CONICS ON A GENERIC HYPERSURFACE

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ABSTRACT. In this paper, we compute the contributions from double cover maps to genus 0 degree 2 Gromov-Witten invariants of general type projective hypersurfaces. Our results correspond to a generalization of Aspinwall-Morrison formula to general type hypersurfaces in some special cases.
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1. INTRODUCTION

In this paper, we discuss a generalization of the multiple cover formula for rational Gromov-Witten invariants of Calabi-Yau manifolds [AM], [M] to double cover maps of a line $L$ on a degree $k$ hypersurface $M_N^k$ in $\mathbb{P}^{N-1}$. Naïvely, for a given finite set of elements $\alpha_j \in H^*(M_N^k, \mathbb{Z})$, the rational Gromov-Witten invariant $\langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \cdots \mathcal{O}_{\alpha_n} \rangle_{0.d}$ of $M_N^k$ counts the number of degree $d$ (possibly singular and reducible) rational curves on $M_N^k$ that intersect real sub-manifolds of $M_N^k$ that are Poincaré-dual to $\alpha_j$.

Recently, the mirror computation of rational Gromov-Witten invariants of $M_N^k$ with negative first Chern class $(k-N > 0)$ was established in [CG], [Iri], [J]. Using the method presented in these articles, we can compute $\langle \mathcal{O}_{e=1} \mathcal{O}_{e=2} \cdots \mathcal{O}_{e=m} \rangle_{0.d}$ where $e$ is the generator of $H^{1,1}(M_N^k, \mathbb{Z})$. Briefly, mirror computation of $M_N^k$ $(k > N)$ in [J] goes as follows. We start from the following ODE:

$$(1) \quad \left( (\partial_x)^{N-1} - k \cdot \exp(x) \cdot ((k-1)(k-2) \cdots (k-1) + 1) \right) w(x) = 0,$$

and construct the virtual Gauss-Manin system associated with (1):

$$(2) \quad \partial_x \tilde{\psi}_{N-2-m}(x) = \tilde{\psi}_{N-1-2m}(x) + \sum_{d=1}^{\infty} \exp(dx) \cdot \tilde{L}^{N,k,d}_{m} \cdot \tilde{\psi}_{N-1-m-(N-k)d}(x),$$

where $m$ runs through all the integers and $\tilde{L}^{N,k,d}_{m}$ is non-zero only if $0 \leq m \leq N-1+(k-N)d$. From the compatibility of (1) and (2), we can derive the recursive formulas that determine all the $\tilde{L}^{N,k,d}_{m}$:

$$\sum_{n=0}^{k-1} \tilde{L}^{N,k,1}_{n} w^n = k \cdot \prod_{j=1}^{k-1} (jw + (k-j)),$$

$$\sum_{m=0}^{N-1+(k-N)d} \tilde{L}^{N,k,d}_{m} z^m = \sum_{l=2}^{d} (-1)^l \sum_{0=i_0 < \cdots < i_l = d} \times$$

$$\sum_{i_{i_{i + (N-k)(i_{i+1})}}}(i_{i_{i-1}} + (d-i_{i-1})z) \tilde{L}^{N,k,i_{i_{i+1}}}_{d} (i_{i_{i+1}} + (N-k)(i_{i+1})), $$

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With these data, we can construct the formulas that represent rational three point Gromov-Witten invariant \( \langle O_e \circ \Omega_{e,N-2-m} \circ \Omega_{e,m-1-(k-N)} \rangle_d \) in terms of \( \tilde{L}_m^{N,k,d} \). These three point Gromov-Witten invariants are enough for reconstruction of all the rational Gromov-Witten invariants \( \langle O_{e,1} \circ \Omega_{e,2} \cdots \circ \Omega_{e,m} \rangle_{0,d} \) [KM]. In particular, we obtain the following formula in the \( d = 2 \) case:

\[
\langle O_e \circ \Omega_{e,N-2-m} \circ \Omega_{e,m-1-(k-N)} \rangle_2 = k \cdot \left( \tilde{L}_m^{N,k,2} - \tilde{L}_1^{N,k,2} - \tilde{L}_1^{N,k,1} \left( \sum_{j=0}^{k-N} \frac{\tilde{L}_m^{N,k,1} - \tilde{L}_1^{N,k,1}}{1+j} \right) \right).
\]

According to the results of this procedure, rational three point Gromov-Witten invariants can be rational numbers with large denominator if \( k > N \), in contrast to the Calabi-Yau case where rational three point Gromov-Witten invariants are always integers.

One of the reasons of this rationality (non-integrality) comes from the contributions of multiple cover maps to Gromov-Witten invariants. In the Calabi-Yau case \( (N = k) \), for any divisor \( m \) of \( d \) there are some contributions from degree \( m \) multiple cover maps \( \phi \) of a rational curve \( \mathbb{P}^1 \) onto a degree \( \frac{d}{m} \) rational curve \( C \hookrightarrow M_k^b \). The contributions from the multiple cover maps are expressed in terms of the virtual fundamental class of Gromov-Witten invariants. Let \( C \) be a general degree \( d \) rational curve in \( M_k^b \). Its normal bundle \( N_{C/M_k^b} \) is decomposed into a direct sum of line bundles as follows:

\[
N_{C/M_k^b} \cong O_C(-1) \oplus O_C(-1) \oplus O_C^{b(k-5)}.
\]

Let \( \phi : \mathbb{P}^1 \to C \) be a holomorphic map of degree \( m \). Since the pull-back \( \phi^*(N_{C/M_k^b}) \) is given by

\[
\phi^*(N_{C/M_k^b}) \cong O_{\mathbb{P}^1}(-m) \oplus O_{\mathbb{P}^1}(-m) \oplus O_{\mathbb{P}^1}^{b(k-5)},
\]

we obtain \( h^1(\phi^*(N_{C/M_k^b})) = 2m - 2 \). On the other hand, let \( \overline{M}_{0,0}(M,d) \) be the moduli space of 0-pointed stable maps of degree \( d \) from genus 0 curve to \( M \). Then the moduli space of \( \phi \) is the fiber space \( \pi : \overline{M}_{0,0}(M,m) \to \overline{M}_{0,0}(M_k^b, \frac{d}{m}) \), whose fibre \( \overline{M}_{0,0}(C,m) \) over \( C \) (fixed) has complex dimension \( 2m - 2 \). Then the push-forward of the virtual fundamental class \( \pi_*\left( \chi_{\overline{M}_{0,0}}(H^1(\phi^*(N_{C/M_k^b}))) \right) \) can be computed only by intersection theory on the fiber \( \overline{M}_{0,0}(C,m) \), which turns out to be equal to \( \frac{1}{m} \). This depends on neither the structure of the base \( \overline{M}_{0,0}(M_k^b, \frac{d}{m}) \) nor the normal structure of the fibration.

But when \( k < N \), the situation is more complicated than \( M_k^b \) because of negative first Chern class. Let us concentrate on the case of \( d = 2, m = 2 \) that we discuss in this paper. In this case, \( C \) is just a line \( L \) on the hypersurface \( M_k^b \). The moduli space \( \overline{M}_{0,0}(M_k^b, 1) \) is a sub-manifold of \( \overline{M}_{0,0}(\mathbb{P}^{N-1}, 1) \), while \( \overline{M}_{0,0}(\mathbb{P}^{N-1}, 1) \) is the Grassmannian \( G(2,N) \), the moduli space of rank 2 quotients of \( V = \mathbb{C}^N \). As will be shown later, for a generic line \( L \), \( N_{L/M_k^b} \) is decomposed into

\[
N_{L/M_k^b} \cong O_L(-1)^{b(N-2)} \oplus O_L^{b(2N-k-5)}.
\]
By pulling back it by the degree 2 map $\phi : \mathbb{P}^1 \to L$, we obtain,

$$\phi^*N_{L/M_N^k} \simeq O_\mathbb{P}((-2)^{m_k - N + 2} \oplus O_{\mathbb{P}}^{m_{2^k - k - 5}}).$$

Therefore, $h^1(\phi^*(N_{L/M_N^k})) = k - N + 2$, which is strictly greater than two, the complex dimension of the fiber $\overline{M}_{0,0}(L, 2)$. Thus we need to know the global structure of the fibration $\pi$ in order to compute the multiple cover contribution to degree 2 rational Gromov-Witten invariants of $M_N^k$.

