CONICS ON A GENERIC HYPERSURFACE

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ABSTRACT. In this paper, we compute the contributions from double cover maps to genus 0 degree 2 Gromov-Witten invariants of general type projective hypersurfaces. Our results correspond to a generalization of Aspinwall-Morrison formula to general type hypersurfaces in some special cases.
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1. INTRODUCTION

In this paper, we discuss a generalization of the multiple cover formula for rational Gromov-Witten invariants of Calabi-Yau manifolds [AM], [M] to double cover maps of a line $L$ on a degree $k$ hypersurface $M^k_N$ in $\mathbb{P}^{N-1}$. Naively, for a given finite set of elements $\alpha_j \in H^*(M^k_N, \mathbb{Z})$, the rational Gromov-Witten invariant $(\mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \cdots \mathcal{O}_{\alpha_n})_{0,d}$ of $M^k_N$ counts the number of degree $d$ (possibly singular and reducible) rational curves on $M^k_N$ that intersect real sub-manifolds of $M^k_N$ that are Poincaré-dual to $\alpha_j$.

Recently, the mirror computation of rational Gromov-Witten invariants of $M^k_N$ with negative first Chern class $(k-N > 0)$ was established in [CG], [Ir], [J]. Using the method presented in these articles, we can compute $(\mathcal{O}_{e^1} \mathcal{O}_{e^2} \cdots \mathcal{O}_{e^{m+d}})_{0,d}$ where $e$ is the generator of $H^{1,1}(M^k_N, \mathbb{Z})$. Briefly, mirror computation of $M^k_N$ $(k > N)$ in [J] goes as follows. We start from the following ODE:

\begin{equation}
(\partial_x)^{N-1} - k \cdot \exp(x) \cdot (k \partial_x + k - 1)(k \partial_x + k - 2) \cdots (k \partial_x + 1) w(x) = 0,
\end{equation}

and construct the virtual Gauss-Manin system associated with (1):

\begin{equation}
\partial_x \tilde{w}_{N-2-m}(x) = \tilde{w}_{N-1-m}(x) + \sum_{d=1}^{\infty} \exp(dx) \cdot \tilde{L}_{m,k,d} \cdot \tilde{w}_{N-1-m-(N-k)d}(x),
\end{equation}

where $m$ runs through all the integers and $\tilde{L}_{m,k,d}$ is non-zero only if $0 \leq m \leq N-1+(k-N)d$. From the compatibility of (1) and (2), we can derive the recursive formulas that determine all the $\tilde{L}_{m,k,d}$:

\begin{align*}
\sum_{n=0}^{k-1} \tilde{L}_{m,k,1} w^n &= k \cdot \prod_{j=1}^{k-1} (jw + (k - j)), \\
\sum_{m=0}^{N-1+(k-N)d} \tilde{L}_{m,k,d} w^m &= \sum_{l=2}^{d} (-1)^{l-1} \sum_{0=i_0 < \cdots < i_l = m} \times \\
\sum_{j_0=0}^{j_3} \cdots \sum_{j_{l-1}=0}^{j_2} \prod_{n=1}^{l} \left( i_{n+1} + (d - i_n - 1) j_n \right) \tilde{L}_{m,k,j_n-(i_n+1)} \tilde{L}_{m,k,j_n+1} \tilde{L}_{m,k,i_n-(i_n+1)}
\end{align*}

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With these data, we can construct the formulas that represent rational three point Gromov-Witten invariant \( \langle O_{c}O_{e_{N-2-m}}O_{e_{m-1-(k-N)d}} \rangle_{d} \) in terms of \( \tilde{L}_{m}^{N,k,d} \). These three point Gromov-Witten invariants are enough for reconstruction of all the rational Gromov-Witten invariants \( \langle O_{e_{m}}O_{e_{m+2}}O_{e_{m+n}} \rangle_{d} \) [KM]. In particular, we obtain the following formula in the \( d = 2 \) case:

\[
(3) \quad \langle O_{c}O_{e_{N-2-m}}O_{e_{m-1-(k-N)d}} \rangle_{2} = k \cdot \left( \tilde{L}_{m}^{N,k,2} - \tilde{L}_{1+2(k-N)}^{N,k,1} - 2 \tilde{L}_{1+2(k-N)}^{N,k,1} \sum_{j=0}^{k-N} (\tilde{L}_{m-j}^{N,k,1} - \tilde{L}_{1+2(k-N)-j}^{N,k,1}) \right).
\]

According to the results of this procedure, rational three point Gromov-Witten invariants can be rational numbers with large denominator if \( k > N \), in contrast to the Calabi-Yau case where rational three point Gromov-Witten invariants are always integers.

One of the reasons of this rationality (non-integrality) comes from the contributions of multiple cover maps to Gromov-Witten invariants. In the Calabi-Yau case \( (N = k) \), for any divisor \( m \) of \( d \) there are some contributions from degree \( m \) multiple cover maps \( \phi \) of a rational curve \( \mathbf{P}^{1} \) onto a degree \( \frac{d}{m} \) rational curve \( C \hookrightarrow M_{k}^{d} \). The contributions from the multiple cover maps are expressed in terms of the virtual fundamental class of Gromov-Witten invariants. Let \( C \) be a general degree \( d \) rational curve in \( M_{k}^{d} \). Its normal bundle \( N_{C/M_{k}^{d}} \) is decomposed into a direct sum of line bundles as follows:

\[
N_{C/M_{k}^{d}} \cong O_{C}(-1) \oplus O_{C}(-1) \oplus O_{C}^{\oplus (k-5)}.
\]

Let \( \phi : \mathbf{P}^{1} \to C \) be a holomorphic map of degree \( m \). Since the pull-back \( \phi^{*}(N_{C/M_{k}^{d}}) \) is given by

\[
\phi^{*}(N_{C/M_{k}^{d}}) \cong O_{\mathbf{P}^{1}}(-m) \oplus O_{\mathbf{P}^{1}}(-m) \oplus O_{\mathbf{P}^{1}}^{\oplus (k-5)},
\]

we obtain \( h^{1}(\phi^{*}(N_{C/M_{k}^{d}})) = 2m - 2 \). On the other hand, let \( \overline{M}_{0,0}(M,d) \) be the moduli space of 0-pointed stable maps of degree \( d \) from genus 0 curve to \( M \). Then the moduli space of \( \phi \) is the fiber space \( \pi : \overline{M}_{0,0}(C,m) \to \overline{M}_{0,0}(M_{k}^{d}, \frac{d}{m}) \), whose fibre \( \overline{M}_{0,0}(C,m) \) over \( C \) (fixed) has complex dimension \( 2m - 2 \). Then the push-forward of the virtual fundamental class \( \pi_{*}(c_{0,0}(H^{1}(\phi^{*}N_{C/M_{k}^{d}}))) \) can be computed only by intersection theory on the fiber \( \overline{M}_{0,0}(C,m) \), which turns out to be equal to \( \frac{1}{k} \phi^{*}(N_{C/M_{k}^{d}}) \). This depends on neither the structure of the base \( \overline{M}_{0,0}(M_{k}^{d}, \frac{d}{m}) \) nor the global structure of the fibration.

But when \( k < N \), the situation is more complicated than \( M_{k}^{d} \) because of negative first Chern class. Let us concentrate on the case of \( d = 2, m = 2 \) that we discuss in this paper. In this case, \( C \) is just a line \( L \) on the hypersurface \( M_{k}^{d} \). The moduli space \( \overline{M}_{0,0}(M_{k}^{d}, 1) \) is a sub-manifold of \( \overline{M}_{0,0}(P^{N-1}, 1) \), while \( \overline{M}_{0,0}(P^{N-1}, 1) \) is the Grassmannian \( G(2,N) \), the moduli space of rank 2 quotients of \( V = \mathbb{C}^{N} \). As will be shown later, for a generic line \( L \), \( N_{L/M_{k}^{d}} \) is decomposed into

\[
N_{L/M_{k}^{d}} \cong O_{L}(-1)^{2k-N+2} \oplus O_{L}^{\oplus 2N-k-5}.
\]
By pulling back it by the degree 2 map $\phi : \mathbb{P}^1 \to L$, we obtain,

$$\phi^* N_{L/M_N^k} \simeq O_{\mathbb{P}^1}(-2)^{\oplus k} \oplus O_{\mathbb{P}^1}^{\oplus 2N-3-k}.$$ 

Therefore, $h^1(\phi^* (N_{L/M_N^k})) = k - N + 2$, which is strictly greater than two, the complex dimension of the fiber $\overline{M}_{0,0}(L, 2)$. Thus we need to know the global structure of the fibration $\pi$ in order to compute the multiple cover contribution to degree 2 rational Gromov-Witten invariants of $M_N^k$.

