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Abstract. Recently there are so many mathematical models which describe nonlinear phenomena. In some phenomena, the free energy functional is not convex. So, the existence-uniqueness question is sometimes difficult. In order to study such phenomena, let us introduce the new class of abstract nonlinear evolution equations governed by time-dependent operators of subdifferential type. In this paper we shall show the existence and uniqueness of solution to nonlinear evolution equations with time-dependent constraints in a real Hilbert space. Moreover we apply our abstract results to a parabolic variational inequality with time-dependent double obstacles constraints.
1 Introduction

We study an abstract nonlinear evolution equation in a real Hilbert space $H$ of the form

$$u'(t) + \partial \varphi^t(u(t); u(t)) + G(t, u(t)) \ni f(t) \quad \text{in } H, \quad \text{a.e. } t \in (0, T), \quad (1)$$

where $u'(t) := \frac{d}{dt}u(t)$, $G(t, \cdot)$ is a single valued perturbation small relative to $\varphi^t$, and $f$ is a given $H$-valued function. For each $t \in [0, T]$, a function $\varphi^t(\cdot; \cdot) : H \times H \to \mathbb{R} \cup \{\infty\}$ is given such that for all $w \in H$, $\varphi^t(w; \cdot) : H \to \mathbb{R} \cup \{\infty\}$ is a proper, l.s.c. (lower semi-continuous) and convex function, and $\partial \varphi^t(w; \cdot)$ is its subdifferential operator, i.e.,

$$z^* \in \partial \varphi^t(w; z) \quad \text{if and only if} \quad z \in D(\varphi^t(w; \cdot)) \quad \text{and} \quad (z^*, y - z) \leq \varphi^t(w; y) - \varphi^t(w; z) \quad \text{for all } y \in H.$$

For a proper, l.s.c. and convex function $\psi^t(\cdot) : H \to \mathbb{R} \cup \{\infty\}$, many mathematicians studied the nonlinear evolution equation of the form

$$u'(t) + \partial \psi^t(u(t)) \ni f(t) \quad \text{in } H, \quad \text{a.e. } t \in (0, T). \quad (2)$$

For various aspects of (2), we refer to [2, 5, 6, 8, 9, 11, 18, 19]. For instance, Kenmochi [6] showed the existence-uniqueness, stability and convergence of solutions to (2).

For the nonmonotone perturbation $G(t, \cdot)$, Ótani [16] has already shown the existence of solution to

$$u'(t) + \partial \varphi^t(u(t)) + G(t, u(t)) \ni f(t) \quad \text{in } H, \quad \text{a.e. } t \in (0, T). \quad (3)$$

The large-time behavior of solutions for (3) was discussed by [20] from the viewpoint of attractors. For another works of (3), we refer to [10, 16, 17, 20, 21, 22], for instance.

The main object of this paper is to establish abstract results on existence-uniqueness of solutions to (1). Note that the function $\varphi^t(u; u)$ is not convex in $u$, hence we can not apply the theory established by Ótani [16]. So, by using the idea of Kenmochi-Kubo [7] and Kubo-Yamazaki [12, 13], we shall show the existence of solution to (1) in this paper. Namely, for the given function $w : [0, T] \to H$, let us consider the problem

$$u'(t) + \partial \varphi^t(w(t); u(t)) \ni f(t) - G(t, w(t)) \quad \text{in } H, \quad \text{a.e. } t \in (0, T). \quad (4)$$

Assuming some appropriate conditions on the $t$- and $w$-dependence of the function $\varphi^t(w; z)$, we can apply the result of Kenmochi [6]. Then we see that the equation (4) has a unique solution $u$ for each $w$, and that the mapping $w \mapsto u$ has some compactness property. Hence, by using a fixed point argument, we can get the existence of solution to (1).

In Section 2 we present our main results on existence and uniqueness of solution to (1), and then the uniqueness (Theorem 3) is proved. In Section 3 we prove the local existence result (Theorem 1). In Section 4, the global existence result (Theorem 2) is proved. In the final Section 5 we apply our abstract results to a parabolic variational inequality with time-dependent double obstacle constraints.
2 Assumptions and main results

We consider a Cauchy problem CP($u_0$) for (1) of the following form:

$$\text{CP}(u_0) \begin{cases} u'(t) + \partial \varphi^i(u(t); u(t)) + G(t, u(t)) \ni f(t) & \text{in } H \text{ a.e. } t \in (0, T), \\ u(0) = u_0, \end{cases}$$

where $T$ is a given positive number, a function $\varphi^i(u(t); u(t))$ is introduced in Section 1, $G(t, \cdot)$ is a single valued perturbation small relative to $\varphi^i$, $f \in L^2(0, T; H)$ is a given function, and $u_0 \in H$ is given data.

Definition 1. Given $u_0 \in H$ and $f \in L^2(0, T; H)$, the function $u : [0, T] \to H$ will be called a solution to CP($u_0$), if $u \in W^{1,2}(0, T; H)$, $u(0) = u_0$, $u(t) \in D(\partial \varphi^i(u(t); \cdot))$ and $f(t) - u'(t) - G(t, u(t)) \in \partial \varphi^i(u(t); u(t))$ for a.e. $t \in [0, T]$, namely

$$(f(t) - u'(t) - G(t, u(t)), y - u(t)) \leq \varphi^i(u(t); y) - \varphi^i(u(t); u(t))$$

for any $y \in H$, a.e. $t \in [0, T]$.

For a given positive number $T$, let $\{\alpha_r\} := \{\alpha_r; \ r > 0\}$ be a family of functions $\alpha_r \in W^{1,2}(0, T)$, with parameter $r > 0$. With this family $\{\alpha_r\}$, we specify a class $\Phi(\{\alpha_r\})$ of all families $\{\varphi^i\} := \{\varphi^i; t \in [0, T]\}$ of time-dependent functions $\varphi^i(\cdot; \cdot)$ on $H \times H$ as follows.

