CONICS ON A GENERIC HYPERSURFACE

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Abstract. In this paper, we compute the contributions from double cover maps to genus 0 degree 2 Gromov-Witten invariants of general type projective hypersurfaces. Our results correspond to a generalization of Aspinwall-Morrison formula to general type hypersurfaces in some special cases.

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1. Introduction

In this paper, we discuss a generalization of the multiple cover formula for rational Gromov-Witten invariants of Calabi-Yau manifolds [AM], [M] to double cover maps of a line $L$ on a degree $k$ hypersurface $M^k_N$ in $\mathbb{P}^{N-1}$. Naively, for a given finite set of elements $\alpha_j \in H^*(M^k_N, \mathbb{Z})$, the rational Gromov-Witten invariant $\langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \cdots \mathcal{O}_{\alpha_n} \rangle_{0, d}$ of $M^k_N$ counts the number of degree $d$ (possibly singular and reducible) rational curves on $M^k_N$ that intersect real sub-manifolds of $M^k_N$ that are Poincaré-dual to $\alpha_j$.

Recently, the mirror computation of rational Gromov-Witten invariants of $M^k_N$ with negative first Chern class $(k-N > 0)$ was established in [CG, [I], [J]. Using the method presented in these articles, we can compute $\langle \mathcal{O}_{e^{c_1}} \cdots \mathcal{O}_{e^{c_m}} \rangle_{0, d}$ where $e$ is the generator of $H^{1,1}(M^k_N, \mathbb{Z})$. Briefly, mirror computation of $M^k_N$ $(k > N)$ in [J] goes as follows. We start from the following ODE:

\begin{equation}
(\partial_x)^{N-1} - k \cdot \exp(x) \cdot (k\partial_x + k - 1)(k\partial_x + k - 2) \cdots (k\partial_x + 1) w(x) = 0,
\end{equation}

and construct the virtual Gauss-Manin system associated with (1):

\begin{equation}
\partial_x \tilde{\psi}_{N-2-m}(x) = \tilde{\psi}_{N-1-m}(x) + \sum_{d=1}^{\infty} \exp(dx) \cdot \tilde{L}^{N,k,d}_m \cdot \tilde{\psi}_{N-1-m-(N-k)d}(x),
\end{equation}

where $m$ runs through all the integers and $\tilde{L}^{N,k,d}_m$ is non-zero only if $0 \leq m \leq N-1+(k-N)d$. From the compatibility of (1) and (2), we can derive the recursive formulas that determine all the $\tilde{L}^{N,k,d}_m$:

$$
\sum_{n=0}^{k-1} \tilde{L}^{N,k,1}_n w^n = k \cdot \prod_{j=1}^{k-1} (jw + (k-j)),
$$

$$
\sum_{m=0}^{N-1+(k-N)d} \tilde{L}^{N,k,d,z}_m = \sum_{l=2}^{d} (-1)^l \sum_{0=i_0 < \cdots < i_l = d} \times
$$

$$
\times \sum_{j_0=0}^{\frac{j_0}{j_0}} \cdots \sum_{j_l=0}^{\frac{j_l}{j_l}} \prod_{m=1}^{l} \left( \frac{j_{i_{m-1}} + (d-i_{m-1})z}{d} j_{i_m} - j_{i_{m-1}} \right) \tilde{L}^{N,k,j_{i_m}-i_{m-1}}_{j_{i_m}+(N-k)i_{m-1}},
$$

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With these data, we can construct the formulas that represent rational three point Gromov-Witten invariant \( \langle \mathcal{O}_c \mathcal{O}_{c,-2-m} \mathcal{O}_{c, -1-(k-N)} \rangle_0 \) in terms of \( \tilde{L}^{N,k,d}_m \). These three point Gromov-Witten invariants are enough for reconstruction of all the rational Gromov-Witten invariants \( \langle \mathcal{O}_{c_1} \mathcal{O}_{c_2} \cdots \mathcal{O}_{c_n} \rangle_{0,d} \) [KM]. In particular, we obtain the following formula in the \( d = 2 \) case:

\[
\langle \mathcal{O}_c \mathcal{O}_{c,-2-m} \mathcal{O}_{c, -1-(k-N)} \rangle_2 = k \cdot \left( \tilde{L}^{N,k,2}_m - \tilde{L}^{N,k,2}_{1+2(k-N)} - 2 \tilde{L}^{N,k,1}_{1+(k-N)} \left( \sum_{j=0}^{k-N} (\tilde{L}^{N,k,1}_{m-j} - \tilde{L}^{N,k,1}_{i+2(k-N)-j}) \right) \right).
\]

According to the results of this procedure, rational three point Gromov-Witten invariants can be rational numbers with large denominator if \( k > N \), in contrast to the Calabi-Yau case where rational three point Gromov-Witten invariants are always integers.

One of the reasons of this rationality (non-integrality) comes from the contributions of multiple cover maps to Gromov-Witten invariants. In the Calabi-Yau case \( (N = k) \), for any divisor \( m \) of \( d \) there are some contributions from degree \( m \) multiple cover maps \( \phi \) of a rational curve \( \mathbb{P}^1 \) onto a degree \( \frac{d}{m} \) rational curve \( C \hookrightarrow M^k \). The contributions from the multiple cover maps are expressed in terms of the virtual fundamental class of Gromov-Witten invariants. Let \( C \) be a general degree \( d \) rational curve in \( M^k \). Its normal bundle \( N_{C/M^k} \) is decomposed into a direct sum of line bundles as follows:

\[
N_{C/M^k} \simeq O_C(-1) \oplus O_C(-1) \oplus O_C^{(k-5)}.
\]

Let \( \phi : \mathbb{P}^1 \to C \) be a holomorphic map of degree \( m \). Since the pull-back \( \phi^*(N_{C/M^k}) \) is given by

\[
\phi^*(N_{C/M^k}) \simeq O_{\mathbb{P}^1}(-m) \oplus O_{\mathbb{P}^1}(-m) \oplus O_{\mathbb{P}^1}^{(k-5)},
\]

we obtain \( h^1(\phi^*(N_{C/M^k})) = 2m - 2 \). On the other hand, let \( \overline{M}_{0,0}(M,d) \) be the moduli space of 0-pointed stable maps of degree \( d \) from genus 0 curve to \( M \). Then the moduli space of \( \phi \) is the fiber space \( \pi : \overline{M}_{0,0}(C,m) \to \overline{M}_{0,0}(M^k, \frac{d}{m}) \), whose fibre \( \overline{M}_{0,0}(C,m) \) over \( C \) (fixed) has complex dimension \( 2m - 2 \). Then the push-forward of the virtual fundamental class \( \pi_* (c_{top}(H^1(\phi^*(N_{C/M^k})))) \) can be computed only by intersection theory on the fiber \( \overline{M}_{0,0}(C,m) \), which turns out to be equal to \( \frac{1}{2} \). This depends on neither the structure of the base \( \overline{M}_{0,0}(M^k, \frac{d}{m}) \) nor the global structure of the fibration.

But when \( k < N \), the situation is more complicated than \( M^k \) because of negative first Chern class. Let us concentrate on the case of \( d = 2, m = 2 \) that we discuss in this paper. In this case, \( C \) is just a line \( L \) on the hypersurface \( M^k \). The moduli space \( \overline{M}_{0,0}(M^k,1) \) is a sub-manifold of \( \overline{M}_{0,0}(\mathbb{P}^{N-1},1) \), while \( \overline{M}_{0,0}(\mathbb{P}^{N-1},1) \) is the Grassmannian \( G(2,N) \), the moduli space of rank 2 quotients of \( V = \mathbb{C}^N \). As will be shown later, for a generic line \( L \), \( N_{L/M^k} \) is decomposed into

\[
N_{L/M^k} \simeq O_L(-1)^{2k-N+2} \oplus O_L^{2N-k-5}.
\]
By pulling back it by the degree 2 map \( \phi : \mathbb{P}^1 \to L \), we obtain,
\[
\phi^*N_{L/M_N^k} \simeq O_{\mathbb{P}^1}((-2)^{k-N+2} + O_{\mathbb{P}^1}^{2N-k-5}).
\]
Therefore, \( h^1(\phi^*(N_{L/M_N^k})) = k-N+2 \), which is strictly greater than two, the complex dimension of the fiber \( \overline{M}_{0,0}(L,2) \). Thus we need to know the global structure of the fibration \( \pi \) in order to compute the multiple cover contribution to degree 2 rational Gromov-Witten invariants of \( M_N^k \).

