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Invariant Subspaces And Hankel Type Operators On A Bergman Space

by

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Abstract. Let $L^2 = L^2(D, r dr d\theta / \pi)$ be the Lebesgue space on the open unit disc D and let $L_a^2 = L^2 \cap Hol(D)$ be a Bergman space on D . In this paper, we are interested in a closed subspace \mathcal{M} of L^2 which is invariant under the multiplication by the coordinate function z , and a Hankel type operator from L_a^2 to \mathcal{M}^\perp . In particular, we study an invariant subspace \mathcal{M} such that there does not exist a finite rank Hankel type operator except a zero operator.

§1. Introduction

Let D be the open unit disc in \mathcal{C} and $Hol(D)$ be the set of all holomorphic functions on D . Let $d\mu = r dr d\theta / \pi$ and $L^2 = L^2(D, d\mu)$ the Lebesgue space. The Bergman space L_a^2 on D is defined by $L_a^2 = L^2 \cap Hol(D)$. Then L_a^2 is the closed subspace of L^2 . When \mathcal{M} is a closed subspace of L^2 and $z\mathcal{M} \subseteq \mathcal{M}$, \mathcal{M} is called an invariant subspace. For φ in $L^\infty = L^\infty(D, d\mu)$, a Hankel type operator is defined by

$$H_\varphi^\mathcal{M} f = (I - P^\mathcal{M})(\varphi f) \quad (f \in L_a^2)$$

where $P^\mathcal{M}$ is the orthogonal projection from L^2 onto \mathcal{M} . When $\mathcal{M} = L_a^2$, $H_\varphi^\mathcal{M}$ is called a big Hankel operator and when $\mathcal{M} = (\bar{z}L_a^2)^\perp$, $H_\varphi^\mathcal{M}$ is called a small Hankel operator. When $L_a^2 \subseteq \mathcal{M} \subseteq (\bar{z}L_a^2)^\perp$, $H_\varphi^\mathcal{M}$ is called an intermediate Hankel operator.

It is easy to see that there does not exist a finite rank big Hankel operator except a zero one (see [5], [6]). On the other hand, there exist a lot of finite rank nonzero small Hankel operators (see [5]). In fact, it is easy to see the results. E. Strouse [7] described completely all finite rank intermediate Hankel operators for some invariant subspace. In the previous paper [5], we began to study finite rank intermediate Hankel operators for arbitrary invariant subspace. In [5, Theorem 3.2], we gave three necessary and sufficient conditions for \mathcal{M} such that does not exist a finite rank intermediate Hankel operator except a zero one. In this paper, without the hypothesis on an invariant subspace \mathcal{M} , we give a new necessary and sufficient condition for \mathcal{M} which have a finite rank Hankel type operator except a zero one.

For an invariant subspace \mathcal{M} in L^2 , $\ker H_\varphi^\mathcal{M}$ denotes the kernel of $H_\varphi^\mathcal{M}$ and then $\ker H_\varphi^\mathcal{M} = \{f \in L_a^2; \varphi f \in \mathcal{M}\}$. Hence $\ker H_\varphi^\mathcal{M}$ is also an invariant subspace in L_a^2 . Thus each invariant subspace \mathcal{M} in L^2 is related to an invariant subspace in L_a^2 by a Hankel type operator. In this paper, the following property of invariant subspaces in L^2 is important.

Definition. Let \mathcal{M} be an invariant subspace of L^2 . \mathcal{M} is called weakly divisible if whenever $f \in \mathcal{M}$ and $|f(z)| \leq \gamma |z - a|$ for some $a \in D$ and some $\gamma \geq 0$ then $f(z) = (z - a)g(z)$ and g is a function in \mathcal{M} .

In Section 2, we generalize a theorem of S. Axler and P. Bourdon [1]. Then they will be used in the latter sections. In Section 3, we show that there does not exist a finite rank Hankel type operator $H_\varphi^\mathcal{M}$ except a zero one if and only if \mathcal{M} is weakly divisible. In Section 4, we give several examples of weakly divisible invariant subspaces.

In this paper $[S]_*$ denotes the weak* closed linear span of a subset S in L^∞ and $[S]_2$ denotes the closed linear span of a subset S in L^2 .

