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Invariant Subspaces And Hankel Type Operators
On A Bergman Space

by

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and

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Abstract. Let $L^2 = L^2(D, rdrd\theta / \pi)$ be the Lebesgue space on the open unit disc $D$ and let $L^2_a = L^2 \cap Hol(D)$ be a Bergman space on $D$. In this paper, we are interested in a closed subspace $\mathcal{M}$ of $L^2$ which is invariant under the multiplication by the coordinate function $z$, and a Hankel type operator from $L^2_a$ to $\mathcal{M}^\perp$. In particular, we study an invariant subspace $\mathcal{M}$ such that there does not exist a finite rank Hankel type operator except a zero operator.
§1. Introduction

Let $D$ be the open unit disc in $\mathbb{C}$ and $Hol(D)$ be the set of all holomorphic functions on $D$. Let $d\mu = rd\theta / \pi$ and $L^2 = L^2(D, d\mu)$ the Lebesgue space. The Bergman space $L^2_a$ on $D$ is defined by $L^2_a = L^2 \cap Hol(D)$. Then $L^2_a$ is the closed subspace of $L^2$. When $\mathcal{M}$ is a closed subspace of $L^2$ and $z \mathcal{M} \subseteq \mathcal{M}$, $\mathcal{M}$ is called an invariant subspace. For $\varphi$ in $L^\infty = L^\infty(D, d\mu)$, a Hankel type operator is defined by

$$H_{\varphi}f = (I - P_{\mathcal{M}}) (\varphi f) \quad (f \in L^2_a)$$

where $P_{\mathcal{M}}$ is the orthogonal projection from $L^2$ onto $\mathcal{M}$. When $\mathcal{M} = L^2_a$, $H_{\varphi}$ is called a big Hankel operator and when $\mathcal{M} = (\overline{zL^2_a})^\perp$, $H_{\varphi}$ is called a small Hankel operator. When $L^2_a \subseteq \mathcal{M} \subseteq (\overline{zL^2_a})^\perp$, $H_{\varphi}$ is called an intermediate Hankel operator.

It is easy to see that there does not exist a finite rank big Hankel operator except a zero one (see [5], [6]). On the other hand, there exist a lot of finite rank nonzero small Hankel operators (see [5]). In fact, it is easy to see the results. E. Strouse [7] described completely all finite rank intermediate Hankel operators for some invariant subspace. In the previous paper [5], we began to study finite rank intermediate Hankel operators for arbitrary invariant subspace. In [5, Theorem 3.2], we gave three necessary and sufficient conditions for $\mathcal{M}$ such that does not exist a finite rank intermediate Hankel operator except a zero one. In this paper, without the hypothesis on an invariant subspace $\mathcal{M}$, we give a new necessary and sufficient condition for $\mathcal{M}$ which have a finite rank Hankel type operator except a zero one.

For an invariant subspace $\mathcal{M}$ in $L^2$, $ker H_{\varphi}$ denotes the kernel of $H_{\varphi}$ and then $ker H_{\varphi} = \{ f \in L^2_a; \varphi f \in \mathcal{M} \}$. Hence $ker H_{\varphi}$ is also an invariant subspace in $L^2_a$. Thus each invariant subspace $\mathcal{M}$ in $L^2$ is related to an invariant subspace in $L^2_a$ by a Hankel type operator. In this paper, the following property of invariant subspaces in $L^2$ is important.

**Definition.** Let $\mathcal{M}$ be an invariant subspace of $L^2$. $\mathcal{M}$ is called weakly divisible if whenever $f \in \mathcal{M}$ and $|f(z)| \leq \gamma |z - a|$ for some $a \in D$ and some $\gamma \geq 0$ then $f(z) = (z - a)g(z)$ and $g$ is a function in $\mathcal{M}$.

