Decomposition of Green polynomials of type $A$ and DeConcini-Procesi-Tanisaki algebras of certain types

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Abstract

A class of graded representations of the symmetric group, concerning with the cohomology ring of the corresponding flag variety, are considered. We point out a certain combinatorial property of the Poincaré polynomial of these graded representations, and interpret it in the language of representation theory of the symmetric group.

1 Introduction

Let $W$ be a finite reflection group, and $R = \bigoplus_d R^d$ a natural graded representation of $W$ such as the coinvariant algebra of $W$. Our problem is to understand a certain combinatorial property of the Hilbert polynomial of $R$ in the language of representation theory. The property of the Hilbert polynomial we consider is the following. Let $l$ be some positive integer. If $c(k;l)$ $(k = 0, 1, \ldots, l-1)$ denotes the sum of the coefficients of the Hilbert polynomial whose degrees are congruent to $k$ modulo $l$, then it often occurs that these $c(k;l)$’s are all equal to each other. In other words, if we consider for each $k = 0, 1, \ldots, l-1$ the direct sum $R(k;l)$ of homogeneous components of $R$ of which degrees are congruent to $k$ modulo $l$, then the dimension of these submodules do not depend on $k$, i.e., $\dim R(k;l) = (\dim R)/l$ for every $k$. The problem is to explain this phenomenon in the language of representation theory of $W$ in the following manner, i.e., find a subgroup $H(l)$ of $W$, and find $H(l)$-modules $Z(k;l)$ for each $k = 0, 1, \ldots, l-1$ of which dimensions are equal to each other, satisfying

$$R(k;l) \cong_W \text{Ind}_{H(l)}^W Z(k;l), \quad k = 0, 1, \ldots, l-1,$$

where $\text{Ind}$ denotes the induced representation.

The problem in the present article refers to the case where $W = S_n$ the symmetric group of $n$ letters, and $R = R_\mu$ a graded representation of $S_n$ corresponding to each partition $\mu$ of $n$. The algebra $R_\mu$ is a natural generalization of the coinvariant algebra of $S_n$, a graded version of the left regular representation $C[S_n]$. In the case of the coinvariant algebra, we have found those $l$’s and established the isomorphisms [MN]. Let $R_n$ denote the coinvariant algebra of $S_n$. The problems in [MN] are the followings:
• Determine all the integers \( l \) such that the dimensions of \( R_n(k; l) \) do not depend on \( k \), i.e.,

\[
\dim R_n(k; l) = \frac{\dim R_n}{l}
\]

for all \( k = 0, 1, \ldots, l - 1 \).

• For each integer \( l \) satisfying the condition above, find a subgroup \( H = H_n(l) \) of \( S_n \) and \( H \)-modules \( Z_n(k; l) \) \((k = 0, 1, \ldots, l - 1)\) of equal dimension, satisfying

\[
R_n(k; l) \cong_{S_n} \text{Ind}^{S_n}_H Z_n(k; l)
\]

for each \( k = 0, 1, \ldots, l - 1 \).

The answer for the first problem is given in general for the coinvariant algebras of finite reflection groups \( W \). We can see that each fundamental degrees \([H] \) of \( W \) is available. (Recall that the set of fundamental degrees of \( S_n \) is \( \{2, 3, \ldots, n\} \).) The answer for the second problem is as follows. Let \( l \) be a fundamental degree of the symmetric groups \( S_n \). Then we can choose the subgroup \( H_n(l) \) as

\[
H_n(l) \cong C_l \times S_r,
\]

where \( n = dl + r \), \( 0 \leq r \leq l - 1 \), and the \( H_n(l) \)-modules \( Z_n(k; l) \) \((k = 0, 1, \ldots, l - 1)\), defined by twisting each homogeneous component of the coinvariant algebra \( R_r \) of \( S_r \) by an irreducible representation of \( C_l \). Remark that, since the irreducible representations of \( C_l \) are all one dimensional, each \( Z_n(k; l) \) has dimension \( r! = \dim R_r \). Thus these isomorphisms give a representation theoretical interpretation of the phenomenon “coincidence of dimension” of \( R_n \).

The problem is motivated by a result of Kraśkiewicz and Weyman \([KW]\). They show that each submodules \( R_n(k; n) \), the case where \( l = n \), is induced from the corresponding irreducible representation of a cyclic subgroup generated by a Coxeter element \([H] \) of \( S_n \). Precisely, let \( c = (1, 2, \ldots, n) \) be a cyclic permutation of full length (this is a typical example of Coxeter elements), and let \( \psi_n^{(k)} : C_n \to \mathbb{C} : c \mapsto \zeta_n^k \) the irreducible representation of the cyclic subgroup \( C_n = \langle c \rangle \) sending the generator \( c \) to \( \zeta_n^k \) for each \( k = 0, 1, \ldots, n - 1 \), where \( \zeta_n = e^{2\pi i/n} \) is the primitive \( n \)-th root of unity. Then they show that there exists an \( S_n \)-module isomorphism

\[
R_n(k; n) \cong_{S_n} \text{Ind}^{S_n}_{C_n} \psi_n^{(k)},
\]

for each \( k = 0, 1, \ldots, n - 1 \). It immediately follows from this fact that each submodules \( R_n(k; n) \) have the same dimension \((n - 1)!\). They also show similar results for the Weyl groups of type \( B_n \) and \( D_n \). Stembridge considers this problem for a wider setting. In \([Stm]\), he considers any finite complex reflection group \( W \) and its coinvariant algebra \( R_W \). In this case, we can take \( c \) to be a Springer’s regular element \([Sp]\), and \( l \) its order. (The number \( l \) is also called a regular number of \( W \).) It can be shown that each submodules \( R_W(k; l) \) of equal dimension is obtained by inducing up the corresponding irreducible representation \( \psi_l^{(k)} \) of the cyclic subgroup \( \langle a \rangle \) (hence all \( \psi_l^{(k)} \) are one dimensional).
We will consider a generalization of the result in [MN] for a class of graded representations of the symmetric groups, which we call DeConcini-Procesi-Tanisaki algebras [DP][T]. A DeConcini-Procesi-Tanisaki algebra $R_\mu$ is a graded $S_n$-module defined for each partition $\mu$. It is a natural generalization of the coinvariant algebra, which is isomorphic to the representation obtained by inducing up the trivial representation of the corresponding Young subgroup $S_\mu$. It is well known that the coinvariant algebra $R_n$ is isomorphic to the cohomology ring $H^*(X_n, C)$ of the corresponding flag variety $X_n = GL_n/B$, where $B$ is a Borel subgroup of $GL_n$. On the other hand, the algebra $R_\mu$ is isomorphic to the cohomology ring $H^*(X_\mu, C)$ of the fixed point subvariety $X_\mu$ [DP] of the flag variety $X_n$. The difference of coinvariant algebras and DeConcini-Procesi-Tanisaki algebras we should note here is that the corresponding varieties are smooth or not. Needless to say, the flag varieties are non-singular, and its Poincaré polynomial

$$P_{X_n}(q) = \sum_{d \geq 0} q^d \dim H^{2d}(X_n, C)$$

has a nice property, the unimodal symmetry. This is a consequence of the hard Lefschetz theorem [St] for non-singular varieties, or the strong Lefschetz property [W] for graded Artinian algebras. For example, if we consider the case where $n = 4$, the Poincaré polynomial of $X_4$ is

$$P_{X_4}(q) = 1 + 3q + 5q^2 + 6q^3 + 5q^4 + 3q^5 + q^6.$$  

It is obviously seen that the coefficients form a unimodal symmetric sequence. However, it is possible to find that the “coincidence of dimensions” can occur for cohomology rings $H^*(X, C)$ even if a variety $X$ are singular. In fact, in the case of $\mu = (2, 1, 1, 1)$, we can see that the “mod 3 sums” of the coefficients of the Poincaré polynomial $P_{X_{(2,1,1,1)}}(q)$ coincide each other as follows:

$$1 + 15 + 4 = 4 + 16 = 9 + 11 = 20.$$ 

Similarly, the “mod 2 sums” are also all equal:

$$1 + 9 + 16 + 4 = 4 + 15 + 11 = 30.$$
It is easy to confirm that the same is true for other examples $\mu = (2, 2), (2, 1, 1)$ for the mod 2 sums. In general, if we denote by $M_\mu$ the maximum value of the multiplicities of a partition $\mu = (1^{m_1}2^{m_2} \cdots n^{m_n})$, it is possible to show that, for each fixed $l \in \{1, 2, \ldots, M_\mu\}$, the partial sum modulo $l$ (mod $l$ sum, for short)

$$c_\mu(k; l) = \sum_{d \equiv k \mod l} c_d, \quad k = 0, 1, \ldots, l - 1$$

of the coefficients of the Poincaré polynomial $P_{R_\mu}(q) = \sum_d c_d q^d$ of $R_\mu$ do not depend on its residue $k$. In other words, for each fixed integer $l \in \{1, 2, \ldots, M_\mu\}$, the partial sum modulo $l$ coincides each other. Thus, as in the previous works [KW, Stem, MN], it is natural to understand this phenomenon on the algebra $R_\mu$ in the language of the representation theory of symmetric groups.

In this paper, we find the representation theoretical interpretation for some special two cases: one is for a hook, a partition of the form $(n - h, 1^h)$, and the other is for a rectangle, a partition of the form $(r^k)$. Let $\mu = (n - h, 1^h)$ be a hook. Suppose that $n - h > 1$ for excluding the case of the coinvariant algebra. Then we have $M_\mu = h$. Let an integer $l$, $1 \leq l \leq h$, be fixed. Then the submodules $R_\mu(k; l)$ have the same dimension for $k = 0, 1, \ldots, l - 1$. The problem is to construct a subgroup $H_\mu(l)$ of $S_n$ and its modules $Z_\mu(k; l)$ ($k = 0, 1, \ldots, l - 1$) of the equal dimension, satisfying $R_\mu(k; l) \cong S_n \text{Ind}_{H_\mu(l)}^S Z_\mu(k; l)$. In this case, a similar construction of the subgroup and the modules is possible for each $l$ as in the case of coinvariant algebras. We will show that the subgroup $H_\mu(l)$ can be constructed isomorphic to the direct product

$$H_\mu(l) \cong C_l \times S_{n-dl},$$

where $h = dl + r$, $0 \leq r \leq l - 1$, and the modules are defined by twisting each homogeneous components of some “smaller” algebra $R_\overline{\mu}$ by the corresponding irreducible representation of $C_l$. In the rectangle case, it will be needed some restriction to prove the problem at present. Let a partition $\mu = (r^p)$ be rectangle, whose multiplicity $p$ is a prime. We consider the problem in this situation only for the case $l = p$. In this case, it is possible to prove that the subgroup $H_\mu(p)$ can be taken isomorphic to the semidirect product

$$H_\mu(p) \cong (S_r \times \cdots \times S_r) \rtimes C_p,$$

and the $H_\mu(p)$-modules are one dimensional representations, deeply related to the irreducible representations of $C_p$. These one dimensional representations are, in fact, considered as a smaller DeConcini-Procesi-Tanisaki algebra $R_\overline{\mu} = R_\emptyset$, where $\emptyset$ denotes the empty partition.

In both cases, our problem is reformulated to show the following isomorphism of $(S_n \times C_l)$-modules between the algebra $R_\mu$ and a $S_n$-module induced from a smaller algebra $R_\overline{\mu}$: in
the case of the hook case for example, it is enough to show

\[ R_\mu \cong_{S_n \times C_l} \text{Ind}^{S_n}_{S^\mu} R_{\bar{\mu}}. \]

(This is pointed out by T. Shoji, Nagoya University.) In this point of view, each isomorphism \( R_\mu(k;l) \cong_{S_n} \text{Ind}^{S_n}_{H(k;l)} Z_\mu(k;l) \) is realized as correspondence of \( \zeta^k \)-eigenspaces of the action of a generator of \( C_l \) in the \( (S_n \times C_l) \)-module isomorphism.

We briefly sketch the idea of the proof. We show that the character values of the \( S_n \times C_l \)-modules agree. The main tool in the proof is the Green polynomials \( Q_\mu^\rho(q) \) \([Gr, Mac]\) (of type \( A \)). The Green polynomials are introduced by J. A. Green \([Gr]\) in his argument to determine the irreducible characters of the finite general linear groups \( GL(F_q) \). They also have formulation in terms of symmetric functions. The Green polynomials \( Q_\mu^\rho(q) \) are defined to be the coefficients with some modification of the linear expansion of the power-sum symmetric function \( p_\rho(x) \) by the Hall-Littlewood symmetric functions \( P_\mu(x; q) \) \([Mc]\). It is known that the Green polynomial \( Q_\mu^\rho(q) \) gives the graded character of the DeConcini-Procesi-Tanisaki algebra \( R_\mu \). Then we can see that the proof of the \( (S_n \times C_l) \)-module isomorphism amounts to describe a behavior of the Green polynomial at roots of unity \( \zeta^j \). We should describe the following two points. First, we need a necessary condition for a partition \( \rho \) satisfying

\[ Q_\mu^\rho(\zeta^j) \neq 0. \]

Secondly, for such a partition \( \rho \), we will see that the ratio

\[ \frac{Q_\mu^\rho(q)}{Q_{\bar{\rho}}(q)} \bigg|_{q=\zeta^j} \]

of the values of Green polynomials \( Q_\mu^\rho(q) \) and \( Q_{\bar{\rho}}(q) \) at the roots of unity \( \zeta^j \) is precisely the number of permutations in \( S_n \) satisfying some conditions, where \( \bar{\rho} \) is the partition of the same size with \( \bar{\mu} \), obtained by deleting some cycles corresponding to \( \mu \) and \( j \). This will be done by using some recursive relations and a canonical decomposition form of the Green polynomials. Results of Lascoux-Leclerc-Thibon \([LLT]\) also plays an essential role in this paper.