§2. An invariant subspace and the index

In this section, for a given invariant subspace \mathcal{M} we are interested in two invariant subspaces \mathcal{M}' and \mathcal{M}'' such that $\mathcal{M}' \subseteq \mathcal{M} \subseteq \mathcal{M}''$, $\dim \mathcal{M} \ominus \mathcal{M}' < \infty$ and $\dim \mathcal{M}'' \ominus \mathcal{M} < \infty$. Under some conditions on \mathcal{M} , \mathcal{M}' and \mathcal{M}'' , we describe \mathcal{M}' and \mathcal{M}'' using \mathcal{M} . Corollary 3 will be used in Sections 3 and 4. (1) of Corollary 3 is known in [1].

When \mathcal{M} is an invariant subspace of L^2 , for $a \in \mathbb{C}$ put $\text{ind}_a \mathcal{M} = \dim \{ \mathcal{M} \ominus (z-a)\mathcal{M} \}$. $\text{ind}_a \mathcal{M}$ is called index of \mathcal{M} at a . It is known (cf. [1]) that for each n ($0 \leq n \leq \infty$) and for any $a \in D$ there exists an invariant subspace \mathcal{M} with $\text{ind}_a \mathcal{M} = n$.

Theorem 1. Let \mathcal{M} , \mathcal{M}_1 and \mathcal{M}_2 be invariant subspaces of L^2 and $\mathcal{M}_1 \subseteq \mathcal{M}_2$.

(1) $\text{ind}_a \mathcal{M} = 0$ for any $a \notin D$.

(2) If $\dim \mathcal{M}_2 \ominus \mathcal{M}_1 < \infty$ then there exists a polynomial b such that $b\mathcal{M}_2 \subseteq \mathcal{M}_1$, $Z(b) \subset D$ and the degree of $b \leq \dim \mathcal{M}_2 \ominus \mathcal{M}_1$ and $\sum (\text{ind}_a \mathcal{M}_2; a \in Z(b)) \geq \dim \mathcal{M}_2 \ominus \mathcal{M}_1$.

Proof. (1) If $|a| > 1$ then $(z-a)^{-1} \in H^\infty$ and $\mathcal{M} = (z-a)\mathcal{M}$. Hence $\text{ind}_a \mathcal{M} = 0$. If $|a| = 1$ then $(z-a)\mathcal{M} = (z-a)\{z-a(1+\varepsilon)\}^{-1}\mathcal{M}$. For any $f \in \mathcal{M}$, it is easy to see that

$$\int_D \left| \frac{z-a}{z-a(1+\varepsilon)} f - f \right|^2 d\mu \longrightarrow 0 \quad (\varepsilon \rightarrow 0)$$

by a Lebesgue's convergence theorem. This implies that $(z-a)\mathcal{M}$ is dense in \mathcal{M} and so $\text{ind}_a \mathcal{M} = 0$ for $|a| = 1$. (2) Put $\mathcal{N} = \mathcal{M}_2 \ominus \mathcal{M}_1$ and $\mathcal{S}_z = PM_z|_{\mathcal{N}}$ where M_z is a multiplication operator on L^2 by the coordinate function z and P is the orthogonal projection from L^2 to \mathcal{N} . If $n = \dim \mathcal{N} < \infty$, then there exists a polynomial b of degree n such that $\mathcal{S}_b = b(\mathcal{S}_z) = 0$ and so $b\mathcal{M}_2 \subseteq \mathcal{M}_1$. By (1), we may assume that $Z(b) \subset D$. We will prove that $\sum (\text{ind}_a \mathcal{M}_2; a \in Z(b)) \geq n$. We can write that $b = a_0 \prod_{j=1}^n (z-a_j)$ and so

$Z(b) = \{a_1, a_2, \dots, a_n\}$ where $a_0 \in \mathbb{C}$. If $\sum (\text{ind}_a \mathcal{M}_2; a \in Z(b)) \leq n-1$ then we may assume $\text{ind}_{a_1} \mathcal{M}_2 = 0$. Since $[(z-a_1)\mathcal{M}_2]_2 = \mathcal{M}_2$, $\prod_{j=2}^n (z-a_j)\mathcal{M}_2 \subseteq \mathcal{M}_1 \subset \mathcal{M}_2$. Then it is

easy to see that $\dim \mathcal{M}_2 \ominus [\prod_{j=2}^n (z-a_j)\mathcal{M}_2]_2 \leq n-1$ because $\text{ind}_{a_j} \mathcal{M}_2 \leq 1$ for $2 \leq j \leq n$.

