A MAXIMAL INEQUALITY ASSOCIATED TO SCHRÖDINGER TYPE EQUATION

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Abstract. In this note, we consider a maximal operator $\sup_{t\in\mathbb{R}}|u(x,t)| = \sup_{t\in\mathbb{R}}|e^{it\Omega(D)}f(x)|$, where u is the solution to the initial value problem $u_t = i\Omega(D)u$, $u(0) = f$ for a C^2 function Ω with some growth rate at infinity. We prove that the operator $\sup_{t\in\mathbb{R}}|u(x,t)|$ has a mapping property from a fractional Sobolev space $H^{\frac{1}{4}}$ with additional angular regularity to L^2_{loc}.

1. Introduction

We consider the almost everywhere convergence problem on the free Schrödinger type equation:

$$\frac{\partial}{\partial t}u(x,t) = i\Omega(D)u(x,t) \quad \text{in} \quad \mathbb{R}^{n+1}(n\ge 2), \quad u(x,0) = f(x),$$

where $\Omega(D)$ is a generalized differential operator defined by a C^2 function Ω and $D = (-\Delta)^{\frac{1}{2}}$. For smooth initial data f, the solution $u(x,t) = e^{it\Omega(D)}f$ can be written as

$$u(x,t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi + t\Omega(\xi))}\widehat{f}(\xi)\,d\xi \quad f\in\mathcal{S}(\mathbb{R}^n),$$

where $\widehat{f}(\xi) = \int e^{-ix\cdot\xi}f(x)\,dx$. In this note, we assume that the initial data f has H^s regularity for some $s > 0$ as well as some regularity in the angular direction. For $\alpha,\beta\ge 0$, we define

$$||f||_{H^\alpha H^\beta_\omega} := ||(1-\Delta)^{\frac{\alpha}{2}}f||_{L^2_r H^\beta_\omega} < \infty,$$

where $||g||^2_{L^2_r} = \int_0^\infty |g|^2 r^{n-1}\,dr$, $||g||_{L^2_r H^\beta_\omega} = ||||(1-\Delta_\omega)^{\frac{\beta}{2}}f(r\omega)||_{L^2_\omega}||_{L^2_r}$ (here, $(r,\omega)\in\mathbb{R}_+\times S^{n-1}$ is the spherical coordinates), and Δ_ω is the Laplace-Beltrami operator on S^{n-1}. Since Δ_ω commutes with Δ, one can readily check that $||g||_{H^\alpha H^\beta_\omega}\sim ||(1-\Delta_\omega)^{\frac{\beta}{2}}g||_{H^\alpha}$ (for instance, see [9]). Since not every function in $H^\alpha H^\beta_\omega$ has radial regularity higher than α, there is no embedding from or into a usual Sobolev space. In particular, it should be noted that $H^\alpha H^\beta_\omega \not\subseteq H^{\alpha+\gamma}$ $(0<\gamma<\beta)$ and $H^\alpha H^\beta_\omega \not\supseteq H^{\alpha+\gamma}$ $(\gamma\ge\beta)$.

2000 *Mathematics Subject Classification.* Primary 42B25; Secondary 42A45.

Keywords and phrases. Schrödinger type equation, maximal operator, angular regularity.

The first author was supported by Japan Society for the Promotion of Science under JSPS Postdoctoral Fellowship for Foreign Researchers and the second author was partially supported by the Post-doctoral Fellowship Program of Korea Science & Engineering Foundation (KOSEF).

We also assume that Ω is radially symmetric and satisfies

$$c_1|\rho|^{a-k} \leq |\Omega^{(k)}(\rho)| \leq c_2|\rho|^{a-k} \ (k=0,1,2), \quad \text{if} \quad |\rho| \geq N$$

for some $c_1, c_2, a > 0$ with $a \neq 1$ and a large $N > 0$. For the point-wise convergence for the averaging on the sphere, it is sufficient to consider boundedness of maximal operator $u^*(x) = \sup_{t\in\mathbb{R}} |u(x,t)|$. We prove

