GLOBAL ESTIMATES OF MAXIMAL OPERATORS GENERATED BY DISPERSIVE EQUATIONS

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Abstract. Let $Tf(x,t) = e^{2\pi it\phi(D)}f$ be the solution of of the general dispersive equation with the phase function ϕ and initial data f in the Schwartz class. In case that the phase ϕ has a suitable growth rate at the infinity and the origin and f is a finite linear combination of radial and spherical harmonic functions, we have global L^p estimates of maximal operator defined by taking the supremum w.r.t. t. In particular, we obtain a global estimate at the end point left open.

1. Introduction

The general dispersive equation is defined by

$$iu_t(x) = -2\pi\phi(D)u(x), \quad \text{on} \quad \mathbb{R}^n \times \mathbb{R}, \quad u(x,0) = f(x) \in \mathcal{S}(\mathbb{R}^n)(n \geq 3),$$

where $D = \frac{1}{2\pi i}\nabla$ and ϕ is a smooth phase function . The solution of this equation can be formulated formally by

$$u(x,t) = Tf(x,t) = \int e^{2\pi i(x\cdot\xi + t\phi(\xi))}\widehat{f}(\xi)\,d\xi, \tag{1.1}$$

where $\widehat{f}(\xi) = \int e^{-2\pi ix\cdot\xi}f(x)\,dx$. From the formulation, we define maximal operators defined by

$$T^*f(x) := \sup_{-1\leq t\leq 1}|Tf(x,t)|, \qquad T^{**}f(x) := \sup_{t\in\mathbb{R}}|Tf(x,t)|.$$

In this paper we are mainly concerned with a mapping property of T^{**} defined with the various phase function, whose typical model is $|\xi|^a (a \neq 0)$, from Sobolev space to L^p such that

$$||T^*f||_{L^p} \quad \text{or} \quad ||T^{**}f||_{L^p} \leq C||f||_{X^s}, \tag{1.2}$$

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where X^s is denoted by $\dot{H}^s$ and H^s. Here the Sobolev spaces are defined by the norms:

$$||f||_{\dot{H}^s} = \left(\int |\xi|^{2s}|\widehat{f}(\xi)|^2\, d\xi\right)^{\frac{1}{2}}, \quad ||f||_{H^s} = \left(\int (1+|\xi|)^{2s}|\widehat{f}(\xi)|^2\, d\xi\right)^{\frac{1}{2}}.$$

To control the global estimate of maximal operator T^* or T^{**}, it seems inevitable to assume the growth rate and regularity of the phase function. Therefore we impose the following assumption on the phase ϕ.

A. Let ϕ be a radial function such that for some $a \in \mathbb{R}$ $(\neq 0, 1)$, $\phi \in C^2(\mathbb{R}^n \setminus \{0\})$ and there exist positive constants c_1, c_2 such that

$$c_1|\xi|^{a-k} \le |\phi^{(k)}(\xi)| \le c_2|\xi|^{a-k}\ (k = 0, 1, 2), \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}.$$

The maximal inequality (1.2) is motivated from the well-known pointwise convergence problem: $\lim_{t\to 0} u(x,t) = f(x)$ a.e. x, for $f \in H^{\frac{1}{4}}(\mathbb{R}^n)$, during the three decades, the local and global L^p estimate of the maximal operators have been studied by many authors [2, 3, 4, 5, 7, 9, 16] and [17] etc. P. Sjölin [13] and L. Vega[17] obtained the strong necessary condition ($s \ge \frac{1}{4}$) on the pointwise convergence problem. In particular, P. Sjölin showed that the maximal operator T^{**} cannot have the global L^2 boundedness (see[10]), and that $||T^*f||_{L^2} \le C||f||_{H^s}$ holds for $s > \frac{a}{4}$ and fails for $s < \frac{a}{4}$ $(a > 1)$ (see [13]). But if the data f is a finite linear combination of radial and spherical harmonic functions, then we prove that the inequality (1.2) holds for $s = \frac{1}{4}$ and $\frac{a}{4}$ with suitable weight.

Now we state our main results.

Theorem 1.1. *Let $f_0 \in C_0^\infty(\mathbb{R})$ and Y_k be a spherical harmonic function of order $k \ge 0$. If $f(x) = |x|^k f_0(|x|)Y_k(x')$,*

1. *then $||T^{**}f||_{L^p} \le C_k \frac{||Y_k||_{L^p(S^{n-1})}}{||Y_k||_{L^2(S^{n-1})}}||f||_{\dot{H}^s}$ holds for $s \in [\frac{1}{4}, \frac{1}{2})$, $p = \frac{2n}{n-2s}$ and $0 < a \neq 1$, where $C_k = \mathcal{O}((n+2k)^{n+2k})$ as $k \to \infty$.*
2. *then $||T^*f||_{L^2} \le C_k||f||_{H^s}$ holds for any $s > \frac{a}{4}$ and $0 < a \neq 1$.*
3. *then $||T^*f||_{L^2(|x|^{-\varepsilon}\,dx)} \le C_{\varepsilon,k}||f||_{H^{\frac{a}{4}}}$ holds for any $a > 1$ and $\varepsilon > 0$.*
4. *then $||T^*f||_{L^p((1+|x|)^{-b}\,dx)} \le C_k \frac{||Y_k||_{L^p(S^{n-1})}}{||Y_k||_{L^2(S^{n-1})}}||f||_{\dot{H}^s}$ holds for $0 < a < 1$, $\frac{a}{4} < s < \frac{1}{4}$, $p = \frac{4n(1-a)}{2n(1-a)+a-4s}$ and $b = sp - n(\frac{p}{2} - 1)$.*

We obtain optimal global estimates except for the end point $(s,p) = (\frac{a}{4}, 2)$ $(0 < a < 1)$ which was studied by B. G. Walther [18]. These are obtained by using a bound of one dimensional oscillatory integral of the form $\int_{\mathbb{R}} e^{2\pi i(t\phi(\xi)+x\xi)}|\xi|^{-s}\,d\xi$. Many authors referred in this paper have tried to handle such an integral and obtained various bounds according to the value of growth rate a of ϕ (i.e. $a > 1$ or $a < 1$). Here, we provide that the integral is bounded by $C|x|^{-(1-s)}$ for any $a \neq 1$ and the constant C is independent of t (see Lemma 2.3). Thanks to the time independency of the integral, the proof of main results are much more simplified.