In Section 2, we generalize a theorem of S. Axler and P. Bourdon [1]. Then they will be used in the latter sections. In Section 3, we show that there does not exist a finite rank Hankel type operator $H_{\varphi}$ except a zero one if and only if $\mathcal{M}$ is weakly divisible. In Section 4, we give several examples of weakly divisible invariant subspaces.

In this paper $[S]$ denotes the weak$^*$ closed linear span of a subset $S$ in $L^\infty$ and $[S]_2$ denotes the closed linear span of a subset $S$ in $L^2$. 

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§2. An invariant subspace and the index

In this section, for a given invariant subspace $\mathcal{M}$ we are interested in two invariant subspaces $\mathcal{M}'$ and $\mathcal{M}''$ such that $\mathcal{M}' \subseteq \mathcal{M} \subseteq \mathcal{M}''$, $\dim \mathcal{M} \ominus \mathcal{M}' < \infty$ and $\dim \mathcal{M}'' \ominus \mathcal{M} < \infty$. Under some conditions on $\mathcal{M}$, $\mathcal{M}'$ and $\mathcal{M}''$, we describe $\mathcal{M}'$ and $\mathcal{M}''$ using $\mathcal{M}$. Corollary 3 will be used in Sections 3 and 4. (1) of Corollary 3 is known in [1].

When $\mathcal{M}$ is an invariant subspace of $L^2$, for $a \in \mathbb{C}$ put $\text{ind}_a \mathcal{M} = \text{dim} \{ \mathcal{M} \ominus (z-a)\mathcal{M} \}$. $\text{ind}_a \mathcal{M}$ is called index of $\mathcal{M}$ at $a$. It is known (cf. [1]) that for each $n (0 \leq n \leq \infty)$ and for any $a (\in D)$ there exists an invariant subspace $\mathcal{M}$ with $\text{ind}_a \mathcal{M} = n$.

Theorem 1. Let $\mathcal{M}$, $\mathcal{M}_1$ and $\mathcal{M}_2$ be invariant subspaces of $L^2$ and $\mathcal{M}_1 \subseteq \mathcal{M}_2$.

(1) $\text{ind}_a \mathcal{M} = 0$ for any $a \notin D$.

(2) If $\dim \mathcal{M}_2 \ominus \mathcal{M}_1 < \infty$ then there exists a polynomial $b$ such that $b\mathcal{M}_2 \subseteq \mathcal{M}_1$, $Z(b) \subseteq D$ and the degree of $b \leq \dim \mathcal{M}_2 \ominus \mathcal{M}_1$ and $\sum (\text{ind}_a \mathcal{M}_2; a \in Z(b)) \geq \dim \mathcal{M}_2 \ominus \mathcal{M}_1$.

Proof. (1) If $|a| > 1$ then $(z-a)^{-1} \in H^\infty$ and $\mathcal{M} = (z-a)\mathcal{M}$. Hence $\text{ind}_a \mathcal{M} = 0$. If $|a| = 1$ then $(z-a)\mathcal{M} = (z-a)\{z-(a+1)\}^{-1}\mathcal{M}$. For any $f \in \mathcal{M}$, it is easy to see that

$$\int_D \left| \frac{z-a}{z-(a+1)} f - f \right|^2 d\mu \longrightarrow 0 \quad (\varepsilon \to 0)$$