Theorem 1.1. *For any $\varepsilon > 0$, if $f \in H^{\frac14} H_\omega^{\frac34+\varepsilon}$, then there exists a constant C, depending only on Ω, n, R, such that*

$$||u^*||_{L^2(B_R)} \leq C||f||_{H^{\frac14} H_\omega^{\frac34+\varepsilon}}.$$

Let χ_R be a radial and smooth cut-off function such that $\chi_R = 1$ on B_R and 0 on B_{2R}^c. Let us define for a fixed $s \in [1/4, 1/2]$,

$$Tf(x,t) = \chi_R(x) \int e^{i(x\cdot\xi + t\Omega(\xi))} \widehat{f}(\xi) \, \frac{d\xi}{(1+|\xi|^2)^{\frac{s}{2}}},$$

$$T^*f(r) = \sup_{t\in\mathbb{R}} |Tf(x,t)|.$$

Then Theorem 1.1 follows immediately from

Theorem 1.2. *For any $f \in L_r^2 H_\omega^{1+\varepsilon-s}$, there exists a constant C, depending only on Ω, n, R, s, such that*

$$||T^*f||_{L^2} \leq C||f||_{L_r^2 H_\omega^{1+\varepsilon-s}}.$$

The maximal function u^* and operator T^* have been studied extensively by many authors ([1, 2, 3, 4, 5, 7, 8, 10, 11, 12, 13, 14, 18, 19, 21]). P. Sjölin [14] and L. Vega [19] showed that

$$\|u^*\|_{L^2(B_R)} \leq C\|f\|_{H^s}, \tag{1.1}$$

only if $s \geq \frac14$. However, the sufficiency remains open. Up to now, it is known that (1.1) is true if $n = 1$ ([5, 8]) or the initial data is radial ([4, 12]), or $s > \frac12$ and $n \geq 2$ ([11, 19]). Recently, T. Tao [18] obtained (1.1) for $s > \frac25$ and $n = 2$.

On the other hand, Theorem 1.1 shows that it is true for $s = \frac14$ if we assume the additional angular regularity. If the initial data is a finite linear combination of radial functions and spherical harmonics, it was proved by the first and third authors in [4] that the conjecture is true for $s = \frac14$ but has a dependency on on the order k of spherical harmonics like $O((n+2k)^{n+2k})$. In this connection, Theorem 1.1 improves the dependency on the order up to $k^{\frac34+\varepsilon}$ (see (2.2 below). In case of a local maximal operator $\sup_{|t|\leq 1} |u(x,t)|$, then it was shown by G. Gigante and F. Soria [6] that the angular regularity assumption can be slightly weakened up to $k^{\frac12+\varepsilon}$. However, we don't know yet whether the angular regularity must be imposed or not.

From the assumption on Ω, we treat Ω not only of the form $|\xi|^a$ but also $\sum_{i=1}^{l} a_i|\xi|^{m_i}$ for any number $m_l > m_{l-1} > \cdots > m_1$, $m_l \neq 1$ and $a_i \in \mathbb{R}$. For another use of angular regularity, we refer to [9] in which endpoint Strichartz estimates of 3-d wave and Klein-Gordon equations are considered.

For the proof of our results, we estimate the maximal operators locally in L^2 since there is neither a global L^2 estimate for $s \geq \frac{1}{4}$ ([11]), nor a local estimate in $L^p (p > 2)$ for $s = \frac{1}{4}$. We use the asymptotic property of Bessel functions J_ν and the spherical harmonic expansion $f(r\omega) = \sum_k f_k(r)Y_k(\omega)$ where Y_k are spherical harmonic functions of order k. As to be shown in the next section, the angular regularity is related to the order of spherical harmonic function $Y_k(\omega)$ (see (2.1)). By the orthogonality among spherical harmonic functions of different orders, to get Theorem 1.2, the matters are reduced to obtaining uniform estimate along k. To control the dependency on k, the angular regularity is used.

If not specified, throughout this paper, C denotes a generic constant that depends on Ω, n, R, s. We use the notation $A \lesssim B$ and $A \sim B$ to denote $|A| \leq CB$ and $C^{-1}B \leq |A| \leq CB$ respectively.