In order to estimate the contributions from double cover maps $\phi : \mathbb{P}^1 \to \langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,2}$, we first computed the number of conics, that intersect cycles Poarcă dual to $e^a$, $e^b$ and $e^c$, on $M_N^k$ (whose normal bundle are of the same type) by using the method in [K2]. Then we found the following formula by comparing these integers with the results obtained from (3):

$$\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,2} = \text{(number of corresponding conics)} + \int_{\overline{M}(2, 2)} c_{\text{top}}(S^k Q) \wedge \left[ \frac{c(S^k - 1)}{1 - \frac{1}{2}c_1(Q)} \right]_{k - N} \wedge \sigma_a \wedge \sigma_b \wedge \sigma_c,
$$

where $Q$ is the universal rank 2 quotient bundle of $G(2, N)$, $\sigma_a$ is a Schubert cycle defined by $\sum_{a=0}^{N} \sigma_a := \frac{1}{\prod_{j=1}^{N}}$ and $[\sigma^k_{j-k}]$ is the operation of picking up degree $2k - N$ part of Chern classes.

On the other hand, we have the following formula which directly follows from the definition of the virtual fundamental class of $\overline{M}_{0,0}(M_N^k, 2)$:

$$\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,2} = \text{(number of corresponding conics)} + 8 \int_{\overline{M}(2, 2)} c_{\text{top}}(S^k Q) \wedge \left[ \pi_*(c_{\text{top}}(H^1(\phi^*N_{L/M_N^k}))) \right]_{k - N} \wedge \sigma_a \wedge \sigma_b \wedge \sigma_c,
$$

where $\pi : \overline{M}_{0,0}(L, 2) \to \overline{M}_{0,0}(M_N^k, 1)$ is the natural projection. Here, the factor 8 comes from the divisor axiom of Gromov-Witten invariants.

In this paper, we prove the following formula

$$\pi_*(c_{\text{top}}(H^1(\phi^*N_{L/M_N^k}))) = \frac{1}{8} \left[ \frac{c(S^k - 1)}{1 - \frac{1}{2}c_1(Q)} \right]_{k - N}.$$

By combining (5) with (6), we can derive the formula (4) immediately.

From (4), we see that $\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,2}$ of $M_N^k$ is a rational number with denominator at most $2^{k - N}$. Therefore rationality (non-integrality) of the Gromov-Witten invariant $\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,2}$ is caused by the effect of multiple cover map in this case.

We note here that the total moduli space of double cover maps of lines is isomorphic to $\mathbb{P}(S^2 Q)$ over $G := \overline{M}_{0,0}(M_N^k, 1) \leftrightarrow G(2, N)$, which is an algebraic \textbf{Q}-stack $\mathbb{P}(S^2 Q)^{\text{stack}}$ (in the sense of Mumford). As a consequence, the union of all $H^1(\phi^*N_{L/M_N^k})$ turns out to be a coherent sheaf on $\mathbb{P}(S^2 Q)^{\text{stack}}$ with fractional Chern class in (6), as was suggested in [BT]. See [V, Section 9].

We also did some numerical experiments on degree 3 Gromov-Witten invariants of $M_N^k$ by using the results of [ES]. For $k - N > 0$, there is a new contribution from multiple cover maps to nodal conics in $M_N^k$ that did not appear in the Calabi-Yau case. Therefore, multiple cover map contributions are far more complicated than Calabi-Yau, and we leave general analysis on this problem to future works.
This paper is organized as follows. In Section 1, we analyze characteristics of moduli space of lines in $M^k_N$ and derive $N_{L/M^k_N} \simeq O_L(-1)^{g_k-2N+2} \oplus O_L^{g_k-2N-k-5}$. In Section 2, we study the moduli space $\mathcal{M}_{0,0}(\mathbb{P}^1, 2)$ from the point of view of stability and identify it with $\mathbb{P}^2$ and show that the moduli space $\mathcal{M}_{0,0}(\mathbb{P}^1, 2)$ is isomorphic to $\mathbb{P}(S^2 Q)$ over $G$. In section 4, we describe $H^1(\phi^* N_{L/M^k_N})$ as a coherent sheaf over $\mathbb{P}(S^2 Q)^{\text{stack}}$. In section 5, we derive the main theorem (6) of this paper by using Segre-Witten classes. In Section 6, we mention some generalization to degree 3 Gromov-Witten invariants.

2. Lines on a hypersurface

Let $M$ be a generic hypersurface of degree $k$ of the projective space $\mathbb{P}^{N-1} = \mathbb{P}(V)$. We assume $2N - 5 \geq k \geq N - 2 \geq 2$ throughout this note. In this note we count the number of rational curves of virtual degree two, namely rational curves which doubly cover lines on $M$.

Let $\mathbb{P} = \mathbb{P}(V)$ be the projective space parameterizing all one-dimensional quotients of $V$, which is usually denoted by $\mathbb{P}(V)$ in the standard notation in algebraic geometry. In this notation let $W$ be a subspace of $V$. Then $\mathbb{P}(W)$ is naturally a linear subspace of $\mathbb{P}(V)$ of dimension $\dim W - 1$.

Let $G(2, V)$ be the Grassmann variety of lines in $\mathbb{P}(V)$, the scheme parameterizing all lines of $\mathbb{P} = \mathbb{P}(V)$. This is also the universal scheme parameterizing all one-dimensional quotient linear spaces of $V$. Let $W$ be a two dimensional quotient linear space, $\psi \in G(2, V)$, namely $\psi : \mathbb{P}(W) \to \mathbb{P}(V)$ the natural immersion and $i^*_\psi : V \to W$ the quotient homomorphism. The space $W$ is denoted by $W(\psi)$ when necessary.

There exists the universal bundle $Q_{G(2, V)}$ over $G(2, V)$ and a homomorphism $i^{\text{univ}}_* : O_{G(2, V)} \otimes V \to Q_{G(2, V)}$ whose fiber $i^{\text{univ}}_* : V \to Q_{G(2, V)}$ is the quotient $i^*_\psi : V \to W(\psi)$ of $V$ corresponding to $\psi$.

2.1. Existence of a line on $M$. Let $L = \mathbb{P}(W)$ be a line of $\mathbb{P}$, equivalently $W \in G(2, V)$. Then the condition $L \subseteq M$ imposes at most $k + 1$ conditions on $W$, while the number of moduli of lines of $\mathbb{P}$ equals $\dim G(2, V) = 2N - 4$. Hence we infer

**Lemma 2.2.** If $2N \geq k + 5$, then there exists at least a line on $M$.

See also [Katz.p.152]. Let $G$ be the subscheme of $G(2, V)$ parameterizing all lines of $\mathbb{P}(V)$ lying on $M$, $Q = (Q_{G(2, V)})_{|G}$ the restriction of $Q_{G(2, V)}$ to $G$. By Lemma 2.2, $G$ is nonempty. Let $i^* : O_G \otimes V \to Q$ be the restriction of $i^{\text{univ}}$ to $G$. Let $P = \mathbb{P}(Q)$ and $\pi : P \to G$ the natural projection. Then $\pi$ is the universal line of $M$ over $G$, to be more exact, the universal family over $G$ of lines lying on $M$. In other words, the natural epimorphism $i^* : O_G \otimes V \to Q$ induces a morphism $i : P \to \mathbb{P}_G(V) := G \times \mathbb{P}(V)$, which is a closed immersion into $\mathbb{P}_G(V)$, thus $P$ is a subscheme of $\mathbb{P}_G(V)$ such that $\pi = (\pi_1)_P$. Let $L_\psi = \mathbb{P}(Q_\psi)$. Note that

$L_\psi = P_\psi := \pi^{-1}(\psi) \simeq \mathbb{P}(Q_\psi) \subset \{\psi\} \times \mathbb{P}(V) \simeq \mathbb{P}(V)$.
2.3. The normal bundle $N_{L/M}$. The argument of this section is standard and well known. Let $P = P(V)$, $L = P(W)$ and $i^*_W : V \to W \in G$. Let us recall the following exact sequence:

$$0 \to O_P \to O_P(1) \otimes V^\vee \xrightarrow{D} T_P \to 0$$

where the homomorphism $D$ is defined by

$$D(a \otimes v^\vee) := aD(v^\vee) \quad (a \in O_P(1))$$

$$(D_{v^\vee}F)(u^\vee) := \left(\frac{d}{dt} F(u^\vee + tv^\vee)\right)|_{t=0}$$

for a homogeneous polynomial $F \in S(V)$ and $u^\vee, v^\vee \in V^\vee$. We note $H^0(O_P(1)) \otimes V^\vee = V \otimes V^\vee = \text{End}(V, V)$ and that the image of $H^0(O_P)$ in $\text{End}(V, V)$ is $\text{Cid}_V$.

