A MAXIMAL INEQUALITY ASSOCIATED TO SCHRÖDINGER TYPE EQUATION

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Abstract. In this note, we consider a maximal operator \( \sup_{t \in \mathbb{R}} |u(x, t)| = \sup_{t \in \mathbb{R}} |e^{it\Omega(D)}f(x)| \), where \( u \) is the solution to the initial value problem \( u_t = i\Omega(D)u, \ u(0) = f \) for a \( C^2 \) function \( \Omega \) with some growth rate at infinity. We prove that the operator \( \sup_{t \in \mathbb{R}} |u(x, t)| \) has a mapping property from a fractional Sobolev space \( H^{s/2} \) with additional angular regularity to \( L^2_{\text{loc}} \).

1. Introduction

We consider the almost everywhere convergence problem on the free Schrödinger type equation:
\[
\frac{\partial}{\partial t} u(x, t) = i\Omega(D)u(x, t) \quad \text{in} \quad \mathbb{R}^{n+1} (n \geq 2), \quad u(x, 0) = f,
\]
where \( \Omega(D) \) is a generalized differential operator defined by a \( C^2 \) function \( \Omega \) and \( D = (-\Delta)^{1/2} \). For smooth initial data \( f \), the solution \( u(x, t) = e^{it\Omega(D)}f \) can be written as
\[
u(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi + t\Omega(\xi))} \hat{f}(\xi) \, d\xi \quad f \in \mathcal{S}(\mathbb{R}^n),
\]
where \( \hat{f}(\xi) = \int e^{-ix \cdot \xi} f(x) \, dx \). In this note, we assume that the initial data \( f \) has \( H^s \) regularity for some \( s > 0 \) as well as some regularity in the angular direction.

For \( \alpha, \beta \geq 0 \), we define \(||f|||_{H^s H^\beta_\omega} := ||(1 - \Delta)^{\beta/2} f||_{L^2_{\omega} H^s_\omega} < \infty||\), where \(||g|||_{L^2_{\omega} H^s_\omega} := \int_0^\infty |g|^2 r^{n-1}dr, ||g|||_{L^2_{\omega} H^s_\omega} = ||||(1 - \Delta_\omega)^{s/2} f(r\omega)|||_{L^2_{\omega}} \) (here, \((r, \omega) \in \mathbb{R}_+ \times S^{n-1}\) is the spherical coordinates), and \( \Delta_\omega \) is the Laplace-Beltrami operator on \( S^{n-1} \). Since \( \Delta_\omega \) commutes with \( \Delta \), one can readily check that \( ||g|||_{H^s H^\beta_\omega} \sim ||(1 - \Delta_\omega)^{\beta/2} g|||_{H^s_{\omega}} \) (for instance, see [9]). Since not every function in \( H^s H^\beta_\omega \) has radial regularity higher than \( \alpha \), there is no embedding from or into a usual Sobolev space. In particular, it should be noted that \( H^s H^\beta_\omega \nsubseteq H^{s+\gamma} \) (\( 0 < \gamma < \beta \)) and \( H^s H^\beta_\omega \nsubseteq H^{s+\gamma} \) (\( \gamma \geq \beta \)).

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We also assume that $\Omega$ is radially symmetric and satisfies
\[ c_1|\rho|^{a-k} \leq |\Omega^{(k)}(\rho)| \leq c_2|\rho|^{a-k} \quad (k = 0, 1, 2), \]
for some $c_1, c_2, a > 0$ with $a \neq 1$ and a large $N > 0$. For the point-wise convergence for the averaging on the sphere, it is sufficient to consider boundedness of maximal operator $u^*(x) = \sup_{t \in \mathbb{R}}|u(x, t)|$. We prove

**Theorem 1.1.** For any $\varepsilon > 0$, if $f \in H^{1/2}H^{3/2+\varepsilon}$, then there exists a constant $C$, depending only on $\Omega, n, R, s$, such that
\[ \|u^*\|_{L^2(B_R)} \leq C\|f\|_{H^{1/2}H^{3/2+\varepsilon}}. \]

