GLOBAL ESTIMATES OF MAXIMAL OPERATORS
GENERATED BY DISPERSIVE EQUATIONS

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Abstract. Let \( T f(x,t) = e^{2\pi i \phi(D)} f \) be the solution of the general dispersive equation with the phase function \( \phi \) and initial data \( f \) in the Schwartz class. In case that the phase \( \phi \) has a suitable growth rate at the infinity and the origin and \( f \) is a finite linear combination of radial and spherical harmonic functions, we have global \( L^p \) estimates of maximal operator defined by taking the supremum w.r.t. \( t \). In particular, we obtain a global estimate at the end point left open.

1. Introduction

The general dispersive equation is defined by

\[ iu_t(x) = -2\pi \phi(D)u(x), \quad \text{on} \quad \mathbb{R}^n \times \mathbb{R}, \quad u(x,0) = f(x) \in \mathcal{S}(\mathbb{R}^n)(n \geq 3), \]

where \( D = \frac{1}{2\pi i} \nabla \) and \( \phi \) is a smooth phase function. The solution of this equation can be formulated formally by

\[ u(x,t) = T f(x,t) = \int e^{2\pi i (x\cdot\xi + t\phi(\xi))} \hat{f}(\xi) \, d\xi, \tag{1.1} \]

where \( \hat{f}(\xi) = \int e^{-2\pi i x\cdot\xi} f(x) \, dx \). From the formulation, we define maximal operators defined by

\[ T^* f(x) := \sup_{-1 \leq t \leq 1} |T f(x,t)|, \quad T^{**} f(x) := \sup_{t \in \mathbb{R}} |T f(x,t)|. \]

In this paper we are mainly concerned with a mapping property of \( T^{**} \) defined with the various phase function, whose typical model is \( |\xi|^a(a \neq 0) \), from Sobolev space to \( L^p \) such that

\[ ||T^* f||_{L^p} \quad \text{or} \quad ||T^{**} f||_{L^p} \leq C||f||_{X^s}, \tag{1.2} \]

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where $X^s$ is denoted by $\dot{H}^s$ and $H^s$. Here the Sobolev spaces are defined by the norms:

$$
\|f\|_{\dot{H}^s} = \left( \int |\xi|^{2s} |\hat{f}(|\xi|)^2 d\xi \right)^{\frac{1}{2}},
$$

$$
\|f\|_{H^s} = \left( \int (1 + |\xi|)^{2s} |\hat{f}(|\xi|)^2 d\xi \right)^{\frac{1}{2}}.
$$

To control the global estimate of maximal operator $T^*$ or $T^{**}$, it seems inevitable to assume the growth rate and regularity of the phase function. Therefore we impose the following assumption on the phase $\phi$.

A. Let $\phi$ be a radial function such that for some $a \in \mathbb{R}$ ($\neq 0, 1$), $\phi \in C^2(\mathbb{R}^n \setminus \{0\})$ and there exist positive constants $c_1, c_2$ such that

$$
c_1|x|^{a-k} \leq |\phi^{(k)}(\xi)| \leq c_2|x|^{a-k} \quad (k = 0, 1, 2), \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}.
$$

The maximal inequality (1.2) is motivated from the well-known pointwise convergence problem: $\lim_{t \rightarrow 0} u(x, t) = f(x)$ a.e. $x$, for $f \in H^s(\mathbb{R}^n)$, during the three decades, the local and global $L^p$ estimate of the maximal operators have been studied by many authors [2, 3, 4, 5, 7, 9, 16] and [17] etc. P. Sjölin [13] and L. Vega [17] obtained the strong necessary condition ($s \geq \frac{1}{4}$) on the pointwise convergence problem. In particular, P. Sjölin showed that the maximal operator $T^{**}$ cannot have the global $L^2$ boundedness (see [10]), and that $\|T^* f\|_{L^2} \leq C \|f\|_{H^s}$ holds for $s > \frac{a}{4}$ and fails for $s < \frac{a}{4}$ ($a > 1$) (see [13]). But if the data $f$ is a finite linear combination of radial and spherical harmonic functions, then we prove that the inequality (1.2) holds for $s = \frac{1}{4}$ and $\frac{a}{4}$ with suitable weight.

Now we state our main results.

**Theorem 1.1.** Let $f_0 \in C_0^\infty(\mathbb{R})$ and $Y_k$ be a spherical harmonic function of order $k \geq 0$. If $f(x) = |x|^k f_0(|x|)Y_k(x')$,

1. then $\|T^{**} f\|_{L^p} \leq C_k \|Y_k\|_{L^p(S^{n-1})} \|f\|_{H^s}$ holds for $s \in [\frac{1}{4}, \frac{1}{2})$, $p = \frac{2n}{n-2s}$ and $0 < a \neq 1$, where $C_k = O((n + 2k)^{n+2k})$ as $k \rightarrow \infty$.
2. then $\|T^* f\|_{L^2} \leq C_k \|f\|_{H^s}$ holds for any $s > \frac{a}{4}$ and $0 < a \neq 1$.
3. then $\|T^* f\|_{L^2(|x|^{-\varepsilon} dx)} \leq C_{\varepsilon, k} \|f\|_{H^s}$ holds for any $a > 1$ and $\varepsilon > 0$.
4. then $\|T^* f\|_{L^p(|1+|x|)^{-b} dx)} \leq C_k \|Y_k\|_{L^p(S^{n-1})} \|f\|_{H^s}$ holds for $0 < a < 1, \frac{a}{4} < s < \frac{1}{4}$, $p = \frac{4n(1-a)}{2n(1-a)+a-4s}$ and $b = sp - n(\frac{b}{2} - 1)$.

