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A CLASS OF NONLINEAR EVOLUTION EQUATIONS GOVERNED BY TIME-DEPENDENT OPERATORS OF SUBDIFFERENTIAL TYPE

Dedicated to Professor N. Kenmochi on the Occasion of His 60th Birthday

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Abstract. Recently there are so many mathematical models which describe nonlinear phenomena. In some phenomena, the free energy functional is not convex. So, the existence-uniqueness question is sometimes difficult. In order to study such phenomena, let us introduce the new class of abstract nonlinear evolution equations governed by time-dependent operators of subdifferential type. In this paper we shall show the existence and uniqueness of solution to nonlinear evolution equations with time-dependent constraints in a real Hilbert space. Moreover we apply our abstract results to a parabolic variational inequality with time-dependent double obstacles constraints.

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1 Introduction

We study an abstract nonlinear evolution equation in a real Hilbert space H of the form

$$u'(t) + \partial\varphi^t(u(t); u(t)) + G(t, u(t)) \ni f(t) \quad \text{in } H, \quad \text{a.e. } t \in (0, T), \quad (1)$$

where $u'(t) := \frac{d}{dt}u(t)$, $G(t, \cdot)$ is a single valued perturbation small relative to φ^t , and f is a given H -valued function. For each $t \in [0, T]$, a function $\varphi^t(\cdot; \cdot) : H \times H \rightarrow \mathbb{R} \cup \{\infty\}$ is given such that for all $w \in H$, $\varphi^t(w; \cdot) : H \rightarrow \mathbb{R} \cup \{\infty\}$ is a proper, l.s.c. (lower semi-continuous) and convex function, and $\partial\varphi^t(w; \cdot)$ is its subdifferential operator, i.e., $z^* \in \partial\varphi^t(w; z)$ if and only if

$$z \in D(\varphi^t(w; \cdot)) \quad \text{and} \quad (z^*, y - z) \leq \varphi^t(w; y) - \varphi^t(w; z) \quad \text{for all } y \in H.$$

For a proper, l.s.c. and convex function $\psi^t(\cdot) : H \rightarrow \mathbb{R} \cup \{\infty\}$, many mathematicians studied the nonlinear evolution equation of the form

$$u'(t) + \partial\psi^t(u(t)) \ni f(t) \quad \text{in } H, \quad \text{a.e. } t \in (0, T). \quad (2)$$

For various aspects of (2), we refer to [2, 5, 6, 8, 9, 11, 18, 19]. For instance, Kenmochi [6] showed the existence-uniqueness, stability and convergence of solutions to (2).

For the nonmonotone perturbation $G(t, \cdot)$, Ôtani [16] has already shown the existence of solution to

$$u'(t) + \partial\psi^t(u(t)) + G(t, u(t)) \ni f(t) \quad \text{in } H, \quad \text{a.e. } t \in (0, T). \quad (3)$$

The large-time behavior of solutions for (3) was discussed by [20] from the view-point of attractors. For another works of (3), we refer to [10, 16, 17, 20, 21, 22], for instance.

The main object of this paper is to establish abstract results on existence-uniqueness of solutions to (1). Note that the function $\varphi^t(u; u)$ is not convex in u , hence we can not apply the theory established by Ôtani [16]. So, by using the idea of Kenmochi-Kubo [7] and Kubo-Yamazaki [12, 13], we shall show the existence of solution to (1) in this paper. Namely, for the given function $w : [0, T] \rightarrow H$, let us consider the problem

$$u'(t) + \partial\varphi^t(w(t); u(t)) \ni f(t) - G(t, w(t)) \quad \text{in } H, \quad \text{a.e. } t \in (0, T). \quad (4)$$

Assuming some appropriate conditions on the t - and w -dependence of the function $\varphi^t(w; z)$, we can apply the result of Kenmochi [6]. Then we see that the equation (4) has a unique solution u for each w , and that the mapping $w \mapsto u$ has some compactness property. Hence, by using a fixed point argument, we can get the existence of solution to (1).

In Section 2 we present our main results on existence and uniqueness of solution to (1), and then the uniqueness (Theorem 3) is proved. In Section 3 we prove the local existence result (Theorem 1). In Section 4, the global existence result (Theorem 2) is proved. In the final Section 5 we apply our abstract results to a parabolic variational inequality with time-dependent double obstacle constraints.

Notation

Throughout this paper, let H be a real Hilbert space with norm $|\cdot|_H$ and inner product (\cdot, \cdot) . For a proper l.s.c. convex function ψ on H we use the notation $D(\psi)$, $\partial\psi$ and $D(\partial\psi)$ to indicate the effective domain, subdifferential and its domain of $\partial\psi$, respectively. For their precise definitions and basic properties, see a monograph by Brézis [4].

2 Assumptions and main results

We consider a Cauchy problem $\text{CP}(u_0)$ for (1) of the following form:

$$\text{CP}(u_0) \begin{cases} u'(t) + \partial\varphi^t(u(t); u(t)) + G(t, u(t)) \ni f(t) & \text{in } H \text{ a.e. } t \in (0, T), \\ u(0) = u_0, \end{cases}$$

where T is a given positive number, a function $\varphi^t(u(t); u(t))$ is introduced in Section 1, $G(t, \cdot)$ is a single valued perturbation small relative to φ^t , $f \in L^2(0, T; H)$ is a given function, and $u_0 \in H$ is given data.

Definition 1. Given $u_0 \in H$ and $f \in L^2(0, T; H)$, the function $u : [0, T] \rightarrow H$ will be called a solution to $\text{CP}(u_0)$, if $u \in W^{1,2}(0, T; H)$, $u(0) = u_0$, $u(t) \in D(\partial\varphi^t(u(t); \cdot))$ and $f(t) - u'(t) - G(t, u(t)) \in \partial\varphi^t(u(t); u(t))$ for a.e. $t \in [0, T]$, namely

$$(f(t) - u'(t) - G(t, u(t)), y - u(t)) \leq \varphi^t(u(t); y) - \varphi^t(u(t); u(t))$$

$$\text{for any } y \in H, \text{ a.e. } t \in [0, T].$$

For a given positive number T , let $\{\alpha_r\} := \{\alpha_r; r > 0\}$ be a family of functions $\alpha_r \in W^{1,2}(0, T)$, with parameter $r > 0$. With this family $\{\alpha_r\}$, we specify a class $\Phi(\{\alpha_r\})$ of all families $\{\varphi^t\} := \{\varphi^t; t \in [0, T]\}$ of time-dependent functions $\varphi^t(\cdot; \cdot)$ on $H \times H$ as follows.

