GLOBAL ESTIMATES OF MAXIMAL OPERATORS
GENERATED BY DISPERSIVE EQUATIONS

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Abstract. Let $T f(x, t) = e^{2\pi i t \phi(D)} f$ be the solution of the general dispersive equation with the phase function $\phi$ and initial data $f$ in the Schwartz class. In case that the phase $\phi$ has a suitable growth rate at the infinity and the origin and $f$ is a finite linear combination of radial and spherical harmonic functions, we have global $L^p$ estimates of maximal operator defined by taking the supremum w.r.t. $t$. In particular, we obtain a global estimate at the end point left open.

1. Introduction

The general dispersive equation is defined by

$$iu_t(x) = -2\pi \phi(D) u(x), \quad \text{on } \mathbb{R}^n \times \mathbb{R}, \quad u(x, 0) = f(x) \in \mathcal{S}(\mathbb{R}^n)(n \geq 3),$$

where $D = \frac{1}{2\pi i} \nabla$ and $\phi$ is a smooth phase function. The solution of this equation can be formulated formally by

$$u(x, t) = T f(x, t) = \int e^{2\pi i(x \cdot \xi + t \phi(\xi))} \hat{f}(\xi) \, d\xi,$$

where $\hat{f}(\xi) = \int e^{-2\pi i x \cdot \xi} f(x) \, dx$. From the formulation, we define maximal operators defined by

$$T^* f(x) := \sup_{-1 \leq t \leq 1} |T f(x, t)|, \quad T^{**} f(x) := \sup_{t \in \mathbb{R}} |T f(x, t)|.$$

In this paper we are mainly concerned with a mapping property of $T^{**}$ defined with the various phase function, whose typical model is $|\xi|^a (a \neq 0)$, from Sobolev space to $L^p$ such that

$$\|T^* f\|_{L^p} \quad \text{or} \quad \|T^{**} f\|_{L^p} \leq C\|f\|_{X^s},$$

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where $X^s$ is denoted by $\dot{H}^s$ and $H^s$. Here the Sobolev spaces are defined by the norms:

$$\|f\|_{\dot{H}^s} = \left(\int |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}, \quad \|f\|_{H^s} = \left(\int (1 + |\xi|)^{2s} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$ 

To control the global estimate of maximal operator $T^s$ or $T^{**}$, it seems inevitable to assume the growth rate and regularity of the phase function. Therefore we impose the following assumption on the phase $\phi$.

**A.** Let $\phi$ be a radial function such that for some $a \in \mathbb{R} (\neq 0, 1)$, $\phi \in C^2(\mathbb{R}^n \setminus \{0\})$ and there exist positive constants $c_1, c_2$ such that

$$c_1 |\xi|^{a-k} \leq |\phi^{(k)}(\xi)| \leq c_2 |\xi|^{a-k} \quad (k = 0, 1, 2), \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}.$$ 

The maximal inequality (1.2) is motivated from the well-known pointwise convergence problem: $\lim_{t \to 0} u(x, t) = f(x)$ a.e. $x$, for $f \in H^\frac{n}{2}(\mathbb{R}^n)$, during the three decades, the local and global $L^p$ estimate of the maximal operators have been studied by many authors [2, 3, 4, 5, 7, 9, 16] and [17] etc. P. Sjölin [13] and L. Vega [17] obtained the strong necessary condition ($s \geq \frac{1}{4}$) on the pointwise convergence problem. In particular, P. Sjölin showed that the maximal operator $T^{**}$ cannot have the global $L^2$ boundedness (see [10]), and that $\|T^s f\|_{L^2} \leq C \|f\|_{H^s}$ holds for $s > \frac{a}{4}$ and fails for $s < \frac{a}{4} (a > 1)$ (see [13]). But if the data $f$ is a finite linear combination of radial and spherical harmonic functions, then we prove that the inequality (1.2) holds for $s = \frac{1}{4}$ and $\frac{a}{4}$ with suitable weight.

Now we state our main results.

**Theorem 1.1.** Let $f_0 \in C_0^\infty(\mathbb{R})$ and $Y_k$ be a spherical harmonic function of order $k \geq 0$. If $f(x) = |x|^ka f_0(|x|) Y_k(x')$, then

1. $\|T^{**} f\|_{L^p} \leq C_k \|Y_k\|_{L^p(S^{n-1})} \|f\|_{H^s}$ holds for $s \in \left[\frac{1}{4}, \frac{1}{2}\right)$, $p = \frac{2n}{n-2s}$ and $0 < a \neq 1$, where $C_k = O((n + 2k)^{n+2k})$ as $k \to \infty$.

2. $\|T^s f\|_{L^2} \leq C_k \|f\|_{H^s}$ holds for any $s > \frac{a}{4}$ and $0 < a \neq 1$.

3. $\|T^s f\|_{L^2(|x|^{-\varepsilon}dx)} \leq C_{s, k} \|f\|_{H^\frac{n}{2}}$ holds for any $a > 1$ and $\varepsilon > 0$.

4. $\|T^s f\|_{L^p(1+|x|^{-b}dx)} \leq C_k \|Y_k\|_{L^p(S^{n-1})} \|f\|_{H^s}$ holds for $0 < a < 1$, $\frac{a}{4} < s < \frac{1}{4}$, $p = \frac{4n(1-a)}{2n(1-a)+a-4s}$ and $b = sp - n(\frac{p}{2}-1)$.

We obtain optimal global estimates except for the end point $(s, p) = (\frac{a}{4}, 2)$ $(0 < a < 1)$ which was studied by B. G. Walther [18]. These are obtained by using a bound of one dimensional oscillatory integral of the form $\int_{\mathbb{R}} e^{2\pi i (\lambda \phi(x) + x \xi)} |\xi|^s d\xi$. Many authors referred in this paper have tried to handle such an integral and obtained various bounds according to the value of growth rate $a$ of $\phi$ (i.e. $a > 1$ or $a < 1$). Here, we provide that the integral is bounded by $C|x|^{-(1-s)}$ for any $a \neq 1$ and the constant $C$ is
indipendent of \( t \) (see Lemma 2.3). Thanks to the time independency of the integral, the proof of main results are much more simplified.