2. Proof of Theorem 1.2

We begin with reviewing some properties of the spherical harmonic expansion. If $f(r\omega) = g(r)Y_k(\omega)$ for a radial function g and a spherical harmonic Y_k of order k, then we have

$$\widehat{f}(\rho\theta) = G(\rho)Y_k(\theta), \quad ||g||_{L^2_r} = ||G||_{L^2_r},$$

where

$$G(\rho) = c_{n,k} \int_0^\infty g(r)r^{n-1}(r\rho)^{-\frac{n-2}{2}} J_\nu(r\rho)\,dr,\ |c_{n,k}| \leq C,\ \nu = \frac{2k+n-2}{2}.$$

For the representation of G, see e.g. [16] or [22]. Since $\Delta_\omega Y_k = k(k+n-2)Y_k$, we also have $||f||_{L^2_r H^\beta_\omega} \sim (1+k^2)^{\frac{\beta}{2}} ||g||_{L^2_r} ||Y_k||_{L^2_\omega}$. Furthermore, if $h \in L^2_r H^\beta_\omega$, then there exist radial functions $\{h^l_k\}$ and spherical harmonics $\{Y^l_k\}$ such that

$$h(r\omega) = \sum_{k\geq 0} \sum_{1\leq l\leq d(k)} h^l_k(r)Y^l_k(\omega) \quad \text{in} \quad L^2_r H^\beta_\omega,$$

where $d(k)$ is the dimension of the space of spherical harmonics of degree k, and

$$||h||^2_{L^2_r H^\beta_\omega} \sim \sum_{k\geq 0} \sum_{1\leq l\leq d(k)} (1+k^2)^\beta ||h^l_k||^2_{L^2_r} ||Y^l_k||^2_{L^2_\omega}. \tag{2.1}$$

Thus from the orthogonality of spherical harmonic functions, we have only to consider the case that $f(r\omega) = g(r)Y_k(\omega)$. It is also sufficient for the proof of theorem to show for large k

$$||T^* f||_{L^2} \lesssim k^{\frac{1}{2}-s} ||g||_{L^2_r} ||Y_k||_{L^2_\omega}, \tag{2.2}$$

since $k^{\frac{1}{2}-s}||g||_{L^2_r}||Y_k||_{L^2_\omega} = k^{1-s+\varepsilon}||g||_{L^2_r}||Y_k||_{L^2_\omega} \cdot k^{-\frac{1}{2}-\varepsilon}$.

By scaling and translation, we may assume that $B_R = B(0,1)$, ball with radius 1 centered at the origin. Since $\widehat{f}(\rho\omega) = G(\rho)Y_k(\omega)$, from the definition of T, we have

$$\begin{aligned} Tf(r\omega,t) &= \chi_1(r)\int_{S^{n-1}}\int_0^\infty e^{i(r\omega\cdot\rho\theta + t\Omega(\rho))}G(\rho)Y_k(\theta)\rho^{n-1}\frac{d\rho}{(1+\rho^2)^{\frac{s}{2}}}d\theta \\ &= c_{n,k}\chi_1(r)\int_0^\infty e^{it\Omega(\rho)}(r\rho)^{-\frac{n-2}{2}}J_\nu(r\rho)\rho^{n-1}G(\rho)\,\frac{d\rho}{(1+\rho^2)^{\frac{s}{2}}}Y_k(-\omega). \end{aligned}$$

Let us define an operator S by

$$SG(r,t) = c_{n,k}r^{\frac{n-1}{2}}\chi_1(r)\int_0^\infty e^{it\Omega(\rho)}(r\rho)^{-\frac{n-2}{2}}J_\nu(r\rho)\rho^{n-1}G(\rho)\,\frac{d\rho}{(1+\rho^2)^{\frac{s}{2}}}.$$

Let us denote by $||F||_{L^pL^q}$ the mixed norm $\|(\|F(r,t)\|_{L^q(dt)})\|_{L^p(dr)}$. To prove (2.2) it suffices to show that

$$\|S\tilde{G}\|_{L^2L^\infty} \lesssim k^{\frac12 - s}||\tilde{G}||_{L^2}, \tag{2.3}$$

where $\tilde{G}(\rho) = \rho^{\frac{n-1}{2}}G(\rho)$. Now we define the dual operator S^d of S by

$$S^dF(\rho) = \frac{c_{n,k}}{(1+\rho^2)^{\frac{s}{2}}}\int_{\mathbb{R}}\int_0^\infty e^{-it\Omega(\rho)}(r\rho)^{\frac12}J_\nu(r\rho)\chi_1(r)F(r,t)\,drdt$$