In order to estimate the contributions from double cover maps $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ to $\langle \mathcal{O}_{c^a} \mathcal{O}_{c^b} \mathcal{O}_{c^c} \rangle_{0,2}$, we first computed the number of conics, that intersect cycles Poicaré dual to $e^a, e^b$ and $e^c$, on $M_N^k$ (whose normal bundle are of the same type) by using the method in [K2]. Then we found the following formula by comparing these integers with the results obtained from (3):

$$\langle \mathcal{O}_{c^a} \mathcal{O}_{c^b} \mathcal{O}_{c^c} \rangle_{0,2} = \text{(number of corresponding conics)} + \int_{G(2,N)} c_{\text{top}}(S^k Q) \wedge \left[ \frac{c(S^k Q - 1)}{1 - 2c_1(Q)} \right]_{k-N} \wedge \sigma_{a-1} \wedge \sigma_{b-1} \wedge \sigma_{c-1},$$

where $Q$ is the universal rank 2 quotient bundle of $G(2,N)$, $\sigma_a$ is a Schubert cycle defined by $\sum_{a=0}^{\infty} \sigma_a := \frac{1}{\prod_{j=1}^a (1-j)}$ and $[\ast]_{k-N}$ is the operation of picking up degree $2(k-N)$ part of Chern classes.

On the other hand, we have the following formula which directly follows from the definition of the virtual fundamental class of $\overline{M}_{0,0}(M_N^k, 2)$:

$$\langle \mathcal{O}_{c^a} \mathcal{O}_{c^b} \mathcal{O}_{c^c} \rangle_{0,2} = \text{(number of corresponding conics)} + 8 \int_{G(2,N)} c_{\text{top}}(S^k Q) \wedge \left[ \pi_*(c_{\text{top}}(H^1(\phi^* (N_{L/M_N^k})))) \right]_{k-N} \wedge \sigma_{a-1} \wedge \sigma_{b-1} \wedge \sigma_{c-1},$$

where $\pi : \overline{M}_{0,0}(L, 2) \to \overline{M}_{0,0}(M_N^k, 1)$ is the natural projection. Here, the factor 8 comes from the divisor axiom of Gromov-Witten invariants.

In this paper, we prove the following formula

$$\pi_*(c_{\text{top}}(H^1(\phi^* (N_{L/M_N^k})))) = \frac{1}{8} \left[ \frac{c(S^k Q - 1)}{1 - 2c_1(Q)} \right]_{k-N}.$$ 

By combining (5) with (6), we can derive the formula (4) immediately.

From (4), we see that $\langle \mathcal{O}_{c^a} \mathcal{O}_{c^b} \mathcal{O}_{c^c} \rangle_{0,2}$ of $M_N^k$ is a rational number with denominator at most $2^k - N$. Therefore rationality (non-integrality) of the Gromov-Witten invariant $\langle \mathcal{O}_{c^a} \mathcal{O}_{c^b} \mathcal{O}_{c^c} \rangle_{0,2}$ is caused by the effect of multiple cover map in this case.

We note here that the total moduli space of double cover maps of lines is isomorphic to $\mathbb{P}(S^2 Q)$ over $G := \overline{M}_{0,0}(M_N^k, 1) \hookrightarrow G(2,N)$, which is an algebraic Q-stack $\mathbb{P}(S^2 Q)^{\text{stack}}$ (in the sense of Mumford). As a consequence, the union of all $H^1(\phi^* (N_{L/M_N^k}))$ turns out to be a coherent sheaf on $\mathbb{P}(S^2 Q)^{\text{stack}}$ with fractional Chern class in (6), as was suggested in [BT]. See [V, Section 9].

We also did some numerical experiments on degree 3 Gromov-Witten invariants of $M_N^k$ by using the results of [ES]. For $k - N > 0$, there is a new contribution from multiple cover maps to nodal conics in $M_N^k$ that did not appear in the Calabi-Yau case. Therefore, multiple cover map contributions are far more complicated than Calabi-Yau, and we leave general analysis on this problem to future works.
This paper is organized as follows. In Section 1, we analyze characteristics of moduli space of lines in $M_Y$ and derive $N_{k/M_Y}$ \simeq $O_L(-1)^{g-k-2}$ $\oplus$ $O_L^{2N-k-5}$. In Section 2, we study the moduli space $M_{0,0}(\mathbb{P}^1, 2)$ from the point of view of stability and identify it with $\mathbb{P}^2$ and show that the moduli space $M_{0,0}(\mathbb{P}^1, 2)$ is isomorphic to $\mathbb{P}(S^2Q)$ over $G$. In section 4, we describe $H^1(\phi^*N_{k/M_Y})$ as a coherent sheaf over $\mathbb{P}(S^2Q)^{stack}$. In section 5, we derive the main theorem (6) of this paper by using Segre-Witten classes. In Section 6, we mention some generalization to degree 3 Gromov-Witten invariants.

2. Lines on a hypersurface

Let $M$ be a generic hypersurface of degree $k$ of the projective space $\mathbb{P}^{N-1} = \mathbb{P}(V)$. We assume $2N - 5 \geq k \geq N - 2 \geq 2$ throughout this note. In this note we count the number of rational curves of virtual degree two, namely rational curves which doubly cover lines on $M$.

Let $\mathcal{P} = \mathbb{P}(V)$ be the projective space parameterizing all one-dimensional quotients of $V$, which is usually denoted by $\mathbb{P}(V)$ in the standard notation in algebraic geometry. In this notation let $W$ be a subspace of $V$. Then $\mathbb{P}(W)$ is naturally a linear subspace of $\mathbb{P}(V)$ of dimension $\dim W - 1$.

Let $G(2, V)$ be the Grassmann variety of lines in $\mathbb{P}(V)$, the scheme parameterizing all lines of $\mathbb{P} = \mathbb{P}(V)$. This is also the universal scheme parameterizing all one-dimensional quotient linear spaces of $V$. Let $W$ be a two dimensional quotient linear space, $\psi \in G(2, V)$, namely $\psi : \mathbb{P}(W) \to \mathbb{P}(V)$ the natural immersion and $i^*: V \to W$ the quotient homomorphism. The space $W$ is denoted by $W(\psi)$ when necessary.

There exists the universal bundle $Q_{G(2, V)}$ over $G(2, V)$ and a homomorphism $i^{univ}_*: O_{G(2, V)} \otimes V \to Q_{G(2, V)}$, whose fiber $i^{univ}_*: V \to Q_{G(2, V)_\psi}$ is the quotient $i^*: V \to W(\psi)$ of $V$ corresponding to $\psi$.

2.1. Existence of a line on $M$. Let $L = \mathbb{P}(W)$ be a line of $\mathcal{P}$, equivalently $W \in G(2, V)$. Then the condition $L \subseteq M$ imposes at most $k + 1$ conditions on $W$, while the number of moduli of lines of $\mathcal{P}$ equals $\dim G(2, V) = 2N - 4$. Hence we infer

Lemma 2.2. If $2N \geq k + 5$, then there exists at least a line on $M$. 

See also [Katz,p.152]. Let $G$ be the subscheme of $G(2, V)$ parameterizing all lines of $\mathbb{P}(V)$ lying on $M$, $Q = (Q_{G(2, V)})|_G$ the restriction of $Q_{G(2, V)}$ to $G$. By Lemma 2.2, $G$ is nonempty. Let $i^*: O_G \otimes V \to Q$ be the restriction of $i^{univ}_*$ to $G$. Let $P = \mathbb{P}(Q)$ and $\pi : P \to G$ the natural projection. Then $\pi$ is the universal line of $M$ over $G$, to be more exact, the universal family over $G$ of lines lying on $M$. In other words, the natural epimorphism $i^*: O_G \otimes V \to Q$ induces a morphism $i: P \to \mathbb{P}_G(V) := G \times \mathbb{P}(V)$, which is a closed immersion into $\mathbb{P}_G(V)$, thus $P$ is a subscheme of $\mathbb{P}_G(V)$ such that $\pi = (p_1)_p$. Let $L_\psi = \mathbb{P}(Q_\psi)$. Note that 

$L_\psi = P_\psi := \pi^{-1}(\psi) \simeq \mathbb{P}(Q_\psi) \subset \{\psi\} \times \mathbb{P}(V) \simeq \mathbb{P}(V)$. 

2.3. The normal bundle $N_{L/M}$. The argument of this section is standard and well known. Let $P = P(V)$, $L = P(W)$ and $i_W^*: V \to W \in G$. Let us recall the following exact sequence:

$$0 \to O_P \to O_P(1) \otimes V^\vee \xrightarrow{D} T_P \to 0$$

where the homomorphism $D$ is defined by

$$D(a \otimes v^\vee) := a D_{(v^\vee)} \quad (a \in O_P(1))$$

$$(D_{v^\vee} F)(u^\vee) := \left( \frac{d}{dt} F(u^\vee + tv^\vee) \right)_{t=0}$$

for a homogeneous polynomial $F \in S(V)$ and $u^\vee, v^\vee \in V^\vee$. We note $H^0(O_P(1)) \otimes V^\vee = V \otimes V^\vee = \text{End}(V, V)$ and that the image of $H^0(O_P)$ in $\text{End}(V, V)$ is $\text{Cid}_V$.