Definition 2. We denote by $\{\varphi^i\} \in \Phi(\{\alpha_r\})$ the set of all time-dependent functions $\varphi^i(\cdot; \cdot)$ from $H \times H$ into $\mathbb{R} \cup \{\infty\}$ satisfying the following seven conditions:

(\Phi1) For each $w \in H$ and $t \in [0, T]$, $\varphi^i(w; \cdot) : H \to \mathbb{R} \cup \{\infty\}$ is a proper l.s.c. convex function;

(\Phi2) There exists a positive constant $C_1 > 0$ such that

$$\varphi^i(w; z) \geq C_1|z|^2_H, \quad \forall t \in [0, T], \forall w \in H, \forall z \in D(\varphi^i(w; \cdot)),$$

(\Phi3) For each $t \in [0, T]$, $w \in H$ and $k > 0$, the level set $\{z \in H; \varphi^i(w; z) \leq k\}$ is compact in $H$;

(\Phi4) $D(\varphi^i(w; \cdot))$ is independent of $w \in H$ for any $t \in [0, T]$;
(Φ5) For each \( r > 0, s, t \in [0, T] \) with \( s \leq t, w \in D(\varphi^s(0; \cdot)) \) with \( |w|_H \leq r \) and \( z \in D(\varphi^s(w; \cdot)) \) with \( |z|_H \leq r \) there exists an element \( \tilde{z} \in D(\varphi^s(w; \cdot)) \) such that

\[
|\tilde{z} - z|_H \leq |\alpha_r(t) - \alpha_r(s)| \left( 1 + \varphi^s(0; z) \right) 
\]

and

\[
\varphi'(w; \tilde{z}) - \varphi'(w; z) \leq |\alpha_r(t) - \alpha_r(s)| \left( 1 + \varphi^s(0; z) + \varphi^s(0; w) \right) \left( \frac{1}{2} \varphi^s(0; z) \frac{1}{2} + \varphi^s(0; w) \right) ;
\]

(Φ6) For each \( r > 0 \) there is a positive constant \( C_r > 0 \) such that

\[
|\varphi'(w_1; z) - \varphi'(w_2; z)| \leq C_r|w_1 - w_2|_H \varphi'(0; z) \left( 1 + \varphi^s(0; z) + \varphi^s(0; w) \right) ,
\]

\( \forall t \in [0, T], \forall w_i \in H \) with \( |w_i|_H \leq r, (i = 1, 2) \), and \( \forall z \in D(\varphi^s(0; \cdot)) \);

(Φ7) There is a function \( h \in W^{1,2}(0, T; H) \) with \( C_h := \sup_{t \in [0, T]} \varphi'(0; h(t)) < +\infty \).

Next, we introduce the class \( \mathcal{G}(\{\varphi^t\}) \) of time-dependent perturbation \( G(t, \cdot) \) associated with \( \{\varphi^t\} \in \Phi(\{\alpha_r\}) \).

**Definition 3.** \( \{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^t\}) \) if and only if \( G(t, \cdot) \) is a single valued operator from \( D(G(t, \cdot)) \subset H \) into \( H \) which fulfills the following conditions (G1)-(G3):

(G1) \( D(\varphi'(0; \cdot)) \subset D(G(t, \cdot)) \subset H \) for all \( t \in [0, T] \) and \( G(\cdot, v(\cdot)) \) is (strongly) measurable on \( J \) for any interval \( J \subset [0, T] \) and \( v \in L^2_{\text{loc}}(J; H) \) with \( v(t) \in D(\varphi'(0; \cdot)) \) for a.e. \( t \in J \).

(G2) There are positive constants \( C_2 > 0, C_3 > 0 \) such that

\[
|G(t, z)|_H^2 \leq C_2 \varphi'(0; z) + C_3, \quad \forall t \in [0, T], \forall z \in D(\varphi'(0; \cdot)).
\]

(G3) (Demi-closedness) If \( \{t_n\} \subset [0, T], \{z_n\} \subset H, t_n \to t, z_n \to z \) in \( H \) (as \( n \to +\infty \)) and \( \{\varphi'(0; z_n)\} \) is bounded, then \( G(t_n, z_n) \to G(t, z) \) weakly in \( H \) as \( n \to +\infty \).

Now let us mention our main local existence result in this paper. In Section 3 we shall prove Theorem 1.

**Theorem 1.** Let \( T \) be any positive number. Assume \( \{\varphi^t\} \in \Phi(\{\alpha_r\}), \{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^t\}) \) and \( f \in L^2(0, T; H) \). Then, for each \( u_0 \in D(\varphi^0(0; \cdot)) \) there exists a positive constant \( T_0(\leq T) \) such that \( CP(u_0) \) has at least one solution \( u \) on \( [0, T_0] \).

The next main theorem is concerned with the global existence result in this paper. In Section 4 we shall prove Theorem 2.

**Theorem 2.** Let \( T \) be any positive number. Assume \( \{\varphi^t\} \in \Phi(\{\alpha_r\}), \{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^t\}) \) and \( f \in L^2(0, T; H) \). Additionally, assume that
Theorem 3. There exists a positive constant $C_4 > 0$ such that

$$\varphi^t(w; z) \leq C_4(1 + |w|^2_H + \varphi^t(0; z)), \quad \forall t \in [0, T], \quad \forall w \in H, \quad \forall z \in D(\varphi^t(0; \cdot)).$$

Then, for each $u_0 \in D(\varphi^0(0; \cdot))$ there exists at least one solution $u$ to $CP(u_0)$ on $[0, T]$.

To show the uniqueness of solution to $CP(u_0)$, we shall introduce subclasses of $\Phi(\{\alpha_r\})$ and $\mathcal{G}(\{\varphi^t\})$.

Definition 4. Let $\gamma$ be a non-negative continuous and convex function on $H$ such that $\gamma(z) + \gamma(-z) = 0$ if and only if $z = 0$. Then

1. $\{\varphi^t\} \in \Phi_\gamma(\{\alpha_r\})$ if and only if $\{\varphi^t\} \in \Phi(\{\alpha_r\})$ satisfies the $\gamma$-accretiveness ($\ast$) for $\varphi^t$ as follows:

   ($\ast$) For any $z_i \in D(\partial \varphi^t(z_i; \cdot))$ and $z^*_i \in \partial \varphi^t(z_i; z_i) \quad (i = 1, 2)$, there is an element $w_0 \in \partial \gamma(z_i - z_2)$ so that $(z^*_1 - z^*_2, w_0) \geq 0$, where $\partial \gamma$ is the subdifferential of $\gamma$ in $H$.

2. $\{G(t, \cdot)\} \in \mathcal{G}_\gamma(\{\varphi^t\})$ if and only if for any positive number $\varepsilon > 0$, there is a positive constant $C_\varepsilon > 0$ such that

   $$|(G(t, z_1) - G(t, z_2), w_0)| \leq \varepsilon(z^*_1 - z^*_2, w_0) + C_\varepsilon\{\gamma(z_1 - z_2) + \gamma(z_2 - z_1)\},$$

   whenever $t \in [0, T]$, $z_i \in D(\partial \varphi^t(z_i; \cdot))$, $z^*_i \in \partial \varphi^t(z_i; z_i) \quad (i = 1, 2)$, and

   $$w_0 \in \partial \gamma(z_1 - z_2) \quad \text{with} \quad (z^*_1 - z^*_2, w_0)_H \geq 0.$$

Now let us mention our main uniqueness result in this paper.