S. Wang [19] showed that if $p > 2$, then there exist f_0 and Y_k such that if $f(x) = |x|^k f_0(|x|)Y_k(x')$, then $\lim_{k\to\infty} ||T^*f||_{L^p(B)}/||f||_{H^{\frac{1}{4}}} = \infty$ for any ball B. Thus it will be interesting to prove a local L^p estimate of T^*f holds uniformly on k for some $p \in [1, 2]$. For the more general initial data f, by using a bilinear estimate, T. Tao in [16] showed that the global estimate for T^{**} holds for $p > \frac{2(n+3)}{n+1}$ and $s > n(\frac{1}{2} - \frac{1}{p})$, if the phase ϕ is an elliptic type. As another global estimates, there are several results about the weighted estimates for $s > \frac{1}{2}$. For these results, one can refer [5, 17, 18]. But it remains still open question whether even a local estimate of T^* holds for $s = \frac{1}{4}$.

If not specified, throughout this paper, C denotes by a generic constant that depends on c_1, c_2, a, s, n. We use the notation $A \lesssim B$ and $A \sim B$ to denote $|A| \le CB$ and $C^{-1}B \le |A| \le CB$ respectively.

2. Preliminary lemmas

We begin with the weighted inequality for the Fourier transform.

Lemma 2.1 (see [8])**.** *If $1 \le q \le 2$, $0 \le \alpha < \frac{1}{2}$, $0 \le \alpha_1 < \frac{1}{q'}$ and $\alpha_1 = \alpha + \frac{1}{2} - \frac{1}{q}$, then the following inequality holds*

$$\left(\int_{\mathbb{R}} |\xi|^{-2\alpha}|\widehat{f}(\xi)|^2\, d\xi\right)^{\frac{1}{2}} \lesssim \left(\int_{\mathbb{R}} |f(x)|^q|x|^{\alpha_1 q}\, dx\right)^{\frac{1}{q}}.$$

Now we introduce some estimates of oscillatory integrals. Let us first state a stationary phase lemma which can be found in [7] etc..

Lemma 2.2. *Let ψ be a monotone function and $I = \int_\alpha^\beta e^{i\varphi(\xi)}\psi(\xi)\, d\xi$. Then if $|\frac{d\varphi}{d\xi}| \ge \lambda > 0$ in $[\alpha, \beta]$ and $\frac{d\varphi}{d\xi}$ is monotone, $|I| \le C\lambda^{-1} \sup\limits_{[\alpha,\beta]} |\psi(\xi)|$, and if $|\frac{d^2\varphi}{d\xi^2}| \ge \lambda > 0$, then $|I| \le C\lambda^{-\frac{1}{2}} \sup\limits_{[\alpha,\beta]} |\psi(\xi)|$. The constant C doesn't depend on $\alpha, \beta, \lambda, \varphi$ and ψ.*

Utilizing the lemma above, we get the following lemma.

Lemma 2.3. *Suppose ϕ satisfies the assumption **A** for $0 < a \ne 1$. Let A, B, s be the real numbers such that $A, B \ne 0$, $\frac{1}{2} \le s < 1$. Consider the following integral:*

$$I = \int_{\xi\in\mathbb{R}} e^{2\pi i(A\phi(\xi)+B\xi)}\, |\xi|^{-s}\, d\xi.$$

Then $|I| \le C(a, s, c_1, c_2)\, |B|^{-(1-s)}$.

If $0 < a < 1$, $\frac{a}{2} < s < \frac{1}{2}$ and $|A| \le 2$, then $I \lesssim |B|^{-(1-s)} + |B|^{-(1-s)-\frac{a(1-2s)}{2(1-a)}}$.

Proof of Lemma 2.3. **(Case $a > 1$)** Without loss of generality, we may assume that $A > 0$ and $B > 0$. Let $D = \frac{B}{A^{\frac{1}{a}}}$. Then by the change of variable, we have

$$I = A^{-\frac{1-s}{a}} \int e^{2\pi i (A\phi(A^{-\frac{1}{a}}\xi) + D\xi)} |\xi|^{-s}\, d\xi = \int_{\xi<0} + \int_{\xi>0} = I_- + I_+.$$

We have only to consider I_+ and we denote it I again.

Now we first consider the case when $\phi' > 0$. Observe that

$$E \equiv (A\phi(A^{-\frac{1}{a}}\xi) + D\xi)' \geq c_1\xi^{a-1} + D.$$

Let M be a large positive number depending only on a, s, c_1, c_2. If $D \leq M$, then

$$I = A^{-\frac{1-s}{a}} \left(\int_0^1 + \int_1^\infty \right) = I_1 + I_2$$

For I_1, by direct integration, we have $|I_1| \lesssim A^{-\frac{1-s}{a}} \lesssim B^{-(1-s)}$. For I_2, since $E \gtrsim 1$, by the first part of (2.2), we have $|I_2| \lesssim A^{-\frac{1-s}{a}} \lesssim B^{-(1-s)}$. If $D > M$, then since $E \geq D$, by the first part of Lemma 2.2, we have $|I_2| \lesssim A^{-\frac{1-s}{a}} D^{-1} \leq A^{\frac{s}{a}} B^{-1} \leq B^{-(1-s)}$. For I_1, using the change of variable, we have

$$I_1 = A^{-\frac{1-s}{a}} D^{-(1-s)} \int_0^D e^{2\pi i (A\phi(D^{-1}A^{-\frac{1}{a}}\xi) + \xi)} \xi^{-s} d\xi.$$

Thus $I_1 = \int_0^1 + \int_1^D = I_{1,1} + I_{1,2}$. By the integration, $|I_{1,1}| \lesssim B^{-(1-s)}$. For $I_{1,2}$, since $(A\phi(D^{-1}A^{-\frac{1}{a}}\xi) + \xi)' \geq 1$, from the first part of Lemma 2.2, we have $|I_{1,2}| \lesssim B^{-(1-s)}$ and hence $|I_1| \lesssim B^{-(1-s)}$.