Definition 2. We denote by $\{\varphi^t\} \in \Phi(\{\alpha_r\})$ the set of all time-dependent functions $\varphi^t(\cdot; \cdot)$ from $H \times H$ into $\mathbb{R} \cup \{\infty\}$ satisfying the following seven conditions:

($\Phi 1$) For each $w \in H$ and $t \in [0, T]$, $\varphi^t(w; \cdot) : H \rightarrow \mathbb{R} \cup \{\infty\}$ is a proper l.s.c. convex function;

($\Phi 2$) There exists a positive constant $C_1 > 0$ such that

$$\varphi^t(w; z) \geq C_1 |z|_H^2, \quad \forall t \in [0, T], \forall w \in H, \forall z \in D(\varphi^t(w; \cdot));$$

($\Phi 3$) For each $t \in [0, T]$, $w \in H$ and $k > 0$, the level set $\{z \in H; \varphi^t(w; z) \leq k\}$ is compact in H ;

($\Phi 4$) $D(\varphi^t(w; \cdot))$ is independent of $w \in H$ for any $t \in [0, T]$;

(Φ5) For each $r > 0$, $s, t \in [0, T]$ with $s \leq t$, $w \in D(\varphi^s(0; \cdot))$ with $|w|_H \leq r$ and $z \in D(\varphi^s(w; \cdot))$ with $|z|_H \leq r$ there exists an element $\tilde{z} \in D(\varphi^t(w; \cdot))$ such that

$$|\tilde{z} - z|_H \leq |\alpha_r(t) - \alpha_r(s)| \left(1 + \varphi^s(0; z)^{\frac{1}{2}}\right)$$

and

$$\varphi^t(w; \tilde{z}) - \varphi^s(w; z) \leq |\alpha_r(t) - \alpha_r(s)| \left(1 + \varphi^s(0; z) + \varphi^s(0; w)^{\frac{1}{2}} \varphi^s(0; z)^{\frac{1}{2}} + \varphi^s(0; w)^{\frac{1}{2}}\right);$$

(Φ6) For each $r > 0$ there is a positive constant $C_r > 0$ such that

$$|\varphi^t(w_1; z) - \varphi^t(w_2; z)| \leq C_r |w_1 - w_2|_H \varphi^t(0; z)^{\frac{1}{2}},$$

$$\forall t \in [0, T], \forall w_i \in H \text{ with } |w_i|_H \leq r, (i = 1, 2), \text{ and } \forall z \in D(\varphi^t(0; \cdot));$$

(Φ7) There is a function $h \in W^{1,2}(0, T; H)$ with $C_h := \sup_{t \in [0, T]} \varphi^t(0; h(t)) < +\infty$.

Next, we introduce the class $\mathcal{G}(\{\varphi^t\})$ of time-dependent perturbation $G(t, \cdot)$ associated with $\{\varphi^t\} \in \Phi(\{\alpha_r\})$.

Definition 3. $\{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^t\})$ if and only if $G(t, \cdot)$ is a single valued operator from $D(G(t, \cdot)) \subset H$ into H which fulfills the following conditions (G1)-(G3):

(G1) $D(\varphi^t(0; \cdot)) \subset D(G(t, \cdot)) \subset H$ for all $t \in [0, T]$ and $G(\cdot, v(\cdot))$ is (strongly) measurable on J for any interval $J \subset [0, T]$ and $v \in L^2_{loc}(J; H)$ with $v(t) \in D(\varphi^t(0; \cdot))$ for a.e. $t \in J$.

(G2) There are positive constants $C_2 > 0$, $C_3 > 0$ such that

$$|G(t, z)|_H^2 \leq C_2 \varphi^t(z; z) + C_3, \quad \forall t \in [0, T], \forall z \in D(\varphi^t(0; \cdot)).$$

(G3) (Demi-closedness) If $\{t_n\} \subset [0, T]$, $\{z_n\} \subset H$, $t_n \rightarrow t$, $z_n \rightarrow z$ in H (as $n \rightarrow +\infty$) and $\{\varphi^{t_n}(0, z_n)\}$ is bounded, then $G(t_n, z_n) \rightarrow G(t, z)$ weakly in H as $n \rightarrow +\infty$.

Now let us mention our main local existence result in this paper. In Section 3 we shall prove Theorem 1.

Theorem 1. *Let T be any positive number. Assume $\{\varphi^t\} \in \Phi(\{\alpha_r\})$, $\{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^t\})$ and $f \in L^2(0, T; H)$. Then, for each $u_0 \in D(\varphi^0(0; \cdot))$ there exists a positive constant $T_0(\leq T)$ such that $CP(u_0)$ has at least one solution u on $[0, T_0]$.*

The next main theorem is concerned with the global existence result in this paper. In Section 4 we shall prove Theorem 2.

Theorem 2. *Let T be any positive number. Assume $\{\varphi^t\} \in \Phi(\{\alpha_r\})$, $\{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^t\})$ and $f \in L^2(0, T; H)$. Additionally, assume that*

(Φ8) There is a positive constant $C_4 > 0$ such that

$$\varphi^t(w; z) \leq C_4(1 + |w|_H^2 + \varphi^t(0; z)), \quad \forall t \in [0, T], \quad \forall w \in H, \quad \forall z \in D(\varphi^t(0; \cdot)).$$

Then, for each $u_0 \in D(\varphi^0(0; \cdot))$ there exists at least one solution u to $CP(u_0)$ on $[0, T]$.

To show the uniqueness of solution to $CP(u_0)$, we shall introduce subclasses of $\Phi(\{\alpha_r\})$ and $\mathcal{G}(\{\varphi^t\})$.

Definition 4. Let γ be a non-negative continuous and convex function on H such that $\gamma(z) + \gamma(-z) = 0$ if and only if $z = 0$. Then

(1) $\{\varphi^t\} \in \Phi_\gamma(\{\alpha_r\})$ if and only if $\{\varphi^t\} \in \Phi(\{\alpha_r\})$ satisfies the γ -accretiveness (\star) for φ^t as follows:

(\star) For any $z_i \in D(\partial\varphi^t(z_i; \cdot))$ and $z_i^* \in \partial\varphi^t(z_i; z_i)$ ($i = 1, 2$), there is an element $w_0 \in \partial\gamma(z_1 - z_2)$ so that $(z_1^* - z_2^*, w_0) \geq 0$, where $\partial\gamma$ is the subdifferential of γ in H .

(2) $\{G(t, \cdot)\} \in \mathcal{G}_\gamma(\{\varphi^t\})$ if and only if for any positive number $\varepsilon > 0$, there is a positive constant $C_\varepsilon > 0$ such that

$$|(G(t, z_1) - G(t, z_2), w_0)| \leq \varepsilon(z_1^* - z_2^*, w_0) + C_\varepsilon\{\gamma(z_1 - z_2) + \gamma(z_2 - z_1)\},$$

whenever $t \in [0, T]$, $z_i \in D(\partial\varphi^t(z_i; \cdot))$, $z_i^* \in \partial\varphi^t(z_i; z_i)$ ($i = 1, 2$), and

$$w_0 \in \partial\gamma(z_1 - z_2) \text{ with } (z_1^* - z_2^*, w_0)_H \geq 0.$$

Now let us mention our main uniqueness result in this paper.