We also have the following exact sequences:

$$0 \to T_L \to (T_P)_L \to N_{L/P} \to 0$$

$$0 \to O_L \to O_L(1) \otimes V^\vee \xrightarrow{D_L} (T_P)_L \to 0.$$ 

Lemma 2.4. Let $L = P(W)$. Then

$N_{L/P} \simeq O_L(1) \otimes (V^\vee / W^\vee)$, $H^0(N_{L/P}) \simeq W \otimes (V^\vee / W^\vee)$.

Proof. The assertion is clear from the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccc}
0 & \to & O_L & \to & O_L(1) \otimes W^\vee & \xrightarrow{(D_L)_{W^\vee}} & T_L & \to & 0 \\
\downarrow & & \downarrow & & \downarrow \text{id} \otimes v^\vee & & \downarrow & & \\
0 & \to & O_L & \to & O_L(1) \otimes V^\vee & \xrightarrow{D_L} & (T_P)_L & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & O_L(1) \otimes (V^\vee / W^\vee) & \to & N_{L/P} & \to & 0
\end{array}
$$

The second assertion is clear from $H^0(L, O_L(1)) = W$. Since $T_L \simeq O_L(2)$, there follow exact sequences

$$0 \to H^0(T_L) \to H^0((T_P)_L) \to H^0(N_{L/P}) \to 0$$

$$0 \to H^0(O_L) \to H^0(O_L(1)) \otimes V^\vee \xrightarrow{H^0(D_L)} H^0((T_P)_L) \to 0.$$

We also note

$$H^0(T_L) = \text{LieAut}^0(L) = \text{End}(W, W) / \text{center} = \text{End}(W, W) / \text{Cid}_W.$$

Since $H^0(O_L(1)) = W$, we see

$$H^0((T_P)_L) = W \otimes V^\vee / \text{Im} H^0(O_L) = \text{Hom}(V, W) / \text{Cid}_W.$$

Hence we again see

$$H^0((N_{L/P})) = (\text{Hom}(V, W) / \text{Cid}_W)^0 / (\text{Hom}(W, W) / \text{Cid}_W)$$

$$= W \otimes (V^\vee / W^\vee) = \text{Hom}(V / W, W).$$

For any line $L = P(W)$ of $P$ the following sequence is exact:

$$0 \to N_{L/M} \to N_{L/P} \to (N_{M/P})_L \simeq O_L(k) \to 0.$$
Hence so is the following sequence as well:

\[ 0 \rightarrow H^0(N_{L/M}) \rightarrow H^0(N_{L/P}) \xrightarrow{H^0(D_L)} H^0(O_L(k)) \rightarrow H^1(N_{L/M}) \rightarrow 0. \]

Hence we have

**Lemma 2.5.** The following is exact:

\[ (8) \quad 0 \rightarrow H^0(N_{L/M}) \rightarrow W \otimes (V^\vee/W^\vee) \xrightarrow{H^0(D_L)} S^kW \rightarrow H^1(N_{L/M}) \rightarrow 0. \]

**Corollary 2.6.** \( \dim G \geq 2N - k - 5 \), equality holding if \( H^1(N_{L/M}) = 0. \)

**Proof.** As is well-known, \( \dim G \geq h^0(N_{L/M}) - h^1(N_{L/M}) \). Note \( \dim W \otimes (V^\vee/W^\vee) = 2(N - 2) \) and \( \dim S^kW = k + 1 \). Hence the corollary follows from Lemma 2.5. \( \square \)

**Lemma 2.7.** For a generic line \( L \) on a generic hypersurface \( M \) of degree \( k \)

(i) \( N_{L/M} \cong O_L^{ab} \oplus O_L(-1)^{ab} \), where \( a = 2N - k - 5 \) and \( b = k - N + 2 \),

(ii) \( \text{Coker} H^0(D_L) \cong S^{k-1}W/(V^\vee/W^\vee) \) where \( D_L^- := D_L \otimes O_L(-1). \)

**Proof.** Let \( M \) be a generic hypersurface of degree \( k \) and \( L \) a generic line on \( M \). Without loss of generality we may assume that \( W^\vee \) is generated by \( e_1^\vee \) and \( e_2^\vee \), in other words, \( \psi : L \rightarrow P \) is given by

\[ \psi : [s : t] \rightarrow [x_1, \ldots, x_N] = [s, t, 0, \ldots, 0]. \]

Then \( F \), the polynomial of degree \( k \) defining \( M \), is written as

\[ F = x_3F_3 + x_4F_4 + \cdots + x_NF_N \]

for some polynomials \( F_j \) of degree \( k - 1 \). Let \( f_j = \psi^*F_j = F_j(s, t, 0, \ldots, 0). \)

Now we consider the exact sequence

\[ 0 \rightarrow H^0(N_{L/M}(-1)) \rightarrow H^0(N_{L/P}(-1)) \xrightarrow{H^0(D_L^-)} H^0(O_L(k - 1)) \rightarrow H^1(N_{L/M}(-1)) \rightarrow 0, \]

where we note \( H^0(N_{L/P}(-1)) = V^\vee/W^\vee. \) Hence the following is exact:

\[ (9) \quad 0 \rightarrow H^0(N_{L/M}(-1)) \rightarrow V^\vee/W^\vee \xrightarrow{H^0(D_L^-)} S^{k-1}W \rightarrow H^1(N_{L/M}(-1)) \rightarrow 0, \]

where \( H^0(D_L^-) \) is given by \( H^0(D_L^-)(e_j^\vee) = f_j \quad (j = 3, 4, \ldots, N). \)

A generic choice of \( F \) implies a generic choice of degree \( k - 1 \) polynomials \( f_j \) \((j = 3, 4, \ldots, N)\) in \( s \) and \( t \). By the assumptions

\[ \dim S^{k-1}W = k \geq N - 2 = \dim V^\vee/W^\vee, \]

\[ \dim W \otimes V^\vee/W^\vee = 2(N - 2) \geq k + 1 = \dim S^kW, \]

the generic choice of \( F \) implies that we can choose \( f_j \in S^{k-1}W \quad (j = 3, 4, \ldots, N) \)

(and fix once for all) such that

(iii) \( f_j \quad (j = 3, 4, \ldots, N) \) are linearly independent,

(iv) \( Wf_3 + Wf_4 + \cdots + Wf_N = S^kW. \)
Hence $H^0(D_7)$ is injective by (iii). It follows that $H^0(N_{L/M}(-1)) = 0$. Hence (ii) is clear. Next we consider $H^0(D_L)$. By (iv), we see

\[ S^kW = W \cdot H^0(D_L)(V^\vee/W^\vee) = H^0(D_L)(W \oplus V^\vee/W^\vee), \]

whence $H^0(D_L)$ is surjective. It follows that $H^1(N_{L/M}) = 0$. Hence $N_{L/M} \simeq O_L^{gb} \oplus O_L(-1)^{gb}$ for some $a$ and $b$. Since $a + b = \text{rank}(N_{L/M}) = N - 3$ and $-b = \deg(N_{L/M}) = N - 2 - k$, we have (i).

2.8. Lines on a quintic hypersurface in $\mathbb{P}^4$. See [Katz, Appendix A] for the subsequent examples. Let $N = 5$ and $k = 5$. Hence $M$ is a hypersurface of degree 5 in $\mathbb{P}^4$, a Calabi-Yau 3-fold. Let

\[ F = x_4x_1^4 + x_5x_2^5 + x_3^5 + x_4^5 + x_5^5. \]

First we note that $M = \{ F = 0\}$ is nonsingular. Let $L = \{ x_3 = x_4 = x_5 = 0 \} = \{ [s, t, 0, 0, 0] \}$. In this case $f_3 = 0$, $f_4 = s^4$ and $f_5 = t^4$. In the exact sequence (1) we see $H^0(N_{L/M}(-1)) = \text{ker} H^0(D_7) = \mathcal{C}_L' \text{ and } H^1(N_{L/M}(-1)) = \text{coker} H^0(D_7)$ is 3-dimensional. Hence $N_{L/M} = \mathcal{O}_L(1) \oplus \mathcal{O}_L(-3)$.

We summarize the above. If $\dim \text{ker} H^0(D_7) = 1$ and if $M$ is nonsingular, then $N_{L/M} = \mathcal{O}_L(1) \oplus \mathcal{O}_L(-3)$. Hence $H^0(N_{L/M}) = \text{ker} H^0(D_L) = W \oplus \text{ker} H^0(D_7)$ is 2-dimensional. Therefore we can choose $f_3 = 0$ and a linearly independent pair $f_4$ and $f_5 \in S^4W$ so that $Wf_4 + Wf_5$ is 4-dimensional. The choice $f_4 = s^4$ and $f_5 = t^4$ satisfies the conditions. This enables us to find a nonsingular hypersurface $M$ as above. However if we choose $f_3 = 0$, $f_4 = s^4$ and $f_5 = s^4t$, then $Wf_4 + Wf_5$ is 3-dimensional. Hence $M$ is singular.