We also have the following exact sequences:

$$0 \to T_L \to (T_P)_L \to N_{L/P} \to 0$$

$$0 \to O_L \to O_L(1) \otimes V^\vee \xrightarrow{D_L} (T_P)_L \to 0.$$

**Lemma 2.4.** Let $L = P(W)$. Then

$$N_{L/P} \simeq O_L(1) \otimes (V^\vee / W^\vee), \quad H^0(N_{L/P}) \simeq W \otimes (V^\vee / W^\vee).$$

**Proof.** The assertion is clear from the following commutative diagram with exact rows and columns:

$$\begin{array}{cccccc}
0 & \to & O_L & \to & O_L(1) \otimes W^\vee & \xrightarrow{(D_L)_{W^\vee}} & T_L & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & O_L & \to & O_L(1) \otimes V^\vee & \xrightarrow{D_L} & (T_P)_L & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & O_L(1) \otimes (V^\vee / W^\vee) & \to & N_{L/P} & \to & 0
\end{array}$$

The second assertion is clear from $H^0(L, O_L(1)) = W$. \qed

Since $T_L \simeq O_L(2)$, there follow exact sequences

$$0 \to H^0(T_L) \to H^0((T_P)_L) \to H^0(N_{L/P}) \to 0$$

$$0 \to H^0(O_L) \to H^0(O_L(1)) \otimes V^\vee \xrightarrow{H^0(D_L)} H^0((T_P)_L) \to 0.$$

We also note

$$H^0(T_L) = \text{LieAut}^0(L) = \text{End}(W, W)/\text{center} = \text{End}(W, W)/\text{Cid}_W.$$

Since $H^0(O_L(1)) = W$, we see

$$H^0((T_P)_L) = W \otimes V^\vee / \text{Im} H^0(O_L) = \text{Hom}(V, W)/\text{Cid}_W.$$  

Hence we again see

$$H^0((N_{L/P})) = (\text{Hom}(V, W)/\text{Cid}_W)/(\text{Hom}(W, W)/\text{Cid}_W) = W \otimes (V^\vee / W^\vee) = \text{Hom}(V/W, W).$$

For any line $L = P(W)$ of $P$ the following sequence is exact:

$$(7) \quad 0 \to N_{L/M} \to N_{L/P} \to (N_{M/P})_L(\simeq O_L(k)) \to 0.$$
Hence so is the following sequence as well:

\[ 0 \rightarrow H^0(N_{L/M}) \rightarrow H^0(N_{L/P}) \xrightarrow{\theta^0(D_L)} H^0(O_L(k)) \rightarrow H^1(N_{L/M}) \rightarrow 0. \]

Hence we have

**Lemma 2.5.** The following is exact:

\[(8) \quad 0 \rightarrow H^0(N_{L/M}) \rightarrow W \oplus (V^\vee/W^\vee) \xrightarrow{\theta^0(D_L)} S^k W \rightarrow H^1(N_{L/M}) \rightarrow 0.\]

**Corollary 2.6.** \( \dim G \geq 2N - k - 5 \), equality holding if \( H^1(N_{L/M}) = 0 \).

**Proof.** As is well-known, \( \dim G \geq h^0(N_{L/M}) - h^1(N_{L/M}) \). Note \( \dim W \oplus (V^\vee/W^\vee) = 2(N - 2) \) and \( \dim S^k W = k + 1 \). Hence the corollary follows from Lemma 2.5. \( \square \)

**Lemma 2.7.** For a generic line \( L \) on a generic hypersurface \( M \) of degree \( k \)

(i) \( N_{L/M} \cong O_L^{2n} \oplus O_L(-1)^b \), where \( a = 2N - k - 5 \) and \( b = k - N + 2 \),

(ii) \( \text{Coker } H^0(D_L^{-}) \cong S^{k-1}W/\text{W}^\vee/(W^\vee/W^\vee) \) where \( D_L^{-} := D_L \oplus O_L(-1) \).

**Proof.** Let \( M \) be a generic hypersurface of degree \( k \) and \( L \) a generic line \( L \) on \( M \). Without loss of generality we may assume that \( W^\vee \) is generated by \( e_1^\vee \) and \( e_2^\vee \), in other words, \( \psi : L \rightarrow \mathbb{P} \) is given by

\[ \psi : [s : t] \rightarrow [x_1, \ldots, x_N] = [s, t, 0, \ldots, 0]. \]

Then \( F \), the polynomial of degree \( k \) defining \( M \), is written as

\[ F = x_3F_3 + x_4F_4 + \cdots + x_N F_N \]

for some polynomials \( F_j \) of degree \( k - 1 \). Let \( f_j = \psi^* F_j = F_j(s, t, 0, \ldots, 0) \).

Now we consider the exact sequence

\[ 0 \rightarrow H_0^0(N_{L/M}(-1)) \rightarrow H^0(N_{L/P}(-1)) \xrightarrow{\theta^0(D_L^{-})} H^0(O_L(k - 1)) \rightarrow H^1(N_{L/M}(-1)) \rightarrow 0. \]

where we note \( H^0(N_{L/P}(-1)) = V^\vee/W^\vee \). Hence the following is exact:

\[(9) \quad 0 \rightarrow H_0^0(N_{L/M}(-1)) \rightarrow V^\vee/W^\vee \xrightarrow{\theta^0(D_L^{-})} S^{k-1}W \rightarrow H^1(N_{L/M}(-1)) \rightarrow 0.\]

where \( H^0(D_L^{-}) \) is given by \( H^0(D_L^{-})(e_j^\vee) = f_j \quad (j = 3, 4, \ldots, N) \).

A generic choice of \( F \) implies a generic choice of degree \( k - 1 \) polynomials \( f_j \) \((j = 3, 4, \ldots, N)\) in \( s \) and \( t \). By the assumptions

\[ \dim S^{k-1}W = k \geq N - 2 = \dim V^\vee/W^\vee, \]

\[ \dim W \oplus V^\vee/W^\vee = 2(N - 2) \geq k + 1 = \dim S^k W, \]

the generic choice of \( F \) implies that we can choose \( f_j \in S^{k-1}W \quad (j = 3, 4, \ldots, N) \)

(and fix once for all) such that

(iii) \( f_j \quad (j = 3, 4, \ldots, N) \) are linearly independent,

(iv) \( W f_3 + W f_4 + \cdots + W f_N = S^k W \).
Hence $H^0(D_L)$ is injective by (iii). It follows that $H^0(N_{L/M}(-1)) = 0$. Hence (ii) is clear. Next we consider $H^0(D_L)$. By (iv), we see

$$S^k W = W \cdot H^0(D_L)(V^*/W^*) = H^0(D_L)(W \otimes V^*/W^*),$$

whence $H^0(D_L)$ is surjective. It follows that $H^1(N_{L/M}) = 0$. Hence $N_{L/M} \cong O_L^{b_a} \oplus O_L(-1)^{b_b}$ for some $a$ and $b$. Since $a + b = \text{rank}(N_{L/M}) = N - 3$ and $-b = \deg(N_{L/M}) = N - 2 - k$, we have (i).

\[ \square \]

2.8. **Lines on a quintic hypersurface in $\mathbb{P}^4$**. See [Katz, Appendix A] for the subsequent examples. Let $N = 5$ and $k = 5$. Hence $M$ is a hypersurface of degree 5 in $\mathbb{P}^4$, a Calabi-Yau 3-fold. Let

$$F = x_0 x_1^5 + x_2 x_3^3 + x_4 x_5 + x_0^5.$$  

First we note that $M = \{ F = 0 \}$ is nonsingular. Let $L = \{ x_3 = x_4 = x_5 = 0 \} = \{ [s, t, 0, 0, 0] \}$. In this case $f_3 = 0$, $f_4 = s^4$ and $f_5 = t^4$. In the exact sequence (1) we see $H^0(N_{L/M}(-1)) = \text{Ker} H^0(D_L) = Cc^3_f$ and $H^1(N_{L/M}(-1)) = \text{Coker} H^0(D_L)$ is 3-dimensional. Hence $N_{L/M} = O_L(1) \oplus O_L(-3)$.

We summarize the above. If $\dim \text{Ker} H^0(D_L) = 1$ and if $M$ is nonsingular, then $N_{L/M} = O_L(1) \oplus O_L(-3)$. Hence $H^0(N_{L/M}) = \text{Ker} H^0(D_L) = W \otimes \text{Ker} H^0(D_L)$ is 2-dimensional. Therefore we can choose $f_3 = 0$ and a linearly independent pair $f_4$ and $f_5 \in S^1 W$ so that $W f_4 + W f_5$ is 4-dimensional. The choice $f_3 = s^4$ and $f_4 = t^4$ satisfies the conditions. This enables us to find a nonsingular hypersurface $M$ as above. However if we choose $f_3 = 0$, $f_4 = s^4$ and $f_5 = s^4 t$, then $W f_4 + W f_5$ is 3-dimensional. Hence $M$ is singular.