This contradicts that $\dim \mathcal{M}_2 \ominus \mathcal{M}_1 = n$. \square

Corollary 1. Let \mathcal{M}_1 and \mathcal{M}_2 be invariant subspaces of L^2 and $\mathcal{M}_1 \subseteq \mathcal{M}_2$.

If $\dim \mathcal{M}_2 \ominus \mathcal{M}_1 = 1$ then $(z-a)\mathcal{M}_2 \subseteq \mathcal{M}_1 \subsetneq \mathcal{M}_2$ for some $a \in D$ and $\text{ind}_a \mathcal{M}_2 \geq 1$. If $\text{ind}_a \mathcal{M}_1 = 1$ or $\text{ind}_a \mathcal{M}_2 = 1$ then $\mathcal{M}_1 = [(z-a)\mathcal{M}_2]_2$.

Proof. By Theorem 1, $(z-a)\mathcal{M}_2 \subseteq \mathcal{M}_1$ for some $a \in D$ and so $\text{ind}_a \mathcal{M}_2 \geq 1$. Since $(z-a)\mathcal{M}_1 \subseteq (z-a)\mathcal{M}_2 \subseteq \mathcal{M}_1 \subsetneq \mathcal{M}_2$, $\mathcal{M}_1 = [(z-a)\mathcal{M}_2]_2$ if $\text{ind}_a \mathcal{M}_1 = 1$ or $\text{ind}_a \mathcal{M}_2 = 1$. \square

Corollary 2. Let \mathcal{M}_1 and \mathcal{M}_2 be invariant subspaces such that $\mathcal{M}_1 \subsetneq \mathcal{M}_2$ and $\dim \mathcal{M}_2 \ominus \mathcal{M}_1 = n < \infty$. Suppose that $(z - a)\mathcal{M}_j$ is closed for any a in D when $j = 1, 2$. If $\text{ind}_a \mathcal{M}_1 = 1$ for any a in D or $\text{ind}_a \mathcal{M}_2 = 1$ for any a in D then $\mathcal{M}_1 = b\mathcal{M}_2$ and $\mathcal{M}_2 = \langle f_1/b, \dots, f_n/b \rangle \oplus \mathcal{M}_1$ where $b = \prod_{j=1}^n (z - a_j)$, $\{a_j\} \subset D$ and $\{f_j\} \subset \mathcal{M}_1$.

Proof. By Theorem 1 there exists a polynomial b such that $b\mathcal{M}_2 \subseteq \mathcal{M}_1$ and $Z(b) \subset D$ and the degree of $b \leq n$. Hence $b = \prod_{j=1}^{\ell} (z - a_j)$ and $\{a_j\} \subset D$ and $\ell \leq n$. When $\text{ind}_a \mathcal{M}_2 = 1$ for any a in D , $\dim \mathcal{M}_2 \ominus b\mathcal{M}_2 = \ell$ because $(z - a_j)\mathcal{M}_2$ is closed for $1 \leq j \leq \ell$ and so $\ell = n$. Hence $\mathcal{M}_1 = b\mathcal{M}_2$. When $\text{ind}_a \mathcal{M}_1 = 1$ for any a in D , $\dim \mathcal{M}_1 \ominus b\mathcal{M}_1 = \ell$ by the same reason. Since $b\mathcal{M}_1 \subseteq b\mathcal{M}_2 \subseteq \mathcal{M}_1$ and $\dim b\mathcal{M}_2 \ominus b\mathcal{M}_1 = n$, $\ell = n$ and so $\mathcal{M}_1 = b\mathcal{M}_2$. Put $\mathcal{M}_2 = \langle \varphi_1, \dots, \varphi_n \rangle \oplus \mathcal{M}_1$ where $\{\varphi_j\}$ are orthogonal to \mathcal{M}_1 . What was just proved above, $b\mathcal{M}_2 = \mathcal{M}_1$ and so $b\mathcal{M}_2 = \langle b\varphi_1, \dots, b\varphi_n \rangle \oplus b\mathcal{M}_1 = \mathcal{M}_1$. Put $f_j = b\varphi_j$ for $j = 1, \dots, n$ then $\{f_j\}$ are in \mathcal{M}_1 and $\mathcal{M}_2 = \langle f_1/b, \dots, f_n/b \rangle \oplus \mathcal{M}_1$. \square

Corollary 3. Let \mathcal{M} be an invariant subspace of L^2 .