In order to estimate the contributions from double cover maps \( \phi : \mathbb{P}^1 \to \langle O_c, O_d, O_e \rangle_{0,2} \), we first computed the number of conics, that intersect cycles Poiccaré dual to \( e^a, e^b \) and \( e^c \), on \( M_N^k \) (whose normal bundle are of the same type) by using the method in [K2]. Then we found the following formula by comparing these integers with the results obtained from (3):
\[
\langle O_c, O_d, O_e \rangle_{0,2} = (\text{number of corresponding conics}) + \int_{\overline{M}(2,N)} c_{top}(S^k Q) \wedge \left[ \frac{c(S^{k-1} Q)}{1 - \frac{1}{2} c_1(Q)} \right]_{k-N} \wedge \sigma_{a-1} \wedge \sigma_{b-1} \wedge \sigma_{c-1},
\]
where \( Q \) is the universal rank 2 quotient bundle of \( G(2,N) \), \( \sigma_a \) is a Schubert cycle defined by \( \sum_{a=0}^\infty \sigma_a := \frac{1}{\prod_{i=1}^n} \) and \( [\ast]_{k-N} \) is the operation of picking up degree \( 2(k-N) \) part of Chern classes.

On the other hand, we have the following formula which directly follows from the definition of the virtual fundamental class of \( \overline{M}_{0,0}(M_N^k,2) \):
\[
\langle O_c, O_d, O_e \rangle_{0,2} = (\text{number of corresponding conics}) + 8 \int_{\overline{M}(2,N)} c_{top}(S^k Q) \wedge \left[ \pi_\ast(c_{top}(H^1(\phi^*N_{L/M_N^k}))) \right]_{k-N} \wedge \sigma_{a-1} \wedge \sigma_{b-1} \wedge \sigma_{c-1},
\]
where \( \pi : \overline{M}_{0,0}(L,2) \to \overline{M}_{0,0}(M_N^k,1) \) is the natural projection. Here, the factor 8 comes from the divisor axiom of Gromov-Witten invariants.

In this paper, we prove the following formula
\[
\pi_\ast(c_{top}(H^1(\phi^*N_{L/M_N^k}))) = \frac{1}{8} \left[ \frac{c(S^{k-1} Q)}{1 - \frac{1}{2} c_1(Q)} \right]_{k-N}.
\]
By combining (5) with (6), we can derive the formula (4) immediately.

From (4), we see that \( \langle O_c, O_d, O_e \rangle_{0,2} \) of \( M_N^k \) is a rational number with denominator at most \( 2^{k-N} \). Therefore rationality (non-integrality) of the Gromov-Witten invariant \( \langle O_c, O_d, O_e \rangle_{0,2} \) is caused by the effect of multiple cover map in this case.

We note here that the total moduli space of double cover maps of lines is isomorphic to \( \mathbb{P}(S^2 Q) \) over \( G := \overline{M}_{0,0}(M_N^k,1) \hookrightarrow G(2,N) \), which is an algebraic \( Q \)-stack \( \mathbb{P}(S^2 Q)^{\text{stack}} \) (in the sense of Mumford). As a consequence, the union of all \( H^1(\phi^*N_{C/M_N^k}) \) turns out to be a coherent sheaf on \( \mathbb{P}(S^2 Q)^{\text{stack}} \) with fractional Chern class in (6), as was suggested in [BT]. See [V, Section 9].

We also did some numerical experiments on degree 3 Gromov-Witten invariants of \( M_N^k \) by using the results of [ES]. For \( k-N > 0 \), there is a new contribution from multiple cover maps to nodal conics in \( M_N^k \) that did not appear in the Calabi-Yau case. Therefore, multiple cover map contributions are far more complicated than Calabi-Yau, and we leave general analysis on this problem to future works.
This paper is organized as follows. In Section 1, we analyze characteristics of moduli space of lines in $M^k_N$ and derive $N_{L/M^k_N} \simeq O_L(-1)^{g(L)-N+2} \oplus O_L^{g(2N-k-5)}$. In Section 2, we study the moduli space $M_{0,0}(\mathbb{P}^1, 2)$ from the point of view of stability and identify it with $\mathbb{F}^2$ and show that the moduli space $M_{0,0}(\mathbb{P}^1, 2)$ is isomorphic to $\mathbb{P}(S^2 Q)$ over $G$. In section 4, we describe $H^1(\phi^* N_{L/M^k_N})$ as an coherent sheaf over $\mathbb{P}(S^2 Q)^{\text{stack}}$. In section 5, we derive the main theorem (6) of this paper by using Segre-Witten classes. In section 6, we mention some generalization to degree 3 Gromov-Witten invariants.

2. Lines on a Hypersurface

Let $M$ be a generic hypersurface of degree $k$ of the projective space $\mathbb{P}^{N-1} = \mathbb{P}(V)$. We assume $2N - 5 \geq k \geq N - 2 \geq 2$ throughout this note. In this note we count the number of rational curves of virtual degree two, namely rational curves which doubly cover lines on $M$.

Let $\mathcal{P} = \mathbb{P}(V)$ be the projective space parameterizing all one-dimensional quotients of $V$, which is usually denoted by $\mathbb{P}(V)$ in the standard notation in algebraic geometry. In this notation let $W$ be a subspace of $V$. Then $\mathcal{P}(W)$ is naturally a linear subspace of $\mathcal{P}(V)$ of dimension $\dim W - 1$.

Let $G(2, V)$ be the Grassmann variety of lines in $\mathcal{P}(V)$, the scheme parameterizing all lines of $\mathcal{P} = \mathbb{P}(V)$. This is also the universal scheme parameterizing all one-dimensional quotient linear spaces of $V$. Let $W$ be a two dimensional quotient linear space, $\psi \in G(2, V)$, namely $\psi : \mathcal{P}(W) \to \mathcal{P}(V)$ the natural immersion and $i_\psi : V \to W$ the quotient homomorphism. The space $W$ is denoted by $W(\psi)$ when necessary.

There exists the universal bundle $Q_{G(2, V)}$ over $G(2, V)$ and a homomorphism $i^{\text{univ}}:* : O_{G(2, V)} \otimes V \to Q_{G(2, V)}$ whose fiber $i^{\text{univ}}:* : V \to Q_{G(2, V)}$ is the quotient $i_\psi^* : V \to W(\psi)$ of $V$ corresponding to $\psi$.

2.1. Existence of a Line on $M$. Let $L = \mathcal{P}(W)$ be a line of $\mathcal{P}$, equivalently $W \in G(2, V)$. Then the condition $L \subseteq M$ imposes at most $k + 1$ conditions on $W$, while the number of moduli of lines of $\mathcal{P}$ equals $\dim G(2, V) = 2N - 4$. Hence we infer

Lemma 2.2. If $2N \geq k + 5$, then there exists at least a line on $M$.

See also [Katz.p.152]. Let $G$ be the subscheme of $G(2, V)$ parameterizing all lines of $\mathcal{P}(V)$ lying on $M$, $Q = (Q_{G(2, V)})_G$ the restriction of $Q_{G(2, V)}$ to $G$. By Lemma 2.2, $G$ is nonempty. Let $i^*: O_G \otimes V \to Q$ be the restriction of $i^{\text{univ}}:*$ to $G$. Let $P = \mathcal{P}(Q)$ and $\pi : P \to G$ the natural projection. Then $\pi$ is the universal line of $M$ over $G$, to be more exact, the universal family over $G$ of lines lying on $M$. In other words, the natural epimorphism $i^*: O_G \otimes V \to Q$ induces a morphism $i : P \to \mathcal{P}_G(V) := G \times \mathcal{P}(V)$, which is a closed immersion into $\mathcal{P}_G(V)$, thus $P$ is a subscheme of $\mathcal{P}_G(V)$ such that $\pi = (p_1)_P$. Let $L_\psi = \mathcal{P}(Q_\psi)$. Note that

$L_\psi = P_\psi := \pi^{-1}(\psi) \cong \mathcal{P}(Q_\psi) \subset \{\psi\} \times \mathcal{P}(V) \cong \mathcal{P}(V).$
2.3. The normal bundle $N_{L/M}$. The argument of this section is standard and well known. Let $P = P(V)$, $L = P(W)$ and $i^*_W : V \to W \in G$. Let us recall the following exact sequence:

$$
0 \longrightarrow O_P \longrightarrow O_P(1) \otimes V^\vee \overset{D}{\longrightarrow} T_P \longrightarrow 0
$$

where the homomorphism $D$ is defined by

$$
D(a \otimes v^\vee) := aD_{L}(v) \quad (a \in O_P(1))
$$

$$(D_{V^\vee}F)(u^\vee) := \frac{d}{dt}F(u^\vee + tv^\vee)|_{t=0}
$$

for a homogeneous polynomial $F \in S(V)$ and $u^\vee, v^\vee \in V^\vee$. We note $H^0(O_P(1)) \otimes V^\vee = V \otimes V^\vee \in \text{End}(V, V)$ and that the image of $H^0(O_P)$ in $\text{End}(V, V)$ is $\text{CId}_V$.

