



Title	$L^p$ - $L^q$ estimates for convolutions with distribution kernels having singularities on the light cone
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**$L^p$ - $L^q$  ESTIMATES FOR CONVOLUTIONS WITH DISTRIBUTION  
Kernels Having Singularities on the Light Cone**

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ABSTRACT. We study the convolution operator  $T^z$  with the distribution kernel given by analytic continuation from the function

$$\tilde{K}^z(y, s, t) = \begin{cases} (t^2 - s^2 - |y|^2)_+^z / \Gamma(z+1) & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}, \quad \operatorname{Re}(z) > -1$$

where  $(y, s, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$ . We obtain some improvement upon the previous known estimates for  $T^z$ . Then we extend the result of the cone multiplier of negative order on  $\mathbb{R}^3$  [8] to the case of general  $\mathbb{R}^{n+1}$ ,  $n \geq 2$ .

1. INTRODUCTION AND STATEMENT OF RESULTS

Let  $\tilde{K}^z$  be the family of distributions on  $\mathbb{R}^{n+1}$ ,  $n \geq 2$  defined by analytic continuation from the equation

$$\tilde{K}^z(y, s, t) = \begin{cases} (t^2 - s^2 - |y|^2)_+^z / \Gamma(z+1) & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}, \quad \operatorname{Re}(z) > -1$$

where  $(y, s, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$ , and  $r_+ = r$  if  $r \geq 0$  and  $r_+ = 0$  if  $r < 0$ . The  $L^p$ - $L^q$  estimates for the convolution operator  $f \rightarrow \tilde{K}^z * f$  were studied by Oberlin [12]. Oberlin [12] showed that for  $-\frac{n}{2} \leq \operatorname{Re}(z) \leq 0$ ,  $f \rightarrow \tilde{K}^z * f$  is bounded from  $L^p(\mathbb{R}^{n+1})$  to  $L^q(\mathbb{R}^{n+1})$  provided

$$(1.1) \quad \frac{1}{p} - \frac{1}{q} = 1 + \frac{2\operatorname{Re}(z)}{n+1}, \quad 1 + \frac{\operatorname{Re}(z)}{n} < \frac{1}{p} < 1 + \frac{\operatorname{Re}(z)(n-1)}{n^2+n}.$$

He also showed when  $z$  is non-integer, for  $-\frac{n+1}{2} < z < 0$ ,  $f \rightarrow \tilde{K}^z * f$  is bounded from  $L^p(\mathbb{R}^{n+1})$  to  $L^q(\mathbb{R}^{n+1})$  only if (1.1) is satisfied (by simple modification of his argument one can see the necessary condition is also valid for integer  $z$ ). Using the  $L^2$  boundedness of  $f \rightarrow \tilde{K}^z * f$  when  $\operatorname{Re}(z) = -(n+1)/2$ , by interpolation it can be shown that  $f \rightarrow \tilde{K}^z * f$  is bounded from  $L^p(\mathbb{R}^{n+1})$  to  $L^q(\mathbb{R}^{n+1})$  if  $1 \leq p < 2 < q \leq \infty$  and (1.1) is satisfied. The same results were also obtained by Harmse [7] using the method based on the interpolation along analytic family of operators. These and the previously known results are obtained by some variants of  $T^*T$  method

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which is basically an  $L^2$ -theory. The aim of this paper is to extend the range of  $p, q$  for which  $T^z$  is bounded beyond the region mentioned above.

By a simple linear transformation, the distribution  $\tilde{K}^z$  is essentially equivalent to  $K^z$  given by analytic continuation from the function

$$K^z(y, s, t) = \begin{cases} (ts - \frac{|y|^2}{4})_+^z / \Gamma(z+1) & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}, \quad \operatorname{Re}(z) > -1.$$

It is easy to see that the  $L^p$ - $L^q$  boundedness properties of  $f \rightarrow K^z * f$  and  $f \rightarrow \tilde{K}^z * f$  are equivalent. It is also possible to handle  $f \rightarrow \tilde{K}^z * f$  directly but computations involved can be simplified by considering  $f \rightarrow K^z * f$ . Thus from now on, we study  $f \rightarrow K^z * f$  instead of  $f \rightarrow \tilde{K}^z * f$ .

Our results are stated in terms of Besov spaces. Let  $\phi \in C_0^\infty(1/2, 2)$  satisfying  $\sum_k \phi(|\cdot|/2^k) = 1$ . We use a kind of homogeneous Besov space  $\tilde{B}_{p,r}^s$ , which is equipped with the norm:

$$(1.2) \quad \|f\|_{\tilde{B}_{p,r}^s} := \left( \sum_k 2^{srk} \|\Delta_k f\|_p^r \right)^{\frac{1}{r}}, \quad 1 \leq p, r \leq \infty, \quad s \in \mathbb{R},$$

where  $\widehat{\Delta_k f}(\eta, \rho, \tau) = \phi(|\rho|/2^k) \widehat{f}(\eta, \rho, \tau)$ ,  $(\eta, \rho, \tau) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$ ,  $\widehat{\cdot}$  stands for the Fourier transform defined by  $\widehat{f}(\xi) = \int e^{-2\pi i x \cdot \xi} f(x) dx$  and  $\|\cdot\|_p = \|\cdot\|_{L^p(\mathbb{R}^{n+1})}$ . The following is our main result.

**Theorem 1.1.** *For  $-\frac{n(n+1)^2}{2(n^2+2n-1)} \leq \operatorname{Re}(z) < 0$ , there is a constant  $C$  such that*

$$(1.3) \quad \|K^z * f\|_{\tilde{B}_{q,r}^s} \leq C \|f\|_{\tilde{B}_{p,r}^s}$$

for all  $f \in \mathcal{S}(\mathbb{R}^{n+1})$ , provided  $p, q$  satisfies (1.1).

For real  $z$ , the conditions (1.1) are necessary for (1.3). Indeed, the first is a simple consequence of homogeneity. For the second, consider a positive function  $f$ , whose Fourier transform is supported in  $\{(\eta, \rho, \tau) : \rho \sim 1\}$ . Here for  $A, B > 0$ ,  $A \sim B$  means that  $A/2 \leq B \leq 2A$ . Then choosing a suitable  $\phi$ , we have  $\Delta_0 T^z f = T^z f$ . Employing the argument of Oberlin [12], we see  $T^z f \in L^q$  only if  $-z/n < 1/q$ .

If  $|s| \leq \frac{1}{p}$ , then by virtue of the density of Schwartz class in  $\tilde{B}_{p,r}^s$  [13], one can replace the condition that  $f \in \mathcal{S}(\mathbb{R}^{n+1})$  with  $f \in \tilde{B}_{p,r}^s$  in Theorem 1.1. On the other hand, from Littlewood-Paley theory, we have  $\|f\|_q \leq C \|f\|_{\tilde{B}_{q,2}^0}$  and  $\|f\|_{\tilde{B}_{p,2}^0} \leq C_2 \|f\|_p$  when  $1 \leq p \leq 2 \leq q \leq \infty$  (see [3], p.152). Setting  $s = 0$  and  $r = 2$ , the estimates (1.3) implies the previously known  $L^p$ - $L^q$  estimates for  $f \rightarrow K^z * f$  when  $1 \leq p \leq 2 \leq q \leq \infty$ . However, when  $2 \leq p, q \leq \infty$ , the estimates (1.3) seems to be slightly weaker than  $L^p$ - $L^q$  boundedness.

Unlike the approaches in [12], [7], our study of  $f \rightarrow K^z * f$  is carried out by analyzing Fourier transform of  $K^z * f$  in a more direct way. To obtain (1.3), we decompose the distribution  $K^z$  into functions whose Fourier transforms are

supported in the sets  $\{(\eta, \rho, \tau) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} : \rho \sim 2^l\}$ . Then a simple re-scaling argument reduces the problem to evaluating the sharp  $L^p$ - $L^q$  norm of the Fourier multiplier operator  $T_\delta$  whose Fourier multiplier is essentially supported in a  $\delta$ -neighborhood of the light cone. More precisely, for  $0 < \delta \ll 1$ , let us define

$$(1.4) \quad T_\delta f(y, s, t) = \int_{\mathbb{R}^{n+1}} \phi(\rho) \psi\left(\frac{\tau - |\eta|^2/\rho}{\delta}\right) b(\eta) \widehat{f}(\eta, \rho, \tau) d\eta d\rho d\tau$$

where  $b \in C_0^\infty(B(0, 1))$ ,  $\phi \in C_0^\infty(1/2, 2)$  and  $\psi \in \mathcal{S}(\mathbb{R})$ .

Using the  $L^{2(n+1)/(n+3)} \rightarrow L^2$  restriction estimate for the light cone in  $\mathbb{R}^{n+1}$  (see [15], p. 365-367) and its dual estimate, it follows that  $\|T_\delta f\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^{n+1})} \leq C\delta^{1/2} \|f\|_{L^2(\mathbb{R}^{n+1})}$ . Interpolating this with *false* estimate  $\|T_\delta f\|_{\frac{2n}{n-1}} \leq C \|f\|_{\frac{2n}{n-1}}$  corresponding to the cone multiplier conjecture, one might expect that if  $(n-1)(1 - \frac{1}{p}) = (n+1)\frac{1}{q}$  and  $\frac{2(n+1)}{n-1} \geq q > \frac{2n}{n-1}$ , then there is a constant  $C$  such that

$$(1.5) \quad \|T_\delta f\|_q \leq C\delta^{\frac{n+1}{2}(\frac{1}{p}-\frac{1}{q})} \|f\|_p.$$

Using a smooth bump function adapted to a rectangle of size  $(\delta^{1/2})^{n-1} \times \delta \times 1$  contained in  $\delta$ -neighborhood of the light cone, the sharpness of the bound  $C\delta^{\frac{n+1}{2}(\frac{1}{p}-\frac{1}{q})}$  can easily be seen.