for $F \in C_0^\infty(\mathbb{R}_+\times\mathbb{R})$. Then, by duality (2.3) follows from

$$||S^dF||_{L^2} \le Ck^{\frac12 - s}||F||_{L^2L^1}. \tag{2.4}$$

Choose smooth cut-off functions ϕ_0, ϕ_1 and ϕ_3 so that $\phi_0 = 1$ on $\{|s| < \frac12\}$, $\phi_0 = 0$ on $\{|s| > 1\}$, $\phi_1 = 1$ on $\{|s| \sim 1\}$, $\phi_1 = 0$ otherwise, $\phi_2 = 0$ on $\{|s| < 2\}$, $\phi_2 = 1$ on $\{|s| > 3\}$, and $\phi_0 + \phi_1 + \phi_2 = 1$. Then we decompose S^d as

$$S^dF(\rho) = S_0F + S_1F + S_2F,$$

where for $i = 0,1,2$,

$$S_iF(\rho) = \frac{c_{n,k}}{(1+\rho^2)^{\frac{s}{2}}}\int_{\mathbb{R}}\int_0^\infty e^{-it\Omega(\rho)}(r\rho)^{\frac12}J_\nu(r\rho)\phi_i\left(\frac{r\rho}{\nu}\right)\chi_1(r)F(r,t)\,drdt.$$

Now we need to show each S_i satisfies (2.4) in the place of S^d. Each estimate is to be shown using the following asymptotic behavior of Bessel functions:

$$|J_\nu(t)| \le C\exp(-C\nu), \quad \text{if} \quad t \le \frac{\nu}{2}, \tag{2.5}$$

$$\frac{1}{r}\int_0^r |J_\nu(t)|^2t\,dt \le C \quad \text{for all } r > 0, \tag{2.6}$$

$$J_\nu(t)\phi_2(\frac{t}{\nu}) = t^{-\frac12}(b_+e^{it} + b_-e^{-it})\phi_2(\frac{t}{\nu}) + \Phi_\nu(t)\phi_2(\frac{t}{\nu}), \tag{2.7}$$

where $|\Phi_\nu(t)| \le \frac{C}{t}$, $|b_\pm| \le C$ and the constant C is independent of ν. For the proof of (2.5), see [17]. The mean value estimate (2.6) can be found in Section 4.10 of [20]. Invoking the Schläfli's integral representation (see p.176 in [23]):

$$J_\nu(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(t\sin\theta - \nu\theta)}\,d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-\nu\tau - t\sinh\tau}\,d\tau,$$

the last asymptotic behavior follows from the easy estimate

$$\left|\frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-\nu\tau-t\sinh\tau}\,d\tau\right|\le\frac{C}{\nu+t}$$

and the method of stationary phase such that

$$\frac{1}{2\pi}\int_0^{2\pi}e^{i(t\sin\theta-\nu\theta)}\,d\theta\sim(b_+e^{it}+b_-e^{-it})t^{-\frac12}+O(t^{-\frac32})\quad\text{for}\quad t>2\nu.$$

Using (2.5), we now see

$$\begin{aligned}S_0F(\rho)&\lesssim\nu^{\frac12}e^{-C\nu}(1+\rho^2)^{-\frac{s}{2}}\int_0^{\min(\frac{\nu}{\rho},2)}||F(r,\cdot)||_{L^1}\,dr\\&\lesssim\nu^{\frac12}e^{-C\nu}(1+\rho^2)^{-\frac{s}{2}}(\min(\frac{\nu}{\rho},\,2))^{\frac12}||F||_{L^2L^1}.\end{aligned}$$

Thus we have

$$\begin{aligned}||S_0F||_{L^2}&\lesssim\nu^{\frac12}e^{-C\nu}\left(\int_0^\infty(1+\rho^2)^{-s}\min(\frac{\nu}{\rho},\,2)\,d\rho\right)^{\frac12}||F||_{L^2L^1}\\&\lesssim\nu^{\frac12-s}e^{-C\nu}||F||_{L^2L^1}.\end{aligned}\tag{2.8}$$

For S_1, we have

$$|S_1F(\rho)|\lesssim(1+\rho^2)^{-\frac{s}{2}}\left(\int_0^2J_\nu^2(r\rho)r\rho\phi_1^2\left(\frac{r\rho}{\nu}\right)\,dr\right)^{\frac12}||F||_{L^2L^1}.$$

