Mixed amplitude solutions of semilinear systems of 3-dimensional wave equations *

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Abstract

In this paper we study a global in time existence of classical solutions of semilinear systems of 3-dim. wave equations. We see that one component of the global in time solution can be arbitrarily large if another component is small enough according to some balance of each amplitude. Also its sharpness is discussed. This is a specific nature of strongly coupled systems.

1 Introduction

For the initial value problem of semilinear wave equations

$$\begin{cases} \partial_t^2 u - \Delta u = F(u), \\ u(x,0) = \varepsilon f(x),\ \partial_t u(x,0) = \varepsilon g(x), \end{cases}$$

the general theory has been studied by many works. Under the smallness assumption on the data such as $\varepsilon > 0$ is "small", we can discuss conditions which guarantee the global existence of small solutions when the nonlinearity is a large class of power-type, for example,

$$F(u) = O(|u|^p) \quad \text{as } u \to 0.$$

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Most of all the results of C^∞ category in H^s framework are summarized in Li & Chen [5]. More precise argument on the power was started by John[4] in C^2 framework.

On the other hand, we have a different story when $\varepsilon > 0$ is "large". Only the case where the energy is conserved, for example,

$$F(u) \sim -|u|^{p-1}u \quad \text{for large } u,$$

can be studied for the global existence. See Shatah & Struwe[6] for this direction. In this way we have a big gap in each analysis between toward to small solutions and to large one for the single equation. But for the system of equations the situation need not be the case.

We now condiser the following system of real-valued unknowns u, v especially in three space dimensions.

$$\begin{cases} \partial_t^2 u - \Delta u = A u^\alpha v^\beta \\ \partial_t^2 v - \Delta v = B u^\gamma v^\delta \end{cases} \text{in } \mathbf{R}^3 \times [0, \infty) \\ \begin{cases} u(x,0) = \varepsilon f_1(x), \ \partial_t u(x,0) = \varepsilon g_1(x) \\ v(x,0) = M f_2(x), \ \partial_t v(x,0) = M g_2(x), \end{cases} \tag{1.1}$$

where both A and B are non-zero constants and all the indices $\alpha, \beta, \gamma, \delta$ are non-negative integers. The initial data f_i, g_i $(i = 1, 2)$ are smooth with positive parameters ε, M. In this paper we will see what happens on the global existence of the solution (u, v) when M is large. For weakly coupled case, i.e. $\alpha = \delta = 0$, at low powers $\beta = \gamma = 2$, we have no global solution for any data even with small M. See Agemi & Kurokawa & Takamura[1] which was initiated by DelSanto & Georgiev & Mitidieri[2]. In order to avoid such a kind of blow-up case, we shall assume the following strongly coupled condition.

$$\begin{array}{l} AB \neq 0, \ \alpha, \beta, \gamma, \delta \in \mathbf{N}, \\ \alpha + \beta, \gamma + \delta = 3, 4, 5, \cdots > 2, \\ f_i \in C_0^3(\mathbf{R}^3), \ g_i \in C_0^2(\mathbf{R}^3) \ (i = 1, 2). \end{array} \tag{1.2}$$

In the system(1.1), roughly speaking, we may put $u = O(\varepsilon), v = O(M)$ as $\varepsilon \to 0, M \to \infty$. According to this point of view, we need a balance such that both $\varepsilon^{\alpha-1} M^\beta$ and $\varepsilon^\gamma M^{\delta-1}$ are small. Actually in the next section we have a global existence under such a condition. The rest of this paper is devoted to its sharpness. Roughly speaking again, we have no hope to prove the global existence result for any data when one of $\varepsilon^{\alpha-1} M^\beta$, $\varepsilon^\gamma M^{\delta-1}$ is large.

2 Global existence theorem for $\alpha \neq 1$

It is well-known that the solution (u, v) of (1.1) has to satisfy

$$\begin{cases} u = L(Au^\alpha v^\beta) + \varepsilon u^0, \\ v = L(Bu^\gamma v^\delta) + M v^0, \end{cases}$$

where the linear operator L for a function $F(x,t)$ is defined by

$$L(F)(x,t) = \frac{1}{4\pi} \int_{\Gamma(x,t)} (t-s) F(y,s) dy ds$$

and u^0, v^0 have the following expressions.

$$u^0 = \partial_t L_0(f_1) + L_0(g_1), \ v^0 = \partial_t L_0(f_2) + L_0(g_2).$$

Here we set

$$L_0(f)(x,t) = \frac{t}{4\pi} \int_{|\omega|=1} f(x + t\omega) dS_\omega$$

and $\Gamma(x,t)$ is a backward cone with a vertix (x,t) defined by

$$\Gamma(x,t) = \{(y,s) \in \mathbf{R}^3 \times [0,\infty) \ : \ |x-y| \leq t-s\}.$$

Let us define a function space W by

$$\begin{aligned} W = \ & \{(u,v) \in \{C^2(\mathbf{R}^3 \times [0,\infty))\}^2 \ : \\ & \mathrm{supp}(u,v) \subset \{|x| \leq t+k\}, \ |||(u,v)||| < \infty\}. \end{aligned}$$