Next in the same manner we find $L$ on a nonsingular hypersurface $M$ with $N_{L/M} = O_L \oplus O_L(-2)$ or $N_{L/M} = O_L(-1)^{\oplus 2}$. Let

$$F = x_3 x_1^4 + x_4 x_2^3 x_2 + x_5 x_3^2 + x_4^3 + x_5^2.$$  

Then we have $f_3 = s^4$, $f_4 = s^3 t$ and $f_5 = t^4$. Since $W f_3 + W f_4 + W f_5$ is 3-dimensional, $H^0(N_{L/M}(-1)) = \text{Ker} H^0(D_L) = 0$, $H^0(N_{L/M}) = \text{Ker} H^0(D_L) = C(c^3_f - s c_f^4)$. We see also that $\dim H^1(N_{L/M}) = \dim \text{Coker} H^0(D_L) = 1$ and $N_{L/M} = O_L \oplus O_L(-2)$. The hypersurface $M = \{ F = 0 \}$ is easily shown to be nonsingular.

If $F = x_3 x_1^4 + x_4 x_2^2 x_2 + x_3 x_2^3 + x_4^5 + x_5^3 + x_6^5$ and $M = \{ F = 0 \}$, then $N_{L/M} = O_L(-1)^{\oplus 2}$.

2.9. **Lines on a generic hypersurface $M^g_r$ of $\mathbb{P}^6$**. Let $N = 7$ and $k = 8$. In view of Lemma 2.2 there exists a line $L$ on any generic hypersurface $M$ of degree 8 in $\mathbb{P}(V) = \mathbb{P}^6$. In view of Lemma 2.7, $a = 1, b = 3$ and $N_{L/M} \cong O_L \oplus O_L(-1)^{\oplus 3}$. For example let $L : x_j = 0 \ (j \geq 3)$ and we take

$$F_3 = 8 x_1^2, F_4 = 8 x_2 x_3, F_5 = 8 x_1^2 x_2, F_6 = 8 x_1^2 x_5, F_7 = 8 x_2^2,$$

$$F = x_3 F_3 + x_4 F_4 + x_5 F_5 + x_6 F_6 + x_7 F_7.$$

and let $M = M^g_r : F = 0$. We see that $M$ is nonsingular near $L$ and has at most isolated singularities. However it is still unclear to us whether $M = M^g_r$ is
nonsingular everywhere. The space $H^0(N_{L/M})$ is spanned by $te_N^r - se_N^r$, hence an infinitesimal deformation $L_e$ of $L$ is given by

$$[s,t] \mapsto [s,t,\varepsilon s,0,0,0]$$

which yields $F_{L_e} = \varepsilon^8(s^8 + t^8) \equiv 0 \mod \varepsilon^8$. Since $H^1(N_{L/M}) = 0$, this infinitesimal deformation is integrable and $G$ (the moduli of lines of $P^6$ contained in $M$) is nonsingular and one dimensional at the point $[L]$.

We note that $M$ also contains 8 lines

$$L' := L_{ss} : \varepsilon s x_1 - x_2 = x_3 + \varepsilon s x_4 = x_j = 0 \ (j \geq 5),$$

with $N_{L/M} = O_L(1)^{\mathbb{R}^3} \oplus O_L(-6)$ where $\varepsilon^8 = -1$.

3. Stability

**Definition 3.1.** Suppose that a reductive algebraic group $G$ acts on a vector space $V$. Let $v \in V$, $v \neq 0$.

1. the vector $v$ is said to be semistable if there exists a $G$-invariant homogeneous polynomial $F$ on $V$ such that $F(v) \neq 0$,
2. the vector $v$ is said to be stable if $p$ has a closed $G$-orbit in $X_{ss}$ and the stabilizer subgroup of $v$ in $G$ is finite.

Let $\pi : V \setminus \{0\} \to \mathbb{P}(V^\vee)$ be the natural surjection. Then $v \in V$ is semistable (resp. stable) if and only if $\pi(v)$ is semistable (resp. stable).

3.2. Grassmann variety. Let $V$ be an $N$-dimensional vector space, and $G(r,N)$ the Grassmann variety parameterizing all $r$-dimensional quotient spaces of $V$. Here is a natural way of understanding $G(r,N)$ via GIT-stability. Let $U$ be an $r$-dimensional vector space, $X = \text{Hom}(V,U)$ and $\pi : X \setminus \{0\} \to \mathbb{P}(X^\vee)$ the natural map. Then $\text{SL}(U)$ acts on $X$ from the left by:

$$(g \cdot \phi^*)(v) = g \cdot (\phi^*(v)) \quad \text{for } \phi^* \in X, \ v \in V.$$  

We see that for $\phi^* \in X$

$$\phi^* \text{ is } \text{SL}(U)\text{-stable } \iff \text{rank } \phi^* = r,$$

$$\phi^* \text{ is } \text{SL}(U)\text{-semistable } \iff \phi^* \text{ is } \text{SL}(U)\text{-stable}.$$  

In fact, if $\text{rank } \phi^* = r - 1$, then there is a one-parameter torus $T$ of $\text{SL}(U)$ such that the closure of the orbit $T \cdot \phi$ contains the zero vector as the following simple example $(r = 2)$ shows

$$\lim_{t \to 0} \left( \begin{array}{cc} t & 0 \\ 0 & t^{-1} \end{array} \right) \left( \begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1N} \\ 0 & 0 & \cdots & 0 \end{array} \right) = \lim_{t \to 0} \left( \begin{array}{cccc} ta_{11} & ta_{12} & \cdots & ta_{1N} \\ 0 & 0 & \cdots & 0 \end{array} \right).$$

Let $X_s$ be the set of all (semi)stable points and $P_s$ the image of $X_s$ by $\pi$. It is, as we saw above, just the set of all $\phi \in X$ with $\text{rank } \phi^* = r$. Therefore the GIT-orbit space $P_s//\text{SL}(U)$ is the orbit space $P_s/\text{SL}(U)$ by the free action, the Grassmann variety $G(r,N)$. 
3.3. Moduli of double coverings of $\mathbb{P}^1$ (1). Let $W$ and $U$ be a pair of two dimensional vector spaces, $X = \text{Hom}(W, S^2U)$, and $\pi : X \setminus \{0\} \to \mathbb{P}(X^*)$ the natural morphism. Note that $\text{SL}(U)$ acts on $S^2U$ from the left via the natural action: $\sigma(u_1u_2) = \sigma(u_1)\sigma(u_2)$ for $u_1, u_2 \in U$. Thus $\text{SL}(U)$ acts on $X$ from the left in the same manner in the subsection 3.2.

Lemma 3.4. Let $\phi^* \in X$.

(i) $\phi^*$ is unstable iff $\phi^*(w)$ has a double root for any $w \in W$.
(ii) $\phi^*$ is semistable iff $\phi^*(w)$ has no double roots for some nonzero $w \in W$.
(iii) $\phi^*$ is stable iff $\phi^*(W)$ is a base-point free linear subsystem of $S^2U$ on $\mathbb{P}(U)$.

Proof. We note that $\phi^*$ is unstable iff there is a suitable basis $s$ and $t$ of $U$ such that $\phi^*(w) = a(w)s^2$ for any $w \in W$ since a torus orbit $T \cdot \phi^*$ contains the zero vector. This proves (i). This also proves (iii). Next we prove (iii). If $\phi^*(W)$ has a base point, then it is clear that $\phi^*$ is not stable. If $\phi^*$ is semistable and it is not stable, then we choose a basis $s$, $t$ of $U$ and a basis $w_1$, $w_2$ of $W$ such that $\phi^*(w_1) = st$. If $\phi(w_2) = as^2 + bst$, then $\phi^*$ is not stable. This proves the lemma. □

Theorem 3.5. Let $X_{ss}$ be the Zariski open subset of $X$ consisting of all semistable points of $X$, $\pi(X_{ss})$ the image of $X_{ss}$ by $\pi$, and $Y := \pi(X_{ss})//\text{SL}(U)$. Then $Y \simeq \mathbb{P}^2$.

Proof. First consider a simplest case. We choose a basis $s$, $t$ of $U$. Let $w_1$ and $w_2$ be a basis of $W$, $T$ the subgroup of $\text{SL}(U)$ of diagonal matrices and $X' = \{\phi^* \in X; \phi^*(w_1) = 2st\}$. Let $Z' = \text{SL}(U) \cdot X'$.