In order to obtain Theorem 1.1, it is important to obtain the sharp  $L^p$ - $L^q$  bound for  $T_\delta$  in terms of  $\delta$ . The estimate (1.3) is actually a consequence of establishing (1.5) for  $q > \frac{2(n^2+2n-1)}{n^2-1}$  (see Proposition 2.5). This will be done using the bilinear restriction estimates due to Wolff [21] and Tao [16], for which we refer the readers to [9] and [6]. Combining the known local smoothing estimates ([10], [20]) with the argument in this note (also see [8]), one can extend the range of  $q$  for which (1.5) holds with bound  $C\delta^{\frac{n+1}{2}(\frac{1}{p}-\frac{1}{q})-\epsilon}$ . But these estimates can not be used to obtain (1.3) because of the homogeneity condition  $1/p - 1/q = 1 + \frac{2\text{Re}(z)}{n+1}$ .

The method used in this paper is similar to the argument used by the third author to get some sharp estimates for the cone multiplier of negative order in  $\mathbb{R}^3$  ([8]). However, exploiting the parabolic structure of the cone in a more direct way, the arguments here are simpler than those in [8]. The convolution estimate for  $K^z$  is essentially equivalent to the cone multiplier problem of negative order, since for  $-\frac{n+1}{2} < \text{Re}(z) < -\frac{n-1}{2}$ , Fourier transform of  $\widetilde{K}^z$  is

$$C(z) \left\{ (\tau^2 - \rho^2 - |\eta|^2)_-^{-z-\frac{n+1}{2}} - \sin \pi(z + \frac{n}{2}) (\tau^2 - \rho^2 - |\eta|^2)_+^{-z-\frac{n+1}{2}} \right\}$$

where  $C(z) = \pi^{-2z-\frac{n+3}{2}} \Gamma(z + \frac{n+1}{2})$  (see [4], p.284). Consequently, our estimates for  $T_\delta$  provide some new sharp bounds for the cone multiplier of negative order in  $\mathbb{R}^{n+1}$ ,  $n \geq 3$ . Some remarks on the cone multiplier operator are in the last section.

Throughout this paper,  $C$  is a positive constant which may vary line to line. In addition to the symbol  $\widehat{\cdot}$ , we use  $\mathcal{F}(\cdot)$ ,  $\mathcal{F}^{-1}(\cdot)$  to denote Fourier transform, inverse Fourier transform, respectively.

## 2. PROOF OF THEROEM 1.1

2.1. **Dyadic decomposition of the distribution  $K^z$ .** Let's denote by  $D^z$  the distribution analytically continued from the function

$$D^z = t_+^z / \Gamma(z+1), \quad \operatorname{Re}(z) > -1.$$

We decompose  $K^z$  so that the Fourier transforms of the decomposed distributions have dyadically disjoint supports.

**Lemma 2.1.** *Let  $z$  be a complex number which is not integer and  $-(n+1)/2 < \operatorname{Re}(z) < 0$ . Then there are smooth functions  $\psi_z, \phi_z$  with  $\psi_z \in C_0^\infty(1/2, 2)$ ,  $\widehat{\phi}_z \in C_0^\infty((-2, -1/2) \cup (1/2, 2))$  such that for all  $h \in \mathcal{S}(\mathbb{R}^{n+1})$ ,*

$$(2.1) \quad \langle K^z, h \rangle = \sum_l \sum_k 2^{z(k+l)} \iint \int \psi_z(t/2^k) \phi_z\left(\frac{s - |y|^2/4t}{2^l}\right) h(y, s, t) ds dy dt.$$

*Proof.* First we prove Lemma 2.1 for  $-1 < \operatorname{Re}(z) < 0$ . Since  $K^z$  is a function for  $-1 < \operatorname{Re}(z) < 0$ , we have

$$\langle K^z, h \rangle = \iint \int t^z \int \frac{(s - |y|^2/4t)_+^z}{\Gamma(z+1)} h(y, s, t) ds dy dt.$$

Let us choose  $\psi_z \in C_0^\infty(1/2, 2)$  such that  $t^z = \sum_k 2^{kz} \psi(t/2^k)$  and we use the following fact from [4](p.173). If  $z$  is not integer

$$\widehat{D}_z(\rho) = i(2\pi)^{-z-1} (e^{iz\pi/2} \rho_-^{-z-1} - e^{-iz\pi/2} \rho_+^{-z-1}).$$

Let  $g \in C_0^\infty((-2, -1/2) \cup (1/2, 2))$  satisfying  $\sum g(2^l \cdot) = 1$ . Applying Parseval's formula to the inner integral (in  $s$ ) and dominated convergence theorem in  $\rho$  (note  $\widehat{D}_z(\rho)$  is locally integrable), we get

$$\langle K^z, h \rangle = \sum_k \sum_l 2^{zk} \iint \int \psi_z(t/2^k) \int e^{-i2\pi\rho|y|^2/4t} \widehat{D}_z(\rho) g(2^l \rho) \mathcal{F}_s^{-1}(h)(y, \rho, t) d\rho dy dt$$

where  $\mathcal{F}_s^{-1}(h)(y, \cdot, t)$  is the inverse Fourier transform of  $h(y, \cdot, t)$  in  $s$  variable.

Now set

$$\phi_z(s) = i(2\pi)^{-z-1} \mathcal{F}^{-1}(g(\rho)(e^{iz\pi/2} \rho_-^{-z-1} - e^{-iz\pi/2} \rho_+^{-z-1}))(s).$$

By Parseval's formula one can see

$$\int e^{-i2\pi\rho|y|^2/4t} \widehat{D}_z(\rho) g(2^l \rho) \mathcal{F}_s^{-1}(h)(y, \rho, t) d\rho = 2^{lz} \int \phi_z\left(\frac{s - |y|^2/4t}{2^l}\right) h(y, s, t) ds.$$

Therefore we have

$$\langle K^z, h \rangle = \sum_k \sum_l 2^{z(k+l)} \int \psi_z(t/2^k) \phi_z\left(\frac{s - |y|^2/4t}{2^l}\right) h(y, s, t) dy ds dt.$$

By the rapid decay of  $h$  and the fact that  $-1 < \operatorname{Re}(z) < 0$  one can see the right hand side of the above absolutely converges. Thus the order of summation can be interchanged.

Now we prove Lemma 2.1 for  $-(n+1)/2 < \operatorname{Re}(z) < -1$ . Let us set a differential operator  $L = \frac{\partial^2}{\partial s \partial t} - \sum_{i=1}^{n-1} \frac{\partial^2}{\partial y_i^2}$ . Then  $LK^{z+1} = 4(z + (n+1)/2)K^z$ . Using this, we see  $K^z$  is analytically continued by the formula

$$(2.2) \quad \langle K^z, h \rangle = \frac{1}{C(z, k, n)} \langle K^{z+k}, L^k h \rangle$$

where  $C(z, k, n) = 4^k(z + (n+1)/2)(z + 1 + (n+1)/2) \cdots (z + k - 1 + (n+1)/2)$ . Suppose  $-2 < \operatorname{Re}(z) < -1$ . Then, using above formula, we have

$$\langle K^z, h \rangle = \frac{1}{4(z + (n+1)/2)} \langle K^{z+1}, Lh \rangle$$

Since  $\operatorname{Re}(z+1) > -1$ , from the above results we have  $\phi_{z+1}$  and  $\psi_{z+1}$  such that

$$(2.3) \quad \langle K^{z+1}, Lh \rangle = \sum_{l, k} 2^{(z+1)(k+l)} \iiint \psi_{z+1}(t/2^k) \phi_{z+1} \left( \frac{s - |y|^2/4t}{2^l} \right) Lh(y, s, t) dy ds dt.$$

Set

$$\psi_z = \frac{\psi'_{z+1}(t) + \frac{(n-1)}{2t} \psi_{z+1}(t)}{4(z + (n+1)/2)}, \quad \phi_z = \frac{\phi'_{z+1}}{4(z + (n+1)/2)}$$

and note that that  $\psi_z \in C_0^\infty(1/2, 2)$ ,  $\widehat{\phi}_z \in C_0^\infty((-2, -1/2) \cup (1/2, 2))$ . Since

$$L \left[ \psi_{z+1} \left( \frac{t}{2^k} \right) \phi_{z+1} \left( \frac{s - |y|^2/4t}{2^l} \right) \right] = 2^{-k-l} \left[ \psi'_{z+1} \left( \frac{t}{2^k} \right) + \frac{(n-1)2^k}{2t} \psi_{z+1} \left( \frac{t}{2^k} \right) \right] \phi'_{z+1} \left( \frac{s - |y|^2/4t}{2^l} \right),$$

by integration by parts in (2.3) one can easily see (2.1) holds. One can repeat this argument using (2.2) for  $\operatorname{Re}(z) < -\frac{n+1}{2}$ . This completed the proof.  $\square$