Let $\chi_R$ be a radial and smooth cut-off function such that $\chi_R = 1$ on $B_R$ and 0 on $B_{2R}^c$. Let us define for a fixed $s \in [1/4, 1/2]$,
\[ Tf(x, t) = \chi_R(x) \int e^{i(x \cdot \xi + t\Omega(\xi))} \hat{f}(\xi) \frac{d\xi}{(1 + |\xi|^2)^{s}}, \]
\[ T^* f(r) = \sup_{t \in \mathbb{R}} |Tf(x, t)|. \]

Then Theorem 1.1 follows immediately from

**Theorem 1.2.** For any $f \in L^2_sH^{1+s-\varepsilon}$, there exists a constant $C$, depending only on $\Omega, n, R, s$, such that
\[ \|T^* f\|_{L^2} \leq C\|f\|_{L^2_sH^{1+s-\varepsilon}}. \]

The maximal function $u^*$ and operator $T^*$ have been studied extensively by many authors ([1, 2, 3, 4, 5, 7, 8, 10, 11, 12, 13, 14, 18, 19, 21]). P. Sjölin [14] and L. Vega [19] showed that (1.1)\[ \|u^*\|_{L^2(B_R)} \leq C\|f\|_{H^s}, \]
only if $s \geq \frac{1}{4}$. However, the sufficiency remains open. Up to now, it is known that (1.1) is true if $n = 1$ ([5, 8]) or the initial data is radial ([4, 12]), or $s > \frac{1}{4}$ and $n \geq 2$ ([11, 19]). Recently, T. Tao [18] obtained (1.1) for $s > \frac{2}{5}$ and $n = 2$.

On the other hand, Theorem 1.1 shows that it is true for $s = \frac{1}{4}$ if we assume the additional angular regularity. If the initial data is a finite linear combination of radial functions and spherical harmonics, it was proved by the first and third authors in [4] that the conjecture is true for $s = \frac{1}{4}$ but has a dependency on on the order $k$ of spherical harmonics like $O((n + 2k)^{n+2k})$. In this connection, Theorem 1.1 improves the dependency on the order up to $k^{4/3+\varepsilon}$ (see (2.2 below). In case of a local maximal operator $\sup_{|t| \leq 1}|u(x, t)|$, then it was shown by G. Gigante and F. Soria [6] that the angular regularity assumption can be slightly weakened up to $k^{4/3+\varepsilon}$. However, we don’t know yet whether the angular regularity must be imposed or not.
From the assumption on $\Omega$, we treat $\Omega$ not only of the form $|\xi|^a$ but also $\sum_{i=1}^l a_i |\xi|^{m_i}$ for any number $m_l > m_{l-1} > \cdots > m_1$, $m_l \neq 1$ and $a_i \in \mathbb{R}$. For another use of angular regularity, we refer to [9] in which endpoint Strichartz estimates of 3-d wave and Klein-Gordon equations are considered.

For the proof of our results, we estimate the maximal operators locally in $L^2$ since there is neither a global $L^2$ estimate for $s \geq \frac{1}{4}$ ([11]), nor a local estimate in $L^p(p > 2)$ for $s = \frac{1}{2}$. We use the asymptotic property of Bessel functions $J_\nu$ and the spherical harmonic expansion $f(r\omega) = \sum_k f_k(r) Y_k(\omega)$ where $Y_k$ are spherical harmonic functions of order $k$. As to be shown in the next section, the angular regularity is related to the order of spherical harmonic function $Y_k(\omega)$ (see (2.1)). By the orthogonality among spherical harmonic functions of different orders, to get Theorem 1.2, the matters are reduced to obtaining uniform estimate along $k$. To control the dependency on $k$, the angular regularity is used.

If not specified, throughout this paper, $C$ denotes a generic constant that depends on $\Omega, n, R, s$. We use the notation $A \lesssim B$ and $A \sim B$ to denote $|A| \leq CB$ and $C^{-1}B \leq |A| \leq CB$ respectively.

