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Weighted $L^p$ Sobolev-Lieb-Thirring inequalities

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Abstract: We give a weighted $L^p$ version of the Sobolev-Lieb-Thirring inequality for suborthonormal functions.

Key words: Sobolev-Lieb-Thirring inequalities; $A_p$-weights.

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1 Introduction

In 1976 Lieb and Thirring proved the following inequality.

Theorem 1.1 ([4]). Let $n \in \mathbb{N}$. Then there exists a positive constant $c_n$ such that for every family $\{\phi_i\}_{i=1}^N$ in $H^1(\mathbb{R}^n)$ which is orthonormal in $L^2(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} \left( \sum_{i=1}^N |\phi_i(x)|^2 \right)^{1+2/n} \, dx \leq c_n \sum_{i=1}^N \|\nabla \phi_i\|^2.$$

In this theorem $H^1(\mathbb{R}^n)$ denotes the Sobolev space and $\|\cdot\|$ is the norm of $L^2(\mathbb{R}^n)$. In [4] Lieb and Thirring applied this inequality to the problem of the stability of matter. Ghidaglia, Marion, and Temam proved a generalization of (1) under the suborthonormal condition on $\{\phi_i\}$, where $\{\phi_i\}_{i=1}^N$ in $L^2(\mathbb{R}^n)$ is called suborthonormal if the inequality

$$\sum_{i,j=1}^N \xi_i \xi_j (\phi_i, \phi_j) \leq \sum_{i=1}^N |\xi_i|^2$$

holds for all $\xi_i \in \mathbb{C}, i = 1, \ldots, N$, where $(\cdot, \cdot)$ means the $L^2$ inner product([2]). They applied the inequality (1) to the estimate of the dimension of attractors associated with partial differential equations. In this paper we shall give a weighted $L^p$ version of (1) under the suborthonormal condition on $\{\phi_i\}$.
For the statement of our result we need to recall the definition of $A_p$-weights (c.f. [3], [5]). By a cube in $\mathbb{R}^n$ we mean a cube which sides are parallel to coordinate axes. Let $w$ be a non-negative, locally integrable function on $\mathbb{R}^n$. We say that $w$ is an $A_p$-weight for $1 < p < \infty$ if there exists a positive constant $C$ such that

$$\frac{1}{|Q|} \int_Q w(x) \, dx \left( \frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} \, dx \right)^{p-1} \leq C$$

for all cubes $Q \subset \mathbb{R}^n$. For example, $w(x) = |x|^\alpha$ is an $A_p$-weight when $-n < \alpha < n(p-1)$.

We say that $w$ is an $A_1$-weight if there exists a positive constant $C$ such that

$$\frac{1}{|Q|} \int_Q w(y) \, dy \leq Cw(x) \quad \text{a.e. } x \in Q$$

for all cubes $Q \subset \mathbb{R}^n$. If $-n < \alpha \leq 0$, then $w(x) = |x|^\alpha$ is an $A_1$-weight. Let $A_p$ be the class of $A_p$-weights. The inclusion $A_p \subset A_q$ holds for $p < q$.

A nonnegative, locally integrable function $w$ on $\mathbb{R}^n$ is called a weight function. For a weight function $w$ we define

$$L^p(w) = \left\{ f : \text{measurable on } \mathbb{R}^n, \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx < \infty \right\}.$$

The following is a conclusion of [7, Theorem 1.2] and [6, Lemma 3.2].

**Theorem 1.2.** Let $n \in \mathbb{N}$, $3 \leq n$, $w \in A_2$, and $w^{-n/2} \in A_{n/2}$. Then there exists a positive constant $c$ such that for every family $\{\phi_i\}_{i=1}^N$ in $L^2(\mathbb{R}^n)$ which is suborthonormal in $L^2(\mathbb{R}^n)$ and $|\nabla \phi_i| \in L^2(w)$, $i = 1, \ldots, N$, we have

$$\int_{\mathbb{R}^n} \left( \sum_{i=1}^N |\phi_i(x)|^2 \right)^{1+2/n} w(x) \, dx \leq c \sum_{i=1}^N \int_{\mathbb{R}^n} |\nabla \phi_i(x)|^2 w(x) \, dx,$$

where $c$ depends only on $n$ and $w$.

By using this theorem we can prove the following weighted $L^p$ version of the Sobolev-Lieb-Thirring inequality.