by a Lebesgue’s convergence theorem. This implies that $(z-a)\mathcal{M}$ is dense in $\mathcal{M}$ and so $\text{ind}_a \mathcal{M} = 0$ for $|a| = 1$. (2) Put $\mathcal{N} = \mathcal{M}_2 \ominus \mathcal{M}_1$ and $\mathcal{S}_z = PM_z | \mathcal{N}$ where $M_z$ is a multiplication operator on $L^2$ by the coordinate function $z$ and $P$ is the orthogonal projection from $L^2$ to $\mathcal{N}$. If $n = \dim \mathcal{N} < \infty$, then there exists a polynomial $b$ of degree $n$ such that $\mathcal{S}_b = b(\mathcal{S}_z) = 0$ and so $b\mathcal{M}_2 \subseteq \mathcal{M}_1$. By (1), we may assume that $Z(b) \subseteq D$. We will prove that $\sum (\text{ind}_a \mathcal{M}_2; a \in Z(b)) \geq n$. We can write that $b = a_0 \prod_{j=1}^n (z-a_j)$ and so $Z(b) = \{a_1, a_2, \ldots, a_n\}$ where $a_0 \in \mathbb{C}$. If $\sum (\text{ind}_a \mathcal{M}_2; a \in Z(b)) \leq n - 1$ then we may assume $\text{ind}_a \mathcal{M}_2 = 0$. Since $[(z-a)\mathcal{M}_2]_2 = \mathcal{M}_2 \ominus \prod_{j=2}^n (z-a_j)\mathcal{M}_2 \subseteq \mathcal{M}_1 \subset \mathcal{M}_2$. Then it is easy to see that $\dim \mathcal{M}_2 \ominus \prod_{j=2}^n (z-a_j)\mathcal{M}_2 \leq n - 1$ because $\text{ind}_a \mathcal{M}_2 \leq 1$ for $2 \leq j \leq n$. This contradicts that $\dim \mathcal{M}_2 \ominus \mathcal{M}_1 = n$. □

Corollary 1. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be invariant subspaces of $L^2$ and $\mathcal{M}_1 \subseteq \mathcal{M}_2$. If $\dim \mathcal{M}_2 \ominus \mathcal{M}_1 = 1$ then $(z-a)\mathcal{M}_2 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2$ for some $a \in D$ and $\text{ind}_a \mathcal{M}_2 \geq 1$. If $\text{ind}_a \mathcal{M}_1 = 1$ or $\text{ind}_a \mathcal{M}_2 = 1$ then $\mathcal{M}_1 = [(z-a)\mathcal{M}_2]_2$.

Proof. By Theorem 1, $(z-a)\mathcal{M}_2 \subseteq \mathcal{M}_1$ for some $a \in D$ and so $\text{ind}_a \mathcal{M}_2 \geq 1$. Since $(z-a)\mathcal{M}_1 \subseteq (z-a)\mathcal{M}_2 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2$, $1 = [(z-a)\mathcal{M}_2]_2$ if $\text{ind}_a \mathcal{M}_1 = 1$ or $\text{ind}_a \mathcal{M}_2 = 1$. □
Corollary 2. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be invariant subspaces such that $\mathcal{M}_1 \subseteq \mathcal{M}_2$ and $\dim \mathcal{M}_2 \ominus \mathcal{M}_1 = n < \infty$. Suppose that $(z - a)\mathcal{M}_1$ is closed for any $a$ in $D$ when $j = 1, 2$. If $\text{ind}_a \mathcal{M}_1 = 1$ for any $a$ in $D$ or $\text{ind}_a \mathcal{M}_2 = 1$ for any $a$ in $D$ then $\mathcal{M}_1 = b. \mathcal{M}_2$ and $\mathcal{M}_2 = \langle f_1/b, \ldots, f_n/b \rangle \oplus \mathcal{M}_1$ where $b = \prod_{j=1}^{n} (z - a_j), \{a_j\} \subset D$ and $\{f_j\} \subset \mathcal{M}_1$.