Let us set

$$K_{l,k}^z(y, s, t) = 2^{z k + z l} \psi_z(t/2^k) \phi_z((s - |y|^2/t)/2^l)$$

with  $\psi_z, \phi_z$  given in Lemma 2.1. Then by Lemma 2.1 for complex  $z$  which is not integer we can write  $K^z * f$  as

$$(2.4) \quad K^z * f = \sum_k \sum_l K_{l,k}^z * f$$

provided  $-(n+1)/2 < \operatorname{Re}(z) < 0$ . By the presence of quadratic term  $|y|^2/t$ , the Fourier transform of  $K_{l,k}^z$  can be computed in an exact form. We state this in the following:

**Lemma 2.2.** *Let  $\psi \in C_0^\infty(1/2, 2)$  and  $\phi \in \mathcal{S}(\mathbb{R})$  with  $\widehat{\phi} = 0$  on  $(-\epsilon, \epsilon)$  for some  $\epsilon > 0$ . Then*

$$(2.5) \quad \mathcal{F}(\psi(t)\phi(s - |y|^2/4t))(\eta, \rho, \tau) = \widetilde{\phi}(\rho)\widetilde{\psi}(\tau - |\eta|^2/\rho)$$

where  $\widetilde{\phi}(\rho) = (\operatorname{sgn}(-\rho))^{\frac{n-1}{2}} |\rho|^{\frac{1-n}{2}} \widehat{\phi}(\rho)$ ,  $\widetilde{\psi}(\tau) = c_n \mathcal{F}(t^{\frac{n-1}{2}} \psi(t))(\tau)$ , where  $c_n = (2i)^{\frac{(n-1)}{2}}$ .

Since  $\widehat{\phi}_z$  is supported in  $(-2, -1/2) \cup (1/2, 2)$ , using this and re-scaling, we see

$$(2.6) \quad \widehat{K_{l,k}^z}(\eta, \rho, \tau) = 2^{(z+\frac{n+1}{2})k} 2^{(z+\frac{n+1}{2})l} \widetilde{\phi}_z(2^l \rho) \widetilde{\psi}_z(2^k(\tau - |\eta|^2/\rho)).$$

Note that  $\widetilde{\phi}_z(2^l \cdot)$  is supported on the set  $\{|\rho| \sim 2^{-l}\}$ .

*Proof of Lemma 2.2.* Note that

$$\begin{aligned} & \mathcal{F}(\psi(t)\phi(s - |y|^2/4t))(\eta, \rho, \tau) \\ &= \iiint \psi(t) \int \widehat{\phi}(a) e^{2\pi i a(s - |y|^2/4t)} da e^{-2\pi i(t\tau + s\rho + y\cdot\eta)} dy ds dt. \end{aligned}$$

Integrating the right hand side of the above in  $y$ , we see

$$\begin{aligned} & \mathcal{F}(\psi(t)\phi(s - |y|^2/4t))(\eta, \rho, \tau) \\ &= c_n \iint \psi(t) t^{\frac{n-1}{2}} \widehat{\phi}(a) (\text{sgn}(-a))^{\frac{n-1}{2}} |a|^{\frac{1-n}{2}} e^{2\pi i t(|\eta|^2/a - \tau)} \int e^{2\pi i s(a-\rho)} ds dt da. \end{aligned}$$

Since the inner integral is the delta function, one can easily see (2.5). In fact, these calculations are not rigorous. But it can be made so in sense of tempered distribution.  $\square$

**2.2. A summation method.** Now we prove Theorem 1.1. To do this we need to establish the sharp  $L^p$ - $L^q$  estimates for  $f \rightarrow K_{l,k}^z * f$ . The following will be shown in the next Subsection 2.3.

**Proposition 2.3.** *If  $(1/p, 1/q)$  is contained in the open quadrangle  $\mathcal{Q}$  with vertices  $(1/2, 1/2)$ ,  $(1/p_0, 1/q_0)$ ,  $(1, 0)$ ,  $(1 - 1/q_0, 1 - 1/p_0)$  where*

$$(2.7) \quad p_0 = \frac{2(n^2 + 2n - 1)}{n^2 + 2n - 3}, \quad q_0 = \frac{2(n^2 + 2n - 1)}{n^2 - 1},$$

*then there is a constant  $C$  such that*

$$(2.8) \quad \|K_{l,k}^z * f\|_q \leq C 2^{(\text{Re}(z) + \frac{n+1}{2}(1 - \frac{1}{p} + \frac{1}{q}))(k+l)} \|f\|_p.$$

Note that the point  $(1/p_0, 1/q_0)$  is on the line  $(n-1)(1 - \frac{1}{p}) = (n+1)\frac{1}{q}$ . Once Proposition 2.3 has been established, the proof of Theorem 1.1 is rather straightforward. Define points  $Q_z, Q'_z \in [0, 1] \times [0, 1]$  by

$$Q_z = \left(1 + \frac{\text{Re}(z)}{n}, \frac{(1-n)\text{Re}(z)}{n^2+n}\right), \quad Q'_z = \left(1 + \frac{(n-1)\text{Re}(z)}{n^2+n}, -\frac{\text{Re}(z)}{n}\right)$$

which are on the line  $\frac{1}{p} - \frac{1}{q} = 1 + \frac{2\text{Re}(z)}{n+1}$ . By a simple computation, one can see that the conditions (1.1) are equivalent to  $(1/p, 1/q) \in (Q_z, Q'_z)$ . Set

$$K_l^z * f := \sum_k K_{l,k}^z * f.$$

We claim that if  $(1/p, 1/q) \in (Q_z, Q'_z) \cap \mathcal{Q}$ , then

$$(2.9) \quad \|K_l^z * f\|_q \leq C \|f\|_p.$$

One can verify that if  $\text{Re}(z) \leq -\frac{n(n+1)^2}{2(n^2+2n-1)}$ , then  $(Q_z, Q'_z) \cap \mathcal{Q} = (Q_z, Q'_z)$ . To show (2.9), we use an elementary interpolation lemma to be used several times

throughout this paper. We use a simple extension of the Lemma in [8] which is implicit in [2](also see [5]).

**Lemma 2.4** (Interpolation lemma). *Let  $\varepsilon_1, \varepsilon_2 > 0$ . Suppose that  $\{T_j\}_{j \in \mathbb{Z}}$  be  $l$ -linear operators satisfying that*

$$\begin{aligned} \|T_j(f^1, \dots, f^l)\|_{q_1, \infty} &\leq M_1 2^{\varepsilon_1 j} \prod_{i=1}^l \|f^i\|_{p_1^i, 1} \quad \text{and} \\ \|T_j(f^1, \dots, f^l)\|_{q_2, \infty} &\leq M_2 2^{-\varepsilon_2 j} \prod_{i=1}^l \|f^i\|_{p_2^i, 1} \end{aligned}$$

for  $1 \leq p_1^i, p_2^i < \infty$ ,  $i = 1, \dots, l$  (here, the superscript  $i$  is not an exponent, but an index) and  $1 < q_1, q_2 < \infty$ . Then  $T := \sum T_j$  is bounded from  $L^{p^1, 1} \times \dots \times L^{p^l, 1}$  to  $L^{q, \infty}$  with

$$\|T(f^1, \dots, f^l)\|_{L^{q, \infty}} \leq C M_1^\theta M_2^{1-\theta} \prod_{i=1}^l \|f^i\|_{p^i, 1}$$

where  $\theta = \varepsilon_2/(\varepsilon_1 + \varepsilon_2)$ ,  $1/q = \theta/q_1 + (1 - \theta)/q_2$ ,  $1/p^i = \theta/p_1^i + (1 - \theta)/p_2^i$ , for  $i = 1, \dots, l$ .

*Proof of Lemma 2.4.* Let  $N \in \mathbb{Z}$ , which will be chosen later. Set  $T_N = \sum_{-\infty}^N T_j$ ,  $T^N = \sum_{N+1}^{\infty} T_j$ . Since  $q_1, q_2 > 1$ , the spaces  $L^{q_1, \infty}$ ,  $L^{q_2, \infty}$  are Banach spaces. By summation we have

$$\|T_N\|_{L^{p_1^1, 1} \times \dots \times L^{p_1^l, 1} \rightarrow L^{q_1, \infty}} \leq C M_1 2^{N\varepsilon_1}, \quad \|T_N\|_{L^{p_2^1, 1} \times \dots \times L^{p_2^l, 1} \rightarrow L^{q_2, \infty}} \leq C M_2 2^{-N\varepsilon_2}.$$

Let  $E_1, \dots, E_l$  be measurable sets and let  $\lambda > 0$ . By Chebyshev's inequality, the measure of the set  $\{x : |T(\chi_{E_1}, \dots, \chi_{E_l})(x)| > \lambda\}$  is bounded above by

$$\begin{aligned} &|\{x : |T_N(\chi_{E_1}, \dots, \chi_{E_l})(x)| > \frac{1}{2}\lambda\}| + |\{x : |T^N(\chi_{E_1}, \dots, \chi_{E_l})(x)| > \frac{1}{2}\lambda\}| \\ &\leq C(M_1^{q_1} 2^{\varepsilon_1 N q_1} \prod_{i=1}^l |E_i|^{q_1/p_1^i} \lambda^{-q_1} + M_2^{q_2} 2^{-\varepsilon_2 N q_2} \prod_{i=1}^l |E_i|^{q_2/p_2^i} \lambda^{-q_2}). \end{aligned}$$