By the change of variable $r\mapsto r/\rho$, the integral in the RHS of the above estimate is bounded by $\frac{1}{\rho}\int_0^{2\rho}J_\nu^2(r)r\phi_1^2(r/\nu)\,dr$. Since $\rho\ge\frac{\nu}{4}$ from the support condition of ϕ_1, by (2.6) we have

$$|S_1F(\rho)|\le C\nu^{\frac12}(1+\rho^2)^{-\frac{s}{2}}\rho^{-\frac12}\chi_{\{\rho\ge\frac{\nu}{4}\}}||F||_{L^2L^1}.$$

We thus obtain

$$||S_1F||_{L^2}\lesssim\nu^{\frac12-s}||F||_{L^2L^1}.\tag{2.9}$$

Now we estimate S_2F. Let us set $S_2F=S_+F+S_-F+S_3F$, where

$$\begin{aligned}S_\pm F(\rho)&=\frac{c_{n,k}b_\pm}{(1+\rho^2)^{\frac{s}{2}}}\int_{\mathbb{R}}\int_0^\infty e^{i(\pm r\rho-t\Omega(\rho))}\phi_2(\frac{r\rho}{\nu})\chi_1(r)F(r,t)\,drdt\\S_3F(\rho)&=\frac{c_{n,k}}{(1+\rho^2)^{\frac{s}{2}}}\int_{\mathbb{R}}\int_0^\infty e^{-it\Omega(\rho)}(r\rho)^{\frac12}\Phi_\nu(r\rho)\phi_2\left(\frac{r\rho}{\nu}\right)\chi_1(r)F(r,t)\,drdt.\end{aligned}$$

For the estimate $S_\pm F$, it suffices to consider S_+F. We decompose it into two parts as follows:

$$S_+F(\rho)=I+II$$

where

$$\begin{aligned}I&=\frac{c_{n,k}b_+}{(1+\rho^2)^{\frac{s}{2}}}\int_{\mathbb{R}}\int_0^\infty e^{i(r\rho-t\Omega(\rho))}\chi_1(r)F(r,t)\,drdt,\\II&=\frac{c_{n,k}b_+}{(1+\rho^2)^{\frac{s}{2}}}\int_{\mathbb{R}}\int_0^\infty e^{i(r\rho-t\Omega(\rho))}(1-\phi(\frac{r\rho}{\nu}))\chi_1(r)F(r,t)\,drdt.\end{aligned}$$

For II, we have

$$|II| \lesssim (1+\rho^2)^{-\frac{s}{2}} \int_0^{\min(\frac{3\nu}{\rho},2)} ||F(r,\cdot)||_{L^1}\,dr$$
$$\lesssim (1+\rho^2)^{-\frac{s}{2}} \left(\min(\frac{3\nu}{\rho},2)\right)^{\frac{1}{2}} ||F||_{L^2L^1}$$

and hence

$$||II||_{L^2} \lesssim \nu^{\frac{1}{2}-s}||F||_{L^2L^1}. \tag{2.10}$$

Now we estimate I. Since F is in $C_0^\infty(\mathbb{R}_+\times\mathbb{R})$, obviously we may assume

$$I = \frac{c_{n,k}b_+}{(1+\rho^2)^{\frac{s}{2}}} \int_{\mathbb{R}^2} e^{i(r\rho - t\Omega(\rho))}\chi_1(|r|)F(r,t)\,drdt.$$

Squaring and integrating I over $\{|\rho|\le N\}$, we have

$$\int_{|\rho|<N} |I|^2\,d\rho \le C||F||_{L^2L^1}^2. \tag{2.11}$$

Since $\frac{1}{4}\le s\le\frac{1}{2}$, it is easy to see

$$\int_{|\rho|>N}|I|^2 d\rho$$
$$\le C\iiiint |K(r-r',t-t')|\chi_1(|r|)|F(r,t)|\chi_1(|r'|)|F(r',t')|\,drdr'dtdt',$$

where

$$K(r,t) = \int_{|\rho|>N} e^{i(r\rho-t\Omega(\rho))}\frac{d\rho}{|\rho|^{\frac{1}{2}}}.$$

For the kernel estimate, we introduce a lemma which shows uniform bound of kernel K on t.