S. Wang [19] showed that if \( p > 2 \), then there exist \( f_0 \) and \( Y_k \) such that if \( f(x) = |x|^k f_0(|x|) Y_k(x') \), then \( \lim_{k \to \infty} ||T^* f||_{L^p(B)} / ||f||_{L^q} = \infty \) for any ball \( B \). Thus it will be interesting to prove a local \( L^p \) estimate of \( T^* f \) holds uniformly on \( k \) for some \( p \in [1, 2] \). For the more general initial data \( f \), by using a bilinear estimate, T. Tao in [16] showed that the global estimate for \( T^* f \) holds for \( p > \frac{2(n+3)}{n+1} \) and \( s > n(\frac{1}{2} - \frac{1}{p}) \), if the phase \( \phi \) is an elliptic type.

As another global estimates, there are several results about the weighted estimates for \( s > \frac{1}{2} \). For these results, one can refer [5, 17, 18]. But it remains still open question whether even a local estimate of \( T^* f \) holds for \( s = \frac{1}{2} \).

If not specified, throughout this paper, \( C \) denotes by a generic constant that depends on \( c_1, c_2, a, s, n \). We use the notation \( A \lesssim B \) and \( A \sim B \) to denote \( |A| \leq CB \) and \( C^{-1}B \leq |A| \leq CB \) respectively.

2. Preliminary lemmas

We begin with the weighted inequality for the Fourier transform.

**Lemma 2.1** (see [8]). If \( 1 \leq q \leq 2 \), \( 0 \leq \alpha < \frac{1}{2} \), \( 0 \leq \alpha_1 < \frac{1}{q} \) and \( \alpha_1 = \alpha + \frac{1}{2} - \frac{1}{q} \), then the following inequality holds

\[
\left( \int_{\mathbb{R}} |\xi|^{-2\alpha_1} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \lesssim \left( \int_{\mathbb{R}} |f(x)|^q |x|^\alpha_1 dx \right)^{\frac{1}{q}}.
\]

Now we introduce some estimates of oscillatory integrals. Let us first state a stationary phase lemma which can be found in [7] etc..

**Lemma 2.2.** Let \( \psi \) be a monotone function and \( I = \int_{\alpha}^{\beta} e^{i\varphi(x)} \psi(x) dx \). Then if \( \left| \frac{d\varphi}{dx} \right| \geq \lambda > 0 \) in \( [\alpha, \beta] \) and \( \frac{d\varphi}{dx} \) is monotone, \( |I| \leq C \lambda^{-1} \sup_{[\alpha, \beta]} |\psi(\xi)| \), and if \( \left| \frac{d^2\varphi}{dx^2} \right| \geq \lambda > 0 \), then \( |I| \leq C \lambda^{-\frac{3}{2}} \sup_{[\alpha, \beta]} |\psi(\xi)| \). The constant \( C \) doesn’t depend on \( \alpha, \beta, \lambda, \varphi \) and \( \psi \).

Utilizing the lemma above, we get the following lemma.

**Lemma 2.3.** Suppose \( \phi \) satisfies the assumption \( A \) for \( 0 < a \neq 1 \). Let \( A, B, s \) be the real numbers such that \( A, B \neq 0 \), \( \frac{1}{2} \leq s < 1 \). Consider the following integral:

\[
I = \int_{\xi \in \mathbb{R}} e^{2\pi i (A\phi(\xi) + B\xi)} |\xi|^{-s} d\xi.
\]

Then \( |I| \leq C(a, s, c_1, c_2) |B|^{-(1-s)} \).

If \( 0 < a < 1, \frac{a}{2} < s < \frac{1}{2} \) and \( |A| \leq 2 \), then \( I \lesssim |B|^{-(1-s)} + |B|^{-(1-s) - \frac{a(1-2s)}{2(1-s)}} \).
Proof of Lemma 2.3. (Case \( a > 1 \)) Without loss of generality, we may assume that \( A > 0 \) and \( B > 0 \). Let \( D = \frac{B}{A^a} \). Then by the change of variable, we have
\[
I = A^{-\frac{1-a}{a}} \int e^{2\pi i (A\phi(A^{-\frac{1}{a}} \xi) + D\xi)} |\xi|^{-s} \, d\xi = \int_{\xi < 0} + \int_{\xi > 0} = I_+ + I_-.
\]
We have only to consider \( I_+ \) and we denote it \( I \) again.

Now we first consider the case when \( \phi' > 0 \). Observe that
\[
E \equiv (A\phi(A^{-\frac{1}{a}} \xi) + D\xi)' \geq c_1 \xi^{a-1} + D.
\]
Let \( M \) be a large positive number depending only on \( a, s, c_1, c_2 \). If \( D \leq M \), then
\[
I = A^{-\frac{1-a}{a}} \left( \int_0^1 + \int_1^\infty \right) = I_1 + I_2.
\]
For \( I_1 \), by direct integration, we have \( |I_1| \lesssim A^{-\frac{1-a}{a}} \lesssim B^{-(1-s)} \). For \( I_2 \), since \( E \gtrsim 1 \), by the first part of (2.2), we have \( |I_2| \lesssim A^{-\frac{1-a}{a}} \lesssim B^{-(1-s)} \). If \( D > M \), then since \( E \geq D \), by the first part of Lemma 2.2, we have \( |I_2| \lesssim A^{-\frac{1-a}{a}} D^{-1} \lesssim A^s B^{-1} \lesssim B^{-(1-s)} \). For \( I_1 \), using the change of variable, we have
\[
I_1 = A^{-\frac{1-a}{a}} D^{-(1-s)} \int_0^D e^{2\pi i (A\phi(D^{-1} A^{-\frac{1}{a}} \xi) + \xi)} \xi^{-s} \, d\xi.
\]
Thus \( I_1 = \int_0^1 + \int_1^D = I_{1,1} + I_{1,2} \). By the integration, \( |I_{1,1}| \lesssim B^{-(1-s)} \). For \( I_{1,2} \), since \( (A\phi(D^{-1} A^{-\frac{1}{a}} \xi) + \xi)' \geq 1 \), from the first part of Lemma 2.2, we have \( |I_{1,2}| \lesssim B^{-(1-s)} \) and hence \( |I_1| \lesssim B^{-(1-s)} \).

