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Decomposition of Green polynomials of type A and DeConcini-Procesi-Tanisaki algebras of certain types

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Abstract

A class of graded representations of the symmetric group, concerning with the cohomology ring of the corresponding flag variety, are considered. We point out a certain combinatorial property of the Poincaré polynomial of these graded representations, and interpret it in the language of representation theory of the symmetric group.

1 Introduction

Let W be a finite reflection group, and $R = \bigoplus_d R^d$ a natural graded representation of W such as the coinvariant algebra of W . Our problem is to understand a certain combinatorial property of the Hilbert polynomial of R in the language of representation theory. The property of the Hilbert polynomial we consider is the following. Let l be some positive integer. If $c(k; l)$ ($k = 0, 1, \dots, l-1$) denotes the sum of the coefficients of the Hilbert polynomial whose degrees are congruent to k modulo l , then it often occurs that these $c(k; l)$'s are all equal to each other. In other words, if we consider for each $k = 0, 1, \dots, l-1$ the direct sum $R(k; l)$ of homogeneous components of R of which degrees are congruent to k modulo l , then the dimension of these submodules do not depend on k , i.e., $\dim R(k; l) = (\dim R)/l$ for every k . The problem is to explain this phenomenon in the language of representation theory of W in the following manner, i.e., find a subgroup $H(l)$ of W , and find $H(l)$ -modules $Z(k; l)$ for each $k = 0, 1, \dots, l-1$ of which dimensions are equal to each other, satisfying

$$R(k; l) \cong_W \text{Ind}_{H(l)}^W Z(k; l), \quad k = 0, 1, \dots, l-1,$$

where Ind denotes the induced representation.

The problem in the present article refers to the case where $W = S_n$ the symmetric group of n letters, and $R = R_\mu$ a graded representation of S_n corresponding to each partition μ of n . The algebra R_μ is a natural generalization of the coinvariant algebra of S_n , a graded version of the left regular representation $\mathbf{C}[S_n]$. In the case of the coinvariant algebra, we have found those l 's and established the isomorphisms [MN]. Let R_n denote the coinvariant algebra of S_n . The problems in [MN] are the followings:

- Determine all the integers l such that the dimensions of $R_n(k; l)$ do not depend on k , i.e.,

$$\dim R_n(k; l) = \frac{\dim R_n}{l}$$

for all $k = 0, 1, \dots, l - 1$.

- For each integer l satisfying the condition above, find a subgroup $H = H_n(l)$ of S_n and H -modules $Z_n(k; l)$ ($k = 0, 1, \dots, l - 1$) of equal dimension, satisfying

$$R_n(k; l) \cong_{S_n} \text{Ind}_H^{S_n} Z_n(k; l)$$

for each $k = 0, 1, \dots, l - 1$.

The answer for the first problem is given in general for the coinvariant algebras of finite reflection groups W . We can see that each *fundamental degrees* [H] of W is available. (Recall that the set of fundamental degrees of S_n is $\{2, 3, \dots, n\}$.) The answer for the second problem is as follows. Let l be a fundamental degree of the symmetric groups S_n . Then we can choose the subgroup $H_n(l)$ as

$$H_n(l) \cong C_l \times S_r,$$

where $n = dl + r$, $0 \leq r \leq l - 1$, and the $H_n(l)$ -modules $Z_n(k; l)$ ($k = 0, 1, \dots, l - 1$), defined by twisting each homogeneous component of the coinvariant algebra R_r of S_r by an irreducible representation of C_l . Remark that, since the irreducible representations of C_l are all one dimensional, each $Z_n(k; l)$ has dimension $r! = \dim R_r$. Thus these isomorphisms give a representation theoretical interpretation of the phenomenon “coincidence of dimension” of R_n .

The problem is motivated by a result of Krařkiewicz and Weyman [KW]. They show that each submodules $R_n(k; n)$, the case where $l = n$, is induced from the corresponding irreducible representation of a cyclic subgroup generated by a *Coxeter element* [H] of S_n . Precisely, let $c = (1, 2, \dots, n)$ be a cyclic permutation of full length (this is a typical example of Coxeter elements), and let $\psi_n^{(k)} : C_n \rightarrow \mathbf{C} : c \mapsto \zeta_n^k$ the irreducible representation of the cyclic subgroup $C_n = \langle c \rangle$ sending the generator c to ζ_n^k for each $k = 0, 1, \dots, n - 1$, where $\zeta_n = e^{2\pi i/n}$ is the primitive n -th root of unity. Then they show that there exists an S_n -module isomorphism

$$R_n(k; n) \cong_{S_n} \text{Ind}_{C_n}^{S_n} \psi_n^{(k)},$$

for each $k = 0, 1, \dots, n - 1$. It immediately follows from this fact that each submodules $R_n(k; n)$ have the same dimension $(n - 1)!$. They also show similar results for the Weyl groups of type B_n and D_n . Stembridge considers this problem for a wider setting. In [Stm], he considers any finite complex reflection group W and its coinvariant algebra R_W . In this case, we can take c to be a *Springer’s regular element* [Sp], and l its order. (The number l is also called a *regular number* of W .) It can be shown that each submodules $R_W(k; l)$ of equal dimension is obtained by inducing up the corresponding irreducible representation $\psi_l^{(k)}$ of the cyclic subgroup $\langle a \rangle$ (hence all $\psi_l^{(k)}$ are one dimensional).

We will consider a generalization of the result in [MN] for a class of graded representations of the symmetric groups, which we call *DeConcini-Procesi-Tanisaki algebras* [DP][T]. A DeConcini-Procesi-Tanisaki algebra R_μ is a graded S_n -module defined for each partition μ . It is a natural generalization of the coinvariant algebra, which is isomorphic to the representation obtained by inducing up the trivial representation of the corresponding Young subgroup S_μ . It is well known that the coinvariant algebra R_n is isomorphic to the cohomology ring $H^*(X_n, \mathbf{C})$ of the corresponding flag variety $X_n = GL_n/B$, where B is a Borel subgroup of GL_n . On the other hand, the algebra R_μ is isomorphic to the cohomology ring $H^*(X_\mu, \mathbf{C})$ of the *fixed point subvariety* X_μ [DP] of the flag variety X_n . The difference of coinvariant algebras and DeConcini-Procesi-Tanisaki algebras we should note here is that the corresponding varieties are smooth or not. Needless to say, the flag varieties are non-singular, and its Poincaré polynomial

$$P_{X_n}(q) = \sum_{d \geq 0} q^d \dim H^{2d}(X_n, \mathbf{C})$$

has a nice property, the unimodal symmetry. This is a consequence of the *hard Lefschetz theorem* [St] for non-singular varieties, or the *strong Lefschetz property* [W] for graded Artinian algebras. For example, if we consider the case where $n = 4$, the Poincaré polynomial of X_4 is

$$P_{X_4}(q) = 1 + 3q + 5q^2 + 6q^3 + 5q^4 + 3q^5 + q^6.$$

It is obviously seen that the coefficients form a unimodal symmetric sequence. Contrary, fixed point subvarieties are singular in general. Thus it seems difficult to expect that the Poincaré polynomials have a nice combinatorial property, such as unimodal symmetry. Some examples of Poincaré polynomials of fixed point subvarieties, corresponding to partitions $\mu = (2, 2)$, $(2, 1, 1)$ and $(2, 1, 1, 1)$, can be seen below:

$$\begin{aligned} P_{X_{(2,2)}}(q) &= 1 + 3q + 2q^2, \\ P_{X_{(2,1,1)}}(q) &= 1 + 3q + 5q^2 + 3q^3, \\ P_{X_{(2,1,1,1)}}(q) &= 1 + 4q + 9q^2 + 15q^3 + 16q^4 + 11q^5 + 4q^6, \end{aligned}$$

and the coefficients of these polynomials are not symmetric.

However, it is possible to find that the “coincidence of dimensions” can occur for cohomology rings $H^*(X, \mathbf{C})$ even if a variety X are singular. In fact, in the case of $\mu = (2, 1, 1, 1)$, we can see that the “mod 3 sums” of the coefficients of the Poincaré polynomial $P_{X_{(2,1,1,1)}}(q)$ coincide each other as follows:

$$1 + 15 + 4 = 4 + 16 = 9 + 11 = 20.$$

Similarly, the “mod 2 sums” are also all equal:

$$1 + 9 + 16 + 4 = 4 + 15 + 11 = 30.$$

It is easy to confirm that the same is true for other examples $\mu = (2, 2), (2, 1, 1)$ for the mod 2 sums. In general, if we denote by M_μ the maximum value of the multiplicities of a partition $\mu = (1^{m_1} 2^{m_2} \cdots n^{m_n})$, it is possible to show that, for each fixed $l \in \{1, 2, \dots, M_\mu\}$, the *partial sum modulo l* (mod l sum, for short)

$$c_\mu(k; l) = \sum_{d \equiv k \pmod{l}} c_d, \quad k = 0, 1, \dots, l-1$$

of the coefficients of the Poincaré polynomial $P_{R_\mu}(q) = \sum_d c_d q^d$ of R_μ do not depend on its residue k . In other words, for each fixed integer $l \in \{1, 2, \dots, M_\mu\}$, the dimensions of the submodules

$$R_\mu(k; l) = \bigoplus_{d \equiv k \pmod{l}} R_\mu^d, \quad k = 0, 1, \dots, l-1$$

coincide each other. Thus, as in the previous works [KW, Stem, MN], it is natural to understand this phenomenon on the algebra R_μ in the language of the representation theory of symmetric groups.

In this paper, we find the representation theoretical interpretation for some special two cases: one is for a *hook*, a partition of the form $(n-h, 1^h)$, and the other is for a *rectangle*, a partition of the form (r^k) . Let $\mu = (n-h, 1^h)$ be a hook. Suppose that $n-h > 1$ for excluding the case of the coinvariant algebra. Then we have $M_\mu = h$. Let an integer l , $1 \leq l \leq h$, be fixed. Then the submodules $R_\mu(k; l)$ have the same dimension for $k = 0, 1, \dots, l-1$. The problem is to construct a subgroup $H_\mu(l)$ of S_n and its modules $Z_\mu(k; l)$ ($k = 0, 1, \dots, l-1$) of the equal dimension, satisfying $R_\mu(k; l) \cong_{S_n} \text{Ind}_{H_\mu(l)}^{S_n} Z_\mu(k; l)$. In this case, a similar construction of the subgroup and the modules is possible for each l as in the case of coinvariant algebras. We will show that the subgroup $H_\mu(l)$ can be constructed isomorphic to the direct product

$$H_\mu(l) \cong C_l \times S_{n-dl},$$

where $h = dl + r$, $0 \leq r \leq l-1$, and the modules are defined by twisting each homogeneous components of some “smaller” algebra $R_{\bar{\mu}}$ by the corresponding irreducible representation of C_l . In the rectangle case, it will be needed some restriction to prove the problem at present. Let a partition $\mu = (r^p)$ be rectangle, whose multiplicity p is a prime. We consider the problem in this situation only for the case $l = p$. In this case, it is possible to prove that the subgroup $H_\mu(p)$ can be taken isomorphic to the semidirect product

$$H_\mu(p) \cong \underbrace{(S_r \times \cdots \times S_r)}_p \rtimes C_p,$$

and the $H_\mu(p)$ -modules are one dimensional representations, deeply related to the irreducible representations of C_p . These one dimensional representations are, in fact, considered as a smaller DeConcini-Procesi-Tanisaki algebra $R_{\bar{\mu}} = R_\emptyset$, where \emptyset denotes the empty partition.

In both cases, our problem is reformulated to show the following isomorphism of $(S_n \times C_l)$ -modules between the algebra R_μ and a S_n -module induced from a smaller algebra $R_{\bar{\mu}}$: in

the case of the hook case for example, it is enough to show

$$R_\mu \cong_{S_n \times C_l} \text{Ind}_{S_r}^{S_n} R_{\bar{\mu}}.$$

(This is pointed out by T. Shoji, Nagoya University.) In this point of view, each isomorphism $R_\mu(k; l) \cong_{S_n} \text{Ind}_{H_\mu(l)}^{S_n} Z_\mu(k; l)$ is realized as correspondence of ζ_l^k -eigenspaces of the action of a generator of C_l in the $(S_n \times C_l)$ -module isomorphism.

We briefly sketch the idea of the proof. We show that the character values of the $S_n \times C_l$ -modules agree. The main tool in the proof is the *Green polynomials* $Q_\rho^\mu(q)$ [Gr, Mac] (of type A). The Green polynomials are introduced by J. A. Green [Gr] in his argument to determine the irreducible characters of the finite general linear groups $GL(\mathbf{F}_q)$. They also have formulation in terms of symmetric functions. The Green polynomials $Q_\rho^\mu(q)$ are defined to be the coefficients with some modification of the linear expansion of the *power-sum symmetric function* $p_\rho(x)$ by the *Hall-Littlewood symmetric functions* $P_\mu(x; q)$ [Mc]. It is known that the Green polynomial $Q_\rho^\mu(q)$ gives the graded character of the DeConcini-Procesi-Tanisaki algebra R_μ . Then we can see that the proof of the $(S_n \times C_l)$ -module isomorphism amounts to describe a behavior of the Green polynomial at roots of unity ζ_l^j . We should describe the following two points. First, we need a necessary condition for a partition ρ satisfying

$$Q_\rho^\mu(\zeta_l^j) \neq 0.$$

Secondly, for such a partition ρ , we will see that the ratio

$$\left. \frac{Q_\rho^\mu(q)}{Q_{\bar{\rho}}^{\bar{\mu}}(q)} \right|_{q=\zeta_l^j}$$

of the values of Green polynomials $Q_\rho^\mu(q)$ and $Q_{\bar{\rho}}^{\bar{\mu}}(q)$ at the roots of unity ζ_l^j is precisely the number of permutations in S_n satisfying some conditions, where $\bar{\rho}$ is the partition of the same size with $\bar{\mu}$, obtained by deleting some cycles corresponding to μ and j . This will be done by using some recursive relations and a canonical decomposition form of the Green polynomials. Results of Lascoux-Leclerc-Thibon [LLT] also plays an essential role in this paper.

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2 DeConcini-Procesi-Tanisaki algebras

In this section, we recall the definition and fundamental facts about the algebras R_μ .