Proof. By Theorem 1 there exists a polynomial $b$ such that $b. \mathcal{M}_2 \subseteq \mathcal{M}_1$ and $Z(b) \subset D$ and the degree of $b \leq n$. Hence $b = \prod_{j=1}^{\ell} (z - a_j)$ and $\{a_j\} \subset D$ and $\ell \leq n$. When $\text{ind}_a \mathcal{M}_2 = 1$ for any $a$ in $D$, $\dim \mathcal{M}_2 \ominus b. \mathcal{M}_2 = \ell$ because $(z - a_j). \mathcal{M}_2$ is closed for $1 \leq j \leq \ell$ and so $\ell = n$. Hence $\mathcal{M}_1 = b. \mathcal{M}_2$. When $\text{ind}_a \mathcal{M}_1 = 1$ for any $a$ in $D$, $\dim b. \mathcal{M}_2 \ominus b. \mathcal{M}_1 = \ell$ by the same reason. Since $b. \mathcal{M}_1 \subseteq b. \mathcal{M}_2 \subseteq 1$ and $\dim b. \mathcal{M}_2 \ominus b. \mathcal{M}_1 = n$, $\ell = n$ and so $\mathcal{M}_1 = b. \mathcal{M}_2$. Put $\mathcal{M}_2 = \langle \varphi_1, \ldots, \varphi_n \rangle \oplus \mathcal{M}_1$ where $\{\varphi_j\}$ are orthogonal to $\mathcal{M}_1$. What was just proved above, $b. \mathcal{M}_2 = \mathcal{M}_1$ and so $b. \mathcal{M}_2 = \langle b\varphi_1, \ldots, b\varphi_n \rangle \oplus b. \mathcal{M}_1 = \mathcal{M}_1$. Put $f_j = b\varphi_j$ for $j = 1, \ldots, n$ then $\{f_j\}$ are in $\mathcal{M}_1$ and $\mathcal{M}_2 = \langle f_1/b, \ldots, f_n/b \rangle \oplus \mathcal{M}_1$. □

Corollary 3. Let $\mathcal{M}$ be an invariant subspace of $L^2$.

1. If $\dim L^2_a \ominus \mathcal{M} = n < \infty$ and $n \neq 0$ then $\mathcal{M} = bL^2_a$ where $b = \prod_{j=1}^{n} (z - a_j)$ and $\{a_j\} \subset D$.

2. If $\dim L^2_a \ominus \mathcal{M} = n < \infty$ then $\mathcal{M} = L^2_a$.

Proof. It is known that $\text{ind}_a L^2_a = 1$ and $(z - a)L^2_a$ is closed for each $a$ in $D$. Hence we can apply Corollary 2. $\mathcal{M}_1 = L^2_a$ or $\mathcal{M}_2 = L^2_a$. If $\mathcal{M}_1 = \mathcal{M}$ and $\mathcal{M}_2 = L^2_a$ then (1) follows. If $\mathcal{M}_1 = L^2_a$ and $\mathcal{M}_2 = \mathcal{M}$ then $\mathcal{M} = \langle f_1/b, \ldots, f_n/b \rangle \oplus L^2_a$ where $b = \prod_{j=1}^{n} (z - a_j), \{a_j\} \subset D$ and $\{f_j\} \subset L^2_a$. For each $1 \leq \ell \leq n$, $f_\ell/b \in L^2_a$ and so $f_\ell(a_j) = 0$ for $1 \leq j \leq n$. Then $f_\ell/b$ belongs to $L^2_a$ and so $f_\ell/b = 0$ for each $\ell$. Thus $\mathcal{M} = L^2_a$ and so (2) follows. □

§3. Finite rank Hankel type operators

In this section, we study the relation between finite rank Hankel type operators and invariant subspaces.

Theorem 2. Let $\mathcal{M}$ be an invariant subspace of $L^2$. Then there does not exist a finite rank Hankel type operator $H^\#_\varphi$ except a zero one if and only if $\mathcal{M}$ is weakly divisible.