Theorem 3. Let $T$ be any positive number. Assume $\{\varphi^t\} \in \Phi_\gamma(\{\alpha_r\})$, $\{G(t, \cdot)\} \in \mathcal{G}_\gamma(\{\varphi^t\})$ and $f \in L^2(0, T; H)$. Then, for each $u_0 \in H$ the solution $u$ to $CP(u_0)$ is unique.

Proof. Let $u$ and $v$ be solutions to $CP(u_0)$. By the $\gamma$-accretiveness of $\varphi^t$, for a.e. $\tau \in [0, T]$ there exists $z^*(\tau) \in \partial \gamma(u(\tau) - v(\tau))$ such that

$$(u^*(\tau) - v^*(\tau), z^*(\tau)) \geq 0 \quad (5)$$

for any $u^*(\tau) \in \partial \varphi^\tau(u(\tau); u(\tau))$ and $v^*(\tau) \in \partial \varphi^\tau(v(\tau); v(\tau))$.

By $\{G(t, \cdot)\} \in \mathcal{G}_\gamma(\{\varphi^t\})$, for a number $\varepsilon \in (0, 1]$ there is a constant $C_\varepsilon > 0$ such that

$$|(G(\tau, u(\tau)) - G(\tau, v(\tau)), z^*(\tau))| \leq \varepsilon(u^*(\tau) - v^*(\tau), z^*(\tau)) + C_\varepsilon\{\gamma(u(\tau) - v(\tau)) + \gamma(v(\tau) - u(\tau))\} \quad (6)$$

for a.e. $\tau \in [0, T]$.

From (5) and (6) it follows that

$$0 \leq (u^*(\tau) - v^*(\tau), z^*(\tau))$$

$$= ([f(\tau) - u'(\tau) - G(\tau, u(\tau))] - [f(\tau) - v'(\tau) - G(\tau, v(\tau))], z^*(\tau))$$

$$\leq (-u'(\tau) + v'(\tau), z^*(\tau)) + \{(-G(\tau, u(\tau)) + G(\tau, v(\tau)), z^*(\tau))\}$$

$$\leq -\frac{d}{d\tau}\gamma(u(\tau) - v(\tau)) + \varepsilon(u^*(\tau) - v^*(\tau), z^*(\tau)) + C_\varepsilon\{\gamma(u(\tau) - v(\tau)) + \gamma(v(\tau) - u(\tau))\},$$
which implies that
\[
\frac{d}{d\tau} \gamma(u(\tau) - v(\tau)) \leq C \{ \gamma(u(\tau) - v(\tau)) + \gamma(v(\tau) - u(\tau)) \} \quad \text{for a.e. } \tau \in [0, T].
\]

Similarly we have
\[
\frac{d}{d\tau} \gamma(v(\tau) - u(\tau)) \leq C \{ \gamma(u(\tau) - v(\tau)) + \gamma(v(\tau) - u(\tau)) \},
\]
hence we have
\[
\frac{d}{d\tau} \{ \gamma(u(\tau) - v(\tau)) + \gamma(v(\tau) - u(\tau)) \} \leq 2C \{ \gamma(u(\tau) - v(\tau)) + \gamma(v(\tau) - u(\tau)) \},
\]
for a.e. \( \tau \in [0, T] \).

Now, applying Gronwall’s inequality to (7), we get
\[
e^{-2C_t} \{ \gamma(u(t) - v(t)) + \gamma(v(t) - u(t)) \} \leq 0 \quad \text{for any } 0 \leq t \leq T;
\]
which implies that \( u(t) = v(t) \) for all \( t \in [0, T] \). Thus Theorem 3 has been proved. \( \square \)

3 Proof of Theorem 1

In this section we shall show Theorem 1 by the fixed point argument. To do so, for a given positive number \( T > 0 \), we put a Banach space
\[
E(T) \equiv \left\{ w \in W^{1,2}(0, T; H) ; \sup_{t \in [0, T]} \varphi^1(0; w(t)) < +\infty \right\}.
\]

By the assumption (Φ7) we note that \( E(T) \neq \emptyset \).

Now, for each \( w \in E(T) \) let us consider a following Cauchy problem \( CP(w; u_0) \):
\[
CP(w; u_0) \left\{ \begin{array}{l}
u'(t) + \partial \varphi^1(w(t); u(t)) \ni f(t, w(t)) \quad \text{in } H, \text{ a.e. } t \in (0, T),
\end{array} \right.
\]
\[
u(0) = u_0.
\]

To show the existence-uniqueness of solution to \( CP(w; u_0) \), we prepare the key lemma.

Lemma 1. For each \( w \in E(T) \) we take a positive constant \( R > 0 \) such that
\[
\sup_{t \in [0, T]} |w(t)|_H \leq R. \text{ Put } 
\]
\[
\psi^1_w(z) := \varphi^1(w(t); z) \text{ for } z \in H.
\]

Then, there is a positive constant \( N_1 > 0 \) independent of \( w \) satisfying the following: for any \( s, t \in [0, T] \) with \( s \leq t \) and \( z \in D(\psi^1_w) \) with \( |z|_H \leq R \), there exists \( \tilde{z} \in D(\psi^1_w) \) such that
\[
|\tilde{z} - z|_H \leq N_1(1 + C_R)^4(1 + R)^6|\alpha_R(t) - \alpha_R(s)| \left( 1 + \psi^1_w(z) \right)^{\frac{1}{2}},
\]

\[
\psi^1_w(\tilde{z}) - \psi^1_w(z)
\]
\[
\leq N_1(1 + C_R)^4(1 + R)^6 \left\{ |\alpha_R(t) - \alpha_R(s)| \left( |1 + \psi^1_w(z)| + |w(t) - w(s)|_H \right) (1 + \psi^1_w(z))^{\frac{1}{2}} 
\]
\[
+ |\alpha_R(t) - \alpha_R(s)| \varphi^1(0; w(s))^{\frac{1}{2}} (1 + \psi^1_w(z))^{\frac{1}{2}} \right\} .
\]
Proof. Taking \( w = w(s) \) in (Φ5), then for any \( s, t \in [0, T] \) with \( s \leq t \) and \( z \in \text{D}(\varphi^s(w(s); \cdot)) \) with \( |z|_H \leq R \), there exists \( \tilde{z} \in \text{D}(\varphi^s(w(s); \cdot)) \) such that

\[
|\tilde{z} - z|_H \leq |\alpha_R(t) - \alpha_R(s)| \left( 1 + \varphi^s(0; z) \right)^{\frac{1}{2}},
\]

