<table>
<thead>
<tr>
<th>Instructions for use</th>
<th>Title: Weighted $L^p$ Sobolev-Lieb-Thirring inequalities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Tachizawa, Kazuya</td>
</tr>
<tr>
<td>Citation</td>
<td>Hokkaido University Preprint Series in Mathematics, 702, 1-6</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2005</td>
</tr>
<tr>
<td>DOI</td>
<td>10.14943/83853</td>
</tr>
<tr>
<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/69507">http://hdl.handle.net/2115/69507</a></td>
</tr>
<tr>
<td>Type</td>
<td>bulletin (article)</td>
</tr>
<tr>
<td>File Information</td>
<td>pre702.pdf</td>
</tr>
</tbody>
</table>

Hokkaido University Collection of Scholarly and Academic Papers: HUSCAP
Weighted $L^p$ Sobolev-Lieb-Thirring inequalities

By Kazuya Tachizawa

Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060-0810

Abstract: We give a weighted $L^p$ version of the Sobolev-Lieb-Thirring inequality for suborthonormal functions.

Key words: Sobolev-Lieb-Thirring inequalities; $A_p$-weights.

2000 Mathematics Subject Classification. Primary 26D15; Secondary 42B25.

1 Introduction

In 1976 Lieb and Thirring proved the following inequality.

**Theorem 1.1 ([4]).** Let $n \in \mathbb{N}$. Then there exists a positive constant $c_n$ such that for every family $\{\phi_i\}_{i=1}^N$ in $H^1(\mathbb{R}^n)$ which is orthonormal in $L^2(\mathbb{R}^n)$, we have

$$
\int_{\mathbb{R}^n} \left( \sum_{i=1}^N |\phi_i(x)|^2 \right)^{1+2/n} \, dx \leq c_n \sum_{i=1}^N \|\nabla \phi_i\|^2.
$$

In this theorem $H^1(\mathbb{R}^n)$ denotes the Sobolev space and $\|\cdot\|$ is the norm of $L^2(\mathbb{R}^n)$. In [4] Lieb and Thirring applied this inequality to the problem of the stability of matter. Ghidaglia, Marion, and Temam proved a generalization of (1) under the suborthonormal condition on $\{\phi_i\}$, where $\{\phi_i\}_{i=1}^N$ in $L^2(\mathbb{R}^n)$ is called suborthonormal if the inequality

$$
\sum_{i,j=1}^N \xi_i \bar{\xi}_j (\phi_i, \phi_j) \leq \sum_{i=1}^N |\xi_i|^2
$$

holds for all $\xi_i \in \mathbb{C}, i = 1, \ldots, N$, where $(\cdot, \cdot)$ means the $L^2$ inner product([2]). They applied the inequality (1) to the estimate of the dimension of attractors associated with partial differential equations. In this paper we shall give a weighted $L^p$ version of (1) under the suborthonormal condition on $\{\phi_i\}$.
For the statement of our result we need to recall the definition of $A_p$-weights (c.f. [3], [5]). By a cube in $\mathbb{R}^n$ we mean a cube which sides are parallel to coordinate axes. Let $w$ be a non-negative, locally integrable function on $\mathbb{R}^n$. We say that $w$ is an $A_p$-weight for $1 < p < \infty$ if there exists a positive constant $C$ such that

$$\frac{1}{|Q|} \int_Q w(x) \, dx \left( \frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} \, dx \right)^{p-1} \leq C$$

for all cubes $Q \subset \mathbb{R}^n$. For example, $w(x) = |x|^\alpha$ is an $A_p$-weight when $-n < \alpha < n(p-1)$.

We say that $w$ is an $A_1$-weight if there exists a positive constant $C$ such that

$$\frac{1}{|Q|} \int_Q w(y) \, dy \leq C w(x) \quad \text{a.e. } x \in Q$$

for all cubes $Q \subset \mathbb{R}^n$. If $-n < \alpha \leq 0$, then $w(x) = |x|^\alpha$ is an $A_1$-weight. Let $A_p$ be the class of $A_p$-weights. The inclusion $A_p \subset A_q$ holds for $p < q$.

A nonnegative, locally integrable function $w$ on $\mathbb{R}^n$ is called a weight function. For a weight function $w$ we define

$$L^p(w) = \left\{ f : \text{measurable on } \mathbb{R}^n, \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx < \infty \right\}.$$

The following is a conclusion of [7, Theorem 1.2] and [6, Lemma 3.2].

