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Invariant Subspaces And Hankel Type Operators
On A Bergman Space

by

Takahiko Nakazi*

and

Tomoko Osawa

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Abstract. Let $L^2 = L^2(D, rdrd\theta / \pi)$ be the Lebesgue space on the open unit disc $D$ and let $L^2_a = L^2 \cap \text{Hol}(D)$ be a Bergman space on $D$. In this paper, we are interested in a closed subspace $\mathcal{M}$ of $L^2$ which is invariant under the multiplication by the coordinate function $z$, and a Hankel type operator from $L^2_a$ to $\mathcal{M}^\perp$. In particular, we study an invariant subspace $\mathcal{M}$ such that there does not exist a finite rank Hankel type operator except a zero operator.
§1. Introduction

Let $D$ be the open unit disc in $\mathbb{C}$ and $Hol(D)$ be the set of all holomorphic functions on $D$. Let $d\mu = rdrd\theta / \pi$ and $L^2 = L^2(D,d\mu)$ the Lebesgue space. The Bergman space $L^2_a$ on $D$ is defined by $L^2_a = L^2 \cap Hol(D)$. Then $L^2_a$ is the closed subspace of $L^2$. When $\mathcal{M}$ is a closed subspace of $L^2$ and $z\mathcal{M} \subseteq \mathcal{M}$, $\mathcal{M}$ is called an invariant subspace. For $\varphi$ in $L^\infty = L^\infty(D,d\mu)$, a Hankel type operator is defined by

$$H_\varphi f = (I - P_\varphi)(\varphi f) \quad (f \in L^2_a)$$

where $P_\varphi$ is the orthogonal projection from $L^2$ onto $\mathcal{M}$. When $\mathcal{M} = L^2_a$, $H_\varphi$ is called a big Hankel operator and when $\mathcal{M} = \left(\overline{zL^2_a}\right)^\perp$, $H_\varphi$ is called a small Hankel operator. When $L^2_a \subseteq \mathcal{M} \subseteq \left(\overline{zL^2_a}\right)^\perp$, $H_\varphi$ is called an intermediate Hankel operator.

It is easy to see that there does not exist a finite rank big Hankel operator except a zero one (see [5], [6]). On the other hand, there exist a lot of finite rank nonzero small Hankel operators (see [5]). In fact, it is easy to see the results. E. Strouse [7] described completely all finite rank intermediate Hankel operators for some invariant subspace. In the previous paper [5], we began to study finite rank intermediate Hankel operators for arbitrary invariant subspace. In [5, Theorem 3.2], we gave three necessary and sufficient conditions for $\mathcal{M}$ such that does not exist a finite rank intermediate Hankel operator except a zero one. In this paper, without the hypothesis on an invariant subspace $\mathcal{M}$, we give a new necessary and sufficient condition for $\mathcal{M}$ which have a finite rank Hankel type operator except a zero one.

For an invariant subspace $\mathcal{M}$ in $L^2$, $\ker H_\varphi$ denotes the kernel of $H_\varphi$ and then $\ker H_\varphi = \{ f \in L^2_a; \varphi f \in \mathcal{M} \}$. Hence $\ker H_\varphi$ is also an invariant subspace in $L^2_a$. Thus each invariant subspace $\mathcal{M}$ in $L^2$ is related to an invariant subspace in $L^2_a$ by a Hankel type operator. In this paper, the following property of invariant subspaces in $L^2$ is important.

**Definition.** Let $\mathcal{M}$ be an invariant subspace of $L^2$. $\mathcal{M}$ is called weakly divisible if whenever $f \in \mathcal{M}$ and $|f(z)| \leq \gamma \ |z - a|$ for some $a \in D$ and some $\gamma \geq 0$ then $f(z) = (z - a)g(z)$ and $g$ is a function in $\mathcal{M}$.

In Section 2, we generalize a theorem of S. Axler and P. Bourdon [1]. Then they will be used in the latter sections. In Section 3, we show that there does not exist a finite rank Hankel type operator $H_\varphi$ except a zero one if and only if $\mathcal{M}$ is weakly divisible. In Section 4, we give several examples of weakly divisible invariant subspaces.