We obtain optimal global estimates except for the end point $(s, p) = (\frac{a}{4}, 2)$ ($0 < a < 1$) which was studied by B. G. Walther [18]. These are obtained by using a bound of one dimensional oscillatory integral of the form $\int_{\mathbb{R}} e^{2\pi i \phi(t) + \alpha t} \xi^{-s} d\xi$. Many authors referred in this paper have tried to handle such an integral and obtained various bounds according to the value of growth rate $a$ of $\phi$ (i.e. $a > 1$ or $a < 1$). Here, we provide that the integral is bounded by $C|x|^{-(1-s)}$ for any $a \neq 1$ and the constant $C$ is
independent of $t$ (see Lemma 2.3). Thanks to the time independency of the integral, the proof of main results are much more simplified.

S. Wang [19] showed that if $p > 2$, then there exist $f_0$ and $Y_k$ such that if $f(x) = |x|^k f_0(|x|) Y_k(x')$, then $\lim_{k \to \infty} \|T^* f\|_{L^p(B)/\|f\|_{H^n_k}} = \infty$ for any ball $B$. Thus it will be interesting to prove a local $L^p$ estimate of $T^*$ holds uniformly on $k$ for some $p \in [1, 2]$. For the more general initial data $f$, by using a bilinear estimate, T. Tao in [16] showed that the global estimate for integral, the proof of main results are much more simplified.

As another global estimates, there are several results about the weighted estimates for \textit{As another global estimates, there are several results about the weighted estimates for} $s > \frac{1}{2}$. For these results, one can refer [5, 17, 18]. But it remains still open question whether even a local estimate of $T^*$ holds for $s = \frac{1}{2}$.

If not specified, throughout this paper, $C$ denotes by a generic constant that depends on $c_1, c_2, a, s, n$. We use the notation $A \lesssim B$ and $A \sim B$ to denote $|A| \leq CB$ and $C^{-1}B \leq |A| \leq CB$ respectively.

2. Preliminary Lemmas

We begin with the weighted inequality for the Fourier transform.

**Lemma 2.1** (see [8]). If $1 \leq q \leq 2$, $0 \leq \alpha < \frac{1}{2}$, $0 \leq \alpha_1 < \frac{1}{q}$ and $\alpha_1 = \alpha + \frac{1}{2} - \frac{1}{q}$, then the following inequality holds

$$\left( \int_{\mathbb{R}} |\xi|^{-2\alpha} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \lesssim \left( \int_{\mathbb{R}} |f(x)|^q |x|^\alpha q dx \right)^{\frac{1}{q}}.$$

Now we introduce some estimates of oscillatory integrals. Let us first state a stationary phase lemma which can be found in [7] etc..

**Lemma 2.2.** Let $\psi$ be a monotone function and $I = \int_{[\alpha, \beta]} e^{i\varphi(\xi)} \psi(\xi) d\xi$. Then if $|\frac{d\varphi}{d\xi}| \geq \lambda > 0$ in $[\alpha, \beta]$ and $\frac{d\varphi}{d\xi}$ is monotone, $|I| \leq C\lambda^{-1} \sup_{[\alpha, \beta]} |\psi(\xi)|$, and if $|\frac{d^2\varphi}{d\xi^2}| \geq \lambda > 0$, then $|I| \leq C\lambda^{-\frac{1}{2}} \sup_{[\alpha, \beta]} |\psi(\xi)|$. The constant $C$ doesn’t depend on $\alpha, \beta, \lambda, \varphi$ and $\psi$.

Utilizing the lemma above, we get the following lemma.

**Lemma 2.3.** Suppose $\phi$ satisfies the assumption $A$ for $0 < a \neq 1$. Let $A, B, s$ be the real numbers such that $A, B \neq 0$, $\frac{1}{2} \leq s < 1$. Consider the following integral:

$$I = \int_{\xi \in \mathbb{R}} e^{2\pi i (A\phi(\xi) + B\xi)} |\xi|^{-s} d\xi.$$

Then $|I| \leq C(a, s, c_1, c_2) |B|^{-(1-s)}$.

If $0 < a < 1, \frac{a}{2} < s < \frac{1}{2}$ and $|A| \leq 2$, then $I \lesssim |B|^{-(1-s)} + |B|^{-(1-s)-\frac{a(1-2s)}{2(1-s)}}$.
Proof of Lemma 2.3. (Case $a > 1$) Without loss of generality, we may assume that $A > 0$ and $B > 0$. Let $D = \frac{B}{A^s}$. Then by the change of variable, we have

$$I = A^{-\frac{1-a}{a}} \int \epsilon^{2\pi i (A\phi(A^{-\frac{1}{a}}\xi)+D\xi)} |\xi|^{-s} d\xi = \int_{\xi<0} + \int_{\xi>0} = I_- + I_+. $$

We have only to consider $I_+$ and we denote it $I$ again.

Now we first consider the case when $\phi' > 0$. Observe that

$$E \equiv (A\phi(A^{-\frac{1}{a}}\xi) + D\xi)' \geq c_1 \xi^{a-1} + D. $$

Let $M$ be a large positive number depending only on $a, s, c_1, c_2$. If $D \leq M$, then

$$I = A^{-\frac{1-a}{a}} \left( \int_0^1 + \int_{1}^{\infty} \right) = I_1 + I_2. $$

For $I_1$, by direct integration, we have $|I_1| \lesssim A^{-\frac{1-a}{a}} \lesssim B^{-(1-s)}$. For $I_2$, since $E \gtrsim 1$, by the first part of (2.1), we have $|I_2| \lesssim A^{-\frac{1-a}{a}} \lesssim B^{-(1-s)}$.