Choosing  $N$  that optimizes the above, we have

$$|\{x \in \mathbb{R}^n : |T(\chi_{E_1}, \dots, \chi_{E_l})(x)| > \lambda\}| \leq C(M_1^\theta M_2^{1-\theta} \prod_{i=1}^l |E_i|^{1/p^i} \lambda^{-1})^q.$$

This completes the proof.  $\square$

Fix  $z$ ,  $0 > \operatorname{Re}(z) > -\frac{n+1}{2}$ . Let  $(1/p, 1/q) \in (Q_z, Q'_z) \cap \mathcal{Q}$ . Then we can find two points  $(1/p_1, 1/q_1)$ ,  $(1/p_2, 1/q_2)$  contained in  $\mathcal{Q}$  such that the line joining  $(1/p_1, 1/q_1)$  and  $(1/p_2, 1/q_2)$  passes through  $(1/p, 1/q)$ , and  $1/p_1 - 1/q_1 < 1 + \frac{2\operatorname{Re}(z)}{n+1}$  and  $1/p_2 - 1/q_2 > 1 + \frac{2\operatorname{Re}(z)}{n+1}$ . Then from Proposition 2.3, we have two estimates

$$\|K_{l,k}^z * f\|_{q_i} \leq C 2^{\left(\operatorname{Re}(z) + \frac{n+1}{2} \left(1 - \frac{1}{p_i} + \frac{1}{q_i}\right)\right)(k+l)} \|f\|_{p_i}, \quad i = 1, 2.$$



Using this and applying Lemma 2.4 to  $K_l^z * f = \sum_k K_{l,k}^z * f$ , we see  $\|K_l^z * f\|_{q,\infty} \leq C\|f\|_{p,1}$  if  $(1/p, 1/q) \in (Q_z, Q'_z) \cap \mathcal{Q}$ . By real interpolation among these, (for fixed  $z$ ) we obtain (2.9).

From (2.6), it follows that  $\widehat{K}_l$  is supported in the set  $\{(\eta, \rho, \tau) : 2^{-1-l} \leq |\rho| \leq 2^{1-l}\}$ . Therefore, invoking the definition of Besov spaces  $\widetilde{B}_{p,r}^s$  and  $K^z = \sum K_l^z$ , and using (2.9), we see that

$$\|K^z * f\|_{\widetilde{B}_{q,r}^s} \leq C \left( \sum_l 2^{-srl} \|K_l^z * \Delta_{(-l)} f\|_q^r \right)^{1/r} \leq C \left( \sum_l 2^{-srl} \|\Delta_{(-l)} f\|_p^r \right)^{1/r}$$

provided  $(1/p, 1/q) \in (Q_z, Q'_z) \cap \mathcal{Q}$ . This completes the proof of Theorem 1.1.

**2.3. Convolution estimates for  $K_{l,k}^z$ ; proof of Proposition 2.3.** Let  $\widetilde{T}_\delta$  be the multiplier operator given by

$$\mathcal{F}(\widetilde{T}_\delta f)(\eta, \rho, \tau) = \widetilde{\phi}_z(\rho) \widetilde{\psi}_z \left( \frac{\tau - |\eta|^2/\rho}{\delta} \right) \widehat{f}(\eta, \rho, \tau).$$

Since the Fourier transform of  $K_{l,k}^z$  is given as (2.6), re-scaling  $(\eta, \rho) \rightarrow (2^{-l/2}\eta, 2^{-l}\rho)$  in the Fourier transform side, for (2.8) it suffices to show that  $\|\widetilde{T}_\delta\|_{p \rightarrow q} \leq C\delta^{-\frac{n+1}{2}(\frac{1}{p}-\frac{1}{q})}$  for  $(\frac{1}{p}, \frac{1}{q}) \in \mathcal{Q}$ . Now consider  $\widetilde{T}_\delta^R$  defined by

$$\mathcal{F}(\widetilde{T}_\delta^R f)(\eta, \rho, \tau) = \mathcal{F}(\widetilde{T}_\delta)(\eta, \rho, \tau) b\left(\frac{\eta}{R}\right) \widehat{f}(\eta, \rho, \tau)$$

where  $b$  is a smooth function supported in  $B(0,1) \in \mathbb{R}^{n-1}$  with  $b(0) = 1$ . Then, it suffices to show that if  $(\frac{1}{p}, \frac{1}{q}) \in \mathcal{Q}$ , then  $\|\widetilde{T}_\delta^R\|_{p \rightarrow q} \leq C\delta^{-\frac{n+1}{2}(\frac{1}{p}-\frac{1}{q})}$ , uniformly in  $R$ . Again by re-scaling  $(\eta, \rho) \rightarrow (R\eta, R^2\rho)$ , to prove (2.8), it is sufficient to show that if  $(\frac{1}{p}, \frac{1}{q}) \in \mathcal{Q}$ , then  $\|\widetilde{T}_\delta^1\|_{p \rightarrow q} \leq C\delta^{-\frac{n+1}{2}(\frac{1}{p}-\frac{1}{q})}$ . Since  $L^2$  boundedness for  $\widetilde{T}_\delta$  is trivial, by duality, we need only to show  $\|\widetilde{T}_\delta^1\|_{p \rightarrow q} \leq C\delta^{-\frac{n+1}{2}(\frac{1}{p}-\frac{1}{q})}$  for  $(\frac{1}{p}, \frac{1}{q}) \in ((1/p_0, 1/q_0), (\infty, \infty)]$ . Therefore Proposition 2.3 follows from

**Proposition 2.5.** *Let  $\delta > 0$  and let  $T_\delta$  be a multiplier operator given by*

$$\widehat{T_\delta f}(\eta, \rho, \tau) = \phi(\rho) \psi \left( \frac{\tau - |\eta|^2/\rho}{\delta} \right) b(\eta) \widehat{f}(\eta, \rho, \tau)$$

where  $\phi \in C_0^\infty(1/2, 2)$ ,  $\psi \in \mathcal{S}(\mathbb{R})$  and  $b$  is a smooth function supported in  $C_0^\infty(B(0,1))$ ,  $B(0,1) \subset \mathbb{R}^{n-1}$ . Then if  $(n-1)(1-\frac{1}{p}) = (n+1)\frac{1}{q}$  and  $q > q_0 = \frac{2(n^2+2n-1)}{n^2-1}$ , then there is a constant  $C$  such that

$$(2.10) \quad \|T_\delta f\|_q \leq C\delta^{-\frac{n+1}{2}(\frac{1}{p}-\frac{1}{q})} \|f\|_p.$$

And when  $(p, q) = (p_0, q_0)$ ,  $T_\delta f$  is of restricted weak type  $(p_0, q_0)$ . Namely,

$$(2.11) \quad \|T_\delta f\|_{q_0, \infty} \leq C\delta^{-\frac{n+1}{2}(\frac{1}{p_0}-\frac{1}{q_0})} \|f\|_{p_0, 1}.$$

When  $\delta \geq 1$ , (2.10) follows from direct kernel estimate. Indeed, by differentiation,  $\partial^\alpha (\phi(\rho) \psi(\frac{\tau-|\eta|^2/\rho}{\delta}) b(\eta)) \in L^1(\mathbb{R}^{n+1})$  for any multi-indices  $\alpha$ . Hence the

kernel of  $T_\delta$  is contained in any  $L^p$  spaces,  $1 \leq p \leq \infty$  with its norm  $O(1)$ . Therefore  $\|T_\delta\|_{p \rightarrow q} \leq C$  for any  $1 \leq p \leq q$ . Hence it is sufficient to show Proposition 2.5 for the case  $0 < \delta < 1$ .

2.3.1. *Proof of Proposition 2.5.* Since from Lemma 2.2 we have

$$\mathcal{F}(\delta^{-\frac{n+1}{2}} \psi(t\delta) \phi((s - |y|^2/4t))) = \tilde{\phi}(\tau) \tilde{\psi}\left(\frac{\tau - |\eta|^2/\rho}{\delta}\right),$$

it follows that

$$(2.12) \quad \|T_\delta f\|_\infty \leq C \delta^{\frac{n+1}{2}} \|f\|_1.$$

Interpolation (real) between this and (2.11) gives (2.10). Therefore it is sufficient to show (2.11).