The symmetric group S_n of n letters acts on the polynomial ring $P_n = \mathbf{C}[x_1, x_2, \dots, x_n]$ by permuting the variables: For $\sigma \in S_n$ and $f(x_1, x_2, \dots, x_n) \in P_n$, define

$$(\sigma f)(x_1, x_2, \dots, x_n) := f(x_{\sigma_1}, x_{\sigma_2}, \dots, x_{\sigma_n}).$$

Let μ be a partition of n . In the sequel, we use the symbol $\mu \vdash n$ for depicting that μ is a partition of n . We consider a homogeneous ideal I_μ of P_n corresponding to the partition μ , generated by the set [T]

$$\left\{ e_m(x_{i_1}, \dots, x_{i_{n-k+1}}) \mid \begin{array}{l} k = 1, \dots, \mu_1, \\ n - k + 1 \geq m \geq n - k + 1 - (\mu'_k + \mu'_{k+1} + \dots + \mu'_{\mu_1}) + 1 \end{array} \right\},$$

where $e_m(x_{i_1}, \dots, x_{i_{n-k+1}})$ denotes the m -th elementary symmetric polynomial in the variables $x_{i_1}, \dots, x_{i_{n-k+1}}$. For example, if we take μ to be $(2, 1) \vdash 3$, then I_μ is generated by

$$\begin{array}{l} e_3(x_1, x_2, x_3), e_2(x_1, x_2, x_3), e_1(x_1, x_2, x_3), \\ e_2(x_1, x_2), e_2(x_1, x_3), e_2(x_2, x_3) \end{array}.$$

By definition, the *DeConcini-Procesi-Tanisaki algebra* R_μ is the quotient algebra

$$R_\mu := P_n / I_\mu.$$

Since the ideal I_μ is homogeneous and S_n -invariant, the algebra R_μ affords a graded representation of S_n :

$$R_\mu = \bigoplus_{d=0}^{t_\mu} R_\mu^d,$$

where each homogeneous component R_μ^d is also an S_n -module. Its highest degree t_μ is known to be

$$n(\mu) = \sum_{i \geq 1} (i-1)\mu_i.$$

It is well known that R_μ is isomorphic as an S_n -module to the representation induced from the trivial representation of the Young subgroup S_μ :

$$R_\mu \cong_{S_n} \text{Ind}_{S_\mu}^{S_n} 1. \quad (2.1)$$

Our first problem is to determine which positive integers l satisfy the condition that the subspaces

$$R_\mu(k; l) := \bigoplus_{d \equiv k \pmod{l}} R_\mu^d, \quad k = 0, 1, \dots, l-1,$$

have the same dimension. We call the subspace $R_\mu(k; l)$ the k -th *partial sum modulo l* , or k -th *mod l sum* of the graded algebra R_μ for each $k = 0, 1, \dots, l-1$. To find such l 's, it is necessary to consider the *Hilbert polynomial*

$$\text{Hilb}_{R_\mu}(q) := \sum_{d \geq 0} q^d \dim R_\mu^d$$

for the algebra R_μ . In fact, we consider more generally the “graded character” of the algebra R_μ , since the Hilbert polynomial is a special case of it. Note that each homogeneous component R_μ^d is an S_n -module. We denote its character by $\text{char}R_\mu^d$, and then the *graded character* char_qR_μ of the graded S_n -module R_μ is defined by

$$\text{char}_qR_\mu := \sum_{d \geq 0} q^d \text{char}R_\mu^d.$$

If we evaluate the graded character at the identity element e , we obtain the Hilbert polynomial

$$\text{Hilb}_{R_\mu}(q) = \text{char}_qR_\mu(e),$$

since we have $\text{char}R_\mu^d(e) = \dim R_\mu^d$.

The following lemma, due to T. Oshima (Tokyo University), is fundamental [MN, Lemma 3].

Lemma 1 Let t be an indeterminate and consider a polynomial $f(t) = a_0 + a_1t + \cdots$ with coefficients in a field of characteristic zero. Let $m \geq 2$ be an integer, and ζ a primitive m -th root of unity. Then the following two conditions are equivalent:

1. For each $k = 1, \dots, m - 1$, we have $f(\zeta^k) = 0$.
2. The mod l sums $c_\ell(f) = \sum_{j \geq 1} a_{mj+\ell}$ of the coefficients of $f(t)$ coincide each other.

□

By this lemma, it is sufficient to show which roots of unity are zeros of the Hilbert polynomial of the algebra R_μ . For example, in the case of the coinvariant algebra R_n (the case $\mu = (1^n)$), its Hilbert polynomial $\text{Hilb}_{R_n}(q)$ is given by

$$\text{Hilb}_{R_n}(q) = \frac{(1-q)(1-q^2) \cdots (1-q^n)}{(1-q)^n}.$$

From this formula, it is easy to see that for every integer $l = 2, \dots, n$, we have

$$\text{Hilb}_{R_n}(\zeta_l^k) = 0$$

for each $k = 0, \dots, l - 1$. Therefore, the preceding lemma shows that the dimensions of the subspaces $R_n(k; l)$ ($k = 0, 1, \dots, l - 1$) do not depend on k for each fixed $l = 1, 2, \dots, n$.

In the next section, we will consider for which positive integers l the dimensions of the mod l sums of R_μ coincide. Our main tool is the graded character of R_μ , which turns out to be a “Green polynomial” of type A .

3 Green polynomials

In this section, we recall the definition of the Green polynomials, and see a canonical decomposition form (Theorem 5) that we use essentially in the proof of the main result. Let ρ be a partition of n and t an indeterminate. Since the *Hall-Littlewood functions* $P_\mu(x; t)$ form a $\mathbf{Z}[t]$ -basis of the ring of the symmetric functions $\Lambda[t] = \Lambda \otimes_{\mathbf{Z}} \mathbf{Z}[t]$, we can express the *power-sum symmetric function* $p_\rho(x)$ as a linear combination of $P_\mu(x; t)$'s with coefficients in $\mathbf{Z}[t]$ [Mc, III, (2.7)]:

$$p_\rho(x) = \sum_{\mu \vdash n} X_\rho^\mu(t) P_\mu(x; t).$$

Then the Green polynomial (of type A) $Q_\rho^\mu(q)$ [Gr] (see also [Mc, III, (7.8)]) is defined by modifying the coefficient $X_\rho^\mu(t)$ as follows:

$$Q_\rho^\mu(q) = q^{n(\mu)} X_\rho^\mu(q^{-1}). \quad (3.1)$$

It is known that the Green polynomials are described by the *Kostka polynomials*. See [Mc, III, Section 6] for the definition of the Kostka polynomials. Let λ and μ be partitions of the same size n . We denote by $K_{\lambda\mu}(q)$ the Kostka polynomial as usual, and define the *modified* one as follows:

$$\tilde{K}_{\lambda\mu}(q) := q^{n(\mu)} K_{\lambda\mu}(q^{-1}).$$

Then it is known [Mc, III, (7.11)] that

$$Q_\rho^\mu(q) = \sum_{\lambda \vdash n} \chi_\rho^\lambda \tilde{K}_{\lambda\mu}(q),$$

where χ_ρ^λ denotes the λ -th irreducible character evaluated at the conjugacy class of cycle type ρ . Moreover, it is also known that the modified Kostka polynomial $\tilde{K}_{\lambda\mu}(q)$ coincide with the Poincaré polynomial of the λ -th irreducible representation V^λ in R_μ (see e.g., [GP]):

$$\tilde{K}_{\lambda\mu}(q) = \sum_{\lambda \vdash n} [R_\mu^d : V^\lambda] q^d$$

Thus we have

$$Q_\rho^\mu(q) = \text{char}_q R_\mu(\rho).$$

In the rest of this section, we consider an explicit form of $Q_\rho^\mu(q)$, together with its property that we use later.

Definition 2 For a partition $\lambda = (1^{m_1} 2^{m_2} \dots n^{m_n}) \vdash n$, we define

$$\begin{aligned} e_\lambda(q) &:= (1-q)^{m_1} (1-q^2)^{m_2} \dots (1-q^n)^{m_n}, \\ b_\lambda(q) &:= \prod_{i \geq 1} (1-q)(1-q^2) \dots (1-q^{m_i}). \end{aligned}$$

Also, we define

$$z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \dots n^{m_n} m_n!$$

as usual.

For example, if $\lambda = (2, 2, 2, 1, 1) = (1^2 2^3)$, then we define

$$\begin{aligned} e_\lambda(q) &= (1-q)^2(1-q^2)^3, \\ b_\lambda(q) &= (1-q)(1-q^2)(1-q^3)(1-q)(1-q^2). \end{aligned}$$

Note that, if a partition λ is a union of two partitions μ and ν , then it is readily seen that $e_\lambda = e_\mu e_\nu$.

Explicit formulas for the Green polynomials $Q_\rho^\mu(q) = \text{char}_q R_\mu$ are known for some special cases. For example, in the case corresponding to the coinvariant algebra, i.e., the case $\mu = (1^n)$, we have the following formula (see e.g., [G, p. 376]).

Proposition 3 For each $\rho = (1^{r_1} 2^{r_2} \cdots n^{r_n}) \vdash n$, it holds that

$$Q_\rho^{(1^n)}(q) = \frac{b_{(1^n)}(q)}{e_\rho(q)} = \frac{(1-q)(1-q^2) \cdots (1-q^n)}{(1-q)^{r_1} (1-q^2)^{r_2} \cdots (1-q^n)^{r_n}}.$$

□

On the other hand, if μ is a hook of which the first component is 2, i.e., a partition of the form $\mu = (2, 1^{n-2})$, then we also have an explicit description of $Q_\rho^\mu(q)$ [Mr, Lemma 3].

Proposition 4 (Morris) For each partition $\rho = (1^{r_1} 2^{r_2} \cdots n^{r_n})$ of n , we have

$$Q_\rho^{(2, 1^{n-2})}(q) = \frac{(1-q) \cdots (1-q^{n-2})}{e_\rho(q)} \{(r_1 - 1)q^n - r_1 q^{n-1} + 1\}.$$

□

Let $\mu = (1^{m_1} 2^{m_2} \cdots n^{m_n})$ be a partition of n . Let M_μ be the maximum value of the multiplicities of μ :

$$M_\mu = \max\{m_1, m_2, \dots, m_n\}.$$

We can see from this formula that each integer $l \in \{(1), 2, \dots, n-2\}$ is available for coincidence of the dimensions of the mod l sums of R_μ . In these two examples, it can be seen that there remains a polynomial when we factoring out the rational part

$$\frac{(1-q) \cdots (1-q^{M_\mu})}{e_\rho(q)}$$

from $Q_\rho^\mu(q)$. In fact, this turns out to be true in general.

Theorem 5 Let μ and ρ be partitions of n . Then there exists a polynomial $G_\rho^\mu(q) \in \mathbf{Z}[q]$ satisfying

$$Q_\rho^\mu(q) = \frac{b_{(1^{M_\mu})}(q)}{e_\rho(q)} G_\rho^\mu(q).$$

Proof. We will show that

$$G_\rho^\mu(q) = \frac{e_\rho(q)}{b_{(1^{M_\mu})}(q)} Q_\rho^\mu(q)$$

is a Laurent polynomial. If this is correct, then the assertion follows from the identity

$$Q_\rho^\mu(q) e_\rho(q) = b_{(1^{M_\mu})}(q) G_\rho^\mu(q)$$

by comparing the coefficients of the lowest degree in each side. (Remark that Q , e , b are polynomials.) Actually, we will show that

$$\tilde{G}_\rho^\mu(q) := \frac{b_{(1^{M_\mu})}(q)}{b_\mu(q^{-1})} G_\rho^\mu(q)$$

is a Laurent polynomial. If this is true, then $G_\rho^\mu(q)$ is a Laurent polynomial since

$$\frac{b_\mu(q^{-1})}{b_{(1^{M_\mu})}(q)}$$

is also a Laurent polynomial.

Let $\bar{\mu}$ be a partition defined by $\bar{\mu}' = (\mu'_1, \dots, \mu'_{a-1})$, where $\mu' = (\mu'_1, \dots, \mu'_a)$ is the conjugate of μ . In other words, $\bar{\mu}$ is the partition whose Young diagram is obtained by deleting the last column from that of μ . Also, let $r = \mu'_a$ and $\nu = (1^r)$. If we express the product of the following two Hall-Littlewood functions $P_{\bar{\mu}} = P_{\bar{\mu}}(x; t)$ and $P_\nu = P_\nu(x; t)$ as a linear combination of Hall-Littlewood functions again, then we have

$$P_{\bar{\mu}} P_\nu = \sum_{\substack{\lambda \vdash n \\ \lambda - \bar{\mu} : \text{vertical } r\text{-strip}}} f_{\bar{\mu}\nu}^\lambda(t) P_\lambda, \quad (3.2)$$

where $f_{\bar{\mu}\nu}^\lambda(t) \in \mathbf{Z}[t]$. Recall that if $\theta = \lambda - \mu$ is a vertical r -strip, then we have [Mc, III (3.2)]

$$f_{\bar{\mu}(1^r)}^\lambda(t) := \prod_{i \geq 1} \begin{bmatrix} \lambda'_i - \lambda'_{i+1} \\ \lambda'_i - \bar{\mu}'_i \end{bmatrix}_t,$$

where $\begin{bmatrix} n \\ r \end{bmatrix}_t$ is a Gaussian polynomial.