If $D > M$, then since $E \geq D$, by the first part of Lemma 2.2, we have $|I_2| \lesssim A^{-\frac{1-a}{a}} D^{-1} \leq A\xi B^{-1} \leq B^{-(1-s)}$. For $I_1$, using the change of variable, we have

$$I_1 = A^{-\frac{1-a}{a}} D^{-1} \int_0^D e^{2\pi i (A\phi(D^{-1}A^{-\frac{1}{a}}\xi)+\xi) \xi^{-s} d\xi}. $$

Thus $I_1 = \int_0^1 + \int_1^D = I_{1,1} + I_{1,2}$. By the integration, $|I_{1,1}| \lesssim B^{-(1-s)}$. For $I_{1,2}$, since $(A\phi(D^{-1}A^{-\frac{1}{a}}\xi) + \xi)' \geq 1$, from the first part of Lemma 2.2, we have $|I_{1,2}| \lesssim B^{-(1-s)}$ and hence $|I_1| \lesssim B^{-(1-s)}$.

Now we consider the case when $\phi' < 0$. We observe that

$$-c_2 \xi^{a-1} + D \leq E = (A\phi(A^{-\frac{1}{a}}\xi) + D\xi)' \leq -c_1 \xi^{a-1} + D. $$

If $D \leq M$, then we split $I$ into two parts as follows:

$$I = A^{-\frac{1-a}{a}} \left( \int_0^{(\frac{D}{2})^{\frac{1}{1-a}}} + \int_{(\frac{D}{2})^{\frac{1}{1-a}}}^{\infty} \right) = I_3 + I_4. $$

For $I_5$, we have by direct integration $|I_5| \lesssim A^{-\frac{1-a}{a}} \lesssim B^{-(1-s)}$. For $I_4$, since $E \leq -c_1 \xi^{a-1} + D \leq -1$, by the first part of Lemma 2.2, we get $|I_4| \lesssim A^{-\frac{1-a}{a}} \lesssim B^{-(1-s)}$. If $D > M$, then we split $I$ into four parts as follows:

$$I = A^{-\frac{1-a}{a}} \left( \int_0^1 + \int_1^{(\frac{D}{2})^{\frac{1}{1-a}}} + \int_{(\frac{D}{2})^{\frac{1}{1-a}}}^{(\frac{2D}{2})^{\frac{1}{1-a}}} + \int_{(\frac{2D}{2})^{\frac{1}{1-a}}}^{\infty} \right) \equiv I_5 + I_6 + I_7 + I_8. $$
For $I_5$, we use the change of variable so that

$$I_5 = A^{-\frac{1-s}{a}} D^{-(1-s)} \int_0^D e^{2\pi i (A\phi(D^{-1}A^{-\frac{1}{2}}\xi)+\xi)} \xi^{-s} d\xi.$$  

We split $I_5$ into two part: $I_5 = A^{-\frac{1-s}{a}} D^{-(1-s)} \left(\int_0^1 + \int_1^D\right) = I_{5,1} + I_{5,2}$.

For $I_{5,1}$ and $I_{5,2}$, we use the direct integration and the first part of Lemma 2.2 respectively, we have $|I_{5,1}| + |I_{5,2}| \lesssim A^{-\frac{1-s}{a}} D^{-(1-s)} = B^{-(1-s)}$. For $I_6$, since $E \geq D \geq D^{1-s}$, using the first part of Lemma 2.2, we have $|I_6| \lesssim A^{-\frac{1-s}{a}} D^{-(1-s)} = B^{-(1-s)}$.

To estimate $I_7$, we use the second derivative $|E'| \sim \xi^{a-2} \sim D^{\frac{a-2}{a-1}}$. Then from the second part of Lemma 2.2, we obtain

$$|I_7| \lesssim A^{-\frac{1-s}{a}} D^{-\frac{a-2}{a-1}} D^{\frac{s}{a}} = A^{-\frac{1-s}{a}} D^{-\frac{a^2+2a}{2(a-1)}}.$$  

Since $a > 1$ and $s \geq \frac{1}{2}$, we have $|I_7| \lesssim A^{-\frac{1-s}{a}} D^{-(1-s)} = B^{-(1-s)}$. Finally, we estimate $I_4$. Since $E \geq D \geq D^{1-s}$, by the first part of Lemma 2.2, we have $|I_4| \lesssim A^{-\frac{1-s}{a}} D^{-(1-s)} D^{-\frac{s}{a}} \lesssim B^{-(1-s)}$.

(Case $a < 1$) We first consider the case $\frac{1}{2} \leq s < 1$. We may assume $A, B > 0$. Let $\tilde{D} = \frac{A}{B}$. Then by the change of variable, we write

$$B^{1-s} I = \int e^{2\pi i (A\phi(D) + \xi)} |\xi|^{-s} d\xi = \int_{-\infty}^{0} + \int_{0}^{\infty} = I_+ + I_-.$$  

As in the previous case ($a > 1$), we only consider $I_+$ and denote it by $I$ again.

In case that $\phi' > 0$, we have $E \equiv (A\phi(D) + \xi)' \geq c_1 \tilde{D} \xi^{a-1} + 1 \geq 1$ for all $\xi > 0$. We divide $I$ into two parts: $I = \int_0^1 + \int_1^{\infty}$. For the first integral, we just integrate and for the second one, we use the first part of Lemma 2.2. Then we can see $|I| \lesssim 1$.

Now we consider the case when $\phi' < 0$. Then we can observe that

$$-c_2 \tilde{D} \xi^{a-1} + 1 \leq E \leq -c_1 \tilde{D} \xi^{a-1} + 1.$$  

If $c_2 \tilde{D} < 2$, then we divide $I$ into two parts: $I = \int_0^{1} + \int_1^{\infty} = I_1 + I_2$.

By the integration, we get $|I_1| \lesssim 1$. And since $c_2 \tilde{D} < 2$ and hence $E \geq 1$, by the first part of Lemma 2.2, we have $|I_2| \lesssim 1$.