We note that $Z'$ is an $\text{SL}(U)$-invariant subset of $X_{ss}$. We prove $\pi(Z')//\text{SL}(U) \simeq \mathbb{C}^2$. Let $\phi^*$ and $\psi^*$ be points of $X'$. Let $\phi^*(w_2) = A s^2 + 2Bst + Ct^2$ and $\psi^*(w_2) = As^2 + 2bst + ct^2$. Then it is easy to check

$$g \cdot \phi^* = \psi^* \text{ for } \exists g \in \text{SL}(U) \iff g \cdot \phi^* = \psi^* \text{ for } \exists g \in T$$

$$\iff A = au^2, B = b, C = u^{-2}c \text{ for } \exists u \neq 0.$$ 

Therefore each equivalence class of $\pi(Z')//\text{SL}(U)$ is represented by the pair $(AC, B)$, which proves $\pi(Z')//\text{SL}(U) \simeq \mathbb{C}^2$.

Now we prove the lemma. Let $\phi^* \in X_{ss}$, $\phi_j = \phi^*(w_j)$ and $\phi_0 = -(\phi_1 + \phi_2)$. Let

$$\phi_0 = r_1s^2 + 2r_2st + r_3t^2,$$
$$\phi_1 = p_1s^2 + 2p_2st + p_3t^2,$$
$$\phi_2 = q_1s^2 + 2q_2st + q_3t^2,$$

and we define

$$D_1 = p_2^2 - p_1p_3, \quad D_2 = q_2^2 - q_1q_3,$$
$$D_0 = r_2^2 - r_1r_3 = D_1 + D_2 + 2p_2q_2 - (p_1q_3 + p_3q_1).$$

To show the lemma, we prove the more precise isomorphism

$$\pi(X_{ss})//\text{SL}(U) = \text{Proj } \mathbb{C}[D_0, D_1, D_2]$$

For this purpose we define $Y_j = \pi(\{\phi^* \in X_{ss}; \phi_j \text{ has no double roots}\})//\text{SL}(U)$. It suffices to prove $Y_1 = \text{Spec } \mathbb{C}[D_0, D_1]$ by reducing it to the first simplest case.
Let $\phi^* \in Y_1$. Let $\alpha$ and $\beta$ be the roots of $\phi_1 = 0$. By the assumption $\phi_1$ has no double roots, hence $\alpha \neq \beta$. Let

$$u = \frac{1}{\gamma} (s - \alpha t), \quad v = \frac{1}{\gamma} (s - \beta t), \quad g = \frac{1}{\gamma} \left( 1 - \frac{\alpha}{\beta} \right)$$

where $\gamma = \sqrt{\alpha - \beta}$. Note that $g \in \text{SL}(U)$. Hence we see

$$(\phi_1(s, t), \phi_2(s, t)) \equiv (p_1^4 uv, A_1 u^2 + 2B_1 uv + C_1 v^2)$$

where

$$A_1 = q_1^2 \beta^2 + 2q_2 \beta + q_3,$$

$$B_1 = q_1 \alpha \beta + q_2 (\alpha + \beta) + q_3,$$

$$C_1 = q_1^2 \alpha^2 + 2q_2 \alpha + q_3.$$

Thus we see

$$(\phi_1(s, t), \phi_2(s, t)) \equiv (2st, As^2 + 2Bst + C t^2)$$

where

$$A = \frac{2A_1}{p_1^4}, \quad B = \frac{2B_1}{p_1^4}, \quad C = \frac{2C_1}{p_1^4}, \quad p_1^4 = 4D_1,$$

$$AC = B^2 - \frac{D_2}{D_1}, \quad B = \frac{D_0 - D_1 - D_2}{2D_1}.$$ 

Therefore by the first half of the proof

$$Y_1 \simeq \text{Spec } C[AC, B] = \text{Spec } C[\frac{D_0}{D_1}, \frac{D_2}{D_1}].$$

This completes the proof of the lemma.

**Corollary 3.6.** Let $Y^s = \pi(X_s) / \text{SL}(U)$. Then $Y \setminus Y^s$ is a conic of $Y$ defined by

$$Y \setminus Y^s : D_0^2 + D_1^2 + D_2^2 - 2D_0D_1 - 2D_1D_2 - 2D_2D_0 = 0.$$ 

**Proof.** In view of Theorem 3.5, $Y_1 \simeq \text{Spec } C[AC, B]$. The complement of $Y_s$ in $Y_1$ is then the curve defined by $AC = 0$, which is easily identified with the above conic. \qed

**Corollary 3.7.** Let $X^0$ be the Zariski open subset of $X$ consisting of all semistable points $\phi^*$ of $X$ with $\text{rank} \phi^* = 2$, and let $Y^0 := \pi(X^0) / \text{SL}(U)$. Then $Y^0 \simeq \pi(X^0) / \text{SL}(U) \simeq Y \simeq \mathbb{P}^2$.

**Proof.** It suffices to compare $Y_1$ and $Y^0 \cap Y_1$. As in the proof of Theorem 3.5 we let $X' = \{ \phi^* \in X; \phi^*(w_1) = 2st \}$. Let $Z = \text{SL}(U) \cdot X'$ and $Z^0 = \text{SL}(U) \cdot (X' \cap X^0)$.

Then with the notation in Theorem 3.5, we recall $X' = \{ \phi^* \in X; \phi^*(w_1) = 2st, \phi^*(w_2) = As^2 + 2Bst + C t^2, \pi(Z) / \text{SL}(U) \simeq \text{Spec } C[AC, B] \}$ where

$$X' \cap X^0 = \{ \phi^* \in X'; \ A \neq 0 \ or \ C \neq 0 \}.$$ 

In the same manner as before we see $\pi(Z^0) / \text{SL}(U) \simeq \text{Spec } C[AC, B]$, whence $\pi(Z^0) / \text{SL}(U) = \pi(Z) / \text{SL}(U)$. This proves $Y^0 \cap Y_1 = Y_1$. This completes the proof of the corollary. \qed
3.8. Moduli of double coverings of $\mathbb{P}(W)$ (2). There is an alternative way of understanding $\pi(X_{\psi})//\text{SL}(U) \simeq \mathbb{P}^2$ by using the isomorphism $S^2\mathbb{P}^1 \simeq \mathbb{P}^2$. We use the following convention to denote a point of $\mathbb{P}(U) = U^\vee \setminus \{0\}/G_m$: $(u : v) = us^\vee + tv^\vee \in U^\vee$ where $s^\vee$ and $t^\vee$ are a basis dual to $s$ and $t$. In what follows we fix a basis $w_1$ and $w_2$ of $W$. Let $P := (a_1 : a_2)$ and $Q := (b_1 : b_2)$ be a pair of points of $\mathbb{P}(W) \simeq \mathbb{P}^1$. If $P \neq Q$, there is a double covering $\phi : \mathbb{P}(U) \to \mathbb{P}(W)$ ramifying at $P$ and $Q$, unique up to isomorphism once we fix the base $w_1$ and $w_2$:

$$\frac{b_2w_1 - b_1w_2}{a_2w_1 - a_1w_2} = \left(\frac{t}{s}\right)^2.$$ 

Thus $\phi$ is given explicitly by

$$\phi_1 := \phi^*(w_1) = b_1s^2 - a_1t^2, \quad \phi_2 := \phi^*(w_2) = b_2s^2 - a_2t^2, \quad \phi_0 = -(\phi_1 + \phi_2)$$

for which we have

$$D_1 = a_1b_1, \quad D_2 = a_2b_2, \quad D_0 = (a_1 + a_2)(b_1 + b_2).$$

The isomorphism $S^2\mathbb{P}^1 \simeq \mathbb{P}^2$ is given by $(P, Q) \mapsto (D_0, D_1, D_2)$. This shows

Corollary 3.9. We have a natural isomorphism: $Y \simeq \mathbb{P}(S^2W)$.

4. The virtual normal bundle of a double covering

4.1. The case $N = 7$ and $k = 8$ revisited. We revisit the example in the subsection 2.9. Let $N = 7$ and $k = 8$. Let $L : x_j = 0$ ($j \geq 3$) and we take

$$F_3 = 8x_1^7, F_4 = 8x_1^6x_2, F_5 = 8x_1^5x_2^2, F_6 = 8x_1^4x_2^3, F_7 = 8x_1^7,$$

$$F = x_3F_3 + x_4F_4 + x_5F_5 + x_6F_6 + x_7F_7 + x_3^8 + x_4^8 + x_5^8 + x_6^8 + x_7^8.$$

and let $M = M^\circ_0 : F = 0$. We often denote $L$ also by $\mathbb{P}(W)$ with $W$ a two dimensional vector space for later convenience. Since $H^0(D_L^-)$ is injective and $H^0(D_L)$ is surjective, we have $N_{L/M} \simeq O_L \oplus O_L(-1)^{35}$. Hence $H^1(N_{L/M}(-1)) \neq H^1(O_L(-2)^{35})$ is 3-dimensional. As we see easily, this follows also from the fact that $\text{Coker} H^0(D_L^-)$ is freely generated by $x_1x_2^2, x_1^2x_2^2$ and $x_1x_2^3$.