Next in the same manner we find $L$ on a nonsingular hypersurface $M$ with $N_{L/M} = \mathcal{O}_L \oplus \mathcal{O}_L(-2)$ or $N_{L/M} = \mathcal{O}_L(-1)^{gb}$. Let

\[ F = x_3x_1^4 + x_4x_1^3x_2 + x_5x_3^4 + x_4^5 + x_5^5. \]

Then we have $f_3 = s^4$, $f_4 = s^4t$ and $f_5 = t^4$. Since $Wf_3 + Wf_4 + Wf_5$ is 5-dimensional, $H^0(N_{L/M}(-1)) = \text{ker} H^0(D_7) = 0$, $H^0(N_{L/M}) = \text{ker} H^0(D_L) = \mathcal{C}(t_3' - s_2t_4')$. We see also that $\dim H^1(N_{L/M}) = \dim \text{coker} H^0(D_L) = 1$ and $N_{L/M} = O_L \oplus O_L(-2)$. The hypersurface $M = \{ F = 0 \}$ is easily shown to be nonsingular.

If $F = x_3x_1^4 + x_4x_1^3x_2 + x_5x_3^4 + x_4^5 + x_5^5$ and $M = \{ F = 0 \}$, then $N_{L/M} = \mathcal{O}_L(-1)^{gb}$.

2.9. Lines on a generic hypersurface $M^8$ of $\mathbb{P}^6$. Let $N = 7$ and $k = 8$. In view of Lemma 2.2 there exists a line $L$ on any generic hypersurface of degree 8 in $\mathbb{P}(V) = \mathbb{P}^6$. In view of Lemma 2.7, $a = 1$, $b = 3$ and $N_{L/M} \simeq O_L \oplus O_L(-1)^{gb}$. For example let $L : x_j = 0$ ($j \geq 3$) and we take

\[ F_3 = 8x_1^2, F_4 = 8x_1x_2, F_5 = 8x_1^2x_2^2, F_6 = 8x_1^2x_5^2, F_7 = 8x_2^3, \]

\[ F = x_3F_3 + x_4F_4 + x_5F_5 + x_6F_6 + x_7F_7 + x_8 + x_9 + x_8 + x_8 + x_8 + x_8. \]

and let $M = M^8 : F = 0$. We see that $M$ is nonsingular near $L$ and has at most isolated singularities. However it is still unclear to us whether $M = M^8$ is
nonsingular everywhere. The space $H^0(N_{L/M})$ is spanned by $te_N^Y - se_Y^Y$, hence an infinitesimal deformation $L_c$ of $L$ is given by
\[ [s, t] \mapsto [s, t, \varepsilon s, 0, 0, 0] \]
which yields $F|_{L_c} = \varepsilon^8 (s^8 + t^8) \equiv 0 \mod \varepsilon^8$. Since $H^1(N_{L/M}) = 0$, this infinitesimal deformation is integrable and $G (:= \text{the moduli of lines of } \mathbb{P}^5 \text{ contained in } M)$ is nonsingular and one dimensional at the point $[L]$.

We note that $M$ also contains 8 lines
\[ L' := L'_{c_8} : \varepsilon_8 x_1 - x_2 = x_3 + \varepsilon_8 x_4 = x_j = 0 \quad (j \geq 5), \]
with $N_{L'/M} = O_{L'}(1) \oplus O_{L'}(-6)$ where $\varepsilon_8 = -1$.

3. Stability

**Definition 3.1.** Suppose that a reductive algebraic group $G$ acts on a vector space $V$. Let $v \in V$, $v \neq 0$.

1. the vector $v$ is said to be semistable if there exists a $G$-invariant homogeneous polynomial $F$ on $V$ such that $F(\varepsilon) \neq 0$,

2. the vector $v$ is said to be stable if $p$ has a closed $G$-orbit in $X_{ss}$ and the stabilizer subgroup of $v$ in $G$ is finite.

Let $\pi : V \setminus \{0\} \to \mathbb{P}(V^\vee)$ be the natural surjection. Then $v \in V$ is semistable (resp. stable) if and only if $\pi(v)$ is semistable (resp. stable).

3.2. Grassmann variety. Let $V$ be an $N$-dimensional vector space, and $G(r, N)$ the Grassmann variety parameterizing all $r$-dimensional quotient spaces of $V$. Here is a natural way of understanding $G(r, N)$ via GIT-stability. Let $U$ be an $r$-dimensional vector space, $X = \text{Hom}(V, U)$ and $\pi : X \setminus \{0\} \to \mathbb{P}(X^\vee)$ the natural map. Then $\text{SL}(U)$ acts on $X$ from the left by:
\[ (g \cdot \phi^*)(v) = g \cdot (\phi^*(v)) \quad \text{for } \phi^* \in X, \ v \in V. \]

We see that for $\phi^* \in X$
\[ \phi^* \text{ is } \text{SL}(U)\text{-stable } \iff \text{rank } \phi^* = r, \]
\[ \phi^* \text{ is } \text{SL}(U)\text{-semistable } \iff \phi^* \text{ is } \text{SL}(U)\text{-stable.} \]

In fact, if $\text{rank } \phi^* = r - 1$, then there is a one-parameter torus $T$ of $\text{SL}(U)$ such that the closure of the orbit $T \cdot \phi$ contains the zero vector as the following simple example $(r = 2)$ shows
\[ \lim_{t \to 0} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \lim_{t \to 0} \begin{pmatrix} ta_{11} & ta_{12} & \cdots & ta_{1N} \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \]

Let $X_s$ be the set of all (semi)stable points and $\mathbb{P}_s$ the image of $X_s$ by $\pi$. It is, as we saw above, just the set of all $\phi \in X$ with $\text{rank } \phi^* = r$. Therefore the GIT-orbit space $\mathbb{P}_s / \text{SL}(U)$ is the orbit space $\mathbb{P}_s / \text{SL}(U)$ by the free action, the Grassmann variety $G(r, N)$. 
3.3. Moduli of double coverings of $\mathbb{P}^1$ (1). Let $W$ and $U$ be a pair of two dimensional vector spaces, $X = \text{Hom}(W, S^2 U)$, and $\pi : X \setminus \{0\} \to \mathbb{P}(X^*)$ the natural morphism. Note that $\text{SL}(U)$ acts on $S^2 U$ from the left via the natural action: $\sigma(u_1 u_2) = \sigma(u_1) \sigma(u_2)$ for $\forall u_1, u_2 \in U$. Thus $\text{SL}(U)$ acts on $X$ from the left in the same manner in the subsection 3.2.

Lemma 3.4. Let $\phi^* \in X$.

(i) $\phi^*$ is unstable iff $\phi^*(w)$ has a double root for any $w \in W$.
(ii) $\phi^*$ is semistable iff $\phi^*(w)$ has no double roots for some nonzero $w \in W$.
(iii) $\phi^*$ is stable iff $\phi^*(W)$ is a base-point free linear subsystem of $S^2 U$ on $\mathbb{P}(U)$.

Proof. We note that $\phi^*$ is unstable iff there is a suitable basis $s$ and $t$ of $U$ such that $\phi^*(w) = a(w)s^2$ for any $w \in W$ since a torus orbit $T \cdot \phi^*$ contains the zero vector. This proves (i). This also proves (ii). Next we prove (iii). If $\phi^*(W)$ has a base point, then it is clear that $\phi^*$ is not stable. If $\phi^*$ is semistable and it is not stable, then we choose a basis $s$, $t$ of $U$ and a basis $w_1$, $w_2$ of $W$ such that $\phi^*(w_1) = st$. If $\phi(w_2) = as^2 + bst$, then $\phi^*$ is not stable. This proves the lemma. \qed

Theorem 3.5. Let $X_{ss}$ be the Zariski open subset of $X$ consisting of all semistable points of $X$, $\pi(X_{ss})$ the image of $X_{ss}$ by $\pi$, and $Y := \pi(X_{ss})/\text{SL}(U)$. Then $Y \simeq \mathbb{P}^2$.

Proof. First consider a simplest case. We choose a basis $s$, $t$ of $U$. Let $w_1$ and $w_2$ be a basis of $W$, $T$ the subgroup of $\text{SL}(U)$ of diagonal matrices and $X' = \{\phi^* \in X; \phi^*(w_1) = 2st\}$. Let $Z' = \text{SL}(U) \cdot X'$.