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2 DeConcini-Procesi-Tanisaki algebras

In this section, we recall the definition and fundamental facts about the algebras \( R_\mu \).
The symmetric group $S_n$ of $n$ letters acts on the polynomial ring $P_n = \mathbb{C}[x_1, x_2, \ldots, x_n]$ by permuting the variables: For $\sigma \in S_n$ and $f(x_1, x_2, \ldots, x_n) \in P_n$, define

$$ (\sigma f)(x_1, x_2, \ldots, x_n) := f(x_{\sigma 1}, x_{\sigma 2}, \ldots, x_{\sigma n}). $$

Let $\mu$ be a partition of $n$. In the sequel, we use the symbol $\mu \vdash n$ for depicting that $\mu$ is a partition of $n$. We consider a homogeneous ideal $I_\mu$ of $P_n$ corresponding to the partition $\mu$, generated by the set $[T]$

$$ \left\{ e_m(x_{i_1}, \ldots, x_{i_{n-k+1}}) \mid k = 1, \ldots, \mu_1, \quad n - k + 1 \geq m \geq n - k + 1 - (\mu'_k + \mu'_{k+1} + \cdots + \mu'_1) + 1 \right\}, $$

where $e_m(x_{i_1}, \ldots, x_{i_{n-k+1}})$ denotes the $m$-th elementary symmetric polynomial in the variables $x_{i_1}, \ldots, x_{i_{n-k+1}}$. For example, if we take $\mu$ to be $(2, 1) \vdash 3$, then $I_\mu$ is generated by

$$ e_3(x_1, x_2, x_3), e_2(x_1, x_2, x_3), e_1(x_1, x_2, x_3), $$

$$ e_2(x_1, x_2), e_2(x_1, x_3), e_2(x_2, x_3). $$

By definition, the DeConcini-Procesi-Tanisaki algebra $R_\mu$ is the quotient algebra

$$ R_\mu := P_n / I_\mu. $$

Since the ideal $I_\mu$ is homogeneous and $S_n$-invariant, the algebra $R_\mu$ affords a graded representation of $S_n$:

$$ R_\mu = \bigoplus_{d=0}^{t_\mu} R^d_\mu, $$

where each homogeneous component $R^d_\mu$ is also an $S_n$-module. Its highest degree $t_\mu$ is known to be

$$ n(\mu) = \sum_{i \geq 1} (i - 1) \mu_i. $$

It is well known that $R_\mu$ is isomorphic as an $S_n$-module to the representation induced from the trivial representation of the Young subgroup $S_\mu$:

$$ R_\mu \cong S_n \text{Ind}_{S_\mu}^S 1. \quad (2.1) $$

Our first problem is to determine which positive integers $l$ satisfy the condition that the subspaces

$$ R_\mu(k; l) := \bigoplus_{d \equiv k \text{ mod } l} R^d_\mu, \quad k = 0, 1, \ldots, l - 1, $$

have the same dimension. We call the subspace $R_\mu(k; l)$ the $k$-th partial sum modulo $l$, or $k$-th mod $l$ sum of the graded algebra $R_\mu$ for each $k = 0, 1, \ldots, l - 1$. To find such $l$’s, it is necessary to consider the Hilbert polynomial

$$ \text{Hilb}_{R_\mu}(q) := \sum_{d \geq 0} q^d \dim R^d_\mu $$

6
for the algebra $R_\mu$. In fact, we consider more generally the “graded character” of the algebra $R_\mu$, since the Hilbert polynomial is a special case of it. Note that each homogeneous component $R^d_\mu$ is an $S_n$-module. We denote its character by $\text{char} R^d_\mu$, and then the graded character $\text{char}_q R_\mu$ of the graded $S_n$-module $R_\mu$ is defined by

$$\text{char}_q R_\mu := \sum_{d \geq 0} q^d \text{char} R^d_\mu.$$ 

If we evaluate the graded character at the identity element $e$, we obtain the Hilbert polynomial

$$\text{Hilb}_{R_\mu}(q) = \text{char}_q R_\mu(e),$$

since we have $\text{char} R^d_\mu(e) = \dim R^d_\mu$.

The following lemma, due to T. Oshima (Tokyo University), is fundamental [MN, Lemma 3].

**Lemma 1** Let $t$ be an indeterminate and consider a polynomial $f(t) = a_0 + a_1 t + \cdots$ with coefficients in a field of characteristic zero. Let $m \geq 2$ be an integer, and $\zeta$ a primitive $m$-th root of unity. Then the following two conditions are equivalent:

1. For each $k = 1, \ldots, m - 1$, we have $f(\zeta^k) = 0$.

2. The mod $l$ sums $c_\ell(f) = \sum_{j \geq 1} a_{mj+\ell}$ of the coefficients of $f(t)$ coincide each other.

□

By this lemma, it is sufficient to show which roots of unity are zeros of the Hilbert polynomial of the algebra $R_\mu$. For example, in the case of the coinvariant algebra $R_n$ (the case $\mu = (1^n)$), its Hilbert polynomial $\text{Hilb}_{R_n}(q)$ is given by

$$\text{Hilb}_{R_n}(q) = \frac{(1 - q)(1 - q^2) \cdots (1 - q^n)}{(1 - q)^n}.$$ 

From this formula, it is easy to see that for every integer $l = 2, \ldots, n$, we have

$$\text{Hilb}_{R_n}(\zeta^k) = 0$$

for each $k = 0, \ldots, l - 1$. Therefore, the preceding lemma shows that the dimensions of the subspaces $R_n(k; l)$ ($k = 0, 1, \ldots, l - 1$) do not depend on $k$ for each fixed $l = 1, 2, \ldots, n$.

In the next section, we will consider for which positive integers $l$ the dimensions of the mod $l$ sums of $R_\mu$ coincide. Our main tool is the graded character of $R_\mu$, which turns out to be a “Green polynomial” of type $A$. 

7
3 Green polynomials

In this section, we recall the definition of the Green polynomials, and see a canonical decomposition form (Theorem 5) that we use essentially in the proof of the main result. Let \( \rho \) be a partition of \( n \) and \( t \) an indeterminate. Since the Hall-Littlewood functions \( P_\mu(x;t) \) form a \( \mathbf{Z}[t] \)-basis of the ring of the symmetric functions \( \Lambda[t] = \Lambda \otimes_\mathbf{Z} \mathbf{Z}[t] \), we can express the power-sum symmetric function \( p_\rho(x) \) as a linear combination of \( P_\mu(x;t) \)'s with coefficients in \( \mathbf{Z}[t] \) \cite[III, (2.7)]{Mc}:

\[
p_\rho(x) = \sum_{\mu \vdash n} X_\rho^\mu(t) P_\mu(x;t).
\]

Then the Green polynomial (of type \( A \)) \( Q_\rho^\mu(q) \) \cite{Gr} (see also \cite[III, (7.8)]{Mc}) is defined by modifying the coefficient \( X_\rho^\mu(t) \) as follows:

\[
Q_\rho^\mu(q) = q^{n(\mu)} X_\rho^\mu(q^{-1}).
\] (3.1)

It is known that the Green polynomials are described by the Kostka polynomials. See \cite[III, Section 6]{Mc} for the definition of the Kostka polynomials. Let \( \lambda \) and \( \mu \) be partitions of the same size \( n \). We denote by \( K_{\lambda \mu}(q) \) the Kostka polynomial as usual, and define the modified one as follows:

\[
\tilde{K}_{\lambda \mu}(q) := q^{n(\mu)} K_{\lambda \mu}(q^{-1}).
\]

Then it is known \cite[(7.11)]{Mc} that

\[
Q_\rho^\mu(q) = \sum_{\lambda \vdash n} \chi_\lambda^\rho \tilde{K}_{\lambda \mu}(q),
\]

where \( \chi_\lambda^\rho \) denotes the \( \lambda \)-th irreducible character evaluated at the conjugacy class of cycle type \( \rho \). Moreover, it is also known that the modified Kostka polynomial \( \tilde{K}_{\lambda \mu}(q) \) coincide with the Poincaré polynomial of the \( \lambda \)-th irreducible representation \( V_\lambda \) in \( R_\mu \) (see e.g., \cite{GP}):

\[
\tilde{K}_{\lambda \mu}(q) = \sum_{\lambda \vdash n} [R_\mu^d : V_\lambda] q^d
\]

Thus we have

\[
Q_\rho^\mu(q) = \text{char}_q R_\mu(\rho).
\]

In the rest of this section, we consider an explicit form of \( Q_\rho^\mu(q) \), together with its property that we use later.

**Definition 2** For a partition \( \lambda = (1^{m_1} 2^{m_2} \cdots n^{m_n}) \vdash n \), we define

\[
e_\lambda(q) := (1 - q)^{m_1}(1 - q^2)^{m_2} \cdots (1 - q^n)^{m_n},
\]

\[
b_\lambda(q) := \prod_{i \geq 1} (1 - q)(1 - q^2) \cdots (1 - q^{m_i}).
\]

Also, we define

\[
z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots n^{m_n} m_n!
\]
as usual.
For example, if $\lambda = (2, 2, 1, 1) = (1^2 2^3)$, then we define
\[
\begin{align*}
e_\lambda(q) &= (1 - q)^2(1 - q^2)^3, \\
b_\lambda(q) &= (1 - q)(1 - q^2)(1 - q^3)(1 - q)(1 - q^2).
\end{align*}
\]
Note that, if a partition $\lambda$ is a union of two partitions $\mu$ and $\nu$, then it is readily seen that $e_\lambda = e_\mu e_\nu$.

Explicit formulas for the Green polynomials $Q^\mu_\rho(q) = \text{char}_q R_\mu$ are known for some special cases. For example, in the case corresponding to the coinvariant algebra, i.e., the case $\mu = (1^n)$, we have the following formula (see e.g., [G, p. 376]).

**Proposition 3** For each $\rho = (1^{r_1} 2^{r_2} \cdots n^{r_n}) \vdash n$, it holds that
\[
Q^{(1^n)}_\rho(q) = \frac{b^{(1^n)}(q)}{e_\rho(q)} = \frac{(1 - q)(1 - q^2) \cdots (1 - q^n)}{(1 - q)^{r_1}(1 - q^2)^{r_2} \cdots (1 - q^n)^{r_n}}.
\]

\[\Box\]

On the other hand, if $\mu$ is a hook of which the first component is 2, i.e., a partition of the form $\mu = (2, 1^{n-2})$, then we also have an explicit description of $Q^\mu_\rho(q)$ [Mr, Lemma 3].

**Proposition 4 (Morris)** For each partition $\rho = (1^{r_1} 2^{r_2} \cdots n^{r_n})$ of $n$, we have
\[
Q^{(2, 1^{n-2})}_\rho(q) = \frac{(1 - q) \cdots (1 - q^{n-2})}{e_\rho(q)} \{(r_1 - 1)q^n - r_1q^{n-1} + 1\}.
\]

\[\Box\]

Let $\mu = (1^{m_1} 2^{m_2} \cdots n^{m_n})$ be a partition of $n$. Let $M_\mu$ be the maximum value of the multiplicities of $\mu$:
\[
M_\mu = \max\{m_1, m_2, \ldots, m_n\}.
\]

We can see from this formula that each integer $l \in \{(1), 2, \ldots, n - 2\}$ is available for coincidence of the dimensions of the mod $l$ sums of $R_\mu$. In these two examples, it can be seen that there remains a polynomial when we factoring out the rational part
\[
\frac{(1 - q) \cdots (1 - q^{M_\mu})}{e_\rho(q)}
\]
from $Q^\mu_\rho(q)$. In fact, this turns out to be true in general.

**Theorem 5** Let $\mu$ and $\rho$ be partitions of $n$. Then there exists a polynomial $G^\mu_\rho(q) \in \mathbb{Z}[q]$ satisfying
\[
Q^\mu_\rho(q) = \frac{b^{(1^{M_\mu})}(q)}{e_\rho(q)} G^\mu_\rho(q).
\]
Proof. We will show that
\[ G^\mu_\rho(q) = \frac{e_\rho(q)}{b_{(1^M)}(q)} Q^\mu_\rho(q) \]
is a Laurent polynomial. If this is correct, then the assertion follows from the identity
\[ Q^\mu_\rho(q)e_\rho(q) = b_{(1^M)}(q) G^\mu_\rho(q) \]
by comparing the coefficients of the lowest degree in each side. (Remark that \( Q, e, b \) are polynomials.) Actually, we will show that\[ \tilde{G}^\mu_\rho(q) := \frac{b_{(1^M)}(q)}{b_\mu(q^{-1})} G^\mu_\rho(q) \]
is a Laurent polynomial. If this is true, then \( G^\mu_\rho(q) \) is a Laurent polynomial since\[ \frac{b_\mu(q^{-1})}{b_{(1^M)}(q)} \]
is also a Laurent polynomial.

Let \( \bar{\mu} \) be a partition defined by \( \bar{\mu}' = (\mu'_1, \ldots, \mu'_{a-1}) \), where \( \mu' = (\mu'_1, \ldots, \mu'_a) \) is the conjugate of \( \mu \). In other words, \( \bar{\mu} \) is the partition whose Young diagram is obtained by deleting the last column from that of \( \mu \). Also, let \( r = \mu'_a \) and \( \nu = (1^r) \). If we express the product of the following two Hall-Littlewood functions \( P_{\bar{\mu}} = P_{\bar{\mu}}(x; t) \) and \( P_\nu = P_\nu(x; t) \) as a linear combination of Hall-Littlewood functions again, then we have
\[ P_{\bar{\mu}} P_{\nu} = \sum_{\lambda \vdash n} f^\lambda_{\bar{\mu}\nu}(t) P_\lambda, \tag{3.2} \]
where \( f^\lambda_{\bar{\mu}\nu}(t) \in \mathbb{Z}[t] \). Recall that if \( \theta = \lambda - \mu \) is a vertical \( r \)-strip, then we have [Mc, III (3.2)]
\[ f^\lambda_{\bar{\mu}(\nu)}(t) := \prod_{i \geq 1} \left[ \frac{\lambda'_i - \lambda'_{i+1}}{\lambda'_i - \bar{\mu}'_i} \right]_t, \]
where \([n]_t\) is a Gaussian polynomial.