In this paper $[S]_2$ denotes the weak* closed linear span of a subset $S$ in $L^\infty$ and $[S]_2$ denotes the closed linear span of a subset $S$ in $L^2$.  

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\section{An invariant subspace and the index}

In this section, for a given invariant subspace \( \mathcal{M} \) we are interested in two invariant subspaces \( \mathcal{M}' \) and \( \mathcal{M}'' \) such that \( \mathcal{M}' \subseteq \mathcal{M} \subseteq \mathcal{M}'' \), \( \dim \mathcal{M} \cap \mathcal{M}' < \infty \) and \( \dim \mathcal{M}'' \cap \mathcal{M} < \infty \). Under some conditions on \( \mathcal{M}, \mathcal{M}' \) and \( \mathcal{M}'' \), we describe \( \mathcal{M}' \) and \( \mathcal{M}'' \) using \( \mathcal{M} \). Corollary 3 will be used in Sections 3 and 4. (1) of Corollary 3 is known in [1].

When \( \mathcal{M} \) is an invariant subspace of \( L^2 \), for \( a \in \mathbb{C} \) put \( \text{ind}_a \mathcal{M} = \dim \{ \mathcal{M} \cap (z-a) \mathcal{M} \} \). \( \text{ind}_a \mathcal{M} \) is called index of \( \mathcal{M} \) at \( a \). It is known (cf. [1]) that for each \( n (0 \leq n \leq \infty) \) and for any \( a (\in D) \) there exists an invariant subspace \( \mathcal{M} \) with \( \text{ind}_a \mathcal{M} = n \).

**Theorem 1.** Let \( \mathcal{M}, \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be invariant subspaces of \( L^2 \) and \( \mathcal{M}_1 \subseteq \mathcal{M}_2 \).

1. \( \text{ind}_a \mathcal{M} = 0 \) for any \( a \notin D \).
2. If \( \dim \mathcal{M}_2 \cap \mathcal{M}_1 < \infty \) then there exists a polynomial \( b \) such that \( b \mathcal{M}_2 \subseteq \mathcal{M}_1 \), \( Z(b) \subseteq D \) and the degree of \( b \leq \dim \mathcal{M}_2 \cap \mathcal{M}_1 \) and \( \sum (\text{ind}_a \mathcal{M}_2 ; a \in Z(b)) \geq \dim \mathcal{M}_2 \cap \mathcal{M}_1 \).

**Proof.** (1) If \( |a| > 1 \) then \( (z-a)^{-1} \in H^\infty \) and \( \mathcal{M} = (z-a)^{-1} \mathcal{M} \). Hence \( \text{ind}_a \mathcal{M} = 0 \). If \( |a| = 1 \) then \( (z-a) \mathcal{M} = (z-a) \{ z-a(1+\varepsilon)^{-1} \} \mathcal{M} \). For any \( f \in \mathcal{M} \), it is easy to see that

\[
\int_D \left| \frac{z-a}{z-a(1+\varepsilon)} f - f \right|^2 \, d\mu \to 0 \quad (\varepsilon \to 0)
\]