Proof. Suppose $\mathcal{M}$ is weakly divisible. If $H^\#_\varphi$ is of finite rank then $\ker H^\#_\varphi$ is an invariant subspace in $L^2_a$ and $\dim L^2_a/\ker H^\#_\varphi < \infty$. By (1) of Corollary 3, $\ker H^\#_\varphi = bL^2_a$ for some polynomial $b$ with $Z(b) \subset D$ and so $b\varphi$ belongs to $\mathcal{M}$. Put $f = b\varphi$ then $| f(z) | \leq \gamma |
$b(z) \mid (z \in D)$ where $\gamma = \|\varphi\|_\infty$. Suppose $b(z) = a_0 \prod_{j=1}^n (z - a_j)$ where $\{a_j\} \subset D$. For any $\ell$ with $1 \leq \ell \leq n$, 

$$\left| \frac{f(z)}{z - a_\ell} \right| \leq \gamma a_0 \prod_{j \neq \ell} |z - a_j| \quad (z \in D)$$

and $f(z)/(z - a_\ell)$ belongs to $M$ because $a_\ell \in D$ and $M$ is weakly divisible. Thus $\varphi(z) = f(z)/b(z)$ belongs to $M$. Hence $H_{\varphi}^{,\ell} = 0$.

Conversely if $M$ is not weakly divisible then there exists a function $f$ in $M$ and a point $a$ in $D$ such that $|f(z)| \leq \gamma |z - a|$ $(z \in D)$ and $f(z)/(z - a)$ does not belong to $M$. Put $\varphi = f(z)/(z - a)$ then $\varphi \in L^\infty$ and $H_{\varphi}^{,\ell}$ is not zero because $\varphi \notin M$. On the other hand, $(z - a)\varphi \in M$ and so the kernel of $H_{\varphi}^{,\ell}$ contains $(z - a)L_a^2$. This implies that $H_{\varphi}^{,\ell}$ is of rank one because $L_a^2/(z - a)L_a^2 = \mathcal{C}$. \(\square\)

**Proposition 1.** If there exists a symbol $\varphi$ such that $r\left(H_{\varphi}^{,\ell}\right) = n \geq 1$ then there exists a symbol $\varphi_j$ such that $r\left(H_{\varphi_j}^{,\ell}\right) = j$ for any $j$ with $0 \leq j \leq n - 1$.

**Proof.** Suppose $1 \leq n = r\left(H_{\varphi}^{,\ell}\right) < \infty$. Then $ker H_{\varphi}^{,\ell}$ is the kernel of $H_{\varphi}^{,\ell}$ and $L_a^2/ker H_{\varphi}^{,\ell}$ is of finite dimension $n$. By Corollary 3, $ker H_{\varphi}^{,\ell} = bL_a^2$ where $b = \prod_{\ell=1}^n (z - a_\ell)$ and $(a_\ell) \subset D$. Hence $b\varphi$ belongs to $M$. Put $\varphi_j = \varphi \prod_{\ell=1}^n (z - a_\ell)$ for $1 \leq j \leq n - 1$ then $\varphi_j \notin M$ for $1 \leq j \leq n - 1$ and $\varphi_0 = b\varphi$.

Since $ker H_{\varphi_j}^{,\ell} = b_jL_a^2$ for $1 \leq j \leq n - 1$ where $b_j = \prod_{\ell=1}^n (z - a_\ell)$, $H_{\varphi_j}^{,\ell}$ is of finite rank $j$ for $0 \leq j \leq n - 1$. \(\square\)

**Corollary 4.** The following (1) and (2) are equivalent for an invariant subspace $M$.

1. If $r\left(H_{\varphi}^{,\ell}\right) < \infty$ then $r\left(H_{\varphi}^{,\ell}\right) = 0$.
2. If $r\left(H_{\varphi}^{,\ell}\right) \leq 1$ then $r\left(H_{\varphi}^{,\ell}\right) = 0$.