If we rewrite (3.2) by
\[ Q_{\bar{\mu}} Q_\nu = \sum_{\lambda} f^\lambda_{\bar{\mu}\nu}(t) \frac{b_\mu(t) b_\nu(t)}{b_\lambda(t)} Q_\lambda. \tag{3.3} \]
then we have
\[ Q_{\bar{\mu}} Q_\nu = \sum_{\rho} f^\rho_{\bar{\mu}\nu}(t) \frac{b_\mu(t) b_\nu(t)}{b_\rho(t)} Q_\lambda. \]
Also, it can be seen [Mc, III (7.5)] that
\[ Q_\lambda(x; t) = \sum_{\rho \vdash n} \frac{e_\rho(t)}{z_\rho} X^\lambda_\rho(t) p_\rho(x). \tag{3.4} \]
By (3.3), it follows from (3.4) that

\[
\sum_{\rho^{(1)}_n \rightarrow r, \rho^{(2)}_{n-r}} \frac{e_{\rho^{(1)}}(t) e_{\rho^{(2)}}(t)}{z_{\rho^{(1)}} z_{\rho^{(2)}}} X^\mu_{\rho^{(1)}} X^\nu_{\rho^{(1)} \cup \rho^{(2)}} p_{\rho^{(1)} \cup \rho^{(2)}}(x)
\]

\[
= \sum_{\rho \neq n} \frac{z_{\rho}}{b_\rho(t)} \left\{ \sum_{\lambda} f^\lambda_{\rho \mu}(t) \frac{b_\mu(t) b_\nu(t)}{b_\lambda(t)} X^\lambda_{\rho}(t) \right\} p_\rho(x).
\]

Comparing the coefficient of \( p_\rho(x) \), we have

\[
X^\mu_{\rho}(t) = \sum_{\rho^{(1)}_n \rightarrow r, \rho^{(2)}_{n-r}} \frac{z_{\rho}}{b_\rho(t)} \frac{b_\mu(q^{-1})}{b_\nu(q^{-1})} X^\mu_{\rho^{(1)}} X^\nu_{\rho^{(1)} \cup \rho^{(2)}}
\]

\[- \sum_{\lambda \neq \mu} f^\lambda_{\rho \mu}(t) \frac{b_\mu(t)}{b_\lambda(t)} X^\lambda_{\rho}(t),
\]

since \( f^\mu_{\rho \mu}(t) \equiv 1 \). Rewriting this identity again by (3.1), we have

\[
Q^\mu_\rho(q) = \sum_{\rho^{(1)}_n \rightarrow r, \rho^{(2)}_{n-r}} \frac{z_{\rho}}{b_\rho(q^{-1})} \frac{b_\mu(q^{-1})}{b_\nu(q^{-1})} Q^\mu_{\rho^{(1)}}(q) Q^\nu_{\rho^{(2)}}(q)
\]

\[- \sum_{\lambda \neq \mu} f^\lambda_{\rho \mu}(q) \frac{b_\mu(q^{-1})}{b_\nu(q^{-1})} q^{(\mu) - n(\lambda)} Q^\lambda_\rho(q)
\]

\[
= \frac{b_{(1)_{\rho \nu}}(q)}{e_\rho(q)} \sum_{\rho^{(1)}_n \rightarrow r, \rho^{(2)}_{n-r}} \frac{z_{\rho}}{b_\rho(q^{-1})} \frac{b_\mu(q^{-1})}{b_\nu(q^{-1})} b_{(1)_{\mu \nu}}(q) G^\mu_{\rho^{(1)}}(q) b_{(1)_{\nu \mu}}(q) G^\nu_{\rho^{(2)}}(q)
\]

\[- \frac{b_{(1)_{\mu \nu}}(q)}{e_\rho(q)} \sum_{\lambda} f^\lambda_{\rho \mu}(q) \frac{b_\mu(q^{-1})}{b_\nu(q^{-1})} q^{n(\mu) - n(\lambda)} b_{(1)_{\mu \nu}}(q) G^\lambda_\rho(q).
\]

Hence it holds that

\[
G^\mu_\rho(q) = \sum_{\rho^{(1)}_n \rightarrow r, \rho^{(2)}_{n-r}} \frac{z_{\rho}}{b_\rho(q^{-1})} \frac{b_\mu(q^{-1})}{b_\nu(q^{-1})} G^\mu_{\rho^{(1)}}(q) G^\nu_{\rho^{(2)}}(q)
\]

\[- \sum_{\lambda} f^\lambda_{\rho \mu}(q) \frac{b_\mu(q^{-1})}{b_\nu(q^{-1})} q^{n(\mu) - n(\lambda)} G^\lambda_\rho(q).
\]

From the definition of \( \tilde{G} \), it follows that

\[
\tilde{G}^\mu_\rho(q) = \sum_{\rho^{(1)}_n \rightarrow r, \rho^{(2)}_{n-r}} \frac{z_{\rho}}{b_\rho(q^{-1})} \frac{b_\mu(q^{-1})}{b_\nu(q^{-1})} \tilde{G}^\mu_{\rho^{(1)}}(q) \tilde{G}^\nu_{\rho^{(2)}}(q)
\]

\[- \sum_{\lambda} f^\lambda_{\rho \mu}(q) \frac{b_\mu(q^{-1})}{b_\nu(q^{-1})} q^{n(\mu) - n(\lambda)} \tilde{G}^\lambda_\rho(q).
\] (3.5)
In the following, we will show that $\tilde{G}_\rho^\mu(q)$ is a Laurent polynomial by induction on a partial order of the partitions. The partial order on the partitions are defined as follows: For any two partitions $\lambda$ and $\mu$, define

$$\lambda \preceq \mu \iff \left\{ \begin{array}{l} |\lambda| < |\mu|, \text{ or} \\ |\lambda| = |\mu|, \text{ and } \lambda \preceq \mu. \end{array} \right.$$ 

where $|\lambda|$ denotes the size of the partition $\lambda$, and $\preceq$ denotes the dominance partial order [Mc, p. 7] of the partitions, which is defined by the following condition:

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \leq \mu_1 + \mu_2 + \cdots + \mu_i$$

for each $i = 1, 2, \ldots, n$, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$. Now it is obvious that $\bar{\mu}, (1^\nu) \preceq \mu$, since the sizes of $\bar{\mu}$ and $(1^\nu)$ are smaller than $\mu$. On the other hand, if $\lambda - \bar{\mu}$ is a vertical $r$-strip, then the size of these partitions $\lambda$, $\mu$ coincide, but $\lambda$ is smaller than $\mu$ in the dominance order if $\lambda \neq \mu$. Thus we may assume that the $\tilde{G}$’s in the right hand side of (3.5) are all Laurent polynomials by the induction hypothesis. Finally, if we remark that, for $\lambda$ such that $\lambda - \bar{\mu}$ is a vertical $r$-strip, $f_{\lambda(1^\nu)}(q^{-1})$ is also a Laurent polynomials, then it follows that $\tilde{G}_\rho^\mu(q)$ is a Laurent polynomial. The initial condition of the induction is satisfied, since $\tilde{G}_\rho^\mu(q) = (-1)^n q^n$ for all $n$ and $\rho \vdash n$. This follows from the fact that $G_{(1^n)}^\nu(q) \equiv 1$ for all $n$ and $\rho \vdash n$ (Proposition 3).}

**Example 6** According to the table of Green polynomials due to Green [Gr], the polynomial $G_{(1^4)}^{2,1,1}(q)$ can be computed for each $\rho$ as follows:

$$G_{(1^4)}^{2,1,1} = 1 - 4q^3 + 3q^4,$$

$$G_{(21^2)}^{2,1,1} = 1 - 2q^3 + q^4,$$

$$G_{(2^2)}^{2,1,1} = 1 - q^4,$$

$$G_{(31)}^{2,1,1} = 1 - q^3,$$

$$G_{(4)}^{2,1,1} = 1 - q^4.$$

**Remark 7** In Example 6, all the polynomials still have the factor $1 - q$. In fact, we can expect more precise statement for the decomposition of Green polynomials. Let $\mu$ be a partition of $n$. Then for each partition $\rho \vdash n$, we conjecture that there exists a polynomial $H_\rho^\mu(q) \in \mathbb{Z}[q]$ such that

$$Q_\rho^\mu(q) = \frac{b_\rho(q)}{e_\rho(q)} H_\rho^\mu(q).$$

But in our present context, this is too much for our sake, since we only have to know what roots of unity are zeros of Green polynomials.

**Corollary 8** Let $\mu$ be a partition of $n$, and an integer $l \in \{2, \ldots, M_\mu\}$ arbitrarily fixed. Then it holds that $Q_\rho^\mu(\zeta_l^k) = 0$ for each $k = 1, \ldots, l - 1$. 

12
If we consider the case where $\mu = (21^3)$ for example, we can see from the table in [Gr] that
\[ Q_{(1^3)}^{(\mu)}(q) = (1 + q)(1 + q + q^3)(1 + 2q + 3q^2 + 4q^3), \]
and it is clear that the assertion of the corollary holds.

Let $\mu$ be a partition, and an integer $l$ such that $1 \leq l \leq M_\mu$ fixed. By Oshima’s lemma, it immediately follows that the dimensions of the mod $l$ sums $R_{\mu}(k;l)$ of $R_{\mu}$ coincide each other. In the next section, we will consider its representation theoretical interpretation for the algebra $R_\mu$.

At the end of this section, we mention to an explicit formula for the Green polynomials for arbitrary hook which generalizes Morris’ formula (Proposition 4), that can be proved by similar argument to Theorem 5.

**Proposition 9** Let $\mu = (n - h, 1^h)$ be a hook, and $\rho = (1^{r_1}2^{r_2} \cdots n^{r_n}) \vdash n$ a partition of $n$. Then we have
\[ Q_\mu^{(\rho)}(q) = \frac{(1 - q) \cdots (1 - q^h)}{e_\rho(q)} G_\mu^{(\rho)}(q), \]
where
\[ G_\mu^{(\rho)}(q) = (1 - q^n) - \sum_{k=1}^{n-h-1} \sum_{\tau \vdash (1^k, k, t_1, \cdots, t_k)} \binom{r_1}{t_1} \cdots \binom{r_k}{t_k} q^{n-k} e_\tau(q). \]

**Proof.** We prove by the induction on $n - h$. For the case $n - h = 1$, it is clear from Proposition 3, from which we can see that $G_\mu^{(\rho)}(q) \equiv 1$. Suppose that $n - h > 1$. Let $\bar{\mu} = (1^{h+1})$ and $\nu = (n - h - 1)$. Consider the product of two Hall-Littlewood functions $P_{\bar{\mu}} = P_{\bar{\mu}}(x; t)$ and $P_\nu = P_\nu(x; t)$, and expand it into a linear combination of $P_\lambda(x; t)$’s again. Then we have [M, III, (2.7) and (3.7)]
\[ P_{\bar{\mu}}P_\nu = f^{(n-h-1, 1^{h+1})}_{\bar{\mu}\nu}(t) P_{(n-h-1, 1^{h+1})} + f^{(n-h-1, 1^{h+1})}_{\bar{\mu}\nu}(t) P_{\mu}. \]
If $\lambda = (n - h - 1, 1^{h+1})$ or $\lambda = \mu$, then $\lambda - \bar{\mu}$ is a horizontal strip [Mc, p. 5]. We have an explicit formula for the polynomials $f^{(n-h-1, 1^{h+1})}_{\bar{\mu}\nu}(t)$ [Mc, III, (3.10)] in this case, and it holds that
\[ f^{(n-h-1, 1^{h+1})}_{\bar{\mu}\nu}(t) = 1, \quad f^{(n-h-1, 1^{h+1})}_{\bar{\mu}\nu}(t) = 1. \]
Therefore we have
\[ P_{\bar{\mu}}P_\nu = P_{(n-h-1, 1^{h+1})} + P_{\mu}. \]
By this identity, the same argument as in the proof of Theorem 5 shows that
\[ Q_\rho^{(\mu)}(q) = \frac{1}{1 - q^{h+1}} \left\{ Q_\rho^{(\mu)}(q) - \sum_{\rho^{(1)} = (n - h - 1), \rho^{(2)} = \rho} q^{h+1} Q_\rho^{(1)}(q) \right\}, \]
where $\eta = (n-h-1, 1^{h+1})$. By the induction hypothesis, it follows that

\[
\begin{align*}
Q_\rho^\mu(q) &= \frac{1}{1-q^{h+1}} \left\{ (1-q) \cdots (1-q^{h+1}) G_\rho^n(q) 
- \sum_{\rho^{(1)}\cap \rho^{(2)} = \rho} \frac{z_\rho}{z_\rho^{(1)} z_\rho^{(2)}} q^{h+1} (1-q) \cdots (1-q^{h+1}) \right\} \\
&= \frac{(1-q) \cdots (1-q^h)}{e_\rho(q)} \left\{ G_\rho^n(q) - \sum_{\rho^{(1)} \cap \rho^{(2)}} \frac{z_\rho}{z_\rho^{(1)} z_\rho^{(2)}} q^n e_\tau(q) \right\} \\
&= \frac{(1-q) \cdots (1-q^h)}{e_\rho(q)} \left\{ \sum_{\tau \subset \rho} \frac{z_\rho}{z_\rho^{(1)} z_\rho^{(2)}} q^n (n-h-1) e_\tau(q) \right\},
\end{align*}
\]

where $\tau \subset \rho$ denotes that $\tau$ is a subpartition of $\rho$. Therefore we have

\[
G_\rho^n(q) = G_\rho^n(q) - \sum_{\tau \subset \rho} \frac{z_\rho}{z_\rho^{(1)} z_\rho^{(2)}} q^n (n-h-1) e_\tau(q).
\]

Remark that, for $\tau = (1^{t_1} 2^{t_2} \cdots)$, we have

\[
\frac{z_\rho}{z_\rho^{(1)} z_\rho^{(2)}} = \left( \begin{array}{cc} r_1 & r_2 \\ t_1 & t_2 \end{array} \right) \cdots.
\]

Then, again by the induction hypothesis, the assertion follows. \hfill \square

**Example 10** Let us consider the case where $\mu = (3,1,1,1)$. Let $\rho = (2^2 1^2)$ for example. By Proposition 7, we have

\[
Q_\rho^\mu(q) = \frac{(1-q)(1-q^2)(1-q^3)}{(1-q)^2(1-q^2)^2} G_\rho^n(q),
\]

and

\[
G_\rho^n(q) = (1-q^6) \sum_{k=0}^{2} \sum_{\tau = (1^{r_1} \cdots (1^{r_k} \cdots q^n k e_\tau(q)}
\]

\[
= (1-q^6) - \left( \begin{array}{c} 2 \\\ 1 \end{array} \right) \left( \begin{array}{c} 2 \\\ 0 \end{array} \right) q^5 (1-q) - \left( \begin{array}{c} 2 \\\ 2 \end{array} \right) \left( \begin{array}{c} 2 \\\ 0 \end{array} \right) q^4 (1-q)^2 - \left( \begin{array}{c} 2 \\\ 0 \end{array} \right) \left( \begin{array}{c} 2 \\\ 1 \end{array} \right) q^4 (1-q^2)
\]

\[
= 1 - 3q^6 + 2q^6.
\]

Thus

\[
Q_\rho^\mu(q) = \frac{(1-q)(1-q^2)(1-q^3)}{(1-q)^2(1-q^2)^2} (1-3q^4 + 2q^6)
\]

\[
= (1+q)(1+q+q^2)(1-q+2q^2-2q^3),
\]

which coincides with the result on the table due to Morris [Mr]. \hfill \square
4 hook case

In this section, we will find a representation theoretical interpretation of ‘coincidence of dimension’ of the algebra $R_\mu$ in the case where $\mu$ is a hook.