(10)

\[
\varphi^s(w(s); \tilde{z}) - \varphi^s(w(s); z) \leq |\alpha_R(t) - \alpha_R(s)| \left( 1 + \varphi^s(0; z) + \varphi^s(0; w(s)) \right)^{\frac{1}{2}} \varphi^s(0; z)^{\frac{1}{2}} + \varphi^s(0; w(s))^{\frac{1}{2}}.
\]

(11)

It follows from (Φ4) that

\[
z \in \text{D}(\varphi^s(w(s); \cdot)) = \text{D}(\psi^s_w), \quad \tilde{z} \in \text{D}(\varphi^t(w(s); \cdot)) = \text{D}(\psi^t_w).
\]

(12)

Note that by (Φ6) and \( \text{E} \leq \text{G} \), we have

\[
\varphi^s(0; z) \leq 2\varphi^s(w(s); z) + C^2_R|w(s)|^2_H \leq 2\psi^s_w(z) + C^2_R R^2.
\]

(13)

Then, by (10) and (13) there is a positive number \( N_2 > 0 \) independent of \( w \) satisfying

\[
|\tilde{z} - z|_H \leq |\alpha_R(t) - \alpha_R(s)| \left( 1 + \sqrt{2}\psi^s_w(z) + C_R R \right)
\]

\[
\leq N_2(1 + C_R R)|\alpha_R(t) - \alpha_R(s)| \left( 1 + \psi^s_w(z) \right) .
\]

(14)

Moreover, we observe that by (11), (13), (Φ6) there is a positive number \( N_3 > 0 \) independent of \( w \) satisfying the following:

\[
\psi^t_w(\tilde{z}) - \psi^s_w(z) = \varphi^s(0; \tilde{z}) - \varphi^s(0; z) + \varphi^t(0; \tilde{z}) - \varphi^t(0; z)
\]

\[
\leq N_3(1 + C_R)^2(1 + R)^2 \left\{ |w(t) - w(s)|_H \psi^t_w(\tilde{z})^{\frac{1}{2}} + |w(t) - w(s)|_H
\]

\[
+|\alpha_R(t) - \alpha_R(s)| \varphi^s(0; w(s))^{\frac{1}{2}} \right\}.
\]

(15)

From \( \alpha_R \in W^{1,2}(0, T) \), \( w \in E(T) \) and (15) it follows that

\[
\psi^t_w(\tilde{z}) \leq N_4(1 + C_R)^4(1 + R)^6 \left\{ 1 + \psi^s_w(z) + |\alpha_R(t) - \alpha_R(s)|^2 \varphi^s(0; w(s))\right\}
\]

(16)

for some constant \( N_4 > 0 \). Therefore, using (16) in the right hand side of (15), and by (12)-(14), we get this Lemma for some constant \( N_1 > 0 \) independent of \( w \).

\[ \Box \]

**Proposition 1.** For each \( w \in E(T) \), \( CP(w; u_0) \) has a unique solution \( u \) on \([0, T]\).

**Proof.** We note that \( CP(w; u_0) \) can be regarded as the Cauchy problem for the nonlinear evolution equation of the form:

\[
\left\{
\begin{array}{ll}
w'(t) + \partial \psi^t_w(u(t)) & \equiv f(t) - G(t, w(t)) \quad \text{in } H \text{ a.e. } t \in (0, T), \\
u(0) & = u_0.
\end{array}
\right.
\]

Here, from (Φ6) and (G2) we see that for each \( w \in E(T) \) with \( \sup_{t \in [0, T]} |w(t)|_H \leq R \)

\[
\int_0^T |G(t, w(t))|^2_H dt \leq \int_0^T \left\{ C_2 \varphi^t(w(t); w(t)) + C_3 \right\} dt
\]

\[
\leq T \left\{ 2C_2 \sup_{t \in [0, T]} \varphi^t(0; w(t)) + \frac{C_2C^2_R R^2}{4} + C_3 \right\} < +\infty,
\]

(17)
which implies that \( f - G(\cdot, w(\cdot)) \in L^2(0, T; H) \). Moreover, by Lemma 1 we get the time-dependence of \( \psi_w^\prime \). Therefore taking account of the assumption \((\Phi 1)\), we can apply the abstract theory established by Kenmochi [6]. Thus we get the existence-uniqueness of solution \( u \) for \( CP(w; u_0) \). For detail proofs, see \([6, Theorems 1.1.1, 1.1.2]\). 

By Proposition 1, the boundedness (cf. \([6, Theorem 1.1.2]\)) of solution to \( CP(w; u_0) \), and \((13)\), we can define a mapping \( \psi \), we can define a mapping \( Q : E(T) \rightarrow E(T) \) by \( Qw = u \) for each \( w \in E(T) \), where \( u \) is a solution for \( CP(w; u_0) \).

**Lemma 2.** There are positive constants \( T_0, M_0 \) and \( R_0 \) such that \( Q \) is a self-mapping on \( E(T_0, M_0, R_0) \), i.e., \( Qw(= u) \in E(T_0, M_0, R_0) \) for any \( w \in E(T_0, M_0, R_0) \), where

\[
E(T_0, M_0, R_0) = \left\{ w \in E(T_0) : \begin{aligned}
&\sup_{t \in [0, T_0]} \varphi^i(0; w(t)) \leq M_0, \\
&\sup_{t \in [0, T_0]} |w(t)|_H \leq R_0, \\
&w(0) = u_0
\end{aligned} \right\}
\]

**Proof.** Fix \( R > 0 \) for a while and take \( w \in E(T) \) with \( \sup_{t \in [0, T]} |w(t)|_H \leq R \). We shall give a boundedness of solution \( u \) to the problem \( CP(w; u_0) \).

Now, multiplying \( CP(w; u_0) \) by \( u(t) - h(t) \), we get

\[
(u'(t), u(t) - h(t)) + \varphi^i(w(t); u(t)) - \varphi^i(w(t); h(t)) \leq (f(t) - G(t, w(t)), u(t) - h(t)) \quad \text{a.e. } t \in (0, T),
\]

where \( h \) is the function in \((\Phi 7)\). Taking account of \((\Phi 2), (\Phi 6) \) and \((G2)\), we have

\[
\frac{d}{dt}|u(t) - h(t)|_H^2 - |u(t) - h(t)|_H^2 \leq N_5 \left( |f(t)|^2_H + |h'(t)|^2_H + \varphi^i(0; h(t)) + \varphi^i(0; w(t)) + C_R R^2 + 1 \right),
\]

for a constant \( N_5 = N_5(C_2, C_3) > 0 \). By applying Gronwall’s inequality to \((18)\), we obtain

\[
\sup_{t \in [0, T]} |u(t)|_H \leq \sup_{t \in [0, T]} |h(t)|_H + e^{\frac{r}{2}} |u_0 - h(0)|_H + e^{\frac{r}{2}} N_5 \left\{ |f|_{L^2(0, T; H)} + |h'|_{L^2(0, T; H)} \right\}
\]

\[
+ e^{\frac{r}{2}} N_5 \left\{ \sup_{t \in [0, T]} \varphi^i(0; h(t))^{\frac{1}{2}} + \sup_{t \in [0, T]} \varphi^i(0; w(t))^{\frac{1}{2}} + C_R R^2 + 1 \right\}.
\]