Even though  $\tilde{\psi}$  is not compactly supported, by the rapid decay of  $\tilde{\psi}$ , it can be replaced by a function with compact support. In fact, Proposition 2.5 can be deduced from:

**Proposition 2.6.** *Let  $0 < \delta \leq 1$  and let  $\psi_\delta$  be a smooth function supported on  $[-\delta, \delta]$  with  $|\frac{d^n}{dt^n} \psi_\delta| \leq C \delta^{-n}$  for any  $n$  and define a multiplier operator  $S_\delta$  by*

$$(2.13) \quad \widehat{S_\delta f}(\eta, \rho, \tau) = \phi(\rho) \psi_\delta(\tau - |\eta|^2/\rho) b(\eta) \widehat{f}(\eta, \rho, \tau)$$

where  $\phi \in C_0^\infty(1/2, 2)$  and  $b$  is a smooth function supported in  $C_0^\infty(B(0, 1))$ ,  $B(0, 1) \subset \mathbb{R}^{n-1}$ . Then there is a constant  $C$  such that for  $0 < \delta \leq 1$ ,

$$(2.14) \quad \|S_\delta f\|_{q_0, \infty} \leq C \delta^{\frac{n+1}{2}(\frac{1}{p_0} - \frac{1}{q_0})} \|f\|_{p_0, 1}.$$

Assuming Proposition 2.6 for a moment, we prove Proposition 2.5. Let  $\chi \in C_0^\infty((-2, -1/2) \cup (1/2, 2))$  with  $\sum_k \chi(\cdot/2^k) = 1$ . Let us set  $\psi_k := \chi(\cdot/2^k) \tilde{\psi}(\cdot/\delta)$  and

$$\begin{aligned} m_1 &:= \phi(\rho) b(\eta) \sum_{2^k \leq \delta} \psi_k(\tau - |\eta|^2/\rho), \\ m_2 &:= \phi(\rho) b(\eta) \sum_{\delta < 2^k \leq 1} \psi_k(\tau - |\eta|^2/\rho), \\ m_3 &:= \phi(\rho) b(\eta) \sum_{1 < 2^k} \psi_k(\tau - |\eta|^2/\rho). \end{aligned}$$

Also let  $T_1, T_2, T_3$  be the multiplier operators given by  $\widehat{T_i f} = m_i \widehat{f}$ ,  $i = 1, 2, 3$ , so that  $S_\delta = T_1 + T_2 + T_3$ . Since  $\sum_{2^k \leq \delta} \psi_k = \psi_0(\cdot/\delta)$  for some  $\psi_0 \in C_0^\infty(-1, 1)$ , one can easily see  $T_1$  falls into the category of  $S_\delta$ . By Proposition 2.6,  $\|T_1\|_{L^{p_0, 1} \rightarrow L^{q_0, \infty}} \leq C \delta^{\frac{n+1}{2}(\frac{1}{p_0} - \frac{1}{q_0})}$ . Next, from the rapid decay of  $\psi$  we see that  $\|T_3\|_{p \rightarrow q} \leq C \delta^M \|f\|_p$  for all  $p \leq q$  and  $M$ . Let  $T^k$  be the multiplier operator with multiplier  $\phi(\rho) b(\eta) \psi_k(\tau - |\eta|^2/\rho)$  so that  $T_2 = \sum_{\delta < 2^k \leq 1} T^k$ . Observe that if  $\delta \leq 2^{-k}$ , then

$$\left| \frac{d^n}{d\tau^n} \psi_k(\tau) \right| \leq C 2^{-kn} \delta^M 2^{-kM}$$

for any  $M > 0$ . Since  $\psi_k$  is supported in  $[-2^k, 2^k]$ , using Proposition 2.6, we see that

$$\|T^k f\|_{q_0, \infty} \leq C \delta^M 2^{-kM} 2^k \frac{n+1}{2} \left(\frac{1}{p_0} - \frac{1}{q_0}\right) \|f\|_{p_0, 1}.$$

Choosing large  $M$ , we get the required estimates for  $T_2$  after summing in  $k$ . From this Proposition 2.5 follows.

**2.3.2. A bilinear type estimates for  $S_\delta$ .** For the proof of 2.6 we use the bilinear restriction estimates for the cone [16], [21]. For the reader's convenience, we give a statement which is slightly different from those in [16], [21]. To exploit the parabolic structure of the cone in effective way, we reformulate it in co-normal coordinates. Let  $Q = [-1, 1]^{n-1}$  and define

$$(2.15) \quad R^*(F)(y, s, t) = \int_{1/2}^2 \int_Q e^{2\pi i(y \cdot \eta + s\rho + t|\eta|^2/\rho)} F(\eta, \rho) d\eta d\rho.$$

And let  $\Theta(F) = \{(\eta/\rho) \in Q : (\eta, \rho) \in \text{supp}(F) \text{ for some } \rho \in [1/2, 2]\}$ . The following is the bilinear restriction theorem for the cone:

Suppose  $\text{dist}(\Theta(F), \Theta(G)) \sim 1$ . Then for  $q \geq \frac{n+3}{n+1}$ , there is a constant  $C$  such that

$$(2.16) \quad \|R^*(F)R^*(G)\|_{L^q(\mathbb{R}^{n+1})} \leq C \|F\|_{L^2(Q \times [1, 2])} \|G\|_{L^2(Q \times [1, 2])}.$$

Using this we obtain a bilinear type estimates for  $T_\delta$ . Let us define  $\Delta(f) = \{(\eta/\rho) \in \mathbb{R}^{n-1} : (\eta, \rho, \tau) \in \text{supp} f, \text{ for some } \rho, \tau\}$ .

**Lemma 2.7.** *Let  $0 < \delta \ll 1$ . If  $\text{dist}(\Delta(\widehat{f}), \Delta(\widehat{g})) \sim 1$ , then for  $\frac{n+3}{n+1} \leq p \leq 2$ , there is a constant  $C$  such that  $\|S_\delta f S_\delta g\|_p \leq C \delta \|f\|_2 \|g\|_2$ .*

*Proof.* Note that the Fourier transforms of  $S_\delta f$  and  $S_\delta g$  are supported in a  $\delta$ -neighborhood  $\mathcal{C}(\delta)$  of the cone  $\{(\eta, \rho, \tau) : \tau = |\eta|^2/\rho, \rho \sim 1\}$ .  $\mathcal{C}(\delta)$  can be written as a union of the surfaces  $C_s = \{(\eta, \rho, \tau) \in \mathcal{C} : \tau = |\eta|^2/\rho + s, \rho \sim 1, \eta \in Q\}$ ,  $s \in (-\delta, \delta)$ . Namely,

$$\mathcal{C}(\delta) \subset \bigcup_{s \in (-\delta, \delta)} C_s.$$

Let  $Q_1 = \Delta(\widehat{f})$ ,  $Q_2 = \Delta(\widehat{g})$ . Set

$$C_s^i := \{(\eta, \rho, \tau) \in C_s : \eta/\rho \in Q_i\}$$

for  $i = 1, 2$ . Let  $d\mu_s^i$  be the surface measure of  $C_s^i$ . Then using the bilinear restriction estimates (2.16), we can see that for  $p \geq \frac{n+3}{n+1}$ , there is constant  $C$ , independent of  $s, t \in (-\delta, \delta)$ , such that

$$(2.17) \quad \left\| \widehat{F d\mu_s^1 G d\mu_t^2} \right\|_p \leq C \|F\|_2 \|G\|_2.$$

(Here we discard harmless factors which come from changing surface measures.)  
Now we set

$$\begin{aligned}\widetilde{f}(\eta, \rho, \tau) &:= \phi(\rho)\psi_\delta(\tau - |\eta|^2/\rho)b(\eta)\widehat{f}(\eta, \rho, \tau), \\ \widetilde{g}(\eta, \rho, \tau) &:= \phi(\rho)\psi_\delta(\tau - |\eta|^2/\rho)b(\eta)\widehat{g}(\eta, \rho, \tau)\end{aligned}$$

and also let  $\widetilde{f}_s = \widetilde{f}|_{\mathcal{C}_s^1}$ ,  $\widetilde{g}_t = \widetilde{g}|_{\mathcal{C}_t^2}$ . Since  $\widetilde{f}, \widetilde{g}$  are supported in  $\mathcal{C}(\delta)$ , it follows that

$$\|S_\delta f S_\delta g\|_p^p = \int \left| \int_{-\delta}^\delta \int_{-\delta}^\delta \widehat{\widetilde{f}_s d\mu_s^1} \widehat{\widetilde{g}_t d\mu_t^2} dt ds \right|^p dx.$$

Since  $\text{dist}(\Theta(\widetilde{f}_s), \Theta(\widetilde{g}_t)) \sim 1$  for all  $s, t \in (-\delta, \delta)$ , using Hölder's inequality and (2.17), we see

$$\begin{aligned}\|S_\delta f S_\delta g\|_p^p &\leq C\delta^{2p-2} \int_{-\delta}^\delta \int_{-\delta}^\delta \int \left| \widehat{\widetilde{f}_s d\mu_s^1} \widehat{\widetilde{g}_t d\mu_t^2} \right|^p dx dt ds \\ &\leq C\delta^{2p-2} \int_{-\delta}^\delta \int_{-\delta}^\delta \|\widetilde{f}_s\|_2^p \|\widetilde{g}_t\|_2^p ds dt.\end{aligned}$$

Since  $p \leq 2$ , Hölder's inequality yields  $\int_{-\delta}^\delta \int_{-\delta}^\delta \|\widetilde{f}_s\|_2^p \|\widetilde{g}_t\|_2^p ds dt \leq C\delta^{2-p} \|\widetilde{f}\|_2^p \|\widetilde{g}\|_2^p$ . Using Plancherel's theorem, we get Lemma 2.7.  $\square$

2.3.3. *Proof of Proposition 2.6.* We use a decomposition technique introduced in [19], which is useful in exploiting bilinear estimate. For each  $j \geq 1$  we dyadically decompose  $Q \subset \mathbb{R}^{n-1}$  into  $\sim 2^{(n-1)j}$  dyadic cubes  $Q_k^j$  with side length  $2^{-(j-1)}$ . We say  $Q_k^j \sim Q_{k'}^j$  to mean that  $Q_k^j, Q_{k'}^j$  are not adjacent but have adjacent parent cubes of diameter  $2^{-j+2}$ . So if  $Q_k^j \sim Q_{k'}^j$ , then  $\text{dist}(Q_k^j, Q_{k'}^j) \sim 2^{-j}$ . By a Whitney decomposition of  $Q \times Q$  away from the diagonal  $D$  of  $Q \times Q$  (e.g. [14], p.16), ignoring some harmless measure zero sets, we have

$$(2.18) \quad Q \times Q \setminus D = \bigcup_{j \geq 1} \bigcup_{Q_k^j \sim Q_{k'}^j} Q_k^j \times Q_{k'}^j.$$