Let $\mu = (n - h, 1^h) \vdash n$ be a hook. Suppose that $n - h > 1$, since we should exclude the case of coinvariant algebras from our argument. In this case, we have $M_\mu = h$. Then for each fixed $l = 1, 2, \ldots, h$, it follows from Proposition 9 and Lemma 1 that the mod $l$ sums $R_\mu(k; l)$ of the algebra $R_\mu$ have the same dimension $\dim R_\mu/l$. Since the case $l = 1$ is trivial, we consider the problem for $l = 2, 3, \ldots, h$ in the sequel. Precisely:

For each fixed $l = 2, 3, \ldots, h$,

- find a subgroup $H_\mu(l)$ of $S_n$, and
- find $H_\mu(l)$-modules $Z_\mu(k; l)$ ($k = 0, 1, \ldots, l - 1$) of equal dimension such that

$$R_\mu(k; l) \cong_{S_n} \text{Ind}_{H_\mu(l)}^{S_n} Z_\mu(k; l)$$

for each $k = 0, 1, \ldots, l - 1$.

Let $h = dl + r$ ($0 \leq r \leq l - 1$), and let $\bar{\mu}$ denote the partition $\bar{\mu} = (n - h, 1^{h-dl}) \vdash n - dl$.

Consider the following product $a$ of $d$ cyclic permutations of order $l$:

$$a = (1, 2, \ldots, l)(l + 1, l + 2, \ldots, 2l) \cdots ((d - 1)l + 1, (d - 1)l + 2, \ldots, dl).$$

It is obvious that $a^l = 1$. Consider the cyclic subgroup $H_1$ of $S_n$ generated by $a$:

$$H_1 := \langle a \rangle \cong C_l,$$

where $C_l$ denotes the cyclic group of order $l$. Consider also the isotropic subgroup $H_2$ of $S_n$ for $\{1, 2, \ldots, dl\}$:

$$H_2 = \{\sigma \in S_n | \sigma(i) = i \text{ for } i = 1, 2, \ldots, dl\}.$$

If we denote by $S_{\{i, j, \ldots, k\}}$ the symmetric group of the letters $\{i, j, \ldots, k\}$, then we have $H_2 = S_{\{dl+1, \ldots, n\}}$, thus $H_2$ is isomorphic to the symmetric group of $n - dl$ letters. Let a subgroup $H_\mu(l)$ of $S_n$ be the direct product of $H_1$ and $H_2$:

$$H_\mu(l) := H_1 \times H_2 \cong C_l \times S_{n-dl}.$$

For each $k = 0, 1, \ldots, l - 1$, we define a representation $Z_\mu(k; l)$ of $H_\mu(l)$ as follows:

$$Z_\mu(k; l) := \bigoplus_{d=0}^{n(\bar{\mu})} \varphi^{(k-d)} \otimes R_{\bar{\mu}}^d,$$
where \( \varphi^{(s)} \) is the irreducible representation \( a \mapsto \zeta^s \) of the cyclic group \( C_l = \langle a \rangle \), and \( R^d_\bar{\mu} \) denotes the \( d \)-th homogeneous component of the DeConcini-Procesi-Tanisaki algebra \( R_\bar{\mu} \) corresponding to the partition \( \bar{\mu} \). Remark that these \( l \) modules \( Z_\mu(k; l) \) have the same dimension which is equal to that of \( R^d_\bar{\mu} \), since the \( H_\mu(l) \)-module \( Z_\mu(k; l) \) is obtained by twisting each homogeneous component \( R^d_\bar{\mu} \) of the graded algebra \( R_\bar{\mu} \) by a one-dimensional representation \( \varphi^{(k-d)} \).

**Example 11** Let \( \mu = (2\, 1^5) \) and \( l = 2 (< 5 = M_\mu) \). In this case, the element \( a \) is defined by \( a = (1, 2)(3, 4) \), and the subgroup \( H_\mu(2) \) is then defined by

\[
H_\mu(2) = \langle a \rangle \times S_{\{5, 6, 7\}} \cong C_2 \times S_3.
\]

The \( H_\mu(2) \)-modules \( Z_\mu(k; 2) \) \((k = 0, 1)\) of equal dimension are defined by

\[
Z_\mu(0; 2) = (\varphi^{(0)} \otimes R^0_\bar{\mu}) \oplus (\varphi^{(1)} \otimes R^1_\bar{\mu}),
\]

\[
Z_\mu(1; 2) = (\varphi^{(1)} \otimes R^0_\bar{\mu}) \oplus (\varphi^{(0)} \otimes R^1_\bar{\mu}),
\]

where \( \bar{\mu} = (2, 1) \) and \( \varphi^{(k)} \) is the irreducible representation of \( C_2 \) sending the generator \( a \) to \( \zeta^k \).

Now our subject is to show

\[
R_\mu(k; l) \cong \text{Ind}_{H_\mu(l)}^{S_n} Z_\mu(k; l)
\]

for each \( k = 0, 1, \ldots, l - 1 \). We will see in the following that these \( S_n \)-isomorphisms amount to an \( S_n \times C_l \)-isomorphism

\[
R_\mu \cong_{S_n \times C_l} \text{Ind}_{S_{n-d}}^{S_n} R_\bar{\mu}.
\]

Here the \( S_n \)-actions on both sides are natural ones. The \( C_l \)-actions in both sides are defined as follows. In the left hand side, the cyclic group \( C_l \) acts on \( R_\mu \) as a scalar multiplication for each homogeneous component:

\[
a.x = \zeta^d_l x, \quad x \in R^d_\mu.
\]

On the other hand, remark that the induced representation \( \text{Ind}_{S_{n-d}}^{S_n} R_\bar{\mu} \) has the following realization:

\[
\text{Ind}_{S_{n-d}}^{S_n} R_\bar{\mu} = \bigoplus_{\sigma \in S_n/S_{n-d}} \sigma \otimes R_\bar{\mu}.
\]

By the identification (4.1), the action of the cyclic group \( C_l \) on \( \text{Ind}_{S_{n-d}}^{S_n} R_\bar{\mu} \) is defined by

\[
a(\sigma \otimes x) := \sigma a^{-1} \otimes ax = \zeta^d_l \sigma a^{-1} \otimes x \quad \text{if} \ x \in R^d_\bar{\mu}.
\]

We can show that these two \( S_n \times C_l \)-module structures are actually isomorphic. It should be remarked here that it can be easily seen from (2.1) that this isomorphism is trivially hold as an \( S_n \)-isomorphism.
**Theorem 12** Let a partition \( \mu = (n-h, 1^h) \vdash n \) \((n-h > 1)\) be a hook, \( l \) a positive integer belonging to \( \{2,3,\ldots,h\} \). Let \( n = dl + r \) \((0 \leq r \leq l-1)\), and \( \bar{\mu} = (n-h, 1^r) \) a partition obtained by deleting \( dl \)’s from \( \mu \). Then, there exists an isomorphism of \( S_n \times C_l \)-modules
\[
R_\mu \cong \text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}.
\]

This theorem is shown in the next section. In the rest of this section, we will verify that the theorem is actually equivalent to our claim.

**Proposition 13** Let \( \mu = (n-h, 1^h) \vdash n \) \((n-h > 1)\) be a hook, and let an integer \( l \in \{2,3,\ldots,h\} \) be fixed. If the isomorphism of \( S_n \times C_l \)-modules in Theorem 12 holds, then there exists an \( S_n \)-module isomorphism
\[
R_\mu(k;l) \cong \text{Ind}_{S_{n-dl}}^{S_n} Z_\mu(k;l)
\]
for each \( k = 0,1,\ldots,l-1 \).

Conversely, if these \( S_n \)-module isomorphisms hold, then \( R_\mu \) and \( \text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}} \) are isomorphic as \( S_n \times C_l \)-modules.

**Proof.** Suppose that there exists an \( S_n \times C_l \)-module isomorphism
\[
R_\mu \cong \text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}. \tag{4.2}
\]
We compare the eigenspaces of the generator \( a \) of the cyclic group \( C_l \) in each side of (4.2). For each \( k = 0,1,\ldots,l-1 \), it is clear from the action of \( C_l \) that the eigenspace of \( a \) with the eigenvalue \( \zeta^k \) in the left hand side coincide with the \( k \)-th mod \( l \) sum \( \text{Ind}_{S_{n-dl}}^{S_n} Z_\mu(k;l) \). In the right hand side, we have
\[
\text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}} = \bigoplus_{\sigma \in S_n/S_{n-dl}} \sigma \otimes R_{\bar{\mu}}
\]
\[
= \bigoplus_{d \geq 0} \bigoplus_{w \in S_n/C_l \times S_{n-dl}} \bigoplus_{j=0}^{l-1} w a^j \otimes R_{\bar{\mu}}^d
\]
\[
= \bigoplus_{d \geq 0} \bigoplus_{w \in S_n/C_l \times S_{n-dl}} \bigoplus_{s=0}^{l-1} w b_s \otimes R_{\bar{\mu}}^d,
\]
where \( b_s = 1 + (\zeta^s) a + (\zeta^{2s}) a^2 + \cdots + (\zeta^{(l-1)s}) a^{l-1} \) for each \( s = 0,1,\ldots,l-1 \). Since it holds that \( b_s a^{-1} = \zeta^s b_s \) for each \( s \), it can readily be seen that the \( \zeta^k \)-eigenspace of \( a \) in the right hand side is
\[
\bigoplus_{d+s = k \mod l} b_s \otimes R_{\bar{\mu}}^d.
\]
Noting that \( C b_s \cong C_l \varphi^{(s)} \) for each \( s = 0,1,\ldots,l-1 \), the necessity part follows. Conversely, the sufficiency part is proved by tracing back the proof of the necessity part. \( \Box \)
5 Proof

In this section, we will verify Theorem 12:

$$R_\mu \cong_{S_n \times C_l} \text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}.$$ 

Since we work on a field of characteristics zero, it suffices to show that

$$\text{char} R_\mu(w, a^j) = \text{char} \text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}(w, a^j)$$  \hspace{1cm} (5.1)

for every $(w, a^j) \in S_n \times C_l$ and $j = 0, 1, \ldots, l-1$. Recall that the graded character $\text{char}_q R_\mu(\rho)$ coincides with the Green polynomial $Q_\mu^{\rho}(q)$ for each $\rho \vdash n$. In this point of view, considering the action of $C_l$, it is not difficult to see that (5.1) is equivalent to the following identity:

$$Q_\mu^{\lambda(w)}(q)_{| q = \zeta_j^l} = \text{char} \text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}(w, a^j).$$  \hspace{1cm} (5.2)

Note that the case $j = 0$ is already proved for any $w \in S_n$. In this case, (5.1) reduces to the identity

$$\text{char} R_\mu(w) = \text{char} \text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}(w),$$

which is assured by the $S_n$-module isomorphism (2.1). Therefore it is enough to show (5.1) for $(w, a^j) \in S_n \times C_l$ with $1 \leq j \leq l-1$. First, we will consider the support of $\text{char} \text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}$, i.e., the elements $(w, a^j)$ of $S_n \times C_l$ satisfying

$$\text{char} \text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}(w, a^j) \neq 0.$$

Actually, we will see the necessary condition for such an element of $S_n \times C_l$.

Suppose that $(w, a^j) \in S_n \times C_l$ is a support of $\text{char} \text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}$. Since we have

$$\text{char} \text{Ind}_{S_{n-dl}}^{S_n} (w, a^j) R_{\bar{\mu}} = \sum_{\sigma \in S_n/S_{n-dl}} \text{char}(\sigma \otimes R_{\bar{\mu}})(w, a^j),$$

there exists an element $\sigma$ of $S_n/S_{n-dl}$ and a basis element $x$ of $R_{\bar{\mu}}^d$ such that

$$(w, a^j)(\sigma \otimes x)|_{\sigma \otimes x} \neq 0,$$

where $(w, a^j)(\sigma \otimes x)|_{\sigma \otimes x}$ denotes the coefficient of $\sigma \otimes x$ in the linear expansion of $(w, a^j)(\sigma \otimes x)$ by the basis $\{\sigma \otimes x\}$ of $\text{char} \text{Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}$. In this case, since we have

$$(w, a^j)(\sigma \otimes x) = w\sigma a^{-j} \otimes a^j x = \zeta^d_{ij} w\sigma a^{-j} \otimes x,$$

it holds that

$$(w, a^j)(\sigma \otimes x)|_{\sigma \otimes x} \neq 0 \implies w\sigma a^{-j} \equiv \sigma \text{ mod } S_{n-dl}.$$ 

This shows that there exists an element $\tau$ of $S_{n-dl}$ such that $w\sigma a^{-j} = \sigma \tau$, i.e., $w = \sigma \tau a^j \sigma^{-1}$.

