Weighted $L^p$ Sobolev-Lieb-Thirring inequalities

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Abstract: We give a weighted $L^p$ version of the Sobolev-Lieb-Thirring inequality for suborthonormal functions.

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1 Introduction

In 1976 Lieb and Thirring proved the following inequality.

Theorem 1.1 ([4]). Let $n \in \mathbb{N}$. Then there exists a positive constant $c_n$ such that for every family $\{\phi_i\}_{i=1}^N$ in $H^1(\mathbb{R}^n)$ which is orthonormal in $L^2(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} \left( \sum_{i=1}^N |\phi_i(x)|^2 \right)^{1+2/n} dx \leq c_n \sum_{i=1}^N \|\nabla \phi_i\|^2.$$  

In this theorem $H^1(\mathbb{R}^n)$ denotes the Sobolev space and $\| \cdot \|$ is the norm of $L^2(\mathbb{R}^n)$. In [4] Lieb and Thirring applied this inequality to the problem of the stability of matter. Ghidaglia, Marion, and Temam proved a generalization of (1) under the suborthonormal condition on $\{\phi_i\}$, where $\{\phi_i\}_{i=1}^N$ in $L^2(\mathbb{R}^n)$ is called suborthonormal if the inequality

$$\sum_{i,j=1}^N \xi_i \xi_j (\phi_i, \phi_j) \leq \sum_{i=1}^N |\xi_i|^2$$

holds for all $\xi_i \in \mathbb{C}, i = 1, \ldots, N$, where $(\cdot, \cdot)$ means the $L^2$ inner product([2]). They applied the inequality (1) to the estimate of the dimension of attractors associated with partial differential equations. In this paper we shall give a weighted $L^p$ version of (1) under the suborthonormal condition on $\{\phi_i\}$. 

\[1\]
For the statement of our result we need to recall the definition of $A_p$-weights (c.f. [3], [5]). By a cube in $\mathbb{R}^n$ we mean a cube which sides are parallel to coordinate axes. Let $w$ be a non-negative, locally integrable function on $\mathbb{R}^n$. We say that $w$ is an $A_p$-weight for $1 < p < \infty$ if there exists a positive constant $C$ such that

$$\frac{1}{|Q|} \int_Q w(x) \, dx \left( \frac{1}{|Q|} \int_Q w(x)^{1/(p-1)} \, dx \right)^{p-1} \leq C$$

for all cubes $Q \subset \mathbb{R}^n$. For example, $w(x) = |x|^\alpha$ is an $A_p$-weight when $-n < \alpha < n(p-1)$.

We say that $w$ is an $A_1$-weight if there exists a positive constant $C$ such that

$$\frac{1}{|Q|} \int_Q w(y) \, dy \leq C w(x) \quad a.e. \ x \in Q$$

for all cubes $Q \subset \mathbb{R}^n$. If $-n < \alpha \leq 0$, then $w(x) = |x|^\alpha$ is an $A_1$-weight. Let $A_p$ be the class of $A_p$-weights. The inclusion $A_p \subset A_q$ holds for $p < q$.

A nonnegative, locally integrable function $w$ on $\mathbb{R}^n$ is called a weight function. For a weight function $w$ we define

$$L^p(w) = \left\{ f : \text{measurable on } \mathbb{R}^n, \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx < \infty \right\}.$$

The following is a conclusion of [7, Theorem 1.2] and [6, Lemma 3.2].

**Theorem 1.2.** Let $n \in \mathbb{N}$, $3 \leq n$, $w \in A_p$, and $w^{-n/2} \in A_{n/2}$. Then there exists a positive constant $c$ such that for every family $\{\phi_i\}_{i=1}^N$ in $L^2(\mathbb{R}^n)$ which is suborthonormal in $L^2(\mathbb{R}^n)$ and $|\nabla \phi_i| \in L^2(w)$, $i = 1, \ldots, N$, we have

$$\int_{\mathbb{R}^n} \left( \sum_{i=1}^N |\phi_i(x)|^2 \right)^{1+2/n} w(x) \, dx \leq c \sum_{i=1}^N \int_{\mathbb{R}^n} |\nabla \phi_i(x)|^2 w(x) \, dx,$$

where $c$ depends only on $n$ and $w$.

By using this theorem we can prove the following weighted $L^p$ version of the Sobolev-Lieb-Thirring inequality.