We also have the following exact sequences:

$$
0 \longrightarrow T_L \longrightarrow (T_P)_L \longrightarrow N_{L/P} \longrightarrow 0
$$

$$
0 \longrightarrow O_L \longrightarrow O_L(1) \otimes V^\vee \overset{D_L}{\longrightarrow} (T_P)_L \longrightarrow 0.
$$

**Lemma 2.4.** Let $L = P(W)$. Then

$$
N_{L/P} \simeq O_L(1) \otimes (V^\vee / W^\vee), \ H^0(N_{L/P}) \simeq W \otimes (V^\vee / W^\vee).
$$

**Proof.** The assertion is clear from the following commutative diagram with exact rows and columns:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & O_L & \longrightarrow & O_L(1) \otimes W^\vee & \overset{(D_L)|_{W^\vee}}{\longrightarrow} & T_L & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \otimes V^\vee & & \downarrow & & \downarrow \\
0 & \longrightarrow & O_L & \longrightarrow & O_L(1) \otimes V^\vee & \overset{D_L}{\longrightarrow} & (T_P)_L & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & O_L(1) \otimes (V^\vee / W^\vee) & \longrightarrow & N_{L/P} & \longrightarrow & 0
\end{array}
\]

The second assertion is clear from $H^0(L, O_L(1)) = W$. \hfill \Box

Since $T_L \cong O_L(2)$, there follow exact sequences

$$
0 \longrightarrow H^0(T_L) \longrightarrow H^0((T_P)_L) \longrightarrow H^0(N_{L/P}) \longrightarrow 0
$$

$$
0 \longrightarrow H^0(O_L) \longrightarrow H^0(O_L(1)) \otimes V^\vee \overset{H^0(D_L)}{\longrightarrow} H^0((T_P)_L) \longrightarrow 0.
$$

We also note

$$
H^0(T_L) = \text{LieAut}^0(L) = \text{End}(W, W)/\text{center} = \text{End}(W, W)/\text{CId}_W.
$$

Since $H^0(O_L(1)) = W$, we see

$$
H^0((T_P)_L) = W \otimes V^\vee / \text{Im} H^0(O_L) = \text{Hom}(V, W)/\text{CId}_W.
$$

Hence we again see

$$
H^0(N_{L/P}) = (\text{Hom}(V, W)/\text{CId}_W^*)/(\text{Hom}(W, W)/\text{CId}_W)
$$

$$
= W \otimes (V^\vee / W^\vee) = \text{Hom}(V/W, W).
$$

For any line $L = P(W)$ of $P$ the following sequence is exact:

$$
0 \to N_{L/M} \to N_{L/P} \to (N_{M/P})_L(z \in O_L(k)) \to 0.
$$
Hence so is the following sequence as well:
\[
\begin{array}{c}
0 \longrightarrow H^0(N_{L/M}) \longrightarrow H^0(N_{L/P}) \overset{H^0(D_L)}{\longrightarrow} H^0(O_L(k)) \\
\longrightarrow H^1(N_{L/M}) \longrightarrow 0.
\end{array}
\]

Hence we have

**Lemma 2.5.** The following is exact:

(8) \[ 0 \rightarrow H^0(N_{L/M}) \rightarrow W \otimes (V^\vee/W^\vee) \overset{H^0(D_L)}{\rightarrow} S^kW \rightarrow H^1(N_{L/M}) \rightarrow 0. \]

**Corollary 2.6.** \( \dim G \geq 2N - k - 5 \), equality holding if \( H^1(N_{L/M}) = 0 \).

**Proof.** As is well-known, \( \dim G \geq h^0(N_{L/M}) - h^1(N_{L/M}) \). Note \( \dim W \otimes (V^\vee/W^\vee) = 2(N - 2) \) and \( \dim S^kW = k + 1 \). Hence the corollary follows from Lemma 2.5. \( \square \)

**Lemma 2.7.** For a generic line \( L \) on a generic hypersurface \( M \) of degree \( k \)

(i) \( N_{L/M} \simeq O_L^{\oplus a} \oplus O_L(-1)^{\oplus b} \), where \( a = 2N - k - 5 \) and \( b = k - N + 2 \),

(ii) \( \text{Coker } H^0(D_L) \simeq S^{k-1}W/(V^\vee/W^\vee) \) where \( D_L \) denotes \( D_L \otimes O_L(-1) \).

**Proof.** Let \( M \) be a generic hypersurface of degree \( k \) and \( L \) a generic line on \( M \). Without loss of generality we may assume that \( W^\vee \) is generated by \( e_1^V \) and \( e_2^V \), in other words, \( \psi : L \rightarrow \mathbb{P} \) is given by

\[ \psi : [s : t] \rightarrow [x_1, \cdots, x_N] = [s, t, 0, \cdots, 0]. \]

Then \( F \), the polynomial of degree \( k \) defining \( M \), is written as

\[ F = x_3F_3 + x_4F_4 + \cdots + x_NF_N \]

for some polynomials \( F_j \) of degree \( k - 1 \). Let \( f_j = \psi^*F_j = F_j(s, t, 0, \cdots, 0) \).

Now we consider the exact sequence

\[
\begin{array}{c}
0 \longrightarrow H^0(N_{L/M}(-1)) \longrightarrow H^0(N_{L/P}(-1)) \overset{H^0(D_L^-)}{\longrightarrow} H^0(O_L(k - 1)) \\
\longrightarrow H^1(N_{L/M}(-1)) \longrightarrow 0,
\end{array}
\]

where we note \( H^0(N_{L/P}(-1)) = V^\vee/W^\vee \). Hence the following is exact:

(9) \[ 0 \longrightarrow H^0(N_{L/M}(-1)) \longrightarrow V^\vee/W^\vee \overset{H^0(D_L^-)}{\longrightarrow} S^{k-1}W \\
\longrightarrow H^1(N_{L/M}(-1)) \longrightarrow 0 \]

where \( H^0(D_L^-) \) is given by \( H^0(D_L^-)(e_j^V) = f_j \quad (j = 3, 4, \cdots, N) \).

A generic choice of \( F \) implies a generic choice of degree \( k - 1 \) polynomials \( f_j \) \((j = 3, 4, \cdots, N)\) in \( s \) and \( t \). By the assumptions

\begin{align*}
\dim S^{k-1}W & = k \geq N - 2 = \dim V^\vee/W^\vee, \\
\dim W \otimes V^\vee/W^\vee & = 2(N - 2) \geq k + 1 = \dim S^kW,
\end{align*}

the generic choice of \( F \) implies that we can choose \( f_j \in S^{k-1}W \quad (j = 3, 4, \cdots, N) \)

and fix once for all) such that

(iii) \( f_j \) \((j = 3, 4, \cdots, N)\) are linearly independent,

(iv) \( Wf_3 + Wf_4 + \cdots + Wf_N = S^kW \).
Hence $H^0(D_L)$ is injective by (iii). It follows that $H^0(N_{L/M}(-1)) = 0$. Hence (ii) is clear. Next we consider $H^0(D_L)$. By (iv), we see

$$S^k W = W \cdot H^0(D_L)(V^*/W^*) = H^0(D_L)(W \otimes V^*/W^*),$$

whence $H^0(D_L)$ is surjective. It follows that $H^1(N_{L/M} = 0$. Hence $N_{L/M} \simeq O_L^{\oplus a} \oplus O_L(-1)^{\oplus b}$ for some $a$ and $b$. Since $a + b = \text{rank}(N_{L/M}) = N - 3$ and $-b = \text{deg}(N_{L/M}) = N - 2 - k$, we have (i).

\[\square\]

2.8. Lines on a quintic hypersurface in $\mathbb{P}^4$. See [Katz, Appendix A] for the subsequent examples. Let $N = 5$ and $k = 5$. Hence $M$ is a hypersurface of degree 5 in $\mathbb{P}^4$, a Calabi-Yau 3-fold. Let

$$F = x_4x_1^4 + x_5x_2^4 + x_3^5 + x_4^3 + x_5^2.$$

First we note that $M = \{F = 0\}$ is nonsingular. Let $L = \{x_3 = x_4 = x_5 = 0\} = \{s, t, 0, 0, 0\}$. In this case $f_3 = 0$, $f_4 = s^4$ and $f_5 = t^4$. In the exact sequence (1) we see $H^0(N_{L/M}(-1)) = \text{Ker} H^0(D_L) = C_{\mathbb{C}}^{\oplus 5}$ and $H^1(N_{L/M}(-1)) = \text{Coker} H^0(D_L)$ is 3-dimensional. Hence $N_{L/M} = O_L(1) \oplus O_L(-3)$.

We summarize the above. If $\text{dim} H^0(D_L) = 1$ and if $M$ is nonsingular, then $N_{L/M} = O_L(1) \oplus O_L(-3)$. Hence $H^0(N_{L/M}) = \text{Ker} H^0(D_L) = W \otimes \text{Ker} H^0(D_L)$ is 2-dimensional. Therefore we can choose $f_3 = 0$ and a linearly independent pair $f_4$ and $f_5 \in S^3W$ so that $W f_4 + W f_5$ is 4-dimensional. The choice $f_3 = s^4$ and $f_5 = t^4$ satisfies the conditions. This enables us to find a nonsingular hypersurface $M$ as above. However if we choose $f_3 = 0$, $f_4 = s^4$ and $f_5 = s^2t$, then $W f_4 + W f_5$ is 3-dimensional. Hence $M$ is singular.