Now we consider the case when \( \phi' < 0 \). We observe that
\[
-c_2 \xi^{a-1} + D \leq E = (A\phi(A^{-\frac{1}{a}} \xi) + D\xi)' \leq -c_1 \xi^{a-1} + D.
\]
If \( D \leq M \), then we split \( I \) into two parts as follows:
\[
I = A^{-\frac{1-a}{a}} \left( \int_0^\infty \left( \frac{a}{2\gamma_1} \right)^{\frac{1}{1-a}} + \int_1^\infty \right) = I_3 + I_4.
\]
For \( I_5 \), we have by direct integration \( |I_5| \lesssim A^{-\frac{1-a}{a}} \lesssim B^{-(1-s)} \). For \( I_4 \), since \( E \leq -c_1 \xi^{a-1} + D \leq -1 \), by the first part of Lemma 2.2, we get \( |I_4| \lesssim A^{-\frac{1-a}{a}} \lesssim B^{-(1-s)} \). If \( D > M \), then we split \( I \) into four parts as follows:
\[
(2.1)
\]
\[
I = A^{-\frac{1-a}{a}} \left( \int_0^1 + \int_1^{(\frac{a}{2\gamma_1})^{\frac{1}{1-a}}} + \int_1^{(\frac{a}{2\gamma_1})^{\frac{1}{1-a}}} + \int_1^\infty \right) \equiv I_5 + I_6 + I_7 + I_8.
\]
For $I_5$, we use the change of variable so that

$$I_5 = A^{-\frac{1-s}{\alpha}} D^{-(1-s)} \int_0^D e^{2\pi i (A\phi(D^{-1}A^{-\frac{1}{\alpha}}) + \xi) \xi^{-s}} \, d\xi.$$ 

We split $I_5$ into two part: $I_5 = A^{-\frac{1-s}{\alpha}} D^{-(1-s)} \left( \int_0^1 + \int_1^D \right) = I_{5,1} + I_{5,2}$. For $I_{5,1}$ and $I_{5,2}$, using the direct integration and the first part of Lemma 2.2 respectively, we have $|I_{5,1}| + |I_{5,2}| \lesssim A^{-\frac{1-s}{\alpha}} D^{-(1-s)} = B^{-(1-s)}$. For $I_6$, since $E \geq D \geq D^{1-s}$, using the first part of Lemma 2.2, we have $|I_6| \lesssim A^{-\frac{1-s}{\alpha}} D^{-(1-s)} = B^{-(1-s)}$.

To estimate $I_7$, we use the second derivative $|E'| \sim \xi^{a-2} \sim D^{\frac{n-2}{n-1}}$. Then from the second part of Lemma 2.2, we obtain

$$|I_7| \lesssim A^{-\frac{1-s}{\alpha}} D^{-\frac{n-2}{n-1}} D^{-\frac{s}{n-1}} = A^{-\frac{1-s}{\alpha}} D^{-\frac{n-2+2s}{2(n-1)}}.$$ 

Since $\alpha > 1$ and $s \geq \frac{1}{2}$, we have $|I_7| \lesssim A^{-\frac{1-s}{\alpha}} D^{-(1-s)} = B^{-(1-s)}$. Finally, we estimate $I_8$. Since $E \geq D \geq D^{1-s}$, by the first part of Lemma 2.2, we have $|I_8| \lesssim A^{-\frac{1-s}{\alpha}} D^{-(1-s)} D^{-\frac{s}{n-1}} \lesssim B^{-(1-s)}$.

**Case $a < 1$** We first consider the case $\frac{1}{2} \leq s < 1$. We may assume $A, B > 0$. Let $\tilde{D} = \frac{A}{B}$. Then by the change of variable, we write

$$B^{1-s} I = \int e^{2\pi i (A\phi(\tilde{D}^{-1}B^{-1}) + \xi) \xi^{-s}} \, d\xi = \int_0^\infty + \int_{-\infty}^0 = I_+ + I_-.$$ 

As in the previous case ($a > 1$), we only consider $I_+$ and denote it by $I$ again.

In case that $\phi' > 0$, we have $E \equiv (A\phi(\tilde{D}^{-1}B^{-1}))' \geq c_1 \tilde{D} \xi^{a-1} + 1 \geq 1$ for all $\xi > 0$. We divide $I$ into two parts: $I = \int_0^1 + \int_1^\infty$. For the first integral, we just integrate and for the second one, we use the first part of Lemma 2.2. Then we can see $|I| \lesssim 1$.

Now we consider the case when $\phi' < 0$. Then we can observe that

$$-c_2 \tilde{D} \xi^{a-1} + 1 \leq E \leq -c_1 \tilde{D} \xi^{a-1} + 1.$$ 

If $c_2 \tilde{D} < 2$, then we divide $I$ into two parts: $I = \int_0^{\frac{1}{c_2 \tilde{D}}} + \int_{\frac{1}{c_2 \tilde{D}}}^\infty = I_1 + I_2$. By the integration, we get $|I_1| \lesssim 1$. And since $c_2 \tilde{D} < 2$ and hence $E \geq 1$, by the first part of Lemma 2.2, we have $|I_2| \lesssim 1$.

If $c_1 \tilde{D} > 2$, then we divide $I$ into four parts:

$$I = \int_0^1 + \int_{1}^{(\frac{1}{c_2 \tilde{D}})^{\frac{1}{1-s}}} + \int_{(\frac{1}{c_2 \tilde{D}})^{\frac{1}{1-s}}}^{(\frac{2}{c_2 \tilde{D}})^{\frac{1}{1-s}}} + \int_{(\frac{2}{c_2 \tilde{D}})^{\frac{1}{1-s}}}^\infty = I_3 + I_4 + I_5 + I_6.$$ 

For $I_3$, by the integration, $|I_3| \lesssim 1$. For $|I_5|$, since $|E'| \sim \tilde{D} \tilde{D}^{-\frac{n-2}{n-1}} = \tilde{D}^{\frac{1}{n-1}}$ and $s \geq \frac{1}{2}$, by the second part of Lemma 2.2, we have $|I_5| \lesssim \tilde{D}^{\frac{2}{n-1}} \lesssim 1$. 