Let $\phi^* = (\phi_1, \phi_2) \in X^0$. Then $\text{Coker} H^0(\phi^*D_L)$ is generated by a single element $\phi_2^Y - \phi_1^W$, while $\text{Coker} H^0(\phi^*D_L)$ is generated by $S^2U : \phi_1^3\phi_2^3, S^2U : \phi_1^3\phi_2^4$ and $S^2U : \phi_1\phi_2^4$. To be more precise, we see

$$\text{Coker} H^0(\phi^*D_L) = \{\phi_1^3\phi_2^3, \phi_1^3\phi_2^4, \phi_1\phi_2^4\} \cap S^2U / \langle \phi_1, \phi_2 \rangle.$$

In fact, this is proved as follows: first we consider the case where $\phi_1$ and $\phi_2$ has no common zeroes. In this case $\phi^*$ gives rise to a double covering $\phi : \mathbb{P}(U) \to \mathbb{P}(W)$ ($\simeq L$), which we denote by $L_\phi$ for brevity. By pulling back by $\phi^*$ the normal sequence

$$0 \to N_{L/M} \to N_{L/P} \to O_L(k) \to 0 \quad (k = 8)$$

for the line $L$ we infer an exact sequence

$$0 \to \phi^*N_{L/M} \to \phi^*N_{L/P} \to O_L(k) \to 0,$$

which yields an exact sequence

$$0 \to H^0(\phi^*N_{L/M}) \to S^2U \oplus (V^\vee / W^\vee) \to H^0(\phi^*D_L) \to H^0(O_L(-2k)) \to 0.$$
Let $\eta = q_3 e_3 + \cdots + q_7 e_7 \in \text{Ker } H^0(\phi^*D_L)$, $q_j \in S^2 U$. Then we have

$$\phi_1^2(q_3 \phi_1 + q_4 \phi_2 + q_5 \phi_3 + q_6 \phi_5) = -q_7 \phi_7.$$ 

Since $\phi_1$ and $\phi_2$ are mutually prime and $q_j$ is of degree two, we have $q_7 = 0$ and

$$\phi_1^2(q_3 \phi_1 + q_4 \phi_2 + q_5 \phi_3) = -q_6 \phi_6.$$ 

Hence $q_6 = 0$ and similarly we infer also $q_5 = 0$. Thus we have $q_3 \phi_1 + q_4 \phi_2 = 0$. This proves that $\text{Ker } H^0(\phi^*D_L)$ is generated by $\phi_2 e_2 - \phi_1 e_1$.

Next we prove that $\text{Coker } H^0(\phi^*D_L)$ is generated by $\phi^* \text{Coker } H^0(D_L)$ over $S^2 U$, in fact over $S^2 U/\phi^* (W)$. Without loss of generality we may assume that $\phi_1 = 2st$ and $\phi_2 = \lambda \nu^2 + 2\nu st + t^2$ for some $\lambda \neq 0$ and $\nu \in \mathbb{C}$. Let $\phi^* W = \{ \phi_1, \phi_2 \}$. Then one checks $U \cdot \phi^* W = S^2 U$, and hence $S^2 U \cdot \phi^* W = S^4 U$, $S^{2m-2} U \cdot \phi^* W = S^{2m} U$ for $m \geq 2$. It follows $S^2 U \cdot \phi^* (S^{m-1} W) = S^{2m} U$ for $m \geq 1$. In fact, by the induction on $m$

$$S^2 U \cdot \phi^* (S^{m} W) = S^2 U \cdot \phi^* (W) \cdot \phi^* (S^{m-1} W)$$

$$= S^4 U \cdot \phi^* (S^{m-1} W)$$

$$= S^2 U \cdot (S^2 U \cdot \phi^* (S^{m-1} W))$$

$$= S^2 U \cdot S^{2m} U = S^{2m+2} U.$$ 

Therefore $H^0(O_{L.M}(2k)) = S^{16} U = S^2 U \cdot \phi^* (S^7 W)$. Hence

$$\text{Coker } H^0(\phi^*D_L) = S^{16} U / \text{Im } H^0(\phi^*D_L)$$

$$= S^2 U \cdot \phi^* (S^7 W) / S^2 U \cdot \phi^* (\text{Im } H^0(D_L^-))$$

$$= (S^2 U / \phi^* (W)) \cdot \phi^* (S^7 W / \text{Im } H^0(D_L^-)).$$

because $\text{Coker } H^0(D_L^-) = S^7 W / \text{Im } H^0(D_L^-)$ and $W \cdot S^7 W \subset W \cdot \text{Im } H^0(D_L^-) = S^8 W$ by the choice of $L$. This proves that $\text{Coker } H^0(\phi^*D_L)$ is generated by $\phi^* \text{Coker } H^0(D_L^-)$ over $S^2 U / \phi^* (W)$. It follows $\text{Coker } H^0(\phi^*D_L) = (\phi^* \text{Coker } H^0(D_L^-)) \otimes (S^2 U / \phi^* W)$.

Finally we consider the case where $\phi_1$ and $\phi_2$ has a common zero. In this case we may assume $\phi_1 = 2st$ and $\phi_2 = 2\nu st + t^2$. In this case $L_\phi$ is a chain of two rational curves $C_\phi$ and $C_\phi'$ where $C_\phi$ is the proper transform of $P(U)$, where the double covering map from $L_\phi$ to $P(W)$ is the union of the isomorphisms $\phi'$ and $\phi''$, say, $\phi' \cup \phi''$. Let $\psi_1 = 2s$ and $\psi_2 = 2\nu s + t$. Then $\phi'$ is induced by the homomorphism $(\phi')^* \in \text{Hom}(W, U)$ such that $(\phi')^*(w_j) = \psi_j$. On the other hand let $U_{\phi}' = C \lambda + C \nu t$, $\psi_1' = 2t$ and $\psi_2' = \lambda + 2t$ where we note $\psi_j'$ is the linear part of $\phi_j$ in $t$ with $s = 1$. Then $C_\phi = P(U_{\phi}')$ and $\phi''$ is induced by the homomorphism $(\phi'')^* \in \text{Hom}(W, U_{\phi}')$ such that $(\phi'')^*(w_j) = \psi_j'$. Furthermore the pull back by $\phi^*$ of the normal sequence for $L$

$$0 \rightarrow \phi^* N_{L/M} \rightarrow \phi^* N_L / P \twoheadrightarrow \phi^* O_L(k) \rightarrow 0,$$
yields exact sequences with natural vertical homomorphisms:

\[ 0 \longrightarrow \phi^*N_{L/M} \longrightarrow (\phi')^*N_{L/M} \oplus (\phi'')^*N_{L/M} \longrightarrow C \longrightarrow 0 \]

\[ 0 \longrightarrow \phi^*N_{L/P} \longrightarrow (\phi')^*N_{L/P} \oplus (\phi'')^*N_{L/P} \longrightarrow V^\vee/W^\vee \longrightarrow 0 \]

\[ 0 \longrightarrow \phi^*O_L(k) \longrightarrow O_{C_\chi}(k) \oplus O_{C_\chi}(k) \longrightarrow C \longrightarrow 0. \]

This yields the following long exact sequences:

\[ 0 \longrightarrow H^0((\phi')^*N_{L/M}) \longrightarrow U \otimes V^\vee/W^\vee \xrightarrow{H^0((\phi')^*D_L)} S^kU \]

\[ \quad \quad \quad \quad \longrightarrow H^1((\phi')^*N_{L/M}) \longrightarrow 0 \]

\[ 0 \longrightarrow H^0((\phi'')^*N_{L/M}) \longrightarrow U'' \otimes V^\vee/W^\vee \xrightarrow{H^0((\phi'')^*D_L)} S^kU'' \]

\[ \quad \quad \quad \quad \longrightarrow H^1((\phi'')^*N_{L/M}) \longrightarrow 0 \]

whence \( H^1((\phi')^*N_{L/M}) = H^1((\phi'')^*N_{L/M}) = 0, \) and both \( H^0((\phi')^*N_{L/M}) \) and \( H^0((\phi'')^*N_{L/M}) \) are one-dimensional. Let \( U^t \) be the subspace of \( U \) consisting of elements vanishing at \( C^t_\phi \cap C^t_\phi \), namely the subspace spanned by \( t \). Then the restriction of \( H^0((\phi')^*D_L) \) to \( U^t \otimes V^\vee/W^\vee \) equals \( t \cdot H^0((\phi')^*D_L^-) \). Hence

\[ \text{Coker } H^0(\phi^*D_L) \simeq t \cdot S^2U/t \cdot \text{Im } H^0((\phi')^*D_L^-) \oplus \text{Coker } H^0((\phi'')^*D_L^-) \]

\[ \simeq S^2U/t \cdot \text{Im } H^0((\phi')^*D_L^-) \simeq \text{Coker } H^0((\phi')^*D_L^-). \]

