Weighted $L^p$ Sobolev-Lieb-Thirring inequalities

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Abstract: We give a weighted $L^p$ version of the Sobolev-Lieb-Thirring inequality for suborthonormal functions.

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1 Introduction

In 1976 Lieb and Thirring proved the following inequality.

Theorem 1.1 ([4]). Let $n \in \mathbb{N}$. Then there exists a positive constant $c_n$ such that for every family $\{\phi_i\}_{i=1}^N$ in $H^1(\mathbb{R}^n)$ which is orthonormal in $L^2(\mathbb{R}^n)$, we have

$$ \left( \int_{\mathbb{R}^n} \left( \sum_{i=1}^N |\phi_i(x)|^2 \right)^{1+2/n} dx \right)^{1/n} \leq c_n \sum_{i=1}^N \|\nabla \phi_i\|^2. $$

In this theorem $H^1(\mathbb{R}^n)$ denotes the Sobolev space and $\| \cdot \|$ is the norm of $L^2(\mathbb{R}^n)$. In [4] Lieb and Thirring applied this inequality to the problem of the stability of matter. Ghidaglia, Marion, and Temam proved a generalization of (1) under the suborthonormal condition on $\{\phi_i\}$, where $\{\phi_i\}_{i=1}^N$ in $L^2(\mathbb{R}^n)$ is called suborthonormal if the inequality

$$ \sum_{i,j=1}^N \xi_i \bar{\xi}_j (\phi_i, \phi_j) \leq \sum_{i=1}^N |\xi_i|^2 $$

holds for all $\xi_i \in \mathbb{C}, i = 1, \ldots, N$, where $(\cdot, \cdot)$ means the $L^2$ inner product([2]). They applied the inequality (1) to the estimate of the dimension of attractors associated with partial differential equations. In this paper we shall give a weighted $L^p$ version of (1) under the suborthonormal condition on $\{\phi_i\}$.
For the statement of our result we need to recall the definition of \( A_p \)-weights (c.f. [3], [5]). By a cube in \( \mathbb{R}^n \) we mean a cube which sides are parallel to coordinate axes. Let \( w \) be a non-negative, locally integrable function on \( \mathbb{R}^n \). We say that \( w \) is an \( A_p \)-weight for \( 1 < p < \infty \) if there exists a positive constant \( C \) such that
\[
\frac{1}{|Q|} \int_Q w(x) \, dx \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} \, dx\right)^{p-1} \leq C
\]
for all cubes \( Q \subset \mathbb{R}^n \). For example, \( w(x) = |x|^\alpha \) is an \( A_p \)-weight when \( -n < \alpha < n(p-1) \).

We say that \( w \) is an \( A_1 \)-weight if there exists a positive constant \( C \) such that
\[
\frac{1}{|Q|} \int_Q w(y) \, dy \leq C w(x) \quad \text{a.e. } x \in Q
\]
for all cubes \( Q \subset \mathbb{R}^n \). If \( -n < \alpha \leq 0 \), then \( w(x) = |x|^\alpha \) is an \( A_1 \)-weight. Let \( A_p \) be the class of \( A_p \)-weights. The inclusion \( A_p \subset A_q \) holds for \( p < q \).

A nonnegative, locally integrable function \( w \) on \( \mathbb{R}^n \) is called a weight function. For a weight function \( w \) we define
\[
L^p(w) = \left\{ f : \text{measurable on } \mathbb{R}^n, \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx < \infty \right\}.
\]

The following is a conclusion of [7, Theorem 1.2] and [6, Lemma 3.2].

**Theorem 1.2.** Let \( n \in \mathbb{N}, 3 \leq n, w \in A_2, \) and \( w^{-n/2} \in A_{n/2} \). Then there exists a positive constant \( c \) such that for every family \( \{\phi_i\}_{i=1}^N \) in \( L^2(\mathbb{R}^n) \) which is suborthonormal in \( L^2(\mathbb{R}^n) \) and \( |\nabla \phi_i| \in L^2(w) \), \( i = 1, \ldots, N \), we have
\[
\int_{\mathbb{R}^n} \left( \sum_{i=1}^N |\phi_i(x)|^2 \right)^{1+2/n} w(x) \, dx \leq c \sum_{i=1}^N \int_{\mathbb{R}^n} |\nabla \phi_i(x)|^2 w(x) \, dx,
\]
where \( c \) depends only on \( n \) and \( w \).

By using this theorem we can prove the following weighted \( L^p \) version of the Sobolev-Lieb-Thirring inequality.