**Proof.** (1) $\Rightarrow$ (2) is clear. (2) $\Rightarrow$ (1). If (1) is not true then there exists a symbol $\varphi$ with $r\left(H_{\varphi}^{,\ell}\right) = n \geq 2$. By Proposition 1 there exists a symbol $\varphi_1$ such that $r\left(H_{\varphi_1}^{,\ell}\right) = 1$. This contradicts (2). \(\square\)
§4. Weakly divisible invariant subspaces

For a function \( f \) in \( L^2_a \), put \( Z(f) = \{ a \in D : f(a) = 0 \} \) and \( Z(G) = \cap \{ Z(f) : f \in G \} \) for a subset \( G \) in \( L^2_a \). For \( 1 \leq p \leq \infty \), if \( E \) is an open set in \( D \), \( H^p_E \) denotes the set of all functions in \( L^p \) that are analytic on \( E \). In Corollary 5, a weakly divisible invariant subspace \( \mathcal{M} \) is described completely when \( \mathcal{M} \) is in \( L^2_a \). There exists a nonzero invariant subspace \( \mathcal{M} \) in \( L^2_a \) such that \( \mathcal{M} \cap L^\infty = \langle 0 \rangle \). For it is known (see [4]) that there exists a nonzero function \( f \) in \( L^2_a \) such that \( Z(f) \) does not satisfy the Blaschke condition.

**Theorem 3.** Let \( \mathcal{M} \) be an invariant subspace of \( L^2 \).

1. If \( \mathcal{M} \cap L^\infty \subseteq H^\infty \) and \( Z(\mathcal{M} \cap L^\infty) = \emptyset \) then \( \mathcal{M} \) is weakly divisible.
2. If \( \mathcal{M} \cap L^\infty = H^\infty_E \) for some open set \( E \), then \( \mathcal{M} \) is weakly divisible.
3. If \( \mathcal{M} \cap L^\infty = \langle 0 \rangle \) then \( \mathcal{M} \) is weakly divisible.

**Proof.** (1) If \( \{ f_n \} \) is a sequence in \( \mathcal{M} \cap L^\infty \) which converges pointwise boundedly to \( f \), then \( f \in \mathcal{M} \). By the Krein-Schmulian criterion (see [3, IV 2.1]), \( \mathcal{M} \cap L^\infty \) is weak-star closed. Hence, by a well known theorem of Beurling [2] \( \mathcal{M} \cap L^\infty = qH^\infty \) for some inner function \( q \). Hence if \( f \in \mathcal{M} \) and \( | f(z) | \leq \gamma \) \( | z - a | \) \( (z \in D) \) for some \( a \in D \) then \( f = qh \) for some \( h \in H^\infty \). Since \( Z(\mathcal{M} \cap L^\infty) = \emptyset \), \( | q(z) | > 0 \) \( (z \in D) \) and so \( h(a) = 0 \). Hence \( f(z)/(z-a) = q(z) \times \langle h(z)/(z-a) \rangle \in qH^\infty \). Thus \( f(z)/(z-a) \) belongs to \( \mathcal{M} \). (2) If \( f \in H^\infty_E \) and \( | f(z) | \leq \gamma \) \( | z - a | \) \( (z \in D) \) for some \( a \in D \) then \( f(z)/(z-a) \in L^\infty \) and \( f(z)/(z-a) \) is analytic on \( E \). Hence \( f(z)/(z-a) \) belongs to \( H^\infty_E \) and so \( \mathcal{M} \) is weakly divisible. (3) is clear. \( \square \)

**Corollary 5.** Let \( \mathcal{M} \) be an invariant subspace of \( L^2_a \). Then \( \mathcal{M} \) is weakly divisible if and only if \( \mathcal{M} \cap L^\infty = \langle 0 \rangle \) or \( Z(\mathcal{M} \cap L^\infty) = \emptyset \).

