,1}$ is not necessarily almost periodic though it has mean value zero for any D.

Lemma 4.3. *A function $f \in \dot{B}^0_{\infty,1}$ has D-mean value zero for any D.*

Proof. It suffices to prove that $\varphi_j * f$ has D-mean value zero for all $j \in \mathbf{Z}$ since $f_m \to f$ in $\dot{B}^0_{\infty,1}$ implies $f_m \to f$ in L^∞ for $f_m = \Sigma_{|j|\leq m}\varphi_j * f$. Let $\{\hat{\psi}_\ell\}_{\ell=1}^N \subset C_0^\infty(\mathbf{R}^n)$ be a partition of unity of the support of φ_j and supp $\hat{\psi}_\ell$ does not contain the plane $\{\xi_1 = 0\}$ for some $i = 1, \cdots, n$ (i may depend on ℓ). Then there is $\rho_\ell \in \mathcal{S}$ such that $\hat{\psi}_\ell = j\xi_j\ \hat{\rho}_\ell$. Thus we observe

$$\psi_\ell * \varphi_j * f = \partial_i(\rho_\ell * \varphi_j * f)$$

to get

$$F_R(x) := (\chi_D^R * \psi_\ell * \varphi_j * f)(x) = \frac{1}{R^n|D|}\int_{\partial(RD)} n_i(\rho_\ell * \varphi_j * f)d\mathcal{H}^{n-1}.$$

We estimate F_R to get

$$|F_R(x)| \leq \frac{1}{R^n|D|}\|\rho_\ell * \varphi_j * f\|_\infty \mathcal{H}^{n-1}(\partial(RD)) \to 0$$

since $\rho_\ell * \varphi_j \in L^1$ is independent of R. Thus

$$\varphi_j * f = \Sigma_{\ell=1}^N \psi_\ell * \varphi_j * f$$

has D-mean value zero. The proof is now complete. $\square$

An element of BUC having its mean value for some D but may not have mean value for another D. Here is an example. We consider

$$f(x) = \frac{x}{\sqrt{x^2+1}}(\cos\log\sqrt{x^2+1} - \sin\log\sqrt{x^2+1})$$

which is the derivative of

$$g(x) = \sqrt{x^2+1}\cos\log\sqrt{x^2+1}.$$

If $D = (-1,1)$, the mean value exists and equals zero by uniform continuity of f. If $D = (-1/2, 3/2)$, the mean value does not exist. Indeed, $(\chi_D^R * f)(0) = \frac{1}{R}(g(3R/2) - g(R/2))$ does not converge as $R \to \infty$.

Our goal in this section is to prove that existence of mean value is preserved for the Navier-Stokes flow.

Theorem 4.4. *Assume that u_0 has D-mean value $c \in \mathbf{R}^n$. Then the solution u of the Navier-Stokes equation with initial data u_0 (in Proposition 2.1) has D-mean value c for all $t \in (0, T_0)$.*

Proof. Since u solves (2.1), i.e.,

$$u(t) = e^{t\Delta}u_0 - \int_0^t \mathrm{div}(e^{(t-s)\Delta}\ \mathbf{P}(u \otimes u))ds, \tag{4.1}$$

it suffices to prove that $e^{t\Delta}u_0$ has D-mean value c and that the second term $F(t)$ of (4.1) belongs to $\dot{B}^0_{\infty,1}$ if we notice that any element of $\dot{B}^0_{\infty,1}$ has D-mean value zero (Lemma 4.3). We shall prove these facts in the next lemmas.

Lemma 4.5. *(i) If $a \in BUC(\mathbf{R}^n)$ has D-mean value $c \in \mathbf{R}$, then $f * a$ has D-mean value $c\int_{\mathbf{R}^n} f dx$ provided that $f \in L^1(\mathbf{R}^n)$. In particular, $e^{t\Delta}a$ has D-mean value c.*
(ii) If $u \in L^\infty(\mathbf{R}^n \times (0,T))$, then $F(t) \in \dot{B}^0_{\infty,1}(\mathbf{R}^n)$.

Proof. (i) This is easy since

$$\|\chi_D^R * f * a - f * c\|_\infty = \|(\chi_D^R * a - c) * f\|_\infty \leq \|f\|_{L^1}\|\chi_D^R * a - c\|_\infty.$$

$\rightarrow\ 0$ as $R \rightarrow \infty$.