2. Proof of Theorem 1.2

We begin with reviewing some properties of the spherical harmonic expansion. If $f(r\omega) = g(r) Y_k(\omega)$ for a radial function $g$ and a spherical harmonic $Y_k$ of order $k$, then we have

$$\hat{f}(\rho \theta) = G(\rho) Y_k(\theta), \quad ||g||_{L^2} = ||G||_{L^2},$$

where

$$G(\rho) = c_{n,k} \int_0^\infty g(r) r^{n-1} (r\rho)^{-\frac{n+2}{2}} J_\nu(r\rho) \, dr, \quad |c_{n,k}| \leq C, \quad \nu = \frac{2k + n - 2}{2}.$$ 

For the representation of $G$, see e.g. [16] or [22]. Since $\Delta_\omega Y_k = k(k+n-2)Y_k$, we also have $||f||_{L^2 H^\beta_\omega} \sim (1+k^2)^\beta ||g||_{L^2} ||Y_k||_{L^2}$. Furthermore, if $h \in L^2 H^\beta_\omega$, then there exist radial functions $\{h_k\}$ and spherical harmonics $\{Y_k\}$ such that

$$h(r\omega) = \sum_{k \geq 0} \sum_{l=1}^{d(k)} h_k(r) Y_k(\omega) \quad \text{in} \quad L^2 H^\beta_\omega,$$

where $d(k)$ is the dimension of the space of spherical harmonics of degree $k$, and

$$(2.1) \quad ||h||^2_{L^2 H^\beta_\omega} \sim \sum_{k \geq 0} \sum_{l=1}^{d(k)} (1 + k^2)^\beta ||h_k||^2_{L^2} ||Y_k||^2_{L^2}.$$ 

Thus from the orthogonality of spherical harmonic functions, we have only to consider the case that $f(r\omega) = g(r) Y_k(\omega)$. It is also sufficient for the proof of theorem to show for large $k$

$$(2.2) \quad ||T^s f||_{L^2} \lesssim k^{\frac{s}{2} - \epsilon} ||g||_{L^2} ||Y_k||_{L^2},$$

since $k^{\frac{s}{2} - \epsilon} ||g||_{L^2} ||Y_k||_{L^2} = k^{1-s+\epsilon} ||g||_{L^2} ||Y_k||_{L^2} \cdot k^{-\frac{s}{2} - \epsilon}.$
By scaling and translation, we may assume that $B_R = B(0, 1)$, ball with radius 1 centered at the origin. Since $\hat{f}(\rho \omega) = G(\rho)Y_k(\omega)$, from the definition of $T$, we have

$$Tf(r\omega, t) = \chi_1(r) \int_{S^{n-1}} e^{i(r\omega \cdot \rho \Theta + iT \rho)} G(\rho)Y_k(\Theta) d\Theta \rho_0 (1 + \rho^2)^{\frac{3}{2}} d\Theta$$

$$= c_{n,k} \chi_1(r) \int_0^\infty e^{it\Omega(\rho)} (r \rho)^{-\frac{\nu-2}{2}} J_{\nu}(r \rho) \rho_0 (1 + \rho^2)^{\frac{3}{2}} Y_k(\omega).$$

Let us define an operator $S$ by

$$SG(r, t) = c_{n,k} \chi_1(r) \int_0^\infty e^{it\Omega(\rho)} (r \rho)^{-\frac{\nu-2}{2}} J_{\nu}(r \rho) \rho_0 (1 + \rho^2)^{\frac{3}{2}} Y_k(\omega).$$

Let us denote by $\|F\|_{L^p L^q}$ the mixed norm $\|(F(r, t))\|_{L^p(\Omega)}\|_{L^q(d\rho)}$. To prove (2.2) it suffices to show that

$$\|SG\|_{L^p L^q} \lesssim k^{\frac{1}{2} - s} \|G\|_{L^2},$$

where $G(\rho) = \rho^{\frac{\nu-2}{2}} G(\rho)$. Now we define the dual operator $S^d$ of $S$ by

$$S^d F(\rho) = \frac{c_{n,k}}{(1 + \rho^2)^{\frac{3}{2}}} \int_0^\infty e^{-it\Omega(\rho)} (r \rho)^{\frac{\nu-2}{2}} J_{\nu}(r \rho) \chi_1(r) F(r, t) dr dt$$

for $F \in C^0_0(\mathbb{R}_+ \times \mathbb{R})$. Then, by duality (2.3) follows from

$$\|S^d F\|_{L^2 L^1} \leq C k^{\frac{s}{2} - s} \|F\|_{L^2 L^1}. $$

Choose smooth cut-off functions $\phi_0$, $\phi_1$ and $\phi_3$ so that $\phi_0 = 1$ on $|s| < \frac{1}{2}$, $\phi_0 = 0$ on $|s| > 1$, $\phi_1 = 1$ on $|s| \sim 1$, $\phi_1 = 0$ otherwise, $\phi_2 = 0$ on $|s| < 2$, $\phi_2 = 1$ on $|s| > 3$, and $\phi_0 + \phi_1 + \phi_2 = 1$. Then we decompose $S^d$ as