by a Lebesgue’s convergence theorem. This implies that \( (z-a) \mathcal{M} \) is dense in \( \mathcal{M} \) and so \( \text{ind}_a \mathcal{M} = 0 \) for \( |a| = 1 \). (2) Put \( \mathcal{N} = \mathcal{M}_2 \cap \mathcal{M}_1 \) and \( \mathcal{I} = PM_z | \mathcal{N} \) where \( M_z \) is a multiplication operator on \( L^2 \) by the coordinate function \( z \) and \( P \) is the orthogonal projection from \( L^2 \) to \( \mathcal{N} \). If \( n = \dim \mathcal{N} < \infty \), then there exists a polynomial \( b \) of degree \( n \) such that \( \mathcal{I} b = b \mathcal{I} \mathcal{N} = 0 \) and so \( b \mathcal{M}_2 \subseteq \mathcal{M}_1 \). By (1), we may assume that \( Z(b) \subseteq D \). We will prove that \( \sum (\text{ind}_a \mathcal{M}_2 ; a \in Z(b)) \geq n \). We can write that \( b = a_0 \prod_{j=1}^n (z-a_j) \) and so \( Z(b) = \{ a_1, a_2, \ldots, a_n \} \) where \( a_0 \in \mathbb{C} \). If \( \sum (\text{ind}_a \mathcal{M}_2 ; a \in Z(b)) \leq n - 1 \) then we may assume \( \text{ind}_a \mathcal{M}_2 = 0 \). Since \( (z-a) \mathcal{M}_2 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \mathcal{M}_2 \subseteq \mathcal{M}_2 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2 \). Then it is easy to see that \( \dim \mathcal{M}_2 \cap \prod_{j=2}^n (z-a_j) \mathcal{M}_2 \leq n - 1 \) because \( \text{ind}_a \mathcal{M}_2 \leq 1 \) for \( 2 \leq j \leq n \).

This contradicts that \( \dim \mathcal{M}_2 \cap \mathcal{M}_1 = n \). \( \square \)

**Corollary 1.** Let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be invariant subspaces of \( L^2 \) and \( \mathcal{M}_1 \subseteq \mathcal{M}_2 \). If \( \dim \mathcal{M}_2 \cap \mathcal{M}_1 = 1 \) then \( (z-a) \mathcal{M}_2 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2 \) for some \( a \in D \) and \( \text{ind}_a \mathcal{M}_2 \geq 1 \). If \( \text{ind}_a \mathcal{M}_1 = 1 \) or \( \text{ind}_a \mathcal{M}_2 = 1 \) then \( \mathcal{M}_1 = [(z-a) \mathcal{M}_2]_2 \).

**Proof.** By Theorem 1, \( (z-a) \mathcal{M}_2 \subseteq \mathcal{M}_1 \) for some \( a \in D \) and so \( \text{ind}_a \mathcal{M}_2 \geq 1 \). Since \( (z-a) \mathcal{M}_1 \subseteq (z-a) \mathcal{M}_2 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2, \mathcal{M}_1 = [(z-a) \mathcal{M}_2]_2 \) if \( \text{ind}_a \mathcal{M}_1 = 1 \) or \( \text{ind}_a \mathcal{M}_2 = 1 \). \( \square \)
Corollary 2. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be invariant subspaces such that $\mathcal{M}_1 \subseteq \mathcal{M}_2$ and $\dim \mathcal{M}_2 \ominus \mathcal{M}_1 = n < \infty$. Suppose that $(z-a)\mathcal{M}_j$ is closed for any $a \in D$ when $j = 1, 2$. If $\text{ind}_a \mathcal{M}_1 = 1$ for any $a \in D$ or $\text{ind}_a \mathcal{M}_2 = 1$ for any $a \in D$ then $\mathcal{M}_1 = b_1 \mathcal{M}_2$ and $\mathcal{M}_2 = \langle f_1/b, \ldots, f_n/b \rangle \oplus \mathcal{M}_1$ where $b = \prod_{j=1}^{n}(z-a_j), \{a_j\} \subset D$ and $\{f_j\} \subset \mathcal{M}_1$.