(1) If $\dim L_a^2 \ominus \mathcal{M} = n < \infty$ and $n \neq 0$ then $\mathcal{M} = bL_a^2$ where $b = \prod_{j=1}^n (z - a_j)$ and $\{a_j\} \subset D$.

(2) If $\dim \mathcal{M} \ominus L_a^2 = n < \infty$ then $\mathcal{M} = L_a^2$.

Proof. It is known that $\text{ind}_a L_a^2 = 1$ and $(z - a)L_a^2$ is closed for each $a \in D$. Hence we can apply Corollary 2 $\mathcal{M}_1 = L_a^2$ or $\mathcal{M}_2 = L_a^2$. If $\mathcal{M}_1 = \mathcal{M}$ and $\mathcal{M}_2 = L_a^2$ then (1) follows. If $\mathcal{M}_1 = L_a^2$ and $\mathcal{M}_2 = \mathcal{M}$ then $\mathcal{M} = \langle f_1/b, \dots, f_n/b \rangle \oplus L_a^2$ where $b = \prod_{j=1}^n (z - a_j)$, $\{a_j\} \subset D$ and $\{f_j\} \subset L_a^2$. For each $1 \leq \ell \leq n$, $f_\ell/b \in L^2$ and so $f_\ell(a_j) = 0$ for $1 \leq j \leq n$. Then f_ℓ/b belongs to L_a^2 and so $f_\ell/b = 0$ for each ℓ . Thus $\mathcal{M} = L_a^2$ and so (2) follows. \square

§3. Finite rank Hankel type operators

In this section, we study the relation between finite rank Hankel type operators and invariant subspaces.

Theorem 2. Let \mathcal{M} be an invariant subspace of L^2 . Then there does not exist a finite rank Hankel type operator $H_\varphi^{\mathcal{M}}$ except a zero one if and only if \mathcal{M} is weakly divisible.

Proof. Suppose \mathcal{M} is weakly divisible. If $H_\varphi^{\mathcal{M}}$ is of finite rank then $\ker H_\varphi^{\mathcal{M}}$ is an invariant subspace in L_a^2 and $\dim L_a^2 / \ker H_\varphi^{\mathcal{M}} < \infty$. By (1) of Corollary 3, $\ker H_\varphi^{\mathcal{M}} = bL_a^2$ for some polynomial b with $Z(b) \subset D$ and so $b\varphi$ belongs to \mathcal{M} . Put $f = b\varphi$ then $|f(z)| \leq \gamma |$

$b(z) \mid (z \in D)$ where $\gamma = \|\varphi\|_\infty$. Suppose $b(z) = a_0 \prod_{j=1}^n (z - a_j)$ where $\{a_j\} \subset D$. For any ℓ with $1 \leq \ell \leq n$,

$$\left| \frac{f(z)}{z - a_\ell} \right| \leq \gamma |a_0| \prod_{j \neq \ell} |z - a_j| \quad (z \in D)$$

and $f(z)/(z - a_\ell)$ belongs to \mathcal{M} because $a_\ell \in D$ and \mathcal{M} is weakly divisible. Thus $\varphi(z) = f(z)/b(z)$ belongs to \mathcal{M} . Hence $H_\varphi^{\mathcal{M}} = 0$.

Conversely if \mathcal{M} is not weakly divisible then there exists a function f in \mathcal{M} and a point a in D such that $|f(z)| \leq \gamma |z - a|$ ($z \in D$) and $f(z)/(z - a)$ does not belong to \mathcal{M} . Put $\varphi = f(z)/(z - a)$ then $\varphi \in L^\infty$ and $H_\varphi^{\mathcal{M}}$ is not zero because $\varphi \notin \mathcal{M}$. On the other hand, $(z - a)\varphi \in \mathcal{M}$ and so the kernel of $H_\varphi^{\mathcal{M}}$ contains $(z - a)L_a^2$. This implies that $H_\varphi^{\mathcal{M}}$ is of rank one because $L_a^2/(z - a)L_a^2 = \mathcal{C}$. \square

Proposition 1. *If there exists a symbol φ such that $r(H_\varphi^{\mathcal{M}}) = n \geq 1$ then there exists a symbol φ_j such that $r(H_{\varphi_j}^{\mathcal{M}}) = j$ for any j with $0 \leq j \leq n - 1$.*