Now we consider the case when $\phi' < 0$. We observe that

$$-c_2\xi^{a-1} + D \leq E = (A\phi(A^{-\frac{1}{a}}\xi) + D\xi)' \leq -c_1\xi^{a-1} + D.$$

If $D \leq M$, then we split I into two parts as follows:

$$I = A^{-\frac{1-s}{a}} \left(\int_0^{(\frac{2M}{c_2})^{\frac{1}{a-1}}} + \int_{(\frac{2M}{c_2})^{\frac{1}{a-1}}}^\infty \right) = I_3 + I_4.$$

For I_5, we have by direct integration $|I_3| \lesssim A^{-\frac{1-s}{a}} \leq B^{-(1-s)}$. For I_4, since $E \leq -c_1\xi^{a-1} + D \leq -1$, by the first part of Lemma 2.2, we get $|I_4| \lesssim A^{-\frac{1-s}{a}} \leq B^{-(1-s)}$. If $D > M$, then we split I into four parts as follows:

$$I = A^{-\frac{1-s}{a}} \left(\int_0^1 + \int_1^{(\frac{D}{2c_2})^{\frac{1}{a-1}}} + \int_{(\frac{D}{2c_2})^{\frac{1}{a-1}}}^{(\frac{2D}{c_1})^{\frac{1}{a-1}}} + \int_{(\frac{2D}{c_1})^{\frac{1}{a-1}}}^\infty \right) \tag{2.1}$$
$$\equiv I_5 + I_6 + I_7 + I_8.$$

For I_5, we use the change of variable so that

$$I_5 = A^{-\frac{1-s}{a}} D^{-(1-s)} \int_0^D e^{2\pi i (A\phi(D^{-1}A^{-\frac{1}{a}}\xi)+\xi)} \xi^{-s}\, d\xi.$$

We split I_5 into two part: $I_5 = A^{-\frac{1-s}{a}} D^{-(1-s)} \left(\int_0^1 + \int_1^D\right) = I_{5,1} + I_{5,2}$. For $I_{5,1}$ and $I_{5,2}$, using the direct integration and the first part of Lemma 2.2 respectively, we have $|I_{5,1}| + |I_{5,2}| \lesssim A^{-\frac{1-s}{a}} D^{-(1-s)} = B^{-(1-s)}$. For I_6, since $E \gtrsim D \geq D^{1-s}$, using the first part of Lemma 2.2, we have $|I_6| \lesssim A^{-\frac{1-s}{a}} D^{-(1-s)} = B^{-(1-s)}$.

To estimate I_7, we use the second derivative $|E'| \sim \xi^{a-2} \sim D^{\frac{a-2}{a-1}}$. Then from the second part of Lemma 2.2, we obtain

$$|I_7| \lesssim A^{-\frac{1-s}{a}} D^{-\frac{a-2}{2(a-1)}} D^{-\frac{s}{a-1}} = A^{-\frac{1-s}{a}} D^{-\frac{a-2+2s}{2(a-1)}}.$$

Since $a > 1$ and $s \geq \frac{1}{2}$, we have $|I_7| \lesssim A^{-\frac{1-s}{a}} D^{-(1-s)} = B^{-(1-s)}$. Finally, we estimate I_8. Since $E \gtrsim D \geq D^{1-s}$, by the first part of Lemma 2.2, we have $|I_8| \lesssim A^{-\frac{1-s}{a}} D^{-(1-s)} D^{-\frac{s}{a-1}} \lesssim B^{-(1-s)}$.

(Case $a < 1$**)** We first consider the case $\frac{1}{2} \leq s < 1$. We may assume $A, B > 0$. Let $\tilde{D} = \frac{A}{B^a}$. Then by the change of variable, we write

$$B^{1-s} I = \int e^{2\pi i (A\phi(\frac{\xi}{B})+\xi)} |\xi|^{-s} d\xi = \int_0^\infty + \int_{-\infty}^0 = I_+ + I_-.$$

As in the previous case ($a > 1$), we only consider I_+ and denote it by I again.

In case that $\phi' > 0$, we have $E \equiv (A\phi(\frac{\xi}{B}) + \xi)' \geq c_1 \tilde{D} \xi^{a-1} + 1 \geq 1$ for all $\xi > 0$. We divide I into two parts: $I = \int_0^1 + \int_1^\infty$. For the first integral, we just integrate and for the second one, we use the first part of Lemma 2.2. Then we can see $|I| \lesssim 1$.

Now we consider the case when $\phi' < 0$. Then we can observe that

$$-c_2 \tilde{D} \xi^{a-1} + 1 \leq E \leq -c_1 \tilde{D} \xi^{a-1} + 1.$$

If $c_2 \tilde{D} < 2$, then we divide I into two parts: $I = \int_0^{(\frac{1}{4})^{\frac{1}{a-1}}} + \int_{(\frac{1}{4})^{\frac{1}{a-1}}}^\infty = I_1 + I_2$. By the integration, we get $|I_1| \lesssim 1$. And since $c_2 \tilde{D} < 2$ and hence $E \gtrsim 1$, by the first part of Lemma 2.2, we have $|I_2| \lesssim 1$.

If $c_1 \tilde{D} > 2$, then we divide I into four parts:

$$I = \int_0^1 + \int_1^{(\frac{2}{c_1 \tilde{D}})^{\frac{1}{a-1}}} + \int_{(\frac{2}{c_1 \tilde{D}})^{\frac{1}{a-1}}}^{(\frac{1}{2c_2 \tilde{D}})^{\frac{1}{a-1}}} + \int_{(\frac{1}{2c_2 \tilde{D}})^{\frac{1}{a-1}}}^\infty = I_3 + I_4 + I_5 + I_6.$$

For I_3, by the integration, $|I_3| \lesssim 1$. For $|I_5|$, since $|E'| \sim \tilde{D} \tilde{D}^{-\frac{a-2}{a-1}} = \tilde{D}^{\frac{1}{a-1}}$ and $s \geq \frac{1}{2}$, by the second part of Lemma 2.2, we have $|I_5| \lesssim \tilde{D}^{\frac{2s-1}{2(a-1)}} \lesssim 1$.

And since $E \lesssim -1$ on $[1, (\frac{2}{c_1\tilde{D}})^{\frac{1}{a-1}}]$ and $E \gtrsim 1$ on $[(\frac{1}{2c_2\tilde{D}})^{\frac{1}{a-1}}, \infty)$, we also have $|I_4|, |I_6| \lesssim 1$.