Let  $f_k^j$  be defined by

$$(2.19) \quad \widehat{f}_k^j(\eta, \rho, \tau) = \chi_{Q_k^j}(\eta/\rho) \widehat{f}(\eta, \rho, \tau).$$

We may assume that the Fourier transform of  $f$  is supported in  $Q \times [1/2, 2] \times [-1, 1]$ . Since  $\sum_{j \geq 1} \sum_{Q_k^j \sim Q_{k'}^j} \chi_{Q_k^j} \chi_{Q_{k'}^j} = 1$  almost everywhere in  $Q \times Q$ , we see that

$$(2.20) \quad S_\delta f(x) \cdot S_\delta f(x) = \sum_{j \geq 1} \sum_{Q_k^j \sim Q_{k'}^j} S_\delta f_k^j(x) \cdot S_\delta f_{k'}^j(x).$$

Fixing  $j$ , we define a bilinear operator by

$$B_j(f, g)(x) = \sum_{Q_k^j \sim Q_{k'}^j} S_\delta f_k^j(x) \cdot S_\delta g_{k'}^j(x).$$

Then, it is easy to see that

$$(2.21) \quad (S_\delta f(x))^2 = \sum_{j \geq 1} B_j(f, f).$$

Now, we compute the operator norm of  $B_j$  from  $L^p \times L^p$  to  $L^{p/2}$ .

**Lemma 2.8.** *Suppose for some  $p, q$  satisfying  $2 \leq p < q$  and  $1/2 < 1/q + 1/p$ , there is a constant  $B$  such that for  $Q_k^j \sim Q_{k'}^j$ ,*

$$(2.22) \quad \|S_\delta f_k^j \cdot S_\delta g_{k'}^j\|_{L^{q/2}(\mathbb{R}^{n+1})} \leq B \|f_k^j\|_{L^p(\mathbb{R}^{n+1})} \|g_{k'}^j\|_{L^p(\mathbb{R}^{n+1})}.$$

*Then there is a constant  $C$ , independent of  $j$  and  $\delta$ , such that*

$$(2.23) \quad \|B_j(f, g)\|_{L^{q/2}(\mathbb{R}^{n+1})} \leq CB \|f\|_{L^p(\mathbb{R}^{n+1})} \|g\|_{L^p(\mathbb{R}^{n+1})}.$$

*Proof.* For a fixed  $j$ , if  $Q_k^j \sim Q_{k'}^j$ , then the Fourier support of  $S_\delta f_k^j \cdot S_\delta g_{k'}^j$  is contained in the set  $\{(\eta, \rho, \tau) : |\eta/\rho - c_k^j| \leq 2^{3-j}, |\eta| \leq C, 2 \leq \rho \leq 4\}$  where  $c_k^j$  is the center of  $Q_k^j$ . So the Fourier transforms of  $\{S_\delta f_k^j \cdot S_\delta g_{k'}^j\}_{Q_k^j \sim Q_{k'}^j}$  are supported in boundedly (at most  $8^n$ ) overlapping (infinite) rectangles. By Plancherel's theorem and a standard argument (see, [TVV, Lemma 6.1]), we have for  $1 \leq q/2 \leq \infty$ ,

$$(2.24) \quad \|B_j(f, g)\|_{q/2} \leq C \left( \sum_{Q_k^j \sim Q_{k'}^j} \|S_\delta f_k^j \cdot S_\delta g_{k'}^j\|_{q/2}^{(q/2)^*} \right)^{1/(q/2)^*}$$

where  $p^* = \min(p, p')$ . Now by the assumption (2.22), we have

$$\|B_j(f, g)\|_{q/2} \leq CB \left( \sum_{Q_k^j \sim Q_{k'}^j} \|f_k^j\|_p^{(q/2)^*} \|g_{k'}^j\|_p^{(q/2)^*} \right)^{1/(q/2)^*}.$$

Since the number of  $Q_k^j$  satisfying  $Q_k^j \sim Q_{k'}^j$  is at most  $4^n$ ,

$$\sum_{Q_k^j \sim Q_{k'}^j} \|f_k^j\|_p^{(q/2)^*} \|g_{k'}^j\|_p^{(q/2)^*} \leq C \left( \sum_k \|f_k^j\|_p^{2(q/2)^*} \right)^{\frac{1}{2}} \left( \sum_k \|g_k^j\|_p^{2(q/2)^*} \right)^{\frac{1}{2}}.$$

Since  $1/2 < 1/q + 1/p$  and  $2 \leq p < q$ , it is easy to see that  $p < 2(q/2)^*$ . We see that the right hand side of the above is bounded above by  $C \|f\|_p^{(q/2)^*} \|g\|_p^{(q/2)^*}$  using the following: If  $2 \leq r < p < \infty$ , then there is a constant  $C = C(p, r)$  such that

$$(2.25) \quad \left( \sum_k \|f_k^j\|_r^p \right)^{1/p} \leq C \|f\|_r.$$

The inequality (2.25) follows from the observations: (i)  $\sup_{j,k} \|f_k^j\|_p \leq C \|f\|_p$  for any  $1 < p < \infty$  by the singular integral theory, especially  $L^p$  boundedness of Hilbert transform ([St1, p.100]), (ii)  $(\sum_k \|f_k^j\|_2^2)^{1/2} \leq \|f\|_2$  by Plancherel theorem, and (iii) interpolation between the above estimates (i) and (ii).  $\square$

Now we state a simple lemma which is to be used for kernel estimates of  $S_\delta$ .

**Lemma 2.9.** *For  $0 < r, \delta \leq 1$ , let  $Q(a, r) \subset Q = [-1, 1]^{n-1}$  be a cube centered at  $a$  with side length  $2r$  and let  $\chi_{Q(a,r)}$  be a smooth function adapted to the cube  $Q(a, r)$ , that is,  $\chi_{Q(a,r)} = \chi(\frac{\cdot - a}{r})$  where  $\chi$  be a smooth function supported in  $Q$ . Set*

$$m_\delta^{a,r} = \tilde{\phi}(\rho) \tilde{\psi}_\delta(\tau - |\eta|^2/\rho) \chi_{Q(a,r)}(\eta/\rho).$$

*Then, there is a constant  $C$ , independent of  $a, \delta, r$ , such that*

$$\|\mathcal{F}^{-1}[m_\delta^{a,r}]\|_\infty \leq C \delta r^{(n-1)}, \quad \|\mathcal{F}^{-1}[m_\delta^{a,r}]\|_1 \leq C \max(1, r^{n-1} \delta^{-\frac{n-1}{2}}).$$

*Proof.* The first inequality is trivial, because  $\int |m_\delta^{a,r}(\eta, \rho, \tau)| d\eta d\rho d\tau \leq C\delta r^{(n-1)}$ . For the second, we consider the cases  $\delta \geq r^2$ ,  $\delta < r^2$ , separately. By the linear map

$$L_{a,r}(\eta, \rho, \tau) = (r\eta + a\rho, \rho, r^2\tau + 2a \cdot r\eta + a^2\rho),$$

we may assume  $a = 0$ , using the facts  $\|\mathcal{F}^{-1}[m_\delta^{a,r}]\|_1 = \|\mathcal{F}^{-1}[m_\delta^{a,r}(L_{a,1})]\|_1$  and  $m_\delta^{a,r}(L_{a,1}(\eta, \rho, \tau)) = m_\delta^{0,r}(\eta, \rho, \tau)$ .

First we consider the case  $\delta \geq r^2$ . It suffices to show that  $\|\mathcal{F}^{-1}[m_\delta^{0,r}(L_{0,r})]\|_1 \leq C$ . Since  $\|(\frac{\partial}{\partial\tau})^{\alpha_1}(\frac{\partial}{\partial\eta})^{\alpha_2}(\frac{\partial}{\partial\rho})^{\alpha_3}m_\delta^{0,r}(L_{0,r})\|_1 \leq C\left(\frac{r^2}{\delta}\right)^{\alpha_1-1}$  for any index  $(\alpha_1, \alpha_2, \alpha_3)$ , by integration by parts one can see that

$$|\mathcal{F}^{-1}[m_\delta^{0,r}(L_{0,r})](y, s, t)| \leq C\frac{\delta}{r^2} \left(1 + |y| + |s| + \left|\frac{\delta}{r^2}t\right|\right)^{-N}$$

for some large  $N$ . By integration, we get  $\|\mathcal{F}^{-1}[m_\delta^{0,r}(L_{0,r})]\|_1 \leq C$ .