**Theorem 1.3.** Let \( n \in \mathbb{N} \) and \( 3 \leq n \). Let \( 2n/(n+2) < p < n, p \neq 2 \), and \( w \) be a weight function. When \( p > 2 \), we assume that \( w^{n/(n-p)} \in A_{p(n-2)/(2(n-p))} \). When \( p < 2 \), we assume that \( w^{n/(n-2)} \in A_1 \).

Then there exists a positive constant \( c \) such that for every family \( \{\phi_i\}_{i=1}^N \) in \( L^2(\mathbb{R}^n) \) which is suborthonormal in \( L^2(\mathbb{R}^n) \) and \( |\nabla \phi_i| \in L^p(w) \), \( i = 1, \ldots, N \), we have
\[
\int_{\mathbb{R}^n} \left( \sum_{i=1}^N |\phi_i(x)|^2 \right)^{(1+2/n)p/2} w(x) \, dx \leq c \int_{\mathbb{R}^n} \left( \sum_{i=1}^N |\nabla \phi_i(x)|^2 \right)^{p/2} w(x) \, dx,
\]
where \( c \) depends only on \( n, p \) and \( w \).
This is a new result even in the case \( w \equiv 1 \). When \( 2 < p < n \), an example of \( w \) is given by \( w(x) = |x|^\alpha, \ -n + p < \alpha < n(p-2)/2 \). When \( 2n/(n+2) < p < 2 \), an example of \( w \) is given by \( w(x) = |x|^\alpha, \ -n + 2 < \alpha \leq 0 \).

## 2 Proof of Theorem 1.3

Let \( M \) be the Hardy-Littlewood maximal operator, that is,
\[
M(f)(x) = \sup_{Q(x)} \frac{1}{|Q|} \int_Q |f(y)| \, dy,
\]
where \( f \) is a locally integrable function on \( \mathbb{R}^n \) and the supremum is taken over all cubes \( Q \) which contain \( x \). The following proposition is proved in [3, Chapter IV] or [5, Chapter V].

Proposition 2.1.

(i) Let \( 1 < p < \infty \) and \( w \) be a weight function on \( \mathbb{R}^n \). Then there exists a positive constant \( c \) such that
\[
\int_{\mathbb{R}^n} M(f)^p w \, dx \leq c \int_{\mathbb{R}^n} |f|^p w \, dx
\]
for all \( f \in L^p(w) \) if and only if \( w \in A_p \).

(ii) Let \( 1 < p < \infty \) and \( w \in A_p \). Then there exists a \( q \in (1, p) \) such that \( w \in A_q \).

(iii) Let \( 0 < \tau < 1 \) and \( f \) be a locally integrable function on \( \mathbb{R}^n \) such that \( M(f)(x) < \infty \) a.e.. Then \( M(f)^\tau \in A_1 \).

(iv) Let \( 1 < p < \infty \). Then \( w \in A_p \) if and only if \( w^{1-p'} \in A_p' \), where \( p^{-1} + p'^{-1} = 1 \).

(v) Let \( 1 < p < \infty \) and \( w_1, w_2 \in A_1 \). Then \( w_1 w_2^{1-p} \in A_p \).

Proof of Theorem 1.3

Our proof is very similar to that of the extrapolation theorem in harmonic analysis (c.f. [1, Theorem 7.8]). In our proof the integral means that over \( \mathbb{R}^n \).

Let \( 2 < p < n \) and \( 2/p + 1/q = 1 \). We remark that the assumption \( w^{n/(n-p)} \in A_{p(n-2)/(2(n-p))} \) leads to \( w \in A_p \) by an easy calculation. Let \( u \in L^q(w), \ u \geq 0, \) and \( \|u\|_{L^q(w)} = 1 \). Since \( w^{n/(n-p)} \in A_{p(n-2)/(2(n-p))} \), we have \( w^{-2/(p-2)} \in A_{p(n-2)/(n(p-2))} \) by (iv) of Proposition 2.1. Hence there exists a \( \gamma \) such that \( n/(n-2) < \gamma < q \) and \( w^{-2/(p-2)} \in A_{p/(\gamma(p-2))} \) by (ii) of Proposition 2.1. Then we have \( uw \leq M((uw)^\gamma)^{1/\gamma} \) a.e.. Because
\[
w^{-2q/p} = w^{-2/(p-2)} \in A_{p/(\gamma(p-2))} = A_{q/\gamma}
\]
(2) \[ \int M((uw)\gamma)^{\eta/\gamma}w^{-2\eta/p}\,dx \leq c \int (uw)^{\eta}w^{-2\eta/p}\,dx = c \int w^\eta w\,dx = c \]