Next in the same manner we find $L$ on a nonsingular hypersurface $M$ with $N_{L/M} = O_L \oplus O_L(-2)$ or $N_{L/M} = O_L(-1)^{\oplus 2}$. Let

$$F = x_3x_1^4 + x_4x_2^4x_2 + x_5x_3^4 + x_4^3 + x_5^2.$$

Then we have $f_3 = s^4$, $f_4 = s^3t$ and $f_5 = t^4$. Since $W f_3 + W f_4 + W f_5$ is 5-dimensional, $H^0(N_{L/M}(-1)) = \text{Ker} H^0(D_L) = 0$, $H^0(N_{L/M}) = \text{Ker} H^0(D_L) = C_{\mathbb{C}}^{\oplus 5}$. We see also that $\text{dim} H^1(N_{L/M}) = \text{dim} \text{Coker} H^0(D_L) = 1$ and $N_{L/M} = O_L \oplus O_L(-2)$. The hypersurface $M = \{F = 0\}$ is easily shown to be nonsingular.

If $F = x_3x_1^4 + x_4x_2^4x_2 + x_5x_3^4 + x_4^3 + x_5^2$ and $M = \{F = 0\}$, then $N_{L/M} = O_L(-1)^{\oplus 2}$.

2.9. Lines on a generic hypersurface $M^8$ of $\mathbb{P}^6$. Let $N = 7$ and $k = 8$. In view of Lemma 2.2 there exists a line $L$ on any generic hypersurface $M$ of degree 8 in $\mathbb{P}(V) = \mathbb{P}^6$. In view of Lemma 2.7, $a = 1$, $b = 3$ and $N_{L/M} \simeq O_L \oplus O_L(-1)^{\oplus 3}$. For example let $L : x_j = 0 \ (j \geq 3)$ and we take

$$F_3 = 8x_1^4, \ F_4 = 8x_1^3x_2, \ F_5 = 8x_2^6, \ F_6 = 8x_1^2x_2^5, \ F_7 = 8x_2^7,$$

$$F = x_3F_3 + x_4F_4 + x_5F_5 + x_6F_6 + x_7F_7 + x_8^8 + x_9^8 + x_1^8 + x_2^8.$$
nonsingular everywhere. The space $H^0(N_{L/M})$ is spanned by $te^X_\xi - se^Y_\eta$, hence an infinitesimal deformation $L_e$ of $L$ is given by

$$[s,t] \mapsto [s,t, \varepsilon s, 0, 0, 0]$$

which yields $F|_{L_e} = \varepsilon^8 (s^8 + t^8) \equiv 0 \mod \varepsilon^8$. Since $H^1(N_{L/M}) = 0$, this infinitesimal deformation is integrable and $G$ (the moduli of lines of $P^0$ contained in $M$) is nonsingular and one dimensional at the point $[L]$.

We note that $M$ also contains 8 lines

$$L' := L'_{e_8} = \varepsilon s x_1 - x_2 = x_3 + \varepsilon s x_4 = x_j = 0 \ (j \geq 5),$$

with $N_{L/M} = O_{L'}(1)^{\mathbb{R}^3} \oplus O_{L'}(-6)$ where $\varepsilon^8 = -1$.

3. Stability

**Definition 3.1.** Suppose that a reductive algebraic group $G$ acts on a vector space $V$. Let $v \in V$, $v \neq 0$.

(1) the vector $v$ is said to be semistable if there exists a $G$-invariant homogeneous polynomial $F$ on $V$ such that $F(v) \neq 0$,

(2) the vector $v$ is said to be stable if $p$ has a closed $G$-orbit in $X_{ss}$ and the stabilizer subgroup of $v$ in $G$ is finite.

Let $\pi : V \setminus \{0\} \to \mathbb{P}(V^\vee)$ be the natural surjection. Then $v \in V$ is semistable (resp. stable) if and only if $\pi(v)$ is semistable (resp. stable).

3.2. Grassmann variety. Let $V$ be an $N$-dimensional vector space, and $G(r,N)$ the Grassmann variety parameterizing all $r$-dimensional quotient spaces of $V$. Here is a natural way of understanding $G(r,N)$ via GIT-stability. Let $U$ be an $r$-dimensional vector space, $X = \text{Hom}(V,U)$ and $\pi : X \setminus \{0\} \to \mathbb{P}(X^\vee)$ the natural map. Then $\text{SL}(U)$ acts on $X$ from the left by:

$$(g \cdot \phi^*)(v) = g \cdot (\phi^*(v)) \quad \text{for } \phi^* \in X, \ v \in V.$$ 

We see that for $\phi^* \in X$

$$\phi^* \text{ is } \text{SL}(U)\text{-stable} \iff \text{rank } \phi^* = r,$$

$$\phi^* \text{ is } \text{SL}(U)\text{-semistable} \iff \phi^* \text{ is } \text{SL}(U)\text{-stable}.$$ 

In fact, if rank $\phi^* = r - 1$, then there is a one-parameter torus $T$ of $\text{SL}(U)$ such that the closure of the orbit $T \cdot \phi$ contains the zero vector as the following simple example ($r = 2$) shows

$$\lim_{t \to 0} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \lim_{t \to 0} \begin{pmatrix} ta_{11} & ta_{12} & \cdots & ta_{1N} \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$ 

Let $X_s$ be the set of all (semi)stable points and $P_s$ the image of $X_s$ by $\pi$. It is, as we saw above, just the set of all $\phi \in X$ with rank $\phi^* = r$. Therefore the GIT-orbit space $P_s/\text{SL}(U)$ is the orbit space $P_s/\text{SL}(U)$ by the free action, the Grassmann variety $G(r,N)$.
3.3. Moduli of double coverings of $\mathbb{P}^1$ (1). Let $W$ and $U$ be a pair of two dimensional vector spaces, $X = \text{Hom}(W, S^2U)$, and $\pi : X \setminus \{0\} \to \mathbb{P}(X^*)$ the natural morphism. Note that $\text{SL}(U)$ acts on $S^2U$ from the left via the natural action: $\sigma(u_1u_2) = \sigma(u_1)\sigma(u_2)$ for $\forall u_1, u_2 \in U$. Thus $\text{SL}(U)$ acts on $X$ from the left in the same manner in the subsection 3.2.

**Lemma 3.4.** Let $\phi^* \in X$.

(i) $\phi^*$ is unstable iff $\phi^*(w)$ has a double root for any $w \in W$.

(ii) $\phi^*$ is semistable iff $\phi^*(w)$ has no double roots for some nonzero $w \in W$.

(iii) $\phi^*$ is stable iff $\phi^*(W)$ is a base-point free linear subsystem of $S^2U$ on $\mathbb{P}(U)$.

**Proof.** We note that $\phi^*$ is unstable iff there is a suitable basis $s$ and $t$ of $U$ such that $\phi^*(w) = a(w)s^2$ for any $w \in W$ since a torus orbit $T \cdot \phi^*$ contains the zero vector. This proves (i). This also proves (ii). Next we prove (iii). If $\phi^*(W)$ has a base point, then it is clear that $\phi^*$ is not stable. If $\phi^*$ is semistable and it is not stable, then we choose a basis $s, t$ of $U$ and a basis $w_1, w_2$ of $W$ such that $\phi^*(w_1) = st$. If $\phi^*(w_2) = as^2 + bst$, then $\phi^*$ is not stable. This proves the lemma. 

**Theorem 3.5.** Let $X_{ss}$ be the Zariski open subset of $X$ consisting of all semistable points of $X$, $\pi(X_{ss})$ the image of $X_{ss}$ by $\pi$, and $Y := \pi(X_{ss})//\text{SL}(U)$. Then $Y \simeq \mathbb{P}^2$.

**Proof.** First consider a simplest case. We choose a basis $s, t$ of $U$. Let $w_1$ and $w_2$ be a basis of $W$, $T$ the subgroup of $\text{SL}(U)$ of diagonal matrices and $X^t = \{ \phi^* \in X; \phi^*(w_1) = 2st \}$. Let $Z' = \text{SL}(U) \cdot X^t$.

We note that $Z'$ is an $\text{SL}(U)$-invariant subset of $X_{ss}$. We prove $\pi(Z')//\text{SL}(U) \simeq C^2$. Let $\phi^*$ and $\psi^*$ be points of $X^t$. Let $\phi^*(w_2) = As^2 + 2Bst + Ct^2$ and $\psi^*(w_2) = as^2 + bst + ct^2$. Then it is easy to check

$$g \cdot \phi^* = \psi^* \text{ for } \exists g \in \text{SL}(U) \iff g \cdot \phi^* = \psi^* \text{ for } \exists g \in T$$

$$\iff A = au^2, \ B = b, \ C = u^{-2}c \text{ for } \exists u \neq 0.$$ Therefore each equivalence class of $\pi(Z)//\text{SL}(U)$ is represented by the pair $(AC, B)$, which proves $\pi(Z)//\text{SL}(U) \simeq \mathbb{P}^2$.