And since $E \lesssim -1$ on $[1, \left(\frac{2}{c_1 D}\right)^{\frac{1}{m-1}}]$ and $E \gtrsim 1$ on $\left(\frac{1}{2c_2 X} D\right)^{\frac{1}{m-1}}, \infty)$, we also have $|I_4|, |I_6| \lesssim 1$.

If $\frac{a}{c_2} \leq D \leq \frac{a}{c_1}$, choose a large number $M$ depending only on $c_1, c_2$, and divide $I$ as follows: $I = \int_0^M + \int_M^\infty$. Then as the estimate of $I_1$ and $I_2$, we can obtain $|I| \lesssim 1$.

If $0 < a < 1$ and $\frac{a}{2} < s < \frac{1}{2}$, then except for the integral $I_5$, we can treat every integral by the same method as above. For $I_5$, since $|E'| \sim D^{-\frac{a-2}{s-1}} = D^{-\frac{1}{m-1}}, |A| \leq 2$ and $s \leq \frac{1}{2}$, by the second part of Lemma 2.2, we have $|I_5| \lesssim B^{-\frac{a(1-2\alpha)}{m-1-2\alpha}}$. This completes the proof of lemma. □

**Lemma 2.4.** Let $\delta$ be a positive number and $\alpha, \beta$ be the real number with $0 < |\alpha| \leq 1, \beta \neq 0$. Let $\varphi$ be a $C_0^\infty(\mathbb{R})$ function with the support away from the origin. Consider the oscillatory integral

$$I_\delta(\alpha, \beta) = \delta \int e^{2\pi i (\alpha \phi(\delta \xi) + \delta \beta \xi)} \varphi(\xi) \, d\xi,$$

where $\phi$ satisfies the assumption A. If $\delta \geq 1$ and $0 < a \neq 1$, then $\int |I_\delta(\alpha, \beta)| \, d\beta \leq C_a \delta^\frac{3}{2}$. The constant $C_a$ doesn’t depend on $\alpha$.

**Proof of Lemma 2.4.** If $|\beta| > C_a \delta^a \alpha$ and $|\delta | \geq 1$, then by the integration by part, we have

$$|I_\delta| \leq C(a, \mu) \delta(1 + |\beta|)^{-\mu}$$

for any positive number $\mu$. By the seconde part of Lemma 2.2, we have

$$|I_\delta| \leq C_a \delta(\delta |\alpha|)^{-\frac{1}{2}} \quad \text{and} \quad |I_\delta| \leq C \delta.$$

We divide the integral $\int |I_\delta| \, d\beta$ into four part as follows.

$$\int |I_\delta| \, d\beta = \int_{|\beta| \leq 1} + \int_{1 \leq |\beta| \leq C_a \delta^a} + \int_{C_a \delta^a |\alpha| \leq |\beta| \leq C_a \delta^a} + \int_{|\beta| > C_a \delta^a} \equiv \sum_{i=1}^4 I_i.$$

Now we estimate each term. At first, by the second part of (2.3), $I_1 \leq C \delta \delta^{-1} = C$. For $I_2$, using the first part of (2.3),

$$I_2 \leq C_a \delta \int_{|\beta| \leq C_a \delta^a - 1 |\alpha|} \left( \delta^a |\alpha| \right)^{-\frac{1}{2}} d\beta \leq C_a \delta^\frac{1}{2} \int_{|\beta| \leq C_a \delta^a - 1} |\beta|^{-\frac{1}{2}} d\beta \leq C_a \delta^\frac{3}{2}.$$

Using (2.2) with $\mu = 1 - \frac{a}{2+2a}$, for $I_3$, we have

$$I_3 \leq C_a \delta^{1-\mu} \int_{C_a \delta^a |\alpha| \leq |\beta| \leq C_a \delta^a} |\beta|^{-\mu} d\beta \leq C_a \delta^{(1-\mu)(1+a)} = C_a \delta^\frac{2}{2}.$$

Finally, for $I_4$, using (2.2) with large $\mu$, we have

$$I_4 \leq C(a, \mu) \delta^{1-\mu} \int_{|\beta| > C_a \delta^a - 1} |\beta|^{-\mu} d\beta \leq C(a, \mu) \delta^{\alpha(1-\mu)} \leq C_a \delta^\frac{3}{2}.$$

This completes the proof of lemma. □
3. Proof of Theorem 1.1

3.1. Proof of (1). Using Fourier transform of the radial function and spherical harmonic function (see [15]),

\[ \hat{f}(\xi) = c_{n,k} |\xi|^{-\nu(k)} |\xi|^k Y_k(\xi') \int_0^\infty f_0(r) J_{\nu(k)}(2\pi r|\xi|) r^{\frac{n+2k}{2}} \, dr, \]

where \( \nu(k) = \frac{n+2k}{2} - 2 \) and \( |c_{n,k}| \leq C \). Let \( G_0(\rho) = \rho^{k+\frac{n+1}{2}} g_0(\rho) \) and \( g_0(\rho) = \rho^{-\nu(k)} \int_0^\infty f_0(r) J_{\nu(k)}(2\pi r \rho) r^{\frac{n+2k}{2}} \, dr \). We define an auxiliary operator \( T_R \) by

\[ T_R f(x, t) = \int e^{2\pi i (x \cdot \xi + t \cdot \phi(\xi))} \hat{f}(\xi) \frac{d\xi}{|\xi|^s}. \]