$$S^d F(\rho) = S_1 F + S_1 F + S_2 F,$$

where for $i = 0, 1, 2$,

$$S_i F(\rho) = \frac{c_{n,k}}{(1 + \rho^2)^{\frac{3}{2}}} \int_0^\infty e^{-it\Omega(\rho)} (r \rho)^{\frac{\nu-2}{2}} J_{\nu}(r \rho) \phi_i \left(\frac{r \rho}{\rho}\right) \chi_1(r) F(r, t) dr dt.$$

Now we need to show each $S_i$ satisfies (2.4) in the place of $S^d$. Each estimate is to be shown using the following asymptotic behavior of Bessel functions:

$$|J_{\nu}(t)| \leq C \exp(-C \tau), \quad \text{if} \quad t \leq \frac{\nu}{2},$$

$$\frac{1}{r} \int_0^r |J_{\nu}(t)|^2 dt \leq C \quad \text{for all} \quad r > 0,$$

$$J_{\nu}(t) \varphi_2 \left(\frac{t}{\nu}\right) = t^{-\frac{1}{2}} (b \nu e^{t \nu} + b \nu e^{-t \nu}) \varphi_1 \left(\frac{t}{\nu}\right) + \Phi_{\nu}(t) \varphi_2 \left(\frac{t}{\nu}\right),$$

where $|\Phi_{\nu}(t)| \leq C \frac{t^{-1}}{\nu}$, $|b \nu| \leq C$ and the constant $C$ is independent of $\nu$. For the proof of (2.5), see [17]. The mean value estimate (2.6) can be found in Section 4.10 of [20]. Invoking the Schl"afli’s integral representation (see p.176 in [23]):

$$J_{\nu}(t) = \frac{1}{2\pi} \int_0^{2\pi} e^{i(t \sin \theta - \nu \theta)} d\theta - \frac{\sin(\nu \pi)}{\pi} \int_0^\infty e^{-\nu \tau - 
u \tau \sin \theta} d\tau,$$
the last asymptotic behavior follows from the easy estimate
\[ \left| \frac{\sin(\nu \tau)}{\tau} \int_{0}^{\infty} e^{-\nu \tau - t \sinh \tau} \, dt \right| \leq \frac{C}{\nu + t} \]
and the method of stationary phase such that
\[ \frac{1}{2\pi} \int_{0}^{2\pi} e^{i(t \sin \theta - \nu \theta)} \, d\theta \sim (b_+ e^{it} + b_- e^{-it}) t^{-\frac{1}{2}} + O(t^{-\frac{3}{2}}) \quad \text{for} \quad t > 2\nu. \]

Using (2.5), we now see
\[ S_0 F(\rho) \lesssim \nu^\frac{1}{2} e^{-C\nu}(1 + \rho^2)^{-\frac{1}{2}} \int_{0}^{\infty} ||F(r, \cdot)||_{L^1} \, dr \]
\[ \lesssim \nu^\frac{1}{2} e^{-C\nu}(1 + \rho^2)^{-\frac{1}{2}} \min\left(\frac{\nu}{\rho}, 2\right) \frac{1}{2} ||F||_{L^2L^1}. \]
Thus we have
\[ (2.8) \quad ||S_0 F||_{L^2} \lesssim \nu^\frac{1}{2} e^{-C\nu} \left( \int_{0}^{\infty} (1 + \rho^2)^{-\frac{1}{2}} \min\left(\frac{\nu}{\rho}, 2\right) \, dp \right)^\frac{1}{2} ||F||_{L^2L^1} \]
\[ \lesssim \nu^\frac{1}{2} e^{-C\nu} ||F||_{L^2L^1}. \]

For $S_1$, we have
\[ |S_1 F(\rho)| \lesssim (1 + \rho^2)^{-\frac{1}{2}} \left( \int_{0}^{2\rho} (r \rho)^2 \left( \frac{r^2}{\rho} \right) \, dr \right)^\frac{1}{2} ||F||_{L^2L^1}. \]