If $\frac{2}{c_2} \le \tilde{D} \le \frac{2}{c_1}$, choose a large number M depending only on c_1, c_2, and divide I as follows: $I = \int_0^M + \int_M^\infty$. Then as the estimate of I_1 and I_2, we can obtain $|I| \lesssim 1$.

If $0 < a < 1$ and $\frac{a}{2} < s < \frac{1}{2}$, then except for the integral I_5, we can treat every integral by the same method as above. For I_5, since $|E'| \sim \tilde{D}\tilde{D}^{-\frac{a-2}{a-1}} = \tilde{D}^{\frac{1}{a-1}}$, $|A| \le 2$ and $s < \frac{1}{2}$, by the second part of Lemma 2.2, we have $|I_5| \lesssim B^{-\frac{a(1-2s)}{2(1-a)}}$. This completes the proof of lemma. $\square$

Lemma 2.4. *Let δ be a positive number and α, β be the real number with $0 < |\alpha| \le 1, \beta \ne 0$. Let φ be a $C_0^\infty(\mathbb{R})$ function with the support away from the orgin. Consider the oscillatory integral*

$$I_\delta(\alpha, \beta) = \delta \int e^{2\pi i(\alpha\phi(\delta\xi) + \delta\beta\xi)}\varphi(\xi)d\xi,$$

*where ϕ satisfies the assumption **A**. If $\delta \ge 1$ and $0 < a \ne 1$, then $\int |I_\delta(\alpha, \beta)|d\beta \le C_a\delta^{\frac{a}{2}}$. The constant C_a doesn't depend on α.*

Proof of Lemma 2.4. If $\delta|\beta| > C_a\delta^a\alpha$ and $\delta|\beta| \ge 1$, then by the integration by part, we have

$$|I_\delta| \le C(a,\mu)\delta(1 + \delta|\beta|)^{-\mu} \tag{2.2}$$

for any positive number μ. By the seconde part of Lemma 2.2, we have

$$|I_\delta| \le C_a\delta(\delta|\alpha|)^{-\frac{1}{2}} \quad \text{and} \quad |I_\delta| \le C\delta. \tag{2.3}$$

We divide the integral $\int |I_\delta|d\beta$ into four part as follows.

$$\int |I_\delta|d\beta = \int_{\delta|\beta|\le 1} + \int_{\substack{1 \le \delta|\beta| \le C_a\delta^a, \\ \delta|\beta| \le C_a\delta^a|\alpha|}} + \int_{\substack{\delta|\beta| \ge 1, \\ C_a\delta^a|\alpha| \le |\beta| \le C_a\delta^a}} + \int_{\delta|\beta| > C_a\delta^a} \equiv \sum_{i=1}^4 II_i.$$

Now we estimate each term. At first, by the second part of (2.3), $II_1 \le C\delta\delta^{-1} = C$. For II_2, using the first part of (2.3),

$$II_2 \le C_a\delta \int_{|\beta| \le C_a\delta^{a-1}|\alpha|} (\delta^a|\alpha|)^{-\frac{1}{2}}d\beta \le C_a\delta^{\frac{1}{2}} \int_{|\beta| \le C_a\delta^{a-1}} |\beta|^{-\frac{1}{2}}d\beta \le C_a\delta^{\frac{a}{2}}.$$

Using (2.2) with $\mu = 1 - \frac{a}{2+2a}$, for II_3, we have

$$II_3 \le C_a\delta^{1-\mu} \int_{\substack{\delta|\beta| \ge 1, \\ C_a\delta^a|\alpha| \le |\beta| \le C_a\delta^a}} |\beta|^{-\mu}d\beta \le C_a\delta^{(1-\mu)(1+a)} = C_a\delta^{\frac{a}{2}}.$$

Finally, for II_4, using (2.2) with large μ, we have

$$II_4 \le C(a,\mu)\delta^{1-\mu} \int_{|\beta| > C_a\delta^{a-1}} |\beta|^{-\mu}d\beta \le C(a,\mu)\delta^{a(1-\mu)} \le C_a\delta^{\frac{a}{2}}.$$

This completes the proof of lemma. $\square$

3. Proof of Theorem 1.1

3.1. **Proof of (1).** Using Fourier transform of the radial function and spherical harmonic function (see [15]),

$$\widehat{f}(\xi) = c_{n,k}|\xi|^{-\nu(k)}|\xi|^k Y_k(\xi') \int_0^\infty f_0(r) J_{\nu(k)}(2\pi r|\xi|) r^{\frac{n+2k}{2}}\, dr,$$

where $\nu(k) = \frac{n+2k-2}{2}$ and $|c_{n,k}| \le C$. Let $G_0(\rho) = \rho^{k+\frac{n-1}{2}} g_0(\rho)$ and $g_0(\rho) = \rho^{-\nu(k)} \int_0^\infty f_0(r) J_{\nu(k)}(2\pi r\rho) r^{\frac{n+2k}{2}}\, dr$. We define an auxiliary operator T_R by

$$T_R f(x,t) = \int e^{2\pi i(x\cdot\xi + t\phi(\xi))} \widehat{f}(\xi) \frac{d\xi}{|\xi|^s}.$$

Then using Fourier transform of the spherical harmonic function again, it can be written as:

$$\begin{aligned}
T_R f(x,t) &= c_{n,k} \int e^{2\pi i(x\cdot\xi + t\phi(\xi))} |\xi|^k Y_k(\xi) g_0(|\xi|) \frac{d\xi}{|\xi|^s} \\
&= c_{n,k} \int_0^\infty e^{2\pi i t\phi(\rho)} \rho^{k+n-1-s} g_0(\rho) \left(\int_{S^{n-1}} e^{2\pi i r\rho x'\cdot\xi'} Y_k(\xi')\, d\xi' \right) d\rho \\
&= c_{n,k} \int_0^\infty e^{2\pi i t\phi(\rho)} \rho^{k+n-1-s} g_0(\rho) (r\rho)^{-\frac{n-2}{2}} J_{\nu(k)}(2\pi r\rho)\, d\rho\, Y_k(-x') \\
&= c_{n,k} r^{-\frac{n-2}{2}} \int_0^\infty e^{2\pi i t\phi(\rho)} \rho^{\frac{1}{2}-s} G_0(\rho) J_{\nu(k)}(2\pi r\rho)\, d\rho\, Y_k(-x') \\
&\equiv c_{n,k} T_{0,k} G_0(r,t) Y_k(-x').
\end{aligned}$$

Then we have $||T_R^{**} f||_{L_x^p} \le C ||T_{o,k}^{**} G_0(\cdot) r^{\frac{n-1}{p}}||_{L_r^p} ||Y_k||_{L^p(S^{n-1})}$.