Here we set

$$\begin{aligned} & |||(u,v)||| = |||u|||_1 + |||v|||_2, \\ & |||u|||_1 = \sum_{|a| \leq 2} \|\nabla_x^a u\|_1, \ |||v|||_2 = \sum_{|b| \leq 2} \|\nabla_x^b v\|_2 \end{aligned}$$

and

$$\|u\|_j = \sup_{(x,t) \in \mathbf{R}^3 \times [0,\infty)} w_j(|x|,t)|u(x,t)|,$$

where

$$\begin{aligned} w_1(r,t) &= \tau_+(r,t)\tau_-(r,t)^{\alpha+\beta-2}, \\ w_2(r,t) &= \tau_+(r,t)\tau_-(r,t)^{\gamma+\delta-2} \end{aligned}$$

and

$$\tau_\pm(r,t) = \frac{t \pm r + 2k}{k}$$

For constants $U, V > 0$ a subspace W_0 of W is defined by

$$W_0(U,V) = \{(u,v) \in W \ : \ |||u|||_1 \leq U, \ |||v|||_2 \leq V\}$$

Theorem 1 *Let $\alpha \neq 1$. Assume (1.2) and $\mathrm{supp} f_i, \mathrm{supp} g_i \subset \{x \in \mathbf{R}^3 : |x| \leq k\}$ $(i = 1, 2)$ for some fixed constant $k > 0$. Then, for any $M > 0$, there exists a unique global in time solution $(u, v) \in W_0(2|||u^0|||_1\varepsilon, 2|||v^0|||_2 M)$ of (1.1) provided ε is small enough to satisfy that*

$$|A|\varepsilon^{\alpha-1}M^{\beta} \leq C_1 \quad \text{and} \quad |B|\varepsilon^{\gamma}M^{\delta-1} \leq C_2, \tag{2.1}$$

where $C_1 = C_1(\alpha, \beta, f_1, f_2, g_1, g_2)$, $C_2 = C_2(\gamma, \delta, f_1, f_2, g_1, g_2)$ are positive constants.

Proof. Recall that the usual uniqueness result yields

$$\mathrm{supp}(u, v) \subset \{|x| \leq t + k\}.$$

See Appendix1 in John[3] for example. We shall start with the following basic estimate.

Proposition 2.1 (John[4]) *Let $p > 2$ and $q > 1$. For any $(x, t) \in \mathbf{R}^3 \times [0, \infty)$ satsifying $|x| \leq t + k$, there exists a positive constant $C_{p,q}$ depending only on p, q such that*

$$\tau_+\tau_-^{p-2}L(\tau_+^{-p}\tau_-^{-q}\chi_{\{|y|\leq s+k\}}) \leq C_{p,q}k^2.$$

Employing this proposition we have that

$$\begin{array}{l} \|L(w_1^{-\alpha}w_2^{-\beta}\chi_{\mathrm{supp}(u,v)})\|_1 \leq D_1, \\ \|L(w_1^{-\gamma}w_2^{-\delta}\chi_{\mathrm{supp}(u,v)})\|_2 \leq D_2, \end{array} \tag{2.2}$$

where $D_1 = D_1(\alpha, \beta, k)$, $D_2 = D_2(\gamma, \delta, k)$ are positive constants. Because simple computation gives us

$$\begin{array}{l} w_1^{\alpha}w_2^{\beta} = \tau_+^{\alpha+\beta}\tau_-^{\alpha(\alpha+\beta-2)+\beta(\gamma+\delta-2)}, \\ w_1^{\gamma}w_2^{\delta} = \tau_+^{\gamma+\delta}\tau_-^{\gamma(\alpha+\beta-2)+\delta(\gamma+\delta-2)} \end{array}$$

and

$$\begin{array}{l} \alpha(\alpha+\beta-2) + \beta(\gamma+\delta-2) \geq \alpha+\beta > 2, \\ \gamma(\alpha+\beta-2) + \delta(\gamma+\delta-2) \geq \gamma+\delta > 2. \end{array}$$

Using this estimate we shall construct a solution of (1.1) by classical iteration argument.

Let us define a sequence $\{(u_n, v_n)\}_{n\in\mathbf{N}}$ by

$$\left\{ \begin{array}{l} u_n = L(Au_{n-1}^{\alpha}v_{n-1}^{\beta}) + u_0, \quad u_0 = \varepsilon u^0, \\ v_n = L(Bu_{n-1}^{\gamma}v_{n-1}^{\delta}) + v_0, \quad v_0 = Mv^0. \end{array} \right. \tag{2.3}$$

Then it follows from (2.2) that

$$\left\{ \begin{array}{l} \|u_n\|_1 \le |A| D_1 \|u_{n-1}\|_1^{\alpha} \|v_{n-1}\|_2^{\beta} + \|u^0\|_1 \varepsilon, \\ \|v_n\|_2 \le |B| D_2 \|u_{n-1}\|_1^{\gamma} \|v_{n-1}\|_2^{\delta} + \|v^0\|_2 M. \end{array} \right. \tag{2.4}$$