By the change of variable $r \mapsto r/\rho$, the integral in the RHS of the above estimate is bounded by $\frac{1}{\rho} \int_{0}^{2\rho} J_0^2(r) r \phi_1^2(r/\rho) \, dr$. Since $\rho \geq \frac{\nu}{4}$ from the support condition of $\phi_1$, by (2.6) we have
\[ |S_1 F(\rho)| \leq C\nu^\frac{1}{2} (1 + \rho^2)^{-\frac{1}{2}} \rho^{-\frac{1}{2}} \chi_{\{\rho \geq \frac{\nu}{4}\}} ||F||_{L^2L^1}. \]

We thus obtain
\[ (2.9) \quad ||S_1 F||_{L^2} \lesssim \nu^\frac{1}{2} e^{-C\nu} ||F||_{L^2L^1}. \]

Now we estimate $S_2 F$. Let us set $S_2 F = S_+ F + S_- F + S_3 F$, where
\[ S_\pm F(\rho) = \frac{c_{n,k} b_\pm}{(1 + \rho^2)^{\frac{1}{2}}} \int_{\mathbb{R}} \int_{0}^{\infty} e^{i(\pm r \rho - t\Omega(\rho))} \phi_2\left( \frac{r \rho}{\rho} \right) \chi_1(r) F(r, t) \, dr dt \]
\[ S_3 F(\rho) = \frac{c_{n,k}}{(1 + \rho^2)^{\frac{1}{2}}} \int_{\mathbb{R}} \int_{0}^{\infty} e^{-it\Omega(\rho)} (r \phi_1(\rho) \phi_2\left( \frac{r \rho}{\rho} \right) \chi_1(r) F(r, t) \, dr dt. \]

For the estimate $S_\pm F$, it suffices to consider $S_+ F$. We decompose it into two parts as follows:
\[ S_+ F(\rho) = I + II \]
where
\[ I = \frac{c_{n,k} b_+}{(1 + \rho^2)^{\frac{1}{2}}} \int_{\mathbb{R}} \int_{0}^{\infty} e^{i(r \rho - t\Omega(\rho))} \chi_1(r) F(r, t) \, dr dt, \]
\[ II = \frac{c_{n,k} b_+}{(1 + \rho^2)^{\frac{1}{2}}} \int_{\mathbb{R}} \int_{0}^{\infty} e^{i(r \rho - t\Omega(\rho))} (1 - \phi\left( \frac{r \rho}{\rho} \right)) \chi_1(r) F(r, t) \, dr dt. \]
For $II$, we have
\[
|II| \lesssim (1 + \rho^2)^{-\frac{s}{2}} \int_0^{\min\left(\frac{3\nu}{\rho}, 2\right)} \|F(r, \cdot)\|_{L^1} \, dr
\]
\[
\lesssim (1 + \rho^2)^{-\frac{s}{2}} \left(\min\left(\frac{3\nu}{\rho}, 2\right)\right)^{\frac{s}{2}} \|F\|_{L^2 L^1}^2
\]
and hence
\[
(2.10) \quad ||II||_{L^2} \lesssim \nu^{-s} \|F\|_{L^2 L^1}^2.
\]
Now we estimate $I$. Since $F$ is in $C^\infty_0(\mathbb{R}_+ \times \mathbb{R})$, obviously we may assume
\[
I = c_{n, k, b} \frac{e^{i(r, t)\Omega(\rho)}}{(1 + \rho^2)^{\frac{s}{2}}} \int_{\mathbb{R}^2} e^{i(r - r') \Omega(\rho)} \chi_1(|r|) F(r, t) \, dr \, dt.
\]
Squaring and integrating $I$ over $\{|\rho| \leq N\}$, we have
\[
(2.11) \quad \int_{|\rho| < N} |I|^2 \, d\rho \leq C \|F\|_{L^2 L^1}^2.
\]
Since $\frac{1}{4} \leq s \leq \frac{1}{2}$, it is easy to see
\[
\int_{|\rho| > N} |I|^2 \, d\rho \leq C \int K(r - r', t - t') |\chi_1(|r|)| F(r, t) |\chi_1(|r'|)| F(r', t') \, dr \, dr' \, dt \, dt',
\]
where
\[
K(r, t) = \int_{|\rho| > N} e^{i(r - r') \Omega(\rho)} \frac{d\rho}{|\rho|^{\frac{s}{2}}}
\]
For the kernel estimate, we introduce a lemma which shows uniform bound of kernel $K$ on $t$.