(ii) We shall recall an estimate

$$\|\nabla e^{t\Delta} f\|_{\dot{B}^0_{\infty,1}} \leq \frac{C}{t^{1/2}} \|f\|_{\dot{\mathcal{B}}^0_{\infty,\infty}} \tag{4.2}$$

found, for example, in [I]. (Here the space $\dot{\mathcal{B}}^0_{\infty,\infty} (\subset \mathcal{S}')$ is defined as the dual space of

$$\dot{\mathcal{B}}^0_{1,1} = \{ f \in \mathcal{S}' | f = \sum_{j=-\infty}^{\infty} f * \varphi_j \text{ in } \mathcal{S}',$$

$$\|f\|_{\dot{\mathcal{B}}^0_{1,1}} = \sum_{j=-\infty}^{\infty} \|\varphi_j * f\|_{L^1} < \infty \}.)$$

Using this estimate for $F(t)$, we observe that

$$\|F(t)\|_{\dot{B}^0_{\infty,1}} \leq \int_0^t \frac{C}{(t-s)^{1/2}} |\mathbf{P}| \ \|u\|^2_\infty (s) ds$$

where $|\mathbf{P}|$ is the operator norm in $\dot{\mathcal{B}}^0_{\infty,\infty}$, which is finite (see e.g. [A]); here we invoked the property that $\|f\|_{\dot{\mathcal{B}}^0_{\infty,\infty}} \leq C' \|f\|_\infty$. This estimate yields that $F(t) \in \dot{B}^0_{\infty,1}$ □

Remark 4.6. (i) The estimate (4.2) also implies (2.3) if we notice that $\|f\|_\infty \leq \|f\|_{\dot{B}^0_{\infty,1}}$.
(ii) A similar result holds for the Navier-Stokes equation in a rotating frame. In this case we have to assume that $\mathcal{W}$ component of the initial data u_0 has D-mean value.

Acknowledgement This work is partly supported by the Grant-in-Aid fund of COE "Mathematics of Nonlinear Structures via Singularities". The work of the first author is partly supported by the Grant-in-Aid for Scientific Research, No.14204011, the Japan Society of the Promotion of Science. Much of the work of the first author was done when he was in the Department of Mathematics of the Hokkaido University whose support is gratefully acknowledged.

References

[AG] N. I. Akhiezer and I. M. Glazman (1961), Theory of linear operators in Hilbert space, *F. Ungar Pub. Co., New York.*

[A] H. Amann (1997), Operator-Valued Fourier Multipliers, Vector-Valued Besov Spaces, and Applications, *Math. Nachr.*, **186**, p. 5-56.

[BMN1] A. Babin, A. Mahalov and B. Nicolaenko (2001), 3D Navier-Stokes and Euler Equations with initial data characterized by uniformly large vorticity, *Indiana University Mathematics Journal*, **50**, p. 1-35.

[BMN2] A. Babin, A. Mahalov and B. Nicolaenko (1999), Global regularity of the 3D Rotating Navier-Stokes Equations for resonant domains, *Indiana University Mathematics Journal*, **48**, No. 3, p. 1133-1176.

[BMN3] A. Babin, A. Mahalov and B. Nicolaenko (1997), Regularity and integrability of the 3D Euler and Navier-Stokes Equations for uniformly rotating fluids, *Asympt. Anal.*, **15**, No. 2, p. 103-150.

[CK] J.R. Cannon and G.H.Knightly (1970), A note on the Cauchy problem for the Navier-Stokes equations, *SIAM J. Appl. Math.*, **18**, p. 641-644.

[Ca] M. Cannone (1995), Ondelettes, Praproduits et Navier-Stokes, Diderot Editeur, *Arts et Sciences Paris-New York-Amsterdam*.

[CaM] M. Cannone and Y. Meyer (1995), Littlewood-Paley decomposition and Navier-Stokes equations, *Methods and Applications of Analysis*, **2**, p. 307-319.

[Co] C. Corduneanu (1968), Almost Periodic Functions, *Interscience Publishers, New York*,

[FMRT] C. Foias, O. Manley, R. Rosa and R. Temam (2001), Navier-Stokes Equations and Turbulence, Encyclopedia of Mathematics and Its Applications, vol. **83**, *Cambridge University Press*.

[GIMM] Y. Giga, K. Inui, A. Mahalov and S. Matsui (2004), Navier-Stokes equations in a rotating frame in $\mathbb{R}^3$ with initial data nondecaying at infinity, Hokkaido Univ. Preprint Series in Math. **#664**

[GIM] Y. Giga, K. Inui and S. Matsui (1999), On the Cauchy problem for the Navier-Stokes equations with nondecaying initial data, *Quaderni di Matematika*, **4**, p. 28-68.

[GMS] Y. Giga, S. Matsui and O. Sawada (2001), Global existence of two-dimensional Navier-Stokes flow with nondecaying initial velocity, *J. Math. Fluid Mech.*, **3**, p. 302-315.

[I] K. Inui, Rotating Navier-Stokes Equations with Initial Data Nondecreasing at Infinity, *Sūrikaisekikenkyūsho Kōkyūroku*, to appear.

[KT] H. Koch and D. Tataru (2001), Well-posedness for the Navier-Stokes equations, *Advances in Mathematics*, **157**, p. 22-35.

[MN] A. Mahalov and B. Nicolaenko (2003), Global regularity of the 3D Navier-Stokes Equations with weakly aligned large initial vorticity, *Russian Math. Surveys*, **58**, No. 2 (350), p. 287-318.

[S] O. Sawada (2004), The Navier-Stokes flow with linearly growing initial velocity in the whole space, *Bol. Soc. Paran. Mat.*, to appear.

[VF] M.I. Vishik and A.V. Fursikov (1988), Mathematical Problems of Statistical Hydromechanics, Kluwer, *Academic Publisher*.