One could understand the above isomorphism as

\[ \text{Coker } H^0(\phi^*D_L) = \text{Coker } (\phi^*H^0(D_L^-)) \oplus (S^2U/\phi^*W). \]

Thus \( H^0(\phi^*N_{L/M}) \) is one-dimensional, while \( H^1(\phi^*N_{L/M}) \) is 3-dimensional. This is immediately generalized into the following

**Lemma 4.2.** For any \( \phi^* \in X^0 \) we have

\[ \text{Ker } H^0(\phi^*D_L) = \phi^* \text{Ker } H^0(D_L), \]

\[ \text{Coker } H^0(\phi^*D_L) = (\phi^* \text{Coker } H^0(D_L^-)) \oplus (S^2U/\phi^*W). \]

**Lemma 4.3.** We define a line bundle \( L_0 \) (resp. \( L_1 \)) on \( Y \) (resp. \( \mathbb{P}(S^2W) \)) by the assignment:

\[ X^0 \ni \phi^* \mapsto \phi^* \text{Ker } H^0(D_L) \quad (\text{resp. } \phi^* \text{Coker } H^0(D_L^-)). \]

Then \( L_k \simeq O_{\mathbb{P}(S^2W)}. \)

**Proof.** We know that \( \phi^* \text{Ker } H^0(D_L) \) is generated by \( \phi_2e^\gamma_2 - \phi_1e^\gamma_1 \). By the \( \text{SL}(2) \)-variable change of \( s \) and \( t \), \( \phi_j \) is transformed into a new quadratic polynomial, which is however the same as the first \( \phi_j \). This shows the generator is unchanged, whence \( L_0 \simeq O_{\mathbb{P}(S^2W)}. \) The proof for \( L_1 \) is the same. \( \square \)
Lemma 4.4. We define a coherent sheaf \( L \) on the stack \( Y \) (\( \simeq P(S^2 W) \)) (See Remark below) by the assignment:

\[
X^0 \ni \phi^* \mapsto S^2 U/\phi^* W.
\]

Then \( L^2 \simeq O_{P(S^2 W)}(-1) \).

Proof. The GIT-quotient \( Y^0 \) is covered with the images of \( X'_i \):

\[
X'_1 = \{(\phi_1, \phi_2) \in X^0; \phi_1 = 2st, \phi_2 = \lambda s^2 + 2\nu st + t^2, \lambda, \nu \in \mathbb{C} \},
\]

\[
X'_2 = \{(\phi_1, \phi_2) \in X^0; \phi_1 = ps^2 + 2qst + t^2, \phi_2 = 2st, p, q \in \mathbb{C} \}.
\]

It is clear that the natural image of \( X'_j \) in \( Y \) is \( Y_j \). The map \( \phi \) given by \( \phi^* = (\phi_1, \phi_2) \in Y_1 \) has natural \( \mathbb{Z}_2 \) involution generated by,

\[
r : (\sqrt{\lambda} s + t, \sqrt{\lambda} s - t) \mapsto (\sqrt{\lambda} s + t, -(\sqrt{\lambda} s - t)).
\]

Since

\[
2st = \frac{1}{2\sqrt{\lambda}}((\sqrt{\lambda} s + t)^2 - (\sqrt{\lambda} s - t)^2),
\]

\[
\lambda s^2 + 2\nu st + t^2 = \frac{\nu}{2\sqrt{\lambda}}((\sqrt{\lambda} s + t)^2 - (\sqrt{\lambda} s - t)^2) + \frac{1}{\lambda}((\sqrt{\lambda} s + t)^2 + (\sqrt{\lambda} s - t)^2),
\]

it is clear that,

\[
r^*(\phi_1) = \phi_1, \quad r^*(\phi_2) = \phi_2, \quad r^*(\lambda s^2 - t^2) = - (\lambda s^2 - t^2).
\]

Therefore, we can decompose \( S^2 U \) into \( \langle \lambda s^2 - t^2 \rangle \mathbb{C} \oplus \langle \phi_1, \phi_2 \rangle \mathbb{C} \) with respect to eigenvalue of \( r^* \) and take \( \lambda s^2 - t^2 \) as canonical generator of \( S^2 U/\phi^* W \). Similarly \( S^2 U/\phi^* W \) is generated by \( ps^2 - t^2 \) on \( Y_2 \). The problem is therefore to write \( \lambda s^2 - t^2 \) as an \( \Gamma(Y_1; \mathcal{O}_Y) \)-multiple of \( pu^2 - v^2 \) when we write \( \phi_2 = 2uv \) by a variable change in \( GL(2) \). The following variable change \( (s, t) \mapsto (u, v) \) is in \( GL(2) \):

\[
s = \frac{\sqrt{2\alpha}}{(\beta - \alpha)^2}(2u - \frac{(\beta - \alpha)^2}{2\alpha}v), \quad t = \frac{\sqrt{2\alpha}}{(\beta - \alpha)^2}(2\beta u - \frac{(\beta - \alpha)^2}{2\alpha}v),
\]

where \( \alpha, \beta \) are roots of the equation \( \lambda s^2 + 2\nu st + t^2 = 0 \). Under this coordinate change, \( \phi_1 \) and \( \phi_2 \) is rewritten as follows:

\[
\phi_1 = \frac{\lambda}{(\nu^2 - \lambda)^2}u^2 + 2\frac{\nu}{\nu^2 - \lambda}uv + v^2 \equiv pu^2 + 2quv + v^2, \quad \phi_2 = 2uv.
\]

Then we have

\[
pu^2 - v^2 = -\frac{2}{\beta - \alpha}(\lambda s^2 - t^2) = -\frac{1}{\sqrt{\nu^2 - \lambda}}(\lambda s^2 - t^2) = -\sqrt{\frac{D_1}{D_2}}(\lambda s^2 - t^2).
\]

Similarly by computing the effect on \( S^2 U/\phi^* W \) by the variable change from \( X'_1 \) into \( X'_0 \), we see that \( L^2 \) is isomorphic to \( O_{P(S^2 W)}(-1) \). This completes the proof. \( \Box \)

Remark 4.5. We remark that the space \( X \) must be regarded as a \( \mathbb{Q} \)-stack \( Y^{stack} \) as follows: First we define \( \phi_0 = -\phi_1 - \phi_2 \). For each atlas \( X'_\alpha \) we define an atlas \( Y^{stack}_\alpha \)
(α = 0, 1, 2) by
\[ Y_0^{stack} = \{(φ_0, φ_1, φ_2, ±φ_0) ∈ X^0 × S^2 U; φ_0 = 2st, φ_1 = as^2 + 2bst + t^2, \]
\[ \psi_0 = as^2 - t^2, a, b ∈ C\}, \]
\[ Y_1^{stack} = \{(φ_0, φ_1, φ_2, ±ψ_0) ∈ X^0 × S^2 U; φ_1 = 2st, φ_2 = λs^2 + 2νst + t^2, \]
\[ ψ_1 = λs^2 - t^2, λ, ν ∈ C\}, \]
\[ Y_2^{stack} = \{(φ_0, φ_1, φ_2, ±ψ_2) ∈ X^0 × S^2 U; φ_1 = ps^2 + 2qst + t^2, φ_2 = 2st, \]
\[ ψ_2 = ps^2 - t^2, p, q ∈ C\} \]

Since \( L^2 \simeq O_{P(S^2 W)}(-1) \) we have \( c_1(L) = -\frac{1}{2} c_1(O_{P(S^2 W)}(1)) \) in the Chow ring \( A(Y^{stack})_Q = A(X)_Q = A(P(S^2 W))_Q \).

5. Proof of the main theorem

**Theorem 5.1.**

\[ \pi_*(c_{top}(H^1)) = \frac{1}{8} \left[ \frac{c(S^{k-1} Q)}{1 - \frac{1}{2} c_1(Q)} \right]_{k-N}, \]

where \( \pi \) is the natural projection from \( \tilde{M}_{0,0}(L, 2) \) to \( G \) and \([s]_{k-N} \) is the operation of picking up the degree \( k - N \) part of Chern classes.