**Theorem 1.2.** Let $n \in \mathbb{N}$, $3 \leq n$, $w \in A_2$, and $w^{-n/2} \in A_{n/2}$. Then there exists a positive constant $c$ such that for every family $\{\phi_i\}_{i=1}^N$ in $L^2(\mathbb{R}^n)$ which is suborthonormal in $L^2(\mathbb{R}^n)$ and $|\nabla \phi_i| \in L^2(w)$, $i = 1, \ldots, N$, we have

$$\int_{\mathbb{R}^n} \left( \sum_{i=1}^N |\phi_i(x)|^2 \right)^{1+2/n} w(x) \, dx \leq c \sum_{i=1}^N \int_{\mathbb{R}^n} |\nabla \phi_i(x)|^2 w(x) \, dx,$$

where $c$ depends only on $n$ and $w$.

By using this theorem we can prove the following weighted $L^p$ version of the Sobolev-Lieb-Thirring inequality.

**Theorem 1.3.** Let $n \in \mathbb{N}$ and $3 \leq n$. Let $2n/(n+2) < p < n$, $p \neq 2$, and $w$ be a weight function. When $p > 2$, we assume that $w^{n/(n-p)} \in A_{p(n-2)/(2(n-p))}$. When $p < 2$, we assume that $w^{n/(n-2)} \in A_1$.

Then there exists a positive constant $c$ such that for every family $\{\phi_i\}_{i=1}^N$ in $L^2(\mathbb{R}^n)$ which is suborthonormal in $L^2(\mathbb{R}^n)$ and $|\nabla \phi_i| \in L^p(w)$, $i = 1, \ldots, N$, we have

$$\int_{\mathbb{R}^n} \left( \sum_{i=1}^N |\phi_i(x)|^2 \right)^{(1+2/n)p/2} w(x) \, dx \leq c \int_{\mathbb{R}^n} \left( \sum_{i=1}^N |\nabla \phi_i(x)|^2 \right)^{p/2} w(x) \, dx,$$

where $c$ depends only on $n, p$ and $w$. 

2
This is a new result even in the case \( w \equiv 1 \). When \( 2 < p < n \), an example of \( w \) is given by \( w(x) = |x|^\alpha, -n+p < \alpha < n(p-2)/2 \). When \( 2n/(n+2) < p < 2 \), an example of \( w \) is given by \( w(x) = |x|^{\alpha}, -n+2 < \alpha \leq 0 \).

## 2 Proof of Theorem 1.3

Let \( M \) be the Hardy-Littlewood maximal operator, that is,

\[
M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \, dy,
\]

where \( f \) is a locally integrable function on \( \mathbb{R}^n \) and the supremum is taken over all cubes \( Q \) which contain \( x \). The following proposition is proved in [3, Chapter IV] or [5, Chapter V].

**Proposition 2.1.**

(i) Let \( 1 < p < \infty \) and \( w \) be a weight function on \( \mathbb{R}^n \). Then there exists a positive constant \( c \) such that

\[
\int_{\mathbb{R}^n} M(f)^p w \, dx \leq c \int_{\mathbb{R}^n} |f|^p w \, dx
\]

for all \( f \in L^p(w) \) if and only if \( w \in A_p \).

(ii) Let \( 1 < p < \infty \) and \( w \in A_p \). Then there exists a \( q \in (1, p) \) such that \( w \in A_q \).

(iii) Let \( 0 < \tau < 1 \) and \( f \) be a locally integrable function on \( \mathbb{R}^n \) such that \( M(f)(x) < \infty \) a.e. Then \( M(f)^\tau \in A_1 \).

(iv) Let \( 1 < p < \infty \). Then \( w \in A_p \) if and only if \( w^{1-p'} \in A_{p'} \), where \( p^{-1} + p'^{-1} = 1 \).

(v) Let \( 1 < p < \infty \) and \( w_1, w_2 \in A_1 \). Then \( w_1 w_2^{1-p} \in A_p \).

**Proof of Theorem 1.3**

Our proof is very similar to that of the extrapolation theorem in harmonic analysis (c.f. [1, Theorem 7.8]). In our proof the integral means that over \( \mathbb{R}^n \).

Let \( 2 < p < n \) and \( 2/p + 1/q = 1 \). We remark that the assumption \( w^{n/(n-p)} \in A_{p(n-2)/(2(n-p))} \) leads to \( w \in A_p \) by an easy calculation. Let \( u \in L^q(w), u \geq 0, \) and \( \|u\|_{L^q(w)} = 1 \). Since \( w^{n/(n-p)} \in A_{p(n-2)/(2(n-p))} \), we have \( w^{-2/(p-2)} \in A_{p(n-2)/(n(p-2))} \) by (iv) of Proposition 2.1. Hence there exists a \( \gamma \) such that \( n/(n-2) < \gamma < q \) and \( w^{-2/(p-2)} \in A_{p/(\gamma(p-2))} \) by (ii) of Proposition 2.1. Then we have \( uw \leq M((uw)^\gamma)^{1/\gamma} \) a.e.. Because

\[
w^{-2q/p} = w^{-2/(p-2)} \in A_{p/(\gamma(p-2))} = A_{q/\gamma}
\]
and
\[ (2) \quad \int M((uw)\gamma)^{q/\gamma} w^{-2q/p} dx \leq c \int (uw)^q w^{-2q/p} dx = c \int u^q w dx = c \]