Lemma 2.1. *For any real number $A, B (A\neq 0)$ and $s\in[\frac{1}{2},1)$, there exists a constant C independent of A and B such that*

$$\left|\int_{|\rho|>N} e^{i(A\Omega(\rho)+B\rho)}\frac{d\rho}{|\rho|^s}\right| \le C|B|^{-(1-s)}.$$

Applying Lemma 2.1 with $s=\frac{1}{2}$ and $B=r-r'$, from fractional integration it follows

$$\begin{aligned}
&\int_{|\rho|>N}|I|^2 d\rho\\
&\lesssim \iint |r-r'|^{-\frac{1}{2}}\chi_1(|r|)||F(r,\cdot)||_{L^1}\chi_1(|r'|)||F(r',\cdot)||\,drdr'\\
&\lesssim ||\mathcal{I}_{\frac{1}{2}}(\chi_1||F||_{L^1})||_{L^4}||\chi_1||F||_{L^1}||_{L^{\frac{4}{3}}}\\
&\lesssim ||F||_{L^2L^1}^2,
\end{aligned} \tag{2.12}$$

where $\mathcal{I}_{\frac{1}{2}}$ is the Riesz potential of order $\frac{1}{2}$.

Finally, we estimate S_3F. From the uniform bound of Φ_ν on ν, for small $\varepsilon > 0$, we have

$$\begin{aligned}
|S_3F(\rho)| &\le \frac{C}{(1+\rho^2)^{\frac{s}{2}}} \int (r\rho)^{-\frac12}\phi_2\left(\frac{r\rho}{\nu}\right)\chi_1(r)||F(r,\cdot)||_{L^1}\,dr \\
&\le C\rho^{-s-\frac12}\chi_{\{\rho\ge\nu\}}\int_{\frac{2\nu}{\rho}}^2 r^{-\frac12}||F(r,\cdot)||_{L^1}\,dr \\
&\le C_\varepsilon \nu^{-\varepsilon}\rho^{-s-\frac12+\varepsilon}\chi_{\{\rho\ge\nu\}}\int_{\frac{2\nu}{\rho}}^2 r^{-\frac12+\varepsilon}||F(r,\cdot)||_{L^1}\,dr \\
&\le C_\varepsilon \nu^{-\varepsilon}\rho^{-s-\frac12+\varepsilon}\chi_{\{\rho\ge\nu\}}||F||_{L^2L^1}.
\end{aligned}$$

Choosing ε as $\frac18$, we obtain

$$||S_3F||_{L^2} \lesssim \nu^{-s}||F||_{L^2L^1}. \tag{2.13}$$

Combining all the estimates (2.8) to (2.13) and recalling $\nu = \frac{2k+n-2}{2}$, we get (2.4) and hence Theorem 1.2.

Proof of Lemma 2.1. To prove Lemma 2.1, we need the following (see e.g. [8] and [15])

Lemma 2.2. *Let ψ be a monotone function and $I = \int_\alpha^\beta e^{i\varphi(\rho)}\psi(\rho)\,d\rho$. Then if $|\frac{d\varphi}{d\rho}| \ge \lambda > 0$ in $[\alpha,\beta]$ and $\frac{d\varphi}{d\rho}$ is monotone, $|I| \le C\lambda^{-1}\sup_{[\alpha,\beta]}|\psi(\rho)|$, and if $|\frac{d^2\varphi}{d\rho^2}| \ge \lambda > 0$, then $|I| \le C\lambda^{-\frac12}\sup_{[\alpha,\beta]}|\psi(\rho)|$. The constant C doesn't depend on $\alpha,\beta,\lambda,\varphi$ and ψ.*

We may assume that $N = 0$ because there is no harm to the entire estimates. And by symmetry, we also assume that $A > 0$ and $B > 0$.

(Case $a > 1$**)** Let $D = \frac{B}{A^{\frac1a}}$. Then by the change of variable, we have

$$I = A^{-\frac{1-s}{a}}\int e^{i(A\Omega(A^{-\frac1a}\rho)+D\rho)}|\rho|^{-s}\,d\rho = \int_{\rho<0} + \int_{\rho>0} = I_- + I_+.$$

We have only to consider I_+ and we denote it I again.