Suppose we prove that

$$||T_{o,k}^{**} G_0(\cdot) r^{\frac{n-1}{p}}||_{L_r^p} \le A_k ||G_0||_{L^2}, \quad ||G_0||_{L^2} \le B_k ||r^k f_0(\cdot) r^{\frac{n-1}{2}}||_{L^2}, \tag{3.1}$$

where $|A_k|, |B_k| \le C(n+2k)^{\frac{n+2k}{2}}$. Then we have

$$||T_R^{**} f||_{L_x^p} \le C_k ||r^k f_0(\cdot) r^{\frac{n-1}{2}}||_{L^2} ||Y_k||_{L^p(S^{n-1})} = C_k \frac{||Y_k||_{L^p(S^{n-1})}}{||Y_k||_{L^2(S^{n-1})}} ||f||_{L^2}.$$

This proves the theorem.

Now we first prove that $||G_0||_{L^2} \le B_k ||r^k f_0(\cdot) r^{\frac{n-1}{2}}||_{L^2}$. If we recall the definition of G_0, then since $G_0(\rho) = \int_0^\infty F_0(r) J_{\nu(k)}(2\pi r\rho)(r\rho)^{\frac{1}{2}}\, dr$, where $F_0(r) = r^{k+\frac{n-1}{2}} f_0(r)$. Thus we have only to show that $||G_0||_{L_r^2} \le B_k ||F_0||_{L_r^2}$. Dividing the integral region into two parts: $G_0 = \int_0^{\frac{1}{\rho}} + \int_{\frac{1}{\rho}}^\infty \equiv G_1 + G_2$. For G_1, we have $\frac{1}{\rho}\left|G_1\left(\frac{1}{\rho}\right)\right| \le C\frac{1}{\rho}\int_0^\rho |F_0(r)|\, dr \le C\mathcal{M}(F_0)\left(\frac{\rho}{2}\right)$, where $\mathcal{M}$ is the Hardy-Littlewood maximal function. Therefore $||G_1||_{L^2} \le C||F_0||_{L^2}$.

Using the asymptotic behavior (3.2) of Bessel function (see Lemma 2.3 in [19]):

$$\begin{cases} |J_\nu(r)| \le Cr^\nu \quad \text{for} \quad r \le 1, \\ J_\nu(r) = r^{-\frac{1}{2}}(b_+ e^{ir} + b_- e^{-ir}) + \Psi_\nu(r) r^{-\frac{3}{2}} \quad \text{for} \quad r \ge 1, \\ |b_\pm| \le C \quad \text{and} \quad |\Psi_\nu(r)| \le C(2\nu))^{\nu+1} \quad \text{for} \quad \nu \ge \frac{3}{2}, \end{cases} \tag{3.2}$$

we have

$$\begin{aligned} G_2(\rho) &= \int_{\frac{1}{\rho}}^\infty F_0(r)(b_+ 2^{2\pi i r\rho} + b_- e^{-2\pi i r\rho})\, dr + \int_{\frac{1}{\rho}}^\infty F_0(r)\Psi_{\nu(k)}(r\rho)^{-1}\, dr \\ &\equiv G_{2,+} + G_{2,-} + G_3. \end{aligned}$$

For G_3, we have $\frac{1}{\rho}\left|G_3\left(\frac{1}{\rho}\right)\right| \le C(n+2k)^{\frac{n+2k}{2}} \int_\rho^\infty \frac{F_0(r)}{r}\, dr$ and

$$\begin{aligned} \left\|\int_\rho^\infty \frac{F_0(r)}{r}\, dr\right\|_{L^2} &= \sup_{||g||_{L^2}\le 1}\left|\int_0^\infty \int_\rho^\infty \frac{F_0(r)}{r}\, dr\, g(\rho)\, d\rho\right| \\ &= \sup_{||g||_{L^2}\le 1}\left|\int_0^\infty F_0(r)\frac{1}{r}\int_0^r g(\rho)\, d\rho\, dr\right| \\ &\le C \sup_{||g||_{L^2}\le 1} ||F_0||_{L^2} ||\mathcal{M}(g)||_{L^2} \le C||F_0||_{L^2}. \end{aligned}$$

We write $G_{2,\pm}$ as $G_{2,\pm}(\rho) = b_\pm \int_0^\infty e^{\pm 2\pi i r\rho} F_0(r)\, dr - b_\pm \int_0^{\frac{1}{\rho}} e^{\pm 2\pi i r\rho} F_0(r)\, dr$. By the Plancheral theorem and the similar estimate of G_1, we get $||G_{2,\pm}||_{L^2} \le ||F_0||_{L^2}$.

Next we prove the first part of (3.1). Let $S_{0,k}G = r^{\frac{n-1}{p}} T_{0,k}G$ and $S^d_{0,k}$ be the dual operator of $S_{0,k}$. Then for any $F \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$, we may write $S^d_{0,k}F$ as follows:

$$\begin{aligned} S^d_{0,k}F(\rho) &= \rho^{\frac{1}{2}-s} \iint e^{-2\pi i t\phi(\rho)} J_{\nu(k)}(2\pi r\rho) r^{\frac{1}{2}-\gamma} F(r,t)\, drdt \\ &= \rho^{\frac{1}{2}-s} \iint_0^{\frac{1}{\rho}} e^{-2\pi i t\phi(\rho)} \mathcal{O}((r\rho)^{\nu(k)}) r^{\frac{1}{2}-\gamma} F(r,t)\, drdt \\ &\quad + \rho^{-s} \iint_{\frac{1}{\rho}}^\infty e^{-2\pi i t\phi(\rho)} (b_+ e^{2\pi i r\rho} + b_- e^{-2\pi i r\rho}) r^{-\gamma} F(r,t)\, drdt \\ &\quad + \rho^{-1-s} \iint_{\frac{1}{\rho}}^\infty e^{-2\pi i t\phi(\rho)} \Psi_{\nu(k)}(r\rho) r^{-1-\gamma} F(r,t)\, drdt \\ &\equiv \mathcal{A} + \mathcal{B}_+ + \mathcal{B}_- + \mathcal{C}, \end{aligned}$$

where $\gamma = (n-1)(\frac{1}{2} - \frac{1}{p})$.