If we rewrite (3.2) by

$$Q_\lambda = Q_\lambda(x; t) := b_\lambda(t) P_\lambda(x; t), \quad [\text{Mc, III (2.11)}]$$

then we have

$$Q_{\bar{\mu}} Q_\nu = \sum_\lambda f_{\bar{\mu}\nu}^\lambda(t) \frac{b_{\bar{\mu}}(t) b_\nu(t)}{b_\lambda(t)} Q_\lambda. \quad (3.3)$$

Also, it can be seen [Mc, III (7.5)] that

$$Q_\lambda(x; t) = \sum_{\rho \vdash n} \frac{e_\rho(t)}{z_\rho} X_\rho^\lambda(t) p_\rho(x). \quad (3.4)$$

By (3.3), it follows from (3.4) that

$$\begin{aligned} & \sum_{\substack{\rho^{(1)} \vdash n-r \\ \rho^{(2)} \vdash r}} \frac{e_{\rho^{(1)}}(t)e_{\rho^{(2)}}(t)}{z_{\rho^{(1)}}z_{\rho^{(2)}}} X_{\rho^{(1)}}^{\bar{\mu}} X_{\rho^{(2)}}^{\nu} p_{\rho^{(1)} \cup \rho^{(2)}}(x) \\ &= \sum_{\rho \vdash n} \frac{e_{\rho}(t)}{z_{\rho}} \left\{ \sum_{\lambda} f_{\bar{\mu}\nu}^{\lambda}(t) \frac{b_{\bar{\mu}}(t)b_{\nu}(t)}{b_{\lambda}(t)} X_{\rho}^{\lambda}(t) \right\} p_{\rho}(x). \end{aligned}$$

Comparing the coefficient of $p_{\rho}(x)$, we have

$$\begin{aligned} X_{\rho}^{\mu}(t) &= \sum_{\substack{\rho^{(1)} \vdash n-r \\ \rho^{(2)} \vdash r}} \frac{z_{\rho}}{z_{\rho^{(1)}}z_{\rho^{(2)}}} \frac{b_{\lambda}(t)}{b_{\bar{\mu}}(t)b_{\nu}(t)} X_{\rho^{(1)}}^{\bar{\mu}} X_{\rho^{(2)}}^{\nu} \\ &\quad - \sum_{\lambda \neq \mu} f_{\bar{\mu}\nu}^{\lambda}(t) \frac{b_{\mu}(t)}{b_{\lambda}(t)} X_{\rho}^{\lambda}(t), \end{aligned}$$

since $f_{\bar{\mu}\nu}^{\mu}(t) \equiv 1$. Rewriting this identity again by (3.1), we have

$$\begin{aligned} Q_{\rho}^{\mu}(q) &= \sum_{\substack{\rho^{(1)} \vdash n-r, \rho^{(2)} \vdash r, \\ \rho^{(1)} \cup \rho^{(2)} = \rho}} \frac{z_{\rho}}{z_{\rho^{(1)}}z_{\rho^{(2)}}} \frac{b_{\mu}(q^{-1})}{b_{\bar{\mu}}(q^{-1})b_{(1^r)}(q^{-1})} Q_{\rho^{(1)}}^{\bar{\mu}}(q) Q_{\rho^{(2)}}^{(1^r)}(q) \\ &\quad - \sum_{\substack{\lambda \vdash n \\ \lambda - \bar{\mu}: \text{vertical } r\text{-strip}}} f_{\bar{\mu}(1^r)}^{\lambda}(q^{-1}) \frac{b_{\mu}(q^{-1})}{b_{\lambda}(q^{-1})} q^{n(\mu)-n(\lambda)} Q_{\rho}^{\lambda}(q) \\ &= \frac{b_{(1^{M\mu})}(q)}{e_{\rho}(q)} \sum_{\rho^{(1)}, \rho^{(2)}} \frac{z_{\rho}}{z_{\rho^{(1)}}z_{\rho^{(2)}}} \frac{b_{\mu}(q^{-1})}{b_{(1^{M\mu})}(q)} \frac{b_{(1^{M\mu})}(q)}{b_{\bar{\mu}}(q^{-1})} G_{\rho^{(1)}}^{\bar{\mu}}(q) \frac{b_{(1^r)}(q)}{b_{(1^r)}(q^{-1})} G_{\rho^{(2)}}^{(1^r)}(q) \\ &\quad - \frac{b_{(1^{M\mu})}(q)}{e_{\rho}(q)} \sum_{\lambda} f_{\bar{\mu}(1^r)}^{\lambda}(q^{-1}) \frac{b_{\mu}(q^{-1})}{b_{(1^{M\mu})}(q)} q^{n(\mu)-n(\lambda)} \frac{b_{(1^{M\mu})}(q)}{b_{\lambda}(q^{-1})} G_{\rho}^{\lambda}(q). \end{aligned}$$

Hence it holds that

$$\begin{aligned} G_{\rho}^{\mu}(q) &= \sum_{\rho^{(1)}, \rho^{(2)}} \frac{z_{\rho}}{z_{\rho^{(1)}}z_{\rho^{(2)}}} \frac{b_{\mu}(q^{-1})}{b_{(1^{M\mu})}(q)} \frac{b_{(1^{M\mu})}(q)}{b_{\bar{\mu}}(q^{-1})} G_{\rho^{(1)}}^{\bar{\mu}}(q) \frac{b_{(1^r)}(q)}{b_{(1^r)}(q^{-1})} G_{\rho^{(2)}}^{(1^r)}(q) \\ &\quad - \sum_{\lambda} f_{\bar{\mu}(1^r)}^{\lambda}(q^{-1}) \frac{b_{\mu}(q^{-1})}{b_{(1^{M\mu})}(q)} q^{n(\mu)-n(\lambda)} \frac{b_{(1^{M\mu})}(q)}{b_{\lambda}(q^{-1})} G_{\rho}^{\lambda}(q). \end{aligned}$$

From the definition of \tilde{G} , it follows that

$$\begin{aligned} \tilde{G}_{\rho}^{\mu}(q) &= \sum_{\rho^{(1)}, \rho^{(2)}} \frac{z_{\rho}}{z_{\rho^{(1)}}z_{\rho^{(2)}}} \frac{b_{\mu}(q^{-1})}{b_{(1^{M\mu})}(q)} \tilde{G}_{\rho^{(1)}}^{\bar{\mu}}(q) \tilde{G}_{\rho^{(2)}}^{(1^r)}(q) \\ &\quad - \sum_{\lambda} f_{\bar{\mu}(1^r)}^{\lambda}(q^{-1}) \frac{b_{\mu}(q^{-1})}{b_{(1^{M\mu})}(q)} q^{n(\mu)-n(\lambda)} \tilde{G}_{\rho}^{\lambda}(q). \end{aligned} \tag{3.5}$$

In the following, we will show that $\tilde{G}_\rho^\mu(q)$ is a Laurent polynomial by induction on a partial order of the partitions. The partial order on the partitions are defined as follows: For any two partitions λ and μ , define

$$\lambda \leq \mu \iff \begin{cases} |\lambda| < |\mu|, & \text{or} \\ |\lambda| = |\mu|, & \text{and } \lambda \preceq \mu. \end{cases}$$

where $|\lambda|$ denotes the size of the partition λ , and \preceq denotes the *dominance partial order* [Mc, p. 7] of the partitions, which is defined by the following condition:

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \leq \mu_1 + \mu_2 + \cdots + \mu_i$$

for each $i = 1, 2, \dots, n$, where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_n)$. Now it is obvious that $\bar{\mu}, (1^r) \leq \mu$, since the sizes of $\bar{\mu}$ and (1^r) are smaller than μ . On the other hand, if $\lambda - \bar{\mu}$ is a vertical r -strip, then the size of these partitions λ, μ coincide, but λ is smaller than μ in the dominance order if $\lambda \neq \mu$. Thus we may assume that the \tilde{G} 's in the right hand side of (3.5) are all Laurent polynomials by the induction hypothesis. Finally, if we remark that, for λ such that $\lambda - \mu$ is a vertical r -strip, $f_{\bar{\mu}(1^r)}^\lambda(q^{-1})$ is also a Laurent polynomials, then it follows that $\tilde{G}_\rho^\mu(q)$ is a Laurent polynomial. The initial condition of the induction is satisfied, since $\tilde{G}_\rho^{(1^n)}(q) = (-1)^n q^n$ for all n and $\rho \vdash n$. This follows from the fact that $G_\rho^{(1^n)}(q) \equiv 1$ for all n and $\rho \vdash n$ (Proposition 3). \square

Example 6 According to the table of Green polynomials due to Green [Gr], the polynomial $G_\rho^{(2,1,1)}(q)$ can be computed for each ρ as follows:

$$\begin{aligned} G_{(1^4)}^\mu &= 1 - 4q^3 + 3q^4, \\ G_{(21^2)}^\mu &= 1 - 2q^3 + q^4, \\ G_{(2^2)}^\mu &= 1 - q^4, \\ G_{(31)}^\mu &= 1 - q^3, \\ G_{(4)}^\mu &= 1 - q^4. \end{aligned}$$

\square

Remark 7 In Example 6, all the polynomials still have the factor $1 - q$. In fact, we can expect more precise statement for the decomposition of Green polynomials. Let μ be a partition of n . Then for each partition $\rho \vdash n$, we conjecture that there exists a polynomial $H_\rho^\mu(q) \in \mathbf{Z}[q]$ such that

$$Q_\rho^\mu(q) = \frac{b_\mu(q)}{e_\rho(q)} H_\rho^\mu(q).$$

But in our present context, this is too much for our sake, since we only have to know what roots of unity are zeros of Green polynomials. \square

Corollary 8 Let μ be a partition of n , and an integer $l \in \{2, \dots, M_\mu\}$ arbitrarily fixed. Then it holds that $Q_{(1^n)}^\mu(\zeta_l^k) = 0$ for each $k = 1, \dots, l - 1$.

If we consider the case where $\mu = (21^3)$ for example, we can see from the table in [Gr] that

$$Q_{(1^5)}^\mu(q) = (1+q)(1+q+q^2)(1+2q+3q^2+4q^3),$$

and it is clear that the assertion of the corollary holds.

Let μ be a partition, and an integer l such that $1 \leq l \leq M_\mu$ fixed. By Oshima's lemma, it immediately follows that the dimensions of the mod l sums $R_\mu(k; l)$ of R_μ coincide each other. In the next section, we will consider its representation theoretical interpretation for the algebra R_μ .

At the end of this section, we mention to an explicit formula for the Green polynomials for arbitrary hook which generalizes Morris' formula (Proposition 4), that can be proved by similar argument to Theorem 5.

Proposition 9 Let $\mu = (n-h, 1^h)$ be a hook, and $\rho = (1^{r_1} 2^{r_2} \cdots n^{r_n}) \vdash n$ a partition of n . Then we have

$$Q_\rho^\mu(q) = \frac{(1-q) \cdots (1-q^h)}{e_\rho(q)} G_\rho^\mu(q),$$

where

$$G_\rho^\mu(q) = (1-q^n) - \sum_{k=1}^{n-h-1} \sum_{\tau=(1^{t_1} \cdots k^{t_k}) \vdash k} \binom{r_1}{t_1} \cdots \binom{r_k}{t_k} q^{n-k} e_\tau(q).$$

Proof. We prove by the induction on $n-h$. For the case $n-h=1$, it is clear from Proposition 3, from which we can see that $G_\rho^\mu(q) \equiv 1$. Suppose that $n-h > 1$. Let $\bar{\mu} = (1^{h+1})$ and $\nu = (n-h-1)$. Consider the product of two Hall-Littlewood functions $P_{\bar{\mu}} = P_{\bar{\mu}}(x; t)$ and $P_\nu = P_\nu(x; t)$, and expand it into a linear combination of $P_\lambda(x; t)$'s again. Then we have [M, III, (2.7) and (3.7)]

$$P_{\bar{\mu}} P_\nu = f_{\bar{\mu}\nu}^{(n-h-1, 1^{h+1})}(t) P_{(n-h-1, 1^{h+1})} + f_{\bar{\mu}\nu}^\mu(t) P_\mu.$$

If $\lambda = (n-h-1, 1^{h+1})$ or $\lambda = \mu$, then $\lambda - \bar{\mu}$ is a *horizontal strip* [Mc, p. 5]. We have an explicit formula for the polynomials $f_{\bar{\mu}\nu}^\lambda(t)$ [Mc, III, (3.10)] in this case, and it holds that

$$f_{\bar{\mu}\nu}^{(n-h-1, 1^{h+1})}(t) = 1, \quad f_{\bar{\mu}\nu}^\mu(t) = 1.$$

Therefore we have

$$P_{\bar{\mu}} P_\nu = P_{(n-h-1, 1^{h+1})} + P_\mu.$$

By this identity, the same argument as in the proof of Theorem 5 shows that

$$Q_\rho^\mu(q) = \frac{1}{1-q^{h+1}} \left\{ Q_\rho^\eta(q) - \sum_{\substack{\rho^{(1)} \vdash h+1 \\ \rho^{(2)} \vdash n-h-1 \\ \rho^{(1)} \cup \rho^{(2)} = \rho}} q^{h+1} Q_{\rho^{(1)}}^{\bar{\mu}}(q) \right\},$$

where $\eta = (n - h - 1, 1^{h+1})$. By the induction hypothesis, it follows that

$$\begin{aligned}
Q_\rho^\mu(q) &= \frac{1}{1 - q^{h+1}} \left\{ \frac{(1 - q) \cdots (1 - q^{h+1})}{e_\rho(q)} G_\rho^\eta(q) \right. \\
&\quad \left. - \sum_{\substack{\rho^{(1)} \vdash h+1 \\ \rho^{(2)} \vdash n-h-1 \\ \rho^{(1)} \cup \rho^{(2)} = \rho}} \frac{z_\rho}{z_{\rho^{(1)}} z_{\rho^{(2)}}} q^{h+1} \frac{(1 - q) \cdots (1 - q^{h+1})}{e_{\rho^{(1)}}(q)} \right\} \\
&= \frac{(1 - q) \cdots (1 - q^h)}{e_\rho(q)} \left\{ G_\rho^\eta(q) - \sum_{\rho^{(1)}, \rho^{(2)}} \frac{z_\rho}{z_{\rho^{(1)}} z_{\rho^{(2)}}} q^{h+1} \frac{e_\rho(q)}{e_{\rho^{(1)}}(q)} \right\} \\
&= \frac{(1 - q) \cdots (1 - q^h)}{e_\rho(q)} \left\{ G_\rho^\eta(q) - \sum_{\substack{\tau \vdash n-h-1 \\ \tau \subset \rho}} \frac{z_\rho}{z_{\rho-\tau} z_\tau} q^{n-(n-h-1)} e_\tau(q) \right\},
\end{aligned}$$

where $\tau \subset \rho$ denotes that τ is a subpartition of ρ . Therefore we have

$$G_\rho^\mu(q) = G_\rho^\eta(q) - \sum_{\tau} \frac{z_\rho}{z_{\rho-\tau} z_\tau} q^{n-(n-h-1)} e_\tau(q).$$

Remark that, for $\tau = (1^{t_1} 2^{t_2} \cdots)$, we have

$$\frac{z_\rho}{z_{\rho^{(1)}} z_{\rho^{(2)}}} = \binom{r_1}{t_1} \binom{r_2}{t_2} \cdots.$$

Then, again by the induction hypothesis, the assertion follows. \square

Example 10 Let us consider the case where $\mu = (3, 1, 1, 1)$. Let $\rho = (2^2 1^2)$ for example. By Proposition 7, we have

$$Q_\rho^\mu(q) = \frac{(1 - q)(1 - q^2)(1 - q^3)}{(1 - q)^2(1 - q^2)^2} G_\rho^\mu(q),$$

and

$$\begin{aligned}
G_\rho^\mu(q) &= (1 - q^6) \sum_{k=1}^2 \sum_{\tau=(1^{t_1} \dots k^{t_k}) \vdash k} \binom{r_1}{t_1} \cdots \binom{r_k}{t_k} q^{n-k} e_\tau(q) \\
&= (1 - q^6) - \binom{2}{1} \binom{2}{0} q^5 (1 - q) - \binom{2}{2} \binom{2}{0} q^4 (1 - q)^2 - \binom{2}{0} \binom{2}{1} q^4 (1 - q^2) \\
&= 1 - 3q^4 + 2q^6.
\end{aligned}$$

Thus

$$\begin{aligned}
Q_\rho^\mu(q) &= \frac{(1 - q)(1 - q^2)(1 - q^3)}{(1 - q)^2(1 - q^2)^2} (1 - 3q^4 + 2q^6) \\
&= (1 + q)(1 + q + q^2)(1 - q + 2q^2 - 2q^3),
\end{aligned}$$

which coincides with the result on the table due to Morris [Mr]. \square

4 hook case

In this section, we will find a representation theoretical interpretation of ‘coincidence of dimension’ of the algebra R_μ in the case where μ is a hook.