Proof. By Theorem 1 there exists a polynomial $b$ such that $b \mathcal{M}_2 \subseteq \mathcal{M}_1$ and $Z(b) \subset D$ and the degree of $b \leq n$. Hence $b = \prod_{j=1}^{\ell}(z-a_j)$ and $\{a_j\} \subset D$ and $\ell \leq n$. When $\text{ind}_a \mathcal{M}_2 = 1$ for any $a \in D$, $\dim \mathcal{M}_2 \ominus b \mathcal{M}_2 = \ell$ because $(z-a)\mathcal{M}_2$ is closed for $1 \leq j \leq \ell$ and so $\ell = n$. Hence $\mathcal{M}_1 = b \mathcal{M}_2$. When $\text{ind}_a \mathcal{M}_1 = 1$ for any $a \in D$, $\dim \mathcal{M}_1 \ominus b \mathcal{M}_1 = \ell$ by the same reason. Since $b \mathcal{M}_1 \subseteq b \mathcal{M}_2 \subseteq \mathcal{M}_1$ and $\dim b \mathcal{M}_2 \ominus b \mathcal{M}_1 = n$, $\ell = n$ and so $\mathcal{M}_1 = b \mathcal{M}_2$. Put $\mathcal{M}_2 = \langle \varphi_1, \ldots, \varphi_n \rangle \oplus \mathcal{M}_1$ where $\{\varphi_j\}$ are orthogonal to $\mathcal{M}_1$. What was just proved above, $b \mathcal{M}_2 = \mathcal{M}_1$ and so $b \mathcal{M}_2 = \langle b \varphi_1, \ldots, b \varphi_n \rangle \oplus b \mathcal{M}_1 = \mathcal{M}_1$. Put $f_j = b \varphi_j$ for $j = 1, \ldots, n$ then $\{f_j\}$ are in $\mathcal{M}_1$ and $\mathcal{M}_2 = \langle f_1/b, \ldots, f_n/b \rangle \oplus \mathcal{M}_1$. □

Corollary 3. Let $\mathcal{M}$ be an invariant subspace of $L^2$.

(1) If $\dim L^2_a \ominus \mathcal{M} = n < \infty$ and $n \not= 0$ then $\mathcal{M} = bL^2_a$ where $b = \prod_{j=1}^{n}(z-a_j)$ and $\{a_j\} \subset D$.

(2) If $\dim \mathcal{M} \ominus L^2_a = n < \infty$ then $\mathcal{M} = L^2_a$.

Proof. It is known that $\text{ind}_a L^2_a = 1$ and $(z-a)L^2_a$ is closed for each $a \in D$. Hence we can apply Corollary 2. $\mathcal{M}_1 = L^2_a$ or $\mathcal{M}_2 = L^2_a$. If $\mathcal{M}_1 = \mathcal{M}$ and $\mathcal{M}_2 = L^2_a$ then (1) follows. If $\mathcal{M}_1 = L^2_a$ and $\mathcal{M}_2 = \mathcal{M}$ then $\mathcal{M} = \langle f_1/b, \ldots, f_n/b \rangle \oplus L^2_a$ where $b = \prod_{j=1}^{n}(z-a_j), \{a_j\} \subset D$ and $\{f_j\} \subset L^2_a$. For each $1 \leq \ell \leq n$, $f_\ell/b \in L^2_a$ and so $f_\ell(a_j) = 0$ for $1 \leq j \leq n$. Then $f_\ell/b$ belongs to $L^2_a$ and so $f_\ell/b = 0$ for each $\ell$. Thus $\mathcal{M} = L^2_a$ and so (2) follows. □

§3. Finite rank Hankel type operators

In this section, we study the relation between finite rank Hankel type operators and invariant subspaces.

Theorem 2. Let $\mathcal{M}$ be an invariant subspace of $L^2$. Then there does not exist a finite rank Hankel type operator $H^\#_\varphi$ except a zero one if and only if $\mathcal{M}$ is weakly divisible.

Proof. Suppose $\mathcal{M}$ is weakly divisible. If $H^\#_\varphi$ is of finite rank then $\ker H^\#_\varphi$ is an invariant subspace in $L^2_a$ and $\dim L^2_a/\ker H^\#_\varphi = \infty$. By (1) of Corollary 3, $\ker H^\#_\varphi = bL^2_a$ for some polynomial $b$ with $Z(b) \subset D$ and so $b \varphi$ belongs to $\mathcal{M}$. Put $f = b \varphi$ then $|f(z)| \leq \gamma$.
b(z) \mid (z \in D)\text{ where } \gamma = \|\varphi\|_\infty. \text{ Suppose } b(z) = a_0 \prod_{j=1}^{n}(z - a_j) \text{ where } \{a_j\} \subset D. \text{ For any } \ell \text{ with } 1 \leq \ell \leq n, \\
\left| \frac{f(z)}{z - a_\ell} \right| \leq \gamma |a_0| \prod_{j \neq \ell} |z - a_j| \quad (z \in D)
\text{ and } f(z)/(z - a_\ell) \text{ belongs to } \mathcal{M} \text{ because } a_\ell \in D \text{ and } \mathcal{M} \text{ is weakly divisible. Thus } \varphi(z) = f(z)/b(z) \text{ belongs to } \mathcal{M}. \text{ Hence } H_{\varphi}^{\#} = 0.