We note that $Z'$ is an $\text{SL}(U)$-invariant subset of $X_{ss}$. We prove $\pi(Z')/\text{SL}(U) \simeq \mathbb{C}^2$. Let $\phi^*$ and $\psi^*$ be points of $X'$. Let $\phi^*(w_2) = As^2 + 2Bst + Ct^2$ and $\psi^*(w_2) = as^2 + bst + ct^2$. Then it is easy to check

$$g \cdot \phi^* = \psi^* \quad \text{for} \quad \exists g \in \text{SL}(U) \iff g \cdot \phi^* = \psi^* \quad \text{for} \quad \exists g \in T$$

$$\iff A = au^2, \quad B = b, \quad C = u^{-2}c \quad \text{for} \quad \exists u \neq 0.$$

Therefore each equivalence class of $\pi(Z)/\text{SL}(U)$ is represented by the pair $\langle AC, B \rangle$, which proves $\pi(Z)/\text{SL}(U) \simeq \mathbb{C}^2$.

Now we prove the lemma. Let $\phi^* \in X_{ss}$, $\phi_j = \phi^*(w_j)$ and $\phi_0 = -(\phi_1 + \phi_2)$. Let

$$\phi_0 = r_1 s^2 + 2r_2 st + r_3 t^2,$$
$$\phi_1 = p_1 s^2 + 2p_2 st + p_3 t^2,$$
$$\phi_2 = q_1 s^2 + 2q_2 st + q_3 t^2,$$

and we define

$$D_1 = p_2^2 - p_1 p_3, \quad D_2 = q_2^2 - q_1 q_3,$$
$$D_0 = r_2^2 - r_1 r_3 = D_1 + D_2 + 2p_2 q_2 - (p_1 q_3 + p_3 q_1).$$

To show the lemma, we prove the more precise isomorphism

$$\pi(X_{ss})/\text{SL}(U) = \text{Proj} \mathbb{C}[D_0, D_1, D_2]$$

For this purpose we define $Y_j = \pi(\{\phi^* \in X_{ss}; \phi_j \text{ has no double roots}\})/\text{SL}(U)$. It suffices to prove $Y_1 = \text{Spec} \mathbb{C}[D_0, D_1, D_2]$ by reducing it to the first simplest case.
Let $\phi^* \in Y_1$. Let $\alpha$ and $\beta$ be the roots of $\phi_1 = 0$. By the assumption $\phi_1$ has no double roots, hence $\alpha \neq \beta$. Let

$$u = \frac{1}{\gamma}(s - \alpha t), \quad v = \frac{1}{\gamma}(s - \beta t), \quad g = \frac{1}{\gamma} \left( \begin{array}{c} 1 - \alpha \\ 1 - \beta \end{array} \right)$$

where $\gamma = \sqrt{\alpha - \beta}$. Note that $g \in \text{SL}(U)$. Hence we see

$$(\phi_1(s, t), \phi_2(s, t)) \equiv (p_1^4 uv, A_1 u^2 + 2B_1 uv + C_1 v^2)$$

where

$$A_1 = q_1^2 \beta^2 + 2q_2 \beta + q_3,$$

$$-B_1 = q_1 \alpha \beta + q_2 (\alpha + \beta) + q_3,$$

$$C_1 = q_1^2 \alpha^2 + 2q_2 \alpha + q_3.$$

Thus we see

$$(\phi_1(s, t), \phi_2(s, t)) \equiv (2st, As^2 + 2Bst + Ct^2)$$

where

$$A = \frac{2A_1}{p_1^4}, \quad B = \frac{2B_1}{p_1^4}, \quad C = \frac{2C_1}{p_1^4}, \quad p_1^4 = 4D_1,$$

$$AC = B^2 - \frac{D_2}{D_1}, \quad B = \frac{D_0 - D_1 - D_2}{2D_1}.$$

Therefore by the first half of the proof

$$Y_1 \simeq \text{Spec } C[AC, B] = \text{Spec } C[\frac{D_0}{D_1}, \frac{D_2}{D_1}].$$

This completes the proof of the lemma. \hfill \Box

**Corollary 3.6.** Let $Y^s = \pi(X_s)\sslash \text{SL}(U)$. Then $Y \setminus Y^s$ is a conic of $Y$ defined by

$$Y \setminus Y^s : D_0^2 + D_1^2 + D_2^2 - 2D_0D_1 - 2D_1D_2 - 2D_2D_0 = 0.$$  

**Proof.** In view of Theorem 3.5, $Y_1 \simeq \text{Spec } C[AC, B]$. The complement of $Y_s$ in $Y_1$ is then the curve defined by $AC = 0$, which is easily identified with the above conic. \hfill \Box

**Corollary 3.7.** Let $X^0$ be the Zariski open subset of $X$ consisting of all semistable points $\phi^*$ of $X$ with rank $\phi^* = 2$, and let $Y^0 := \pi(X^0)\sslash \text{SL}(U)$. Then $Y^0 \simeq \pi(X^0)/\text{SL}(U) \simeq Y \simeq P^2$.

**Proof.** It suffices to compare $Y_1$ and $Y^0 \cap Y_1$. As in the proof of Theorem 3.5 we let $X' = \{\phi^* \in X; \phi^*(w_1) = 2st\}$. Let $Z = \text{SL}(U) \cdot X'$ and $Z^0 = \text{SL}(U) \cdot (X' \cap X^0)$.

Then with the notation in Theorem 3.5, we recall $X' = \{\phi^* \in X; \phi^*(w_1) = 2st, \phi^*(w_2) = As^2 + 2Bst + Ct^2\}$, $\pi(Z)\sslash \text{SL}(U) \simeq \text{Spec } C[AC, B]$ where

$$X' \cap X^0 = \{\phi^* \in X'; A \neq 0 \text{ or } C \neq 0\}.$$  

In the same manner as before we see $\pi(Z^0)\sslash \text{SL}(U) \simeq \text{Spec } C[AC, B]$, whence $\pi(Z^0)\sslash \text{SL}(U) \simeq \pi(Z)\sslash \text{SL}(U)$. This proves $Y^0 \cap Y_1 = Y_1$. This completes the proof of the corollary. \hfill \Box
3.8. Moduli of double coverings of $\mathbf{P}(W)$ (2). There is an alternative way of understanding $\pi(X_{c_s})//\text{SL}(U) \simeq \mathbf{P}^2$ by using the isomorphism $S^2 \mathbf{P}^1 \simeq \mathbf{P}^2$. We use the following convention to denote a point of $\mathbf{P}(U) = U^\vee \setminus \{0\}/G_m$: $(u : v) = u s^v + v t^v \in U^\vee$ where $s^v$ and $t^v$ are a basis dual to $s$ and $t$. In what follows we fix a basis $w_1$ and $w_2$ of $W$. Let $P := (a_1 : a_2)$ and $Q := (b_1 : b_2)$ be a pair of points of $\mathbf{P}(W) \simeq \mathbf{P}^1$. If $P \neq Q$, there is a double covering $\phi : \mathbf{P}(U) \to \mathbf{P}(W)$ ramifying at $P$ and $Q$, unique up to isomorphism once we fix the base $w_1$ and $w_2$:

$$\frac{b_2 w_1 - b_1 w_2}{a_2 w_1 - a_1 w_2} = \left(\frac{s}{t}\right)^2.$$

Thus $\phi$ is given explicitly by

$$\phi_1 := \phi^*(w_1) = b_1 s^2 - a_1 t^2, \quad \phi_2 := \phi^*(w_2) = b_2 s^2 - a_2 t^2, \quad \phi_0 = -(\phi_1 + \phi_2)$$

for which we have

$$D_1 = a_1 b_1, \quad D_2 = a_2 b_2, \quad D_0 = (a_1 + a_2)(b_1 + b_2).$$

The isomorphism $S^2 \mathbf{P}^1 \simeq \mathbf{P}^2$ is given by $(P, Q) \mapsto (D_0, D_1, D_2)$. This shows