**Lemma 2.1.** For any real number $A, B (A \neq 0)$ and $s \in \left[\frac{1}{2}, 1\right)$, there exists a constant $C$ independent of $A$ and $B$ such that
\[
\left| \int_{|\rho| > N} e^{i(A\Omega(\rho) + B\rho)} \frac{d\rho}{|\rho|^{\frac{s}{2}}} \right| \leq C |B|^{-(1-s)}.
\]
Applying Lemma 2.1 with $s = \frac{1}{2}$ and $B = r - r'$, from fractional integration it follows
\[
(2.12) \quad \int_{|\rho| > N} |I|^2 \, d\rho \lesssim \int |r - r'|^{-\frac{1}{2}} \chi_1(|r|) \|F(r, \cdot)\|_{L^1} \chi_1(|r'|) \|F(r', \cdot)\|_{L^1} \, dr \, dr'
\]
\[
\lesssim \|\mathcal{I}_{\frac{1}{2}}(\chi_1 |F|_{L^1})\|_{L^1} \chi_1 |F|_{L^1} \|\mathcal{I}_{\frac{1}{2}}(\chi_1 |F|_{L^1})\|_{L^1} \|F\|_{L^2 L^1}^2
\]
where $\mathcal{I}_{\frac{1}{2}}$ is the Riesz potential of order $\frac{1}{2}$. 
Finally, we estimate $S_3 F$. From the uniform bound of $\Phi_\nu$ on $\nu$, for small $\varepsilon > 0$, we have

$$|S_3 F(\rho)| \leq \frac{C}{(1 + \rho^2)^2} \int (\rho \rho)^{-\frac{1}{2}} \phi_2 \left( \frac{\rho \rho}{\nu} \right) \chi_1(r) ||F(r, \cdot)||_{L^1} dr$$

$$\leq C \rho^{-s - \frac{1}{2}} \chi_{\{ \rho \geq \nu \}} \int_{\frac{\rho}{\nu}}^{2} r^{-\frac{1}{2}} ||F(r, \cdot)||_{L^1} dr$$

$$\leq C \varepsilon \rho^{-s - \frac{1}{2} + \varepsilon} \chi_{\{ \rho \geq \nu \}} \int_{\frac{\rho}{\nu}}^{2} r^{-\frac{1}{2} + \varepsilon} ||F(r, \cdot)||_{L^1} dr$$

Choosing $\varepsilon$ as $\frac{1}{8}$, we obtain

$$||S_3 F||_{L^2} \lesssim \nu^{-s} ||F||_{L^2 L^1}. \tag{2.13}$$

Combining all the estimates (2.8) to (2.13) and recalling $\nu = \frac{2k + n - 2}{2}$, we get (2.4) and hence Theorem 1.2.

**Proof of Lemma 2.1.** To prove Lemma 2.1, we need the following (see e.g. [8] and [15])

**Lemma 2.2.** Let $\psi$ be a monotone function and $I = \int_{\alpha}^{\beta} e^{i\varphi(\rho)} \psi(\rho) d\rho$. Then if $|\frac{d\varphi}{d\rho}| \leq \lambda > 0$ in $[\alpha, \beta]$ and $\frac{d\psi}{d\rho}$ is monotone, $|I| \leq C \lambda^{-1} \sup_{[\alpha, \beta]} |\psi(\rho)|$, and if $|\frac{d^2 \varphi}{d\rho^2}| \geq \lambda > 0$, then $|I| \leq C \lambda^{-\frac{1}{2}} \sup_{[\alpha, \beta]} |\psi(\rho)|$. The constant $C$ doesn’t depend on $\alpha, \beta, \lambda, \varphi$ and $\psi$.

We may assume that $N = 0$ because there is no harm to the entire estimates. And by symmetry, we also assume that $A > 0$ and $B > 0$.

**Case $a > 1$** Let $D = \frac{B}{A^{\frac{1}{2}}}$. Then by the change of variable, we have

$$I = A^{-\frac{1}{2} + \varepsilon} \int e^{i(\varphi(A^{-\frac{1}{2}} \rho) + D \rho)} |\rho|^{-s} d\rho = \int_{\rho < 0} + \int_{\rho > 0} = I_0 + I_1.$$

We have only to consider $I_0$ and we denote it $I$ again.