To summarize:
Lemma 14: If an element \((w, a^j)\) of \(S_n \times C_l\) satisfies the condition
\[
\text{char Ind}_{S_n \times C_l} \mu \left( w, a^j \right) \neq 0,
\]
then \(w\) is conjugate to \(a^j \tau\) for some \(\tau \in S_{n-dl}\): \(w \sim a^j \tau, \tau \in S_{n-dl}\). □

Our next subject is to compute the character value \(\text{char Ind}_{S_n \times C_l} \mu \left( w, a^j \right)\) for a support \((w, a^j) \in S_n \times C_l\). Let an element \((w, a^j) \in S_{n-dl}\) satisfy
\[
\text{char Ind}_{S_{n-dl}} \mu \left( w, a^j \right) \neq 0.
\]
\[\tag{5.3}\]
By Lemma 14, we may assume that \(w = a^j \tau, \tau \in S_{n-dl}\) without loss of generality. It also follows from (5.3) that there exists \(\sigma \in S_n/S_{n-dl}\) such that \(w \sigma a^{-j} = \sigma \rho\) for some \(\rho \in S_{n-dl}\).

For such an element \(\sigma\), we have
\[
(w, a^j)(\sigma \otimes x) = w \sigma a^{-j} \otimes a^j x = \sigma \rho \otimes a^j x = \sigma \otimes a^j \rho x.
\]
(Remark that \(a\) and \(\rho\) commute.) If we denote by \(B_{\tilde{\mu}}\) a homogeneous basis of \(R_{\tilde{\mu}}\), then it follows that
\[
\text{char}(\sigma \otimes R_{\tilde{\mu}})(w, a^j) = \sum_{x \in B_{\tilde{\mu}}} (w, a^j)(\sigma \otimes x)|_{\sigma \otimes x}
\]
\[
= \sum_{d \geq 0} \sum_{x \in B_{\tilde{\mu}}} \xi_d^j \sigma \otimes \rho_x |_{\sigma \otimes x}
\]
\[
= \text{char}_{qR_{\tilde{\mu}}}(\rho)|_{q = \xi_d^j}.
\]

Let
\[
S^{(j)}(w) := \{ \sigma \in S_n/S_{n-dl} \mid w \sigma a^{-j} \equiv \sigma \text{ mod } S_{n-dl} \},
\]
and
\[
S^{(j)}(w) := \{ \sigma \in S_n/S_{n-dl} \mid w \sigma a^{-j} = \sigma \rho \}
\]
for each \(\rho \in S_{n-dl}\). Note that
\[
S^{(j)}(w) = \bigcup_{\rho \in S_{n-dl}} S^{(j)}(w). \quad \text{(disjoint union)}
\]

Then it holds that
\[
\text{char Ind}_{S_{n-dl}} \mu (w, a^j) = \sum_{\rho \in S_{n-dl}} \sum_{\sigma \in S^{(j)}(w)} \text{char}(\sigma \otimes R_{\tilde{\mu}})(w, a^j)
\]
\[
= \sum_{\rho \in S_{n-dl}} \#S^{(j)}(w) \text{char}_{qR_{\tilde{\mu}}}(\rho)|_{q = \xi_d^j}.
\]

Remark that \(\rho \in S_{n-dl}\) satisfying \(S^{(j)}(w) \neq \phi\) are all conjugate in \(S_{n-dl}\). It is easy to see that:
**Lemma 15** Let us consider an element \( w \in S_n \) of the form \( w = a^j \tau \) (\( \tau \in S_{n-dl} \)). Then, for each \( \rho \in S_{n-dl} \), we have

\[
w \sigma a^{-j} = \sigma \rho \implies \tau \sim \rho.
\]

**Proof.** Suppose that \( w \sigma a^{-j} = \sigma \rho \) for some \( \rho \in S_{n-dl} \). If \( w = a^j \tau \), then we have \( a^j \tau \sigma a^{-j} = \sigma \rho \), hence \( a^j \tau \sigma = \sigma \rho a^j \). Therefore, \( a^j \tau \) is conjugate to \( \rho a^j \). Thus \( \tau \) and \( \rho \) are conjugate since \( C_l \) and \( S_{n-dl} \) commute. \( \square \)

By Lemma 15, we have

\[
\text{char Ind}_{S_{n-dl}}^{S_n} R_{\bar{\mu}}(w, a^j) = \sum_{\rho \in S_{n-dl}} \sharp S_j^\rho(w) \cdot \text{char}_{\bar{\mu}}(\tau) |_{q = \zeta_l}.
\]

The cardinality of \( S_j^\rho(w) \) is obtained as follows.

**Proposition 16** For \( w = a^j \tau \in C_l \times S_{n-dl} = H_{\mu}(l) \), it holds that

\[
\sharp S_j^\rho(a^j \tau) = p^e! \binom{z_p + e}{e},
\]

where \( \lambda(a^j) = (p^e) \) and \( \lambda(\tau) = (1^{z_1}2^{z_2} \cdots) \) are the cycle type of \( a^j \) and \( \tau \) respectively.

**Proof.** First we remark that \( \sharp S_j^\rho(a^j \tau) = \sharp S_j^\rho(a^j \tau') \) if \( \tau \) and \( \tau' \) are conjugate in \( S_{n-dl} \). This follows from the fact that, if \( \tau' = z \tau z^{-1} \) (\( z \in S_{n-dl} \)), then we have a bijection

\[
S_j^\rho(a^j \tau) \rightarrow S_j^\rho(a^j \tau') : \sigma \mapsto z \sigma z^{-1}
\]

Hence it is enough to confirm the assertion for the following special \( \tau \):

\[
\tau = \sigma(1) \sigma(2) \\
\tau(1) = (dl + 1, \ldots, dl + p) \cdots (dl + (z_p - 1)p + 1, dl + z_p p),
\]

where \( \tau(2) \) is a permutation of remaining letters \( \{dl + z_p p + 1, \ldots, n\} \).

Our subject here is enumeration of the elements \( \sigma \in S_n \) satisfying the condition

\[
a^j \tau \sigma a^{-j} = \sigma \rho \quad (5.4)
\]

for some \( \rho \in S_{n-dl} \). Note that, by (5.4), \( \tau \) and \( \rho \) should be conjugate in \( S_{n-dl} \). Hence it is enough to count the number of elements \( \sigma \in S_n \) satisfying

\[
a^j \tau \sigma = \sigma a^j \rho \quad (5.5)
\]

for \( \tau, \rho \in S_{n-dl} \) with \( \lambda(\tau) = \lambda(\rho) \).
Let the cycle type $\lambda(a^j)$ of $a^j$ be $(p^e)$. Note that $pe = dl$. Let $\rho$ decompose into the product of $z_p$ cycles of length $p$ and the remainder as follows:

$$\rho = \rho^{(1)}\rho^{(2)}, \quad \lambda(\rho^{(1)}) = (p^e), \quad \lambda(\rho^{(2)}) = \lambda(v) - (p^e).$$

Since the effect of the action of $\tau^{(2)}$ should be coincide with that of the action of $\rho^{(2)}$ on the positions of the components of $\sigma$, it should hold that

$$\{dl + z_pp + 1, \ldots, n\} \subset \{\sigma_{dl+1}, \ldots, \sigma_n\}.$$ 

Since $\sigma$ is an element of $S_n/S_{n-dl}$, we may assume that $\sigma_{dl+1} < \cdots < \sigma_n$. Therefore the last $n - (e + z_p)p (= n - (dl + z_pp + 1) + 1)$ components of $\sigma$ are uniquely determined as follows:

$$\sigma_{dl+z_pp+1} = dl + z_pp + 1, \ldots, \sigma_n = n.$$ 

Now, it suffice to enumerate the elements

$$\sigma' = [\sigma_1, \sigma_2, \ldots, \sigma_{dl+z_pp}] \in S_{dl+z_pp},$$

satisfying

$$a^j\tau^{(1)}\sigma' = \sigma'a^j\rho^{(2)}, \quad \sigma_{dl+1} < \cdots < \sigma_{dl+z_pp}.$$ 

At the beginning, it is necessary to determine which $e$ cycles of length $p$ come to the first $pe$ positions in $\sigma'$ corresponding to the permutation $a^j$. There are \binom{z_p + e}{e} ways of such a choice. Once we fix such a choice, the number of arranging these components with respect to the condition $a^j[\sigma_1, \ldots, \sigma_{pe}] = [\sigma_1, \ldots, \sigma_{pe}]a^j$ is just the cardinality of the centralizer of $a^j$, which is equal to $p^e e!$. The remaining parts are uniquely determined, since they should obey the condition $\sigma_{pe+1} < \cdots < \sigma_{dl+z_pp}$. Thus the number of elements $\sigma \in S_n/S_{n-dl}$ satisfying (5.5) is

$$\binom{z_p + e}{e} p^e e!.$$ 

\[\square\]

**Example 17** Let us consider the case where $\mu = (3,1^8) \vdash 11$ and $l = 3$. In this case, we have $h = 8$ and our permutation $a$ is

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 3 & 1 & 5 & 6 & 4 & 7 & 8 & 9 & 10 & 11 \end{pmatrix} = (1,2,3)(4,5,6),$$

since $8 = 2 \times 3 + 2 (d = 2)$. Suppose that the cycle type $\lambda(\tau)$ of an element $\tau \in S_{\{7,8,9,10,11\}} \cong S_{n-dl} = S_5$ is $\lambda(\tau) = (3,2)$. We can take $\tau = (7,8,9)(10,11)$ without loss of generality. We consider the case $j = 1$ in this example. Then the elements $\sigma \in S_n/S_{n-dl}$ that we are going to enumerate satisfy the condition $a\tau\sigma a^{-1} = \sigma\rho$ for some $\rho \in S_{\{7,8,9,10,11\}}$ of which cycle type
is $\lambda(\rho) = (3, 2)$. We may also assume that $\sigma_7 < \sigma_8 < \cdots < \sigma_{11}$. Therefore, they should satisfy

$$
(1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11)[\sigma_1, \sigma_2, \ldots, \sigma_{11}]
= [\sigma_1, \sigma_2, \ldots, \sigma_{11}](1, 2, 3)(4, 5, 6)
\begin{pmatrix}
7 & 8 & 9 & 10 & 11 \\
\rho_7 & \rho_8 & \rho_9 & \rho_{10} & \rho_{11}
\end{pmatrix}
= [\sigma_1, \sigma_2, \ldots, \sigma_{11}](1, 2, 3)(4, 5, 6)(i, j, k)(a, b)
= [\{\sigma_1, \sigma_2, \sigma_3\}, \{\sigma_4, \sigma_5, \sigma_6\}, \{\sigma_1, \sigma_j, \sigma_k\}, \{\sigma_a, \sigma_b\}],
$$

where $\rho = (i, j, k)(a, b)$ denotes its cyclic decomposition, and $\{\{\sigma_1, \sigma_2, \sigma_3\} \cdots\}$ denotes that we can ignore the order of these components $\sigma_1, \sigma_2, \sigma_3$ in the permutation. Then, since it should for example coincide the effect of the left action of $(1,2,3)$ on the letters of $\sigma$ with the effect of the right action of one of the cyclic permutations $(1, 2, 3), (4, 5, 6), (i, j, k)$ on the position of letters of $\sigma$, we can conclude that one of the following conditions holds: $\{1, 2, 3\} = \{\sigma_1, \sigma_2, \sigma_3\}, \{\sigma_4, \sigma_5, \sigma_6\}, \{\sigma_1, \sigma_j, \sigma_k\}$. From this observation, we can see that there are $2^{n+1}$ ways of assigning letters for $\{\sigma_1, \sigma_2, \sigma_3\}, \{\sigma_4, \sigma_5, \sigma_6\}$. Once we choose letters for these positions, the remaining parts are uniquely determined by the condition $\sigma_7 < \sigma_8 < \cdots < \sigma_{11}$. Thus there are $3^22!(\frac{2n+1}{2})$ suitable elements in $S_n/\text{S}_{n-d}$.

To summarize:

**Proposition 18** Let $j$ be an integer in $\{1, 2, \ldots, l - 1\}$. Suppose that the cycle type of $a^j$ is given by $(p^e)$. For each $(w, a^j) \in S_n \times C_l$, we have

$$
\text{char Ind}_{S_{n-d}}^{S_n} R_{\bar{\mu}}(w, a^j)
= \begin{cases}
p^e! \left( \begin{array}{c}
z_p + e \\ e
\end{array} \right) \text{char}_q R_{\bar{\mu}}(\tau)|_{q = q^j}, & \text{if } w \sim a^j v \text{ for some } \tau \in S_{n-d} \\
0, & \text{otherwise}.
\end{cases}
$$

Now, by Lemma 14 and Proposition 18, (5.2) is equivalent to the following two conditions:

- $Q_{p^e}(q^j) \neq 0 \implies \rho = \lambda(a^j \tau) \quad (\tau \in S_{n-d}),$
- $Q_{\lambda(a^j \tau)}(q^j) = p^e! \left( \begin{array}{c}
z_p + e \\ e
\end{array} \right) Q_{\lambda(\tau)}(q^j) \quad (a^j \tau \in H_{\mu}(l)).$

First we consider the sufficient condition for a partition $\rho$ satisfying

$$
Q_{p^e}(q^j) \neq 0.
$$

It is known that the Green polynomials can also defined as the scalar product value of power-sum functions and “modified” Hall-Littlewood functions as follows. Let $\mu$ be a partition.
The modified Hall-Littlewood function $Q'_\mu(x; q)$ are defined by changing the variables of Hall-Littlewood functions $Q_\mu(x; q)$ from $x = (x_1, x_2, \ldots)$ to
\[ x/(1-q) = (x_1, qx_1, q^2x_1, \ldots; x_2, qx_2, q^2x_2, \ldots),\]
i.e.,
\[ Q'_\mu(x; q) := Q_\mu \left( \frac{x}{1-q}; q \right). \]
The modified Hall-Littlewood functions $\{Q'_\mu(x; q)\}$ form the dual basis of $\{P_\mu(x; q)\}$ with respect to the ordinary inner product of $\Lambda$, i.e.,
\[ \langle P_\lambda(x; q), Q'_\mu(x; q) \rangle = \delta_{\lambda,\mu}, \]
where $\langle \cdot, \cdot \rangle$ is defined by $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$, and $s_\lambda$ is the Schur function [Mc] corresponding to the partition $\lambda$. Therefore we have
\[ X'_\mu(q) = \langle Q'_\mu(x; q), p_\rho \rangle. \]
Let $\lambda = (1^{m_1}2^{m_2} \cdots n^{m_n})$ be a partition of $n$. Then, with respect to the inner product of $\Lambda$, it is known that the adjoint transformation of
\[ \times p_\lambda : \Lambda \longrightarrow \Lambda : f \mapsto p_\lambda f \]
is given by
\[ z_\lambda \frac{\partial}{\partial p_\lambda} : \Lambda \longrightarrow \Lambda, \]
i.e.,
\[ \langle p_\lambda f, g \rangle = \langle f, z_\lambda \frac{\partial}{\partial p_\lambda} g \rangle, \quad f, g \in \Lambda, \]
where
\[ \frac{\partial}{\partial p_\lambda} = \left( \frac{\partial}{\partial p_1} \right)^{m_1} \left( \frac{\partial}{\partial p_2} \right)^{m_2} \cdots \left( \frac{\partial}{\partial p_n} \right)^{m_n}. \]
Our object is a partition $\rho$ satisfying $Q'_\rho(\zeta_i^l) \neq 0$, which is equivalent to
\[ X'_\rho(\zeta_i^l) \neq 0, \]
since $Q'_\rho(q) = q^{n(\nu)}X'_\rho(q^{-1})$. The following two propositions due to Lascoux-Leclerc-Thibon [LLT] are fundamental for our argument.