Therefore we obtain the boundedness of $\{(u_n, v_n)\}$ such that

$$\|u_n\|_1 \le 2\|u^0\|_1 \varepsilon, \ \ \|v_n\|_2 \le 2\|v^0\|_2 M \tag{2.5}$$

provided

$$\left\{ \begin{array}{l} |A| D_1 (2\|u^0\|_1 \varepsilon)^{\alpha} (2\|v^0\|_2 M)^{\beta} \le \|u^0\|_1 \varepsilon, \\ |B| D_2 (2\|u^0\|_1 \varepsilon)^{\gamma} (2\|v^0\|_2 M)^{\delta} \le \|v^0\|_2 M. \end{array} \right. \tag{2.6}$$

Keeping this situation we have the convergence of $\{(u_n, v_n)\}$. In fact, it follows from (2.3) that

$$\left\{ \begin{array}{l} u_{n+1} - u_n = A \cdot L \left(v_n^{\beta} (u_n^{\alpha} - u_{n-1}^{\alpha}) + u_{n-1}^{\alpha} (v_n^{\beta} - v_{n-1}^{\beta}) \right), \\ v_{n+1} - v_n = B \cdot L \left(v_n^{\delta} (u_n^{\gamma} - u_{n-1}^{\gamma}) + u_{n-1}^{\gamma} (v_n^{\delta} - v_{n-1}^{\delta}) \right). \end{array} \right. \tag{2.7}$$

We note that

$$\begin{array}{l} |u_n^p - u_{n-1}^p| \le p(w_1^{-1} 2\|u^0\|_1 \varepsilon)^{p-1} |u_n - u_{n-1}|, \\ |v_n^p - v_{n-1}^p| \le p(w_2^{-1} 2\|v^0\|_2 M)^{p-1} |v_n - v_{n-1}|, \end{array} \tag{2.8}$$

where p stands for one of $\alpha, \beta, \gamma, \delta$. Hence (2.2) yields that

$$\left\{ \begin{array}{ll} \|u_{n+1} - u_n\|_1 \le & |A| D_1 (2\|v^0\|_2 M)^{\beta} \alpha (2\|u^0\|_1 \varepsilon)^{\alpha-1} \|u_n - u_{n-1}\|_1 \\ & + |A| D_1 (2\|u^0\|_1 \varepsilon)^{\alpha} \beta (2\|v^0\|_2 M)^{\beta-1} \|v_n - v_{n-1}\|_2, \\ \|v_{n+1} - v_n\|_2 \le & |B| D_2 (2\|v^0\|_2 M)^{\delta} \gamma (2\|u^0\|_1 \varepsilon)^{\gamma-1} \|u_n - u_{n-1}\|_1 \\ & + |B| D_2 (2\|u^0\|_1 \varepsilon)^{\gamma} \delta (2\|v^0\|_2 M)^{\delta-1} \|v_n - v_{n-1}\|_2. \end{array} \right.$$

This can be rewritten as

$$\left\{ \begin{array}{l} \|u_{n+1} - u_n\|_1 \le C_{11} \|u_n - u_{n-1}\|_1 + C_{12} \|v_n - v_{n-1}\|_2, \\ \|v_{n+1} - v_n\|_2 \le C_{21} \|u_n - u_{n-1}\|_1 + C_{22} \|v_n - v_{n-1}\|_2, \end{array} \right.$$

where

$$\begin{array}{l} C_{11} = 2^{\alpha+\beta-1} \alpha D_1 \|u^0\|_1^{\alpha-1} \|v^0\|_2^{\beta} |A| \varepsilon^{\alpha-1} M^{\beta} \\ C_{12} = 2^{\alpha+\beta-1} \beta D_1 \|u^0\|_1^{\alpha} \|v^0\|_2^{\beta-1} |A| \varepsilon^{\alpha} M^{\beta-1} \\ C_{21} = 2^{\gamma+\delta-1} \gamma D_2 \|u^0\|_1^{\gamma-1} \|v^0\|_2^{\delta} |B| \varepsilon^{\gamma-1} M^{\delta} \\ C_{22} = 2^{\gamma+\delta-1} \delta D_2 \|u^0\|_1^{\gamma} \|v^0\|_2^{\delta-1} |B| \varepsilon^{\gamma} M^{\delta-1}. \end{array} \tag{2.9}$$

If we assume further that

$$\left\{\begin{array}{l} 2^{\alpha+\beta-3}\max\{\alpha,\beta\}D_1\|u^0\|_1^{\alpha-1}\|v^0\|_2^{\beta}|A|\varepsilon^{\alpha-1}M^{\beta}\le 1,\\ 2^{\gamma+\delta-3}\max\{\gamma,\delta\}D_2\|u^0\|_1^{\gamma}\|v^0\|_2^{\delta-1}|B|\varepsilon^{\gamma}M^{\delta-1}\le 1,\end{array}\right. \tag{2.10}$$

we have

$$C_{11}, C_{22}\le \frac{1}{4},\ C_{12}C_{21}\le\frac{1}{16}. \tag{2.11}$$

Therefore we obtain

$$\|u_{n+1}-u_n\|_1\le\frac{E_1}{2^{n+1}},\quad \|v_{n+1}-v_n\|_2\le\frac{E_2}{2^{n+1}}, \tag{2.12}$$

where

$$\begin{array}{l} E_1=\|u_1-u_0\|_1+4C_{12}\|v_1-v_0\|_2,\\ E_2=4C_{21}\|u_1-u_0\|_1+\|v_1-v_0\|_2.\end{array}$$