Theorem 3. Let T be any positive number. Assume $\{\varphi^t\} \in \Phi_\gamma(\{\alpha_r\})$, $\{G(t, \cdot)\} \in \mathcal{G}_\gamma(\{\varphi^t\})$ and $f \in L^2(0, T; H)$. Then, for each $u_0 \in H$ the solution u to $CP(u_0)$ is unique.

Proof. Let u and v be solutions to $CP(u_0)$. By the γ -accretiveness of φ^t , for a.e. $\tau \in [0, T]$ there exists $z^*(\tau) \in \partial\gamma(u(\tau) - v(\tau))$ such that

$$(u^*(\tau) - v^*(\tau), z^*(\tau)) \geq 0 \tag{5}$$

for any $u^*(\tau) \in \partial\varphi^\tau(u(\tau); u(\tau))$ and $v^*(\tau) \in \partial\varphi^\tau(v(\tau); v(\tau))$.

By $\{G(t, \cdot)\} \in \mathcal{G}_\gamma(\{\varphi^t\})$, for a number $\varepsilon \in (0, 1]$ there is a constant $C_\varepsilon > 0$ such that

$$\begin{aligned} & |(G(\tau, u(\tau)) - G(\tau, v(\tau)), z^*(\tau))| \\ & \leq \varepsilon(u^*(\tau) - v^*(\tau), z^*(\tau)) + C_\varepsilon\{\gamma(u(\tau) - v(\tau)) + \gamma(v(\tau) - u(\tau))\} \end{aligned} \tag{6}$$

for a.e. $\tau \in [0, T]$.

From (5) and (6) it follows that

$$\begin{aligned} 0 & \leq (u^*(\tau) - v^*(\tau), z^*(\tau)) \\ & = ([f(\tau) - u'(\tau) - G(\tau, u(\tau))] - [f(\tau) - v'(\tau) - G(\tau, v(\tau))], z^*(\tau)) \\ & \leq (-u'(\tau) + v'(\tau), z^*(\tau)) + |(-G(\tau, u(\tau)) + G(\tau, v(\tau)), z^*(\tau))| \\ & \leq -\frac{d}{d\tau}\gamma(u(\tau) - v(\tau)) \\ & \quad + \varepsilon(u^*(\tau) - v^*(\tau), z^*(\tau)) + C_\varepsilon\{\gamma(u(\tau) - v(\tau)) + \gamma(v(\tau) - u(\tau))\}, \end{aligned}$$

which implies that

$$\frac{d}{d\tau}\gamma(u(\tau) - v(\tau)) \leq C_\varepsilon\{\gamma(u(\tau) - v(\tau)) + \gamma(v(\tau) - u(\tau))\} \quad \text{for a.e. } \tau \in [0, T].$$

Similarly we have

$$\frac{d}{d\tau}\gamma(v(\tau) - u(\tau)) \leq C_\varepsilon\{\gamma(u(\tau) - v(\tau)) + \gamma(v(\tau) - u(\tau))\},$$

hence we have

$$\frac{d}{d\tau}\{\gamma(u(\tau) - v(\tau)) + \gamma(v(\tau) - u(\tau))\} \leq 2C_\varepsilon\{\gamma(u(\tau) - v(\tau)) + \gamma(v(\tau) - u(\tau))\}, \quad (7)$$

for a.e. $\tau \in [0, T]$.

Now, applying Gronwall's inequality to (7), we get

$$e^{-2C_\varepsilon t}\{\gamma(u(t) - v(t)) + \gamma(v(t) - u(t))\} \leq 0 \quad \text{for any } 0 \leq t \leq T,$$

which implies that $u(t) = v(t)$ for all $t \in [0, T]$. Thus Theorem 3 has been proved. \square

3 Proof of Theorem 1

In this section we shall show Theorem 1 by the fixed point argument. To do so, for a given positive number $T > 0$, we put a Banach space

$$E(T) \equiv \left\{ w \in W^{1,2}(0, T; H) ; \sup_{t \in [0, T]} \varphi^t(0; w(t)) < +\infty \right\}.$$

By the assumption $(\Phi 7)$ we note that $E(T) \neq \emptyset$.

Now, for each $w \in E(T)$ let us consider a following Cauchy problem $\text{CP}(w; u_0)$:

$$\text{CP}(w; u_0) \begin{cases} u'(t) + \partial\varphi^t(w(t); u(t)) \ni f(t) - G(t, w(t)) & \text{in } H, \text{ a.e. } t \in (0, T), \\ u(0) = u_0. \end{cases}$$

To show the existence-uniqueness of solution to $\text{CP}(w; u_0)$, we prepare the key lemma.

Lemma 1. *For each $w \in E(T)$ we take a positive constant $R > 0$ such that*

$\sup_{t \in [0, T]} |w(t)|_H \leq R$. Put

$$\psi_w^t(z) := \varphi^t(w(t); z) \text{ for } z \in H.$$

Then, there is a positive constant $N_1 > 0$ independent of w satisfying the following: for any $s, t \in [0, T]$ with $s \leq t$ and $z \in D(\psi_w^s)$ with $|z|_H \leq R$, there exists $\tilde{z} \in D(\psi_w^t)$ such that

$$|\tilde{z} - z|_H \leq N_1(1 + C_R)^4(1 + R)^6|\alpha_R(t) - \alpha_R(s)| \left(1 + \psi_w^s(z)^{\frac{1}{2}}\right), \quad (8)$$

$$\begin{aligned} & \psi_w^t(\tilde{z}) - \psi_w^s(z) \\ \leq & N_1(1 + C_R)^4(1 + R)^6 \left\{ |\alpha_R(t) - \alpha_R(s)| (1 + \psi_w^s(z)) + |w(t) - w(s)|_H (1 + \psi_w^s(z))^{\frac{1}{2}} \right. \\ & \left. + |\alpha_R(t) - \alpha_R(s)| \varphi^s(0; w(s))^{\frac{1}{2}} (1 + \psi_w^s(z))^{\frac{1}{2}} \right\}. \end{aligned} \quad (9)$$

Proof. Taking $w = w(s)$ in $(\Phi 5)$, then for any $s, t \in [0, T]$ with $s \leq t$ and $z \in D(\varphi^s(w(s); \cdot))$ with $|z|_H \leq R$, there exists $\tilde{z} \in D(\varphi^t(w(s); \cdot))$ such that

$$|\tilde{z} - z|_H \leq |\alpha_R(t) - \alpha_R(s)| \left(1 + \varphi^s(0; z)^{\frac{1}{2}}\right), \quad (10)$$

$$\begin{aligned} & \varphi^t(w(s); \tilde{z}) - \varphi^s(w(s); z) \\ & \leq |\alpha_R(t) - \alpha_R(s)| \left(1 + \varphi^s(0; z) + \varphi^s(0; w(s))^{\frac{1}{2}} \varphi^s(0; z)^{\frac{1}{2}} + \varphi^s(0; w(s))^{\frac{1}{2}}\right). \end{aligned} \quad (11)$$