**Theorem 1.3.** Let $n \in \mathbb{N}$ and $3 \leq n$. Let $2n/(n+2) < p < n$, $p \neq 2$, and $w$ be a weight function. When $p > 2$, we assume that $w^{n/(n-p)} \in A_{p(n-2)/(2(n-p))}$. When $p < 2$, we assume that $w^{n/(n-2)} \in A_1$.

Then there exists a positive constant $c$ such that for every family $\{\phi_i\}_{i=1}^N$ in $L^2(\mathbb{R}^n)$ which is suborthonormal in $L^2(\mathbb{R}^n)$ and $|\nabla \phi_i| \in L^p(w)$, $i = 1, \ldots, N$, we have

$$\int_{\mathbb{R}^n} \left( \sum_{i=1}^N |\phi_i(x)|^2 \right)^{(1+2/n)p/2} w(x) \, dx \leq c \int_{\mathbb{R}^n} \left( \sum_{i=1}^N |\nabla \phi_i(x)|^2 \right)^{p/2} w(x) \, dx,$$

where $c$ depends only on $n, p$ and $w$.  

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This is a new result even in the case $w \equiv 1$. When $2 < p < n$, an example of $w$ is given by $w(x) = |x|^\alpha$, $-n + p < \alpha < n(p - 2)/2$. When $2n/(n+2) < p < 2$, an example of $w$ is given by $w(x) = |x|^\alpha$, $-n + 2 < \alpha \leq 0$.

\section{Proof of Theorem 1.3}

Let $M$ be the Hardy-Littlewood maximal operator, that is,
\[ M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \, dy, \]
where $f$ is a locally integrable function on $\mathbb{R}^n$ and the supremum is taken over all cubes $Q$ which contain $x$. The following proposition is proved in \cite[Chapter IV]{3} or \cite[Chapter V]{5}.

\textbf{Proposition 2.1.}

(i) \ Let $1 < p < \infty$ and $w$ be a weight function on $\mathbb{R}^n$. Then there exists a positive constant $c$ such that
\[ \int_{\mathbb{R}^n} M(f)^p w \, dx \leq c \int_{\mathbb{R}^n} |f|^p w \, dx \]
for all $f \in L^p(w)$ if and only if $w \in A_p$.

(ii) \ Let $1 < p < \infty$ and $w \in A_p$. Then there exists a $q \in (1, p)$ such that $w \in A_q$.

(iii) \ Let $0 < \tau < 1$ and $f$ be a locally integrable function on $\mathbb{R}^n$ such that $M(f)(x) < \infty$ a.e.. Then $M(f)^\tau \in A_1$.

(iv) \ Let $1 < p < \infty$. Then $w \in A_p$ if and only if $w^{1-\tau} \in A_p$, where $p^{-1} + p^{-1} = 1$.

(v) \ Let $1 < p < \infty$ and $1 < w_1, w_2 < 2$. Then $w_1 w_2^{1-p} \in A_p$.

\textbf{Proof of Theorem 1.3}

Our proof is very similar to that of the extrapolation theorem in harmonic analysis (c.f.\cite[Theorem 7.8]{1}). In our proof the integral means that over $\mathbb{R}^n$.

Let $2 < p < n$ and $2/p + 1/q = 1$. We remark that the assumption $w^{n/(n-p)} \in A_{p(n-2)/(2(n-p))}$ leads to $w \in A_p$ by an easy calculation. Let $u \in L^q(w)$, $u \geq 0$, and $\|u\|_{L^q(w)} = 1$. Since $w^{n/(n-p)} \in A_{p(n-2)/(2(n-p))}$, we have $w^{-2/(p-2)} \in A_{p(n-2)/(n(p-2))}$ by (iv) of Proposition 2.1. Hence there exists a $\gamma$ such that $n/(n - 2) < \gamma < q$ and $w^{-2/(p-2)} \in A_{p/(\gamma(p-2))}$ by (ii) of Proposition 2.1. Then we have $uw \leq M((uw)^\gamma)^{1/\gamma}$ a.e.. Because
\[ w^{-2q/p} = w^{-2/(p-2)} \in A_{p/(\gamma(p-2))} = A_{q/\gamma} \]
and
\( (2) \quad \int M((uw)^\gamma)^{n/\gamma}w^{-2q/p} \, dx \leq c \int (uw)^q w^{-2q/p} \, dx = c \int w^q \, dx = c \)