If $c_1 \tilde{D} > 2$, then we divide $I$ into four parts:

$$I = \int_0^1 + \int_1^{\frac{1}{c_1 \tilde{D}}} + \int_{\frac{1}{c_1 \tilde{D}}}^{\frac{1}{2\tilde{D}}} + \int_{\frac{1}{2\tilde{D}}}^{\infty} = I_3 + I_4 + I_5 + I_6.$$  

For $I_3$, by the integration, $|I_3| \lesssim 1$. For $|I_5|$, since $|E'| \sim \tilde{D} \tilde{D}^{-\frac{a^2-2}{a}} = \tilde{D}^{\frac{1}{a-1}}$ and $s \geq \frac{1}{2}$, by the second part of Lemma 2.2, we have $|I_5| \lesssim \tilde{D}^{\frac{a-1}{2(a-1)}} \lesssim 1$.  

And since $E \lesssim -1$ on $[1, \left(\frac{2}{c_2 D}\right)^{\frac{1}{a-1}}]$ and $E \gtrsim 1$ on $[\left(\frac{1}{2c_2 D}\right)^{\frac{1}{a-1}}, \infty)$, we also have $|I_4|, |I_6| \lesssim 1$.

If $\frac{a}{c_2} \leq \tilde{D} \leq \frac{a}{c_1}$, choose a large number $M$ depending only on $c_1, c_2$, and divide $I$ as follows: $I = \int_0^M + \int_M^\infty$. Then as the estimate of $I_1$ and $I_2$, we can obtain $|I| \lesssim 1$.

If $0 < a < 1$ and $\frac{a}{2} < s < \frac{1}{2}$, then except for the integral $I_5$, we can treat every integral by the same method as above. For $I_5$, since $|E'| \sim \tilde{D}^{-\frac{a-2}{a-1}} = \tilde{D}^{-\frac{1}{a-1}}, |A| \leq 2$ and $s < \frac{1}{2}$, by the second part of Lemma 2.2, we have $|I_5| \lesssim B^{-\frac{a(1-2\beta)}{2(1+a)}}$. This completes the proof of lemma.

□

**Lemma 2.4.** Let $\delta$ be a positive number and $\alpha, \beta$ be the real number with $0 < |\alpha| \leq 1, \beta \neq 0$. Let $\varphi$ be a $C_0^\infty(\mathbb{R})$ function with the support away from the origin. Consider the oscillatory integral

$$I_\delta(\alpha, \beta) = \delta \int e^{2\pi i (\alpha \phi(\delta \xi) + \delta \beta \xi)} \varphi(\xi) d\xi,$$

where $\phi$ satisfies the assumption A. If $\delta \geq 1$ and $0 < a \neq 1$, then $\int |I_\delta(\alpha, \beta)| d\beta \leq C_a \delta^\frac{3}{2}$. The constant $C_a$ doesn’t depend on $\alpha$.

**Proof of Lemma 2.4.** If $|\delta| \beta > C_a \delta^a \alpha$ and $|\delta| \beta \geq 1$, then by the integration by part, we have

$$|I_\delta| \leq C(a, \mu)|\delta| (1 + |\beta|)^{-\mu}$$

for any positive number $\mu$. By the seconde part of Lemma 2.2, we have

$$|I_\delta| \leq C_a \delta^2 |\delta| |\alpha|^{-\frac{a}{2}} \quad \text{and} \quad |I_\delta| \leq C \delta.$$

We divide the integral $\int |I_\delta| d\beta$ into four part as follows.

$$\int |I_\delta| d\beta = \int |\delta| \beta \leq 1 + \int_{|\delta| \beta \leq C_a \delta^a} |\delta| \beta \leq C_a \delta^a \|\alpha\| + \int_{|\delta| \beta \geq 1, C_a \delta^a |\alpha| \leq |\beta| \leq C_a \delta^a} + \int_{|\delta| \beta > C_a \delta^a} \equiv \sum_{i=1}^4 I_i.$$

Now we estimate each term. At first, by the second part of (2.3), $I_1 \leq C \delta \delta^{-1} = C$. For $I_2$, using the first part of (2.3),

$$I_2 \leq C_a \delta \int_{[\delta| \beta| \leq C_a \delta^a - 1 |\alpha|} |\delta| |\alpha|^{-\frac{1}{2}} d\beta \leq C_a \delta^a \int_{|\beta| \leq C_a \delta^a - 1} |\beta|^{-\frac{1}{2}} d\beta \leq C_a \delta^\frac{3}{2}.$$

Using (2.2) with $\mu = 1 - \frac{a}{2 + 2a}$, for $I_3$, we have

$$I_3 \leq C_a \delta^{1-\mu} \int_{C_a \delta^a |\alpha| \leq |\beta| \leq C_a \delta^a} |\beta|^{-\mu} d\beta \leq C_a \delta^{(1-\mu)(1+a)} = C_a \delta^\frac{a}{2}.$$}

Finally, for $I_4$, using (2.2) with large $\mu$, we have

$$I_4 \leq C(a, \mu) \delta^{1-\mu} \int_{|\beta| > C_a \delta^a} |\beta|^{-\mu} d\beta \leq C(a, \mu) \delta^a (1-\mu) \leq C_a \delta^\frac{a}{2}.$$

This completes the proof of lemma. □
3. Proof of Theorem 1.1

3.1. Proof of (1). Using Fourier transform of the radial function and spherical harmonic function (see [15]),

\[ \hat{f}(\xi) = c_{n,k} |\xi|^{-\nu(k)} |\xi|^k Y_k(\xi') \int_0^\infty f_0(r) J_\nu(k)(2\pi r|\xi|) r^{n+2k} \, dr, \]

where \( \nu(k) = \frac{n+2k}{2} \) and \( |c_{n,k}| \leq C \). Let \( G_0(\rho) = \rho^{k+\frac{n-1}{2}} g_0(\rho) \) and \( g_0(\rho) = \rho^{-\nu(k)} \int_0^\infty f_0(r) J_\nu(k)(2\pi r\rho) r^{n+2k} \, dr \). We define an auxiliary operator \( T_R \) by

\[ T_R f(x,t) = \int e^{2\pi i(x,\xi + t\phi(\xi))} \hat{f}(\xi) \frac{d\xi}{|\xi|^s}. \]