We first estimate $\mathcal{A}$. Since $r\rho \le 1$, we have

$$|\mathcal{A}(\rho)| \lesssim \rho^{\frac{n-1}{2}-s} \int_0^{\frac{1}{\rho}} r^{\frac{n-1}{2}-\gamma} ||F(r,\cdot)||_{L^1_t} dr \le \rho^{-s} \int_0^{\frac{1}{\rho}} r^{-\gamma} ||F(r,\cdot)||_{L^1_t} dr.$$

Using the identity $||\mathcal{A}(\rho)||_{L^2} = ||\frac{1}{\rho}\mathcal{A}(\frac{1}{\rho})||_{L^2}$, we estimate $\frac{1}{\rho}\mathcal{A}(\frac{1}{\rho})$ and hence we have

$$\frac{1}{\rho}\left|\mathcal{A}\left(\frac{1}{\rho}\right)\right| \lesssim \int_0^\rho \frac{r^{-\gamma}}{\rho^{1-s}}||F(r,\cdot)||_{L^1_t}dr \lesssim \mathcal{I}_s(r^{-\gamma}||F(r,\cdot)||_{L^1_t})(\rho).$$

From Lemma 2.1 with $\alpha_1 = s+\frac{1}{2}-\frac{1}{p'}$ and $p=\frac{2n}{n-2s}$, we get

$$||\mathcal{A}||^2_{L^2} \lesssim \int |\xi|^{-2s}|(r^{-\gamma}||F(r,\cdot)||_{L^1_t})^\wedge(\xi)|^2 d\xi \lesssim \left(\int (r^{-\gamma}||F(r,\cdot)||_{L^1_t})^{p'} r^{\alpha_1 p'}dr\right)^{\frac{2}{p'}}.$$

Since $\alpha_1=\gamma$, we get $||\mathcal{A}||_{L^2} \lesssim ||F||_{L^{p'}_r L^1_t}$ for $p=\frac{2n}{n-2s}$.

For $\mathcal{C}$, from (3.2), we have $|\mathcal{C}| \lesssim \rho^{-1-s}A_k \int_{\frac{1}{\rho}}^\infty r^{-1-\gamma}||F(r,\cdot)||_{L^1_t}dr$, where $A_k=(n+2k)^{\frac{n+2k}{2}}$. Similarly to the estimate of $\mathcal{A}$ with $p=\frac{2n}{n-2s}$, we obtain

$$\begin{aligned}\frac{1}{\rho}\left|\mathcal{C}\left(\frac{1}{\rho}\right)\right| &\lesssim \rho^s\int_\rho^\infty r^{-1-\gamma}||F(r,\cdot)||_{L^1_t}dr \lesssim \int_\rho^\infty \frac{r^{-\gamma}}{r^{1-s}}||F(r,\cdot)||_{L^1_t}dr\\ &\lesssim \mathcal{I}_s(r^{-\gamma}||F(r,\cdot)||_{L^1_t})(\rho).\end{aligned}$$

Therefore by using Lemma 2.1, we also have $||\mathcal{C}||_{L^2} \lesssim A_k||F||_{L^{p'}_r L^1_t}$.

Now we estimate $\mathcal{B}_\pm$. To do this, we use the extended operator $\mathcal{B}$ such that

$$\begin{aligned}\mathcal{B}F(\rho) &= \rho^{-s}\iint_{|r|>\frac{1}{\rho}} e^{-2\pi i(t\phi(\rho)+r\rho)}|r|^{-\gamma}F(r,t)drdt\\ &= \rho^{-s}\iint_{\mathbb{R}^2} e^{-2\pi i(t\phi(\rho)+r\rho)}|r|^{-\gamma}F(r,t)drdt\\ &\quad + \rho^{-s}\iint_{|r|\le\frac{1}{\rho}} e^{-2\pi i(t\phi(\rho)+r\rho)}|r|^{-\gamma}F(r,t)drdt \equiv \mathcal{B}_1(\rho)+\mathcal{B}_2(\rho).\end{aligned}$$

For $\mathcal{B}_1$, we have formally

$$||\mathcal{B}_1||^2_{L^2} = C\iiiint K(r,r',t,t')r^{-\gamma}F(r,t)r'^{-\gamma}\overline{F(r',t')}drdr'dtdt',$$

where

$$K(r,r',t,t') = \int e^{-2\pi i((t-t')\phi(\rho)+(r-r')\rho)}\rho^{-2s}d\rho.$$

Since $\frac{1}{4}\le s<\frac{1}{2}$, by Lemma 2.3, we have $|K(r,r',t,t')| \lesssim |r-r'|^{-(1-2s)}$. Thus we have

$$||\mathcal{B}_1||^2_{L^2} \lesssim \iint |r-r'|^{-(1-2s)}r^{-\gamma}||F(r,\cdot)||_{L^1_t}r'^{-\gamma}||F(r',\cdot)||_{L^1_{t'}}drdr'.$$

Invoking Lemma 2.1, we can get $||\mathcal{B}_1||^2_{L^2} \lesssim \left(\int (r^{-\gamma}||F||_{L^1_t})^{p'}r^{\alpha p'}dr\right)^{\frac{2}{p'}}$, provided $\alpha = s+\frac{1}{2}-\frac{1}{p'}$. Since $\alpha=\gamma$, we finally have $||\mathcal{B}_1||_{L^2} \lesssim ||F||_{L^{p'}_r L^1_t}$.