Moreover, by Lemma 1 and arguments of \([6, section 1]\), we see that the function \( \psi_w^i(u(t)) = \varphi^i(w(t); u(t)) \) is of bounded variation on \([0, T]\) and satisfies

\[
\psi_w^i(u(t)) - \psi_w^i(u(s)) + \int_s^t (u'(\tau) - f(\tau) + G(\tau, w(\tau)), u'(\tau))d\tau \\ \leq N_1 (1 + C_R)^4 (1 + R)^6 \int_s^t \left| \alpha^\prime_R(\tau) \right| |u'(\tau) - f(\tau) + G(\tau, w(\tau))| \left\{ 1 + \psi_w^i(u(\tau))^\frac{1}{2} \right\} d\tau
\]

\[
+ N_1 (1 + C_R)^4 (1 + R)^6 \int_s^t \left| \alpha^\prime_R(\tau) \right| |u'(\tau) + \psi_w^i(u(\tau)) + |w'(\tau)|_H \left\{ 1 + \psi_w^i(u(\tau))^\frac{1}{2} \right\} d\tau
\]

\[
+ N_1 (1 + C_R)^4 (1 + R)^6 \int_s^t \left| \alpha^\prime_R(\tau) \right| |\varphi^i(0; w(\tau))^{\frac{1}{2}} + 1 + \psi_w^i(u(\tau))^\frac{1}{2} d\tau
\]
for $0 \leq s \leq t \leq T$ and $w \in E(T)$ with $\sup_{t \in [0,T]} |w(t)|_H \leq R$.

Here we notice the following relations:

\[(u'(\tau) - f(\tau) + G(\tau, w(\tau)), u'(\tau)) \geq \frac{1}{2} |u'(\tau)|^2_H - |f(\tau)|^2_H - |G(\tau, w(\tau))|^2_H, \tag{21}\]

\[|\alpha'_R(\tau)| |u'(\tau) - f(\tau) + G(\tau, w(\tau))|^2_H \left\{ 1 + \psi^w_\tau(u(\tau)) + \frac{1}{4} \right\} \]

\[\leq \delta |u'(\tau)|^2_H + 3\delta |f(\tau)|^2_H + 3\delta |G(\tau, w(\tau))|^2_H + \delta^{-1} |\alpha'_{R}(\tau)|^2 \left\{ 1 + \psi^w_\tau(u(\tau)) \right\}, \tag{22}\]

where in (22) we put $\delta := \frac{1}{12N_1(1 + C_R)^4(1 + R)^6}$. Using (21)-(22) in (20), we obtain

\[\psi^t_w(u(t)) - \psi^s_w(u(s)) + \frac{1}{4} \int_s^t |u'(\tau)|^2_H d\tau \leq N_6(1 + C_R)^8(1 + R)^{12} \int_s^t \left\{ X(\tau)(1 + \psi^w_\tau(u(\tau))) + Y(\tau) \left\{ 1 + \psi^w_\tau(u(\tau)) \right\} \right\} d\tau, \tag{23}\]

for $0 \leq s \leq t \leq T$, where the constant $N_6 > 0$ is determined only by $N_1$, and we put

\[X(\tau) := |f(\tau)|^2_H + |\alpha'_{R}(\tau)|^2 + 1, \quad Y(\tau) := |w(\tau)|^2_H + |\alpha'_{R}(\tau)|\varphi^\tau(0; w(\tau))^\frac{1}{2}.\]

By (F6), (G2) and (23), we obtain

\[\psi^t_w(u(t)) - \psi^s_w(u(s)) + \frac{1}{4} \int_s^t |u'(\tau)|^2_H d\tau \leq N_7(1 + C_R)^{10}(1 + R)^{14} \int_s^t \left\{ X(\tau) + Y(\tau) + \varphi^\tau(0; w(\tau)) \right\} \left\{ 1 + \psi^w_\tau(u(\tau)) \right\} d\tau \tag{24}\]

for $0 \leq s \leq t \leq T$, where $N_7 > 0$ depends on $N_6$, $C_2$ and $C_3$.

Applying Gronwall’s inequality to (24), we obtain

\[\sup_{0 \leq t \leq T} \psi^t_w(u(t)) + \frac{1}{4} \int_0^T |u'(t)|^2_H dt \leq e^{N_7(1 + C_R)^{10}(1 + R)^{14} \left( |X|_{L^1(0,T)} + |Y|_{L^1(0,T)} + |\varphi^0(0; w(t))|_{L^1(0,T)} \right)} \times \left\{ \psi^0_w(u_0) + N_7(1 + C_R)^{10}(1 + R)^{14} \left( |X|_{L^1(0,T)} + |Y|_{L^1(0,T)} + |\varphi^0(0; w(t))|_{L^1(0,T)} \right) \right\}. \tag{25}\]

Now we show that $Q$ is the self-mapping on $E(T_0, M_0, R_0)$ for some chosen constants $T_0 > 0$, $M_0 > 0$ and $R_0 > 0$.

Note that by (F6) we have

\[\varphi^t(0; u(t)) \leq 2\varphi^t(w(t); u(t)) < C_R^2 R^2 \left( = 2\psi^t_w(u(t)) + C_R^2 R^2 \right) \tag{26}\]

for any $w \in E(T)$ with $\sup_{t \in [0,T]} |w(t)|_H \leq R$. 9
Here, we take $R_0 > 0$, $M_0 > 0$ so large that

$$2 \left[ \sup_{t \in [0,T]} |h(t)|_H + e^{\frac{n}{4}} |u_0 - h(0)|_H + e^{\frac{n}{4}} N^\frac{1}{2} \left\{ |f|_{L^2(0,T;H)} + |\tilde{h}'|_{L^2(0,T;H)} \right\} \right] \leq R_0,$$

$$4e^{2N(1+C_{R_0})^{10}(1+R_0)^{14}} \left\{ v^0_{w}(u_0) + 2N^7(1 + C_{R_0})^{10} (1 + R_0)^{14} \right\} + C_{R_0}^2 R_0^2 + C_h$$