This means a convergence of $\{(u_n,v_n)\}$ in C^0-norm under the combined condition of (2.6) and (2.10), i.e.,

$$\left\{\begin{array}{l} 2^{\alpha+\beta}\max\{2^{-3}\max\{\alpha,\beta\},1\}D_1\|u^0\|_1^{\alpha-1}\|v^0\|_2^{\beta}|A|\varepsilon^{\alpha-1}M^{\beta}\le 1,\\ 2^{\gamma+\delta}\max\{2^{-3}\max\{\gamma,\delta\},1\}D_2\|u^0\|_1^{\gamma}\|v^0\|_2^{\delta-1}|B|\varepsilon^{\gamma}M^{\delta-1}\le 1.\end{array}\right. \tag{2.13}$$

Next we shall check the convergence in C^1-norm. It follows from (2.3) that

$$\left\{\begin{array}{l} \partial u_n=AL(\alpha u_{n-1}^{\alpha-1}\partial u_{n-1}v_{n-1}^{\beta}+\beta u_{n-1}^{\alpha}v_{n-1}^{\beta-1}\partial v_{n-1})+\partial u_0,\\ \partial v_n=BL(\gamma u_{n-1}^{\gamma-1}\partial u_{n-1}v_{n-1}^{\delta}+\delta u_{n-1}^{\gamma}v_{n-1}^{\delta-1}\partial v_{n-1})+\partial v_0,\end{array}\right. \tag{2.14}$$

where ∂ stands for one of ∂_{x_i} $(i=1,2,3)$. Hence each total number of the power of weight functions is invariant, so that, under the condition (2.13) which yields (2.5) we get

$$\left\{\begin{array}{ll} \|\partial u_n\|_1 & \le |A|D_1\alpha(2\|u^0\|_1\varepsilon)^{\alpha-1}\|\partial u_{n-1}\|_1(2\|v^0\|_2M)^{\beta}\\ & \quad +|A|D_1\beta(2\|u^0\|_1\varepsilon)^{\alpha}(2\|v^0\|_2M)^{\beta-1}\|\partial v_{n-1}\|_2+\|\partial u^0\|_1\varepsilon,\\ \|\partial v_n\|_2 & \le |B|D_2\gamma(2\|u^0\|_1\varepsilon)^{\gamma-1}\|\partial u_{n-1}\|_1(2\|v^0\|_2M)^{\delta}\\ & \quad +|B|D_2\delta(2\|u^0\|_1\varepsilon)^{\gamma}(2\|v^0\|_2M)^{\delta-1}\|\partial v_{n-1}\|_2+\|\partial v^0\|_2M.\end{array}\right.$$

Therefore we obtain the boundedness of $\{(\partial u_n,\partial v_n)\}$ such that

$$\|\partial u_n\|_1\le 2\|\partial u^0\|_1\varepsilon,\ \|\partial v_n\|_2\le 2\|\partial v^0\|_2M \tag{2.15}$$

provided

$$\left\{\begin{array}{ll} \|\partial u^0\|_1\varepsilon & \ge |A|D_1\alpha(2\|u^0\|_1\varepsilon)^{\alpha-1}2\|\partial u^0\|_1\varepsilon(2\|v^0\|_2M)^{\beta}\\ & \quad +|A|D_1\beta(2\|u^0\|_1\varepsilon)^{\alpha}(2\|v^0\|_2M)^{\beta-1}2\|\partial v^0\|_2M,\\ \|\partial v^0\|_2M & \ge |B|D_2\gamma(2\|u^0\|_1\varepsilon)^{\gamma-1}2\|\partial u^0\|_1\varepsilon(2\|v^0\|_2M)^{\delta}\\ & \quad +|B|D_2\delta(2\|u^0\|_1\varepsilon)^{\gamma}(2\|v^0\|_2M)^{\delta-1}2\|\partial v^0\|_2M.\end{array}\right. \tag{2.16}$$

Keeping this situation we have the convergence of $\{(\partial u_n, \partial v_n)\}$. In fact, it follows from (2.7) that

$$\begin{cases} \partial(u_{n+1}-u_n) & = A\cdot L(\beta v_n^{\beta-1}\partial v_n(u_n^\alpha - u_{n-1}^\alpha) + v_n^\beta \partial(u_n^\alpha - u_{n-1}^\alpha) \\ & \quad +\alpha u_{n-1}^{\alpha-1}\partial u_{n-1}(v_n^\beta - v_{n-1}^\beta)) + u_{n-1}^\alpha \partial(v_n^\beta - v_{n-1}^\beta)), \\ \partial(v_{n+1}-v_n) & = B\cdot L(\delta v_n^{\delta-1}\partial v_n(u_n^\gamma - u_{n-1}^\gamma) + v_n^\delta \partial(u_n^\gamma - u_{n-1}^\gamma) \\ & \quad +\gamma u_{n-1}^{\gamma-1}\partial u_{n-1}(v_n^\delta - v_{n-1}^\delta) + u_{n-1}^\gamma \partial(v_n^\delta - v_{n-1}^\delta)). \end{cases} \tag{2.17}$$