Let $\mu = (n - h, 1^h) \vdash n$ be a hook. Suppose that $n - h > 1$, since we should exclude the case of coinvariant algebras from our argument. In this case, we have $M_\mu = h$. Then for each fixed $l = 1, 2, \dots, h$, it follows from Proposition 9 and Lemma 1 that the mod l sums $R_\mu(k; l)$ of the algebra R_μ have the same dimension $\dim R_\mu/l$. Since the case $l = 1$ is trivial, we consider the problem for $l = 2, 3, \dots, h$ in the sequel. Precisely:

For each fixed $l = 2, 3, \dots, h$,

- find a subgroup $H_\mu(l)$ of S_n , and
- find $H_\mu(l)$ -modules $Z_\mu(k; l)$ ($k = 0, 1, \dots, l - 1$) of equal dimension

such that

$$R_\mu(k; l) \cong_{S_n} \text{Ind}_{H_\mu(l)}^{S_n} Z_\mu(k; l)$$

for each $k = 0, 1, \dots, l - 1$.

Let $h = dl + r$ ($0 \leq r \leq l - 1$), and let $\bar{\mu}$ denote the partition

$$\bar{\mu} = (n - h, 1^{h-dl}) \vdash n - dl.$$

Consider the following product a of d cyclic permutations of order l :

$$a = (1, 2, \dots, l)(l + 1, l + 2, \dots, 2l) \cdots ((d - 1)l + 1, (d - 1)l + 2, \dots, dl).$$

It is obvious that $a^l = 1$. Consider the cyclic subgroup H_1 of S_n generated by a :

$$H_1 := \langle a \rangle \cong C_l,$$

where C_l denotes the cyclic group of order l . Consider also the isotropic subgroup H_2 of S_n for $\{1, 2, \dots, dl\}$:

$$H_2 = \{\sigma \in S_n \mid \sigma(i) = i \text{ for } i = 1, 2, \dots, dl\}.$$

If we denote by $S_{\{i, j, \dots, k\}}$ the symmetric group of the letters $\{i, j, \dots, k\}$, then we have $H_2 = S_{\{dl+1, \dots, n\}}$, thus H_2 is isomorphic to the symmetric group of $n - dl$ letters. Let a subgroup $H_\mu(l)$ of S_n be the direct product of H_1 and H_2 :

$$H_\mu(l) := H_1 \times H_2 \cong C_l \times S_{n-dl}.$$

For each $k = 0, 1, \dots, l - 1$, we define a representation $Z_\mu(k; l)$ of $H_\mu(l)$ as follows:

$$Z_\mu(k; l) := \bigoplus_{d=0}^{n(\bar{\mu})} \varphi^{(k-d)} \otimes R_{\bar{\mu}}^d,$$

where $\varphi^{(s)}$ is the irreducible representation $a \mapsto \zeta_l^s$ of the cyclic group $C_l = \langle a \rangle$, and $R_{\bar{\mu}}^d$ denotes the d -th homogeneous component of the DeConcini-Procesi-Tanisaki algebra $R_{\bar{\mu}}$ corresponding to the partition $\bar{\mu}$. Remark that these l modules $Z_{\mu}(k; l)$ have the same dimension which is equal to that of $R_{\bar{\mu}}$, since the $H_{\mu}(l)$ -module $Z_{\mu}(k; l)$ is obtained by twisting each homogeneous component $R_{\bar{\mu}}^d$ of the graded algebra $R_{\bar{\mu}}$ by a one-dimensional representation $\varphi^{(k-d)}$.

Example 11 Let $\mu = (21^5)$ and $l = 2 (< 5 = M_{\mu})$. In this case, the element a is defined by $a = (1, 2)(3, 4)$, and the subgroup $H_{\mu}(2)$ is then defined by

$$H_{\mu}(2) = \langle a \rangle \times S_{\{5,6,7\}} \cong C_2 \times S_3.$$

The $H_{\mu}(2)$ -modules $Z_{\mu}(k; 2)$ ($k = 0, 1$) of equal dimension are defined by

$$\begin{aligned} Z_{\mu}(0; 2) &= (\varphi^{(0)} \otimes R_{\bar{\mu}}^0) \oplus (\varphi^{(1)} \otimes R_{\bar{\mu}}^1), \\ Z_{\mu}(1; 2) &= (\varphi^{(1)} \otimes R_{\bar{\mu}}^0) \oplus (\varphi^{(0)} \otimes R_{\bar{\mu}}^1), \end{aligned}$$

where $\bar{\mu} = (2, 1)$ and $\varphi^{(k)}$ is the irreducible representation of C_2 sending the generator a to ζ_2^k . \square

Now our subject is to show

$$R_{\mu}(k; l) \cong \text{Ind}_{H_{\mu}(l)}^{S_n} Z_{\mu}(k; l)$$

for each $k = 0, 1, \dots, l-1$. We will see in the following that these S_n -isomorphisms amount to an $S_n \times C_l$ -isomorphism

$$R_{\mu} \cong_{S_n \times C_l} \text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}.$$

Here the S_n -actions on both sides are natural ones. The C_l -actions in both sides are defined as follows. In the left hand side, the cyclic group C_l acts on R_{μ} as a scalar multiplication for each homogeneous component:

$$a \cdot x = \zeta_l^d x, \quad x \in R_{\mu}^d.$$

On the other hand, remark that the induced representation $\text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}$ has the following realization:

$$\text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}} = \bigoplus_{\sigma \in S_n/S_{n-dl}} \sigma \otimes R_{\bar{\mu}}. \quad (4.1)$$

By the identification (4.1), the action of the cyclic group C_l on $\text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}$ is defined by

$$a(\sigma \otimes x) := \sigma a^{-1} \otimes ax = \zeta_l^d \sigma a^{-1} \otimes x \quad \text{if } x \in R_{\bar{\mu}}^d.$$

We can show that these two $S_n \times C_l$ -module structures are actually isomorphic. It should be remarked here that it can be easily seen from (2.1) that this isomorphism is trivially hold as an S_n -isomorphism.

Theorem 12 Let a partition $\mu = (n - h, 1^h) \vdash n$ ($n - h > 1$) be a hook, l a positive integer belonging to $\{2, 3, \dots, h\}$. Let $n = dl + r$ ($0 \leq r \leq l - 1$), and $\bar{\mu} = (n - h, 1^r)$ a partition obtained by deleting dl 1's from μ . Then, there exists an isomorphism of $S_n \times C_l$ -modules

$$R_\mu \cong \text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}.$$

This theorem is shown in the next section. In the rest of this section, we will verify that the theorem is actually equivalent to our claim.

Proposition 13 Let $\mu = (n - h, 1^h) \vdash n$ ($n - h > 1$) be a hook, and let an integer $l \in \{2, 3, \dots, h\}$ be fixed. If the isomorphism of $S_n \times C_l$ -modules in Theorem 12 holds, then there exists an S_n -module isomorphism

$$R_\mu(k; l) \cong \text{Ind}_{S_{n-dl}}^{S_n} Z_\mu(k; l)$$

for each $k = 0, 1, \dots, l - 1$.

Conversely, if these S_n -module isomorphisms hold, then R_μ and $\text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}$ are isomorphic as $S_n \times C_l$ -modules.

Proof. Suppose that there exists an $S_n \times C_l$ -module isomorphism

$$R_\mu \cong \text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}. \quad (4.2)$$

We compare the eigenspaces of the generator a of the cyclic group C_l in each side of (4.2). For each $k = 0, 1, \dots, l - 1$, it is clear from the action of C_l that the eigenspace of a with the eigenvalue ζ_l^k in the left hand side coincide with the k -th mod l sum $R_\mu(k; l)$. In the right hand side, we have

$$\begin{aligned} \text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}} &= \bigoplus_{\sigma \in S_n/S_{n-dl}} \sigma \otimes R_{\bar{\mu}} \\ &= \bigoplus_{d \geq 0} \bigoplus_{w \in S_n/C_l \times S_{n-dl}} \bigoplus_{j=0}^{l-1} w a^j \otimes R_{\bar{\mu}}^d \\ &= \bigoplus_{d \geq 0} \bigoplus_{w \in S_n/C_l \times S_{n-dl}} \bigoplus_{s=0}^{l-1} w b_s \otimes R_{\bar{\mu}}^d, \end{aligned}$$

where $b_s = 1 + (\zeta_l^s)a + (\zeta_l^{2s})a^2 + \dots + (\zeta_l^{(l-1)s})a^{l-1}$ for each $s = 0, 1, \dots, l - 1$. Since it holds that $b_s a^{-1} = \zeta_l^s b_s$ for each s , it can readily be seen that the ζ_l^k -eigenspace of a in the right hand side is

$$\bigoplus_{\substack{d,s \\ d+s \equiv k \pmod{l}}} b_s \otimes R_{\bar{\mu}}^d.$$

Noting that $\mathbf{C}b_s \cong_{C_l} \varphi^{(s)}$ for each $s = 0, 1, \dots, l - 1$, the necessity part follows. Conversely, the sufficiency part is proved by tracing back the proof of the necessity part. \square

5 Proof

In this section, we will verify Theorem 12:

$$R_\mu \cong_{S_n \times C_l} \text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}.$$

Since we work on a field of characteristics zero, it suffices to show that

$$\text{char} R_\mu(w, a^j) = \text{char} \text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}(w, a^j) \quad (5.1)$$

for every $(w, a^j) \in S_n \times C_l$ and $j = 0, 1, \dots, l-1$. Recall that the graded character $\text{char}_q R_\mu(\rho)$ coincides with the Green polynomial $Q_\rho^\mu(q)$ for each $\rho \vdash n$. In this point of view, considering the action of C_l , it is not difficult to see that (5.1) is equivalent to the following identity:

$$Q_{\lambda(w)}^\mu(q)|_{q=\zeta_l^j} = \text{char} \text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}(w, a^j). \quad (5.2)$$

Note that the case $j = 0$ is already proved for any $w \in S_n$. In this case, (5.1) reduces to the identity

$$\text{char} R_\mu(w) = \text{char} \text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}(w),$$

which is assured by the S_n -module isomorphism (2.1). Therefore it is enough to show (5.1) for $(w, a^j) \in S_n \times C_l$ with $1 \leq j \leq l-1$. First, we will consider the *support* of $\text{char} \text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}$, i.e., the elements (w, a^j) of $S_n \times C_l$ satisfying

$$\text{char} \text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}(w, a^j) \neq 0.$$

Actually, we will see the necessary condition for such an element of $S_n \times C_l$.

Suppose that $(w, a^j) \in S_n \times C_l$ is a support of $\text{char} \text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}$. Since we have

$$\text{char} \text{Ind}_{S_{n-dl}}^{S_n} (w, a^j) R_{\bar{\mu}} = \sum_{\sigma \in S_n/S_{n-dl}} \text{char}(\sigma \otimes R_{\bar{\mu}})(w, a^j),$$

there exists an element σ of S_n/S_{n-dl} and a basis element x of $R_{\bar{\mu}}^d$ such that

$$(w, a^j)(\sigma \otimes x)|_{\sigma \otimes x} \neq 0,$$

where $(w, a^j)(\sigma \otimes x)|_{\sigma \otimes x}$ denotes the coefficient of $\sigma \otimes x$ in the linear expansion of $(w, a^j)(\sigma \otimes x)$ by the basis $\{\sigma \otimes x\}$ of $\text{char} \text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}$. In this case, since we have

$$(w, a^j)(\sigma \otimes x) = w\sigma a^{-j} \otimes a^j x = \zeta_l^{dj} w\sigma a^{-j} \otimes x,$$

it holds that

$$(w, a^j)(\sigma \otimes x)|_{\sigma \otimes x} \neq 0 \implies w\sigma a^{-j} \equiv \sigma \pmod{S_{n-dl}}.$$

This shows that there exists an element τ of S_{n-dl} such that $w\sigma a^{-j} = \sigma\tau$, i.e., $w = \sigma\tau a^j \sigma^{-1}$.