For $\mathcal{B}_2$, we have $|\mathcal{B}_2| \lesssim \rho^{-s} \int_0^{\frac{1}{\rho}} r^{-\gamma}||F(r,\cdot)||_{L^1_t} dr$. Thus we also have

$$\frac{1}{\rho}|\mathcal{B}_2|\left(\frac{1}{\rho}\right) \lesssim \frac{1}{\rho^{1-s}} \int_0^{\rho} r^{-\gamma}||F(r,\cdot)||_{L^1_t} dr \lesssim \mathcal{I}_s(r^{-\gamma}||F(r,\cdot)||_{L^1_t}).$$

Using Lemma 2.1, we finally have $||\mathcal{B}_2||_{L^2} \lesssim ||F||_{L^{p'}_r L^1_t}$. This completes the proof of **(1)** of the theorem.

3.2. **Proof of (2).** Let us first define an auxiliary operator T_B by

$$T_B f(x,t) = \int e^{2\pi i(t\phi(\xi)+x\cdot\xi)} \widehat{f}(\xi) \frac{d\xi}{(1+|\xi|)^s}.$$

Using Fourier transform of the spherical harmonic function again, the original operator T_B can be written as:

$$\begin{aligned} Tf(x,t) &= c_{n,k} r^{-\frac{n-1}{2}} \int_0^\infty e^{2\pi i t\phi(\rho)} \rho^{\frac{1}{2}} G_0(\rho) J_{\nu(k)}(2\pi r\rho) \frac{d\rho}{(1+\rho)^s} Y_k(-x') \\ &\equiv c_{n,k} \tilde{T}_{0,k} G_0(r,t) Y_k(-x'), \end{aligned}$$

where G_0 is the same function as in the proof of **(1)**. From the the proof of **(1)**, we have only to prove that $||\tilde{T}^*_{o,k} G_0(\cdot) r^{\frac{n-1}{2}}||_{L^2_r} \le A_k ||G_0||_{L^2}$.

Let $\tilde{S}_{0,k} G_0 = r^{\frac{n-1}{2}} \tilde{T}_{0,k} G_0$. Now we divide the integral region of $\tilde{S}_{0,k} G_0$ into two parts: $\rho \le \frac{1}{r}$ and $\rho > \frac{1}{r}$. Then we have

$$\begin{aligned} \tilde{S}_{0,k} G_0(r,t) &= r^{\frac{1}{2}} \int_0^{\frac{1}{r}} e^{2\pi i t\phi(\rho)} \mathcal{O}((r\rho)^{\nu(k)}) \rho^{\frac{1}{2}} G_0(\rho) \frac{d\rho}{(1+\rho)^s} \\ &\quad + r^{\frac{1}{2}} \int_{\frac{1}{r}}^\infty e^{2\pi i t\phi(\rho)} (r\rho)^{-\frac{1}{2}} (b_+ e^{2\pi i r\rho} + b_- e^{-2\pi i r\rho}) \rho^{\frac{1}{2}} G_0(\rho) \frac{d\rho}{(1+\rho)^s} \\ &\quad + r^{\frac{1}{2}} \int_{\frac{1}{r}}^\infty e^{2\pi i t\phi(\rho)} \Psi_{\nu(k)}(r\rho) \rho^{\frac{1}{2}} G_0(\rho) \frac{d\rho}{(1+\rho)^s} \\ &\equiv \tilde{\mathcal{A}} + \tilde{\mathcal{B}}_+ + \tilde{\mathcal{B}}_- + \tilde{\mathcal{C}}. \end{aligned}$$

Let us define the maximal functions

$$\tilde{\mathcal{A}}^*(r) = \sup_{0<t<1} |\tilde{\mathcal{A}}(r,t)| \quad \text{and} \quad \tilde{\mathcal{C}}^*(r) = \sup_{0<t<1} |\mathcal{C}(r,t)|.$$

For these maximal function, we want to show that $||\tilde{\mathcal{A}}^*||_{L^2} + ||\tilde{\mathcal{C}}^*||_{L^2} \lesssim A_k ||G_0||_{L^2}$, where $A_k = (n+2k)^{\frac{n+2k}{2}}$. To prove this, let us first observe from the asymptotic behavior (3.2) that

$$\left|\frac{1}{r}\tilde{\mathcal{A}}^*\left(\frac{1}{r}\right)\right| \lesssim r^{-1} \int_0^r G_0(\rho) d\rho \le 2\mathcal{M}\left(\frac{r}{2}\right), \quad \left|\frac{1}{r}\tilde{\mathcal{C}}^*\left(\frac{1}{r}\right)\right| \lesssim A_k \int_r^\infty \rho^{-1} G_0(\rho)\, d\rho.$$

Thus using the Hardy-Littlewood maximal function, we have

$$||\tilde{\mathcal{A}}^*||_{L^2} = \left\|\frac{1}{r}\tilde{\mathcal{A}}^*\left(\frac{1}{r}\right)\right\|_{L^2} \lesssim ||G_0||_{L^2}.$$

For $\mathcal{C}^*$, we have

$$\begin{aligned}||\tilde{\mathcal{C}}^*||_{L^2} &\lesssim A_k \sup_{||\varphi||_{L^2}\le 1}\left|\int_0^\infty \int_r^\infty \rho^{-1}|G_0(\rho)|\,d\rho\varphi(r)\,dr\right| \\ &\le A_k \sup_{||\varphi||_{L^2}\le 1}\int_0^\infty |G_0(\rho)|\rho^{-1}\int_0^\rho |\varphi(r)|\,drd\rho \\ &\le 2A_k \sup_{||\varphi||_{L^2}\le 1}\int_0^\infty |G_0(\rho)|\mathcal{M}(\varphi)(\rho/2)\,d\rho \lesssim ||G_0||_{L^2}.\end{aligned}$$

To estimate $\tilde{\mathcal{B}}_\pm$, we use the extended operator $\tilde{\mathcal{B}}$ defining G_0 by $G_0(-\rho)$ for $\rho \le 0$ such that

$$\tilde{\mathcal{B}}G_0(r,t) = \int_{|\rho|\ge \frac{1}{r}} e^{2\pi i(t\phi(|\rho|)+r\rho)}G_0(\rho)\frac{d\rho}{(1+|\rho|)^s}.$$

We can rewrite this as $\tilde{\mathcal{B}}G_0(r,t) = \int_{\mathbb{R}} + \int_{|\rho|<\frac{1}{r}} \equiv \tilde{\mathcal{B}}_1G_0 + \tilde{\mathcal{B}}_2G_0$. Let $\tilde{\mathcal{B}}_1^*G_0(r) = \sup_{|t|<1}|\mathcal{B}_1G_0|$ and $\tilde{\mathcal{B}}_2^*G_0(r) = \sup_{|t|<1}|\tilde{\mathcal{B}}_2G_0|$. Then we first have $\tilde{\mathcal{B}}_2^*G(r) \lesssim \int_0^{\frac{1}{r}}|G_0(\rho)|d\rho$ and hence

$$\frac{1}{r}\tilde{\mathcal{B}}_2^*G_0\left(\frac{1}{r}\right) \lesssim \frac{1}{r}\int_0^r |G_0(\rho)|d\rho \lesssim \mathcal{M}(|G_0|)(\frac{r}{2}),$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal function. Thus $||\tilde{\mathcal{B}}_2^*G_0||_{L^2} \lesssim ||G_0||_{L^2}$.