Proof. Suppose $1 \leq n = r(H_\varphi^{\mathcal{M}}) < \infty$. Then $\ker H_\varphi^{\mathcal{M}}$ = the kernel of $H_\varphi^{\mathcal{M}}$ is an invariant subspace of L_a^2 and $L_a^2/\ker H_\varphi^{\mathcal{M}}$ is of finite dimension n . By Corollary 3, $\ker H_\varphi^{\mathcal{M}} = bL_a^2$ where $b = \prod_{\ell=1}^n (z - a_\ell)$ and $(a_\ell) \subset D$. Hence $b\varphi$ belongs to \mathcal{M} . Put $\varphi_j = \varphi \prod_{\ell=j+1}^n (z - a_\ell)$ for $1 \leq j \leq n - 1$ then $\varphi_j \notin \mathcal{M}$ for $1 \leq j \leq n - 1$ and $\varphi_0 = b\varphi$.

Since $\ker H_{\varphi_j}^{\mathcal{M}} = b_j L_a^2$ for $1 \leq j \leq n - 1$ where $b_j = \prod_{\ell=1}^j (z - a_\ell)$, $H_{\varphi_j}^{\mathcal{M}}$ is of finite rank j for $0 \leq j \leq n - 1$. \square

Corollary 4. *The following (1) and (2) are equivalent for an invariant subspace \mathcal{M} .*

(1) *If $r(H_\varphi^{\mathcal{M}}) < \infty$ then $r(H_\varphi^{\mathcal{M}}) = 0$.*

(2) *If $r(H_\varphi^{\mathcal{M}}) \leq 1$ then $r(H_\varphi^{\mathcal{M}}) = 0$.*

Proof. (1) \Rightarrow (2) is clear. (2) \Rightarrow (1). If (1) is not true then there exists a symbol φ with $r(H_\varphi^{\mathcal{M}}) = n \geq 2$. By Proposition 1 there exists a symbol φ_1 such that $r(H_{\varphi_1}^{\mathcal{M}}) = 1$. This contradicts (2). \square

§4. Weakly divisible invariant subspaces

For a function f in L_a^2 , put $Z(f) = \{a \in D; f(a) = 0\}$ and $Z(G) = \cap\{Z(f); f \in G\}$ for a subset G in L_a^2 . For $1 \leq p \leq \infty$, if E is a open set in D , H_E^p denotes the set of all functions in L^p that are analytic on E . In Corollary 5, a weakly divisible invariant subspace \mathcal{M} is described completely when \mathcal{M} is in L_a^2 . There exists a nonzero invariant subspace \mathcal{M} in L_a^2 such that $\mathcal{M} \cap L^\infty = \langle 0 \rangle$. For it is known (see [4]) that there exists a nonzero function f in L_a^2 such that $Z(f)$ does not satisfy the Blaschke condition.

Theorem 3. *Let \mathcal{M} be an invariant subspace of L^2 .*

- (1) *If $\mathcal{M} \cap L^\infty \subseteq H^\infty$ and $Z(\mathcal{M} \cap L^\infty) = \emptyset$ then \mathcal{M} is weakly divisible.*
- (2) *If $\mathcal{M} \cap L^\infty = H_E^\infty$ for some open set E , then \mathcal{M} is weakly divisible.*
- (3) *If $\mathcal{M} \cap L^\infty = \langle 0 \rangle$ then \mathcal{M} is weakly divisible.*

Proof. (1) If $\{f_n\}$ is a sequence in $\mathcal{M} \cap L^\infty$ which converges pointwise boundedly to f , then $f \in \mathcal{M}$. By the Krein-Schmulian criterion (see[3, IV 2.1]), $\mathcal{M} \cap L^\infty$ is weak-star closed. Hence, by a well known theorem of Beurling [2] $\mathcal{M} \cap L^\infty = qH^\infty$ for some inner function q . Hence if $f \in \mathcal{M}$ and $|f(z)| \leq \gamma |z - a|$ ($z \in D$) for some $a \in D$ then $f = qh$ for some $h \in H^\infty$. Since $Z(\mathcal{M} \cap L^\infty) = \emptyset$, $|q(z)| > 0$ ($z \in D$) and so $h(a) = 0$. Hence $f(z)/(z - a) = q(z) \times (h(z)/(z - a)) \in qH^\infty$. Thus $f(z)/(z - a)$ belongs to \mathcal{M} . (2) If $f \in H_E^\infty$ and $|f(z)| \leq \gamma |z - a|$ ($z \in D$) for some $a \in D$ then $f(z)/(z - a) \in L^\infty$ and $f(z)/(z - a)$ is analytic on E . Hence $f(z)/(z - a)$ belongs to H_E^∞ and so \mathcal{M} is weakly divisible. (3) is clear. \square