It follows from $(\Phi 4)$ that

$$z \in D(\varphi^s(w(s); \cdot)) = D(\psi_w^s), \quad \tilde{z} \in D(\varphi^t(w(s); \cdot)) = D(\psi_w^t). \quad (12)$$

Note that by $(\Phi 6)$ and $w \in E(T)$ we have

$$\varphi^s(0; z) \leq 2\varphi^s(w(s); z) + C_R^2 |w(s)|_H^2 \leq 2\psi_w^s(z) + C_R^2 R^2. \quad (13)$$

Then, by (10) and (13) there is a positive number $N_2 > 0$ independent of w satisfying

$$\begin{aligned} |\tilde{z} - z|_H & \leq |\alpha_R(t) - \alpha_R(s)| \left(1 + \sqrt{2}\psi_w^s(z)^{\frac{1}{2}} + C_R R\right) \\ & \leq N_2(1 + C_R R) |\alpha_R(t) - \alpha_R(s)| \left(1 + \psi_w^s(z)^{\frac{1}{2}}\right). \end{aligned} \quad (14)$$

Moreover, we observe that by (11), (13), $(\Phi 6)$ there is a positive number $N_3 > 0$ independent of w satisfying the following:

$$\begin{aligned} & \psi_w^t(\tilde{z}) - \psi_w^s(z) \quad (= \varphi^t(w(t); \tilde{z}) - \varphi^t(w(s); \tilde{z}) + \varphi^t(w(s); \tilde{z}) - \varphi^s(w(s); z)) \\ & \leq N_3(1 + C_R)^2(1 + R)^2 \left\{ |w(t) - w(s)|_H \psi_w^t(\tilde{z})^{\frac{1}{2}} + |w(t) - w(s)|_H \right. \\ & \quad \left. + |\alpha_R(t) - \alpha_R(s)|(1 + \psi_w^s(z)) + |\alpha_R(t) - \alpha_R(s)| \varphi^s(0; w(s))^{\frac{1}{2}} (1 + \psi_w^s(z))^{\frac{1}{2}} \right\}. \end{aligned} \quad (15)$$

From $\alpha_R \in W^{1,2}(0, T)$, $w \in E(T)$ and (15) it follows that

$$\psi_w^t(\tilde{z}) \leq N_4(1 + C_R)^4(1 + R)^6 \left\{ 1 + \psi_w^s(z) + |\alpha_R(t) - \alpha_R(s)|^2 \varphi^s(0; w(s)) \right\} \quad (16)$$

for some constant $N_4 > 0$. Therefore, using (16) in the right hand side of (15), and by (12)-(14), we get this Lemma for some constant $N_1 > 0$ independent of w . \square

Proposition 1. *For each $w \in E(T)$, $CP(w; u_0)$ has a unique solution u on $[0, T]$.*

Proof. We note that $CP(w; u_0)$ can be regarded as the Cauchy problem for the nonlinear evolution equation of the form:

$$\begin{cases} u'(t) + \partial \psi_w^t(u(t)) \ni f(t) - G(t, w(t)) & \text{in } H \text{ a.e. } t \in (0, T), \\ u(0) = u_0. \end{cases}$$

Here, from $(\Phi 6)$ and $(G2)$ we see that for each $w \in E(T)$ with $\sup_{t \in [0, T]} |w(t)|_H \leq R$

$$\begin{aligned} \int_0^T |G(t, w(t))|_H^2 dt & \leq \int_0^T \{C_2 \varphi^t(w(t); w(t)) + C_3\} dt \\ & \leq T \left\{ 2C_2 \sup_{t \in [0, T]} \varphi^t(0; w(t)) + \frac{C_2 C_R^2 R^2}{4} + C_3 \right\} < +\infty, \end{aligned}$$

which implies that $f - G(\cdot, w(\cdot)) \in L^2(0, T; H)$. Moreover, by Lemma 1 we get the time-dependence of ψ_w^t . Therefore taking account of the assumption $(\Phi 1)$, we can apply the abstract theory established by Kenmochi [6]. Thus we get the existence-uniqueness of solution u for $\text{CP}(w; u_0)$. For detail proofs, see [6, Theorems 1.1.1, 1.1.2]. \square

By Proposition 1, the boundedness (cf. [6, Theorem 1.1.2]) of solution to $\text{CP}(w; u_0)$, and (13), we can define a mapping, we can define a mapping $Q : E(T) \rightarrow E(T)$ by $Qw = u$ for each $w \in E(T)$, where u is a solution for $\text{CP}(w; u_0)$.

Lemma 2. *There are positive constants T_0, M_0 and R_0 such that Q is a self-mapping on $E(T_0, M_0, R_0)$, i.e., $Qw (= u) \in E(T_0, M_0, R_0)$ for any $w \in E(T_0, M_0, R_0)$, where*

$$E(T_0, M_0, R_0) \equiv \left\{ w \in E(T_0) ; \begin{array}{l} \sup_{t \in [0, T_0]} \varphi^t(0; w(t)) \leq M_0, \quad |w'|_{L^2(0, T_0; H)} \leq M_0 \\ \sup_{t \in [0, T_0]} |w(t)|_H \leq R_0, \quad w(0) = u_0 \end{array} \right\}.$$

Proof. Fix $R > 0$ for a while and take $w \in E(T)$ with $\sup_{t \in [0, T]} |w(t)|_H \leq R$. We shall give a boundedness of solution u to the problem $\text{CP}(w; u_0)$.

Now, multiplying $\text{CP}(w; u_0)$ by $u(t) - h(t)$, we get

$$\begin{aligned} & (u'(t), u(t) - h(t)) + \varphi^t(w(t); u(t)) - \varphi^t(w(t); h(t)) \\ & \leq (f(t) - G(t, w(t)), u(t) - h(t)) \quad \text{a.e. } t \in (0, T), \end{aligned} \quad (17)$$

where h is the function in $(\Phi 7)$. Taking account of $(\Phi 2)$, $(\Phi 6)$ and $(G 2)$, we have

$$\begin{aligned} & \frac{d}{dt} |u(t) - h(t)|_H^2 - |u(t) - h(t)|_H^2 \\ & \leq N_5 (|f(t)|_H^2 + |h'(t)|_H^2 + \varphi^t(0; h(t)) + \varphi^t(0; w(t)) + C_R^2 R^2 + 1), \end{aligned} \quad (18)$$

for a constant $N_5 = N_5(C_2, C_3) > 0$. By applying Gronwall's inequality to (18), we obtain

$$\begin{aligned} & \sup_{t \in [0, T]} |u(t)|_H \\ & \leq \sup_{t \in [0, T]} |h(t)|_H + e^{\frac{T}{2}} |u_0 - h(0)|_H + e^{\frac{T}{2}} N_5^{\frac{1}{2}} \{ |f|_{L^2(0, T; H)} + |h'|_{L^2(0, T; H)} \} \\ & \quad + e^{\frac{T}{2}} N_5^{\frac{1}{2}} T^{\frac{1}{2}} \left\{ \sup_{t \in [0, T]} \varphi^t(0; h(t))^{\frac{1}{2}} + \sup_{t \in [0, T]} \varphi^t(0; w(t))^{\frac{1}{2}} + C_R R + 1 \right\}. \end{aligned} \quad (19)$$