Conversely if \mathcal{M} \text{ is not weakly divisible then there exists a function } f \text{ in } \mathcal{M} \text{ and a point } a \in D \text{ such that } |f(z)| \leq \gamma |z - a| \quad (z \in D) \text{ and } f(z)/(z - a) \text{ does not belong to } \mathcal{M}. \text{ Put } \varphi = f(z)/(z - a) \text{ then } \varphi \in L^\infty \text{ and } H_{\varphi}^{\#} \text{ is not zero because } \varphi \notin \mathcal{M}. \text{ On the other hand, } (z - a)\varphi \in \mathcal{M} \text{ and so the kernel of } H_{\varphi}^{\#} \text{ contains } (z - a)L_a^2. \text{ This implies that } H_{\varphi}^{\#} \text{ is of rank one because } L_a^2/(z - a)L_a^2 = \mathcal{C}. \quad \square

**Proposition 1.** If there exists a symbol \varphi \text{ such that } r\left(H_{\varphi}^{\#}\right) = n \geq 1 \text{ then there exists a symbol } \varphi_j \text{ such that } r\left(H_{\varphi_j}^{\#}\right) = j \text{ for any } j \text{ with } 0 \leq j \leq n - 1.

**Proof.** Suppose 1 \leq n = r\left(H_{\varphi}^{\#}\right) < \infty. \text{ Then } \ker H_{\varphi}^{\#} = \text{ the kernel of } H_{\varphi}^{\#} \text{ is an invariant subspace of } L_a^2 \text{ and } L_a^2/\ker H_{\varphi}^{\#} \text{ is of finite dimension } n. \text{ By Corollary 3, } \ker H_{\varphi}^{\#} = bL_a^2 \text{ where } b = \prod_{\ell=1}^{n}(z - a_\ell) \text{ and } (a_\ell) \subset D. \text{ Hence } b\varphi \text{ belongs to } \mathcal{M}. \text{ Put } \\
\varphi_j = \varphi \prod_{\ell=1}^{n}(z - a_\ell) \text{ for } 1 \leq j \leq n - 1 \text{ then } \varphi_j \notin \mathcal{M} \text{ for } 1 \leq j \leq n - 1 \text{ and } \varphi_0 = b\varphi.

Since \ker H_{\varphi_j}^{\#} = b_jL_a^2 \text{ for } 1 \leq j \leq n - 1 \text{ where } b_j = \prod_{\ell=1}^{j}(z - a_\ell), \text{ } H_{\varphi_j}^{\#} \text{ is of finite rank } j \text{ for } 0 \leq j \leq n - 1. \quad \square

**Corollary 4.** The following (1) and (2) are equivalent for an invariant subspace \mathcal{M}.

1. If r\left(H_{\varphi}^{\#}\right) < \infty \text{ then } r\left(H_{\varphi}^{\#}\right) = 0.
2. If r\left(H_{\varphi}^{\#}\right) \leq 1 \text{ then } r\left(H_{\varphi}^{\#}\right) = 0.