Corollary 5. *Let \mathcal{M} be an invariant subspace of L_a^2 . Then \mathcal{M} is weakly divisible if and only if $\mathcal{M} \cap L^\infty = \langle 0 \rangle$ or $Z(\mathcal{M} \cap L^\infty) = \emptyset$.*

Proof. The part of ‘ if ’ is a result of (1) and (3) of Theorem 3. Conversely suppose that \mathcal{M} is weakly divisible. If $\mathcal{M} \cap L^\infty \neq \langle 0 \rangle$ then by a theorem of Beurling there exists an inner function q with $\mathcal{M} \cap L^\infty = qH^\infty$. If $q(a) = 0$ for some $a \in D$ then there exists a finite positive constant γ such that $|q(z)| \leq \gamma |z - a|$ ($z \in D$) and $q/(z - a) \notin \mathcal{M}$. This contradicts the weakly divisibility of \mathcal{M} and so $Z(q) = Z(\mathcal{M} \cap L^\infty) = \emptyset$. \square

Corollary 6. *Let \mathcal{M} be an invariant subspace of L^2 .*

- (1) *If $\mathcal{M} \subsetneq L_a^2$ and $\dim L_a^2 / \mathcal{M} < \infty$ then \mathcal{M} is not weakly divisible.*
- (2) *If $\mathcal{M} \supseteq L_a^2$ and $\dim \mathcal{M} / L_a^2 < \infty$ then \mathcal{M} is weakly divisible.*

Proof. (1) If $\mathcal{M} \subsetneq L_a^2$ and $\dim L_a^2 / \mathcal{M} = \ell < \infty$ then by (1) of Corollary 3 $\mathcal{M} = bL_a^2$ where $b = \prod_{j=1}^{\ell} (z - a_j)$ and $a_j \in D$ ($1 \leq j \leq \ell$). Hence $Z(\mathcal{M} \cap L^\infty) = Z(b) \neq \emptyset$ and so by Corollary 5 \mathcal{M} is not weakly divisible. (2) By (2) of Corollary 3 $\mathcal{M} = L_a^2$ and so $\mathcal{M} \cap L^\infty = H^\infty$.

Hence (1) of Theorem 3 implies that \mathcal{M} is weakly divisible. \square

Corollary 7. *If $\mathcal{M} = H_E^2$ for some open set E in D then \mathcal{M} is weakly divisible.*

Proof. It is a result of (2) of Theorem 3. \square

Proposition 2. *Suppose that \mathcal{M}_j is a weakly divisible invariant subspace of L^2 for $j = 1, 2, \dots$ and $\mathcal{M}_j \times \mathcal{M}_\ell = \{fg; f \in \mathcal{M}_j \text{ and } g \in \mathcal{M}_\ell\} = \langle 0 \rangle$ if $j \neq \ell$. If $\mathcal{M} = \sum_{j=1}^{\infty} \oplus \mathcal{M}_j$ then \mathcal{M} is a weakly divisible invariant subspace.*

Proof. If $f \in \mathcal{M}$ then $f = \sum_{j=1}^{\infty} f_j$ and $|f(z)| = \sum_{j=1}^{\infty} |f_j(z)|$ ($z \in D$) by hypothesis.

This implies that \mathcal{M} is weakly divisible. \square

Corollary 8. *Let $1 \leq \ell \leq \infty$. Suppose D_j is an open set in D with $\mu(\partial D_j) = 0$ for $1 \leq j \leq \ell$, $D_i \cap D_j = \emptyset$ ($i \neq j$) and $D = \cup_{j=1}^{\ell} D_j$. Then $\mathcal{M} = \sum_{j=1}^{\ell} \oplus L_a^2(D_j)$ is weakly divisible.*

Proof. This is a result of Corollary 7 and Proposition 2. \square

Proposition 3. *If \mathcal{M} is a weakly divisible invariant subspace of L^2 and φ is a unimodular function in L^∞ then $\varphi \mathcal{M}$ is a weakly divisible invariant subspace.*

Proof. From the definition of weakly divisibility, the proposition follows trivially.

\square

Corollary 9. *If φ is a unimodular function in L^∞ then φL_a^2 is weakly divisible.*

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