Now we prove the lemma. Let $\phi^* \in X_{ss}$, $\phi_j = \phi^*(w_j)$ and $\phi_0 = -2(\phi_1 + \phi_2)$. Let

$$\phi_0 = r_1s^2 + 2r_2st + r_3t^2,$$ $$\phi_1 = p_1s^2 + 2p_2st + p_3t^2,$$ $$\phi_2 = q_1s^2 + 2q_2st + q_3t^2,$$

and we define

$$D_1 = p_2^2 - q_1p_3, \quad D_2 = q_2^2 - q_1q_3,$$ $$D_0 = r_2^2 - r_1r_3 = D_1 + D_2 + 2p_2q_2 - (p_1q_3 + p_3q_1).$$

To show the lemma, we prove the more precise isomorphism

$$\pi(X_{ss})//\text{SL}(U) = \text{Proj} \mathbb{C}[D_0, D_1, D_2]$$

For this purpose we define $Y_j = \pi(\{ \phi^* \in X_{ss}; \phi_j \text{ has no double roots} \})//\text{SL}(U)$. It suffices to prove $Y_1 = \text{Spec} \mathbb{C}[D_0, D_1, D_2]$ by reducing it to the first simplest case.
Let $\phi^* \in Y_1$. Let $\alpha$ and $\beta$ be the roots of $\phi_1 = 0$. By the assumption $\phi_1$ has no double roots, hence $\alpha \neq \beta$. Let

$$u = \frac{1}{\gamma}(s - \alpha t), \quad v = \frac{1}{\gamma}(s - \beta t), \quad g = \frac{1}{\gamma}\left(\begin{array}{c}
-1 \\
-\alpha \\
-\beta
\end{array}\right)$$

where $\gamma = \sqrt{\alpha - \beta}$. Note that $g \in \text{SL}(U)$. Hence we see

$$(\phi_1(s, t), \phi_2(s, t)) \equiv (p_{17}^4uv, A_1u^2 + 2B_1uv + C_1v^2)$$

where

$$A_1 = q_1^2\beta^2 + 2p_2\beta + q_3,$$
$$-B_1 = q_1\alpha + q_2(\alpha + \beta) + q_3,$$
$$C_1 = q_1^2\alpha^2 + 2p_2\alpha + q_3.$$

Thus we see

$$(\phi_1(s, t), \phi_2(s, t)) \equiv (2st, As^2 + 2Bst + Ct^2)$$

where

$$A = \frac{2A_1}{p_{17}^4}, \quad B = \frac{2B_1}{p_{17}^4}, \quad C = \frac{2C_1}{p_{17}^4}, \quad p_{17}^{44} = 4D_1,$$
$$AC = B^2 - \frac{D_2}{D_1}, \quad B = \frac{D_0 - D_1 - D_2}{2D_1}.$$

Therefore by the first half of the proof

$$Y_1 \simeq \text{Spec } \mathbb{C}[AC, B] = \text{Spec } \mathbb{C}\left[\frac{D_0}{D_1}, \frac{D_2}{D_1}\right].$$

This completes the proof of the lemma.

\begin{cor}
Let $Y^\circ = \pi(X^\circ) \sslash \text{SL}(U)$. Then $Y \setminus Y^\circ$ is a conic of $Y$ defined by

$$Y \setminus Y^\circ : D_0^2 + D_1^2 + D_2^2 - 2D_0D_1 - 2D_1D_2 - 2D_2D_0 = 0.$$

\end{cor}

\begin{proof}
In view of Theorem 3.5, $Y_1 \simeq \text{Spec } \mathbb{C}[AC, B]$. The complement of $Y_0$ in $Y_1$ is then the curve defined by $AC = 0$, which is easily identified with the above conic.
\end{proof}

\begin{cor}
Let $X^0$ be the Zariski open subset of $X$ consisting of all semistable points $\phi^*$ of $X$ with rank $\phi^* = 2$, and let $Y^0 := \pi(X^0) \sslash \text{SL}(U)$. Then $Y^0 \simeq \pi(X^0) / \text{SL}(U) \simeq Y \simeq \mathbb{P}^2$.
\end{cor}

\begin{proof}
It suffices to compare $Y_1$ and $Y^0 \cap Y_1$. As in the proof of Theorem 3.5 we let $X^t = \{\phi^* \in X; \phi^*(w_1) = 2st\}$. Let $Z = \text{SL}(U) \cdot X^t$ and $Z^0 = \text{SL}(U) \cdot (X^t \cap X^0)$.

Then with the notation in Theorem 3.5, we recall $X^t = \{\phi^* \in X; \phi^*(w_1) = 2st, \phi^*(w_2) = As^2 + 2Bst + Ct^2\}$, $\pi(Z) \sslash \text{SL}(U) \simeq \text{Spec } \mathbb{C}[AC, B]$ where

$$X^t \cap X^0 = \{\phi^* \in X^t; A \neq 0 \text{ or } C \neq 0\}.$$

In the same manner as before we see $\pi(Z^0) \sslash \text{SL}(U) \simeq \text{Spec } \mathbb{C}[AC, B]$, whence $\pi(Z^0) / \text{SL}(U) = \pi(Z) / \text{SL}(U)$. This proves $Y^0 \cap Y_1 = Y_1$. This completes the proof of the corollary.
\end{proof}
3.8. Moduli of double coverings of $\mathbb{P}(W)$ (2). There is an alternative way of understanding $\pi(X)\big/\big/\Gamma \simeq \mathbb{P}^2$ by using the isomorphism $\mathbb{P}^2 \simeq \mathbb{P}^2$. We use the following convention to denote a point of $\mathbb{P}(U) = U^\vee = us^\vee + tv^\vee \in U^\vee$ where $s^\vee$ and $t^\vee$ are a basis dual to $s$ and $t$. In what follows we fix a basis $w_1$ and $w_2$ of $W$. Let $P := (a_1 : a_2)$ and $Q := (b_1 : b_2)$ be a pair of points of $\mathbb{P}(W) \simeq \mathbb{P}^1$. If $P \neq Q$, there is a double covering $\phi : \mathbb{P}(U) \to \mathbb{P}(W)$ ramifying at $P$ and $Q$, unique up to isomorphism once we fix the base $w_1$ and $w_2$:

$$\frac{b_2w_1 - b_1w_2}{a_2w_1 - a_1w_2} = \left(\frac{s}{t}\right)^2.$$ 

Thus $\phi$ is given explicitly by

$$\phi_1 := \phi^*(w_1) = b_1s^2 - a_1t^2, \quad \phi_2 := \phi^*(w_2) = b_2s^2 - a_2t^2, \quad \phi_0 = -(\phi_1 + \phi_2)$$

for which we have

$$D_1 = a_1b_1, \quad D_2 = a_2b_2, \quad D_0 = (a_1 + a_2)(b_1 + b_2).$$

The isomorphism $S^2\mathbb{P}^1 \simeq \mathbb{P}^2$ is given by $(P, Q) \mapsto (D_0, D_1, D_2)$. This shows

**Corollary 3.9.** We have a natural isomorphism: $Y \simeq \mathbb{P}(S^2W)$.

4. The virtual normal bundle of a double covering

4.1. The case $N = 7$ and $k = 8$ revisited. We revisit the example in the subsection 29. Let $N = 7$ and $k = 8$. Let $L : x_j = 0 \ (j \geq 3)$ and we take

$$F_3 = 8x_1^7, F_4 = 8x_1^6x_2, F_5 = 8x_1^5x_2^2, F_6 = 8x_1^4x_2^3, F_7 = 8x_1^2,$$

$$F = x_3F_3 + x_4F_4 + x_5F_5 + x_6F_6 + x_7F_7 + x_3^8 + x_4^8 + x_5^8 + x_6^8 + x_7^8.$$

and let $M = M^\phi : F = 0$. We often denote $L$ also by $\mathbb{P}(W)$ with $W$ a two dimensional vector space for later convenience. Since $H^0(D_L^\phi)$ is injective and $H^0(D_L)$ is surjective, we have $N_{L/M} \cong O_L \oplus O_L(-1)^{\oplus 2}$. Hence $H^1(N_{L/M}(-1)) = H^1(O_L(-2)^{\oplus 2})$ is 3-dimensional. As we see easily, this follows also from the fact that $\text{Coker } H^0(D_L^\phi)$ is freely generated by $x_1^2x_2^2, x_1x_2^3$ and $x_1x_3^2$.