We note that

$$\begin{aligned} |\partial(u_n^p - u_{n-1}^p)| & \leq p(w_1^{-1}2\|u^0\|_1\varepsilon)^{p-1}|\partial(u_n - u_{n-1})| \\ & \quad +(p-1)(w_1^{-1}2\|u^0\|_1\varepsilon)^{p-2}(2\|\partial u^0\|_1\varepsilon)|u_n - u_{n-1}|, \\ |\partial(v_n^p - v_{n-1}^p)| & \leq p(w_2^{-1}2\|v^0\|_2 M)^{p-1}|\partial(v_n - v_{n-1})| \\ & \quad +(p-1)(w_2^{-1}2\|v^0\|_2 M)^{p-2}(2\|\partial v^0\|_2 M)|v_n - v_{n-1}|, \end{aligned}$$

where p stands for one of $\alpha, \beta, \gamma, \delta$. Therefore it follows from (2.2), (2.8) and (2.12) that

$$\begin{cases} \|\partial(u_{n+1}-u_n)\|_1 \leq C_{11}\|\partial(u_n - u_{n-1})\|_1 + C_{12}\|\partial(v_n - v_{n-1})\|_2 + \dfrac{E_3}{2^n}, \\ \|\partial(v_{n+1}-v_n)\|_2 \leq C_{21}\|\partial(u_n - u_{n-1})\|_1 + C_{22}\|\partial(v_n - v_{n-1})\|_2 + \dfrac{E_4}{2^n}, \end{cases}$$

where each C_{ij} $(i,j=1,2)$ is the one in (2.9) and

$$\begin{aligned} E_3 = \ & |A|D_1\beta(2\|v^0\|_2 M)^{\beta-1}(2\|\partial v^0\|_2 M)\alpha(2\|u^0\|_1\varepsilon)^{\alpha-1}E_1 \\ & +|A|D_1\alpha(2\|u^0\|_1\varepsilon)^{\alpha-1}(2\|\partial u^0\|_1\varepsilon)\beta(2\|v^0\|_2 M)^{\beta-1}E_2 \\ & +|A|D_1(2\|v^0\|_2 M)^\beta(\alpha-1)(2\|u^0\|_1\varepsilon)^{\alpha-2}(2\|\partial u^0\|_1\varepsilon)E_1 \\ & +|A|D_1(2\|u^0\|_1\varepsilon)^\alpha(\beta-1)(2\|v^0\|_2 M)^{\beta-2}(2\|\partial v^0\|_2 M)E_2, \\ E_4 = \ & |B|D_2\delta(2\|v^0\|_2 M)^{\delta-1}(2\|\partial v^0\|_2 M)\gamma(2\|u^0\|_1\varepsilon)^{\gamma-1}E_1 \\ & +|B|D_2\gamma(2\|u^0\|_1\varepsilon)^{\gamma-1}(2\|\partial u^0\|_1\varepsilon)\delta(2\|v^0\|_2 M)^{\delta-1}E_2 \\ & +|B|D_2(2\|v^0\|_2 M)^\delta(\gamma-1)(2\|u^0\|_1\varepsilon)^{\gamma-2}(2\|\partial u^0\|_1\varepsilon)E_1 \\ & +|B|D_2(2\|u^0\|_1\varepsilon)^\gamma(\delta-1)(2\|v^0\|_2 M)^{\delta-2}(2\|\partial v^0\|_2 M)E_2. \end{aligned}$$

Making use of (2.11) again, we finally obtain

$$\begin{cases} \|\partial(u_{n+1}-u_n)\|_1 \leq \dfrac{E_5 + (n-1)E_7}{2^{n+1}}, \\ \|\partial(v_{n+1}-v_n)\|_2 \leq \dfrac{E_6 + (n-1)E_8}{2^{n+1}}, \end{cases}$$

where

$$\begin{aligned} & E_5 = \|\partial(u_1 - u_0)\|_1 + 4C_{12}\|\partial(v_1 - v_0)\|_2 + 2E_3, \\ & E_6 = 4C_{21}\|\partial(u_1 - u_0)\|_1 + \|\partial(v_1 - v_0)\|_2 + 2E_4, \\ & E_7 = E_3 + 4C_{12}E_4, \\ & E_8 = 4C_{21}E_3 + E_4. \end{aligned}$$

This means a convergence of $\{(u_n, v_n)\}$ in C^1-norm. Only the price that we have paid futher than (2.13) is (2.16) which comes from the boundedness in C^1-norm and changes nothing essentially of the balance between ε and M.

In almost the same way one can see a convergence of $\{(u_n, v_n)\}$ in C^2-norm. We have to assume further only a codition to guarantee the boundedness, which is similar to (2.16), but again nothing is changed essentially in the balance between ε and M. Note that (2.16) can be rewritten as (2.1). Therefore we shall omit details and close the proof of the Theorem1.

3 Positiveness of the solution

This section provides preliminaries for the non-existence result which will appear in the next section. The comparison argument of this kind, which is due to the continuity of the solution, can be found in early works for the single equation. For example, see Zhou[7].