Moreover, by Lemma 1 and arguments of [6, section 1], we see that the function $\psi_w^t(u(t)) = \varphi^t(w(t); u(t))$ is of bounded variation on $[0, T]$ and satisfies

$$\begin{aligned} & \psi_w^t(u(t)) - \psi_w^s(u(s)) + \int_s^t (u'(\tau) - f(\tau) + G(\tau, w(\tau)), u'(\tau)) d\tau \\ & \leq N_1 (1 + C_R)^4 (1 + R)^6 \int_s^t |\alpha'_R(\tau)| |u'(\tau) - f(\tau) + G(\tau, w(\tau))| \left\{ 1 + \psi_w^\tau(u(\tau))^{\frac{1}{2}} \right\} d\tau \\ & \quad + N_1 (1 + C_R)^4 (1 + R)^6 \int_s^t \left[|\alpha'_R(\tau)| (1 + \psi_w^\tau(u(\tau)) + |w'(\tau)|_H \{ 1 + \psi_w^\tau(u(\tau)) \}^{\frac{1}{2}}) \right] d\tau \\ & \quad + N_1 (1 + C_R)^4 (1 + R)^6 \int_s^t |\alpha'_R(\tau)| \varphi^\tau(0; w(\tau))^{\frac{1}{2}} \{ 1 + \psi_w^\tau(u(\tau)) \}^{\frac{1}{2}} d\tau \end{aligned} \quad (20)$$

for $0 \leq s \leq t \leq T$ and $w \in E(T)$ with $\sup_{t \in [0, T]} |w(t)|_H \leq R$.

Here we notice the following relations:

$$(u'(\tau) - f(\tau) + G(\tau, w(\tau)), u'(\tau)) \geq \frac{1}{2} |u'(\tau)|_H^2 - |f(\tau)|_H^2 - |G(\tau, w(\tau))|_H^2, \quad (21)$$

$$\begin{aligned} & |\alpha'_R(\tau)| |u'(\tau) - f(\tau) + G(\tau, w(\tau))|_H \left\{ 1 + \psi_w^\tau(u(\tau))^{\frac{1}{2}} \right\} \\ \leq & \delta |u'(\tau) - f(\tau) + G(\tau, w(\tau))|_H^2 + \delta^{-1} |\alpha'_R(\tau)|^2 \{1 + \psi_w^\tau(u(\tau))\} \\ \leq & 3\delta |u'(\tau)|_H^2 + 3\delta |f(\tau)|_H^2 + 3\delta |G(\tau, w(\tau))|_H^2 + \delta^{-1} |\alpha'_R(\tau)|^2 \{1 + \psi_w^\tau(u(\tau))\}, \end{aligned} \quad (22)$$

where in (22) we put $\delta := \frac{1}{12N_1(1+C_R)^4(1+R)^6}$. Using (21)-(22) in (20), we obtain

$$\begin{aligned} & \psi_w^t(u(t)) - \psi_w^s(u(s)) + \frac{1}{4} \int_s^t |u'(\tau)|_H^2 d\tau \\ \leq & N_6(1+C_R)^8(1+R)^{12} \int_s^t \left\{ X(\tau)(1 + \psi_w^\tau(u(\tau))) + Y(\tau) \{1 + \psi_w^\tau(u(\tau))\}^{\frac{1}{2}} \right. \\ & \left. + |G(\tau, w(\tau))|_H^2 \right\} d\tau \end{aligned} \quad (23)$$

for $0 \leq s \leq t \leq T$, where the constant $N_6 > 0$ is determined only by N_1 , and we put

$$X(\tau) := |f(\tau)|_H^2 + |\alpha'_R(\tau)|^2 + 1, \quad Y(\tau) := |w'(\tau)|_H + |\alpha'_R(\tau)| \varphi^\tau(0; w(\tau))^{\frac{1}{2}}.$$

By $(\Phi 6)$, $(G2)$ and (23), we obtain

$$\begin{aligned} & \psi_w^t(u(t)) - \psi_w^s(u(s)) + \frac{1}{4} \int_s^t |u'(\tau)|_H^2 d\tau \\ \leq & N_7(1+C_R)^{10}(1+R)^{14} \int_s^t \{X(\tau) + Y(\tau) + \varphi^\tau(0; w(\tau))\} \{1 + \psi_w^\tau(u(\tau))\} d\tau \end{aligned} \quad (24)$$

for $0 \leq s \leq t \leq T$, where $N_7 > 0$ depends on N_6 , C_2 and C_3 .

Applying Gronwall's inequality to (24), we obtain

$$\begin{aligned} & \sup_{0 \leq t \leq T} \psi_w^t(u(t)) + \frac{1}{4} \int_0^T |u'(t)|_H^2 dt \\ \leq & e^{N_7(1+C_R)^{10}(1+R)^{14} (|X|_{L^1(0,T)} + |Y|_{L^1(0,T)} + |\varphi^t(0; w(t))|_{L^1(0,T)})} \\ & \times \left\{ \psi_w^0(u_0) + N_7(1+C_R)^{10}(1+R)^{14} (|X|_{L^1(0,T)} + |Y|_{L^1(0,T)} + |\varphi^t(0; w(t))|_{L^1(0,T)}) \right\}. \end{aligned} \quad (25)$$

Now we show that Q is the self-mapping on $E(T_0, M_0, R_0)$ for some chosen constants $T_0 > 0$, $M_0 > 0$ and $R_0 > 0$.

Note that by $(\Phi 6)$ we have

$$\varphi^t(0; u(t)) \leq 2\varphi^t(w(t); u(t)) + C_R^2 R^2 (= 2\psi_w^t(u(t)) + C_R^2 R^2) \quad (26)$$

for any $w \in E(T)$ with $\sup_{t \in [0, T]} |w(t)|_H \leq R$.