$$\leq 4e^{2N(1+C_{R_0})^{10}(1+R_0)^{14}} \left\{ 2\varphi^0(0; u_0) + \frac{C_{R_0}^2 R_0^2}{4} + 2N^7(1 + C_{R_0})^{10} (1 + R_0)^{14} \right\}$$

$$+ C_{R_0}^2 R_0^2 + C_h$$

$$\leq M_0.$$  

Next, we choose $T_0 > 0$ so small that $T_0 \leq T$, $|h'|_{L^2(0,T_0;H)} \leq M_0$, $|X|_{L^1(0,T_0)} \leq 1$,

$$|Y|_{L^1(0,T_0)} + |\varphi'(0; w(t))|_{L^1(0,T_0)} \leq T_0^\frac{1}{2} M_0^\frac{1}{2} + M_0^\frac{1}{2} T_0^\frac{1}{2} |\alpha'_{R_0}|_{L^2(0,T_0)} + T_0 M_0 \leq 1,$$

$$\sup_{t \in [0,T_0]} |h(t)|_H + e^{\frac{n}{4}} |u_0 - h(0)|_H + e^{\frac{n}{4}} N^\frac{1}{2} \left\{ |f|_{L^2(0,T_0;H)} + |\tilde{h}'|_{L^2(0,T_0;H)} \right\}$$

$$+ e^{\frac{n}{4}} N^\frac{1}{2} T_0^\frac{1}{2} \left\{ \sup_{t \in [0,T_0]} \varphi'(0; h(t)) \right\} \leq R_0.$$

Then, the estimates (19), (25) with (26) implies that $Qw(=u)$ belongs to the set $E(T_0, M_0, R_0)$ for $w \in E(T_0, M_0, R_0)$, thus $Q$ is the self-mapping on $E(T_0, M_0, R_0)$.

**Lemma 3.** Let $M_0 > 0$, $R_0 > 0$, and $T_0 > 0$ be constants obtained in Lemma 2. Let $\{w_n\} \subseteq E(T_0, M_0, R_0)$, $w \in E(T_0, M_0, R_0)$ and $u_n$ be the solution of $CP(w_n; u_0)$. Suppose $w_n \longrightarrow w$ in $C([0,T_0];H)$ as $n \longrightarrow +\infty$. Then, there is a solution $u$ of $CP(w; u_0)$ on $[0,T_0]$ such that $u \in E(T_0, M_0, R_0)$ and $u_n \longrightarrow u$ in $C([0,T_0];H)$ as $n \longrightarrow +\infty$.

**Proof.** Since $\{w_n\} \subseteq E(T_0, M_0, R_0)$ and Lemma 2, we have

$$\sup_{t \in [0,T_0]} \varphi'(0; u_n(t)) \leq M_0,$$

$$|u_n'_{L^2(0,T_0;H)}| \leq M_0, \quad \forall n = 1,2,\ldots, (27)$$

$$|u_n(t)|_{H} \leq R_0, \quad \forall n = 1,2,\ldots. (28)$$

By (Φ3), (27), (28), there are a subsequence $\{n_k\}$ of $n$ and a function $u \in W^{1,2}(0, T_0; H)$ such that

$$u_{n_k} \longrightarrow u \quad \text{strongly in} \quad C([0,T_0];H),$$

$$u'_{n_k} \rightharpoonup u' \quad \text{weakly in} \quad L^2(0, T_0; H)$$

as $k \rightarrow +\infty$. By (Φ1), (27)-(30) and the uniqueness of $u_n$, we easily observe that $u \in E(T_0, M_0, R_0)$ and $u_n \longrightarrow u$ in $C([0,T_0];H)$ as $n \rightarrow +\infty$.

Now, let us show that $u$ is a solution of $CP(w; u_0)$ on $[0,T_0]$. To do so, we define

$$\Phi(w; z) = \int_0^{T_0} \varphi'(w(t); z(t)) dt.$$  

Then by the assumption (Φ6), we see that

$$\Phi(w_n; z) \longrightarrow \Phi(w; z) \quad \text{as} \quad n \rightarrow +\infty.$$  

(31)
for any \( z \in L^2(0, T_0; H) \) with \( \varphi(0; z(\cdot)) \in L^1(0, T_0) \). From (27), (29), (\Phi 1), (\Phi 2), (\Phi 6) and the Fatou’s lemma, it follows that

\[
\liminf_{k \to +\infty} \Phi(w_{n_k}; u_{n_k}) = \liminf_{k \to +\infty} \{\Phi(w_{n_k}; u_{n_k}) - \Phi(w; u_{n_k}) + \Phi(w; u_{n_k})\} \\
\geq \liminf_{k \to +\infty} \Phi(w; u_{n_k}) \geq \Phi(w; u).
\]

(32)

Moreover, by \( \{w_n\} \subset E(T_0, M_0, R_0) \) and the demi-closedness (G3) we see that

\[ G(\cdot, w_{n_k}(\cdot)) \to G(\cdot, w(\cdot)) \quad \text{weakly in} \quad L^2(0, T_0; H), \]

hence

\[
f - G(\cdot, w_{n_k}(\cdot)) \rightharpoonup f - G(\cdot, w(\cdot)) \quad \text{weakly in} \quad L^2(0, T_0; H)
\]
as \( k \to +\infty \).

Now, let \( z \) be any function in \( L^2(0, T_0; H) \) with \( \varphi(0; z(\cdot)) \in L^1(0, T_0) \). Since \( u_{n_k} \) is the unique solution of \( CP(w_{n_k}; u_0) \), then the following inequality holds:

\[
\int_0^{T_0} (f(t) - G(t, w_{n_k}(t)) - u'_{n_k}(t), z(t) - u_{n_k}(t)) \, dt \leq \Phi(w_{n_k}; z) - \Phi(w_{n_k}; u_{n_k}).
\]

(34)

Taking account of (29)-(33) and letting \( k \to +\infty \) in (34), we get

\[
\int_0^{T_0} (f(t) - G(t, w(t)) - u'(t), z(t) - u(t)) \, dt \leq \Phi(w; z) - \Phi(w; u),
\]

which implies that \( f(t) - G(t, w(t)) - u'(t) \in \partial\varphi(w(t); u(t)) \) for a.e. \( t \in [0, T_0] \) (cf. [1, Proposition 3.3]). Thus \( u \) is the solution of \( CP(w; u_0) \) on \( [0, T_0] \). \( \square \)

**Proof.** [Proof of Theorem 1; Local existence] By Lemma 2, we can define a self-mapping \( Q : E(T_0, M_0, R_0) \to E(T_0, M_0, R_0) \) by \( Qw = u \) for each \( w \in E(T_0, M_0, R_0) \), where \( u \) is a solution of \( CP(w; u_0) \). Clearly, \( E(T_0, M_0, R_0) \) is compact in \( C([0, T_0]; H) \). Moreover, it follows from Lemma 3 that \( Q \) is continuous with respect to the topology of \( C([0, T_0]; H) \). Therefore, the Schauder’s fixed point theorem implies that the self-mapping \( Q \) has a fixed point \( u \) in \( E(T_0, M_0, R_0) \), i.e. \( Qu = u \). Clearly \( u \) is the solution of \( CP(u_0) \), thus we can construct the local solution \( u \) of \( CP(u_0) \) on \( [0, T_0] \). \( \square \)

### 4 Proof of Theorem 2

In this section we shall prove Theorem 2, which is concerned with the global existence of solution to \( CP(u_0) \).