by (i) of Proposition 2.1, we get \( M((uw)\gamma)(x) < \infty \) a.e.. Hence \( M((uw)\gamma)^{1/\gamma} \in A_1 \) by (iii) of Proposition 2.1. Let \( \alpha = \frac{n}{(n-2)\gamma} \). Then \( 0 < \alpha < 1 \) and
\[
M((uw)\gamma)^{-n/(2\gamma)} = \{M((uw)\gamma)^{\alpha}\}^{1-n/2} \in A_{n/2},
\]
where we used \( M((uw)\gamma)^{\alpha} \in A_1 \) and (v) of Proposition 2.1. Let
\[
\rho(x) = \sum_{i=1}^{N} |\phi_i(x)|^2.
\]
Then we have
\[
\int \rho^{1+2/n}uw\,dx \leq \int \rho^{1+2/n}M((uw)\gamma)^{1/\gamma}\,dx \leq c \int \left( \sum_{i=1}^{N} |\nabla \phi_i|^2 \right)^{p/2} w\,dx
\]
\[
\leq c \left( \int \left( \sum_{i=1}^{N} |\nabla \phi_i|^2 \right)^{p/2} w\,dx \right)^{2/p} \left( \int M((uw)\gamma)^{\eta/\gamma}w^{-2\eta/p}\,dx \right)^{1/q}
\]
\[
\leq c \left( \int \left( \sum_{i=1}^{N} |\nabla \phi_i|^2 \right)^{p/2} w\,dx \right)^{2/p}
\]
where we used Theorem 1.2 and (2). If we take the supremum for all \( u \in L^q(w) \), \( u \geq 0 \), and \( \|u\|_{L^q(w)} = 1 \), then we get
\[
\left( \int \rho^{(1+2/n)p/2}w\,dx \right)^{2/p} \leq c \left( \int \left( \sum_{i=1}^{N} |\nabla \phi_i|^2 \right)^{p/2} w\,dx \right)^{2/p}.
\]

Next we consider the case \( 2n/(n+2) < p < 2 \). We remark that \( w \in A_1 \) by the assumption \( w^{n/(n-2)} \in A_1 \). Let
\[
f = \left( \sum_{i=1}^{N} |\nabla \phi_i|^2 \right)^{1/2}.
\]
We can take \( \gamma \) such that \( (2-p)n/2 < \gamma < p \). Then
\[
\int M(f^{\gamma})^{p/\gamma}w\,dx \leq c \int f^{p}w\,dx < \infty,
\]
where we used \( w \in A_1 \subset A_{p/\gamma} \) and (i) of Proposition 2.1. Hence we have \( M(f^{\gamma})(x) < \infty \) a.e. and
\[
M(f^{\gamma})^{(2-p)n/(2\gamma)} \in A_1
\]
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by (iii) of Proposition 2.1. Furthermore we have

$$M(f^\gamma)^{(2-p)/\gamma} w \in A_2,$$

where we used

$$M(f^\gamma)^{(2-p)/\gamma} \in A_1, \quad w \in A_1,$$

and (v) of Proposition 2.1. Moreover

$$\{M(f^\gamma)^{-(2-p)/\gamma}w\}^{-n/2} = M(f^\gamma)^{(2-p)n/(2\gamma)}(w^{n/(n-2)})^{(1-n/2)} \in A_{n/2}$$

because $w^{n/(n-2)} \in A_1$. Therefore

$$\int \rho^{(1+2/n)p/2} w \, dx = \int \rho^{(1+2/n)p/2} w M(f^\gamma)^{-(2-p)p/(2\gamma)} M(f^\gamma)^{(2-p)p/(2\gamma)} \, dx$$

$$\leq \left( \int \rho^{1+2/n} M(f^\gamma)^{-(2-p)/\gamma} w \, dx \right)^{p/2} \left( \int M(f^\gamma)^{p/\gamma} w \, dx \right)^{1-p/2}$$

$$\leq c \left( \int f^2 M(f^\gamma)^{-(2-p)/\gamma} w \, dx \right)^{p/2} \left( \int f^p w \, dx \right)^{1-p/2}$$

$$\leq c \left( \int M(f^\gamma)^{2/\gamma} M(f^\gamma)^{-(2-p)/\gamma} w \, dx \right)^{p/2} \left( \int f^p w \, dx \right)^{1-p/2} \leq c \int f^p w \, dx,$$

where we used Theorem 1.2 in the second inequality.

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References