Let $\phi^* = (\phi_1, \phi_2) \in X^0$. Then $\text{Coker } H^0(\phi^*D_L)$ is generated by a single element $\phi_3\phi_2^2 - \phi_2\phi_1^3$, while $\text{Coker } H^0(\phi^*D_L)$ is generated by $S^2U \cdot \phi_1^3\phi_2^3, S^2U \cdot \phi_1\phi_2^4$ and $S^2U \cdot \phi_1\phi_2^2$. To be more precise, we see

$$\text{Coker } H^0(\phi^*D_L) = \{\phi_1^3\phi_2^3, \phi_1^4\phi_2^2, \phi_2^3\phi_1^2\} \simeq S^2U / 4\phi_1, \phi_2.$$

In fact, this is proved as follows: first we consider the case where $\phi_1$ and $\phi_2$ has no common zeroes. In this case $\phi^*$ gives rise to a double covering $\phi : \mathbb{P}(U) \to \mathbb{P}(W)$ ($= L$), which we denote by $L_{\phi}$ for brevity. By pulling back by $\phi^*$ the normal sequence

$$0 \to N_{L/M} \to N_{L/P} \to O_L(k) \to 0 \ (k = 8)$$

for the line $L$ we infer an exact sequence

$$0 \to \phi^*N_{L/M} \to \phi^*N_{L/P} \xrightarrow{\phi^*D_L} \phi^*O_L(k) \to 0,$$

which yields an exact sequence

$$0 \xrightarrow{} H^0(\phi^*N_{L/M}) \xrightarrow{} S^2U \oplus (V^\vee/W^\vee) \xrightarrow{H^0(\phi^*D_L)} H^0(O_L(2k)) \xrightarrow{} 0.$$
Let $\eta = q_3 e_3^\vee + \cdots + q_7 e_7^\vee \in \text{Ker } H^0(\phi^*D_L)$, $q_j \in S^2U$. Then we have

$$\phi_1^2(q_3 \phi_1^3 + q_4 \phi_1^4 \phi_2 + q_5 \phi_1^2 \phi_2^2 + q_6 \phi_2^3) = -q_7 \phi_2^2.$$ 

Since $\phi_1$ and $\phi_2$ are mutually prime and $q_j$ is of degree two, we have $q_7 = 0$ and

$$\phi_1^2(q_3 \phi_1^3 + q_4 \phi_1^4 \phi_2 + q_5 \phi_1^2 \phi_2^2) = -q_6 \phi_2^3.$$ 

Hence $q_6 = 0$ and similarly we infer also $q_5 = 0$. Thus we have $q_3 \phi_1 + q_4 \phi_2 = 0$. This proves that $\text{Ker } H^0(\phi^*D_L)$ is generated by $\phi_2 e_2^\vee - \phi_1 e_1^\vee$.

Next we prove that $\text{Coker } H^0(\phi^*D_L)$ is generated by $\phi^* \text{Coker } H^0(D_L^\vee)$ over $S^2U$, in fact over $S^2U/\phi^*(W)$. Without loss of generality we may assume that $\phi_1 = 2st$ and $\phi_2 = \alpha s^2 + 2\nu st + t^2$ for some $\alpha \not= 0$ and $\nu \in \mathbb{C}$. Let $\phi^*W = \{\phi_1, \phi_2\}$. Then one checks $U \cdot \phi^*W = S^2U$, and hence $S^2U \cdot \phi^*W = S^4U$, $S^2m-2U \cdot \phi^*W = S^2mU$ for $m \geq 2$. It follows $S^2U \cdot \phi^*(S^mW) = S^2mU$ for $m \geq 1$. In fact, by the induction on $m$

$$S^2U \cdot \phi^*(S^mW) = S^2U \cdot \phi^*(W) \cdot \phi^*(S^{m-1}W)$$

$$= S^4U \cdot \phi^*(S^{m-1}W)$$

$$= S^2U \cdot S^2U \cdot \phi^*(S^{m-1}W)) = S^2U \cdot S^{m+1}U = S^2mU.$$ 

Therefore $H^0(O_{L_\lambda}(2k)) = S^{16}U = S^2U \cdot \phi^*(S^7W)$. Hence

$$\text{Coker } H^0(\phi^*D_L) = S^{16}U/\text{Im } H^0(\phi^*D_L)$$

$$= S^2U \cdot \phi^*(S^7W)/S^2U \cdot \phi^*(\text{Im } H^0(D_L^\vee))$$

$$= (S^2U/\phi^*(W)) \cdot \phi^*(S^7W/\text{Im } H^0(D_L^\vee)).$$

because $\text{Coker } H^0(D_L^\vee) = S^7W/\text{Im } H^0(D_L^\vee)$ and $S^2U \cdot S^7W \subset W \cdot \text{Im } H^0(D_L^\vee) = S^8W$ by the choice of $L$. This proves that $\text{Coker } H^0(\phi^*D_L)$ is generated by $\phi^* \text{Coker } H^0(D_L^\vee)$ over $S^2U/\phi^*(W)$. It follows $\text{Coker } H^0(\phi^*D_L) = (\phi^* \text{Coker } H^0(D_L^\vee)) \otimes (S^2U/\phi^*W)$.

Finally we consider the case where $\phi_1$ and $\phi_2$ has a common zero. In this case we may assume $\phi_1 = 2st$ and $\phi_2 = 2\nu st + t^2$. In this case $L_{\phi}$ is a chain of two rational curves $C_{\phi}'$ and $C_{\phi}''$, where $C_{\phi}$ is the proper transform of $\text{P}(U)$, where the double covering map from $L_{\phi}$ to $\text{P}(W)$ is the union of the isomorphisms $\phi^t$ and $\phi^s$, say, $\phi = \phi^t \cup \phi^s$. Let $\psi_1 = 2s$ and $\psi_2 = 2\nu s + t$. Then $\phi^s$ is induced by the homomorphism $(\phi^s)^* \in \text{Hom}(W, U)$ such that $(\phi^s)^*(w_j) = \psi_j$. On the other hand let $U_{\phi}^t = C\lambda + Ct$, $\psi_1^t = 2t$ and $\psi_2^t = \lambda + 2t$ where we note $\psi_j^t$ is the linear part of $\phi_j$ in $t$ with $s = 1$. Then $C_{\phi}' = \text{P}(U_{\phi}^t)$ and $\phi^t$ is induced by the homomorphism $(\phi^t)^* \in \text{Hom}(W, U_{\phi}^t)$ such that $(\phi^t)^*(w_j) = \psi_j^t$. Furthermore the pull back by $\phi^s$ of the normal sequence for $L$

$$0 \to \phi^s N_{L/M} \to \phi^s N_{L/P} \xrightarrow{\phi^s D_L} \phi^s O_L(k) \to 0,$$
yields exact sequences with natural vertical homomorphisms:

\[
0 \longrightarrow \mathfrak{o}^* N_{L/M} \longrightarrow (\mathfrak{o}')^* N_{L/M} \oplus (\mathfrak{o}'')^* N_{L/M} \longrightarrow \mathbf{C} \longrightarrow 0
\]

This yields the following long exact sequences:

\[
0 \longrightarrow H^0((\mathfrak{o}')^* N_{L/M}) \longrightarrow U \otimes V^\vee / W^\vee \xrightarrow{H^0((\mathfrak{o}')^* D_L)} S^k U
\]

\[
0 \longrightarrow H^1((\mathfrak{o}')^* N_{L/M}) \longrightarrow 0
\]

\[
0 \longrightarrow H^0((\mathfrak{o}'')^* N_{L/M}) \longrightarrow U'' \otimes V^\vee / W^\vee \xrightarrow{H^0((\mathfrak{o}'')^* D_L)} S^k U''
\]

whence \(H^1((\mathfrak{o}')^* N_{L/M}) = H^1((\mathfrak{o}'')^* N_{L/M}) = 0\), and both \(H^0((\mathfrak{o}')^* N_{L/M})\) and \(H^0((\mathfrak{o}'')^* N_{L/M})\) are one-dimensional. Let \(U^t\) be the subspace of \(U\) consisting of elements vanishing at \(C_{\phi}^d \cap C_{\phi'}^d\), namely the subspace spanned by \(t\). Then the restriction of \(H^0((\mathfrak{o}')^* D_L)\) to \(U^t \otimes V^\vee / W^\vee\) equals \(t \cdot H^0((\mathfrak{o}')^* D_L^-)\). Hence

\[
\text{Coker } H^0(\mathfrak{o}^* D_L) \simeq t \cdot S^2 U / t \cdot \text{Im } H^0((\mathfrak{o}')^* D_L^-) \oplus \text{Coker } H^0((\mathfrak{o}'')^* D_L^-)
\]

\[
\simeq S^2 U / \text{Im } H^0((\mathfrak{o}')^* D_L^-) \simeq \text{Coker } H^0((\mathfrak{o}')^* D_L^-).
\]

One could understand the above isomorphism as

\[
\text{Coker } H^0(\mathfrak{o}^* D_L) = \text{Coker } (\mathfrak{o}^* H^0(D_L^-)) \oplus (S^2 U / \mathfrak{o}^* W).
\]

Thus \(H^0(\mathfrak{o}^* N_{L/M})\) is one-dimensional, while \(H^1(\mathfrak{o}^* N_{L/M})\) is 3-dimensional. This is immediately generalized into the following

**Lemma 4.2.** For any \(\mathfrak{o}^* \in X^0\) we have

\[
\text{Ker } H^0(\mathfrak{o}^* D_L) = \mathfrak{o}^* \text{Ker } H^0(D_L),
\]

\[
\text{Coker } H^0(\mathfrak{o}^* D_L) = (\mathfrak{o}^* \text{Coker } H^0(D_L^-)) \oplus (S^2 U / \mathfrak{o}^* W).
\]

**Lemma 4.3.** We define a line bundle \(L_0\) (resp. \(L_1\)) on \(Y \simeq \mathbb{P}(S^2 W)\) by the assignment:

\[
X^0 \ni \mathfrak{o}^* \mapsto \mathfrak{o}^* \text{Ker } H^0(D_L) \text{ (resp. } \mathfrak{o}^* \text{Coker } H^0(D_L^-)).
\]

Then \(L_k \simeq O_{\mathbb{P}(S^2 W)}\).