First, we consider the inequality (17). By the local existence result in Section 3, we can take \( w = u \in E(T_0, M_0, R_0) \), \( u \) being the solution of \( CP(u_0) \) on a small time interval \( [0, T_0] \) with \( 0 < T_0 \leq T \). Hence, by taking \( w = u \) in (17) it follows from (G2) and the additional assumption (\Phi 8) that

\[
\frac{d}{dt}|u(t) - h(t)|_H^2 + \varphi'(u(t); u(t)) \\
\leq N_h|u(t) - h(t)|_H^2 + N_0 \left( |f(t)|_H^2 + |h'(t)|_H^2 + \varphi'(0; h(t)) + 1 \right)
\]

(35)
for some constants \( N_8 > 0 \) and \( N_9 > 0 \) depending only on \( C_1, C_2, C_3, C_4 \). By applying Gronwall’s inequality to (35), we obtain
\[
\sup_{t \in [0,T]} |u(t)|_H \\
\leq \sup_{t \in [0,T]} |h(t)|_H + \sqrt{e^{N_8 T}}|u_0 - h(0)|_H + \sqrt{N_9 e^{N_8 T}} \left\{ |f|_{L^2(0,T;H)} + |h'|_{L^2(0,T;H)} \right\}
\]
\[
+ \sqrt{N_9 T e^{N_8 T}} \left\{ \sup_{t \in [0,T]} \varphi'(0; h(t)) \right\} + 1 \equiv N_{10}. \tag{36}
\]

Next, take a number \( R > 0 \) with \( R \geq N_{10} \), and we now consider the inequality (23). Applying Schwarz inequality to the term \( Y(\tau) \left\{ 1 + \psi^\tau_w(u(\tau)) \right\}^{1/2} \) and using (G2), (Φ8), we obtain
\[
\psi^t_w(u(t)) - \psi^s_w(u(s)) + \frac{1}{4} \int_s^t |u'(\tau)|^2_H d\tau \\
\leq N_{11}(1 + C_R)^{16}(1 + R)^{24} \int_s^t X(\tau)(1 + \psi^\tau_w(u(\tau)))d\tau + \frac{1}{8} \int_s^t |\psi'(\tau)|^2_H d\tau
\]
\[
+ N_{12}(1 + C_R)^8(1 + R)^{12} \int_s^t \varphi^\tau(0; w(\tau))d\tau \tag{37}
\]
for \( 0 \leq s \leq t \leq T \), where \( N_{11} > 0 \) and \( N_{12} > 0 \) depend on \( C_1, C_2, C_3, C_4, N_6 \).

Applying Gronwall’s inequality to (37), we obtain
\[
\psi^t_w(u(t)) + \frac{1}{4} \int_0^t e^{N_{11}(1+C_R)^{16}(1+R)^{24} \int_s^t X(s)ds} |u'(\tau)|^2_H d\tau \\
\leq e^{N_{11}(1+C_R)^{16}(1+R)^{24} \int_0^t X(s)ds} \left\{ \psi^0_w(u_0) + N_{11}(1 + C_R)^{16}(1 + R)^{24} \int_0^T X(s)ds \right\}
\]
\[
+ \frac{1}{8} \int_0^t e^{N_{11}(1+C_R)^{16}(1+R)^{24} \int_s^t X(s)ds} |\psi'(\tau)|^2_H d\tau
\]
\[
+ N_{12}(1 + C_R)^8(1 + R)^{12} \int_0^t e^{N_{11}(1+C_R)^{16}(1+R)^{24} \int_0^s X(s)ds} \varphi^\tau(0; w(\tau))d\tau. \tag{38}
\]

Here, we can take \( w = u \in E(T_0, M_0, R_0) \), \( u \) being the solution of CP\((u_0)\) on a small time interval \([0,T_0]\) with \( 0 < T_0 \leq T \). Then, by using (26), (36), (38) we get
\[
\varphi'(u(t); u(t)) + \frac{1}{8} \int_0^t e^{N_{11}(1+C_R)^{16}(1+R)^{24} \int_s^t X(s)ds} |u'(\tau)|^2_H d\tau
\]
\[
\leq N_{13}(1 + C_R)^{16}(1 + R)^{24} e^{N_{14}(1+C_R)^{16}(1+R)^{24}} \left( 1 + \int_0^t \varphi^\tau(u(t); u(t))d\tau \right), \tag{39}
\]
frozen for \( 0 \leq t \leq T_0 \), where \( N_{13} > 0, N_{14} > 0 \) are dependent only on the given data. By applying Gronwall’s inequality to (39), we conclude that
\[
\varphi'(u(t); u(t)) + \frac{1}{8} \int_0^{T_0} |u'(t)|^2_H dt
\]
\[
\leq N_{15}(1 + C_R)^{32}(1 + R)^{48} \exp(N_{16}(1 + C_R)^{16}(1 + R)^{24} e^{N_{14}(1+C_R)^{16}(1+R)^{24}}), \tag{40}
\]
for \( 0 \leq t \leq T_0 \).
where $N_{15} > 0$ and $N_{16} > 0$ depends only on the given data and are independent of $T_0(\leq T)$ and $R(\geq N_{10})$.

Now we shall prove Theorem 2 by employing the estimates (36) and (40).

**Proof. [Proof of Theorem 2; Global existence]** Assume that $$T^* := \sup\{T_0; \text{CP}(u_0) \text{ has a solution on } [0, T_0]\} < +\infty.$$ By the local existence result in Section 3, we note $T^* > 0$. By the definition of $T^*$, there is a function $u : [0, T^*) \rightarrow H$ such that for any $T_0 (< T^*)$ $u$ is the solution of CP$(u_0)$ on $[0, T_0]$. By (36) and (40) we have

$$u \in W^{1,2}(0, T^*; H), \quad \varphi^{(1)}(u(\cdot); u(\cdot)) \in L^\infty(0, T^*).$$

Hence by assumptions (Φ1), (Φ3), (Φ5), (Φ6), we observe that the limit $u_0^* := \lim_{t \uparrow T^*} u(t)$ exists strongly in $H$ such that

$$u_0^* \in D(\varphi^{T^*}(0; \cdot)).$$

Now, taking $u_0^*$ as the initial value at $t = T^*$, we can get the solution $u$ beyond the time interval $[0, T^*)$. Thus we observe that the solution to CP$(u_0)$ exists on the whole time interval $[0, T]$. \qed

5 Application to a double obstacle problem

In this section we apply our abstract results (Theorems 1, 2, 3) to a parabolic variational inequality with time-dependent double obstacles.