**Corollary 3.9.** We have a natural isomorphism: $Y \simeq \mathbf{P}(S^2 W)$.

4. The virtual normal bundle of a double covering

4.1. The case $N = 7$ and $k = 8$ revisited. We revisit the example in the subsection 2.9. Let $N = 7$ and $k = 8$. Let $L : x_j = 0$ ($j \geq 3$) and we take

$$F_3 = 8x_1^7, \quad F_4 = 8x_1^6x_2, \quad F_5 = 8x_1^5x_2^2, \quad F_6 = 8x_1^4x_2^3, \quad F_7 = 8x_1^3,$$

$$F = x_3 F_3 + x_4 F_4 + x_5 F_5 + x_6 F_6 + x_7 F_7 + x_8^2 + x_9 + x_9^2 + x_9 + x_9^2 + x_9^2 + x_9^2.$$

and let $M = M_0^5 : F = 0$. We often denote $L$ also by $\mathbf{P}(W)$ with $W$ a two dimensional vector space for later convenience. Since $H^0(D_L^-)$ is injective and $H^0(D_L)$ is surjective, we have $N_{L/M} \simeq O_L \oplus O_L(-1)^{83}$. Hence $H^1(N_{L/M}(-1)) = H^1(O_L(-2)^{83})$ is 3-dimensional. As we see easily, this follows also from the fact that $\text{Coker} H^0(D_L^-)$ is freely generated by $x_1^2 x_2^2, x_1^2 x_2^3$ and $x_1 x_2^4$. Let $\phi^* = (\phi_1, \phi_2) \in X^0$. Then $Ker H^0(\phi^* D_L)$ is generated by a single element $\phi_2^* \phi_3^* - \phi_1^* \phi_4^*$, while $\text{Coker} H^0(\phi^* D_L)$ is generated by $S^2 U \cdot \phi_1^* \phi_2^*, S^2 U \cdot \phi_3^* \phi_4^*$ and $S^2 U \cdot \phi_1 \phi_2^*$. To be more precise, we see

$$\text{Coker} H^0(\phi^* D_L) = \{\phi_1^* \phi_2^*, \phi_3^* \phi_4^*, \phi_1^* \phi_2^*\} \otimes S^2 U / \{\phi_1, \phi_2\}.$$ 

In fact, this is proved as follows: first we consider the case where $\phi_1$ and $\phi_2$ has no common zeroes. In this case $\phi^*$ gives rise to a double covering $\phi : \mathbf{P}(U) \to \mathbf{P}(W)$ ($= L$), which we denote by $L_\phi$ for brevity. By pulling back by $\phi^*$ the normal sequence $0 \to N_{L/M} \to N_{L/P} \to O_L(k) \to 0$ ($k = 8$) for the line $L$ we infer an exact sequence

$$0 \to \phi^* N_{L/M} \to \phi^* N_{L/P} \overset{\phi^* D_L}{\to} \phi^* O_L(k) \to 0,$$

which yields an exact sequence

$$0 \to H^0(\phi^* N_{L/M}) \to S^2 U \circ (V^\vee \otimes W^\vee) \overset{H^0(\phi^* D_L)}{\to} H^0(O_L(k)) \to 0.$$
Let \( \eta = q_3 \phi_3^\gamma + \cdots + q_r \phi_r^\gamma \in \text{Ker} \, H^0(\phi^*D_L), \, q_j \in S^2 U \). Then we have

\[
\phi_1^2(q_3 \phi_3^\gamma + q_4 \phi_4^\gamma + q_5 \phi_5^\gamma + q_6 \phi_6^\gamma) = -q_7 \phi_7^\gamma.
\]

Since \( \phi_1 \) and \( \phi_2 \) are mutually prime and \( q_j \) is of degree two, we have \( q_7 = 0 \) and \( q_6 = 0 \). Hence we infer also \( q_6 = 0 \). Thus we have \( q_3 \phi_1 + q_4 \phi_2 = 0 \).

This proves that \( \text{Ker} \, H^0(\phi^*D_L) \) is generated by \( \phi_2 \phi_2^\gamma - \phi_1 \phi_1^\gamma \).

Next we prove that \( \text{Coker} \, H^0(\phi^*D_L) \) is generated by \( \phi^* \text{Coker} \, H^0(D_L^-) \) over \( S^2 U \), in fact over \( S^2 U/\phi^*(W) \). Without loss of generality we may assume that \( \phi_1 = 2st \) and \( \phi_2 = \lambda s^2 + 2\nu st + t^2 \) for some \( \lambda \neq 0 \) and \( \nu \in \mathbb{C} \). Let \( \phi^*W = \{ \phi_1, \phi_2 \} \). Then one checks \( U \cdot \phi^*W = S^2 U \), and hence \( S^2 U \cdot \phi^*W = S^4 U \). Therefore \( S^2 U \cdot \phi^*(S^m-1\, W) = S^{2m} U \) for \( m \geq 1 \). In fact, by the induction on \( m \)

\[
S^2 U \cdot \phi^*(S^m\, W) = S^2 U \cdot \phi^*(W) \cdot \phi^*(S^{m-1}\, W) = S^4 U \cdot \phi^*(S^{m-1}\, W) = S^2 U \cdot (S^2 U \cdot \phi^*(S^{m-1}\, W)) = S^2 U \cdot S^{2m} U = S^{2m+2} U.
\]

Therefore \( H^0(O_{Lx}(2k)) = S^{16} U = S^2 U \cdot \phi^*(S^7 W) \). Hence

\[
\text{Coker} \, H^0(\phi^*D_L) = S^{16} U / \text{Im} \, H^0(\phi^*D_L) = S^2 U \cdot \phi^*(S^7 W) / S^2 U \cdot \phi^*(\text{Im} \, H^0(D_L^-)) = (S^2 U / \phi^*(W)) \cdot \phi^*(S^7 W / \text{Im} \, H^0(D_L^-)).
\]

because \( \text{Coker} \, H^0(D_L^-) = S^7 W / \text{Im} \, H^0(D_L^-) \) and \( W \cdot S^7 W \subset W \cdot \text{Im} \, H^0(D_L^-) = S^8 W \) by the choice of \( L \). This proves that \( \text{Coker} \, H^0(\phi^*D_L) \) is generated by \( \phi^* \text{Coker} \, H^0(D_L^-) \) over \( S^2 U / \phi^*(W) \). It follows \( \text{Coker} \, H^0(\phi^*D_L) = (\phi^* \text{Coker} \, H^0(D_L^-)) \oplus (S^2 U / \phi^*(W)) \).

Finally we consider the case where \( \phi_1 \) and \( \phi_2 \) has a common zero. In this case we may assume \( \phi_1 = 2st \) and \( \phi_2 = 2\nu st + t^2 \). In this case \( L_\phi \) is a chain of two rational curves \( C_\phi^l \) and \( C_\phi^r \) where \( C_\phi \) is the proper transform of \( \text{P}(U) \), where the double covering map from \( L_\phi \) to \( \text{P}(W) \) is the union of the isomorphisms \( \phi' \) and \( \phi'' \), say, \( \phi = \phi' \cup \phi'' \). Let \( \psi_1 = 2s \) and \( \psi_2 = 2\nu s + t \). Then \( \phi' \) is induced by the isomorphism \( (\phi')^* \in \text{Hom}(W, U) \) such that \( (\phi')^*(w_j) = \psi_j \). On the other hand, let \( U_\phi^p = C\lambda + Ct \), \( \psi_1^p = 2t \) and \( \psi_2^p = \lambda + 2\nu t \) where we note \( \psi_j^p \) is the linear part of \( \phi_j \) in \( t \) with \( s = 1 \). Then \( C_\phi^p = \text{P}(U_\phi^p) \) and \( \phi'' \) is induced by the isomorphism \( (\phi'')^* \in \text{Hom}(W, U_\phi^p) \) such that \( (\phi'')^*(w_j) = \psi_j^p \). Furthermore the pull back by \( \phi^* \) of the normal sequence for \( L \)

\[
0 \to \phi^*N_{L/M} \to \phi^*N_{L/P} \xrightarrow{\phi^*D_L} \phi^*O_L(k) \to 0,
\]
yields exact sequences with natural vertical homomorphisms:

\[
\begin{array}{cccccc}
0 & \rightarrow & \phi^*N_{L/M} & \rightarrow & (\phi')^*N_{L/M} \oplus (\phi'')^*N_{L/M} & \rightarrow & C & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \phi^*N_{L/P} & \rightarrow & (\phi')^*N_{L/P} \oplus (\phi'')^*N_{L/P} & \rightarrow & V^\vee/W^\vee & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \phi^*O_L(k) & \rightarrow & O_{C_0}(k) \oplus O_{C_0}(k) & \rightarrow & C & \rightarrow & 0.
\end{array}
\]