**Proposition 19** Let $\mu = (1^{m_1}2^{m_2} \cdots n^{m_n})$ be a partition of $n$, $l$ a positive integer, $\zeta_\rho$ a primitive root of unity. For each $i = 1, 2, \ldots, n$, let $m_i = k_i l + r_i$, $0 \leq r_i \leq l - 1$. Then we have
\[ Q'_\mu(x; \zeta_i) = Q'_\mu(x; \zeta_\rho) \prod_{i=1}^n Q'_{(s^k_i)}(x; \zeta_i). \]
Proposition 20 Let $p_k(h_r(X))$ denote the plethysm [Mc, I, Section 8] of a complete symmetric function $h_r(X)$ by a power-sum $p_k(X)$. Then we have

$$Q'_{(r)}(x; \zeta_k) = (-1)^{(k-1)}r p_k(h_r(X)).$$

□

Let $\mu \vdash n$ be a hook, $h = M_{\mu}$, $1 \leq l \leq h$ and $h = dl + r \ (0 \leq r \leq l - 1)$. Recall that $a$ is the product of $d$ cyclic permutations

$$(1, 2, \ldots, l)(l + 1, l + 2, \ldots, 2l) \cdots ((d - 1)(l + 1, \ldots, dl)).$$

Theorem 21 For an element $w$ of $S_n$ and each $j = 1, 2, \ldots, l - 1$, it holds that

$$X^\mu_{\lambda(w)}(\zeta^j) \neq 0 \Rightarrow w \sim a^j \tau, \quad \tau \in S_{n-dl}.$$ 

Proof. Let $j = 1, 2, \ldots, l - 1$, and suppose that $\zeta^j$ is a primitive $p$-th root of unity. The cycle type of $a^j$ is $\lambda(a^j) = (p^e)$, where $e = dl/p$, and it suffices to show that

$$X^\mu_{\lambda(w)}(\zeta^j) \neq 0 \Rightarrow r_p \geq e, \quad \rho = (1^{r_1} 2^{r_2} \cdots n^{r_n}).$$

Let $\rho'$ be the subpartition consisting of the non-zero components of $\rho$ divisible by $p$, and $\rho''$ the remaining. If we let $\bar{\mu} = \mu - (1^{pe})$, then we have

$$X^\mu_{\lambda(w)}(\zeta^j) = \left. \langle p_{\bar{\mu}}, Q'_\mu(X; q) \rangle \right|_{q = \zeta^j}$$

$$= \left. \langle p_{\bar{\mu}}, Q'_\mu(X; q) (Q'_{(1^p)}(X; q))^e \rangle \right|_{q = \zeta^j} \quad \text{(Proposition 19)}$$

$$= \left. \pm \langle p_{\rho'}p_{\rho''}, Q'_\mu(X; q) (p_p(h_1))^e \rangle \right|_{q = \zeta^j} \quad \text{(Proposition 20)}$$

$$= \left. \pm \langle p_{\rho'}p_{\rho''}, Q'_\mu(X; q) (p_p)^e \rangle \right|_{q = \zeta^j}$$

$$= \left. \langle (D^e_{p_{\rho'}}(p_{\rho'}), Q'_\mu(X; q)) \rangle \right|_{q = \zeta^j}.$$

Therefore, if $X^\mu_{\lambda(w)}(\zeta^j) \neq 0$, then it should hold that $D^e_{p_{\rho'}}(p_{\rho'} \rho'') \neq 0$. Since $D^e_{p_{\rho'}}(p_{\rho'} \rho'') = D^e_{p_{\rho'}}(p_{\rho'} \rho'')$, it immediately follows that $r_p \geq e$. □

The rest of this section is devoted to the proof of

$$Q^\mu_{\lambda(a^j \tau)}(\zeta^j) = p^e e! \left( z_p + e \right) Q^\bar{\mu}_{\lambda(\tau)}(\zeta^j),$$

for $w = a^j \tau \ (\tau \in S_{n-d})$. We begin with the following simple lemma.
Lemma 22 Let $\mu = (n - h, 1^h)$ ($n - h > 1$) be a hook. Let an integer $l \in \{2, \ldots, l - 1\}$ be fixed, and suppose that $h = dl + r$ ($0 \leq r \leq l - 1$). Let $\bar{\mu} = (n - h, 1^r)$ and $\nu = (1^dl)$. Then it holds that

$$f_{\mu \bar{\mu} \nu}(t) = \frac{b_{\mu}(t)}{b_{\bar{\mu}}(t)b_{\nu}(t)}.$$ 

Proof. Recall that

$$f_{\mu \bar{\mu} \nu}(t) = \prod_{i \geq 1} \left[ \frac{\mu_i - \mu_{i+1}}{\mu_i - \bar{\mu}_i} \right].$$

Since $\mu$ is a hook, we have

$$f_{\mu \bar{\mu} \nu}(t) = \left[ \frac{(h+1) - 1}{(h+1) - (r+1)} \right] = \left[ \frac{h}{h - r} \right].$$

On the other hand,

$$\frac{b_{\mu}(t)}{b_{\bar{\mu}}(t)b_{\nu}(t)} = \frac{(1-t)(1-t^2) \cdots (1-t^h)(1-t)}{(1-t) \cdots (1-t^dl)(1-t) \cdots (1-t^r)(1-t)} = \left[ \frac{h}{r} \right],$$

and this completes the proof. \hfill \Box

As in the proof of Theorem 5, expanding the product of two Hall-Littlewood functions $P_\mu P_\nu$ into a linear combination of $P_\lambda$’s again, we have the following recursive formula for Green polynomials:

$$Q_{\mu}(q) = - \sum_{\lambda \vdash n - dl, \lambda - \mu \text{ is vertical dl-strip}} f_{\mu \bar{\mu} \nu}^\lambda(q^{-1}) b_{\mu}(q^{-1}) q^{n(\mu) - n(\lambda)} Q_{\rho}(q)$$

$$+ \frac{1}{f_{\mu \bar{\mu} \nu}(q^{-1}) b_{\mu}(q^{-1}) b_{\nu}(q^{-1})} q^{n(\mu) - n(\bar{\mu}) - n(\nu)} \sum_{\rho \vdash n - dl, \rho(1) = \rho(2) = \rho \geq 1} \frac{z_{\rho}}{z_{\rho(1)} z_{\rho(2)}} Q_{\rho(1)}(q) Q_{\rho(2)}(q)$$

$$= - \sum_{\lambda} f_{\mu \bar{\mu} \nu}^\lambda(q^{-1}) b_{\mu}(q^{-1}) q^{n(\mu) - n(\lambda)} Q_{\rho}(q)$$

$$+ q^{rdl} \sum_{\rho(1), \rho(2)} \frac{z_{\rho}}{z_{\rho(1)} z_{\rho(2)}} Q_{\rho(1)}(q) Q_{\rho(2)}(q),$$

where the last identity follows from Lemma 22. We prepare the following auxiliary result to proceed our computation.
Lemma 23 Let \( \mu = (n-h, 1^h) \) \( (n-h > 1) \) be a hook, \( t \in \{2, \ldots, h-1\} \) fixed and \( h = dl + r \) \( (0 \leq r \leq l-1) \). Let \( \bar{\mu} = (n-h, 1^r) \) and \( \nu = (1^d) \). Define

\[
C_\mu^\lambda(q) := \frac{f_\mu^\lambda(q) b_\mu(q)}{f_{\bar{\mu}\nu}^\mu(q) b_{\bar{\mu}}(q)}
\]

for a partition \( \lambda \vdash n \) such that \( \lambda - \bar{\mu} \) is a vertical \( dl \)-strip. Then, \( C_\mu^\lambda(q) \) is of the form

\[
C_\mu^\lambda(q) = (1 - q^l) H_\mu^\lambda(q),
\]

where \( H_\mu^\lambda(q) = \phi(q)/\psi(q) \in \mathbb{Z}(q) \) is a rational function with no factor in \( \phi(q) \) and \( \psi(q) \) divisible by \( 1 - q^l \).

Proof. Let \( D_l(f) \) denote the number of the factor \( 1 - q^l \) occurring in a polynomial \( f \in \mathbb{Z}[q] \).

For a rational function \( g(q) = \phi(q)/\psi(q) \in \mathbb{Z}(q) \), define \( D_l(g) = D_l(\phi) - D_l(\psi) \). We will show that \( D_l(C_\mu^\lambda(q)) = 1 \).

Recall that

\[
f_\mu^\lambda(q) = \left[ \frac{r_1 - r_2}{r_1 - \bar{r}_1} \right]_q \left[ \frac{r_2 - r_3}{r_2 - \bar{r}_2} \right]_q \left[ \frac{r_3 - r_4}{r_3 - \bar{r}_3} \right]_q \ldots.
\]

Suppose that \( n - h > 2 \) or \( r < l - 1 \). In this case, since \( r_i - r_{i+1} < l \) for all \( i \geq 2 \), we have

\[
D_l \left( \left[ \frac{r_i - r_{i+1}}{r_i - \bar{r}_i} \right]_q \right) = 0.
\]

Therefore, we have

\[
D_l(f_\mu^\lambda(q)) = D_l \left( \left[ \frac{r_1 - r_2}{r_1 - \bar{r}_1} \right]_q \right).
\]

A similar argument shows that

\[
D_l(f_{\bar{\mu}\nu}^\mu(q)) = D_l \left( \left[ \frac{\mu_1 - \mu_2}{\mu_1 - \bar{\mu}_1} \right]_q \right).
\]

Moreover, by the assumption, we have

\[
D_l \left( \left[ \frac{r_i - r_2}{r_i - \bar{r}_i} \right]_q \right) = D_l \left( \left[ \frac{\mu_1 - \mu_2}{\mu_1 - \bar{\mu}_1} \right]_q \right) = 0,
\]

since, in the definition of the Gaussian polynomials \( \left[ \frac{r_i - r_2}{r_i - \bar{r}_i} \right]_q \) and \( \left[ \frac{\mu_1 - \mu_2}{\mu_1 - \bar{\mu}_1} \right]_q \), the number of the factor \( 1 - q^l \) in the denominator and the numerator coincide. Again by the assumption, it is easy to see that

\[
D_l(b_\mu/b_\lambda) = 1.
\]
Thus we have $D_l(C^λ_μ(q)) = 1$ if $n - h > 2$ or $r < l - 1$.