Then using Fourier transform of the spherical harmonic function again, it can be written as:

\[ T_R f(x, t) = c_{n,k} \int e^{2\pi i (x \cdot \xi + t \cdot \phi(\xi))} |\xi|^k Y_k(\xi) g_0(|\xi|) \frac{d\xi}{|\xi|^s} \]
\[ = c_{n,k} \int_0^\infty e^{2\pi i \phi(\rho) \rho^{k+n-1-s}} g_0(\rho) \left( \int_{S^{n-1}} e^{2\pi i r \cdot \xi'} Y_k(\xi') \, d\xi' \right) \, d\rho \]
\[ = c_{n,k} \int_0^\infty e^{2\pi i \phi(\rho) \rho^{k+n-1-s}} g_0(\rho)(r\rho)^{-\frac{n+2k}{2}} J_{\nu(k)}(2\pi r \rho) \, dY_k(-x') \]
\[ = c_{n,k} r^{-\frac{n+2k}{2}} \int_0^1 e^{2\pi i \phi(\rho) \rho^{\frac{1}{2}-s}} G_0(\rho) J_{\nu(k)}(2\pi r \rho) \, dY_k(-x'). \]

Then we have \( ||T_R^* f||_{L^p_k} \leq C ||T_{o,k}^* G_0(\cdot) r^{\frac{n-1}{p}} ||_{L^p} ||Y_k||_{L^p(S^{n-1})} \).

Suppose we prove that

\[(3.1) \quad ||T_{o,k}^* G_0(\cdot) r^{\frac{n-1}{p}} ||_{L^p} \leq A_k ||G_0||_{L^2}, \quad ||G_0||_{L^2} \leq B_k ||r^k f_0(\cdot) r^{\frac{n+2k}{2}} ||_{L^2}, \]

where \( |A_k|, |B_k| \leq C(n + 2k)^{\frac{n+2k}{2}} \). Then we have

\[ ||T_R^* f||_{L^p_k} \leq C_k ||r^k f_0(\cdot) r^{\frac{n+1}{2}} ||_{L^2} ||Y_k||_{L^p(S^{n-1})} = C_k \frac{||Y_k||_{L^p(S^{n-1})} ||f||_{L^p}}{||Y_k||_{L^2(S^{n-1})}}, \]

This proves the theorem.

Now we first prove that \( ||G_0||_{L^2} \leq B_k ||r^k f_0(\cdot) r^{\frac{n+1}{2}} ||_{L^2} \). If we recall the definition of \( G_0 \), then since \( G_0(\rho) = \int_0^\infty F_0(r) J_{\nu(k)}(2\pi r \rho) \rho^\frac{1}{2} \, dr \), where \( F_0(r) = r^{k+\frac{n+1}{2}} f_0(r) \). Thus we have only to show that \( ||G_0||_{L^2} \leq B_k ||F_0||_{L^2} \).

Dividing the integral region into two parts: \( G_0 = \int_0^1 + \int_1^\infty \equiv G_1 + G_2 \). For \( G_1 \), we have \( \frac{1}{\rho} \left| G_1 \left( \frac{1}{\rho} \right) \right| \leq C \frac{1}{\rho} \int_0^\rho |F_0(r)| \, dr \leq CM(F_0) \left( \frac{\rho}{\rho} \right) \), where \( M \) is the Hardy-Littlewood maximal function. Therefore \( ||G_1||_{L^2} \leq C ||F_0||_{L^2}. \)
Using the asymptotic behavior (3.2) of Bessel function (see Lemma 2.3 in [19]):

\[
\begin{aligned}
|J_\nu(r)| &\leq Cr^\nu \quad \text{for} \quad r \leq 1, \\
J_\nu(r) &= r^{-\frac{1}{2}}(b_+ e^{i\nu r} + b_- e^{-i\nu r}) + \Psi_\nu(r)r^{-\frac{3}{2}} \quad \text{for} \quad r \geq 1,
\end{aligned}
\]

we have

\[
G_2(\rho) = \int_0^\infty F_0(r)(b_+ 2e^{2\pi i\rho r} + b_- e^{-2\pi i\rho r}) \, dr + \int_0^\infty F_0(r)\Psi_\nu(k)(r\rho)^{-1} \, dr
\]

\[
\equiv G_{2,+} + G_{2,-} + G_3.
\]

For \(G_3\), we have

\[
\left\| \int_0^\infty F_0(r) \, dr \right\|_{L^2} \leq C(n + 2k)^{\frac{n+2k}{2}} \int_0^\infty F_0(r) \, dr
\]

\[
= \sup_{||g||_{L^2} \leq 1} \left| \int_0^\infty \int_0^\infty \frac{F_0(r)}{r} \, dr \, g(\rho) \, d\rho \right|
\]

\[
\leq C \sup_{||g||_{L^2} \leq 1} \left| \int_0^\infty F_0(r) \frac{1}{r} \int_0^r g(\rho) \, d\rho \, dr \right|
\]

\[
\leq C \sup_{||g||_{L^2} \leq 1} ||F_0||_{L^2} ||M(g)||_{L^2} \leq C ||F_0||_{L^2}.
\]

We write \(G_{2,\pm}\) as \(G_{2,\pm}(\rho) = b_\pm \int_0^\infty e^{2\pi i\rho r} F_0(r) \, dr - b_\pm \int_0^\frac{1}{\rho} e^{2\pi i\rho r} F_0(r) \, dr\).