Now we first consider the case when $\Omega' > 0$. Observe that

$$E \equiv (A\Omega(A^{-\frac1a}\rho) + D\rho)' \ge c_1\rho^{a-1} + D.$$

Let M be a large positive number depending only on a, s, c_1, c_2. If $D \le M$, then

$$I = A^{-\frac{1-s}{a}}\left(\int_0^1 + \int_1^\infty\right) = I_1 + I_2$$

For I_1, by direct integration, we have $|I_1| \lesssim A^{-\frac{1-s}{a}} \lesssim B^{-(1-s)}$. For I_2, since $E \gtrsim 1$, by the first part of (2.2), we have $|I_2| \lesssim A^{-\frac{1-s}{a}} \lesssim B^{-(1-s)}$. If $D > M$, then since

$E \geq D$, by the first part of Lemma 2.2, we have $|I_2| \lesssim A^{-\frac{1-s}{a}} D^{-1} \leq A^{\frac{s}{a}} B^{-1} \leq B^{-(1-s)}$. For I_1, using the change of variable, we have

$$I_1 = A^{-\frac{1-s}{a}} D^{-(1-s)} \int_0^D e^{i(A\Omega(D^{-1}A^{-\frac{1}{a}}\rho)+\rho)} \rho^{-s} d\rho.$$

Thus $I_1 = \int_0^1 + \int_1^D = I_{1,1} + I_{1,2}$. By the integration, $|I_{1,1}| \lesssim B^{-(1-s)}$. For $I_{1,2}$, since $(A\Omega(D^{-1}A^{-\frac{1}{a}}\rho) + \rho)' \geq 1$, from the first part of Lemma 2.2, we have $|I_{1,2}| \lesssim B^{-(1-s)}$ and hence $|I_1| \lesssim B^{-(1-s)}$.

Now we consider the case when $\Omega' < 0$. We observe that

$$-c_2\rho^{a-1} + D \leq E = (A\Omega(A^{-\frac{1}{a}}\rho) + D\rho)' \leq -c_1\rho^{a-1} + D.$$

If $D \leq M$, then we split I into two parts as follows:

$$I = A^{-\frac{1-s}{a}} \left(\int_0^{(\frac{2M}{c_2})^{\frac{1}{a-1}}} + \int_{(\frac{2M}{c_2})^{\frac{1}{a-1}}}^\infty \right) = I_3 + I_4.$$

For I_5, we have by direct integration $|I_3| \lesssim A^{-\frac{1-s}{a}} \leq B^{-(1-s)}$. For I_4, since $E \leq -c_1\rho^{a-1} + D \leq -1$, by the first part of Lemma 2.2, we get $|I_4| \lesssim A^{-\frac{1-s}{a}} \leq B^{-(1-s)}$. If $D > M$, then we split I into four parts as follows:

$$\begin{aligned} I &= A^{-\frac{1-s}{a}} \left(\int_0^1 + \int_1^{(\frac{D}{2c_2})^{\frac{1}{a-1}}} + \int_{(\frac{D}{2c_2})^{\frac{1}{a-1}}}^{(\frac{2D}{c_1})^{\frac{1}{a-1}}} + \int_{(\frac{2D}{c_1})^{\frac{1}{a-1}}}^\infty \right) \\ &\equiv I_5 + I_6 + I_7 + I_8. \end{aligned} \tag{2.14}$$

For I_5, we use the change of variable so that

$$I_5 = A^{-\frac{1-s}{a}} D^{-(1-s)} \int_0^D e^{i(A\Omega(D^{-1}A^{-\frac{1}{a}}\rho)+\rho)} \rho^{-s}\, d\rho.$$

We split I_5 into two part: $I_5 = A^{-\frac{1-s}{a}} D^{-(1-s)} \left(\int_0^1 + \int_1^D\right) = I_{5,1} + I_{5,2}$. For $I_{5,1}$ and $I_{5,2}$, using the direct integration and the first part of Lemma 2.2 respectively, we have $|I_{5,1}| + |I_{5,2}| \lesssim A^{-\frac{1-s}{a}} D^{-(1-s)} = B^{-(1-s)}$. For I_6, since $E \gtrsim D \geq D^{1-s}$, using the first part of Lemma 2.2, we have $|I_6| \lesssim A^{-\frac{1-s}{a}} D^{-(1-s)} = B^{-(1-s)}$.