Now we first consider the case when $\Omega' > 0$. Observe that

$$E \equiv (A \Omega(A^{-\frac{1}{2}} \rho) + D \rho)' \geq c_1 \rho^{a - 1} + D.$$

Let $M$ be a large positive number depending only on $a, s, c_1, c_2$. If $D \leq M$, then

$$I = A^{-\frac{1}{2} + \varepsilon} \left( \int_{0}^{1} + \int_{1}^{\infty} \right) = I_1 + I_2$$

For $I_1$, by direct integration, we have $|I_1| \lesssim A^{-\frac{1}{2} + \varepsilon} \lesssim B^{-(1-s)}$. For $I_2$, since $E \geq 1$, by the first part of (2.2), we have $|I_2| \lesssim A^{-\frac{1}{2} + \varepsilon} \lesssim B^{-(1-s)}$. If $D > M$, then since
\[ E \geq D, \text{ by the first part of Lemma 2.2, we have } |I_2| \lesssim A^{-\frac{1-a}{2}} D^{-1} \leq A^{\frac{a}{2}} B^{-1} \leq B^{-(1-s)}. \] For \( I_1 \), using the change of variable, we have
\[
I_1 = A^{-\frac{1-a}{2}} D^{-(1-s)} \int_0^D e^{i(A\Omega(D^{-1}A^{-\frac{1}{2}} \rho) + \rho) \rho^{-s}} dp.
\]
Thus \( I_1 = I_1^1 + I_1^D = I_{1,1} + I_{1,2} \). By the integration, \( |I_{1,1}| \lesssim B^{-(1-s)}. \) For \( I_{1,2} \), since \((A\Omega(D^{-1}A^{-\frac{1}{2}} \rho) + \rho)' \geq 1\), from the first part of Lemma 2.2, we have \( |I_{1,2}| \lesssim B^{-(1-s)} \) and hence \( |I_1| \lesssim B^{-(1-s)} \).

Now we consider the case when \( \Omega' < 0 \). We observe that
\[ -c_2 \rho^{a-1} + D \leq E = (A\Omega(A^{-\frac{1}{2}} \rho) + D\rho)' \leq -c_1 \rho^{a-1} + D. \]

If \( D \leq M \), then we split \( I \) into two parts as follows:
\[
I = A^{-\frac{1-a}{2}} \left( \int_0^{\left(\frac{2M}{2a}\right)^{\frac{1}{1-s}}} + \int_{\left(\frac{2M}{2a}\right)^{\frac{1}{1-s}}}^{\infty} \right) = I_3 + I_4.
\]
For \( I_5 \), we have by direct integration \( |I_3| \lesssim A^{-\frac{1-a}{2}} \leq B^{-(1-s)}. \) For \( I_4 \), since \( E \leq -c_1 \rho^{a-1} + D \leq -1 \), by the first part of Lemma 2.2, we get \( |I_4| \lesssim A^{-\frac{1-a}{2}} \leq B^{-(1-s)}. \)

If \( D > M \), then we split \( I \) into four parts as follows:
\[
(2.14) \quad I = A^{-\frac{1-a}{2}} \left( \int_0^1 + \int_1^{\frac{2M}{2a}} + \int_{\frac{2M}{2a}}^{\frac{2M}{2a} + \frac{1}{1-s}} + \int_{\frac{2M}{2a} + \frac{1}{1-s}}^{\infty} \right) = I_5 + I_6 + I_7 + I_8.
\]
For \( I_5 \), we use the change of variable so that
\[
I_5 = A^{-\frac{1-a}{2}} D^{-(1-s)} \int_0^D e^{i(A\Omega(D^{-1}A^{-\frac{1}{2}} \rho) + \rho) \rho^{-s}} dp.
\]
We split \( I_5 \) into two parts: \( I_5 = A^{-\frac{1-a}{2}} D^{-(1-s)} \left( \int_0^1 + \int_1^D \right) = I_{5,1} + I_{5,2}. \) For \( I_{5,1} \) and \( I_{5,2} \), using the direct integration and the first part of Lemma 2.2 respectively, we have \( |I_{5,1}| + |I_{5,2}| \lesssim A^{-\frac{1-a}{2}} D^{-(1-s)} = B^{-(1-s)}. \) For \( I_6 \), since \( E \geq D \geq D^{1-s} \), using the first part of Lemma 2.2, we have \( |I_6| \lesssim A^{-\frac{1-a}{2}} D^{-(1-s)} = B^{-(1-s)}. \)