By the Plancheral theorem and the similar estimate of \(G_1\), we get

\[
||G_{2,\pm}||_{L^2} \leq ||F_0||_{L^2}.
\]

Next we prove the first part of (3.1). Let \(S_{0,k}G = r^{\frac{n-1}{\nu}} T_{0,k}G\) and \(S_{0,k}^d\) be the dual operator of \(S_{0,k}\). Then for any \(F \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})\), we may write \(S_{0,k}^d F\) as follows:

\[
S_{0,k}^d F(\rho) = \rho^\frac{1}{2} - s \int_0^\infty e^{-2\pi i\phi(\rho)} J_{\nu(k)}(2\pi r \rho)^{s - \gamma} F(r, t) \, dr \, dt
\]

\[
= \rho^\frac{1}{2} - s \int_0^\infty e^{-2\pi i\phi(\rho)} \mathcal{O}((r\rho)^{\nu(k)}) (2\pi r \rho)^{s - \gamma} F(r, t) \, dr \, dt
\]

\[
+ \rho^{-s} \int_0^\infty e^{-2\pi i\phi(\rho)} (b_+ e^{2\pi i\rho} + b_- e^{-2\pi i\rho}) r^{-\gamma} F(r, t) \, dr \, dt
\]

\[
+ \rho^{-1-s} \int_0^\infty e^{-2\pi i\phi(\rho)} \Psi_\nu(k)(r\rho)^{-1-\gamma} F(r, t) \, dr \, dt
\]

\[
\equiv A + B_+ + B_- + C,
\]

where \(\gamma = (n-1)(\frac{1}{2} - \frac{1}{\nu})\).

We first estimate \(A\). Since \(r \rho \leq 1\), we have

\[
|A(\rho)| \lesssim \rho^\frac{n-1}{2} - s \int_0^\infty \frac{1}{\rho} r^{\frac{n-1}{2} - \gamma} ||F(r, \cdot)||_{L^1} \, dr \leq \rho^{-s} \int_0^\infty r^{-\gamma} ||F(r, \cdot)||_{L^1} \, dr.
\]
Using the identity \( \|A(\rho)\|_{L^2} = \|\frac{1}{\rho} A(\frac{1}{\rho})\|_{L^2} \), we estimate \( \frac{1}{\rho} A(\frac{1}{\rho}) \) and hence we have

\[
\frac{1}{\rho} \left| A \left( \frac{1}{\rho} \right) \right| \lesssim \int_0^\rho \frac{r^{-\gamma}}{\rho^{1-s}} \|F(r, \cdot)\|_{L^1_r} dr \lesssim I_\gamma(r^{-\gamma}) \|F(r, \cdot)\|_{L^1_r}(\rho).
\]

From Lemma 2.1 with \( \alpha_1 = s + \frac{1}{2} - \frac{1}{p} \) and \( p = \frac{2n}{n-2s} \), we get

\[
\|A\|_{L^2_r}^2 \lesssim \int |\xi|^{-2s} |r^{-\gamma} \|F(r, \cdot)\|_{L^1_r}|^\wedge(\xi)^2 d\xi \lesssim \left( \int |r^{-\gamma} \|F(r, \cdot)\|_{L^1_r}| r^{\alpha_1' p'} dr \right)^{\frac{2}{p'}}.
\]

Since \( \alpha_1 = \gamma \), we get \( \|A\|_{L^2_r} \lesssim \|F\|_{L^p_r L^1_t} \) for \( p = \frac{2n}{n-2s} \).

For \( C \), from (3.2), we have \( |C| \lesssim \rho^{-1-s} A_k \int_0^\infty r^{-1-\gamma} \|F(r, \cdot)\|_{L^1_r} dr \), where \( A_k = (n + 2k)^{\frac{n+2k}{2}} \). Similarly to the estimate of \( A \) with \( p = \frac{2n}{n-2s} \), we obtain

\[
\frac{1}{\rho} \left| C \left( \frac{1}{\rho} \right) \right| \lesssim \rho^s \int_\rho^\infty r^{-1-\gamma} \|F(r, \cdot)\|_{L^1_r} dr \lesssim \int_\rho^\infty \frac{r^{-\gamma}}{\rho^{1-s}} \|F(r, \cdot)\|_{L^1_r} dr \lesssim I_\gamma(r^{-\gamma}) \|F(r, \cdot)\|_{L^1_r}(\rho).
\]

Therefore by using Lemma 2.1, we also have \( \|C\|_{L^2_r} \lesssim A_k \|F\|_{L^p_r L^1_t} \).

Now we estimate \( B_\pm \). To do this, we use the extended operator \( B \) such that

\[
BF(\rho) = \rho^{-s} \int_{|r| > \frac{1}{\rho}} e^{-2\pi i (t\phi(\rho) + r\rho)} |r|^{-\gamma} F(r, t) dr dt
\]

\[
= \rho^{-s} \int_{\mathbb{R}^2} e^{-2\pi i (t\phi(\rho) + r\rho)} |r|^{-\gamma} F(r, t) dr dt + \rho^{-s} \int_{|r| \leq \frac{1}{\rho}} e^{-2\pi i (t\phi(\rho) + r\rho)} |r|^{-\gamma} F(r, t) dr dt \equiv B_1(\rho) + B_2(\rho).
\]

For \( B_1 \), we have formally

\[
\|B_1\|_{L^2_r}^2 = C \int \int \int \int K(r, r', t, t') r^{-\gamma} F(r, t) r'^{-\gamma} \overline{F(r', t')} dr dr' dt dt',
\]

where

\[
K(r, r', t, t') = \int e^{-2\pi i ((t-t')\phi(\rho) + (r-r')\rho)} \rho^{-2s} d\rho.
\]

Since \( \frac{1}{4} \leq s < \frac{1}{2} \), by Lemma 2.3, we have \( |K(r, r', t, t')| \lesssim |r - r'|^{-(1-2s)} \).

Thus we have

\[
\|B_1\|_{L^2_r}^2 \lesssim \int |r - r'|^{-(1-2s)} r^{-\gamma} \|F(r, \cdot)\|_{L^1_r} r'^{-\gamma} \|F(r', \cdot)\|_{L^1_{r'}} dr dr'.
\]

Invoking Lemma 2.1, we can get \( \|B_1\|_{L^2_r}^2 \lesssim \left( \int (r^{-\gamma} \|F\|_{L^1_r})^p r^{\alpha' p'} dr \right)^{\frac{2}{p'}} \), provided \( \alpha = s + \frac{1}{2} - \frac{1}{p} \). Since \( \alpha = \gamma \), we finally have \( \|B_1\|_{L^2_r} \lesssim \|F\|_{L^p_r L^1_t} \).
For $B_2$, we have $|B_2| \lesssim \rho^{-s} \int_0^{1/\rho} r^{-\gamma} |F(r, \cdot)|_{L^1_t} dr$. Thus we also have

$$\frac{1}{\rho} |B_2| \lesssim \frac{1}{\rho^{1-s}} \int_0^{\rho} r^{-\gamma} |F(r, \cdot)|_{L^1_t} dr \lesssim I_s(r^{-\gamma} |F(r, \cdot)|_{L^1_t}).$$

Using Lemma 2.1, we finally have $\|B_2\|_{L^2} \lesssim \|F\|_{L'_t L^1}$. This completes the proof of (1) of the theorem.