To summarize:

Lemma 14 If an element (w, a^j) of $S_n \times C_l$ satisfies the condition

$$\text{char Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}(w, a^j) \neq 0,$$

then w is conjugate to $a^j \tau$ for some $\tau \in S_{n-dl}$: $w \sim a^j \tau$, $\tau \in S_{n-dl}$. \square

Our next subject is to compute the character value $\text{char Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}(w, a^j)$ for a support $(w, a^j) \in S_n \times C_l$. Let an element $(w, a^j) \in S_n \times C_l$ satisfy

$$\text{char Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}(w, a^j) \neq 0. \quad (5.3)$$

By Lemma 14, we may assume that

$$w = a^j \tau, \quad \tau \in S_{n-dl}$$

without loss of generality. It also follows from (5.3) that there exists $\sigma \in S_n/S_{n-dl}$ such that

$$w \sigma a^{-j} = \sigma \rho \quad \text{for some } \rho \in S_{n-dl}.$$

For such an element σ , we have

$$(w, a^j)(\sigma \otimes x) = w \sigma a^{-j} \otimes a^j x = \sigma \rho \otimes a^j x = \sigma \otimes a^j \rho x.$$

(Remark that a and ρ commute.) If we denote by $\mathcal{B}_{\bar{\mu}}$ a homogeneous basis of $R_{\bar{\mu}}$, then it follows that

$$\begin{aligned} \text{char}(\sigma \otimes R_{\bar{\mu}})(w, a^j) &= \sum_{x \in \mathcal{B}_{\bar{\mu}}} (w, a^j)(\sigma \otimes x)|_{\sigma \otimes x} \\ &= \sum_{d \geq 0} \sum_{x \in \mathcal{B}_{\bar{\mu}}} \zeta_l^{dj} \sigma \otimes \rho \sigma x|_{\sigma \otimes x} \\ &= \text{char}_q R_{\bar{\mu}}(\rho)|_{q=\zeta_l^j}. \end{aligned}$$

Let

$$\mathcal{S}^{(j)}(w) := \{\sigma \in S_n/S_{n-dl} \mid w \sigma a^{-j} \equiv \sigma \pmod{S_{n-dl}}\},$$

and

$$\mathcal{S}_{\rho}^{(j)}(w) := \{\sigma \in S_n/S_{n-dl} \mid w \sigma a^{-j} = \sigma \rho\}$$

for each $\rho \in S_{n-dl}$. Note that

$$\mathcal{S}^{(j)}(w) = \bigcup_{\rho \in S_{n-dl}} \mathcal{S}_{\rho}^{(j)}(w). \quad (\text{disjoint union})$$

Then it holds that

$$\begin{aligned} \text{char Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}(w, a^j) &= \sum_{\rho \in S_{n-dl}} \sum_{\sigma \in \mathcal{S}_{\rho}^{(j)}(w)} \text{char}(\sigma \otimes R_{\bar{\mu}})(w, a^j) \\ &= \sum_{\rho \in S_{n-dl}} \#\mathcal{S}_{\rho}^{(j)}(w) \text{char} R_{\bar{\mu}}(\rho)|_{q=\zeta_l^j}. \end{aligned}$$

Remark that $\rho \in S_{n-dl}$ satisfying $\mathcal{S}_{\rho}^{(j)}(w) \neq \emptyset$ are all conjugate in S_{n-dl} . It is easy to see that:

Lemma 15 Let us consider an element $w \in S_n$ of the form $w = a^j \tau$ ($\tau \in S_{n-dl}$). Then, for each $\rho \in S_{n-dl}$, we have

$$w \sigma a^{-j} = \sigma \rho \implies \tau \sim \rho.$$

Proof. Suppose that $w \sigma a^{-j} = \sigma \rho$ for some $\rho \in S_{n-dl}$. If $w = a^j \tau$, then we have $a^j \tau \sigma a^{-j} = \sigma \rho$, hence $a^j \tau \sigma = \sigma \rho a^j$. Therefore, $a^j \tau$ is conjugate to ρa^j . Thus τ and ρ are conjugate since C_l and S_{n-dl} commute. \square

By Lemma 15, we have

$$\begin{aligned} \text{char Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}(w, a^j) &= \sum_{\rho \in S_{n-dl}} \#\mathcal{S}_{\rho}^{(j)}(w) \text{char} R_{\bar{\mu}}(\tau)|_{q=\zeta_l^j} \\ &= \#\mathcal{S}^{(j)}(w) \text{char} R_{\bar{\mu}}(\tau)|_{q=\zeta_l^j}. \end{aligned}$$

The cardinality of $\mathcal{S}^{(j)}(w)$ is obtained as follows.

Proposition 16 For $w = a^j \tau \in C_l \times S_{n-dl} = H_{\mu}(l)$, it holds that

$$\#\mathcal{S}^{(j)}(w) = p^e e! \binom{z_p + e}{e},$$

where $\lambda(a^j) = (p^e)$ and $\lambda(\tau) = (1^{z_1} 2^{z_2} \dots)$ are the cycle type of a^j and τ respectively.

Proof. First we remark that

$$\#\mathcal{S}^{(j)}(a^j \tau) = \#\mathcal{S}^{(j)}(a^j \tau')$$

if τ and τ' are conjugate in S_{n-dl} . This follows from the fact that, if $\tau' = z \tau z^{-1}$ ($z \in S_{n-dl}$), then we have a bijection

$$\mathcal{S}^{(j)}(a^j \tau) \longrightarrow \mathcal{S}^{(j)}(a^j \tau') : \sigma \longmapsto z \sigma z^{-1}$$

Hence it is enough to confirm the assertion for the following special τ :

$$\begin{aligned} \tau &= \tau^{(1)} \tau^{(2)} \\ \tau^{(1)} &= (dl + 1, \dots, dl + p) \cdots (dl + (z_p - 1)p + 1, dl + z_p p), \end{aligned}$$

where $\tau^{(2)}$ is a permutation of remaining letters $\{dl + z_p p + 1, \dots, n\}$.

Our subject here is enumeration of the elements $\sigma \in S_n/S_{n-dl}$ satisfying the condition

$$a^j \tau \sigma a^{-j} = \sigma \rho \tag{5.4}$$

for some $\rho \in S_{n-dl}$. Note that, by (5.4), τ and ρ should be conjugate in S_{n-dl} . Hence it is enough to count the number of elements $\sigma \in S_n/S_{n-dl}$ satisfying

$$a^j \tau \sigma = \sigma a^j \rho \tag{5.5}$$

for $\tau, \rho \in S_{n-dl}$ with $\lambda(\tau) = \lambda(\rho)$.

Let the cycle type $\lambda(a^j)$ of a^j be (p^e) . Note that $pe = dl$. Let ρ decompose into the product of z_p cycles of length p and the remainder as follows:

$$\rho = \rho^{(1)}\rho^{(2)}, \quad \lambda(\rho^{(1)}) = (p^{z_p}), \quad \lambda(\rho^{(2)}) = \lambda(v) - (p^{z_p}).$$

Since the effect of the action of $\tau^{(2)}$ should coincide with that of the action of $\rho^{(2)}$ on the positions of the components of σ , it should hold that

$$\{dl + z_p p + 1, \dots, n\} \subset \{\sigma_{dl+1}, \dots, \sigma_n\}.$$

Since σ is an element of S_n/S_{n-dl} , we may assume that $\sigma_{dl+1} < \dots < \sigma_n$. Therefore the last $n - (e + z_p)p (= n - (dl + z_p p + 1) + 1)$ components of σ are uniquely determined as follows:

$$\sigma_{dl+z_p p+1} = dl + z_p p + 1, \dots, \sigma_n = n.$$

Now, it suffices to enumerate the elements

$$\sigma' = [\sigma_1, \sigma_2, \dots, \sigma_{dl+z_p p}] \in S_{dl+z_p p}$$

satisfying

$$a^j \tau^{(1)} \sigma' = \sigma' a^j \rho^{(2)}, \quad \sigma_{dl+1} < \dots < \sigma_{dl+z_p p}.$$

At the beginning, it is necessary to determine which e cycles of length p come to the first pe positions in σ' corresponding to the permutation a^j . There are $\binom{z_p + e}{e}$ ways of such a choice. Once we fix such a choice, the number of arranging these components with respect to the condition $a^j[\sigma_1, \dots, \sigma_{pe}] = [\sigma_1, \dots, \sigma_{pe}]a^j$ is just the cardinality of the centralizer of a^j , which is equal to $p^e e!$. The remaining parts are uniquely determined, since they should obey the condition $\sigma_{pe+1} < \dots < \sigma_{dl+z_p p}$.

Thus the number of elements $\sigma \in S_n/S_{n-dl}$ satisfying (5.5) is

$$\binom{z_p + e}{e} p^e e!.$$

□

Example 17 Let us consider the case where $\mu = (3, 1^8) \vdash 11$ and $l = 3$. In this case, we have $h = 8$ and our permutation a is

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 3 & 1 & 5 & 6 & 4 & 7 & 8 & 9 & 10 & 11 \end{pmatrix} = (1, 2, 3)(4, 5, 6),$$

since $8 = 2 \times 3 + 2$ ($d = 2$). Suppose that the cycle type $\lambda(\tau)$ of an element $\tau \in S_{\{7,8,9,10,11\}} \cong S_{n-dl} = S_5$ is $\lambda(\tau) = (3, 2)$. We can take $\tau = (7, 8, 9)(10, 11)$ without loss of generality. We consider the case $j = 1$ in this example. Then the elements $\sigma \in S_n/S_{n-dl}$ that we are going to enumerate satisfy the condition $a\tau\sigma a^{-1} = \sigma\rho$ for some $\rho \in S_{\{7,8,9,10,11\}}$ of which cycle type

is $\lambda(\rho) = (3, 2)$. We may also assume that $\sigma_7 < \sigma_8 < \dots < \sigma_{11}$. Therefore, they should satisfy

$$\begin{aligned}
& (1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11)[\sigma_1, \sigma_2, \dots, \sigma_{11}] \\
&= [\sigma_1, \sigma_2, \dots, \sigma_{11}](1, 2, 3)(4, 5, 6) \begin{pmatrix} 7 & 8 & 9 & 10 & 11 \\ \rho_7 & \rho_8 & \rho_9 & \rho_{10} & \rho_{11} \end{pmatrix} \\
&= [\sigma_1, \sigma_2, \dots, \sigma_{11}](1, 2, 3)(4, 5, 6)(i, j, k)(a, b) \\
&= [\{\sigma_1, \sigma_2, \sigma_3\}, \{\sigma_4, \sigma_5, \sigma_6\}, \{\sigma_i, \sigma_j, \sigma_k\}, \{\sigma_a, \sigma_b\}],
\end{aligned}$$

where $\rho = (i, j, k)(a, b)$ denotes its cyclic decomposition, and $[\{\sigma_1, \sigma_2, \sigma_3\} \dots]$ denotes that we can ignore the order of these components $\sigma_1, \sigma_2, \sigma_3$ in the permutation. Then, since it should for example coincide the effect of the left action of $(1, 2, 3)$ on the letters of σ with the effect of the right action of one of the cyclic permutations $(1, 2, 3)$, $(4, 5, 6)$, (i, j, k) on the position of letters of σ , we can conclude that one of the following conditions holds: $\{1, 2, 3\} = \{\sigma_1, \sigma_2, \sigma_3\}$, $\{\sigma_4, \sigma_5, \sigma_6\}$, $\{\sigma_i, \sigma_j, \sigma_k\}$. From this observation, we can see that there are $\binom{2+1}{2}$ ways of assigning letters for $\{\sigma_1, \sigma_2, \sigma_3\}$, $\{\sigma_4, \sigma_5, \sigma_6\}$. Once we choose letters for these positions, the remaining parts are uniquely determined by the condition $\sigma_7 < \sigma_8 < \dots < \sigma_{11}$. Thus there are $3^2 2! \binom{2+1}{2}$ suitable elements in S_n/S_{n-dl} . \square

To summarize:

Proposition 18 Let j be an integer in $\{1, 2, \dots, l-1\}$. Suppose that the cycle type of a^j is given by (p^e) . For each $(w, a^j) \in S_n \times C_l$, we have

$$\begin{aligned}
& \text{char Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}(w, a^j) \\
&= \begin{cases} p^e e! \binom{z_p + e}{e} \text{char}_q R_{\bar{\mu}}(\tau)|_{q=\zeta_l^j}, & \text{if } w \sim a^j v \text{ for some } \tau \in S_{n-dl} \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

\square

Now, by Lemma 14 and Proposition 18, (5.2) is equivalent to the following two conditions:

- $Q_\rho^\mu(\zeta_l^j) \neq 0 \implies \rho = \lambda(a^j \tau) \quad (\tau \in S_{n-dl})$,
- $Q_{\lambda(a^j \tau)}^\mu(\zeta_l^j) = p^e e! \binom{z_p + e}{e} Q_{\lambda(\tau)}^{\bar{\mu}}(\zeta_l^j) \quad (a^j \tau \in H_\mu(l))$.

First we consider the sufficient condition for a partition ρ satisfying

$$Q_\rho^\mu(\zeta_l^j) \neq 0.$$

It is known that the Green polynomials can also defined as the scalar product value of power-sum functions and “modified” Hall-Littlewood functions as follows. Let μ be a partition.

The *modified* Hall-Littlewood function $Q'_\mu(x; q)$ are defined by changing the variables of Hall-Littlewood functions $Q_\mu(x; q)$ from $x = (x_1, x_2, \dots)$ to

$$x/(1-q) = (x_1, qx_1, q^2x_1, \dots; x_2, qx_2, q^2x_2, \dots),$$

i.e.,

$$Q'_\mu(x; q) := Q_\mu\left(\frac{x}{1-q}; q\right).$$

The modified Hall-Littlewood functions $\{Q'_\mu(x; q)\}$ form the dual basis of $\{P_\mu(x; q)\}$ with respect to the ordinary inner product of Λ , i.e.,

$$\langle P_\lambda(x; q), Q'_\mu(x; q) \rangle = \delta_{\lambda, \mu},$$

where $\langle \cdot, \cdot \rangle$ is defined by $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda, \mu}$, and s_λ is the *Schur function* [Mc] corresponding to the partition λ . Therefore we have

$$X_\rho^\mu(q) = \langle Q'_\mu(x; q), p_\rho \rangle.$$

Let $\lambda = (1^{m_1} 2^{m_2} \dots n^{m_n})$ be a partition of n . Then, with respect to the inner product of Λ , it is known that the adjoint transformation of

$$\times p_\lambda : \Lambda \longrightarrow \Lambda : f \longmapsto p_\lambda f$$

is given by

$$z_\lambda \frac{\partial}{\partial p_\lambda} : \Lambda \longrightarrow \Lambda,$$

i.e.,

$$\langle p_\lambda f, g \rangle = \langle f, z_\lambda \frac{\partial}{\partial p_\lambda} g \rangle, \quad f, g \in \Lambda,$$

where

$$\frac{\partial}{\partial p_\lambda} = \left(\frac{\partial}{\partial p_1}\right)^{m_1} \left(\frac{\partial}{\partial p_2}\right)^{m_2} \dots \left(\frac{\partial}{\partial p_n}\right)^{m_n}.$$

Our object is a partition ρ satisfying $Q'_\rho(\zeta_l^j) \neq 0$, which is equivalent to

$$X_\rho^\mu(\zeta_l^j) \neq 0,$$

since $Q'_\rho(q) = q^{n(\mu)} X_\rho^\mu(q^{-1})$. The following two propositions due to Lascoux-Leclerc-Thibon [LLT] are fundamental for our argument.