**Proof.** (1) \Rightarrow (2) \text{ is clear. } (2) \Rightarrow (1). \text{ If } (1) \text{ is not true then there exists a symbol } \varphi \text{ with } r\left(H_{\varphi}^{\#}\right) = n \geq 2. \text{ By Proposition 1 there exists a symbol } \varphi_1 \text{ such that } r\left(H_{\varphi_1}^{\#}\right) = 1. \text{ This contradicts } (2). \quad \square
§4. Weakly divisible invariant subspaces

For a function \( f \) in \( L^p_\alpha \), put \( Z(f) = \{ a \in D : f(a) = 0 \} \) and \( Z(G) = \cap \{ Z(f) : f \in G \} \) for a subset \( G \) in \( L^p_\alpha \). For \( 1 \leq p \leq \infty \), if \( E \) is an open set in \( D \), \( H^p_E \) denotes the set of all functions in \( L^p \) that are analytic on \( E \). In Corollary 5, a weakly divisible invariant subspace \( \mathcal{M} \) is described completely when \( \mathcal{M} \) is in \( L^2_\alpha \). There exists a nonzero invariant subspace \( \mathcal{M} \) in \( L^2_\alpha \) such that \( \mathcal{M} \cap L^\infty = \{ 0 \} \). For it is known (see [4]) that there exists a nonzero function \( f \) in \( L^2_\alpha \) such that \( Z(f) \) does not satisfy the Blaschke condition.

**Theorem 3.** Let \( \mathcal{M} \) be an invariant subspace of \( L^2 \).

1. If \( \mathcal{M} \cap L^\infty \subseteq H^\infty \) and \( Z(\mathcal{M} \cap L^\infty) = \emptyset \) then \( \mathcal{M} \) is weakly divisible.
2. If \( \mathcal{M} \cap L^\infty = H^\infty_E \) for some open set \( E \), then \( \mathcal{M} \) is weakly divisible.
3. If \( \mathcal{M} \cap L^\infty = \{ 0 \} \) then \( \mathcal{M} \) is weakly divisible.

**Proof.** (1) If \( \{ f_n \} \) is a sequence in \( \mathcal{M} \cap L^\infty \) which converges pointwise boundedly to \( f \), then \( f \in \mathcal{M} \). By the Krein-Schmulian criterion (see [3, IV 2.1]), \( \mathcal{M} \cap L^\infty \) is weak-star closed. Hence, by a well known theorem of Beurling [2] \( \mathcal{M} \cap L^\infty = qH^\infty \) for some inner function \( q \). Hence if \( f \in \mathcal{M} \) and \( | f(z) | \leq \gamma | z - a | (z \in D) \) for some \( a \in D \) then \( f = qh \) for some \( h \in H^\infty \). Since \( Z(\mathcal{M} \cap L^\infty) = \emptyset \), \( | q(z) | > 0 \) (z \in D) and so \( h(a) = 0 \). Hence \( f(z)/(z-a) = q(z)/(h(z)/(z-a)) \in qH^\infty \). Thus \( f(z)/(z-a) \) belongs to \( \mathcal{M} \). (2) If \( f \in H^\infty_E \) and \( | f(z) | \leq \gamma | z - a | (z \in D) \) for some \( a \in D \) then \( f(z)/(z-a) \in L^\infty \) and \( f(z)/(z-a) \) is analytic on \( E \). Hence \( f(z)/(z-a) \) belongs to \( H^\infty_E \) and so \( \mathcal{M} \) is weakly divisible. (3) is clear.  

**Corollary 5.** Let \( \mathcal{M} \) be an invariant subspace of \( L^2_\alpha \). Then \( \mathcal{M} \) is weakly divisible if and only if \( \mathcal{M} \cap L^\infty = \{ 0 \} \) or \( Z(\mathcal{M} \cap L^\infty) = \emptyset \).

**Proof.** The part of ‘ if ’ is a result of (1) and (3) of Theorem 3. Conversely suppose that \( \mathcal{M} \) is weakly divisible. If \( \mathcal{M} \cap L^\infty \neq \{ 0 \} \) then by a theorem of Beurling there exists an inner function \( q \) with \( \mathcal{M} \cap L^\infty = qH^\infty \). If \( q(a) = 0 \) for some \( a \in D \) then there exists a finite positive constant \( \gamma \) such that \( | q(z) | \leq \gamma | z - a | (z \in D) \) and \( q/(z-a) \notin \mathcal{M} \). This contradicts the weakly divisibility of \( \mathcal{M} \) and so \( Z(q) = Z(\mathcal{M} \cap L^\infty) = \emptyset \).  