### 3.2. Proof of (2)

Let us first define an auxiliary operator $T_B$ by

$$T_B f(x, t) = \int e^{2\pi i (t(\xi) + x \cdot \xi)} \hat{f}(\xi) \frac{d\xi}{(1 + |\xi|)^s}.$$  

Using Fourier transform of the spherical harmonic function again, the original operator $T_B$ can be written as:

$$Tf(x, t) = c_{n,k} r^{-\frac{n+1}{2}} \int_0^\infty e^{2\pi i t (\rho)} \rho^{\frac{1}{2}} G_0(\rho) J_{\nu(k)}(2\pi \rho r) \frac{d\rho}{(1 + \rho)^s} Y_k(-x')$$

$$= c_{n,k} \tilde{T}_{0,k} G_0(r, t) Y_k(-x'),$$

where $G_0$ is the same function as in the proof of (1). From the the proof of (1), we have only to prove that $\|\tilde{T}_{0,k} G_0(\cdot) r^{-\frac{n+1}{2}}\|_{L^2_t} \lesssim A_k \|G_0\|_{L^2}$. 

Let $\tilde{S}_{0,k} G_0 = r^{\frac{n+1}{2}} \tilde{T}_{0,k} G_0$. Now we divide the integral region of $\tilde{S}_{0,k} G_0$ into two parts: $\rho \leq \frac{1}{r}$ and $\rho > \frac{1}{r}$. Then we have

$$\tilde{S}_{0,k} G_0(r, t) = r^{\frac{1}{2}} \int_0^\frac{1}{r} e^{2\pi i t (\rho)} \mathcal{O}((\rho r)^{\nu(k)}) \rho^{\frac{1}{2}} G_0(\rho) \frac{d\rho}{(1 + \rho)^s}$$

$$+ r^{\frac{1}{2}} \int_0^\frac{1}{r} e^{2\pi i t (\rho)} (\rho r)^{\frac{1}{2}} (b_+ e^{2\pi i \rho r} + b_- e^{-2\pi i \rho r}) \rho^{\frac{1}{2}} G_0(\rho) \frac{d\rho}{(1 + \rho)^s}$$

$$+ r^{\frac{1}{2}} \int_\frac{1}{r}^\infty e^{2\pi i t (\rho)} \Psi_{\nu(k)}(\rho r) \rho^{\frac{1}{2}} G_0(\rho) \frac{d\rho}{(1 + \rho)^s}$$

$$\equiv \tilde{A} + \tilde{B}_+ + \tilde{B}_- + \tilde{C}.$$

Let us define the maximal functions

$$\tilde{A}^*(r) = \sup_{0 < t < 1} |\tilde{A}(r, t)| \quad \text{and} \quad \tilde{C}^*(r) = \sup_{0 < t < 1} |\tilde{C}(r, t)|.$$

For these maximal function, we want to show that $\|\tilde{A}^*\|_{L^2_t} + \|\tilde{C}^*\|_{L^2_t} \lesssim A_k \|G_0\|_{L^2}$, where $A_k = (n + 2k)^{\frac{n+2k}{2}}$. To prove this, let us first observe from the asymptotic behavior (3.2) that

$$\left| \frac{1}{r} \tilde{A}^* \left( \frac{1}{r} \right) \right| \lesssim r^{-1} \int_0^r G_0(\rho) d\rho \lesssim 2M \left( \frac{r}{2} \right), \quad \left| \frac{1}{r} \tilde{C}^* \left( \frac{1}{r} \right) \right| \lesssim A_k \int_r^\infty \rho^{-1} G_0(\rho) d\rho.$$

Thus using the Hardy-Littlewood maximal function, we have

$$\|\tilde{A}^*\|_{L^2_t} = \left\| \frac{1}{r} \tilde{A}^* \left( \frac{1}{r} \right) \right\|_{L^2} \lesssim \|G_0\|_{L^2}.$$
For $C^*$, we have
\[
\|C^*\|_{L^2} \lesssim A_k \sup_{\|\varphi\|_{L^2} \leq 1} \left| \int_0^\infty \int_r^\infty \rho^{-1} |G_0(\rho)| \, d\rho \varphi(r) \, dr \right|
\]
\[
\leq A_k \sup_{\|\varphi\|_{L^2} \leq 1} \int_0^\infty |G_0(\rho)| \rho^{-1} \int_0^\rho |\varphi(r)| \, dr \, d\rho
\]
\[
\leq 2A_k \sup_{\|\varphi\|_{L^2} \leq 1} \int_0^\infty |G_0(\rho)| \mathcal{M}(\varphi)(\rho/2) \, d\rho \lesssim \|G_0\|_{L^2}.
\]

To estimate $\tilde{B}_ \pm$, we use the extended operator $\tilde{B}$ defining $G_0$ by $G_0(-\rho)$ for $\rho \leq 0$ such that
\[
\tilde{B}G_0(r, t) = \int_{|\rho| \geq \frac{1}{r}} e^{2\pi i (t(\rho) + r\rho)} G_0(\rho) \frac{d\rho}{(1 + |\rho|)^r}.
\]
We can rewrite this as
\[
\tilde{B}G_0(r, t) = \int_\mathbb{R} + \int_{|\rho| < \frac{1}{r}} \equiv \tilde{B}_1 G_0 + \tilde{B}_2 G_0. \quad \text{Let } \tilde{B}_1^* G_0(r) = \sup_{|t| < 1} |\tilde{B}_1 G_0| \text{ and } \tilde{B}_2^* G_0(r) = \sup_{|t| < 1} |\tilde{B}_2 G_0|. \quad \text{Then we first have } \tilde{B}_2^* G_0(r) \lesssim \int_0^\frac{1}{r} |G_0(\rho)| \, d\rho \text{ and hence}
\]
\[
\frac{1}{r} \tilde{B}_2^* G_0 \left( \frac{1}{r} \right) \lesssim \frac{1}{r} \int_0^r |G_0(\rho)| \, d\rho \lesssim \mathcal{M}(|G_0|)(\frac{r}{2}),
\]
where $\mathcal{M}$ is the Hardy-Littlewood maximal function. Thus $\|\tilde{B}_2^* G_0\|_{L^2} \lesssim \|G_0\|_{L^2}$.