Let us consider the case where $n - h = 2$ and $r = l - 1$. If $λ ≠ (2^l, 1^{n-2l})$, then the same argument shows that $D_l(C^λ_μ(q)) = 1$. If $λ = (2^l, 1^{n-2l})$, then we have

$$D_l(f^λ_μ(q)) = D_l\left(\frac{λ_1 - λ_2}{λ_1 - μ_1}q + D_l\left(\frac{λ_2 - λ_3}{λ_2 - μ_2}q\right)\right) = 0 + 1 = 1$$

and

$$D_l\left(\frac{μ_1 - μ_2}{μ_1 - μ_1}q\right) = 0.$$

It is also easy to see that $D_l(b_μ/b_λ) = 0$. Hence we have $D_l(C^λ_μ(q)) = 1$. □

Remark that $C^λ_μ(q^{-1})$ is obtained, up to a signature, by multiplying $C^λ_μ(q)$ by a $q$-power. Therefore, for $ρ = λ(a^tv)$, it follows from Lemma 23 that

$$Q_ρ^μ(ζ_l^j) = -\sum_λ C^λ_μ(q^{-1})q^{n(ρ) - n(λ)}Q_ρ^λ(q)\bigg|_{q = ζ_l^j} + q^{rdl} \sum_{ρ^{(1)}_μ(ζ_l^j)}\frac{z_ρ}{z_{ρ^{(1)}_μ}z_{ρ^{(2)}_μ}}Q_{ρ^{(1)}_μ}(q)Q_{ρ^{(2)}_μ}(q)\bigg|_{q = ζ_l^j} = \sum_{ρ^{(1)}_μ(ζ_l^j)}\frac{z_ρ}{z_{ρ^{(1)}_μ}z_{ρ^{(2)}_μ}}Q_{ρ^{(1)}_μ}(q)Q_{ρ^{(2)}_μ}(q)\bigg|_{q = ζ_l^j}.$$

By Theorem 5, this equals

$$b_{(1^a)}(q)\sum_{ρ^{(1)}_μ(ζ_l^j)}\frac{z_ρ}{z_{ρ^{(1)}_μ}z_{ρ^{(2)}_μ}}b^{(1^M_ρ)}(q)e_{λ(a^tv)}(q)C^μ_ρ(1)\bigg|_{q = ζ_l^j} = b_{(1^a)}(q)\sum_{ρ^{(1)}_μ(ζ_l^j)}\frac{z_ρ}{z_{ρ^{(1)}_μ}z_{ρ^{(2)}_μ}}b^{(1^M_ρ)}(q)e_{λ(a^tv)}(q)C^μ_ρ(1)\bigg|_{q = ζ_l^j} = p^r(c!\sum_{ρ^{(1)}_μ(ζ_l^j)}\frac{z_ρ}{z_{ρ^{(1)}_μ}z_{ρ^{(2)}_μ}}Q_{ρ^{(1)}_μ}(q)e_{ρ^{(1)}_μ}(q)e_{λ(1)}(q)\bigg|_{q = ζ_l^j}.$$

We remark here that:

**Lemma 24** If $ρ^{(1)} ≠ λ(v)$, then it holds that $\frac{e_{ρ^{(1)}_μ}(q)}{e_{λ(v)}(q)}\bigg|_{q = ζ_l^j} = 0$.

**Proof.** Note that $ρ^{(1)} \vdash |λ(v)|$ and $ρ^{(1)} ⊂ λ(a^tv) ∪ λ(v)$. Since $ρ^{(1)} ≠ λ(v)$, the multiplicity of $p$ in the partition $ρ^{(1)}$ is larger than that of $λ(v)$. □

27
By Lemma 24, we have
\[ p^e! \sum_{\rho(1), \rho(2)} \frac{z_\rho}{z_{\rho(1)} z_{\rho(2)}} Q_{\rho(1)}^\mu(q) \frac{c_{\rho(1)}(q)}{e_{\rho(v)}(q)} \bigg|_{q = \zeta_1^j} = p^e! \frac{z_{\mu(a^i v)}}{z_{\mu(a^i)} z_{\mu(v)}} Q_{\lambda(v)}^\mu(\zeta_1^j) \]
\[ = p^e! \left( \frac{z_p + e}{e} \right) Q_{\lambda(v)}^\mu(\zeta_1^j). \]

Thus, for \( w \sim a^i \tau \ (\tau \in S_{n-di}) \), it holds that
\[ \text{char}_q R_\mu(w) \bigg|_{q = \zeta_1^j} = \frac{Q_{\lambda(v)}^\mu(\zeta_1^j)}{Q_{\mu(\tau)}^\mu(\zeta_1^j)}, \]
which completes the proof of Theorem 12.

**Example 25** Let \( \mu = (21^4) \) and \( l = 3 (M_\mu = 4) \). By Theorem 21, if \( Q_{\mu}^\mu(\zeta_3) \neq 0 \) for \( \rho \vdash 6 \), then \( \rho \) should be of the form \( (3) \cup \nu \) or \( (1^3) \cup \nu \) for some \( \nu \vdash 3 \). Let us consider the case where \( \rho = (3, 2, 1) \ (\nu = (2, 1)) \). We have shown that
\[ Q_{(3,2,1)}^\mu(\zeta_3) = 3^1! \frac{1}{1} Q_{(2,1)}^{(2,1)}(\zeta_3). \]

By Morris’ table, we can see that \( Q_{(3,2,1)}^\mu(\zeta_3) = 3 \). Also, we know that \( Q_{(2,1)}^{(2,1)}(q) \) is identically 1. Then the above identity holds.

If \( \rho = (3, 3) \), then we can see that \( Q_{(3,3)}^\mu(\zeta_3) = (1-q)(1-q^2)(1-q^3) \|_{q = \zeta_3} = 6(1-\zeta_3) \) from the table. On the other hand, we have \( Q_{(3)}^{(3)}(\zeta_3) = 1 - \zeta_3 \) by Green’s table, and the scalar factor is \( 3^1! (\frac{1}{1})^1 = 6 \). Thus the identity holds. \( \Box \)

6 Kraśkiewicz-Weyman type theorem for rectangles

In this section, we will consider the DeConcini-Procesi-Tanisaki algebra corresponding to a rectangle, a partition of the form \( \mu = (r^h) \). In this case, since \( M_\mu = h \), the dimensions of the mod \( l \) sums \( R_\mu(k; l) \), \( k = 0, 1, \ldots, l-1 \), of the algebra \( R_\mu \) coincide for each \( l = 1, 2, \ldots, h \). We will consider the Kraśkiewicz-Weyman type theorem for the algebra \( R_\mu \), that is, a representation theoretical interpretation of coincidence of dimension for the case \( l = M_\mu (= h) \). We consider this problem only for the case where the multiplicity \( h \) is a prime.

Let \( a \) be the following element of \( S_n \):
\[
a = \begin{pmatrix}
1 & \cdots & r & r + 1 & \cdots & 2r & \cdots & (h-1)r + 1 & \cdots & rh \\
r + 1 & \cdots & 2r & 2r + 1 & \cdots & 3r & \cdots & 1 & \cdots & r
\end{pmatrix} = (1, r + 1, \ldots, (h-1)r + 1)(2, r + 2, \ldots, (h-1)r + 2) \cdots (r, 2r, \ldots, hr)
\]

It is obvious that \( a^h = 1 \). Let \( C_h \) be the cyclic subgroup of \( S_n \) generated by \( a \), and
\[
S_\mu = S_{\{1,2,\ldots,r\}} \times \cdots \times S_{\{(h-1)r+1,\ldots,rh\}}
\]

28
the Young subgroup corresponding to the partition $\mu$. Then, define a subgroup $H_\mu(h)$ of $S_n$ by the semidirect product

$$H_\mu(h) = S_\mu \rtimes C_h.$$ 

For each $k = 0, 1, \ldots, h - 1$, define a one-dimensional representation $\varphi^{(k)}$ of $H_\mu(h)$ as follows:

$$\varphi^{(k)} : H_\mu(h) \rightarrow \mathbb{C}^\times : \left\{ \begin{array}{lcl}
 a & \mapsto & \zeta_h, \\
 \tau & \mapsto & 1, \quad \text{for } \tau \in S_\mu,
\end{array} \right. $$

where $\zeta_h$ denotes a primitive $h$-th root of unity.

**Example 26** Let us consider the case where $\mu = (2, 2, 2)$. In this case, we have $h = 3$ and our permutation $a$ is

$$a = \begin{pmatrix}
 1 & 2 & 3 & 4 & 5 & 6 \\
 3 & 4 & 5 & 6 & 1 & 2
\end{pmatrix}. $$

Clearly, its cyclic decomposition is $a = (1, 3, 5)(2, 4, 6)$ and its order 3. Let $S_\mu = S_{\{1, 2\}} \times S_{\{3, 4\}} \times S_{\{5, 6\}}$ be the Young subgroup corresponding to $\mu$. Then our subgroup $H_\mu(3)$ is by definition the semidirect product $H_\mu(3) = S_\mu \rtimes C_3$, where $C_3 = \langle a \rangle$. The one-dimensional representation $\varphi^{(k)}$ of $H_\mu(3)$ is defined by $\varphi^{(k)}(a^j) = \zeta_3^{kj}$ for $j = 0, 1, 2$, and $\varphi^{(k)}(\tau) = 1$ for any $\tau \in S_\mu$. □

In this section, we will prove the following theorem.

**Theorem 27** For each $k = 0, 1, \ldots, h - 1$, it holds that

$$R_\mu(k; h) \cong_{S_n} \text{Ind}_{H_\mu(h)}^{S_n} \varphi^{(k)}. $$

□

By the same argument as in the hook case, it can be seen that this is equivalent to show that the following $S_n \times C_l$-isomorphism holds:

$$R_\mu \cong_{S_n \times C_l} \text{Ind}_{H_\mu(h)}^{S_n} 1. $$ (6.1)

The isomorphisms in Theorem 27 are realized as correspondence of the eigenspaces with respect to the action of $a$ in the $S_n \times C_h$-isomorphism.

The rest of this section is devoted to the proof of (6.1) for the case $h$ is a prime. Let $h = p$ be a prime. The it can be seen that it is sufficient to show, as in the hook case, that:

1. For $w \in S_n$, it holds that

$$Q_\mu^\mu(\zeta_p^j) \neq 0 \Rightarrow w \sim \tau_1 \cdots \tau_p a^j, \quad \tau_1 \cdots \tau_p \in S_\mu,$$

2. If $w \in S_n$ is conjugate to some element $\tau_1 \cdots \tau_p a^j \in H_\mu(p)$, then we have

$$Q_\mu^\mu(\zeta_p^j) = \sharp\{ \sigma \in S_n / S_\mu \mid w\sigma a^{-j} \equiv \sigma \mod S_\mu \}. $$
First, we will consider the necessary condition for \( w \in S_n \) to be a support of \( Q_\lambda^\mu(w)(\zeta_p) \). On behavior of the Green polynomials \( X_\mu^\rho(q) \) corresponding to a rectangle \( \mu \), the following proposition due to Lascoux-Leclerc-Thibon [LLT, Theorem 3.2] is known. In this proposition, \( h \) is not necessarily be prime.

**Proposition 28 (Lascoux-Leclerc-Thibon)** Let \( \mu = (r^h) \) be a rectangle, and \( \zeta_h \) a primitive \( h \)-th root of unity. Then it holds that

\[
X_\mu^\rho(\zeta_h) = \begin{cases} (-1)^{(h-1)j}hX_\mu^{(r-j)^h}(\zeta_h), & \text{for some } i = jh; \\ 0, & \text{otherwise}. \end{cases}
\]

It is clear from Proposition 28 that:

**Corollary 29** Let \( \mu = (r^h) \) be a rectangle, and \( \zeta_h \) a primitive \( h \)-th root of unity. Then, for a partition \( \rho \vdash rh \), the condition \( X_\mu^\rho(\zeta_h) \neq 0 \) implies that all the non-zero components of \( \rho \) are multiple of \( h \).

These partitions in Corollary 29 are realized as cycle types of permutations belonging to the subgroup \( H_\mu(h) \).

**Lemma 30** For a partition \( \rho \) of the form \( \rho = (h^{\alpha_1}(2h)^{\alpha_2} \cdots (rh)^{\alpha_k}) \), there exists an element \( \tau \) of \( S_{(\alpha_1)} \) satisfying \( \rho = \lambda(\tau a) \).

**Proof.** We can choose \( \tau \) as, for example, \( \tau \in S_{\{1, r+1, \ldots, (p-1)r+1\}} \) such that

\[
\lambda(\tau) = (1^{\alpha_1} 2^{\alpha_2} \cdots r^{\alpha_k}).
\]

Since it holds that

\[
Q_\rho^\mu(\zeta_h) \neq 0 \iff X_\rho^\mu(\zeta_h) \neq 0,
\]

together with Corollary 29 and Lemma 30, we have an answer for our present problem.

**Corollary 31** Let \( \mu = (r^h) \) be a rectangle, and \( \zeta_h \) a primitive \( h \)-th root of unity. Then, for \( \rho \vdash rh \), we have

\[
Q_\rho^\mu(\zeta_h) \neq 0 \implies \rho = \lambda(w), \quad w \in H_\mu(h).
\]

Next we will determine the value \( Q_\lambda^\mu(w)(\zeta_h) \) for \( w \in H_\mu(h) \). Let \( \mu = (r^h) \) be a rectangle. If all the non-zero components of \( \rho \) are multiples of \( h \), then the explicit values of the Green polynomial \( X_\rho^\mu(q) \) at \( q = \zeta_k \) are given as follows:
Lemma 32 Let a partition $\mu = (r^h)$ be rectangle. Then, for a partition
$$\rho = (h^{\alpha_h} (2h)^{\alpha_{2h}} \cdots (r^h)^{\alpha_{rh}})$$
whose non-zero components are all divisible by $h$, it holds that
$$X_\rho^{(r^h)}(\zeta_h) = (-1)^{r(h-1)} h^{l(\rho)},$$
where $l(\rho)$ denotes the length of $\rho$. In particular, the value $X_\rho^{(r^h)}(\zeta_h)$ does not depend upon a particular choice of a primitive $h$-th root of unity.

Proof. Applying Lascoux-Leclerc-Thibon’s theorem repeatedly, we have
$$X_\rho^{(r^h)}(\zeta_h) = (-1)^{(h-1)(\alpha_h + \alpha_{2h} + \cdots + \alpha_{rh})} h^{\alpha_h + \alpha_{2h} + \cdots + \alpha_{rh}} X_\emptyset^{(r^h)}(\zeta_h)$$
$$= (-1)^{r(h-1)} h^{l(\rho)},$$
where $\emptyset$ denotes the empty partition, and $X_\emptyset^{(r^h)}(q) \equiv 1$.

The following proposition is easily follows from (2.1).

Proposition 33 For each $j = 0, 1, \ldots, h - 1$, we have
$$Q_\rho^{(r^h)}(\zeta_h^j) = \begin{cases} X_\rho^{(r^h)}(\zeta_h^{-j}), & \text{if } h \geq 3 \\ (-1)^j r X_\rho^{(r^2)}((-1)^{-j}), & \text{if } h = 2. \end{cases}$$

Proof.
$$Q_\rho^{(r^h)}(\zeta_h^j) = q^{\alpha(\mu)} \left| X_\rho^{(r^h)}(q^{-1}) \right|_{q=\zeta_h^j}$$
$$= q^{r(0+1+\cdots+(h-1))} X_\rho^{(r^h)}(q^{-1}) \bigg|_{q=\zeta_h^j}$$
$$= (\zeta_h^j)^{\frac{h(h-1)}{2}} X_\rho^{(r^h)}(\zeta_h^{-1})$$
$$= \begin{cases} X_\rho^{(r^h)}(\zeta_h^{-j}), & \text{if } h \geq 3 \\ (-1)^j r X_\rho^{(r^2)}((-1)^{-j}), & \text{if } h = 2. \end{cases}$$

Especially for the case $h$ is a prime $p$, we have:

Corollary 34
$$Q_\rho^{(r^p)}(\zeta_p) = \begin{cases} X_\rho^{(r^p)}(\zeta_p), & \text{if } p \geq 3 \\ (-1)^r X_\rho^{(r^2)}((-1)), & \text{if } p = 2. \end{cases}$$
Proof. It holds that
\[ Q^{(r^p)}_\rho(\zeta_p) = q^{n(\mu)}X^{(r^p)}_\rho(q^{-1}) \bigg|_{q=\zeta_p} = X^{(r^p)}_\rho(q^{-1}) \bigg|_{q=\zeta_p}, \]

since \( n(\mu) \) is divisible by \( p \). Because \( \zeta_p^{-1} \) is primitive and the value \( X^{(r^p)}_\rho(\zeta_p) \) does not depend on the choice of a primitive \( p \)-th root of unity \( \zeta_p \), the assertion follows immediately from Proposition 33.