Now for the case  $\delta \leq r^2$ , it suffices to show that  $\|\mathcal{F}^{-1}[m_\delta^{0,r}(L_{0,\delta^{1/2}})]\|_1 \leq Cr^{n-1}\delta^{-\frac{n-1}{2}}$  by re-scaling. Note that  $r\delta^{-1/2} \geq 1$  and

$$m_\delta^{0,r}(L_{0,\delta^{1/2}}(\eta, \rho, \tau)) = \tilde{\phi}(\rho)\tilde{\psi}_\delta(\delta(\tau - |\eta|^2/\rho))\chi_{Q(0,r\delta^{-1/2})}(\eta/\rho).$$

Let  $\tilde{\chi}_0$  be smooth function supported in  $Q$  with  $\sum_{j \in \mathbb{Z}^{n-1}} \tilde{\chi}_0(\cdot - j)$ . We decompose  $\chi_{Q(0,r\delta^{-1/2})}$  into  $O((r\delta^{-1/2})^{n-1})$  smooth functions  $\tilde{\chi}_j = \tilde{\chi}_0(\cdot - j)\chi_{Q(0,r\delta^{-1/2})}$ , which are uniformly contained in  $C^\infty$ . Since

$$\left\| \left(\frac{\partial}{\partial\tau}\right)^{\alpha_1} \left(\frac{\partial}{\partial\eta}\right)^{\alpha_2} \left(\frac{\partial}{\partial\rho}\right)^{\alpha_3} m_\delta^{0,r}(L_{0,\delta^{1/2}})\tilde{\chi}_j \right\|_1 \leq C \quad \text{for any } (\alpha_1, \alpha_2, \alpha_3),$$

the integration by parts yields that for any large  $N$ ,

$$|\mathcal{F}^{-1}[m_\delta^{0,r}(L_{0,\delta^{1/2}})\tilde{\chi}_j](y, s, t)| \leq C(1 + |y| + |t| + |s|)^{-N}.$$

Thus we have  $\|\mathcal{F}^{-1}[m_\delta^{0,r}(L_{0,\delta^{1/2}})\tilde{\chi}_j]\|_1 \leq C$  and hence

$$\|\mathcal{F}^{-1}[m_\delta^{a,r}(L_{0,\delta^{1/2}})]\|_1 \leq Cr^{n-1}\delta^{-\frac{n-1}{2}},$$

because the number of  $j$  are  $O(r^{n-1}\delta^{-\frac{n-1}{2}})$ . This completes the proof of lemma.  $\square$

In view of Lemma 2.8, we only need to estimate  $\|S_\delta f_k^j \cdot S_\delta f_{k'}^j\|_{q/2}$  with  $Q_k^j \sim Q_{k'}^j$ . This is to be done by treating the cases  $2^{2j}\delta \geq 1$  and  $2^{2j}\delta < 1$  separately.

*I. Case  $2^{-2j} \geq \delta$ .* We claim that if  $2^{-2j} \geq \delta$ , then for  $(1/p, 1/q)$  contained in the line segment  $[(0, 0), (1/2, (n+1)/2(n+3))]$ , we have

$$(2.26) \quad \|S_\delta f_k^j \cdot S_\delta g_{k'}^j\|_{q/2} \leq C2^{2(\frac{n+1}{q} - (n-1)(1-\frac{1}{p}))j} \delta^{1-n+\frac{2n}{p}} \|f_k^j\|_p \|g_{k'}^j\|_p.$$

By virtue of interpolation, it is sufficient to show (2.26) for  $(p, q) = (\infty, \infty)$  and  $(p, q) = (2, 2(n+3)/(n+1))$ . The estimate (2.26) for  $(p, q) = (\infty, \infty)$  follows from Lemma 2.9, because  $S_\delta f_k^j = \kappa * f_k^j$  for some  $\kappa$  with  $\|\kappa\|_1 \leq C2^{-(n-1)j}\delta^{-\frac{n-1}{2}}$ .

Now we show (2.26) when  $(p, q) = (2, 2(n+3)/(n+1))$ . The cubes  $Q_k^j$  and  $Q_{k'}^j$  are contained in the ball  $B(a, C2^{-j})$  for some  $a$ . Using the linear transformation

$\eta \rightarrow \eta + a\rho$  on the Fourier transform side, we may assume  $Q_k^j$  and  $Q_{k'}^j$  are contained in the ball  $B(0, C2^{-j})$ . By re-scaling  $(\eta, \rho, \tau) \rightarrow (2^{-j}\eta, \rho, 2^{-2j}\tau)$ ,

$$\begin{aligned} S_\delta f_k^j(y, s, t) &= \int e^{i(2^{-j}y \cdot \eta + s\rho + 2^{-2j}t\tau)} \tilde{\phi}(\rho) \tilde{\psi}_\delta \left( \frac{\tau - |\eta|^2/\rho}{2^{2j}} \right) \chi_{Q_1}(\eta/\rho) \hat{f}_j(\eta, \rho, \tau) d\eta d\rho d\tau \end{aligned}$$

where  $\|f_j\|_2 \leq C2^{-(n+1)j/2} \|f_k^j\|_2$  and  $Q_1$  is a cube contained in  $B(0, C)$ . Applying the same re-scaling to  $S_\delta g_{k'}^j$ , we see that

$$(2.27) \quad (S_\delta f_k^j \cdot S_\delta g_{k'}^j)(y, s, t) = (\tilde{S}_{2^{2j}\delta} f_j \cdot \tilde{S}_{2^{2j}\delta} g_j)(2^{-j}y, s, 2^{-2j}t)$$

where  $\|g_j\|_2 \leq C2^{-(n+1)j/2} \|g_{k'}^j\|_2$  and  $\text{dist}(\Delta(\hat{f}_j), \Delta(\hat{g}_j)) \sim 1$ . Here,  $\tilde{S}_{2^{2j}\delta}$  is defined by replacing  $\psi_\delta$  by  $\psi_\delta(\frac{\cdot}{2^{2j}})$  in (2.13). Since  $\delta 2^{2j} \leq 1$  and  $\psi_\delta(\frac{\cdot}{2^{2j}})$  is supported in  $[-\delta 2^{2j}, \delta 2^{2j}]$ , applying Lemma 2.7 to (2.27) with  $\delta$  replaced by  $\delta 2^{2j}$  and re-scaling, we have the required estimate (2.26).

In case that  $2^{2j}\delta < 1$ , using (2.26) and Lemma 2.8, we have

$$\|B_j(f, g)\|_{q/2} \leq C2^{2(\frac{n+1}{q} - (n-1)(1-\frac{1}{p}))j} \delta^{1-n+\frac{2n}{p}} \|f\|_p \|g\|_p$$

for  $p, q$  satisfying  $\frac{1}{2} < \frac{1}{p} + \frac{1}{q}$  and  $(\frac{1}{p}, \frac{1}{q}) \in [(0, 0), (1/2, (n+1)/2(n+3))]$ . Now applying Lemma 2.4 to the bilinear operators  $\{B_j(f, g)\}_{2^{-2j} \geq \delta}$ , using the above, we obtain

$$(2.28) \quad \left\| \sum_{2^{-2j} \geq \delta} B_j(f, g) \right\|_{q_0/2, \infty} \leq C\delta^{1-n+\frac{2n}{p_0}} \|f\|_{p_0, 1} \|g\|_{p_0, 1}.$$

Note that the line  $\frac{n+1}{q} = (n-1)(1-\frac{1}{p})$  intersects the line segment  $[(0, 0), (1/2, (n+1)/2(n+3))]$  at  $(1/p_0, 1/q_0)$ .

*II. Case  $2^{-2j} \leq \delta$ .* In this case, the condition  $Q_k^j \sim Q_{k'}^j$  is unnecessary. From Lemma 2.9, we see that  $S_\delta f_k^j = \kappa * f_k^j$  with  $\|\kappa\|_1 \leq C$  and  $\|\kappa\|_\infty \leq C\delta 2^{-j(n-1)}$ . So  $\|S_\delta f_k^j\|_p \leq C\|f_k^j\|_p$  for  $1 \leq p \leq \infty$  and  $\|S_\delta f_k^j\|_\infty \leq C\delta 2^{-j(n-1)} \|f_k^j\|_1$ . Interpolation between these two estimates gives us

$$\|S_\delta f_k^j\|_q \leq (\delta 2^{-j(n-1)})^{1/p-1/q} \|f_k^j\|_p,$$

provided  $p \leq q$ . Observe that if  $\frac{n+1}{q} = (n-1)(1-\frac{1}{p})$ ,  $(\delta 2^{-j(n-1)})^{2/p-2/q} = 2^{-j\frac{4n(n-1)}{n+1}(\frac{1}{p}-\frac{n-1}{2n})} \delta^{\frac{4n}{n+1}(\frac{1}{p}-\frac{n-1}{2n})}$ . Thus we deduce from Hölder inequality that there is a constant  $C$ , independent of  $Q_k^j, Q_{k'}^j$ , such that

$$(2.29) \quad \|S_\delta f_k^j \cdot S_\delta g_{k'}^j\|_{q/2} \leq C2^{-j\frac{4n(n-1)}{n+1}(\frac{1}{p}-\frac{n-1}{2n})} \delta^{\frac{4n}{n+1}(\frac{1}{p}-\frac{n-1}{2n})} \|f_k^j\|_p \|g_{k'}^j\|_p$$

for  $p, q$  satisfying  $\frac{2n}{n-1} \leq q \leq 4$ ,  $2 \leq p \leq q$  and  $\frac{n+1}{q} = (n-1)(1-\frac{1}{p})$ . And hence from the application of Lemma 2.8 to (2.29) we also deduce that there exists a constant  $C$ , independent of  $j$ , such that

$$\|B_j(f, g)\|_{q/2} \leq C2^{-j\frac{4n(n-1)}{n+1}(\frac{1}{p}-\frac{n-1}{2n})} \delta^{\frac{4n}{n+1}(\frac{1}{p}-\frac{n-1}{2n})} \|f\|_p \|g\|_p$$

for the same exponent  $p, q$  as in (2.29). Since  $p < \frac{2n}{n-1}$ , we conclude from direct summation on  $j$  that if  $\frac{n+1}{q} = (n-1)(1 - \frac{1}{p})$  and  $p < \frac{2n}{n-1}$ ,

$$(2.30) \quad \left\| \sum_{2^{-2j} \leq \delta} B_j(f, g) \right\|_{q/2} \leq C \delta^{1-n+\frac{2n}{p}} \|f\|_p \|g\|_p.$$

Therefore, combining (2.28) and (2.30), we finally get (2.11). This completes the proof of Proposition 2.6.