Proposition 19 Let $\mu = (1^{m_1} 2^{m_2} \dots n^{m_n})$ be a partition of n , l a positive integer, ζ_p a primitive root of unity. For each $i = 1, 2, \dots, n$, let $m_i = k_i l + r_i$, $0 \leq r_i \leq l - 1$. Then we have

$$Q'_\mu(x; \zeta_l) = Q'_{\bar{\mu}}(x; \zeta_l) \prod_{i=1}^n Q'_{(i^{k_i})}(x; \zeta_l).$$

□

Proposition 20 Let $p_k(h_r(X))$ denote the *plethysm* [Mc, I, Section 8] of a complete symmetric function $h_r(X)$ by a power-sum $p_k(X)$. Then we have

$$Q'_{(r^k)}(x; \zeta_k) = (-1)^{(k-1)r} p_k(h_r(X)).$$

□

Let $\mu \vdash n$ be a hook, $h = M_\mu$, $1 \leq l \leq h$ and $h = dl + r$ ($0 \leq r \leq l - 1$). Recall that a is the product of d cyclic permutations

$$(1, 2, \dots, l)(l + 1, l + 2, \dots, 2l) \cdots ((d - 1)l + 1, \dots, dl).$$

Theorem 21 For an element w of S_n and each $j = 1, 2, \dots, l - 1$, it holds that

$$X_{\lambda(w)}^\mu(\zeta_l^j) \neq 0 \Rightarrow w \sim a^j \tau, \quad \tau \in S_{n-dl}.$$

Proof. Let $j = 1, 2, \dots, l - 1$, and suppose that ζ_l^j is a primitive p -th root of unity. The cycle type of a^j is $\lambda(a^j) = (p^e)$, where $e = dl/p$, and it suffices to show that

$$X_{\lambda(w)}^\mu(\zeta_l^j) \neq 0 \implies r_p \geq e, \quad \rho = (1^{r_1} 2^{r_2} \cdots n^{r_n}).$$

Let ρ' be the subpartition consisting of the non-zero components of ρ divisible by p , and ρ'' the remanning. If we let $\tilde{\mu} = \mu - (1^{pe})$, then we have

$$\begin{aligned} X_{\lambda(w)}^\mu(\zeta_l^j) &= \langle p_\rho, Q'_\mu(X; q) \rangle \Big|_{q=\zeta_l^j} \\ &= \langle p_\rho, Q'_{\tilde{\mu}}(X; q) (Q'_{(1^p)}(X; q))^e \rangle \Big|_{q=\zeta_l^j} \quad (\text{Proposition 19}) \\ &= \pm \langle p_{\rho'} p_{\rho''}, Q'_{\tilde{\mu}}(X; q) (p_p(h_1))^e \rangle \Big|_{q=\zeta_l^j} \quad (\text{Proposition 20}) \\ &= \pm \langle p_{\rho'} p_{\rho''}, Q'_{\tilde{\mu}}(X; q) (p_p)^e \rangle \Big|_{q=\zeta_l^j} \\ &= \pm \langle \langle D_{p_p}^e(p_{\rho'} p_{\rho''}), Q'_{\tilde{\mu}}(X; q) \rangle \rangle \Big|_{q=\zeta_l^j}. \end{aligned}$$

Therefore, if $X_{\lambda(w)}^\mu(\zeta_l^j) \neq 0$, then it should hold that $D_{p_p}^e(p_{\rho'} p_{\rho''}) \neq 0$. Since $D_{p_p}^e(p_{\rho'} p_{\rho''}) = D_{p_p}^e(p_{\rho'}) p_{\rho''}$, it immediately follows that $r_p \geq e$. □

The rest of this section is devoted to the proof of

$$Q_{\lambda(a^j \tau)}^\mu(\zeta_l^j) = p^e e! \binom{z_p + e}{e} Q_{\lambda(\tau)}^{\tilde{\mu}}(\zeta_l^j),$$

for $w = a^j \tau$ ($\tau \in S_{n-dl}$). We begin with the following simple lemma.

Lemma 22 Let $\mu = (n - h, 1^h)$ ($n - h > 1$) be a hook. Let an integer $l \in \{2, \dots, l - 1\}$ be fixed, and suppose that $h = dl + r$ ($0 \leq r \leq l - 1$). Let $\bar{\mu} = (n - h, 1^r)$ and $\nu = (1^{dl})$. Then it holds that

$$f_{\bar{\mu}\nu}^\mu(t) = \frac{b_\mu(t)}{b_{\bar{\mu}}(t)b_\nu(t)}.$$

Proof. Recall that

$$f_{\bar{\mu}\nu}^\mu(t) = \prod_{i \geq 1} \begin{bmatrix} \mu_i - \mu_{i+1} \\ \mu_i - \bar{\mu}_i \end{bmatrix}_t.$$

Since μ is a hook, we have

$$f_{\bar{\mu}\nu}^\mu(t) = \begin{bmatrix} (h+1) - 1 \\ (h+1) - (r+1) \end{bmatrix}_t = \begin{bmatrix} h \\ h - r \end{bmatrix}_t.$$

On the other hand,

$$\begin{aligned} \frac{b_\mu(t)}{b_{\bar{\mu}}(t)b_\nu(t)} &= \frac{(1-t)(1-t^2) \cdots (1-t^h)(1-t)}{(1-t) \cdots (1-t^{dl})(1-t) \cdots (1-t^r)(1-t)} \\ &= \frac{(1-t^{dl}) \cdots (1-t^h)}{(1-t) \cdots (1-t^r)} \\ &= \begin{bmatrix} h \\ r \end{bmatrix}_t, \end{aligned}$$

and this completes the proof. \square

As in the proof of Theorem 5, expanding the product of two Hall-Littlewood functions $P_{\bar{\mu}}P_\nu$ into a linear combination of P_λ 's again, we have the following recursive formula for Green polynomials:

$$\begin{aligned} Q_\rho^\mu(q) &= - \sum_{\substack{\lambda \vdash n-dl \\ \lambda - \mu \text{ is vertical } dl\text{-strip}}} \frac{f_{\bar{\mu}\nu}^\lambda(q^{-1}) b_\mu(q^{-1})}{f_{\bar{\mu}\nu}^\mu(q^{-1}) b_\lambda(q^{-1})} q^{n(\mu) - n(\lambda)} Q_\rho^\lambda(q) \\ &\quad + \frac{1}{f_{\bar{\mu}\nu}^\mu(q^{-1})} \frac{b_\mu(q^{-1})}{b_{\bar{\mu}}(q^{-1})b_\nu(q^{-1})} q^{n(\mu) - n(\bar{\mu}) - n(\nu)} \sum_{\substack{\rho^{(1)} \vdash n-dl, \rho^{(2)} \vdash dl \\ \rho^{(1)} \cup \rho^{(2)} = \rho}} \frac{z_\rho}{z_{\rho^{(1)}} z_{\rho^{(2)}}} Q_{\rho^{(1)}}^{\bar{\mu}}(q) Q_{\rho^{(2)}}^\nu(q) \\ &= - \sum_\lambda \frac{f_{\bar{\mu}\nu}^\lambda(q^{-1}) b_\mu(q^{-1})}{f_{\bar{\mu}\nu}^\mu(q^{-1}) b_\lambda(q^{-1})} q^{n(\mu) - n(\lambda)} Q_\rho^\lambda(q) \\ &\quad + q^{rdl} \sum_{\rho^{(1)}, \rho^{(2)}} \frac{z_\rho}{z_{\rho^{(1)}} z_{\rho^{(2)}}} Q_{\rho^{(1)}}^{\bar{\mu}}(q) Q_{\rho^{(2)}}^\nu(q), \end{aligned}$$

where the last identity follows from Lemma 22. We prepare the following auxiliary result to proceed our computation.

Lemma 23 Let $\mu = (n-h, 1^h)$ ($n-h > 1$) be a hook, $l \in \{2, \dots, h-1\}$ fixed and $h = dl + r$ ($0 \leq r \leq l-1$). Let $\bar{\mu} = (n-h, 1^r)$ and $\nu = (1^{dl})$. Define

$$C_\mu^\lambda(q) := \frac{f_{\bar{\mu}\nu}^\lambda(q) b_\mu(q)}{f_{\bar{\mu}\nu}^\mu(q) b_\lambda(q)}$$

for a partition $\lambda \vdash n$ such that $\lambda - \bar{\mu}$ is a vertical dl -strip. Then, $C_\mu^\lambda(q)$ is of the form

$$C_\mu^\lambda(q) = (1 - q^l) H_\mu^\lambda(q),$$

where $H_\mu^\lambda(q) = \phi(q)/\psi(q) \in \mathbf{Z}(q)$ is a rational function with no factor in $\phi(q)$ and $\psi(q)$ divisible by $1 - q^l$.

Proof. Let $D_l(f)$ denote the number of the factor $1 - q^l$ occurring in a polynomial $f \in \mathbf{Z}[q]$. For a rational function $g(q) = \phi(q)/\psi(q) \in \mathbf{Z}(q)$, define $D_l(g) = D_l(\phi) - D_l(\psi)$. We will show that $D_l(C_\mu^\lambda(q)) = 1$.

Recall that

$$f_{\bar{\mu}\nu}^\lambda(q) = \left[\begin{array}{c} \lambda'_1 - \lambda'_2 \\ \lambda'_1 - \bar{\mu}'_1 \end{array} \right]_q \left[\begin{array}{c} \lambda'_2 - \lambda'_3 \\ \lambda'_2 - \bar{\mu}'_2 \end{array} \right]_q \left[\begin{array}{c} \lambda'_3 - \lambda'_4 \\ \lambda'_3 - \bar{\mu}'_3 \end{array} \right]_q \dots$$

Suppose that $n - h > 2$ or $r < l - 1$. In this case, since $\lambda'_i - \lambda'_{i+1} < l$ for all $i \geq 2$, we have

$$D_l \left(\left[\begin{array}{c} \lambda'_i - \lambda'_{i+1} \\ \lambda'_i - \bar{\mu}'_i \end{array} \right]_q \right) = 0.$$

Therefore, we have

$$D_l(f_{\bar{\mu}\nu}^\lambda(q)) = D_l \left(\left[\begin{array}{c} \lambda'_1 - \lambda'_2 \\ \lambda'_1 - \bar{\mu}'_1 \end{array} \right]_q \right).$$

A similar argument shows that

$$D_l(f_{\bar{\mu}\nu}^\mu(q)) = D_l \left(\left[\begin{array}{c} \mu'_1 - \mu'_2 \\ \mu'_1 - \bar{\mu}'_1 \end{array} \right]_q \right).$$

Moreover, by the assumption, we have

$$D_l \left(\left[\begin{array}{c} \lambda'_1 - \lambda'_2 \\ \lambda'_1 - \bar{\mu}'_1 \end{array} \right]_q \right) = D_l \left(\left[\begin{array}{c} \mu'_1 - \mu'_2 \\ \mu'_1 - \bar{\mu}'_1 \end{array} \right]_q \right) = 0,$$

since, in the definition of the Gaussian polynomials $\left[\begin{array}{c} \lambda'_1 - \lambda'_2 \\ \lambda'_1 - \bar{\mu}'_1 \end{array} \right]_q$ and $\left[\begin{array}{c} \mu'_1 - \mu'_2 \\ \mu'_1 - \bar{\mu}'_1 \end{array} \right]_q$, the number of the factor $1 - q^l$ in the denominator and the numerator coincide. Again by the assumption, it is easy to see that

$$D_l(b_\mu/b_\lambda) = 1.$$

Thus we have $D_l(C_\mu^\lambda(q)) = 1$ if $n - h > 2$ or $r < l - 1$.