Then using Fourier transform of the spherical harmonic function again, it can be written as:

\[
T_R f(x,t) = c_{n,k} \int e^{2\pi i(x,\xi + t\phi(\xi))} |\xi|^k Y_k(\xi) g_0(|\xi|) \frac{d\xi}{|\xi|^s} \\
= c_{n,k} \int_0^\infty e^{2\pi i t \phi(\rho)} \rho^{k+n-1-s} g_0(\rho) \left( \int_{S^{n-1}} e^{2\pi i r \rho \xi' \cdot \xi'} Y_k(\xi') \, d\xi' \right) \, d\rho \\
= c_{n,k} \int_0^\infty e^{2\pi i t \phi(\rho)} \rho^{k+n-1-s} g_0(\rho) (r \rho)^{-\frac{n-2}{2}} J_\nu(k)(2\pi r \rho) dY_k(\rho) \\
= c_{n,k} r^{-\frac{n-2}{2}} \int_0^\infty e^{2\pi i t \phi(\rho)} (r \rho)^{\frac{1}{2} - s} G_0(\rho) J_\nu(k)(2\pi r \rho) dY_k(\rho) \\
= c_{n,k} T_{0,k} G_0(r,t) Y_k(-x').
\]

Then we have \( ||T_R^{**} f||_{L^q} \leq C ||T_{0,k}^{**} G_0(\cdot) r^{\frac{n-1}{2}} ||_{L^p} ||Y_k||_{L^p(S^{n-1})} \).

Suppose we prove that

\[ ||T_{0,k}^{**} G_0(\cdot) r^{\frac{n-1}{2}} ||_{L^p} \leq A_k ||G_0||_{L^2}, \quad ||G_0||_{L^2} \leq B_k ||r^k f_0(\cdot) r^{\frac{n-1}{2}} ||_{L^2}, \]

where \( |A_k|, |B_k| \leq C(n + 2k)^{\frac{n+2k}{2}} \). Then we have

\[ ||T_R^{**} f||_{L^q} \leq C_k ||r^k f_0(\cdot) r^{\frac{n-1}{2}} ||_{L^2} ||Y_k||_{L^p(S^{n-1})} = C_k \frac{||Y_k||_{L^p(S^{n-1})}}{||Y_k||_{L^2(S^{n-1})}} ||f||_{L^2}. \]

This proves the theorem.

Now we first prove that \( ||G_0||_{L^2} \leq B_k ||r^k f_0(\cdot) r^{\frac{n-1}{2}} ||_{L^2}. \) If we recall the definition of \( G_0 \), then since \( G_0(\rho) = \int_0^\infty F_0(r) J_\nu(k)(2\pi r \rho) (r \rho)^{\frac{1}{2}} \, dr \), where \( F_0(r) = r^{k+\frac{n-1}{2}} f_0(r) \). Thus we have only to show that \( ||G_0||_{L^2} \leq B_k ||F_0||_{L^2}. \)

Dividing the integral region into two parts: \( G_0 = \int_0^1 \frac{1}{\rho} + \int_1^\infty \equiv G_1 + G_2. \) For \( G_1 \), we have \( \frac{1}{\rho} |G_1 \left( \frac{1}{\rho} \right)| \leq C \frac{1}{\rho} \int_0^\rho |F_0(r)| \, dr \leq CM(F_0) \left( \frac{1}{\rho} \right), \) where \( M \) is the Hardy-Littlewood maximal function. Therefore \( ||G_1||_{L^2} \leq C ||F_0||_{L^2}. \)
Using the asymptotic behavior (3.2) of Bessel function (see Lemma 2.3 in [19]):
\[
J_\nu(r) \leq Cr^\nu \quad \text{for} \quad r \leq 1,
\]
\[
J_\nu(r) = r^{-\frac{1}{2}}(b_+e^{ir} + b_-e^{-ir}) + \Psi_\nu(r)r^{-\frac{3}{2}} \quad \text{for} \quad r \geq 1,
\]
we have
\[
G_2(\rho) = \int_0^\infty F_0(r)(b_+2\pi ir\rho + b_-e^{-2\pi ir\rho}) dr + \int_0^\infty F_0(r)\Psi_\nu(\rho) r^{-1} dr
\]
\[
\equiv G_{2,+} + G_{2,-} + G_3.
\]
For $G_3$, we have
\[
\left| \int_0^\infty \frac{F_0(r)}{r} dr \right|_{L^2} \leq C(n + 2k)^\frac{n+2k}{2} \int_0^\infty \frac{F_0(r)}{r} dr\]
and
\[
\left| \int_0^\infty \frac{F_0(r)}{r} \right|_{L^2} \leq C(n + 2k)^\frac{n+2k}{2} \int_0^\infty \frac{F_0(r)}{r} dr.
\]
We write $G_{2,\pm}$ as $G_{2,\pm}(\rho) = b_{\pm} \int e^{\pm 2\pi ir\rho} F_0(r) dr - b_{\pm} \int_0^1 e^{\pm 2\pi ir\rho} F_0(r) dr$. By the Plancheral theorem and the similar estimate of $G_1$, we get $\|G_{2,\pm}\|_{L^2} \leq \|F_0\|_{L^2}$.