Next, we consider global $L^2$ estimate of the local maximal operator $\tilde{B}_1^*$. To do this, we employ the Kolmogorv-Seliverstov-Plessner method. Let us defined an operator $T$ as
\[
TG_0(r) = \int e^{2\pi i (t(r) \varphi(|\rho|))} G_0(\rho) \frac{d\rho}{(1 + |\rho|)^s},
\]
where $t(r)$ is any measurable function with $|t(r)| < 1$ on $\mathbb{R}$. Then we may write the operator $T$ by $T_j$ as $TG_0(r) = \sum_{j \geq 0} T_j G_0(r)$, where
\[
T_j G_0(r) = \int e^{2\pi i (t(r) \varphi(|\rho|))} G_0(\rho) \varphi_j(\rho) \frac{d\rho}{(1 + |\rho|)^s} \quad \text{for } j \in \mathbb{Z},
\]
where $\varphi_j$ are Littlewood-Paley functions such that $\varphi_0$ is supported in unit ball $B(0, 1)$, $\varphi_j(\cdot) = \varphi(\frac{\cdot}{2^j})$ is supported in $B(0, 2^{j+1}) \setminus B(0, 2^{j-1})$ and $\sum_{j \geq 0} \varphi_j = 1$. We claim that $\|T_j G_0\|_{L^2} \lesssim 2^{\frac{1}{2}j} \|\Delta_j G_0\|_{L^2}$, where $\Delta_j \varphi = \varphi_j \varphi_j$. To show that, let $T_j^d$ be the dual operator of $T_j$. Then for any $F(r) \in C_0^\infty(\mathbb{R})$ and $j \geq 1$,
\[
\|T_j^d F\|_{L^2}^2 = \iiint K_j(r, r') F(r) \overline{F(r')} \, dr \, dr'.
\]
where

\[ K_j(r, r') = 2^{(1-2s)j} \int e^{-2\pi i((t(r)-t(r'))\phi(2^j|\rho|)+2^j(r-r')\rho)} 2^s \varphi_j^2(\rho) \frac{d\rho}{(1+2|\rho|)^{2s}}. \]

Since \( \frac{2^s \varphi_j^2(\rho)}{(1+2|\rho|)^{2s}} \) and its derivatives are uniformly bounded on \( j \), from Lemma 2.4 replacing \( \delta \) by \( 2^j \), we have

\[ \sup_{r' \in \mathbb{R}} \int |K_j(r, r')|dr, \quad \sup_{r \in \mathbb{R}} \int |K_j(r, r')|dx' \lesssim 2^{\frac{2}{3}(2-2s)j}. \]

It follows from the Schur’s lemma (see the lemma in p.284 of [14]) that

\[ ||T^d_j F||_{L^2} \lesssim 2^{\left(\frac{2}{3}-s\right)j} ||F||_{L^2}. \]

If \( j = 0 \), then

\[ ||T^d_0 F||_{L^2}^2 = \iint K_0(r, r') F(r) \overline{F(r')} drdr' \]

where

\[ K_0(r, r') = \int e^{-2\pi i((t(r)-t(r'))\phi(|\rho|)+(r-r')\rho)} \varphi_0^2(\rho) \frac{d\rho}{(1+|\rho|)^{2s}}. \]

Since \( |t-t'| \leq 2 \), using the integration by part several times, for any \( \mu \), we have

\[ \sup_{(t, t') \in [0,1]^2} |K_0(r, r')| \lesssim (1 + |r - r'|)^{-\mu}. \]

Thus choosing a large \( \mu \) and using Shur’s lemma again, we have \( ||T^d_0 F||_{L^2}^2 \lesssim ||F||_{L^2} \). Therefor combining this and (3.3), we prove the part (2) of theorem.

3.3. Proof of (3) and (4). Let us define an operator \( S_{0,k} \) by \( S_{0,k}G_0(r, t) = r^{-\frac{k}{4}}T_{0,k}G_0(r, t) \), where \( T_{0,k} \) is the same operator with \( s = \frac{1}{4} \) as in the proof of (1). Using the spherical coordinate and following exactly the same argument as in the proof of Theorem 1.1 with \( s = \frac{1}{4} \) and \( p = 2 \), one can easily obtain the following estimate

\[ ||T^{*s} f||_{L^2(|x|^{-\frac{1}{2}} dx)} \leq C_k ||f||_{H^{\frac{1}{4}}}. \]

Since from the result (2) we have

\[ ||T^{*s} f||_{L^2} \leq ||f||_{H^{\gamma}} \quad \text{for any} \quad s > \frac{\alpha}{4}, \]

by the complex interpolation [1] between (3.4) and (3.5) with \( s \) such that \( (1 - \theta)s + \frac{\theta}{4} = \frac{\alpha}{4} \) for \( \theta = 2\varepsilon \), we can obtain the desired result.

For the proof of (4), replacing \( r^{-\gamma} \) in the definition of \( S_{0,k}G_0 \) with \( r^{-\gamma}(1 + r)^{-\frac{b}{2}} \) and using the second part of Lemma 2.3, by the same method as the one in the proof of (1).

**Remark 3.1.** For one dimensional global or higher dimensional local estimate similar to (3.4), see [18] and [11].
GLOBAL ESTIMATES OF MAXIMAL OPERATORS

REFERENCES


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