**Theorem 1.3.** Let $n \in \mathbb{N}$ and $3 \leq n$. Let $2n/(n+2) < p < n$, $p \neq 2$, and $w$ be a weight function. When $p > 2$, we assume that $w^{n/(n-p)} \in A_{p(n-2)/(2(n-p))}$. When $p < 2$, we assume that $w^{n/(n-2)} \in A_1$.

Then there exists a positive constant $c$ such that for every family $\{\phi_i\}_{i=1}^N$ in $L^2(\mathbb{R}^n)$ which is suborthonormal in $L^2(\mathbb{R}^n)$ and $|\nabla \phi_i| \in L^p(w)$, $i = 1, \ldots, N$, we have

$$\int_{\mathbb{R}^n} \left( \sum_{i=1}^N |\phi_i(x)|^2 \right)^{(1+2/n)p/2} w(x) \, dx \leq c \int_{\mathbb{R}^n} \left( \sum_{i=1}^N |\nabla \phi_i(x)|^2 \right)^{p/2} w(x) \, dx,$$

where $c$ depends only on $n, p$ and $w$.  

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This is a new result even in the case $w \equiv 1$. When $2 < p < n$, an example of $w$ is given by $w(x) = |x|^\alpha$, $-n + p < \alpha < n(p-2)/2$. When $2n/(n+2) < p < 2$, an example of $w$ is given by $w(x) = |x|^\alpha$, $-n + 2 < \alpha \leq 0$.

## 2 Proof of Theorem 1.3

Let $M$ be the Hardy-Littlewood maximal operator, that is,

$$M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \, dy,$$

where $f$ is a locally integrable function on $\mathbb{R}^n$ and the supremum is taken over all cubes $Q$ which contain $x$. The following proposition is proved in [3, Chapter IV] or [5, Chapter V].

**Proposition 2.1.**

(i) Let $1 < p < \infty$ and $w$ be a weight function on $\mathbb{R}^n$. Then there exists a positive constant $c$ such that

$$\int_{\mathbb{R}^n} M(f)^pw \, dx \leq c \int_{\mathbb{R}^n} |f|^pw \, dx$$

for all $f \in L^p(w)$ if and only if $w \in A_p$.

(ii) Let $1 < p < \infty$ and $w \in A_p$. Then there exists a $q \in (1, p)$ such that $w \in A_q$.

(iii) Let $0 < \tau < 1$ and $f$ be a locally integrable function on $\mathbb{R}^n$ such that $M(f)(x) < \infty$ a.e.. Then $M(f)^\tau \in A_1$.

(iv) Let $1 < p < \infty$. Then $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$, where $p^{-1} + p'^{-1} = 1$.

(v) Let $1 < p < \infty$ and $w_1, w_2 \in A_1$. Then $w_1w_2^{1-p} \in A_p$.

### Proof of Theorem 1.3

Our proof is very similar to that of the extrapolation theorem in harmonic analysis (c.f. [1, Theorem 7.8]). In our proof the integral means that over $\mathbb{R}^n$.

Let $2 < p < n$ and $2/p + 1/q = 1$. We remark that the assumption $w^{n/(n-p)} \in A_{p(n-2)/(2(n-p))}$ leads to $w \in A_p$ by an easy calculation. Let $u \in L^q(w)$, $u \geq 0$, and $\|u\|_{L^q(w)} = 1$. Since $w^{n/(n-p)} \in A_{p(n-2)/(2(n-p))}$, we have $w^{-2/(p-2)} \in A_{p(n-2)/(n(p-2))}$ by (iv) of Proposition 2.1. Hence there exists a $\gamma$ such that $n/(n-2) < \gamma < q$ and $w^{-2/(p-2)} \in A_{p/(\gamma(p-2))}$ by (ii) of Proposition 2.1. Then we have $uw \leq M((uw)^\gamma)^{1/\gamma}$ a.e.. Because

$$w^{-2q/p} = w^{-2/(p-2)} \in A_{p/(\gamma(p-2))} = A_{q/\gamma}$$
and

$$\int M((uw)^\gamma)q^2 dx \leq c \int \rho^{1/\gamma}w^{-2a/p} dx = c \int w^q dx = c$$