In what follows, we assume that \( \mu = (r^p) \) is a rectangle with prime multiplicity \( p \).

**Proposition 35** Let \( \mu = (r^p) \) be a rectangle, whose multiplicity \( p \) is prime. Then, for each \( w = \tau_1 \cdots \tau_p a^j \in S_\mu C_p \) and \( j = 1, 2, \ldots, p - 1 \), it holds that
\[ Q^\mu_\rho(\zeta_p^j) = Q^\mu_\rho(\zeta_p) = p^{l(\rho)}, \]
where \( \rho = \lambda(w) \) is the cycle type of \( w \), \( l(\rho) \) its length.

Proof. Since \( \zeta_p^j \) is primitive for each \( j = 1, 2, \ldots, p - 1 \), it is enough to show only for the case \( j = 1 \). Let \( w = \tau_1 \cdots \tau_p a^j \in S_\mu C_p \) and \( \rho \) its cycle type. Note that each non-zero components of \( \rho \) is divisible by \( p \). Then it follows from Corollary 34 and Lemma 32 that
\[
Q^{(r^p)}_\rho(\zeta_p) = \begin{cases} X^{(r^p)}_\rho(\zeta_p), & \text{if } p: \text{odd} \\ (-1)^r X^{(r^2)}_\rho(-1), & \text{if } p = 2. \end{cases}
\]
\[ = \begin{cases} (-1)^r p^{l(\rho)} & \text{if } p: \text{odd} \\ (-1)^r (-1)^{r(p-1)} p^{l(\rho)}, & \text{if } p = 2 \end{cases} = p^{l(\rho)}. \]

\[ \square \]

**Example 36** Let \( \mu = (2^2) \). By Corollary 31, the support \( \rho \) of \( Q^\mu_\rho(\zeta_2) \) is \( \rho = (2, 2), (4) \), which can be seen directly from the table [Gr]. Also from the table, we see that \( Q^{(2,2)}(q) = 1 - q + q^2 \) and \( Q^{(4)}(q) = 1 - q \). Thus \( Q^{(2,2)}(-1) = 4 = 2^{l(2(2))} \), and \( Q^{(4)}(-1) = 2 = 2^{l(4)} \).

Our subject here is to show
\[ Q^\mu_{\lambda(w)}(\zeta_p^j) = \{ \sigma \in S_n/S_\mu \mid w\sigma a^{-j} \equiv \sigma \mod S_\mu \} \]
for each \( w \sim \tau_1 \cdots \tau_p a^j \in S_\mu C_p \) and \( j = 1, 2, \ldots, p - 1 \). By Proposition 35, noting that \( p \) is prime, it suffices to show
\[ \sharp \{ \sigma \in S_n/S_\mu \mid w\sigma a^{-1} \equiv \sigma \mod S_\mu \} = p^{l(\lambda(w))}. \quad (6.2) \]

Let \( S_w \) denote the set in the left hand side of (6.2):
\[ S_w = \{ \sigma \in S_n/S_\mu \mid w\sigma a^{-1} \equiv \sigma \mod S_\mu \}. \]
The cyclic subgroup \( C_p \) generated by \( a \) acts on the set \( S_w \) from the right.
Lemma 37 If \( \sigma \) is an element of \( S_w \), then so is \( \sigma a \).

Proof. Let \( \sigma \) be an element of \( S_w \). Let \( \eta_1 \cdots \eta_p \) be an element of the Young subgroup \( S_\mu \) satisfying \( w\sigma a^{-1} = \sigma \eta_1 \cdots \eta_p \). Then we have

\[
w(\sigma a)^{-1} = w\sigma = \sigma \eta_1 \cdots \eta_p a = \sigma a(\eta_1 a) \cdots (\eta_p a).
\]

Thus we have \( \sigma a \in S_w \), since \( a^{-1} \eta_i a \in S_\mu \) \((i = 1, \ldots, p)\). \(\square\)

Let \( \sigma = [\sigma_1 \sigma_2 \cdots \sigma_n] \) be an element of \( S_w \). Since \( \sigma \in S_n/S_\mu \), we may assume

\[
\sigma_1 < \sigma_2 < \cdots < \sigma_r \\
\sigma_{r+1} < \sigma_{r+2} < \cdots < \sigma_{2r} \\
\ldots \\
\sigma_{(p-1)r+1} < \sigma_{(p-1)r+2} < \cdots < \sigma_{pr}.
\]

By Lemma 37, we may also assume that

\[
\sigma_1 = 1.
\]

By the condition

\[
w[\sigma_1, \sigma_2, \ldots, \sigma_n] \\
= [\sigma_1, \sigma_2, \ldots, \sigma_n] \eta_1 \eta_2 \cdots \eta_p a \quad (\eta_1 \eta_2 \cdots \eta_p \in S_\mu) \\
= [\{\sigma_{r+1}, \sigma_{r+2}, \ldots, \sigma_{2r}\}, \{\sigma_{2r+1}, \sigma_{2r+2}, \ldots, \sigma_{3r}\}, \ldots, \{\sigma_1, \sigma_2, \ldots, \sigma_r\}],
\]

it can be seen that the components

\[
\sigma_1, \sigma_2, \ldots, \sigma_r \quad (r = \frac{n}{p})
\]

completely determine the remaining parts of \( \sigma \).

Since \( w \) is assume to be an element of the form \( \tau_1 \cdots \tau_j a^j \in S_\mu C_p \) \((j > 1)\), its cycle type is of the form

\[
\lambda(w) = (p^{\alpha_p} (2p)^{\alpha_{2p}} \cdots (rp)^{\alpha_{rp}}) \vdash rp = n.
\]

In this case, by Lemma 30, the element \( w \) is realized by \( w = \eta a \) if we choose \( \eta \in S_{\{1, 2, \ldots, r\}} \) as follows:

\[
\lambda(\eta) = (1^{\alpha_p} 2^{\alpha_{2p}} \cdots r^{\alpha_{rp}}).
\]
Example 38 In the case of $\mu = (5^3)$, we have

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

$$= (1,6,11)(2,7,12)(3,8,13)(4,9,14)(5,10,15).$$

The partitions which satisfy the condition $Q_{\rho}(\zeta_3) \neq 0$ are

$$(3^5), (6,3^3), (6^2,3), (9,3^2), (9,6), (12,3), (15).$$

The corresponding elements $\eta \in S_5$ for these partitions are

$id, (12), (12)(34), (123), (123)(45), (1234), (12345)$ respectively.

If we choose $\eta$ from these permutations, then the representatives

$$\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_n] \in S_n/S_\mu$$

satisfying the condition

$$w\sigma a^{-1} \equiv \sigma \mod S_\mu \quad (6.3)$$

are constructed as follows. For this construction, we have seen that it is enough to determine the first $r$ letters $\sigma_1, \ldots, \sigma_r$ of $\sigma$. If we rewrite the cycle type of $\eta$ as

$$\lambda(\eta) = (1^{\alpha_1}2^{\alpha_2} \cdots r^{\alpha_r})$$

$$= (\nu_1, \nu_2, \ldots, \nu_s), \quad \nu_1 \geq \nu_2 \geq \ldots \geq \nu_s > 0,$$

then we have

$$\lambda(w) = (p\nu_1, p\nu_2, \ldots, p\nu_s).$$

Let

$$w = (v_1, v_2, \ldots, v_{p\nu_1})$$

$$(v_{p\nu_1+1}, v_{p\nu_1+2}, \ldots, v_{p\nu_1+p\nu_2})$$

$$\cdots$$

$$(v_{p\nu_1+\cdots+p\nu_{s-1}+1}, \ldots, v_{p\nu_1+\cdots+p\nu_s}),$$

where $v_1 = 1, v_1 < v_{p\nu_1+1} < \cdots < v_{p\nu_1+\cdots+p\nu_{s-1}+1}$, be the corresponding cyclic decomposition of $w$. By the assumption $\sigma_1 = 1$ and (6.3), it follows that

$$\sigma_1 = 1, \sigma_2 = 2, \ldots, \sigma_{\nu_1} = \nu_1.$$
Therefore, it is enough to enumerate the remaining $l(\rho) - 1$ blocks

$\sigma_{\nu_1+1}, \sigma_{\nu_1+2}, \ldots, \sigma_{\nu_1+\nu_2},$

$\sigma_{\nu_1+\nu_2+1}, \sigma_{\nu_1+\nu_2+2}, \ldots, \sigma_{\nu_1+\nu_2+\nu_3},$

$\ldots$

$\sigma_{\nu_1+\nu_2+\ldots+\nu_s-1+1}, \ldots, \sigma_{\nu_1+\nu_2+\ldots+\nu_s}.$

For example, in the case of $\sigma_{\nu_1+1}, \sigma_{\nu_1+2}, \ldots, \sigma_{\nu_1+\nu_2}$, (6.3) implies that there are $p$ such possible choices

$$(\sigma_{\nu_1+1}, \sigma_{\nu_1+2}, \ldots, \sigma_{\nu_1+\nu_2}) = (v_{\nu_1+\nu_2+1}, v_{\nu_1+\nu_2+1+p}, v_{\nu_1+\nu_2+1+2p}, \ldots),$$

$$(v_{\nu_1+\nu_2+2}, v_{\nu_1+\nu_2+2+p}, v_{\nu_1+\nu_2+2+2p}, \ldots),$$

$$(v_{\nu_1+\nu_2+p}, v_{\nu_1+\nu_2+p+p}, v_{\nu_1+\nu_2+p+2p}, \ldots),$$

and each of these are available. The same enumeration can be done for other blocks, and there are exactly $p$ ways for each. Since these choice of the blocks are mutually independent, there are $p^{l(\rho)-1}$ ways in total for assigning letters 1, 2, $\ldots$, $|\lambda(\eta)|$ to $\sigma_1, \sigma_2, \ldots, \sigma_{|\lambda(\eta)|}$.

Now we have enumerated the representatives modulo the right action of the cyclic subgroup $C_p$. Thus there are $p^{l(\rho)}$ elements in $S_n/S_\mu$ satisfying (6.3) in total, which completes the proof of Theorem 27.

**Example 39** Let $n = 15$ and $\mu = (5^3)$. If we consider the case

$\rho = (6, 6, 3),$

then we can take

$\eta = (12)(34)$

for the element of $S_{\{1,2,3,4,5\}}$ satisfying

$\lambda(\eta a) = (6, 6, 3).$

Remark that

$w = \eta a = (1, 6, 11, 2, 7, 12)(3, 8, 13, 4, 9, 14)(5, 10, 15).$

Now we are going to enumerate the representatives

$\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_{15}] \in S_{15}/S_{(5^3)}, \quad \sigma_1 = 1,$

satisfying

$w[\sigma_1, \sigma_2, \ldots, \sigma_{15}] = [\sigma_1, \sigma_2, \ldots, \sigma_{15}] u a \quad (u \in S_{(5^3)})$

$= \{[\sigma_6, \sigma_7, \ldots, \sigma_{10}], \{\sigma_{11}, \sigma_{12}, \ldots, \sigma_{15}\}, \{\sigma_1, \sigma_2, \ldots, \sigma_5\}\}.$
For example, it is easy to see that $\sigma = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]$ is an appropriate representative, since we have

$$\{w\sigma_1, w\sigma_2, \ldots, w\sigma_5\} = \{6, 7, 8, 9, 10\} = \{\sigma_6, \sigma_7, \ldots, \sigma_{10}\},$$

$$\{w\sigma_6, w\sigma_7, \ldots, w\sigma_{10}\} = \{11, 12, 13, 14, 15\} = \{\sigma_{11}, \sigma_{12}, \ldots, \sigma_{15}\},$$

$$\{w\sigma_{11}, w\sigma_{12}, \ldots, w\sigma_{15}\} = \{1, 2, 3, 4, 5\} = \{\sigma_1, \sigma_2, \ldots, \sigma_5\}.$$

Thanks to the right action of $C_3$, we only have to determine

$$\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5.$$

The condition satisfied by $\sigma$ allows us to choose

$$\{\sigma_1, \sigma_2\}, \{\sigma_3, \sigma_4\}, \{\sigma_5\}$$

independently. By the assumption $\sigma_1 = 1$, $\sigma_2$ is uniquely determined by $\sigma_2 = 2$, since the pair $(\sigma_1, \sigma_2)$ should have the order 3 as a set under the action of $w$, i.e., it should satisfy

$$w^3\{\sigma_1, \sigma_2\} = \{\sigma_1, \sigma_2\}, \quad w^i\{\sigma_1, \sigma_2\} \neq \{\sigma_1, \sigma_2\} \quad (i = 1, 2).$$

Similarly, since $(\sigma_3, \sigma_4)$ has the order 3 as a set under the action of $w$, the pair $(\sigma_3, \sigma_4)$ should be chosen from the following three pairs

$$(\sigma_3, \sigma_4) = (3, 4), (8, 9), (13, 14),$$

and each of these are available. The same argument shows that there are three ways of choice for $\sigma_5$. $\sigma_5$ should be chosen from 5, 10, 15, and each of these are available. Thus there are $3^2 = 3^{(6,6,3)-1}$ ways of choice modulo the right action of $C_3$. Hence there $3^3 = 3^{(6,6,3)}$ suitable elements in $S_n/S_\mu$.

□

References