Proposition 3.1 *Let $A, B > 0$. Assume that for $i = 1, 2$, $f_i \equiv 0$ and $suppg_i \subset \{x \in \mathbf{R}^3 \ : \ |x| \leq k\}$ with a fixed constant $k > 0$ and*

$$g_i(x) > 0 \text{ for } |x| < k. \tag{3.1}$$

Then the classical solution (u, v) of (1.1) satisfies that $supp(u, v) \subset \{(x, t) \in \mathbf{R}^3 \times [0, \infty) \ : \ |x| \leq t + k\}$ and

$$u(x,t), \ v(x,t) > 0 \text{ for } |x| < t + k. \tag{3.2}$$

Proof. In the previous section we know that $\text{supp}(u, v) \subset \{|x| \leq t + k\}$. Note that

$$t_1 = \inf\{t > 0 \ : \ u = 0 \text{ or } v = 0 \text{ in } \{|x| < t + k\}\} > 0$$

due to the assumption on the data and the continuity of the solution. If the conclusion is false, meaning that there exists a point (x_1, t_1) with $|x_1| < t_1 + k$, we immediatly reach to a contradiction. To see this we may assume that $u(x_1, t_1) = 0$ without loss of the generality. Then it follows from the representation formula and the assumption on the initial data that

$$u^0 = L_0(g_1) \geq 0,$$

so that we have

$$0 = u(x_1, t_1) \geq L(Au^\alpha v^\beta)(x_1, t_1) > 0.$$

Because the definition of t_1 says that

$$u, v > 0 \text{ in the interior of } \Gamma(x_1, t_1) \cap \{|x| \le t + k\}$$

and $|\Gamma(x_1, t_1) \cap \{|x| \le t + k\}| > 0$. This contradiction proves the proposition.

Proposition 3.2 *Assume that all the assumptions in Proposition3.1 are fulfilled. Let (u, v) be a radially symmetric solution of variables (r, t), $r = |x|$ to the system (1.1) with $g_i = g_i(|x|)$ $(i = 1, 2)$. Then (u, v) satisfies*

$$\begin{cases} u(r,t) > \dfrac{A}{4(t+r)} \displaystyle\int_{D(r,t)} (\tau + \lambda) u(\lambda, \tau)^\alpha v(\lambda, \tau)^\beta d\lambda d\tau + \dfrac{C(g_1)\varepsilon}{t+r}, \\ v(r,t) > \dfrac{B}{4(t+r)} \displaystyle\int_{D(r,t)} (\tau + \lambda) u(\lambda, \tau)^\gamma v(\lambda, \tau)^\delta d\lambda d\tau + \dfrac{C(g_2)M}{t+r} \end{cases} \tag{3.3}$$

for

$$(r, t) \in R = \left\{ 0 \le t - r \le \frac{k}{2},\ k \le t + r \right\}, \tag{3.4}$$

where

$$C(g_i) = \int_{k/2}^{k} \lambda g_i(\lambda) d\lambda > 0 \ (i = 1, 2)$$

and

$$D(r, t) = \{(\lambda, \tau) \in [0, \infty)^2 \ : \ 0 \le \tau - \lambda \le t - r,\ k \le \tau + \lambda \le t + r\}.$$

Proof. It is well-known that the radial symmetricity yields

$$L_0(f)(x, t) = L_0(f)(r, t) = \frac{1}{2r} \int_{|t-r|}^{t+r} \lambda f(\lambda) d\lambda \quad \text{for } f(x) = f(r).$$

Then it follows from the assumption on g_i $(i = 1, 2)$ that

$$u^0(r, t) = L_0(g_i)(r, t) \ge \frac{C(g_i)}{t + r}.$$

Similarly, the classical formula of the radially symmetric version

$$L(Au^\alpha v^\beta)(r, t) = \frac{A}{2r} \int_{\widetilde{\Gamma}(r,t)} \lambda u(\lambda, \tau)^\alpha v(\lambda, \tau)^\beta d\lambda,$$

where

$$\widetilde{\Gamma}(r, t) = \{(\lambda, \tau) \in [0, \infty)^2 \ : \ 0 \le \tau \le t,\ |t - \tau - r| \le \lambda \le t - \tau + r\},$$

shows that

$$L(Au^\alpha v^\beta)(r,t) > \frac{A}{2r}\int_{D(r,t)} \lambda u(\lambda,\tau)^\alpha v(\lambda,\tau)^\beta d\lambda d\tau$$

for $(r,t) \in R$. Because it is clear that

$$D(r,t) \subset (\neq)\widetilde{\Gamma}(r,t) \quad \text{when } (r,t) \in R$$

and Proposition3.1 says, when $\tau - \lambda > -k$,

$$u(\lambda,\tau), v(\lambda,\tau) > 0 \quad \text{in the interior of } \widetilde{\Gamma}(r,t) \setminus D(r,t).$$

Therefore the proposition follows from a simple estimate in $D(r,t)$,

$$\begin{aligned} \lambda &= \frac{\tau+\lambda}{2} - \frac{\tau-\lambda}{2} \geq \frac{\tau+\lambda}{2} - \frac{t-r}{2} \\ &\geq \frac{\tau+\lambda}{2} - \frac{k}{4} \geq \frac{\tau+\lambda}{4}. \end{aligned}$$

The proposition is now established.