To estimate I_7, we use the second derivative $|E'| \sim \rho^{a-2} \sim D^{\frac{a-2}{a-1}}$. Then from the second part of Lemma 2.2, we obtain

$$|I_7| \lesssim A^{-\frac{1-s}{a}} D^{-\frac{a-2}{2(a-1)}} D^{-\frac{s}{a-1}} = A^{-\frac{1-s}{a}} D^{-\frac{a-2+2s}{2(a-1)}}.$$

Since $a > 1$ and $s \geq \frac{1}{2}$, we have $|I_7| \lesssim A^{-\frac{1-s}{a}} D^{-(1-s)} = B^{-(1-s)}$. Finally, we estimate I_8. Since $E \gtrsim D \geq D^{1-s}$, by the first part of Lemma 2.2, we have $|I_8| \lesssim A^{-\frac{1-s}{a}} D^{-(1-s)} D^{-\frac{s}{a-1}} \lesssim B^{-(1-s)}$.

(Case $a < 1$) Let $\widetilde{D} = \frac{A}{B^a}$. Then by the change of variable, we write

$$B^{1-s}\, I = \int e^{i(A\Omega(\frac{\rho}{B})+\rho)} |\rho|^{-s} d\rho = \int_0^\infty + \int_{-\infty}^0 = I_+ + I_-.$$

Similarly to the case $a > 1$, we only consider I_+ and denote it by I again.

In case that $\Omega' > 0$, we have $E \equiv (A\Omega(\frac{\rho}{B}) + \rho)' \ge c_1\widetilde{D}\rho^{a-1} + 1 \ge 1$ for all $\rho > 0$. We divide I into two parts: $I = \int_0^1 + \int_1^\infty$. For the first integral, we just integrate and for the second one, we use the first part of Lemma 2.2. Then we can see $|I| \lesssim 1$.

Now we consider the case when $\Omega' < 0$. Then we can observe that

$$-c_2\widetilde{D}\rho^{a-1} + 1 \le E \le -c_1\widetilde{D}\rho^{a-1} + 1.$$

If $c_2\widetilde{D} < 2$, then we divide I into two parts: $I = \int_0^{(\frac{1}{4})^{\frac{1}{a-1}}} + \int_{(\frac{1}{4})^{\frac{1}{a-1}}}^\infty = I_1 + I_2$. By the integration, we get $|I_1| \lesssim 1$. And since $c_2\widetilde{D} < 2$ and hence $E \gtrsim 1$, by the first part of Lemma 2.2, we have $|I_2| \lesssim 1$.

If $c_1\widetilde{D} > 2$, then we divide I into four parts:

$$I = \int_0^1 + \int_1^{(\frac{2}{c_1\widetilde{D}})^{\frac{1}{a-1}}} + \int_{(\frac{2}{c_1\widetilde{D}})^{\frac{1}{a-1}}}^{(\frac{1}{2c_2\widetilde{D}})^{\frac{1}{a-1}}} + \int_{(\frac{1}{2c_2\widetilde{D}})^{\frac{1}{a-1}}}^\infty = I_3 + I_4 + I_5 + I_6.$$

For I_3, by the integration, $|I_3| \lesssim 1$. For $|I_5|$, since $|E'| \sim \widetilde{D}\widetilde{D}^{-\frac{a-2}{a-1}} = \widetilde{D}^{\frac{1}{a-1}}$ and $s \ge \frac{1}{2}$, by the second part of Lemma 2.2, we have $|I_5| \lesssim \widetilde{D}^{\frac{2s-1}{2(a-1)}} \lesssim 1$. And since $E \lesssim -1$ on $[1, (\frac{2}{c_1\widetilde{D}})^{\frac{1}{a-1}}]$ and $E \gtrsim 1$ on $[(\frac{1}{2c_2\widetilde{D}})^{\frac{1}{a-1}}, \infty)$, we also have $|I_4|, |I_6| \lesssim 1$.

If $\frac{2}{c_2} \le \widetilde{D} \le \frac{2}{c_1}$, choose a large number M depending only on c_1, c_2, and divide I as follows: $I = \int_0^M + \int_M^\infty$. Then as the estimate of I_1 and I_2, we can obtain $|I| \lesssim 1$. This completes the proof of lemma. □

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