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ ($N \geq 1$) with smooth boundary. Let $g_1, g_2$ be prescribed obstacle functions on $[0, T] \times \Omega$ so that

$$g_i \in L^\infty(0, T; H^1(\Omega)) \cap L^\infty([0, T] \times \Omega), \quad g_i' \in L^2(0, T; H^1(\Omega)) \cap L^2(0, T; L^\infty(\Omega))$$

for $i = 1, 2$, and

$$g_2 - g_1 \geq C_g \quad \text{a.e. on } [0, T] \times \Omega \text{ for some constant } C_g > 0.$$

For each $t \in [0, T]$, we define the convex set $K(t)$ by

$$K(t) := \{z \in H^1(\Omega); g_1(t) \leq z \leq g_2(t) \text{ a.e. on } \Omega\}.$$ 

Now, let us consider the following interior time-dependent double obstacle problem. **Problem (P):** Find a function $u \in W^{1,2}(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$ such that

$$u(t) \in K(t) \quad \text{for a.e. } t \in [0, T],$$

$$(u'(t) + b(t, \cdot, u(t)) - f(t), u(t) - z) + \int_\Omega a(x, u(t), \nabla u(t)) \cdot \nabla (u(t) - z) dx \leq 0$$

for all $z \in K(t),$
u(0) = u_0 \quad \text{in } \Omega,

where \((\cdot, \cdot)\) is a usual inner product of \(L^2(\Omega)\), \(a = (a_1, ..., a_N)\) is an elliptic vector field, \(b\) and \(f\) are given functions.

The aim of this section is to consider the problem (P) as an application of the abstract evolution equation \(CP(u_0)\). To do so, we suppose that

(A1) \(a(x, s, p)\) is continuous on \(\Omega \times \mathbb{R} \times \mathbb{R}^N\) such that \(a(x, s, p) = \partial_p A(x, s, p)\) for some potential function \(A(x, s, p)\). Moreover, there exist constants \(\mu > 0\), \(\nu_1 = \nu_1(a) > 0\) and \(\nu_2 = \nu_2(a) > 0\) such that

\[
[a(x, s, p) - a(x, s, \hat{p})] \cdot (p - \hat{p}) \geq \mu|p - \hat{p}|^2,
\]

\[
[a(x, s, p)]^2 + |A(x, s, p)|^2 + |\partial_s A(x, s, p)|^2 \leq \nu_1(1 + |s|^2 + |p|^2),
\]

\[
|a(x, s, p) - a(x, \hat{s}, p)| \leq \nu_2(1 + |p|)|s - \hat{s}|
\]

for all \(x \in \Omega\), \(s, \hat{s} \in \mathbb{R}\), \(p, \hat{p} \in \mathbb{R}^N\).

(A2) \(b(t, x, s)\) is continuous on \([0, T] \times \Omega \times \mathbb{R}\) satisfying the following properties: there exist a constant \(L_b > 0\) and a function \(d \in L^1(0, T)\) such that

\[
|b(t, x, s) - b(t, x, \hat{s})| \leq L_b|s - \hat{s}|, \quad \forall t \in [0, T], \forall x \in \Omega, \forall s, \hat{s} \in \mathbb{R},
\]

\[
\sup_{x \in \Omega} |\frac{\partial}{\partial t} b(t, x, 0)| \leq d(t) \quad \text{for a.e. } t \geq 0.
\]

As a direct application of Theorems 1, 2 and 3, we have:

**Proposition 2.** Assume (A1) and (A2). Then, for each \(f \in L^2(0, T; L^2(\Omega))\) and \(u_0 \in K(0)\), the problem (P) has a unique solution \(u\) on \([0, T]\).

**Proof.** To apply Theorems 1, 2 and 3 to the problem (P), we choose \(L^2(\Omega)\) as a real Hilbert space \(H\), and define a function \(\varphi^t(\cdot; \cdot) : L^2(\Omega) \times L^2(\Omega) \to \mathbb{R} \cup \{\infty\}\) by

\[
\varphi^t(w; z) := \begin{cases} 
\int_{\Omega} A(x, w(x), \nabla z(x))dx + C_\mu(1 + |w|^2_{L^2(\Omega)}), & \text{if } z \in K(t), \\
+\infty, & \text{otherwise},
\end{cases}
\]

where \(C_\mu > 0\) is a constant such that \(\varphi^t(w; z) \geq \frac{\mu}{4}|z|^2_{L^2(\Omega)} + 1\) for all \(t \geq 0\), \(w \in L^2(\Omega)\) and \(z \in K(t)\) (cf. [13, Lemma 3.1]).

Let us define an operator \(G(t, \cdot) : L^2(\Omega) \to L^2(\Omega)\) by \(G(t, z) := b(t, \cdot, z(\cdot))\) in \(L^2(\Omega)\). And we define a function \(\gamma\) by \(\gamma(z) := \int_{\Omega} z^+(x)dx\) for \(z \in L^2(\Omega)\), where \(z^+ := \max\{z, 0\}\).

Now we put for any \(t \in [0, T]\) and \(r > 0\)

\[
\alpha_r(t) = k \int_0^t \{ |g_1'|_{L^\infty(\Omega)} + |g_2'|_{L^\infty(\Omega)} + |g_1'|_{H^1(\Omega)} + |g_2'|_{H^1(\Omega)} \} \, d\tau,
\]

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where $k > 0$ is a (sufficient large) positive constant. Then, we easily verify $\{\varphi^t\} \in \Phi_\gamma(\{\alpha_r\})$. For instance, we can show (Φ5) by taking

$$\tilde{z} := (z - g_1(s))\frac{g_2(t) - g_1(t)}{g_2(s) - g_1(s)} + g_1(t)$$

for given $z \in K(s)$. Then, by the slight modification of [22, Lemma 5.1], we can show (Φ5).

Moreover we easily see that $G(t, \cdot) \in G_\gamma(\{\varphi^t\})$ and the assumption (Φ8) hold.

Clearly, the problem (P) can be reformulated in the evolution equation $CP(u_0)$. Thus, by applying Theorems 1, 2 and 3, we see that (P) has a unique global solution $u$.

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References