by (i) of Proposition 2.1, we get \( M((uw)^\gamma)(x) < \infty \) a.e.. Hence \( M((uw)^\gamma)^{1/\gamma} \in A_1 \) by (iii) of Proposition 2.1. Let \( \alpha = \frac{n}{(n-2)\gamma} \). Then \( 0 < \alpha < 1 \) and
\[ M((uw)^\gamma)^{-n/(2\gamma)} = \{M((uw)^\gamma)^\alpha\}^{1-n/2} \in A_{n/2}, \]
where we used \( M((uw)^\gamma)^\alpha \in A_1 \) and (v) of Proposition 2.1. Let
\[ \rho(x) = \sum_{i=1}^N |\phi_i(x)|^2. \]

Then we have
\[
\int \rho^{1+2/n}uw \, dx \leq \int \rho^{1+2/n}M((uw)^\gamma)^{1/\gamma} \, dx \leq c \int \left( \sum_{i=1}^N |\nabla \phi_i|^2 \right)^{p/2} w \, dx \left( \int M((uw)^\gamma)^{n/\gamma}w^{-2q/p} \, dx \right)^{1/q} \]
\[
\leq c \left( \int \left( \sum_{i=1}^N |\nabla \phi_i|^2 \right)^{p/2} w \, dx \right)^{2/p} \left( \int M((uw)^\gamma)^{n/\gamma}w^{-2q/p} \, dx \right)^{1/q} \]
where we used Theorem 1.2 and (2). If we take the supremum for all \( u \in L^q(w), \ u \geq 0, \) and \( \|u\|_{L^q(w)} = 1, \) then we get
\[
\left( \int \rho^{(1+2/n)p/2}w \, dx \right)^{2/p} \leq c \left( \int \left( \sum_{i=1}^N |\nabla \phi_i|^2 \right)^{p/2} w \, dx \right)^{2/p}. \]

Next we consider the case \( 2n/(n+2) < p < 2. \) We remark that \( w \in A_1 \) by the assumption \( w^{n/(n-2)} \in A_1. \) Let
\[ f = \left( \sum_{i=1}^N |\nabla \phi_i|^2 \right)^{1/2}. \]

We can take \( \gamma \) such that \( (2 - p)n/2 < \gamma < p. \) Then
\[
\int M(f^{\gamma})^{p/\gamma}w \, dx \leq c \int f^p w \, dx < \infty, \]
where we used \( w \in A_1 \subset A_{p/\gamma} \) and (i) of Proposition 2.1. Hence we have \( M(f^{\gamma})(x) < \infty \) a.e. and
\[ M(f^{\gamma})^{(2-p)n/(2\gamma)} \in A_1. \]

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by (iii) of Proposition 2.1. Furthermore we have

\[ M(f^\gamma)^{-2-p/\gamma}w \in A_2, \]

where we used

\[ M(f^\gamma)^{(2-p)/\gamma} \in A_1, \quad w \in A_1, \]

and (v) of Proposition 2.1. Moreover

\[ \{M(f^\gamma)^{-(2-p)/\gamma}w\}^{-n/2} = M(f^\gamma)^{(2-p)n/(2\gamma)}(w^{n/(n-2)}(1-n/2) \in A_{n/2} \]

because \( w^{n/(n-2)} \in A_1 \). Therefore

\[
\int \rho^{(1+2/n)p/2}w \, dx = \int \rho^{(1+2/n)p/2}wM(f^\gamma)^{-(2-p)p/(2\gamma)}M(f^\gamma)^{(2-p)p/(2\gamma)} \, dx \\
\leq \left( \int \rho^{1+2/n}M(f^\gamma)^{-(2-p)/\gamma}w \, dx \right)^{p/2} \left( \int M(f^\gamma)^{p/\gamma}w \, dx \right)^{1-p/2} \\
\leq c \left( \int f^2M(f^\gamma)^{-(2-p)/\gamma}w \, dx \right)^{p/2} \left( \int f^p w \, dx \right)^{1-p/2} \\
\leq c \left( \int M(f^\gamma)^{2/\gamma}M(f^\gamma)^{-(2-p)/\gamma}w \, dx \right)^{p/2} \left( \int f^p w \, dx \right)^{1-p/2} \leq c \int f^p w \, dx,
\]

where we used Theorem 1.2 in the second inequality.

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References