**Proof.** We know that \(\mathfrak{o}^* \text{Ker } H^0(D_L)\) is generated by \(\mathfrak{o}_2 c_2^\vee - \mathfrak{o}_1 c_4^\vee\). By the SL(2)-variable change of \(s\) and \(t\), \(\mathfrak{o}_j\) is transformed into a new quadratic polynomial, which is however the same as the first \(\mathfrak{o}_j\). This shows the generator is unchanged, whence \(L_0 \simeq O_{\mathbb{P}(S^2 W)}\). The proof for \(L_1\) is the same. \(\square\)
Lemma 4.4. We define a coherent sheaf $L$ on the stack $Y$ ($\simeq \mathbb{P}(S^2 W)$) (See Remark below) by the assignment:

$$X^0 \ni \phi^* \mapsto S^2 U / \phi^* W.$$ 

Then $L^2 \simeq O_{\mathbb{P}(S^2 W)}(-1)$.

Proof. The GIT-quotient $Y^0$ is covered with the images of $X'_j$:

$$X'_1 = \{ (\phi_1, \phi_2) \in X^0 \mid \phi_1 = 2st, \ \phi_2 = \lambda s^2 + 2\nu st + t^2, \ \lambda, \nu \in \mathbb{C} \},$$

$$X'_2 = \{ (\phi_1, \phi_2) \in X^0 \mid \phi_1 = ps^2 + 2qst + t^2, \ \phi_2 = 2st, \ p, q \in \mathbb{C} \}.$$ 

It is clear that the natural image of $X'_j$ in $Y$ is $Y_j$. The map $\phi$ given by $\phi^* = (\phi_1, \phi_2) \in Y_1$ has natural $\mathbb{Z}_2$ involution generated by,

$$r : (\sqrt{\lambda} s + t, \sqrt{\lambda} s - t) \mapsto (\sqrt{\lambda} s + t, -(\sqrt{\lambda} s - t)).$$

Since

$$2st = \frac{1}{2\sqrt{\lambda}}((\sqrt{\lambda} s + t)^2 - (\sqrt{\lambda} s - t)^2),$$

$$\lambda s^2 + 2\nu st + t^2 = \frac{\nu}{2\sqrt{\lambda}}((\sqrt{\lambda} s + t)^2 - (\sqrt{\lambda} s - t)^2) + \frac{1}{2}(\sqrt{\lambda} s + t)^2 + (\sqrt{\lambda} s - t)^2),$$

it is clear that,

$$r^*(\phi_1) = \phi_1, \ r^*(\phi_2) = \phi_2, \ r^*(\lambda s^2 - t^2) = -(\lambda s^2 - t^2).$$

Therefore, we can decompose $S^2 U$ into $\langle \lambda s^2 - t^2 \rangle_{\mathbb{C}} \oplus \langle \phi_1, \phi_2 \rangle_{\mathbb{C}}$ with respect to eigenvalue of $r^*$ and take $\lambda s^2 - t^2$ as canonical generator of $S^2 U / \phi^* W$. Similarly $S^2 U / \phi^* W$ is generated by $ps^2 - t^2$ on $Y_2$. The problem is therefore to write $\lambda s^2 - t^2$ as an $\Gamma(O_{Y_1 \cap Y_2})$-multiple of $pu^2 - v^2$ when we write $\phi_2 = 2uv$ by a variable change in $GL(2)$. The following variable change $(s, t) \mapsto (u, v)$ is in $GL(2)$:

$$s = \frac{\sqrt{2\alpha}}{(\beta - \alpha)^2}(2u - \frac{(\beta - \alpha)^2}{2\alpha}v), \ t = \frac{\sqrt{2\alpha}}{(\beta - \alpha)^2}(2\beta u - \frac{(\beta - \alpha)^2}{2}v),$$

where $\alpha, \beta$ are roots of the equation $\lambda s^2 + 2\nu st + t^2 = 0$. Under this coordinate change, $\phi_1$ and $\phi_2$ are rewritten as follows:

$$\phi_1 = \frac{\lambda}{(\nu^2 - \lambda)}u^2 + 2\frac{\nu}{\nu^2 - \lambda}uv + v^2 = pu^2 + 2quv + v^2, \ \phi_2 = 2uv.$$ 

Then we have

$$pu^2 - v^2 = -\frac{2}{\beta - \alpha}(\lambda s^2 - t^2) = -\frac{1}{\sqrt{\nu^2 - \lambda}}(\lambda s^2 - t^2) = -\sqrt{\frac{D_1}{D_2}}(\lambda s^2 - t^2).$$

Similarly by computing the effect on $S^2 U / \phi^* W$ by the variable change from $X'_1$ into $X'_0$, we see that $L^2$ is isomorphic to $O_{\mathbb{P}(S^2 W)}(-1)$. This completes the proof. □

Remark 4.5. We remark that the space $X$ must be regarded as a $\mathbb{Q}$-stack $Y^{stack}$ as follows: First we define $\phi_0 = -\phi_1 - \phi_2$. For each atlas $X'_a$ we define an atlas $Y'_a^{stack}$
(α = 0, 1, 2) by
\[ Y_0^{\text{stack}} = \{(φ_0, φ_1, φ_2, ± ψ_0) \in X^0 ∗ S^2 U : φ_0 = 2st, φ_1 = as^2 + 2bst + t^2, \]
\[ ψ_0 = as^2 - t^2 a, b ∈ C \}, \]
\[ Y_1^{\text{stack}} = \{(φ_0, φ_1, φ_2, ± ψ_1) \in X^0 ∗ S^2 U : φ_1 = 2st, φ_2 = λs^2 + 2νst + t^2, \]
\[ ψ_1 = λs^2 - t^2 λ, ν ∈ C \}, \]
\[ Y_2^{\text{stack}} = \{(φ_0, φ_1, φ_2, ± ψ_2) \in X^0 ∗ S^2 U : φ_1 = ps^2 + 2qst + t^2, φ_2 = 2st, \]
\[ ψ_2 = ps^2 - t^2 p, q ∈ C \} \]

Since \( L^2 \simeq O_{P(S^2 W)}(-1) \) we have \( c_1(L) = -\frac{1}{2}c_1(O_{P(S^2 W)}(1)) \) in the Chow ring
\( A(Y^{\text{stack}})_Q = A(X)_Q = A(P(S^2 W))_Q \).

5. Proof of the main theorem

Theorem 5.1.
\[ π_∗(c_{\text{top}}(H^1)) = \frac{1}{8} \left[ \frac{c(S^{k-1}Q)}{1 - \frac{1}{2}c_1(Q)} \right]_{k-N}, \]
where \( π \) is the natural projection from \( \tilde{M}_{0,0}(L, 2) \) to \( G \) and \([*]_{k-N} \) is the operation of picking up the degree \( 2(k - N) \) part of Chern classes.