To estimate \( I_7 \), we use the second derivative \( |E'| \sim \rho^{a-2} \sim D^{\frac{1-a}{2}} \). Then from the second part of Lemma 2.2, we obtain
\[
|I_7| \lesssim A^{-\frac{1-a}{2}} D^{-\frac{a-2}{2-s}} D^{-\frac{s}{2-a}} = A^{-\frac{1-a}{2}} D^{-\frac{a-2}{2-s}} D^{-\frac{s}{2-a}}.
\]
Since \( a > 1 \) and \( s \geq \frac{1}{2} \), we have \( |I_7| \lesssim A^{-\frac{1-a}{2}} D^{-(1-s)} = B^{-(1-s)}. \) Finally, we estimate \( I_8 \). Since \( E \geq D \geq D^{1-s} \), by the first part of Lemma 2.2, we have \( |I_8| \lesssim A^{-\frac{1-a}{2}} D^{-(1-s)}D^{-\frac{s}{2-a}} \lesssim B^{-(1-s)}. \)

(Case \( a < 1 \)) Let \( \tilde{D} = \frac{D}{D^{1-s}}. \) Then by the change of variable, we write
\[
B^{1-s} I = \int e^{i(A\Omega(\xi) + \rho)} |\rho|^{-s} dp = \int_{-\infty}^0 + \int_0^\infty = I_+ + I_-.
\]
Similarly to the case \( a > 1 \), we only consider \( I_+ \) and denote it by \( I \) again.
In case that $\Omega' > 0$, we have $E \equiv (A\Omega(\frac{\epsilon}{\rho^2}))' + \rho' \geq c_1 \tilde{D} \rho^{s-1} + 1 \geq 1$ for all $\rho > 0$. We divide $I$ into two parts: $I = \int_0^1 + \int_1^\infty$. For the first integral, we just integrate and for the second one, we use the first part of Lemma 2.2. Then we can see $|I| \lesssim 1$.

Now we consider the case when $\Omega' < 0$. Then we can observe that

$$-c_2 \tilde{D} \rho^{s-1} + 1 \leq E \leq -c_1 \tilde{D} \rho^{s-1} + 1.$$ 

If $c_2 \tilde{D} < 2$, then we divide $I$ into two parts: $I = \int_0^{(\frac{1}{c_1})\frac{1}{s-1}} + \int_1^{(\frac{1}{c_1})\frac{1}{s-1}} = I_1 + I_2$. By the integration, we get $|I_1| \lesssim 1$. And since $c_2 \tilde{D} < 2$ and hence $E \gtrsim 1$, by the first part of Lemma 2.2, we have $|I_2| \lesssim 1$.

If $c_1 \tilde{D} > 2$, then we divide $I$ into four parts:

$$I = \int_0^1 + \int_1^{(\frac{1}{c_1})\frac{1}{s-1}} + \int_1^{(\frac{1}{c_1})\frac{1}{s-1}} + \int_0^\infty \int_1^{(\frac{1}{c_1})\frac{1}{s-1}} = I_3 + I_4 + I_5 + I_6.$$ 

For $I_3$, by the integration, $|I_3| \lesssim 1$. For $|I_5|$, since $|E'| \sim \tilde{D} \tilde{D}^{-\frac{2s-2}{2s-1}}$ and $s \geq \frac{1}{2}$, by the second part of Lemma 2.2, we have $|I_5| \lesssim \tilde{D}^{\frac{2s-2}{2s-1}} \lesssim 1$. And since $E \lesssim -1$ on $[1, (\frac{1}{c_1})\frac{1}{s-1}]$ and $E \gtrsim 1$ on $[(\frac{1}{c_1})\frac{1}{s-1}, \infty)$, we also have $|I_4|, |I_6| \lesssim 1$.

If $\frac{2s}{c_2} \leq \tilde{D} \leq \frac{2s}{c_1}$, choose a large number $M$ depending only on $c_1, c_2$, and divide $I$ as follows: $I = \int_0^M + \int_M^\infty$. Then as the estimate of $I_1$ and $I_2$, we can obtain $|I| \lesssim 1$. This completes the proof of lemma.

References

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