Next we prove the first part of (3.1). Let $S_{0,k}G = r^{-\frac{n-1}{p}}T_{0,k}G$ and $S_{0,k}^d$ be the dual operator of $S_{0,k}$. Then for any $F \in C^\infty_0(\mathbb{R}_+ \times \mathbb{R})$, we may write $S_{0,k}^dF$ as follows:
\[
S_{0,k}^dF(\rho) = \rho^{\frac{1}{2}-s} \int e^{-2\pi it\phi(\rho)} J_\nu(\rho) r^{\frac{1}{2}-\gamma} F(r,t) dr dt
\]
\[
= \rho^{\frac{1}{2}-s} \int_0^\rho e^{-2\pi it\phi(\rho)} \mathcal{O}((\rho\nu(\rho))) r^{\frac{1}{2}-\gamma} F(r,t) dr dt
\]
\[
+ \rho^{-s} \int_\rho^\infty e^{-2\pi it\phi(\rho)}(b_+e^{2\pi ir\rho} + b_-e^{-2\pi ir\rho}) r^{-\gamma} F(r,t) dr dt
\]
\[
+ \rho^{-1-s} \int_\rho^\infty e^{-2\pi it\phi(\rho)}\Psi_\nu(\rho) r^{-1-\gamma} F(r,t) dr dt
\]
\[
\equiv \mathcal{A} + \mathcal{B}_+ + \mathcal{B}_- + \mathcal{C},
\]
where $\gamma = (n-1)(\frac{1}{2} - \frac{1}{p})$.

We first estimate $\mathcal{A}$. Since $r\rho \leq 1$, we have
\[
|\mathcal{A}(\rho)| \lesssim \rho^{\frac{n-1}{2}-s} \int_0^{\frac{1}{\rho}} r^{\frac{n-1}{2}-\gamma} \|F(r,\cdot)\|_{L^1_\mathcal{L}} dr \leq \rho^{-s} \int_0^\frac{1}{\rho} r^{-\gamma} \|F(r,\cdot)\|_{L^1_\mathcal{L}} dr.
\]
Using the identity \( \|A(\rho)\|_{L^2} = \| \frac{1}{\rho} A(\frac{1}{\rho}) \|_{L^2} \), we estimate \( \frac{1}{\rho} A(\frac{1}{\rho}) \) and hence we have

\[
\frac{1}{\rho} \left| A \left( \frac{1}{\rho} \right) \right| \lesssim \int_0^\rho \frac{r^{-\gamma}}{\rho^{1-s}} |F(r, \cdot)|_{L^1_r}^s dr \lesssim \mathcal{I}_s(r^{-\gamma}|F(r, \cdot)|_{L^1_r}(\rho).
\]

From Lemma 2.1 with \( \alpha_1 = s + \frac{1}{2} - \frac{1}{p} \) and \( p = \frac{2n}{n-2s} \), we get

\[
\|A\|_{L^2}^2 \lesssim \int |\xi|^{-2s} |F(r, \cdot)|_{L^1_r}^s(\xi)|^2 d\xi \lesssim \left( \int (r^{-\gamma}|F(r, \cdot)|_{L^1_r})^{2s} \rho^{\alpha} \rho' \right)^{\frac{1}{2}}.
\]

Since \( \alpha_1 = \gamma \), we get \( \|A\|_{L^2} \lesssim \|F\|_{L^p_t L^1_r} \) for \( p = \frac{2n}{n-2s} \).

For \( C \), from (3.2), we have \( |C| \lesssim \rho^{-1-s} A_k \int_0^\rho r^{-1-\gamma}|F(r, \cdot)|_{L^1_r} dr \), where \( A_k = (n+2k)^{\frac{n+2k}{2}} \). Similarly to the estimate of \( A \) with \( p = \frac{2n}{n-2s} \), we obtain

\[
\frac{1}{\rho} \left| C \left( \frac{1}{\rho} \right) \right| \lesssim \rho^s \int_0^\rho r^{-1-\gamma}|F(r, \cdot)|_{L^1_r} dr \lesssim \int_0^\rho \rho^{-\gamma} |F(r, \cdot)|_{L^1_r} dr \lesssim \mathcal{I}_s(r^{-\gamma}|F(r, \cdot)|_{L^1_r}(\rho).
\]

Therefore by using Lemma 2.1, we also have \( \|C\|_{L^2} \lesssim A_k \|F\|_{L^p_t L^1_r} \).

Now we estimate \( B_\pm \). To do this, we use the extended operator \( B \) such that

\[
BF(\rho) = \rho^{-s} \int_{|r| > \frac{1}{\rho}} e^{-2\pi i(\rho \phi(t) + \tau \rho)} |r|^{-\gamma} F(r, t) dr dt
\]

\[
= \rho^{-s} \int_{\mathbb{R}^2} e^{-2\pi i(\rho \phi(t) + \tau \rho)} |r|^{-\gamma} F(r, t) dr dt + \rho^{-s} \int_{|r| \leq \frac{1}{\rho}} e^{-2\pi i(\rho \phi(t) + \tau \rho)} |r|^{-\gamma} F(r, t) dr dt \equiv B_1(\rho) + B_2(\rho).
\]

For \( B_1 \), we have formally

\[
\|B_1\|_{L^2}^2 = C \iiint K(r, r', t, t') r^{-\gamma} F(r, t) r'^{-\gamma} \overline{F(r', t')} dr dr' dt dt',
\]

where

\[
K(r, r', t, t') = \int e^{-2\pi i((t-t')\phi(\rho) + (r-r')\rho)} \rho^{-2s} d\rho.
\]

Since \( \frac{1}{4} \leq s < \frac{1}{2} \), by Lemma 2.3, we have \( |K(r, r', t, t')| \lesssim |r-r'|^{-(1-2s)} \). Thus we have

\[
\|B_1\|_{L^2}^2 \lesssim \int |r-r'|^{-(1-2s)} r^{-\gamma} |F(r, \cdot)|_{L^1_r} r'^{-\gamma} |F(r', \cdot)|_{L^1_r} dr dr'.
\]

Invoking Lemma 2.1, we can get \( \|B_1\|_{L^2}^2 \lesssim \left( \int (r^{-\gamma}|F|_{L^1_r})^{2s} \rho^{\alpha} d\rho \right)^{\frac{1}{2}} \), provided \( s + \frac{1}{2} - \frac{1}{p} \). Since \( \alpha = \gamma \), we finally have \( \|B_1\|_{L^2} \lesssim \|F\|_{L^p_t L^1_r} \).
For $B_2$, we have $|B_2| \lesssim r^{-s} \int_0^r r^{-\gamma} ||F(r, \cdot)||_{L^1_t} dr$. Thus we also have

$$\frac{1}{ho} |B_2| \left( \frac{1}{\rho} \right) \lesssim \frac{1}{\rho^{1-s}} \int_0^\rho r^{-\gamma} ||F(r, \cdot)||_{L^1_t} dr \lesssim I_s(r^{-\gamma} ||F(r, \cdot)||_{L^1_t}).$$

Using Lemma 2.1, we finally have $||B_2||_{L^2} \lesssim ||F||_{L^p_t L^1_t}$. This completes the proof of (1) of the theorem.