**Proof.** From now on we denote the coherent sheaf \( L \) in Lemma 4.4 by \( O_P(-\frac{1}{2}) \). In view of the results from the previous section, what remains is to evaluate the top chern class of \( (S^{k-1} Q/((V^c \otimes O_G)/Q^c)) \otimes O_P(-\frac{1}{2}) \) on \( P(S^2 Q) \). Since double cover maps parametrized by \( P(S^2 Q) \) have natural \( \mathbb{Z}_2 \) involution \( r \) given in the previous section, we have to multiply the result of integration on \( P(S^2 Q) \) by the factor \( \frac{1}{2} [\pi]^* \) [BT], [FP]. With this set-up, let \( π^* : P(S^2 Q) \rightarrow G \) be the natural projection. Then what we have to compute is \( π_*(c_{top}(H^1)) = \frac{1}{2} π_*(c_{top}(H^1)) = \frac{1}{2} π_*(c_{top}((S^{k-1} Q/((V^c \otimes O_G)/Q^c)) \otimes O_P(-\frac{1}{2}))) \). Let \( z \) be \( c_1(O_P(1)) \). Then we obtain,

\[ \frac{1}{2} π_*(c_{top}((S^{k-1} Q/((V^c \otimes O_G)/Q^c)) \otimes O_P(-\frac{1}{2}))) \]
\[ = \frac{1}{2} \sum_{j=0}^{k-N+2} c_{k-N+2-j}((S^{k-1} Q \oplus Q^c) \cdot π^*(z^j)) \cdot (-\frac{1}{2})^j \]
\[ = \frac{1}{8} \sum_{j=0}^{k-N} c_{k-N-j}((S^{k-1} Q \oplus Q^c) \cdot s_j(S^2 Q)) \cdot (-\frac{1}{2})^j \]
\[ = \frac{1}{8} \left[ \frac{c((S^{k-1} Q) \cdot c(Q^c)}{1 - \frac{1}{2} c_1(S^2 Q) + \frac{1}{2} c_2(S^2 Q) - \frac{1}{2} c_3(S^2 Q)} \right]_{k-N}, \]

where \( s_j(S^2 Q) \) is the \( j \)-th Segre class of \( S^2 Q \). But if we decompose \( c(Q) \) into \( (1 + α)(1 + β) \), we can easily see,

\[ \frac{c(Q^c)}{1 - \frac{1}{2} c_1(S^2 Q) + \frac{1}{2} c_2(S^2 Q) - \frac{1}{2} c_3(S^2 Q)} = \frac{(1 - α)(1 - β)}{(1 - α)(1 - \frac{1}{2}(α + β))(1 - β)} \]
\[ = \frac{1}{1 - \frac{1}{2} c_1(Q)}. \]
Finally, by combining the above theorem with the divisor axiom of Gromov-Witten invariants, we can prove the decomposition formula of degree 2 rational Gromov-Witten invariants of $M^k_N$ found from numerical experiments.

**Corollary 5.2.**

$$\langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,2} = \langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,2-2} + 8(\pi_*(c_{top}(H^1))O_{e^a} O_{e^b} O_{e^c})_{0,1},$$

where $\langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,2-2}$ is the number of conics that intersect cycles Poincaré dual to $e^a$, $e^b$ and $e^c$. We also denote by $\langle \pi_*(c_{top}(H^1))O_{e^a} O_{e^b} O_{e^c} \rangle_{0,1}$ the integral:

$$\int_{G(2,\mathcal{V})} c_{top}(S^2 Q) \wedge \pi_*(c_{top}(H^1)) \wedge \sigma_{a-1} \wedge \sigma_{b-1} \wedge \sigma_{c-1}.$$ 

6. **GENERALIZATION TO TWISTED CUBICS**

In this section, we present a decomposition formula of degree 3 rational Gromov-Witten invariants found from numerical experiments using the results of [ES].

**Conjecture 6.1.** If $k - N = 1$, we have the following equality:

$$\pi_*(c_{top}(H^1)) = 
\frac{1}{27} \left( \frac{1}{27} (27k^2 - 55k + 26)k(k - 1) + \frac{2}{9} c_1(Q)^2 + \left( \frac{7}{6} (k + 1)k(k - 1) + \frac{1}{9} \right) c_2(Q) \right).$$

where $\pi : \overline{M}_{0,0}(L, 3) \to \overline{M}_{0,0}(M^k_N, 1)$ is the natural projection.

In the $k - N > 1$ case, we have not found the explicit formula, because in the $d = 3$ case, we have another contribution from multiple cover maps of type $(2+1) \to (1+1)$. Here multiple cover map of type $(2+1) \to (1+1)$ is the map from nodal curve $P^1 \cup P^1$ to nodal conic $L_1 \cup L_2 \subset M^k_N$, that maps the first (resp. the second) $P^1$ to $L_1$ (resp. $L_2$) by two to one (resp. one to one). In the $k - N = 1$ case, we have also determined the contributions from multiple cover maps of $(2+1) \to (1+1)$ to nodal conics.

**Corollary 6.2.** If $k - N = 1$, $\langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,3}$ is decomposed into the following contributions:

$$\langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,3} = \langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,3-3} + \frac{1}{k} \left( \frac{9}{4} \langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,1} \langle O_{e^a} \rangle_{0,1} \right)
+ \frac{3}{2} \langle O_{e^a} O_{e^b} O_{e^c+2} \rangle_{0,1} \langle O_{e^a} \rangle_{0,1} + \frac{3}{2} \langle O_{e^a} O_{e^b+2} O_{e^c} \rangle_{0,1} \langle O_{e^a} \rangle_{0,1}
+ \frac{3}{2} \langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,1} \langle O_{e^a} \rangle_{0,1}
+ 27(\pi_*(c_{top}(H^1))O_{e^a} O_{e^b} O_{e^c})_{0,1},$$

where $\langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,3-3}$ is the number of twisted cubics that intersect cycles Poincaré dual to $e^a$, $e^b$ and $e^c$.

**Proof.** In the $k - N = 1$ case, dimension of moduli space of multiple cover maps of $(2+1) \to (1+1)$ to nodal conics is given by $N - 6 + N - 6 - (N - 4) + 2 = N - 6$, hence the rank of $H^1$ is given by $N - 6 - (N - 5 - 3) = 2$. On the other hand, dimension of moduli space of $d = 2$ multiple cover maps of $P^1 \to P^1$ is 2, the degree of the form of $\pi_*(c_{top}(H^1))$ equals to $2 - 2 = 0$, where $\pi$ is the projection map.
that projects out the fiber locally isomorphic to the moduli space of \(d = 2\) multiple cover maps. This situation is exactly the same as the Calabi-Yau case. Therefore, we can use the well-known result by Aspinwall and Morrison, that says for \(n\)-point rational Gromov-Witten invariants for Calabi-Yau manifold, \(\tilde{\pi}_*(c_{\text{top}}(H^1))\) for degree \(d\) multiple cover map is given by,

\[
\frac{1}{d^{\frac{1}{2}}}. 
\]

With this formula, we add up all the combinatorial possibility of insertion of external operator \(\mathcal{O}_{e^+}, \mathcal{O}_{e^-}\) and \(\mathcal{O}_{e^0}\),

\[
\frac{1}{k} \left( \langle \tilde{\pi}_*(c_{\text{top}}(H^1)) \mathcal{O}_{e^+} \mathcal{O}_{e^-} \mathcal{O}_{e^0} \rangle_{0,1} \langle \mathcal{O}_{e_{N-5}} \rangle_{0,1} 
+ \langle \tilde{\pi}_*(c_{\text{top}}(H^1)) \mathcal{O}_{e^+} \mathcal{O}_{e^-} \mathcal{O}_{e^0} \rangle_{0,1} \langle \mathcal{O}_{e_{N-6}^-} \mathcal{O}_{e^+} \rangle_{0,1} 
+ \langle \tilde{\pi}_*(c_{\text{top}}(H^1)) \mathcal{O}_{e^+} \mathcal{O}_{e^-} \mathcal{O}_{e^0} \rangle_{0,1} \langle \mathcal{O}_{e_{N-6}^-} \mathcal{O}_{e^+} \mathcal{O}_{e^0} \rangle_{0,1} 
+ \langle \tilde{\pi}_*(c_{\text{top}}(H^1)) \mathcal{O}_{e^+} \mathcal{O}_{e^-} \mathcal{O}_{e^0} \rangle_{0,1} \langle \mathcal{O}_{e_{N-7}^-} \mathcal{O}_{e^+} \mathcal{O}_{e^0} \rangle_{0,1}
\right)
\]

\[= \frac{1}{k} \left( \langle \mathcal{O}_{e^+} \mathcal{O}_{e^-} \mathcal{O}_{e^0} \rangle_{0,1} \langle \mathcal{O}_{e_{N-5}} \rangle_{0,1} 
+ \langle \mathcal{O}_{e^+} \mathcal{O}_{e^-} \mathcal{O}_{e^0} \rangle_{0,1} \langle \mathcal{O}_{e_{N-6}^-} \mathcal{O}_{e^+} \rangle_{0,1} 
+ \langle \mathcal{O}_{e^+} \mathcal{O}_{e^-} \mathcal{O}_{e^0} \rangle_{0,1} \langle \mathcal{O}_{e_{N-6}^-} \mathcal{O}_{e^+} \mathcal{O}_{e^0} \rangle_{0,1} 
+ \langle \mathcal{O}_{e^+} \mathcal{O}_{e^-} \mathcal{O}_{e^0} \rangle_{0,1} \langle \mathcal{O}_{e_{N-7}^-} \mathcal{O}_{e^+} \mathcal{O}_{e^0} \rangle_{0,1}
\right)
\]

The last expression is nothing but the formula we want. 

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