### 3. REMARKS ON THE CONE MULTIPLIER OF NEGATIVE ORDER

The cone multiplier operator  $S^\mu$  of order  $\mu$  in  $\mathbb{R}^{n+1}$ ,  $n \geq 2$ , is a multiplier operator defined by

$$(3.1) \quad \widehat{S^\mu f}(\xi, \tau) = \frac{\phi(\tau)}{\Gamma(\mu+1)} (1 - |\xi|^2/\tau^2)_+^\mu \widehat{f}(\xi, \tau), \quad (\xi, \tau) \in \mathbb{R}^n \times \mathbb{R}$$

where  $\phi \in C_0^\infty(1, 2)$ .

We are concerned with  $L^p$ - $L^q$  estimates for  $S^{-\alpha}$  of negative order. For easier description, we introduce some notations. For  $0 < \alpha < (n+1)/2$ , define vertices in square  $[0, 1] \times [0, 1]$  by

$$\begin{aligned} A_\alpha(n) &= \left( \frac{n-1}{2n} + \frac{\alpha}{n}, 0 \right), & B_\alpha(n) &= \left( \frac{n-1}{2n} + \frac{\alpha}{n}, \frac{n-1}{2n} - \frac{\alpha(n-1)}{n^2+n} \right), \\ A'_\alpha(n) &= \left( 0, \frac{n+1}{2n} - \frac{\alpha}{n} \right), & B'_\alpha(n) &= \left( \frac{n+1}{2n} + \frac{\alpha(n-1)}{n^2+n}, \frac{n+1}{2n} - \frac{\alpha}{n} \right). \end{aligned}$$

And let us denote  $\Delta_\alpha(n)$  be the closed pentagon with vertices  $A_\alpha(n), B_\alpha(n), B'_\alpha(n), A'_\alpha(n), (1, 0)$  from which closed line segments  $[A_\alpha(n), B_\alpha(n)], [A'_\alpha(n), B'_\alpha(n)]$  are removed. In [8] it was shown that  $S^{-\alpha}$  is bounded from  $L^p$  to  $L^q$  only if  $(1/p, 1/q) \in \Delta_\alpha(n)$  and that in  $\mathbb{R}^3$  ( $n = 2$ ) the necessary condition is sufficient for  $3/14 < \alpha < 3/2$ .

Our estimates for  $T_\delta$  in the previous section give some new bounds for the cone multiplier of negative order in  $\mathbb{R}^{n+1}$ ,  $n \geq 3$ . By a linear transform, we may replace the multiplier  $\frac{\phi(\tau)}{\Gamma(\mu+1)} (1 - |\xi|^2/\tau^2)_+^\mu$  by  $\tilde{b}(\eta) K^z(2\eta, \rho, \tau)$  for some fixed  $\tilde{b} \in C_0^\infty(B(0, 2) \setminus B(0, \frac{1}{2}))$ , since  $L^p - L^q$  boundedness of both multiplier operators defined by these two distributions are equivalent to each other.

Using Lemma 2.1, one can have the decomposition

$$b(\eta) K^z(2\eta, \rho, \tau) = \sum_{l, k} b(\eta) K_{l, k}^z(2\eta, \rho, \tau),$$

where  $K_{l, k}^z(2\eta, \rho, \tau) = 2^{z(k+l)} \psi_z(\rho/2^k) \phi_z(\frac{\tau - |\eta|^2/\rho}{2^l})$ , and  $\phi_z$  and  $\psi_z$  is defined in Lemma 2.1. Then using Lemma 2.2, one can obtain

$$\mathcal{F}^{-1}(K^z)(y, s, t) = \sum_{l, k} 2^{(z+\frac{n+1}{2})(l+k)} \tilde{\phi}_z(2^l s) \tilde{\psi}_z(2^k(t - |\eta|^2/s)).$$



Denote by  $T_l^z$  the convolution operator with kernel

$$K_l^z = \sum_k 2^{(z+\frac{n+1}{2})(l+k)} \tilde{\phi}_z(2^l s) \tilde{\psi}_z(2^k(t-|\eta|^2/s)).$$

Using re-scaling and Proposition 2.5 and interpolation, the resulting estimates with simple estimates from Hölder's inequality, one can see that for  $p, q$  satisfying  $\frac{n+1}{q} \leq (n-1)(1-\frac{1}{p})$  and  $q \geq \frac{2(n^2+2n-1)}{n^2-1}$ ,

$$(3.2) \quad \|T_l^z f\|_q \leq C 2^{(z+\frac{n+1}{2}-\frac{n(n+1)}{(n-1)q})l} \|f\|_p.$$

Using Lemma 2.4 and some summation method in [8], [9] or [6], the following can be obtained from (3.2).

**Theorem 3.1.** *Let  $S^{-\alpha}$  be defined by (3.1),  $n \geq 2$ . Then, the followings hold whenever  $\frac{n^2-1}{2(n^2+2n-1)} \leq \alpha < (n+1)/2$ .*

- a) *If  $(1/p, 1/q) = B_\alpha(n)$ , or  $B'_\alpha(n)$ , then  $S^{-\alpha}$  is of restricted weak type  $(p, q)$ .*
- b) *If  $(1/p, 1/q) \in (B'_\alpha(n), A'_\alpha(n))$ , then  $S^{-\alpha}$  is of weak type  $(p, q)$ .*
- c) *If  $(1/p, 1/q) \in \Delta_\alpha(n)$ , then there is a constant  $C$  such that*

$$\|S^{-\alpha} f\|_{L^q(\mathbb{R}^{n+1})} \leq C \|f\|_{L^p(\mathbb{R}^{n+1})}.$$

Apart from sharp estimates, it is possible to get some bound for  $S^{-\alpha}$  beyond Theorem 3.1 if one use the recently proven local smoothing estimates due to Wolff and Laba [10] and the argument in [8] based on some scaling method. For the related results to the cone multiplier, one may refer to [1], [11], [17], [18] and [21].

#### REFERENCES

- [1] J. Bourgain, *Estimates for cone multiplier*, Operator Theory: Advances and Application **79** (1995), 41-60.
- [2] J. Bourgain, *Estimations de certaines fonctions maximales*, C.R. Acad. Sci. Paris **310** (1985), 499-502.
- [3] J. Bergh and J. Löfström, *Interpolation spaces: An introduction*, Springer-Verlag, New York, 1976.
- [4] I.M. Gel'fand and G.E. Shilov, *Generalized functions: Properties and operations*, vol. 1, Academic Press, New York and London (1964).
- [5] A. Carbery, A. Seeger, S. Wainger, and J. Wright, *Class of singular integral operators along variable lines*, Journal of Geometric Analysis **9** (1999), 583-605.
- [6] Y. Cho, Y. Kim, S. Lee and Y. Shim, *Sharp  $L^p - L^q$  estimates for Bochner-Riesz operators of negative index in  $\mathbb{R}^n$ ,  $n \geq 3$* , Journ. Func. Anal. **218** (2005), 150-167.
- [7] J. Harmse, *On Lebesgue space estimates for the wave equation*, Indiana Univ. Math. J. **39** (1990), 229-248.
- [8] S. Lee, *Some sharp bounds for the cone multiplier of negative order in  $\mathbb{R}^3$* , Bull. London Math. Soc. **35** (2003), 373-390.
- [9] S. Lee, *Improved bounds for Bochner-Riesz and maximal Bochner-Riesz operators*, Duke Math. Journ. **122** (2004), 205-232.
- [10] I. Laba and T. Wolff, *A local smoothing estimate in higher dimensions*, J. d'Analyse Math. **88** (2002), 149-171.
- [11] G. Mochenhaupt, *A note on the cone multiplier*, Proc. Amer. Math. Soc. **177** (1993), 145-151.
- [12] D. Oberlin, *Convolution estimates for some distribution with singularities on the light cone*, Duke Math. Journ. **59** (1989), 747-757.

- [13] J. Peetre, *New thoughts on Besov spaces*, Duke University Mathematical Series 1, Durham N.C., (1976).
- [14] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, (1970).
- [15] E. M. Stein, *Real variable methods, orthogonality, and oscillatory integrals*, Princeton Univ. Press, Princeton, (1993).
- [16] T. Tao, *Endpoint bilinear restriction theorems for the cone and some sharp null form estimates*, Math. Z. **238** (2001), 215-268.
- [17] T. Tao and A. Vargas, *A bilinear approach to cone multipliers I. Restriction estimates*, Geom. Funct. Anal. **10** (2000), 185-215.
- [18] T. Tao and A. Vargas, *A bilinear approach to cone multipliers. II. Application*, Geom. Funct. Anal. **10** (2000), 216-258.
- [19] T. Tao, A. Vargas and L. Vega *A bilinear approach to the restriction and Keakeya conjecture*, J. Amer. Math. Soc. **11** (1998), 967-1000.
- [20] T. Wolff, *Local smoothing type estimates on  $L^p$  for large  $p$* , Geom. Funct. Anal. **10** (2000), 1237-1288.
- [21] T. Wolff, *A sharp cone restriction estimate*, Annals of Math. **153** (2001), 661-698.

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