Here, we take $R_0 > 0$, $M_0 > 0$ so large that

$$\begin{aligned}
& 2 \left[\sup_{t \in [0, T]} |h(t)|_H + e^{\frac{T}{2}} |u_0 - h(0)|_H + e^{\frac{T}{2}} N_5^{\frac{1}{2}} \{ |f|_{L^2(0, T; H)} + |h'|_{L^2(0, T; H)} \} \right] \leq R_0, \\
& 4e^{2N_7(1+C_{R_0})^{10}(1+R_0)^{14}} \{ \psi_w^0(u_0) + 2N_7(1+C_{R_0})^{10}(1+R_0)^{14} \} + C_{R_0}^2 R_0^2 + C_h \\
& \leq 4e^{2N_7(1+C_{R_0})^{10}(1+R_0)^{14}} \left\{ 2\varphi^0(0; u_0) + \frac{C_{R_0}^2 R_0^2}{4} + 2N_7(1+C_{R_0})^{10}(1+R_0)^{14} \right\} \\
& \quad + C_{R_0}^2 R_0^2 + C_h \\
& \leq M_0.
\end{aligned}$$

Next, we choose $T_0 > 0$ so small that $T_0 \leq T$, $|h'|_{L^2(0, T_0; H)} \leq M_0$, $|X|_{L^1(0, T_0)} \leq 1$,

$$|Y|_{L^1(0, T_0)} + |\varphi^t(0; w(t))|_{L^1(0, T_0)} \leq T_0^{\frac{1}{2}} M_0^{\frac{1}{2}} + M_0^{\frac{1}{2}} T_0^{\frac{1}{2}} |\alpha'_{R_0}|_{L^2(0, T_0)} + T_0 M_0 \leq 1,$$

$$\begin{aligned}
& \sup_{t \in [0, T_0]} |h(t)|_H + e^{\frac{T_0}{2}} |u_0 - h(0)|_H + e^{\frac{T_0}{2}} N_5^{\frac{1}{2}} \{ |f|_{L^2(0, T_0; H)} + |h'|_{L^2(0, T_0; H)} \} \\
& \quad + e^{\frac{T_0}{2}} N_5^{\frac{1}{2}} T_0^{\frac{1}{2}} \left\{ \sup_{t \in [0, T_0]} \varphi^t(0; h(t))^{\frac{1}{2}} + M_0^{\frac{1}{2}} + C_{R_0} R_0 + 1 \right\} \leq R_0.
\end{aligned}$$

Then, the estimates (19), (25) with (26) implies that $Qw(= u)$ belongs to the set $E(T_0, M_0, R_0)$ for $w \in E(T_0, M_0, R_0)$, thus Q is the self-mapping on $E(T_0, M_0, R_0)$. \square

Lemma 3. *Let $M_0 > 0$, $R_0 > 0$ and $T_0 > 0$ be constants obtained in Lemma 2. Let $\{w_n\} \subset E(T_0, M_0, R_0)$, $w \in E(T_0, M_0, R_0)$ and u_n be the solution of $CP(w_n; u_0)$. Suppose $w_n \rightarrow w$ in $C([0, T_0]; H)$ as $n \rightarrow +\infty$. Then, there is a solution u of $CP(w; u_0)$ on $[0, T_0]$ such that $u \in E(T_0, M_0, R_0)$ and $u_n \rightarrow u$ in $C([0, T_0]; H)$ as $n \rightarrow +\infty$.*

Proof. Since $\{w_n\} \subset E(T_0, M_0, R_0)$ and Lemma 2, we have

$$\sup_{t \in [0, T_0]} \varphi^t(0; u_n(t)) \leq M_0, \quad |u'_n|_{L^2(0, T_0; H)} \leq M_0, \quad \forall n = 1, 2, \dots, \quad (27)$$

$$\sup_{t \in [0, T_0]} |u_n(t)|_H \leq R_0, \quad \forall n = 1, 2, \dots. \quad (28)$$

By $(\Phi 3)$, (27), (28) there are a subsequence $\{n_k\}$ of $\{n\}$ and a function $u \in W^{1,2}(0, T_0; H)$ such that

$$u_{n_k} \rightarrow u \quad \text{strongly in } C([0, T_0]; H), \quad (29)$$

$$u'_{n_k} \rightharpoonup u' \quad \text{weakly in } L^2(0, T_0; H) \quad (30)$$

as $k \rightarrow +\infty$. By $(\Phi 1)$, (27)-(30) and the uniqueness of u_n , we easily observe that $u \in E(T_0, M_0, R_0)$ and $u_n \rightarrow u$ in $C([0, T_0]; H)$ as $n \rightarrow +\infty$.

Now, let us show that u is a solution of $CP(w; u_0)$ on $[0, T_0]$. To do so, we define $\Phi(w; z) = \int_0^{T_0} \varphi^t(w(t); z(t)) dt$. Then by the assumption $(\Phi 6)$ we see that

$$\Phi(w_n; z) \rightarrow \Phi(w; z) \quad \text{as } n \rightarrow +\infty \quad (31)$$

for any $z \in L^2(0, T_0; H)$ with $\varphi^{(\cdot)}(0; z(\cdot)) \in L^1(0, T_0)$. From (27), (29), ($\Phi 1$), ($\Phi 2$), ($\Phi 6$) and the Fatou's lemma, it follows that

$$\begin{aligned} \liminf_{k \rightarrow +\infty} \Phi(w_{n_k}; u_{n_k}) &= \liminf_{k \rightarrow +\infty} \{\Phi(w_{n_k}; u_{n_k}) - \Phi(w; u_{n_k}) + \Phi(w; u_{n_k})\} \\ &\geq \liminf_{k \rightarrow +\infty} \Phi(w; u_{n_k}) \geq \Phi(w; u). \end{aligned} \quad (32)$$

Moreover, by $\{w_n\} \subset E(T_0, M_0, R_0)$ and the demi-closedness (G3) we see that

$$G(\cdot, w_{n_k}(\cdot)) \rightharpoonup G(\cdot, w(\cdot)) \quad \text{weakly in } L^2(0, T_0; H),$$

hence

$$f - G(\cdot, w_{n_k}(\cdot)) \rightharpoonup f - G(\cdot, w(\cdot)) \quad \text{weakly in } L^2(0, T_0; H) \quad (33)$$

as $k \rightarrow +\infty$.

Now, let z be any function in $L^2(0, T_0; H)$ with $\varphi^{(\cdot)}(0; z(\cdot)) \in L^1(0, T_0)$. Since u_{n_k} is the unique solution of $\text{CP}(w_{n_k}; u_0)$, then the following inequality holds:

$$\int_0^{T_0} (f(t) - G(t, w_{n_k}(t)) - u'_{n_k}(t), z(t) - u_{n_k}(t)) dt \leq \Phi(w_{n_k}; z) - \Phi(w_{n_k}; u_{n_k}). \quad (34)$$

Taking account of (29)-(33) and letting $k \rightarrow +\infty$ in (34), we get

$$\int_0^{T_0} (f(t) - G(t, w(t)) - u'(t), z(t) - u(t)) dt \leq \Phi(w; z) - \Phi(w; u),$$

which implies that $f(t) - G(t, w(t)) - u'(t) \in \partial\varphi^t(w(t); u(t))$ for a.e. $t \in [0, T_0]$ (cf. [1, Proposition 3.3]). Thus u is the solution of $\text{CP}(w; u_0)$ on $[0, T_0]$. \square