This yields the following long exact sequences:

\[
\begin{array}{cccc}
0 & \rightarrow & H^0((\phi')^*N_{L/M}) & \rightarrow & U \otimes V^\vee/W^\vee & \rightarrow & H^0((\phi')^*D_L) & \rightarrow & S^kU \\
& & \rightarrow & H^1((\phi')^*N_{L/M}) & \rightarrow & 0 \\
0 & \rightarrow & H^0((\phi'')^*N_{L/M}) & \rightarrow & U'' \otimes V^\vee/W^\vee & \rightarrow & H^0((\phi'')^*D_L) & \rightarrow & S^kU'' \\
& & \rightarrow & H^1((\phi'')^*N_{L/M}) & \rightarrow & 0
\end{array}
\]

whence \( H^1((\phi')^*N_{L/M}) = H^1((\phi'')^*N_{L/M}) = 0 \), and both \( H^0((\phi')^*N_{L/M}) \) and \( H^0((\phi'')^*N_{L/M}) \) are one-dimensional. Let \( U^t \) be the subspace of \( U \) consisting of elements vanishing at \( C_0 \cap C_0', \) namely the subspace spanned by \( t \). Then the restriction of \( H^0((\phi')^*D_L) \) to \( U^t \otimes V^\vee/W^\vee \) equals \( t \cdot H^0((\phi')^*D_L^-) \). Hence

\[
\text{Coker } H^0(\phi^*D_L) \simeq t \cdot S^2U/t \cdot \text{Im } H^0((\phi')^*D_L^-) \oplus \text{Coker } H^0((\phi'')^*D_L) \\
\simeq S^2U/t \cdot \text{Im } H^0((\phi')^*D_L^-) \simeq \text{Coker } H^0((\phi')^*D_L^-).
\]

One could understand the above isomorphism as

\[\text{Coker } H^0(\phi^*D_L) = \text{Coker } (\phi^*H^0(D_L^-)) \oplus (S^2U/\phi^*W).\]

Thus \( H^0(\phi^*N_{L/M}) \) is one-dimensional, while \( H^1(\phi^*N_{L/M}) \) is 3-dimensional. This is immediately generalized into the following

**Lemma 4.2.** For any \( \phi^* \in X^0 \) we have

\[
\begin{array}{c}
\text{Ker } H^0(\phi^*D_L) = \phi^* \text{Ker } H^0(D_L), \\
\text{Coker } H^0(\phi^*D_L) = (\phi^* \text{Coker } H^0(D_L^-)) \oplus (S^2U/\phi^*W).
\end{array}
\]

**Lemma 4.3.** We define a line bundle \( L_0 \) (resp. \( L_1 \)) on \( Y \) \((\simeq \mathbb{P}(S^2W))\) by the assignment:

\[ X^0 \ni \phi^* \mapsto \phi^* \text{Ker } H^0(D_L) \quad \text{(resp. } \phi^* \text{Coker } H^0(D_L^-)). \]

Then \( L_k \simeq O_{\mathbb{P}(S^2W)}. \)

**Proof.** We know that \( \phi^* \text{Ker } H^0(D_L) \) is generated by \( \phi_2c_3^Y - \phi_1c_4^Y. \) By the \( \text{SL}(2) \)-variable change of \( s \) and \( t, \phi_j \) is transformed into a new quadratic polynomial, which is however the same as the first \( \phi_j. \) This shows the generator is unchanged, whence \( L_0 \simeq O_{\mathbb{P}(S^2W)}. \) The proof for \( L_1 \) is the same. \( \square \)
Lemma 4.4. We define a coherent sheaf $L$ on the stack $Y (\simeq P(S^2W))$ (See Remark below) by the assignment:

$$X^0 \ni \phi \mapsto S^2U/\phi^*W.$$ 

Then $L^2 \simeq O_{P(S^2W)}(-1)$.

Proof. The GIT-quotient $Y^0$ is covered with the images of $X'_i$:

$$X'_1 = \{(\phi_1, \phi_2) \in X^0; \phi_1 = 2st, \phi_2 = \lambda s^2 + 2\nu st + t^2, \lambda, \nu \in \mathbb{C} \},$$  

$$X'_2 = \{(\phi_1, \phi_2) \in X^0; \phi_1 = ps^2 + 2qst + t^2, \phi_2 = 2st, p, q \in \mathbb{C} \}.$$  

It is clear that the natural image of $X'_i$ in $Y$ is $Y'_j$. The map $\phi$ given by $\phi^* = (\phi_1, \phi_2) \in Y_1$ has natural $Z_2$ involution generated by,

$$r : \ (\sqrt{\lambda} s + t, \sqrt{\lambda} s - t) \rightarrow (\sqrt{\lambda} s + t, -(\sqrt{\lambda} s - t)).$$

Since

$$2st = \frac{1}{2\sqrt{\lambda}}((\sqrt{\lambda} s + t)^2 - (\sqrt{\lambda} s - t)^2),$$
$$\lambda s^2 + 2\nu st + t^2 = \frac{\nu}{2\sqrt{\lambda}}((\sqrt{\lambda} s + t)^2 - (\sqrt{\lambda} s - t)^2) + \frac{1}{2}((\sqrt{\lambda} s + t)^2 + (\sqrt{\lambda} s - t)^2),$$

it is clear that,

$$r^* (\phi_1) = \phi_1, \ r^* (\phi_2) = \phi_2, \ r^* (\lambda s^2 - t^2) = - (\lambda s^2 - t^2).$$

Therefore, we can decompose $S^2U$ into $\langle \lambda s^2 - t^2 \rangle_C \oplus \langle \phi_1, \phi_2 \rangle_C$ with respect to eigenvalue of $r^*$ and take $\lambda s^2 - t^2$ as canonical generator of $S^2U/\phi^*W$. Similarly $S^2U/\phi^*W$ is generated by $ps^2 - t^2$ on $Y_2$. The problem is therefore to write $\lambda s^2 - t^2$ as an $\Gamma(O_{Y_1} \cdot Y_2)$-multiple of $pu^2 - v^2$ when we write $\phi_2 = 2uv$ by a variable change in $GL(2)$. The following variable change $(s, t) \mapsto (u, v)$ is in $GL(2)$:

$$s = \frac{\sqrt{2\alpha}}{(\beta - \alpha)^2}(2u - \frac{(\beta - \alpha)^2}{2\alpha}v), \ t = \frac{\sqrt{2\alpha}}{(\beta - \alpha)^2}(2\beta u - \frac{(\beta - \alpha)^2}{2\alpha}v),$$

where $\alpha, \beta$ are roots of the equation $\lambda s^2 + 2\nu st + t^2 = 0$. Under this coordinate change, $\phi_1$ and $\phi_2$ are rewritten as follows:

$$\phi_1 = \frac{\lambda}{(\nu^2 - \lambda)^2}u^2 + 2\frac{\nu}{\nu^2 - \lambda}uv + v^2 = pu^2 + 2qv + v^2, \ \phi_2 = 2uv.$$  

Then we have

$$pu^2 - v^2 = -\frac{2}{(\beta - \alpha)}(\lambda s^2 - t^2) = -\frac{1}{\sqrt{\nu^2 - \lambda}}(\lambda s^2 - t^2) = -\frac{D_1}{D_2}(\lambda s^2 - t^2).$$

Similarly by computing the effect on $S^2U/\phi^*W$ by the variable change from $X'_1$ into $X'_0$, we see that $L^2$ is isomorphic to $O_{P(S^2W)}(-1)$. This completes the proof.

Remark 4.5. We remark that the space $X$ must be regarded as a $\mathbb{Q}$-stack $Y^{\text{stack}}$ as follows: First we define $\phi_0 = -\phi_1 - \phi_2$. For each atlas $X'_\alpha$ we define an atlas $Y^{\text{stack}}_\alpha$.
\((\alpha = 0, 1, 2)\) by

\[
Y^\text{stack}_0 = \{(\phi_0, \phi_1, \phi_2, \pm \psi_0) \in X^0 \times S^2 U; \phi_0 = 2st, \phi_1 = as^2 + 2bst + t^2, \\
\psi_0 = as^2 - t^2 a, b \in C\},
\]

\[
Y^\text{stack}_1 = \{(\phi_0, \phi_1, \phi_2, \pm \psi_1) \in X^0 \times S^2 U; \phi_1 = 2st, \phi_2 = \lambda s^2 + 2\nu st + t^2, \\
\psi_1 = \lambda s^2 - t^2 \lambda, \nu \in C\},
\]

\[
Y^\text{stack}_2 = \{(\phi_0, \phi_1, \phi_2, \pm \psi_2) \in X^0 \times S^2 U; \phi_1 = ps^2 + 2qst + t^2, \phi_2 = 2st, \\
\psi_2 = ps^2 - t^2 p, q \in C\}.
\]

Since \(L^2 \simeq O_{P(S^2 W)}(-1)\) we have \(c_1(L) = -\frac{1}{2} c_1(O_{P(S^2 W)}(1))\) in the Chow ring \(A(Y^\text{stack})_Q = A(X)_Q = A(P(S^2 W))_Q\).