**Corollary 6.** Let \( \mathcal{M} \) be an invariant subspace of \( L^2 \).

1. If \( \mathcal{M} \subseteq L^2_\alpha \) and \( \dim L^2_\alpha / \mathcal{M} < \infty \) then \( \mathcal{M} \) is not weakly divisible.
2. If \( \mathcal{M} \supseteq L^2_\alpha \) and \( \dim \mathcal{M} / L^2_\alpha < \infty \) then \( \mathcal{M} \) is weakly divisible.

**Proof.** (1) If \( \mathcal{M} \subseteq L^2_\alpha \) and \( \dim L^2_\alpha / \mathcal{M} = \ell < \infty \) then by (1) of Corollary 3 \( \mathcal{M} = bL^2_\alpha \) where \( b = \prod_{j=1}^{\ell} (z-a_j) \) and \( a_j \in D \) (\( 1 \leq j \leq \ell \)). Hence \( Z(\mathcal{M} \cap L^\infty) = Z(b) \neq \emptyset \) and so by Corollary 5 \( \mathcal{M} \) is not weakly divisible. (2) By (2) of Corollary 3 \( \mathcal{M} = L^2_\alpha \) and so \( \mathcal{M} \cap L^\infty = H^\infty \).
Hence (1) of Theorem 3 implies that $\mathcal{M}$ is weakly divisible. □

**Corollary 7.** If $\mathcal{M} = H_E^2$ for some open set $E$ in $D$ then $\mathcal{M}$ is weakly divisible.

**Proof.** It is a result of (2) of Theorem 3. □

**Proposition 2.** Suppose that $\mathcal{M}_j$ is a weakly divisible invariant subspace of $L^2$ for $j = 1, 2, \ldots$ and $\mathcal{M}_j \times \mathcal{M}_\ell = \{fg; f \in \mathcal{M}_j \text{ and } g \in \mathcal{M}_\ell\} = \{0\}$ if $j \neq \ell$. If $\mathcal{M} = \sum_{j=1}^{\infty} \oplus \mathcal{M}_j$ then $\mathcal{M}$ is a weakly divisible invariant subspace.

**Proof.** If $f \in \mathcal{M}$ then $f = \sum_{j=1}^{\infty} f_j$ and $|f(z)| = \sum_{j=1}^{\infty} |f_j(z)|$ ($z \in D$) by hypothesis. This implies that $\mathcal{M}$ is weakly divisible. □

**Corollary 8.** Let $1 \leq \ell \leq \infty$. Suppose $D_j$ is an open set in $D$ with $\mu(\partial D_j) = 0$ for $1 \leq j \leq \ell$, $D_i \cap D_j = \emptyset$ ($i \neq j$) and $D = \bigcup_{j=1}^{\ell} D_j$. Then $\mathcal{M} = \sum_{j=1}^{\ell} \oplus L_a^2(D_j)$ is weakly divisible.

**Proof.** This is a result of Corollary 7 and Proposition 2. □

**Proposition 3.** If $\mathcal{M}$ is a weakly divisible invariant subspace of $L^2$ and $\varphi$ is a unimodular function in $L^\infty$ then $\varphi \mathcal{M}$ is a weakly divisible invariant subspace.

**Proof.** From the definition of weakly divisibility, the proposition follows trivially. □

**Corollary 9.** If $\varphi$ is a unimodular function in $L^\infty$ then $\varphi L_a^2$ is weakly divisible.

**References**


Takahiko Nakazi
Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan
E-mail : nakazi@math.sci.hokudai.ac.jp

Tomoko Osawa
Mathematical and Scientific Subjects
Asahikawa National College of Technology
Asahikawa 071-8142, Japan
E-mail : ohsawa@asahikawa-nct.ac.jp