Proof. [**Proof of Theorem 1; Local existence**] By Lemma 2, we can define a self-mapping $Q : E(T_0, M_0, R_0) \rightarrow E(T_0, M_0, R_0)$ by $Qw = u$ for each $w \in E(T_0, M_0, R_0)$, where u is a solution of $\text{CP}(w; u_0)$. Clearly, $E(T_0, M_0, R_0)$ is compact in $C([0, T_0]; H)$. Moreover, it follows from Lemma 3 that Q is continuous with respect to the topology of $C([0, T_0]; H)$. Therefore, the Schauder's fixed point theorem implies that the self-mapping Q has a fixed point u in $E(T_0, M_0, R_0)$, i.e. $Qu = u$. Clearly u is the solution of $\text{CP}(u_0)$, thus we can construct the local solution u of $\text{CP}(u_0)$ on $[0, T_0]$. \square

4 Proof of Theorem 2

In this section we shall prove Theorem 2, which is concerned with the global existence of solution to $\text{CP}(u_0)$.

First, we consider the inequality (17). By the local existence result in Section 3, we can take $w = u \in E(T_0, M_0, R_0)$, u being the solution of $\text{CP}(u_0)$ on a small time interval $[0, T_0]$ with $0 < T_0 \leq T$. Hence, by taking $w = u$ in (17) it follows from (G2) and the additional assumption ($\Phi 8$) that

$$\begin{aligned} &\frac{d}{dt} |u(t) - h(t)|_H^2 + \varphi^t(u(t); u(t)) \\ &\leq N_8 |u(t) - h(t)|_H^2 + N_9 (|f(t)|_H^2 + |h'(t)|_H^2 + \varphi^t(0; h(t)) + 1) \end{aligned} \quad (35)$$

for some constants $N_8 > 0$ and $N_9 > 0$ depending only on C_1, C_2, C_3, C_4 . By applying Gronwall's inequality to (35), we obtain

$$\begin{aligned} & \sup_{t \in [0, T_0]} |u(t)|_H \\ \leq & \sup_{t \in [0, T]} |h(t)|_H + \sqrt{e^{N_8 T}} |u_0 - h(0)|_H + \sqrt{N_9 e^{N_8 T}} \{ |f|_{L^2(0, T; H)} + |h'|_{L^2(0, T; H)} \} \\ & + \sqrt{N_9 T e^{N_8 T}} \left\{ \sup_{t \in [0, T]} \varphi^t(0; h(t))^{\frac{1}{2}} + 1 \right\} \equiv N_{10}. \end{aligned} \quad (36)$$

Next, take a number $R > 0$ with $R \geq N_{10}$, and we now consider the inequality (23). Applying Schwarz inequality to the term $Y(\tau) \{1 + \psi_w^\tau(u(\tau))\}^{\frac{1}{2}}$ and using (G2), ($\Phi 8$), we obtain

$$\begin{aligned} & \psi_w^t(u(t)) - \psi_w^s(u(s)) + \frac{1}{4} \int_s^t |u'(\tau)|_H^2 d\tau \\ \leq & N_{11}(1 + C_R)^{16}(1 + R)^{24} \int_s^t X(\tau)(1 + \psi_w^\tau(u(\tau))) d\tau + \frac{1}{8} \int_s^t |w'(\tau)|_H^2 d\tau \\ & + N_{12}(1 + C_R)^8(1 + R)^{12} \int_s^t \varphi^\tau(0; w(\tau)) d\tau \end{aligned} \quad (37)$$

for $0 \leq s \leq t \leq T$, where $N_{11} > 0$ and $N_{12} > 0$ depend on C_1, C_2, C_3, C_4, N_6 .

Applying Gronwall's inequality to (37), we obtain

$$\begin{aligned} & \psi_w^t(u(t)) + \frac{1}{4} \int_0^t e^{N_{11}(1+C_R)^{16}(1+R)^{24} \int_\tau^t X(s) ds} |u'(\tau)|_H^2 d\tau \\ \leq & e^{N_{11}(1+C_R)^{16}(1+R)^{24} \int_0^T X(s) ds} \left\{ \psi_w^0(u_0) + N_{11}(1 + C_R)^{16}(1 + R)^{24} \int_0^T X(s) ds \right\} \\ & + \frac{1}{8} \int_0^t e^{N_{11}(1+C_R)^{16}(1+R)^{24} \int_\tau^t X(s) ds} |w'(\tau)|_H^2 d\tau \\ & + N_{12}(1 + C_R)^8(1 + R)^{12} \int_0^t e^{N_{11}(1+C_R)^{16}(1+R)^{24} \int_\tau^t X(s) ds} \varphi^\tau(0; w(\tau)) d\tau. \end{aligned} \quad (38)$$

Here, we can take $w = u \in E(T_0, M_0, R_0)$, u being the solution of $CP(u_0)$ on a small time interval $[0, T_0]$ with $0 < T_0 \leq T$. Then, by using (26), (36), (38) we get

$$\begin{aligned} & \varphi^t(u(t); u(t)) + \frac{1}{8} \int_0^t e^{N_{11}(1+C_R)^{16}(1+R)^{24} \int_\tau^t X(s) ds} |u'(\tau)|_H^2 d\tau \\ \leq & N_{13}(1 + C_R)^{16}(1 + R)^{24} e^{N_{14}(1+C_R)^{16}(1+R)^{24}} \left(1 + \int_0^t \varphi^\tau(u(\tau); u(\tau)) d\tau \right), \end{aligned} \quad (39)$$

for $0 \leq t \leq T_0$, where $N_{13} > 0, N_{14} > 0$ are dependent only on the given data. By applying Gronwall's inequality to (39), we conclude that

$$\begin{aligned} & \varphi^t(u(t); u(t)) + \frac{1}{8} \int_0^{T_0} |u'(t)|_H^2 dt \\ \leq & N_{15}(1 + C_R)^{32}(1 + R)^{48} \exp(N_{16}(1 + C_R)^{16}(1 + R)^{24} e^{N_{14}(1+C_R)^{16}(1+R)^{24}}), \end{aligned} \quad (40)$$

where $N_{15} > 0$ and $N_{16} > 0$ depends only on the given data and are independent of $T_0(\leq T)$ and $R(\geq N_{10})$.

Now we shall prove Theorem 2 by employing the estimates (36) and (40).