Next, we consider global L^2 estimate of the local maximal operator $\tilde{\mathcal{B}}_1^*$. To do this, we employ the Kolmogorv-Seliverstov-Plessner method. Let us defined an operator $\mathcal{T}$ as

$$\mathcal{T}G_0(r) = \int e^{2\pi i(r\rho+t(r)\phi(|\rho|))}G_0(\rho)\,\frac{d\rho}{(1+|\rho|)^s},$$

where $t(r)$ is any measurable function with $|t(r)| < 1$ on $\mathbb{R}$. Then we may write the operator $\mathcal{T}$ by $\mathcal{T}_j$ as $\mathcal{T}G_0(r) = \sum_{j\ge 0}\mathcal{T}_jG_0(r)$, where

$$\mathcal{T}_jG_0(r) = \int e^{2\pi i(r\cdot\rho+t(r)\phi(|\rho|))}G_0(\rho)\varphi_j(\rho)\,\frac{d\rho}{(1+|\rho|)^s} \quad \text{for} \quad j\in\mathbb{Z},$$

where φ_j are Littlewood-Paley functions such that φ_0 is supported in unit ball $B(0,1)$, $\varphi_j(\cdot) = \varphi(\frac{\cdot}{2^j})$ is supported in $B(0,2^{j+1}) \setminus B(0,2^{j-1})$ and $\sum_{j\ge 0}\varphi_j = 1$. We claim that $||\mathcal{T}_jG_0||_{L^2_r} \lesssim 2^{\frac{aj}{4}}||\Delta_j\check{G}_0||_{L^2}$, where $\widehat{\Delta_j g} = \varphi_j\widehat{g}$. To show that, let $\mathcal{T}_j^d$ be the dual operator of $\mathcal{T}_j$. Then for any $F(r)\in C_0^\infty(\mathbb{R})$ and $j\ge 1$,

$$||\mathcal{T}_j^dF||_{L^2}^2 = \iint K_j(r,r')\,F(r)\overline{F(r')}\,drdr'$$

where

$$K_j(r,r') = 2^{(1-2s)j} \int e^{-2\pi i((t(r)-t(r'))\phi(2^j|\rho|)+2^j(r-r')\rho)} 2^{2sj}\varphi^2(\rho)\frac{d\rho}{(1+2^j|\rho|)^{2s}}.$$

Since $\frac{2^{2sj}\varphi^2(\rho)}{(1+2^j|\rho|)^{2s}}$ and its derivatives are uniformly bounded on j, from Lemma 2.4 replacing δ by 2^j, we have

$$\sup_{r'\in\mathbb{R}} \int |K_j(r,r')|dr, \quad \sup_{r\in\mathbb{R}} \int |K_j(r,r')|dx' \lesssim 2^{(\frac{a}{2}-2s)j}.$$

It follows from the Schur's lemma (see the lemma in p.284 of [14]) that

$$||\mathcal{T}_j^d F||_{L^2} \lesssim 2^{(\frac{a}{4}-s)j}||F||_{L^2}. \tag{3.3}$$

If $j=0$, then

$$||\mathcal{T}_0^d F||_{L^2}^2 = \iint K_0(r,r')\, F(r)\overline{F(r')}\, drdr'$$

where

$$K_0(r,r') = \int e^{-2\pi i((t(r)-t(r'))\phi(|\rho|)+(r-r')\rho)}\varphi_0^2(\rho)\frac{d\rho}{(1+|\rho|)^{2s}}.$$

Since $|t-t'| \le 2$, using the integration by part several times, for any μ, we have

$$\sup_{(t,t')\in[0,1]^2} |K_0(r,r')| \lesssim (1+|r-r'|)^{-\mu}.$$

Thus choosing a large μ and using Shur's lemma again, we have $||\mathcal{T}_0^d F||_{L^2}^2 \lesssim ||F||_{L^2}$. Therefor combining this and (3.3), we prove the part **(2)** of theorem.

3.3. **Proof of (3) and (4).** Let us define an operator $S_{0,k}$ by $S_{0,k}G_0(r,t) = r^{-\frac14}T_{0,k}G_0(r,t)$, where $T_{0,k}$ is the same operator with $s=\frac14$ as in the proof of **(1)**. Using the spherical coordinate and following exactly the same argument as in the proof of Theorem 1.1 with $s=\frac14$ and $p=2$, one can easily obtain the following estimate

$$||T^{**}f||_{L^2(|x|^{-\frac12}\,dx)} \le C_k||f||_{\dot H^{\frac14}}. \tag{3.4}$$

Since from the result **(2)** we have

$$||T^*f||_{L^2} \le ||f||_{H^s} \quad \text{for any} \quad s > \frac{a}{4}, \tag{3.5}$$

by the complex interpolation [1] between (3.4) and (3.5) with s such that $(1-\theta)s + \frac{\theta}{4} = \frac{a}{4}$ for $\theta = 2\varepsilon$, we can obtain the desired result.

For the proof of **(4)**, replacing $r^{-\gamma}$ in the definition of $S_{0,k}G_0$ with $r^{-\gamma}(1+r)^{-\frac{b}{p}}$ and using the second part of Lemma 2.3, by the same method as the one in the proof of **(1)**.

Remark 3.1. *For one dimensional global or higher dimensional local estimate similar to* (3.4)*, see* [18] *and* [11].

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