3.2. Proof of (2). Let us first define an auxiliary operator $T_B$ by

$$T_B f(x, t) = \int e^{2\pi i (t\phi(x) + x \cdot \xi)} \hat{f}(\xi) \frac{d\xi}{(1 + |\xi|)^s}.$$ 

Using Fourier transform of the spherical harmonic function again, the original operator $T_B$ can be written as:

$$T f(x, t) = c_{n,k} r^{\frac{n-1}{2}} \int_0^\infty e^{2\pi i t\rho} \frac{1}{2} G_0(\rho) J_{\nu(k)}(2\pi \rho r) \frac{d\rho}{(1 + \rho)^s} Y_k(-x')$$

$$\equiv c_{n,k} T_{0, k} G_0(r, t) Y_k(-x'),$$

where $G_0$ is the same function as in the proof of (1). From the the proof of (1), we have only to prove that $||T_{0, k} G_0 ||_{L^2} \lesssim A_k ||G_0||_{L^2}$.

Let $\tilde{S}_{0, k} G_0 = r^{\frac{n-1}{2}} T_{0, k} G_0$. Now we divide the integral region of $\tilde{S}_{0, k} G_0$ into two parts: $\rho \leq \frac{1}{r}$ and $\rho > \frac{1}{r}$. Then we have

$$\tilde{S}_{0, k} G_0(r, t) = r^{\frac{1}{2}} \int_0^1 e^{2\pi i t\rho} \rho^{\frac{1}{2}} G_0(\rho) \frac{d\rho}{(1 + \rho)^s}$$

$$+ r^{\frac{1}{2}} \int_1^\infty e^{2\pi i t\rho} \rho^{\frac{1}{2}} (b_+ e^{2\pi i \rho} + b_- e^{-2\pi i \rho}) G_0(\rho) \frac{d\rho}{(1 + \rho)^s}$$

$$+ r^{\frac{1}{2}} \int_1^\infty e^{2\pi i t\rho} \Psi_{\nu(k)}(\rho) \rho^{\frac{1}{2}} G_0(\rho) \frac{d\rho}{(1 + \rho)^s}$$

$$\equiv \tilde{A} + \tilde{B}_+ + \tilde{B}_- + \tilde{C}.$$

Let us define the maximal functions

$$\tilde{A}^*(r) = \sup_{0 \leq t < 1} |\tilde{A}(r, t)| \quad \text{and} \quad \tilde{C}^*(r) = \sup_{0 \leq t < 1} |\tilde{C}(r, t)|.$$ 

For these maximal function, we want to show that $||\tilde{A}^*||_{L^2} + ||\tilde{C}^*||_{L^2} \lesssim A_k ||G_0||_{L^2}$, where $A_k = (n + 2k)^{\frac{n-2k}{2}}$. To prove this, let us first observe from the asymptotic behavior (3.2) that

$$\left| \frac{1}{r} \tilde{A}^* \left( \frac{1}{r} \right) \right| \lesssim r^{-1} \int_0^r G_0(\rho) d\rho \lesssim 2 M \left( \frac{r}{2} \right), \quad \left| \frac{1}{r} \tilde{C}^* \left( \frac{1}{r} \right) \right| \lesssim A_k \int_r^\infty \rho^{-1} G_0(\rho) d\rho.$$

Thus using the Hardy-Littlewood maximal function, we have

$$||\tilde{A}^*||_{L^2} = \left\| \frac{1}{r} \tilde{A}^* \left( \frac{1}{r} \right) \right\|_{L^2} \lesssim ||G_0||_{L^2}.$$
For $C^*$, we have

$$||\mathcal{C}^*||_{L^2} \lesssim A_k \sup_{||\varphi||_{L^2} \leq 1} \left| \int_0^{\infty} \int_r^{\infty} \rho^{-1}|G_0(\rho)| \, d\rho \varphi(r) \, dr \right|$$

$$\leq A_k \sup_{||\varphi||_{L^2} \leq 1} \int_0^{\infty} |G_0(\rho)| \rho^{-1} \int_0^\rho |\varphi(r)| \, dr \, d\rho$$

$$\leq 2A_k \sup_{||\varphi||_{L^2} \leq 1} \int_0^{\infty} |G_0(\rho)| \mathcal{M}(\varphi)(\rho/2) \, d\rho \lesssim ||G_0||_{L^2}.$$

To estimate $\mathcal{B}_\pm$, we use the extended operator $\mathcal{B}$ defining $G_0$ by $G_0(-\rho)$ for $\rho \leq 0$ such that

$$\mathcal{B}G_0(r,t) = \int_{|\rho| \geq \frac{1}{2}} e^{2\pi i(t\phi(|\rho|)+r\rho)} G_0(\rho) \frac{d\rho}{(1+|\rho|)^r}.$$

We can rewrite this as $\mathcal{B}G_0(r,t) = \int_\mathbb{R} + \int_{|\rho| < \frac{1}{2}} \equiv \mathcal{B}_1 G_0 + \mathcal{B}_2 G_0$. Let $\mathcal{B}_1^* G_0(r) = \sup_{|t| < 1} |\mathcal{B}_1 G_0|$ and $\mathcal{B}_2^* G_0(r) = \sup_{|t| < 1} |\mathcal{B}_2 G_0|$. Then we first have $\mathcal{B}_2^* G(r) \lesssim \int_0^{\frac{1}{2}} |G_0(\rho)| d\rho$ and hence

$$\frac{1}{r} \mathcal{B}_2^* G_0 \left( \frac{1}{r} \right) \lesssim \frac{1}{r} \int_0^{r} |G_0(\rho)| \, d\rho \lesssim \mathcal{M}(||G_0||)(\frac{r}{2}),$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal function. Thus $||\mathcal{B}_2^* G_0||_{L^2} \lesssim ||G_0||_{L^2}$.