by (i) of Proposition 2.1, we get $M((uw)^\gamma)(x) < \infty$ a.e.. Hence $M((uw)^\gamma)^{1/\gamma} \in A_1$ by (iii) of Proposition 2.1. Let $\alpha = \frac{n}{(n-2)_\gamma}$. Then $0 < \alpha < 1$ and

$$M((uw)^\gamma)^{-n/(2\gamma)} = \{M((uw)^\gamma)^\alpha\}^{1-n/2} \in A_{n/2},$$

where we used $M((uw)^\gamma)^\alpha \in A_1$ and (v) of Proposition 2.1. Let

$$\rho(x) = \sum_{i=1}^N |\phi_i(x)|^2.$$

Then we have

$$\int \rho^{1+2/n}uw dx \leq \int \rho^{1+2/n}M((uw)^\gamma)^{1/\gamma} dx \leq c \int \left( \sum_{i=1}^N |\nabla \phi_i|^2 \right) M((uw)^\gamma)^{1/\gamma} dx$$

$$\leq c \left( \int \left( \sum_{i=1}^N |\nabla \phi_i|^2 \right)^{p/2} w dx \right)^{2/p} \left( \int M((uw)^\gamma)^{q/\gamma}w^{-2a/p} dx \right)^{1/q}$$

$$\leq c \left( \int \left( \sum_{i=1}^N |\nabla \phi_i|^2 \right)^{p/2} w dx \right)^{2/p}$$

where we used Theorem 1.2 and (2). If we take the supremum for all $u \in L^q(w)$, $u \geq 0$, and $\|u\|_{L^q(w)} = 1$, then we get

$$\left( \int \rho^{(1+2/n)p/2}w dx \right)^{2/p} \leq c \left( \int \left( \sum_{i=1}^N |\nabla \phi_i|^2 \right)^{p/2} w dx \right)^{2/p}.$$

Next we consider the case $2n/(n+2) < p < 2$. We remark that $w \in A_1$ by the assumption $w^{n/(n-2)} \in A_1$. Let

$$f = \left( \sum_{i=1}^N |\nabla \phi_i|^2 \right)^{1/2}.$$

We can take $\gamma$ such that $(2-p)n/2 < \gamma < p$. Then

$$\int M(f^\gamma)^{p/\gamma}w dx \leq c \int f^pw dx < \infty,$$

where we used $w \in A_1 \subset A_{p/\gamma}$ and (i) of Proposition 2.1. Hence we have $M(f^\gamma)(x) < \infty$ a.e. and

$$M(f^\gamma)^{(2-p)n/(2\gamma)} \in A_1,$$

Next, we consider the case $2n/(n+2) < p < 2$. We remark that $w \in A_1$ by the assumption $w^{n/(n-2)} \in A_1$. Let

$$f = \left( \sum_{i=1}^N |\nabla \phi_i|^2 \right)^{1/2}.$$
by (iii) of Proposition 2.1. Furthermore we have

$$M(f^\gamma)^{-\frac{2-p}{\gamma}}w \in A_2,$$

where we used

$$M(f^\gamma)^{(2-p)/\gamma} \in A_1, \quad w \in A_1,$$

and (v) of Proposition 2.1. Moreover

$$\{M(f^\gamma)^{-(2-p)/\gamma}w\}^{-n/2} = M(f^\gamma)^{(2-p)n/(2\gamma)}(w^{n/(n-2)})^{(1-n/2)} \in A_{n/2}$$

because $w^{n/(n-2)} \in A_1$. Therefore

$$\int \rho^{(1+2/n)p/2}w dx = \int \rho^{(1+2/n)p/2}wM(f^\gamma)^{-(2-p)p/(2\gamma)}M(f^\gamma)^{(2-p)p/(2\gamma)} dx$$

$$\leq \left( \int \rho^{1+2/n}M(f^\gamma)^{-(2-p)/\gamma}w dx \right)^{p/2} \left( \int M(f^\gamma)^{p/\gamma}w dx \right)^{1-p/2}$$

$$\leq c \left( \int f^2M(f^\gamma)^{-(2-p)/\gamma}w dx \right)^{p/2} \left( \int f^p w dx \right)^{1-p/2}$$

$$\leq c \left( \int M(f^\gamma)^{2/\gamma}M(f^\gamma)^{-(2-p)/\gamma}w dx \right)^{p/2} \left( \int f^p w dx \right)^{1-p/2} \leq c \int f^p w dx,$$

where we used Theorem 1.2 in the second inequality.

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