Let us consider the case where $n - h = 2$ and $r = l - 1$. If $\lambda \neq (2^l, 1^{n-2l})$, then the same argument shows that $D_l(C_\mu^\lambda(q)) = 1$. If $\lambda = (2^l, 1^{n-2l})$, then we have

$$D_l(f_{\bar{\mu}\nu}^\lambda(q)) = D_l\left(\begin{bmatrix} \lambda'_1 - \lambda'_2 \\ \lambda'_1 - \bar{\mu}'_1 \end{bmatrix}_q\right) + D_l\left(\begin{bmatrix} \lambda'_2 - \lambda'_3 \\ \lambda'_2 - \bar{\mu}'_2 \end{bmatrix}_q\right) = 0 + 1 = 1$$

and

$$D_l\left(\begin{bmatrix} \mu'_1 - \mu'_2 \\ \mu'_1 - \bar{\mu}'_1 \end{bmatrix}_q\right) = 0.$$

It is also easy to see that $D_l(b_\mu/b_\lambda) = 0$. Hence we have $D_l(C_\mu^\lambda(q)) = 1$. \square

Remark that $C_\mu^\lambda(q^{-1})$ is obtained, up to a signature, by multiplying $C_\mu^\lambda(q)$ by a q -power. Therefore, for $\rho = \lambda(a^j v)$, it follows from Lemma 23 that

$$\begin{aligned} Q_\rho^\mu(\zeta_l^j) &= -\sum_\lambda C_\mu^\lambda(q^{-1}) q^{n(\mu)-n(\lambda)} Q_\rho^\lambda(q) \Big|_{q=\zeta_l^j} \\ &\quad + q^{rdl} \sum_{\rho^{(1)}, \rho^{(2)}} \frac{z_\rho}{z_{\rho^{(1)}} z_{\rho^{(2)}}} Q_{\rho^{(1)}}^{\bar{\mu}}(q) Q_{\rho^{(2)}}^\nu(q) \Big|_{q=\zeta_l^j} \\ &= \sum_{\rho^{(1)}, \rho^{(2)}} \frac{z_\rho}{z_{\rho^{(1)}} z_{\rho^{(2)}}} Q_{\rho^{(1)}}^{\bar{\mu}}(q) Q_{\rho^{(2)}}^\nu(q) \Big|_{q=\zeta_l^j}. \end{aligned}$$

By Theorem 5, this equals

$$\begin{aligned} &b_{(1^{dl})}(q) \sum_{\rho^{(1)}, \rho^{(2)}} \frac{z_\rho}{z_{\rho^{(1)}} z_{\rho^{(2)}}} \frac{b_{(1^{M\bar{\mu}})}(q)}{e_{\lambda(a^j v)}(q)} G_{\rho^{(1)}}^{\bar{\mu}}(q) \Big|_{q=\zeta_l^j} \\ &= \frac{b_{(1^{dl})}(q)}{e_{\lambda(a^j)}(q)} \sum_{\rho^{(1)}, \rho^{(2)}} \frac{z_\rho}{z_{\rho^{(1)}} z_{\rho^{(2)}}} \frac{b_{(1^{M\bar{\mu}})}(q)}{e_{\lambda(v)}(q)} G_{\rho^{(1)}}^{\bar{\mu}}(q) \Big|_{q=\zeta_l^j} \\ &= p^e e! \sum_{\rho^{(1)}, \rho^{(2)}} \frac{z_\rho}{z_{\rho^{(1)}} z_{\rho^{(2)}}} Q_{\rho^{(1)}}^{\bar{\mu}}(q) \frac{e_{\rho^{(1)}}(q)}{e_{\lambda(v)}(q)} \Big|_{q=\zeta_l^j}. \end{aligned}$$

We remark here that:

Lemma 24 *If $\rho^{(1)} \neq \lambda(v)$, then it holds that $\frac{e_{\rho^{(1)}}(q)}{e_{\lambda(v)}(q)} \Big|_{q=\zeta_l^j} = 0$.*

Proof. Note that $\rho^{(1)} \vdash |\lambda(v)|$ and $\rho^{(1)} \subset \lambda(a^j) \cup \lambda(v)$. Since $\rho^{(1)} \neq \lambda(v)$, the multiplicity of p in the partition $\rho^{(1)}$ is larger than that of $\lambda(v)$. \square

By Lemma 24, we have

$$\begin{aligned} p^e e! \sum_{\rho^{(1)}, \rho^{(2)}} \frac{z_\rho}{z_{\rho^{(1)}} z_{\rho^{(2)}}} Q_{\rho^{(1)}}^{\bar{\mu}}(q) \frac{e_{\rho^{(1)}}(q)}{e_{\lambda^{(v)}}(q)} \Big|_{q=\zeta_l^j} &= p^e e! \frac{z_{\lambda^{(ajv)}}}{z_{\lambda^{(aj)}} z_{\lambda^{(v)}}} Q_{\lambda^{(v)}}^{\bar{\mu}}(\zeta_l^j) \\ &= p^e e! \binom{z_p + e}{e} Q_{\lambda^{(v)}}^{\bar{\mu}}(\zeta_l^j). \end{aligned}$$

Thus, for $w \sim a^j \tau$ ($\tau \in S_{n-dl}$), it holds that

$$\text{char}_q R_\mu(w) \Big|_{q=\zeta_l^j} = Q_{\lambda^{(w)}}^\mu(\zeta_l^j) = p^e e! \binom{z_p + e}{e} Q_{\lambda^{(\tau)}}^{\bar{\mu}}(\zeta_l^j),$$

which completes the proof of Theorem 12.

Example 25 Let $\mu = (21^4)$ and $l = 3 (< M_\mu = 4)$. By Theorem 21, if $Q_\rho^\mu(\zeta_3) \neq 0$ for $\rho \vdash 6$, then ρ should be of the form $(3) \cup \nu$ or $(1^3) \cup \nu$ for some $\nu \vdash 3$. Let us consider the case where $\rho = (3, 2, 1)$ ($\nu = (2, 1)$). We have shown that

$$Q_{(3,2,1)}^\mu(\zeta_3) = 3^1 1! \binom{1+0}{1} Q_{(2,1)}^{(2,1)}(\zeta_3).$$

By Morris' table, we can see that $Q_{(3,2,1)}^\mu(\zeta_3) = 3$. Also, we know that $Q_{(2,1)}^{(2,1)}(q)$ is identically 1. Then the above identity holds.

If $\rho = (3, 3)$, then we can see that $Q_{(3,3)}^\mu(\zeta_3) = (1-q)(1-q^2)(1-q^4)(1+q^3) \Big|_{q=\zeta_3} = 6(1-\zeta_3)$ from the table. On the other hand, we have $Q_{(3)}^{(2,1)}(\zeta_3) = 1 - \zeta_3$ by Green's table, and the scalar factor is $3^1 1! \binom{1+1}{1} = 6$. Thus the identity holds. \square

6 Kraśkiewicz-Weyman type theorem for rectangles

In this section, we will consider the DeConcini-Procesi-Tanisaki algebra corresponding to a rectangle, a partition of the form $\mu = (r^h)$. In this case, since $M_\mu = h$, the dimensions of the mod l sums $R_\mu(k; l)$, $k = 0, 1, \dots, l-1$, of the algebra R_μ coincide for each $l = 1, 2, \dots, h$. We will consider the Kraśkiewicz-Weyman type theorem for the algebra R_μ , that is, a representation theoretical interpretation of coincidence of dimension for the case $l = M_\mu (= h)$. We consider this problem only for the case where the multiplicity h is a prime.

Let a be the following element of S_n :

$$\begin{aligned} a &= \left(\begin{array}{cccccccc} 1 & \cdots & r & r+1 & \cdots & 2r & \cdots & (h-1)r+1 & \cdots & rh \\ r+1 & \cdots & 2r & 2r+1 & \cdots & 3r & \cdots & 1 & \cdots & r \end{array} \right) \\ &= (1, r+1, \dots, (h-1)r+1)(2, r+2, \dots, (h-1)r+2) \cdots (r, 2r, \dots, hr) \end{aligned}$$

It is obvious that $a^h = 1$. Let C_h be the cyclic subgroup of S_n generated by a , and

$$S_\mu = S_{\{1,2,\dots,r\}} \times \cdots \times S_{\{(h-1)r+1,\dots,rh\}}$$

the Young subgroup corresponding to the partition μ . Then, define a subgroup $H_\mu(h)$ of S_n by the semidirect product

$$H_\mu(h) = S_\mu \rtimes C_h.$$

For each $k = 0, 1, \dots, h-1$, define a one-dimensional representation $\varphi^{(k)}$ of $H_\mu(h)$ as follows:

$$\varphi^{(k)} : H_\mu(h) \longrightarrow \mathbf{C}^\times : \begin{cases} a \longmapsto \zeta_h^k, \\ \tau \longmapsto 1, \quad \text{for } \tau \in S_\mu, \end{cases}$$

where ζ_h denotes a primitive h -th root of unity.

Example 26 Let us consider the case where $\mu = (2, 2, 2)$. In this case, we have $h = 3$ and our permutation a is

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix}.$$

Clearly, its cyclic decomposition is $a = (1, 3, 5)(2, 4, 6)$ and its order 3. Let $S_\mu = S_{\{1,2\}} \times S_{\{3,4\}} \times S_{\{5,6\}}$ be the Young subgroup corresponding to μ . Then our subgroup $H_\mu(3)$ is by definition the semidirect product $H_\mu(3) = S_\mu \rtimes C_3$, where $C_3 = \langle a \rangle$. The one-dimensional representation $\varphi^{(k)}$ of $H_\mu(3)$ is defined by $\varphi^{(k)}(a^j) = \zeta_3^{kj}$ for $j = 0, 1, 2$, and $\varphi^{(k)}(\tau) = 1$ for any $\tau \in S_\mu$. \square

In this section, we will prove the following theorem.

Theorem 27 For each $k = 0, 1, \dots, h-1$, it holds that

$$R_\mu(k; h) \cong_{S_n} \text{Ind}_{H_\mu(h)}^{S_n} \varphi^{(k)}.$$

\square

By the same argument as in the hook case, it can be seen that this is equivalent to show that the following $S_n \times C_l$ -isomorphism holds:

$$R_\mu \cong_{S_n \times C_l} \text{Ind}_{H_\mu(h)}^{S_n} 1. \quad (6.1)$$

The isomorphisms in Theorem 27 are realized as correspondence of the eigenspaces with respect to the action of a in the $S_n \times C_h$ -isomorphism.

The rest of this section is devoted to the proof of (6.1) for the case h is a prime. Let $h = p$ be a prime. Then it can be seen that it is sufficient to show, as in the hook case, that:

1. For $w \in S_n$, it holds that

$$Q_\rho^\mu(\zeta_p^j) \neq 0 \implies w \sim \tau_1 \cdots \tau_p a^j, \quad \tau_1 \cdots \tau_p \in S_\mu,$$

2. If $w \in S_n$ is conjugate to some element $\tau_1 \cdots \tau_p a^j \in H_\mu(p)$, then we have

$$Q_{\lambda(w)}^\mu(\zeta_p^j) = \#\{\sigma \in S_n/S_\mu \mid w\sigma a^{-j} \equiv \sigma \pmod{S_\mu}\}.$$

First, we will consider the necessary condition for $w \in S_n$ to be a support of $Q_{\lambda(w)}^\mu(\zeta_p^j)$. On behavior of the Green polynomials $X_\rho^\mu(q)$ corresponding to a rectangle μ , the following proposition due to Lascoux-Leclerc-Thibon [LLT, Theorem 3.2] is known. In this proposition, h is not necessarily be prime.

Proposition 28 (Lascoux-Leclerc-Thibon) Let $\mu = (r^h)$ be a rectangle, and ζ_k a primitive h -th root of unity. Then it holds that

$$X_\rho^\mu(\zeta_h) = \begin{cases} (-1)^{(h-1)j} h X_{\rho-\{i\}}^{(r-j)^h}(\zeta_h), & \text{for some } i = jh, \\ 0, & \text{otherwise.} \end{cases}$$

□

It is clear from Proposition 28 that:

Corollary 29 Let $\mu = (r^h)$ be a rectangle, and ζ_h a primitive h -th root of unity. Then, for a partition $\rho \vdash rh$, the condition $X_\rho^\mu(\zeta_h) \neq 0$ implies that all the non-zero components of ρ are multiple of h . □

These partitions in Corollary 29 are realized as cycle types of permutations belonging to the subgroup $H_\mu(h)$.

Lemma 30 For a partition ρ of the form $\rho = (h^{\alpha_h} (2h)^{\alpha_{2h}} \dots (rh)^{\alpha_{rh}})$, there exists an element τ of $S_{(r^k)}$ satisfying $\rho = \lambda(\tau a)$.

Proof. We can choose τ as, for example, $\tau \in S_{\{1, r+1, \dots, (p-1)r+1\}}$ such that

$$\lambda(\tau) = (1^{\alpha_k} 2^{\alpha_{2k}} \dots r^{\alpha_{rk}}).$$

□

Since it holds that

$$Q_\rho^\mu(\zeta_h) \neq 0 \iff X_\rho^\mu(\zeta_h) \neq 0,$$

together with Corollary 29 and Lemma 30, we have an answer for our present problem.

Corollary 31 Let $\mu = (r^h)$ be a rectangle, and ζ_h a primitive h -th root of unity. Then, for $\rho \vdash rh$, we have

$$Q_\rho^\mu(\zeta_h) \neq 0 \implies \rho = \lambda(w), \quad w \in H_\mu(h).$$

□

Next we will determine the value $Q_{\lambda(w)}^\mu(\zeta_h)$ for $w \in H_\mu(h)$. Let $\mu = (r^h)$ be a rectangle. If all the non-zero components of ρ are multiples of h , then the explicit values of the Green polynomial $X_\rho^\mu(q)$ at $q = \zeta_k$ are given as follows:

Lemma 32 Let a partition $\mu = (r^h)$ be rectangle. Then, for a partition

$$\rho = (h^{\alpha_h}(2h)^{\alpha_{2h}} \cdots (rh)^{\alpha_{rh}})$$

whose non-zero components are all divisible by h , it holds that

$$X_\rho^{(r^h)}(\zeta_h) = (-1)^{r(h-1)} h^{l(\rho)},$$

where $l(\rho)$ denotes the length of ρ . In particular, the value $X_\rho^{(r^h)}(\zeta_h)$ does not depend upon a particular choice of a primitive h -th root of unity.

Proof. Applying Lascoux-Leclerc-Thibon's theorem repeatedly, we have

$$\begin{aligned} X_\rho^{(r^h)}(\zeta_h) &= (-1)^{(h-1)(\alpha_h+2\alpha_{2h}+\cdots+r\alpha_{rh})} h^{\alpha_h+\alpha_{2h}+\cdots+\alpha_{rh}} X_\emptyset^\emptyset(\zeta_h) \\ &= (-1)^{r(h-1)} h^{l(\rho)}, \end{aligned}$$

where \emptyset denotes the empty partition, and $X_\emptyset^\emptyset(q) \equiv 1$. □

The following proposition easily follows from (2.1).

Proposition 33 For each $j = 0, 1, \dots, h-1$, we have

$$Q_\rho^{(r^h)}(\zeta_h^j) = \begin{cases} X_\rho^{(r^h)}(\zeta_h^{-j}), & \text{if } h \geq 3 \\ (-1)^{jr} X_\rho^{(r^2)}((-1)^{-j}), & \text{if } h = 2. \end{cases}$$

Proof.

$$\begin{aligned} Q_\rho^{(r^h)}(\zeta_h k^j) &= q^{n(\mu)} X_\rho^{(r^h)}(q^{-1}) \Big|_{q=\zeta_h^j} \\ &= q^{r(0+1+\cdots+(h-1))} X_\rho^{(r^h)}(q^{-1}) \Big|_{q=\zeta_h^j} \\ &= (\zeta_h^j)^{r \frac{h(h-1)}{2}} X_\rho^{(r^h)}(\zeta_h^{-1}) \\ &= \begin{cases} X_\rho^{(r^h)}(\zeta_h^{-j}), & \text{if } h \geq 3 \\ (-1)^{jr} X_\rho^{(r^2)}((-1)^{-j}), & \text{if } h = 2. \end{cases} \end{aligned}$$

□

Especially for the case h is a prime p , we have:

Corollary 34

$$Q_\rho^{(r^p)}(\zeta_p) = \begin{cases} X_\rho^{(r^p)}(\zeta_p), & \text{if } p \geq 3 \\ (-1)^r X_\rho^{(r^2)}(-1), & \text{if } p = 2. \end{cases}$$

Proof. It holds that

$$Q_\rho^{(r^p)}(\zeta_p) = q^{n(\mu)} X_\rho^{(r^p)}(q^{-1}) \Big|_{q=\zeta_p} = X_\rho^{(r^p)}(q^{-1}) \Big|_{q=\zeta_p},$$

since $n(\mu)$ is divisible by p . Because ζ_p^{-1} is primitive and the value $X_\rho^{(r^p)}(\zeta_p)$ does not depend on the choice of a primitive p -th root of unity ζ_p , the assertion follows immediately from Proposition 33. \square

In what follows, we assume that $\mu = (r^p)$ is a rectangle with prime multiplicity p .