Next, we consider global $L^2$ estimate of the local maximal operator $\mathcal{B}_1^*$. To do this, we employ the Kolmogorov-Seliverstov-Plessner method. Let us defined an operator $T$ as

$$T G_0(r) = \int e^{2\pi i(t\rho+r\phi(|\rho|))} G_0(\rho) \frac{d\rho}{(1+|\rho|)^r},$$

where $t(r)$ is any measurable function with $|t(r)| < 1$ on $\mathbb{R}$. Then we may write the operator $T$ by $T_j G_0(r) = \sum_{j \geq 0} T_j G_0(r)$, where

$$T_j G_0(r) = \int e^{2\pi i(t\rho+r\phi(|\rho|))} G_0(\rho) \varphi_j(\rho) \frac{d\rho}{(1+|\rho|)^r} \quad \text{for} \quad j \in \mathbb{Z},$$

where $\varphi_j$ are Littlewood-Paley functions such that $\varphi_0$ is supported in unit ball $B(0,1)$, $\varphi_j(\cdot) = \varphi(\frac{\cdot}{2^j})$ is supported in $B(0,2^{j+1}) \setminus B(0,2^{j-1})$ and $\sum_{j \geq 0} \varphi_j = 1$. We claim that $||T_j G_0||_{L^2} \lesssim 2^{\frac{q}{2}} ||\Delta_j G_0||_{L^2}$, where $\Delta_j \hat{g} = \varphi_j \hat{g}$. To show that, let $T_j^d$ be the dual operator of $T_j$. Then for any $F(r) \in C_0^\infty(\mathbb{R})$ and $j \geq 1$, $||T_j^d F||_{L^2}^2 = \iint K_j(r,r') F(r) \overline{F(r')} \, dr \, dr'$.
where
\[ K_j(r, r') = 2^{(1 - 2s)j} \int e^{-2\pi i ((t(r) - t(r'))(\phi(2^j|\rho|) + 2^j(r - r')\rho) + 2^j\rho^2)} \frac{d\rho}{(1 + 2|\rho|)^{2s}}. \]

Since \( \frac{2^{2j}x^2}{(1 + 2|\rho|)^{2s}} \) and its derivatives are uniformly bounded on \( j \), from Lemma 2.4 replacing \( \delta \) by \( 2^j \), we have
\[ \sup_{r' \in \mathbb{R}} \int |K_j(r, r')|dr, \sup_{r \in \mathbb{R}} \int |K_j(r, r')|dx' \lesssim 2^{\frac{2}{j} - 2s}j. \]

It follows from the Schur’s lemma (see the lemma in p.284 of [14]) that
\[ ||T^d_j F||_{L^2} \lesssim 2^{\left( \frac{q}{j} - s \right)j} ||F||_{L^2}. \]

If \( j = 0 \), then
\[ ||T^d_0 F||_{L^2}^2 = \iint K_0(r, r') F(r) F(r') dr dr'. \]

where
\[ K_0(r, r') = \int e^{-2\pi i ((t(r) - t(r'))(\phi(\rho) + (r - r')\rho) + \rho^2)} \frac{d\rho}{(1 + |\rho|)^{2s}}. \]

Since \( |t - t'| \leq 2 \), using the integration by part several times, for any \( \mu \), we have
\[ \sup_{(t, t') \in [0, 1]^2} |K_0(r, r')| \lesssim (1 + |r - r'|)^{-\mu}. \]

Thus choosing a large \( \mu \) and using Shur’s lemma again, we have \( ||T^d_0 F||_{L^2}^2 \lesssim ||F||_{L^2} \). Therefor combining this and (3.3), we prove the part (2) of theorem.

3.3. Proof of (3) and (4). Let us define an operator \( S_{0,k} \) by \( S_{0,k} G_{0}(r, t) = r^{-\frac{1}{4}} T_{0,k} G_{0}(r, t) \), where \( T_{0,k} \) is the same operator with \( s = \frac{1}{4} \) as in the proof of (1). Using the spherical coordinate and following exactly the same argument as in the proof of Theorem 1.1 with \( s = \frac{1}{4} \) and \( p = 2 \), one can easily obtain the following estimate
\[ ||T^{**} f||_{L^2(|x|^{-\frac{1}{4}} dx)} \leq C_k ||f||_{H^{\frac{1}{4}}} \]

Since from the result (2) we have
\[ ||T^* f||_{L^2} \leq ||f||_{H^s} \quad \text{for any} \quad s > \frac{a}{4}, \]
by the complex interpolation [1] between (3.4) and (3.5) with \( s \) such that \( (1 - \theta)s + \frac{\theta}{4} = \frac{a}{4} \) for \( \theta = 2\varepsilon \), we can obtain the desired result.

For the proof of (4), replacing \( r^{-\gamma} \) in the definition of \( S_{0,k} G_0 \) with \( r^{-\gamma}(1 + r)^{-\frac{b}{r}} \) and using the second part of Lemma 2.3, by the same method as the one in the proof of (1).

Remark 3.1. For one dimensional global or higher dimensional local estimate similar to (3.4), see [18] and [11].
REFERENCES


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