Proof. [**Proof of Theorem 2; Global existence**] Assume that

$$T^* := \sup\{T_0; \text{CP}(u_0) \text{ has a solution on } [0, T_0]\} < +\infty.$$

By the local existence result in Section 3, we note $T^* > 0$. By the definition of T^* , there is a function $u : [0, T^*) \rightarrow H$ such that for any $T_0 (< T^*)$ u is the solution of $\text{CP}(u_0)$ on $[0, T_0]$. By (36) and (40) we have

$$u \in W^{1,2}(0, T^*; H), \quad \varphi^{(\cdot)}(u(\cdot); u(\cdot)) \in L^\infty(0, T^*).$$

Hence by assumptions $(\Phi 1)$, $(\Phi 3)$, $(\Phi 5)$, $(\Phi 6)$, we observe that the limit $u_0^* := \lim_{t \uparrow T^*} u(t)$ exists strongly in H such that

$$u_0^* \in D(\varphi^{T^*}(0; \cdot)).$$

Now, taking u_0^* as the initial value at $t = T^*$, we can get the solution u beyond the time interval $[0, T^*]$. Thus we observe that the solution to $\text{CP}(u_0)$ exists on the whole time interval $[0, T]$. \square

5 Application to a double obstacle problem

In this section we apply our abstract results (Theorems 1, 2, 3) to a parabolic variational inequality with time-dependent double obstacles.

Let Ω be a bounded domain in \mathbb{R}^N ($N \geq 1$) with smooth boundary. Let g_1, g_2 be prescribed obstacle functions on $[0, T] \times \Omega$ so that

$$g_i \in L^\infty(0, T; H^1(\Omega)) \cap L^\infty([0, T] \times \Omega), \quad g'_i \in L^2(0, T; H^1(\Omega)) \cap L^2(0, T; L^\infty(\Omega))$$

for $i = 1, 2$, and

$$g_2 - g_1 \geq C_g \quad \text{a.e. on } [0, T] \times \Omega \text{ for some constant } C_g > 0.$$

For each $t \in [0, T]$, we define the convex set $K(t)$ by

$$K(t) := \{z \in H^1(\Omega); g_1(t) \leq z \leq g_2(t) \text{ a.e. on } \Omega\}.$$

Now, let us consider the following interior time-dependent double obstacle problem.

Problem (P): Find a function $u \in W^{1,2}(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$ such that

$$u(t) \in K(t) \quad \text{for a.e. } t \in [0, T],$$

$$(u'(t) + b(t, \cdot, u(t)) - f(t), u(t) - z) + \int_{\Omega} a(x, u(t), \nabla u(t)) \cdot \nabla(u(t) - z) dx \leq 0$$

$$\text{for all } z \in K(t),$$

$$u(0) = u_0 \quad \text{in } \Omega,$$

where (\cdot, \cdot) is a usual inner product of $L^2(\Omega)$, $a = (a_1, \dots, a_N)$ is an elliptic vector field, b and f are given functions.

The aim of this section is to consider the problem (P) as an application of the abstract evolution equation CP(u_0). To do so, we suppose that

- (A1) $a(x, s, p)$ is continuous on $\Omega \times \mathbb{R} \times \mathbb{R}^N$ such that $a(x, s, p) = \partial_p A(x, s, p)$ for some potential function $A(x, s, p)$. Moreover, there exist constants $\mu > 0$, $\nu_1 = \nu_1(a) > 0$ and $\nu_2 = \nu_2(a) > 0$ such that

$$\begin{aligned} [a(x, s, p) - a(x, s, \hat{p})] \cdot (p - \hat{p}) &\geq \mu |p - \hat{p}|^2, \\ |a(x, s, p)|^2 + |A(x, s, p)| + |\partial_s A(x, s, p)|^2 &\leq \nu_1 (1 + |s|^2 + |p|^2), \\ |a(x, s, p) - a(x, \hat{s}, p)| &\leq \nu_2 (1 + |p|) |s - \hat{s}| \end{aligned}$$

for all $x \in \Omega$, $s, \hat{s} \in \mathbb{R}$, $p, \hat{p} \in \mathbb{R}^N$.

- (A2) $b(t, x, s)$ is continuous on $[0, T] \times \Omega \times \mathbb{R}$ satisfying the following properties: there exist a constant $L_b > 0$ and a function $d \in L^1(0, T)$ such that

$$|b(t, x, s) - b(t, x, \hat{s})| \leq L_b |s - \hat{s}|, \quad \forall t \in [0, T], \forall x \in \Omega, \forall s, \hat{s} \in \mathbb{R},$$

$$\sup_{x \in \Omega} \left| \frac{\partial}{\partial t} b(t, x, 0) \right| \leq d(t) \quad \text{for a.e. } t \geq 0.$$

As a direct application of Theorems 1, 2 and 3, we have:

Proposition 2. *Assume (A1) and (A2). Then, for each $f \in L^2(0, T; L^2(\Omega))$ and $u_0 \in K(0)$, the problem (P) has a unique solution u on $[0, T]$.*

Proof. To apply Theorems 1, 2 and 3 to the problem (P), we choose $L^2(\Omega)$ as a real Hilbert space H , and define a function $\varphi^t(\cdot; \cdot) : L^2(\Omega) \times L^2(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$ by

$$\varphi^t(w; z) := \begin{cases} \int_{\Omega} A(x, w(x), \nabla z(x)) dx + C_{\mu} (1 + |w|_{L^2(\Omega)}^2), & \text{if } z \in K(t), \\ +\infty, & \text{otherwise,} \end{cases}$$

where $C_{\mu} > 0$ is a constant such that $\varphi^t(w; z) \geq \frac{\mu}{4} |z|_{L^2(\Omega)}^2 + 1$ for all $t \geq 0$, $w \in L^2(\Omega)$ and $z \in K(t)$ (cf. [13, Lemma 3.1]).

Let us define an operator $G(t, \cdot) : L^2(\Omega) \rightarrow L^2(\Omega)$ by $G(t, z) := b(t, \cdot, z(\cdot))$ in $L^2(\Omega)$. And we define a function γ by $\gamma(z) := \int_{\Omega} z^+(x) dx$ for $z \in L^2(\Omega)$, where $z^+ := \max\{z, 0\}$.

Now we put for any $t \in [0, T]$ and $r > 0$

$$\alpha_r(t) = k \int_0^t \{ |g'_1|_{L^\infty(\Omega)} + |g'_2|_{L^\infty(\Omega)} + |g'_1|_{H^1(\Omega)} + |g'_2|_{H^1(\Omega)} \} d\tau,$$

where $k > 0$ is a (sufficient large) positive constant. Then, we easily verify $\{\varphi^t\} \in \Phi_\gamma(\{\alpha_r\})$. For instance, we can show $(\Phi 5)$ by taking

$$\tilde{z} := (z - g_1(s)) \frac{g_2(t) - g_1(t)}{g_2(s) - g_1(s)} + g_1(t)$$

for given $z \in K(s)$. Then, by the slight modification of [22, Lemma 5.1], we can show $(\Phi 5)$.

Moreover we easily see that $G(t, \cdot) \in \mathcal{G}_\gamma(\{\varphi^t\})$ and the assumption $(\Phi 8)$ hold.

Clearly, the problem (P) can be reformulated in the evolution equation $CP(u_0)$. Thus, by applying Theorems 1, 2 and 3, we see that (P) has a unique global solution u . \square

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