by (i) of Proposition 2.1, we get \( M((uw)\gamma)(x) < \infty \) a.e. Hence \( M((uw)\gamma)^{1/\gamma} \in A_1 \) by (iii) of Proposition 2.1. Let \( \alpha = \frac{n}{(n-2)\gamma} \). Then \( 0 < \alpha < 1 \) and
\[ M((uw)\gamma)^{-n/(2\gamma)} = \{ M((uw)\gamma)^{\alpha} \}^{1-n/2} \in A_{n/2}, \]
where we used \( M((uw)\gamma)^{\alpha} \in A_1 \) and (v) of Proposition 2.1. Let
\[ \rho(x) = \sum_{i=1}^N |\phi_i(x)|^2. \]

Then we have
\[ \int \rho^{1+2/n} uw dx \leq \int \rho^{1+2/n} M((uw)^{\gamma})^{1/\gamma} dx \leq c \int \left( \sum_{i=1}^N |\nabla \phi_i|^2 \right)^{p/2} M((uw)^{\gamma})^{1/\gamma} dx \]
\[ \leq c \left( \int \left( \sum_{i=1}^N |\nabla \phi_i|^2 \right)^{p/2} w dx \right)^{2/p} \left( \int M((uw)^{\gamma})^{q/\gamma} w^{-2q/p} dx \right)^{1/q} \]
\[ \leq c \left( \int \left( \sum_{i=1}^N |\nabla \phi_i|^2 \right)^{p/2} w dx \right)^{2/p} \]

where we used Theorem 1.2 and (2). If we take the supremum for all \( u \in L^q(w) \), \( u \geq 0 \), and \( \|u\|_{L^q(w)} = 1 \), then we get
\[ \left( \int \rho^{(1+2/n)p/2} w dx \right)^{2/p} \leq c \left( \int \left( \sum_{i=1}^N |\nabla \phi_i|^2 \right)^{p/2} w dx \right)^{2/p} \]

Next we consider the case \( 2n/(n+2) < p < 2 \). We remark that \( w \in A_1 \) by the assumption \( w^{n/(n-2)} \in A_1 \). Let
\[ f = \left( \sum_{i=1}^N |\nabla \phi_i|^2 \right)^{1/2}. \]

We can take \( \gamma \) such that \( (2-p)n/2 < \gamma < p \). Then
\[ \int M(f^\gamma) w dx \leq c \int f^p w dx < \infty, \]

where we used \( w \in A_1 \subset A_{p/\gamma} \) and (i) of Proposition 2.1. Hence we have \( M(f^\gamma)(x) < \infty \) a.e. and
\[ M(f^\gamma)^{(2-p)n/(2\gamma)} \in A_1. \]

4
by (iii) of Proposition 2.1. Furthermore we have

\[ M(f^{\gamma})^{-(2-p)/\gamma} w \in A_2, \]

where we used

\[ M(f^{\gamma})^{(2-p)/\gamma} \in A_1, \quad w \in A_1, \]

and (v) of Proposition 2.1. Moreover

\[ \{M(f^{\gamma})^{-(2-p)/\gamma} w\}^{-n/2} = M(f^{\gamma})^{(2-p)n/(2\gamma)} (w^{n/(n-2)})^{(1-n/2)} \in A_{n/2} \]

because \( w^{n/(n-2)} \in A_1 \). Therefore

\[
\int \rho^{(1+2/n)p/2} w \, dx = \int \rho^{(1+2/n)p/2} w M(f^{\gamma})^{-(2-p)/\gamma w} M(f^{\gamma})^{(2-p)/p(2\gamma)} dx \\
\leq \left( \int \rho^{1+2/n} M(f^{\gamma})^{-(2-p)/\gamma w} dx \right)^{p/2} \left( \int M(f^{\gamma})^{p/\gamma w} dx \right)^{1-p/2} \\
\leq c \left( \int f^2 M(f^{\gamma})^{-(2-p)/\gamma w} dx \right)^{p/2} \left( \int f^p w dx \right)^{1-p/2} \\
\leq c \left( \int M(f^{\gamma})^{2/\gamma} M(f^{\gamma})^{-(2-p)/\gamma w} dx \right)^{p/2} \left( \int f^p w dx \right)^{1-p/2} \\
\leq c \left( \int M(f^{\gamma})^{p/\gamma w} dx \right)^{p/2} \left( \int f^p w dx \right)^{1-p/2} \leq c \int f^p w dx,
\]

where we used Theorem 1.2 in the second inequality.

**Acknowledgment**

The author was partly supported by the Grants-in-Aid for formation of COE and for Scientific Research, The Ministry of Education, Science, Sports and Culture, Japan.

**References**