5. Proof of the main theorem

**Theorem 5.1.**

\[
\pi_* (c_{\text{top}}(H^1)) = \frac{1}{8} \left[ \frac{c(S^k - 1)}{1 - \frac{1}{2} c_1(Q)} \right]_{k-N},
\]

where \(\pi\) is the natural projection from \(\tilde{M}_{0,0}(L, 2)\) to \(G\) and \([*]_{k-N}\) is the operation of picking up the degree \(2(k - N)\) part of Chern classes.

**Proof.** From now on we denote the coherent sheaf \(L\) in Lemma 4.4 by \(O_P(-\frac{1}{2})\). In view of the results from the previous section, what remains is to evaluate the top chern class of \((S^k - 1)/(\nu^\vee \otimes O_G)/(\nu^\vee)\) \(\otimes O_P(-\frac{1}{2})\) on \(P(S^2 Q)\). Since double cover maps parametrized by \(P(S^2 Q)\) have natural \(Z_2\) involution \(r\) given in the previous section, we have to multiply the result of integration on \(P(S^2 Q)\) by the factor \(\frac{1}{2}\) [BT], [FP]. With this set-up, let \(\pi' : P(S^2 Q) \rightarrow G\) be the natural projection. Then what we have to compute is \(\pi_* (c_{\text{top}}(H^1)) = \frac{1}{2} \pi'_* (c_{\text{top}}(H^1)) = \frac{1}{2} \pi'_* (c_{\text{top}}((S^k - 1)/(\nu^\vee \otimes O_G)/(\nu^\vee)) \otimes O_P(-\frac{1}{2}))\). Let \(z\) be \(c_1(O_P(1))\). Then we obtain,

\[
\frac{1}{2} \pi'_* (c_{\text{top}}((S^k - 1)/(\nu^\vee \otimes O_G)/(\nu^\vee)) \otimes O_P(-\frac{1}{2}))
\]

\[
= \frac{1}{2} \sum_{j=0}^{k-N+2} c_{k-N+2-j}(S^k - 1 \oplus \nu^\vee) \cdot \pi'_*(z^j) \cdot (-\frac{1}{2})^j
\]

\[
= \frac{1}{8} \sum_{j=0}^{k-N} c_{k-N-j}(S^k - 1 \oplus \nu^\vee) \cdot s_j(S^2 Q) \cdot (-\frac{1}{2})^j
\]

\[
= \frac{1}{8} \left[ \frac{c(S^k - 1) \cdot c(Q^\vee)}{1 - \frac{1}{2} c_1(S^2 Q) + \frac{1}{3} c_2(S^2 Q) - \frac{1}{6} c_3(S^2 Q)} \right]_{k-N},
\]

where \(s_j(S^2 Q)\) is the \(j\)-th Segre class of \(S^2 Q\). But if we decompose \(c(Q)\) into \((1 + \alpha)(1 + \beta)\), we can easily see,

\[
\frac{c(Q^\vee)}{1 - \frac{1}{2} c_1(S^2 Q) + \frac{1}{3} c_2(S^2 Q) - \frac{1}{6} c_3(S^2 Q)} = \frac{(1 - \alpha)(1 - \beta)(1 - \frac{1}{2}(\alpha + \beta)(1 - \beta))}{1 - \frac{1}{2} c_1(Q)}.
\]
Finally, by combining the above theorem with the divisor axiom of Gromov-Witten invariants, we can prove the decomposition formula of degree 2 rational Gromov-Witten invariants of $M^k_V$ found from numerical experiments.

**Corollary 5.2.**

$$\langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,2} = \langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,2} + 8(\pi_*(c_{top}(H^1))O_{e^a} O_{e^b} O_{e^c})_{0,1},$$

where $\langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,2}$ is the number of conics that intersect cycles Poincaré dual to $e^a$, $e^b$ and $e^c$. We also denote by $\langle \pi_*(c_{top}(H^1))O_{e^a} O_{e^b} O_{e^c} \rangle_{0,1}$ the integral:

$$\int_{G(2,V)} c_{top}(S^k Q) \land \pi_*(c_{top}(H^1)) \land \sigma_{a-1} \land \sigma_{b-1} \land \sigma_{c-1}.$$

### 6. Generalization to Twisted Cubics

In this section, we present a decomposition formula of degree 3 rational Gromov-Witten invariants found from numerical experiments using the results of [ES].

**Conjecture 6.1.** If $k - N = 1$, we have the following equality:

$$\pi_*(c_{top}(H^1)) = \frac{1}{27} \left( \left( \frac{1}{24} (27k^2 - 55k + 26)k(k - 1) + \frac{2}{9} c_1(Q)^2 + \left( \frac{7}{6} (k + 1)k(k - 1) + \frac{1}{9} c_2(Q) \right) \right).$$

where $\pi : \overline{M}_{0,0}(L, 3) \to \overline{M}_{0,0}(M^k_V, 1)$ is the natural projection.

In the $k-N > 1$ case, we have not found the explicit formula, because in the $d = 3$ case, we have another contribution from multiple cover maps of type $(2+1) \to (1+1)$. Here multiple cover map of type $(2+1) \to (1+1)$ is the map from nodal curve $P^1 \cup P^1$ to nodal conic $L_1 \cup L_2 \subset M^k_N$, that maps the first (resp. the second) $P^1$ to $L_1$ (resp. $L_2$) by two to one (resp. one to one). In the $k-N = 1$ case, we have also determined the contributions from multiple cover maps of $(2+1) \to (1+1)$ to nodal conics.

**Corollary 6.2.** If $k - N = 1$, $\langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,3}$ is decomposed into the following contributions:

$$\langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,3} = \langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,3} + \frac{1}{k} \left( \left( \frac{9}{4} \langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,1} \langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,1} + \frac{3}{2} \langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,1} \langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,1} + \frac{3}{2} \langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,1} \langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,1} + \frac{27}{2} \pi_*(c_{top}(H^1))O_{e^a} O_{e^b} O_{e^c} \rangle_{0,1},
$$

where $\langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,3} = 3$ is the number of twisted cubics that intersect cycles Poincaré dual to $e^a$, $e^b$ and $e^c$.

**Proof.** In the $k-N = 1$ case, dimension of moduli space of multiple cover maps of $(2+1) \to (1+1)$ to nodal conics is given by $N-6+N-6-(N-4)+2 = N-6$, hence the rank of $H^4$ is given by $N-6-(N-5-3) = 2$. On the other hand, dimension of moduli space of $d = 2$ multiple cover maps of $P^1 \to P^1$ is 2, the degree of the form of $\pi_*(c_{top}(H^1))$ equals to $2 - 2 = 0$, where $\pi$ is the projection map.
that projects out the fiber locally isomorphic to the moduli space of $d = 2$ multiple cover maps. This situation is exactly the same as the Calabi-Yau case. Therefore, we can use the well-known result by Aspinwall and Morrison, that says for $n$-point rational Gromov-Witten invariants for Calabi-Yau manifold, $\pi_*(c_{\text{top}}(H^1))$ for degree $d$ multiple cover map is given by,

$$\pi_*(c_{\text{top}}(H^1)) = \frac{1}{d^{2n}}.$$ 

With this formula, we add up all the combinatorial possibility of insertion of external operator $O_{\mathcal{E}}$, $O_{\mathcal{E}'}$ and $O_{\mathcal{E}''}$,

$$\frac{1}{k} \left( \langle \pi_*(c_{\text{top}}(H^1))O_{\mathcal{E}}O_{\mathcal{E}'}O_{\mathcal{E}''}O_{\mathcal{E}'''} O_{\mathcal{E}''''} O_{\mathcal{E}'''''} O_{\mathcal{E}''''''}, O_{\mathcal{E}''''''}, 0, 1, \langle O_{\mathcal{E}''''''}, \pi_*(c_{\text{top}}(H^1)) \rangle \rangle \right)$$

$$= \frac{1}{k} \left( 2 \langle O_{\mathcal{E}}O_{\mathcal{E}'}O_{\mathcal{E}''}O_{\mathcal{E}'}, O_{\mathcal{E}''}, 0, 1, \langle O_{\mathcal{E}''}, \pi_*(c_{\text{top}}(H^1)) \rangle \rangle \right)$$

$$+ \langle O_{\mathcal{E}}O_{\mathcal{E}'}O_{\mathcal{E}''}O_{\mathcal{E}'}, O_{\mathcal{E}''}, 0, 1, \langle O_{\mathcal{E}''}, \pi_*(c_{\text{top}}(H^1)) \rangle \rangle \right)$$

The last expression is nothing but the formula we want.

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