4 Non-existence theorem for $\alpha \neq 1$ or $\delta \neq 1$

We are now in a position to see the sharpness of Theorem1. Actually we prove the following theorem.

Theorem 2 *Let $\alpha \neq 1$ or $\delta \neq 1$. Assume that all the assumptions in Proposition3.2 are fulfilled, i.e., $f_i \equiv 0$, $g_i(x) = g_i(r)$ $(r = |x|)$ and $g_i > 0$ when $r < k$, $g_i = 0$ when $r \geq k$ for $i = 1,2$. Then, for any $\varepsilon > 0$, the system (1.1) admits no radially symmetric solution $(u(r,t), v(r,t))$ for $t + r > k$ provided M is large enough to satisfy that*

$$\begin{aligned} A\varepsilon^{\alpha-1}M^\beta &\geq \frac{C_3}{k^{2-\alpha-\beta} - (t+r)^{2-\alpha-\beta}} \quad \textit{when } \alpha \neq 1, \\ \textit{or } B\varepsilon^\gamma M^{\delta-1} &\geq \frac{C_4}{k^{2-\gamma-\delta} - (t+r)^{2-\gamma-\delta}} \quad \textit{when } \delta \neq 1, \end{aligned}$$

where $C_3 = C_3(\alpha,\beta,g_1,g_2)$ and $C_4 = C_4(\gamma,\delta,g_1,g_2,)$ are positive constants.

This theorem needs the following comparison argument.

Proposition 4.1 *Assume that all the assumptions in Proposition3.2 are fulfilled. Then the radially symmetric solution (u,v) of (1.1) satisfies*

$$\left\{\begin{array}{l}(t+r)u(r,t) > X(t+r) > 0,\\ (t+r)v(r,t) > Y(t+r) > 0\end{array}\right. \quad \textit{for } (r,t) \in R, \tag{4.1}$$

where X and Y are solutions of

$$\begin{aligned} X(\xi) &= \frac{(t-r)C(g_2)^\beta AM^\beta}{8}\int_k^\xi a^{1-\alpha-\beta}X(a)^\alpha da + C(g_1)\varepsilon,\\ Y(\xi) &= \frac{(t-r)C(g_1)^\gamma B\varepsilon^\gamma}{8}\int_k^\xi a^{1-\delta-\gamma}Y(a)^\delta da + C(g_2)M. \end{aligned} \tag{4.2}$$

Proof. First we note that $X > 0$ and $Y > 0$. It is easy to see, for example, $X > 0$. In fact we have $X(k) > 0$. If there is a point ξ_0 such that

$$\xi_0 = \min\{\xi \in (k,\infty) \ : \ X(\xi) = 0\},$$

then the definition of X in (4.2) yields the desired contradiction, $X(\xi_0) > 0$ because of

$$X(a) > 0 \text{ for } a \in [k, \xi_0).$$

We shall turn to the proof. At a point (r_0, t_0) such that

$$t_0 + r_0 = k \text{ and } t_0 - r_0 = 0,$$

we have, by Proposition3.2,

$$\left\{\begin{array}{l}(t_0+r_0)u(r_0,t_0) > X(t_0+r_0) = C(g_1)\varepsilon\\ (t_0+r_0)v(r_0,t_0) > Y(t_0+r_0) = C(g_2)M.\end{array}\right.$$

If the conclusion is false, we immediately get a contradiction. In fact, suppose that there is a point (r_1, t_1) nearest from (r_0, t_0) such that

$$\begin{array}{l}(t_1+r_1)u(r_1,t_1) = X(t_1+r_1),\\ \text{or } (t_1+r_1)v(r_1,t_1) = Y(t_1+r_1).\end{array}$$

Without loss of the generality, we may assume that the first case is hold. We note $(r_0, t_0) \neq (r_1, t_1)$ by continuity of u and X, which implies that an area of $D(r_1, t_1)$ is positive. Then it follows from Proposition3.1 and 3.2 that

$$u(r_1,t_1) > \frac{C(g_2)^\beta AM^\beta}{4(t_1+r_1)}\int_{D(r_1,t_1)}(\tau+\lambda)^{1-\beta}u(\lambda,\tau)^\alpha d\lambda d\tau + \frac{C(g_1)\varepsilon}{t_1+r_1}.$$

But the definition of (r_1, t_1) yields

$$(\tau+\lambda)u(\lambda,\tau) > X(\tau+\lambda) \quad \text{in the interior of } D(r_1,t_1).$$

Therefore we get a desired contradiction to the definition of (r_1, t_1),

$$u(r_1,t_1) > \frac{C(g_2)^\beta AM^\beta}{4(t_1+r_1)} \int_{D(r_1,t_1)} \xi^{1-\alpha-\beta} X(\xi)^\alpha\big|_{\xi=\tau+\lambda}\, d\lambda d\tau + \frac{C(g_1)\varepsilon}{t_1+r_1}.$$

Because introducing a characteristic variable

$$a=\tau+\lambda,\ b=\tau-\lambda,$$

we have

$$(t_1+r_1)u(r_1,t_1) > X(t_1+r_1).$$

Hence the first inequality follows. The second one can be obtained in the same manner. The proof is now completed.