Proposition 35 Let $\mu = (r^p)$ be a rectangle, whose multiplicity p is prime. Then, for each $w = \tau_1 \cdots \tau_p a^j \in S_\mu C_p$ and $j = 1, 2, \dots, p-1$, it holds that

$$Q_\rho^\mu(\zeta_p^j) = Q_\rho^\mu(\zeta_p) = p^{l(\rho)},$$

where $\rho = \lambda(w)$ is the cycle type of w , $l(\rho)$ its length.

Proof. Since ζ_p^j is primitive for each $j = 1, 2, \dots, p-1$, it is enough to show only for the case $j = 1$. Let $w = \tau_1 \cdots \tau_p a^j \in S_\mu C_p$ and ρ its cycle type. Note that each non-zero components of ρ is divisible by p . Then it follows from Corollary 34 and Lemma 32 that

$$\begin{aligned} Q_\rho^{(r^p)}(\zeta_p) &= \begin{cases} X_\rho^{(r^p)}(\zeta_p), & \text{if } p:\text{odd} \\ (-1)^r X_\rho^{(r^2)}(-1), & \text{if } p = 2. \end{cases} \\ &= \begin{cases} (-1)^{r(p-1)} p^{l(\rho)} & \text{if } p:\text{odd} \\ (-1)^r (-1)^{r(p-1)} p^{l(\rho)}, & p = 2 \end{cases} \\ &= p^{l(\rho)}. \end{aligned}$$

\square

Example 36 Let μ be (2^2) . By Corollary 31, the support ρ of $Q_\rho^\mu(\zeta_2)$ is $\rho = (2, 2), (4)$, which can be seen directly from the table [Gr]. Also from the table, we see that $Q_{(2,2)}^\mu(q) = 1 - q + q^2$ and $Q_{(4)}^\mu(q) = 1 - q$. Thus $Q_{(2,2)}^\mu(-1) = 4 = 2^{l((2,2))}$, and $Q_{(4)}^\mu(-1) = 2 = 2^{l((4))}$. \square

Our subject here is to show

$$Q_{\lambda(w)}^\mu(\zeta_p^j) = \{\sigma \in S_n/S_\mu \mid w\sigma a^{-j} \equiv \sigma \pmod{S_\mu}\}$$

for each $w \sim \tau_1 \cdots \tau_p a^j \in S_\mu C_p$ and $j = 1, 2, \dots, p-1$. By Proposition 35, noting that p is prime, it suffices to show

$$\#\{\sigma \in S_n/S_\mu \mid w\sigma a^{-1} \equiv \sigma \pmod{S_\mu}\} = p^{l(\lambda(w))}. \quad (6.2)$$

Let \mathcal{S}_w denote the set in the left hand side of (6.2):

$$\mathcal{S}_w = \{\sigma \in S_n/S_\mu \mid w\sigma a^{-1} \equiv \sigma \pmod{S_\mu}\}.$$

The cyclic subgroup C_p generated by a acts on the set \mathcal{S}_w from the right.

Lemma 37 If σ is an element of \mathcal{S}_w , then so is σa .

Proof. Let σ be an element of \mathcal{S}_w . Let $\eta_1 \cdots \eta_p$ be an element of the Young subgroup S_μ satisfying $w\sigma a^{-1} = \sigma\eta_1 \cdots \eta_p$. Then we have

$$\begin{aligned} w(\sigma a)a^{-1} &= w\sigma \\ &= \sigma\eta_1 \cdots \eta_p a \\ &= \sigma a(a^{-1}\eta_1 a) \cdots (a^{-1}\eta_p a). \end{aligned}$$

Thus we have $\sigma a \in \mathcal{S}_w$, since $a^{-1}\eta_i a \in S_\mu$ ($i = 1, \dots, p$). □

Let $\sigma = [\sigma_1 \sigma_2 \cdots \sigma_n]$ be an element of \mathcal{S}_w . Since $\sigma \in S_n/S_\mu$, we may assume

$$\begin{aligned} \sigma_1 &< \sigma_2 < \cdots < \sigma_r \\ \sigma_{r+1} &< \sigma_{r+2} < \cdots < \sigma_{2r} \\ &\dots \\ \sigma_{(p-1)r+1} &< \sigma_{(p-1)r+2} < \cdots < \sigma_{pr}. \end{aligned}$$

By Lemma 37, we may also assume that

$$\sigma_1 = 1.$$

By the condition

$$\begin{aligned} w[\sigma_1, \sigma_2, \dots, \sigma_n] &= [\sigma_1, \sigma_2, \dots, \sigma_n] \eta_1 \eta_2 \cdots \eta_p a \quad (\eta_1 \eta_2 \cdots \eta_p \in S_\mu) \\ &= [\{\sigma_{r+1}, \sigma_{r+2}, \dots, \sigma_{2r}\}, \{\sigma_{2r+1}, \sigma_{2r+2}, \dots, \sigma_{3r}\}, \dots, \{\sigma_1, \sigma_2, \dots, \sigma_r\}], \end{aligned}$$

it can be seen that the components

$$\sigma_1, \sigma_2, \dots, \sigma_r \quad \left(r = \frac{n}{p}\right)$$

completely determine the remaining parts of σ .

Since w is assume to be an element of the form $\tau_1 \cdots \tau_p a^j \in S_\mu C_p$ ($j > 1$), its cycle type is of the form

$$\lambda(w) = (p^{\alpha_p} (2p)^{\alpha_{2p}} \cdots (rp)^{\alpha_{rp}}) \vdash rp = n.$$

In this case, by Lemma 30, the element w is realized by $w = \eta a$ if we choose $\eta \in S_{\{1, 2, \dots, r\}}$ as follows:

$$\lambda(\eta) = (1^{\alpha_p} 2^{\alpha_{2p}} \cdots r^{\alpha_{rp}}).$$

Example 38 In the case of $\mu = (5^3)$, we have

$$\begin{aligned} a &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 1 & 2 & 3 & 4 & 5 \end{pmatrix} \\ &= (1, 6, 11)(2, 7, 12)(3, 8, 13)(4, 9, 14)(5, 10, 15). \end{aligned}$$

The partitions which satisfy the condition $Q_\rho^\mu(\zeta_3) \neq 0$ are

$$(3^5), (6, 3^3), (6^2, 3), (9, 3^2), (9, 6), (12, 3), (15).$$

The corresponding elements $\eta \in S_5$ for these partitions are

$$\text{id}, (12), (12)(34), (123), (123)(45), (1234), (12345)$$

respectively. □

If we choose η from these permutations, then the representatives

$$\sigma = [\sigma_1, \sigma_2, \dots, \sigma_n] \in S_n/S_\mu$$

satisfying the condition

$$w\sigma a^{-1} \equiv \sigma \pmod{S_\mu} \tag{6.3}$$

are constructed as follows. For this construction, we have seen that it is enough to determine the first r letters $\sigma_1, \dots, \sigma_r$ of σ . If we rewrite the cycle type of η as

$$\begin{aligned} \lambda(\eta) &= (1^{\alpha_p} 2^{\alpha_{2p}} \dots r^{\alpha_{rp}}) \\ &= (\nu_1, \nu_2, \dots, \nu_s), \quad \nu_1 \geq \nu_2 \geq \dots \geq \nu_s > 0, \end{aligned}$$

then we have

$$\lambda(w) = (p\nu_1, p\nu_2, \dots, p\nu_s).$$

Let

$$\begin{aligned} w &= (v_1, v_2, \dots, v_{p\nu_1}) \\ &\quad (v_{p\nu_1+1}, v_{p\nu_1+2}, \dots, v_{p\nu_1+p\nu_2}) \\ &\quad \dots \\ &\quad (v_{p\nu_1+\dots+p\nu_{s-1}+1}, \dots, v_{p\nu_1+\dots+p\nu_s}), \end{aligned}$$

where $v_1 = 1, v_1 < v_{p\nu_1+1} < \dots < v_{p\nu_1+\dots+p\nu_{s-1}+1}$, be the corresponding cyclic decomposition of w . By the assumption $\sigma_1 = 1$ and (6.3), it follows that

$$\sigma_1 = 1, \sigma_2 = 2, \dots, \sigma_{\nu_1} = \nu_1.$$

Therefore, it is enough to enumerate the remaining $l(\rho) - 1$ blocks

$$\begin{aligned} & \sigma_{\nu_1+1}, \sigma_{\nu_1+2}, \dots, \sigma_{\nu_1+\nu_2}, \\ & \sigma_{\nu_1+\nu_2+1}, \sigma_{\nu_1+\nu_2+2}, \dots, \sigma_{\nu_1+\nu_2+\nu_3}, \\ & \dots \\ & \sigma_{\nu_1+\nu_2+\dots+\nu_{s-1}+1}, \dots, \sigma_{\nu_1+\nu_2+\dots+\nu_s}. \end{aligned}$$

For example, in the case of $\sigma_{\nu_1+1}, \sigma_{\nu_1+2}, \dots, \sigma_{\nu_1+\nu_2}$, (6.3) implies that there are p such possible choices

$$\begin{aligned} (\sigma_{\nu_1+1}, \sigma_{\nu_1+2}, \dots, \sigma_{\nu_1+\nu_2}) = & (v_{\nu_1+\nu_2+1}, v_{\nu_1+\nu_2+1+p}, v_{\nu_1+\nu_2+1+2p}, \dots), \\ & (v_{\nu_1+\nu_2+2}, v_{\nu_1+\nu_2+2+p}, v_{\nu_1+\nu_2+2+2p}, \dots), \\ & \dots \\ & (v_{\nu_1+\nu_2+p}, v_{\nu_1+\nu_2+p+p}, v_{\nu_1+\nu_2+p+2p}, \dots), \end{aligned}$$

and each of these are available. The same enumeration can be done for other blocks, and there are exactly p ways for each. Since these choice of the blocks are mutually independent, there are $p^{l(\rho)-1}$ ways in total for assigning letters $1, 2, \dots, |\lambda(\eta)|$ to $\sigma_1, \sigma_2, \dots, \sigma_{|\lambda(\eta)|}$.

Now we have enumerated the representatives modulo the right action of the cyclic subgroup C_p . Thus there are

$$p^{l(\rho)}$$

elements in S_n/S_μ satisfying (6.3) in total, which completes the proof of Theorem 27.

Example 39 Let $n = 15$ and $\mu = (5^3)$. If we consider the case

$$\rho = (6, 6, 3),$$

then we can take

$$\eta = (12)(34)$$

for the element of $S_{\{1,2,3,4,5\}}$ satisfying

$$\lambda(\eta a) = (6, 6, 3).$$

Remark that

$$w = \eta a = (1, 6, 11, 2, 7, 12)(3, 8, 13, 4, 9, 14)(5, 10, 15).$$

Now we are going to enumerate the representatives

$$\sigma = [\sigma_1, \sigma_2, \dots, \sigma_{15}] \in S_{15}/S_{(5^3)}, \quad \sigma_1 = 1,$$

satisfying

$$\begin{aligned} w[\sigma_1, \sigma_2, \dots, \sigma_{15}] &= [\sigma_1, \sigma_2, \dots, \sigma_{15}]ua \quad (u \in S_{(5^3)}) \\ &= [\{\sigma_6, \sigma_7, \dots, \sigma_{10}\}, \{\sigma_{11}, \sigma_{12}, \dots, \sigma_{15}\}, \{\sigma_1, \sigma_2, \dots, \sigma_5\}]. \end{aligned}$$

For example, it is easy to see that $\sigma = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]$ is an appropriate representative, since we have

$$\begin{aligned}\{w\sigma_1, w\sigma_2, \dots, w\sigma_5\} &= \{6, 7, 8, 9, 10\} = \{\sigma_6, \sigma_7, \dots, \sigma_{10}\}, \\ \{w\sigma_6, w\sigma_7, \dots, w\sigma_{10}\} &= \{11, 12, 13, 14, 15\} = \{\sigma_{11}, \sigma_{12}, \dots, \sigma_{15}\}, \\ \{w\sigma_{11}, w\sigma_{12}, \dots, w\sigma_{15}\} &= \{1, 2, 3, 4, 5\} = \{\sigma_1, \sigma_2, \dots, \sigma_5\}.\end{aligned}$$

Thanks to the right action of C_3 , we only have to determine

$$\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5.$$

The condition satisfied by σ allows us to choose

$$\{\sigma_1, \sigma_2\}, \{\sigma_3, \sigma_4\}, \{\sigma_5\}$$

independently. By the assumption $\sigma_1 = 1$, σ_2 is uniquely determined by $\sigma_2 = 2$, since the pair (σ_1, σ_2) should have the order 3 as a set under the action of w , i.e, it should satisfy

$$w^3\{\sigma_1, \sigma_2\} = \{\sigma_1, \sigma_2\}, \quad w^i\{\sigma_1, \sigma_2\} \neq \{\sigma_1, \sigma_2\} \quad (i = 1, 2).$$

Similarly, since (σ_3, σ_4) has the order 3 as a set under the action of w , the pair (σ_3, σ_4) should be chosen from the following three pairs

$$(\sigma_3, \sigma_4) = (3, 4), (8, 9), (13, 14),$$

and each of these are available. The same argument shows that there are three ways of choice for σ_5 . σ_5 should be chosen from 5, 10, 15, and each of these are available. Thus there are $3^2 = 3^{l(6,6,3)-1}$ ways of choice modulo the right action of C_3 . Hence there $3^3 = 3^{l(6,6,3)}$ suitable elements in S_n/S_μ . \square

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