**Proof.** The part of ‘ if ’ is a result of (1) and (3) of Theorem 3. Conversely suppose that \( \mathcal{M} \) is weakly divisible. If \( \mathcal{M} \cap L^\infty \neq \langle 0 \rangle \) then by a theorem of Beurling there exists an inner function \( q \) with \( \mathcal{M} \cap L^\infty = qH^\infty \). If \( q(a) = 0 \) for some \( a \in D \) then there exists a finite positive constant \( \gamma \) such that \( | q(z) | \leq \gamma \) \( | z - a | \) \( (z \in D) \) and \( q/(z-a) \notin \mathcal{M} \). This contradicts the weakly divisibility of \( \mathcal{M} \) and so \( Z(q) = Z(\mathcal{M} \cap L^\infty) = \emptyset \). \( \square \)

**Corollary 6.** Let \( \mathcal{M} \) be an invariant subspace of \( L^2 \).

1. If \( \mathcal{M} \subseteq L^2_a \) and \( \dim L_a^2/\mathcal{M} < \infty \) then \( \mathcal{M} \) is not weakly divisible.
2. If \( \mathcal{M} \supseteq L^2_a \) and \( \dim \mathcal{M}/L_a^2 < \infty \) then \( \mathcal{M} \) is weakly divisible.

**Proof.** (1) If \( \mathcal{M} \subseteq L^2_a \) and \( \dim L_a^2/\mathcal{M} = \ell < \infty \) then by (1) of Corollary 3 \( \mathcal{M} = bL^2_a \) where \( b = \prod_{j=1}^{\ell} (z-a_j) \) and \( a_j \in D \) \( (1 \leq j \leq \ell) \). Hence \( Z(\mathcal{M} \cap L^\infty) = Z(b) \neq \emptyset \) and so by Corollary 5 \( \mathcal{M} \) is not weakly divisible. (2) By (2) of Corollary 3 \( \mathcal{M} = L^2_a \) and so \( \mathcal{M} \cap L^\infty = H^\infty \).
Hence (1) of Theorem 3 implies that $\mathcal{M}$ is weakly divisible. □

**Corollary 7.** If $\mathcal{M} = H^2_E$ for some open set $E$ in $D$ then $\mathcal{M}$ is weakly divisible.

**Proof.** It is a result of (2) of Theorem 3. □

**Proposition 2.** Suppose that $\mathcal{M}_j$ is a weakly divisible invariant subspace of $L^2$ for $j = 1, 2, \ldots$ and $\mathcal{M}_j \times \mathcal{M}_\ell = \{ fg; f \in \mathcal{M}_j \text{ and } g \in \mathcal{M}_\ell \} = \{0\}$ if $j \neq \ell$. If $\mathcal{M} = \bigoplus_{j=1}^{\infty} \mathcal{M}_j$ then $\mathcal{M}$ is a weakly divisible invariant subspace.

**Proof.** If $f \in \mathcal{M}$ then $f = \sum_{j=1}^{\infty} f_j$ and $|f(z)| = \sum_{j=1}^{\infty} |f_j(z)|$ $(z \in D)$ by hypothesis. This implies that $\mathcal{M}$ is weakly divisible. □

**Corollary 8.** Let $1 \leq \ell \leq \infty$. Suppose $D_j$ is an open set in $D$ with $\mu(\partial D_j) = 0$ for $1 \leq j \leq \ell$, $D_i \cap D_j = \emptyset$ $(i \neq j)$ and $D = \cup_{j=1}^{\ell} D_j$. Then $\mathcal{M} = \bigoplus_{j=1}^{\ell} L^2_\alpha(D_j)$ is weakly divisible.

**Proof.** This is a result of Corollary 7 and Proposition 2. □

**Proposition 3.** If $\mathcal{M}$ is a weakly divisible invariant subspace of $L^2$ and $\varphi$ is a unimodular function in $L^\infty$ then $\varphi \mathcal{M}$ is a weakly divisible invariant subspace.

**Proof.** From the definition of weakly divisibility, the proposition follows trivially. □

**Corollary 9.** If $\varphi$ is a unimodular function in $L^\infty$ then $\varphi L^2_\alpha$ is weakly divisible.

**References**


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