Proof of Theorem2.

First we consider the case of $\alpha \neq 1$. Let X be the one in Proposition4.1. Restricting variables on the line $t-r=k/2$ in R, we know that X satisfies

$$\begin{cases} X'(\xi) = \dfrac{kC(g_2)^\beta AM^\beta}{16}\xi^{1-\alpha-\beta}X(\xi)^\alpha \\ X(k) = C(g_1)\varepsilon. \end{cases}$$

Then, integrating the inequality over $\xi \in [k, t+r]$, we have

$$\begin{aligned} &\frac{X(k)^{1-\alpha} - X(t+r)^{1-\alpha}}{\alpha-1} \\ &= \frac{kC(g_2)^\beta AM^\beta}{16(\alpha+\beta-2)}\left(k^{2-\alpha-\beta} - (t+r)^{2-\alpha-\beta}\right). \end{aligned}$$

Hence we obtain

$$\begin{aligned} X(t+r)^{1-\alpha} = \ & C(g_1)^{1-\alpha}\varepsilon^{1-\alpha} \\ & -\frac{k(\alpha-1)C(g_2)^\beta AM^\beta}{16(\alpha+\beta-2)}\left(k^{2-\alpha-\beta} - (t+r)^{2-\alpha-\beta}\right). \end{aligned}$$

This inequality shows that the statement in the theorem is ture if we put

$$C_3 = \frac{16(\alpha+\beta-2)}{k(\alpha-1)} C(g_1)^{1-\alpha} C(g_2)^{-\beta} > 0.$$

Because

$$X(t+r) \to \infty \text{ as } \varepsilon^{\alpha-1}AM^{\beta} \to \frac{C_3}{k^{2-\alpha-\beta}-(t+r)^{2-\alpha-\beta}}.$$

Therefore, due to Proposition4.1, u cannot exist after X blows-up.

The second case of $\delta \neq 1$ can be investigated in almost the same manner. In this case, we have to start with Y in Proposition4.1 which satisfies

$$\begin{cases} Y'(\xi) = \dfrac{kC(g_1)^{\gamma}B\varepsilon^{\gamma}}{16}\xi^{1-\delta-\gamma}Y(\xi)^{\delta} \\ Y(k) = C(g_2)M. \end{cases}$$

when $t-r=k/2$. Therefore it is easy to see that the statement in the theorem is ture if we put

$$C_4 = \frac{16(\delta+\gamma-2)}{k(\delta-1)}C(g_2)^{1-\delta}C(g_1)^{-\gamma} > 0.$$

Because this implies

$$Y(t+r) \to \infty \text{ as } B\varepsilon^{\gamma}M^{\delta-1} \to \frac{C_4}{k^{2-\alpha-\beta}-(t+r)^{2-\alpha-\beta}}.$$

Therefore, due to Proposition4.1, v cannot exist after Y blows-up. The theorem is now established.

5 Remark on the case of $\alpha = 1$

In this section we shall take a look of the case of $\alpha = 1$. Due to definition of X in (4.2), we have a lower bound of u under the same assumption of the non-existence result. That is, for $t-r=k/2$ and $t+r>k$, one can obtain the exponential growth of u in M such as

$$(t+r)u(r,t) > C(g_1)\varepsilon \exp\left(\frac{kC(g_2)^{\beta}(k^{1-\beta}-(t+r)^{1-\beta})}{16(\beta-1)}AM^{\beta}\right).$$

This is not enough to reach to the expected result saying that the blow-up of some special solution can be established for any ε provided M is larger than some constant independent of ε.

It seems that it is not so easy to obtain a global existence result for any ε and large M by means of the iteration method in W via John's basic estimate. Actually, according to the first estimate of the sequence $\{(u_n, v_n)\}$ in (2.4), we have

$$\left\{ \begin{array}{l} \|u_n\|_1 \le |A|D_1 U V^\beta + \|u^0\|_1 \varepsilon, \\ \|v_n\|_2 \le |B|D_2 U^\gamma V^\delta + \|v^0\|_2 M \end{array} \right.$$

provided $\|u_{n-1}\|_1 \le U$ and $\|v_{n-1}\|_2 \le V$, where U, V are positive constants may depend on ε and M. In order to get the same bounds for $\|u_n\|_1, \|v_n\|_2$ by means of this inequality, we need

$$\left\{ \begin{array}{l} |A|D_1 U V^\beta + \|u^0\|_1 \varepsilon \le U, \\ |B|D_2 U^\gamma V^\delta + \|v^0\|_2 M \le V. \end{array} \right.$$

Therefore the first line says that we should have

$$|A|D_1 V^\beta < 1.$$

But this is impossible to be established under the same assumption of Theorem2 if M is larger than some constant independent of ε. Because we have $V \ge \|v\|_2$. Then by Proposition4.1, this inequality implies that

$$(|A|D_1)^{-1/\beta} > V \ge w_2(r,t)v(r,t)|_{(r,t)\in R} > \frac{2^{\gamma+\delta-2}C(g_2)}{k} M.$$

This fact indicates that we need a different method from our simple iteration in order to construct a global solution when $\alpha = 1$.

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