Proof. From now on we denote the coherent sheaf \( L \) in Lemma 4.4 by \( O_P(-\frac{1}{2}) \). In view of the results from the previous section, what remains is to evaluate the top chern class of \( (S^{k-1}Q/((V^c ⊗ O_G)/Q^c)) ⊗ O_P(-\frac{1}{2}) \) on \( P(S^2 Q) \). Since double cover maps parametrized by \( P(S^2 Q) \) have natural \( Z_2 \) involution \( r \) given in the previous section, we have to multiply the result of integration on \( P(S^2 Q) \) by the factor \( \frac{1}{2} \) [BT], [FP]. With this set-up, let \( π' : P(S^2 Q) → G \) be the natural projection. Then what we have to compute is \( π_∗(c_{\text{top}}(H^1)) = \frac{1}{2}π'_∗(c_{\text{top}}(H^1)) = \frac{1}{2}π'_∗(c_{\text{top}}(S^{k-1}Q/((V^c ⊗ O_G)/Q^c)) \circ O_P(-\frac{1}{2})) \). Let \( z \) be \( c_1(O(P(1))) \). Then we obtain,
\[ \frac{1}{2}π'_∗(c_{\text{top}}(S^{k-1}Q/((V^c ⊗ O_G)/Q^c)) \circ O_P(-\frac{1}{2})) \]
\[ = \frac{1}{2} \sum_{j=0}^{k-N+2} \binom{k-N+2}{j} (S^{k-1}Q ⊕ Q^c) \cdot π'_∗(z^j) \cdot (-\frac{1}{2})^j \]
\[ = \frac{1}{8} \sum_{j=0}^{k-N} \binom{k-N}{j} (S^{k-1}Q ⊕ Q^c) \cdot s_j(S^2 Q) \cdot (-\frac{1}{2})^j \]
\[ = \frac{1}{8} \left[ \frac{c(S^{k-1}Q) \cdot c(Q^c)}{1 - \frac{1}{2}c_1(S^2 Q) + \frac{1}{8}c_2(S^2 Q) - \frac{1}{8}c_3(S^2 Q)} \right]_{k-N}, \]
where \( s_j(S^2 Q) \) is the \( j \)-th Segre class of \( S^2 Q \). But if we decompose \( c(Q^c) \) into \( (1 + α)(1 + β) \), we can easily see,
\[ \frac{c(Q^c)}{1 - \frac{1}{2}c_1(S^2 Q) + \frac{1}{8}c_2(S^2 Q) - \frac{1}{8}c_3(S^2 Q)} = \frac{(1 - α)(1 - β)}{(1 - α)(1 - \frac{1}{2}(α + β))(1 - β)} \]
\[ = \frac{1}{1 - \frac{1}{2}c_1(Q)}. \]
Finally, by combining the above theorem with the divisor axiom of Gromov-Witten invariants, we can prove the decomposition formula of degree 2 rational Gromov-Witten invariants of $M^k_N$ found from numerical experiments.

**Corollary 5.2.**

$$\langle c_{top}(H^1) \rangle_{0,1} = \langle c_{top}(H^1) \rangle_{0,2} + 8(\pi_*(c_{top}(H^1))O_{e^a}O_{e^b}O_{e^c})_{0,1,2} + 8(\pi_*(c_{top}(H^1))O_{e^a}O_{e^b}O_{e^c})_{0,1},$$

where $\langle c_{top}(H^1) \rangle_{0,2}$ is the number of conics that intersect cycles Poincaré dual to $e^a$, $e^b$ and $e^c$. We also denote by $\langle \pi_*(c_{top}(H^1))O_{e^a}O_{e^b}O_{e^c} \rangle_{0,1}$ the integral:

$$\int_{\mathcal{G}(2, Y)} c_{top}(S^k Q) \wedge \pi_*(c_{top}(H^1)) \wedge \sigma_{a-1} \wedge \sigma_{b-1} \wedge \sigma_{c-1}.$$  

6. GENERALIZATION TO TWISTED CUBICS

In this section, we present a decomposition formula of degree 3 rational Gromov-Witten invariants found from numerical experiments using the results of [ES].

**Conjecture 6.1.** If $k - N = 1$, we have the following equality:

$$\pi_*(c_{top}(H^1)) = \frac{1}{27} \left( \left( \frac{1}{24} (27k^2 - 55k + 26)k(k - 1) + \frac{2}{9} c_1(Q)^2 + \left( \frac{7}{6} (k + 1)k(k - 1) + \frac{1}{9} c_2(Q) \right) \right) \right).$$

where $\pi : \overline{M}_{0,0}(L, 3) \to \overline{M}_{0,0}(M^k_N, 1)$ is the natural projection.

In the $k - N > 1$ case, we have not found the explicit formula, because in the $d = 3$ case, we have another contribution from multiple cover maps of type $(2+1) \to (1+1)$. Here multiple cover map of type $(2+1) \to (1+1)$ is the map from nodal curve $P^1 \cup P^1$ to nodal conic $L_1 \cup L_2 \subset M^k_N$, that maps the first (resp. the second) $P^1$ to $L_1$ (resp. $L_2$) by two to one (resp. one to one). In the $k - N = 1$ case, we have also determined the contributions from multiple cover maps of $(2+1) \to (1+1)$ to nodal conics.

**Corollary 6.2.** If $k - N = 1$, $\langle c_{top}(H^1) \rangle_{0,3}$ is decomposed into the following contributions:

$$\langle c_{top}(H^1) \rangle_{0,3} = \langle c_{top}(H^1) \rangle_{0,3} + \frac{1}{k} \left( \langle c_{top}(H^1) \rangle_{0,1} + \frac{3}{2} \langle c_{top}(H^1) \rangle_{0,1} + \frac{3}{2} \langle c_{top}(H^1) \rangle_{0,1} \right) + \frac{27}{\pi_*(c_{top}(H^1))O_{e^a}O_{e^b}O_{e^c}} \rangle_{0,1},$$

where $\langle c_{top}(H^1) \rangle_{0,3}$ is the number of twisted cubics that intersect cycles Poincaré dual to $e^a$, $e^b$ and $e^c$.

**Proof.** In the $k - N = 1$ case, dimension of moduli space of multiple cover maps of $(2+1) \to (1+1)$ to nodal conics is given by $N - 6 + N - 6 - (N - 4) + 2 = N - 6$, hence the rank of $H^1$ is given by $N - 6 - (N - 5 - 3) = 2$. On the other hand, dimension of moduli space of $d = 2$ multiple cover maps of $P^1 \to P^1$ is 2, the degree of the form of $\pi_*(c_{top}(H^1))$ equals to $2 - 2 = 0$, where $\pi$ is the projection map.
that projects out the fiber locally isomorphic to the moduli space of \( d = 2 \) multiple cover maps. This situation is exactly the same as the Calabi-Yau case. Therefore, we can use the well-known result by Aspinwall and Morrison, that says for \( n \)-point rational Gromov-Witten invariants for Calabi-Yau manifold, \( \pi_*(\mathcal{C}_{top}(H^1)) \) for degree \( d \) multiple cover map is given by,

\[
\pi_*(\mathcal{C}_{top}(H^1)) = \frac{1}{d} \pi_* (\mathcal{C}_{top}(H^1))
\]

With this formula, we add up all the combinatorial possibility of insertion of external operator \( \mathcal{O}_{e_1}, \mathcal{O}_{e_2}, \) and \( \mathcal{O}_{e_3} \),

\[
\frac{1}{k^3} (\pi_*(\mathcal{C}_{top}(H^1)) \mathcal{O}_{e_1} \mathcal{O}_{e_2} \mathcal{O}_{e_3} )_{0,1} (\mathcal{O}_{e}^{-2} )_{0,1} \\
+ (\pi_*(\mathcal{C}_{top}(H^1)) \mathcal{O}_{e_1} \mathcal{O}_{e_2} \mathcal{O}_{e+2} )_{0,1} (\mathcal{O}_{e}^{-2} )_{0,1} \\
+ (\pi_*(\mathcal{C}_{top}(H^1)) \mathcal{O}_{e_1} \mathcal{O}_{e+2} \mathcal{O}_{e} )_{0,1} (\mathcal{O}_{e}^{-2} )_{0,1} \\
+ (\pi_*(\mathcal{C}_{top}(H^1)) \mathcal{O}_{e+2} \mathcal{O}_{e_1} \mathcal{O}_{e} )_{0,1} (\mathcal{O}_{e}^{-2} )_{0,1} \\
+ (\mathcal{O}_{e_1} \mathcal{O}_{e_2} \mathcal{O}_{e} )_{0,1} (\mathcal{O}_{e}^{-2} )_{0,1} \\
+ (\mathcal{O}_{e_1} \mathcal{O}_{e_2} \mathcal{O}_{e+2} )_{0,1} (\mathcal{O}_{e}^{-2} )_{0,1} \\
+ (\mathcal{O}_{e_1} \mathcal{O}_{e+2} \mathcal{O}_{e_2} )_{0,1} (\mathcal{O}_{e}^{-2} )_{0,1} \\
+ (\mathcal{O}_{e+2} \mathcal{O}_{e_1} \mathcal{O}_{e_2} )_{0,1} (\mathcal{O}_{e}^{-2} )_{0,1} \\
+ \frac{1}{2} (\mathcal{O}_{e_1} \mathcal{O}_{e_2} \mathcal{O}_{e+2} )_{0,1} (\mathcal{O}_{e}^{-2} )_{0,1} \\
+ \frac{1}{2} (\mathcal{O}_{e_1} \mathcal{O}_{e_2} \mathcal{O}_{e+2} )_{0,1} (\mathcal{O}_{e}^{-2} )_{0,1} \\
+ \frac{1}{4} (\mathcal{O}_{e_1} \mathcal{O}_{e_2} \mathcal{O}_{e_3} )_{0,1} (\mathcal{O}_{e}^{-2} )_{0,1}
\]

The last expression is nothing but the formula we want.

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