CONICS ON A GENERIC HYPERSURFACE

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Abstract. In this paper, we compute the contributions from double cover maps to genus 0 degree 2 Gromov-Witten invariants of general type projective hypersurfaces. Our results correspond to a generalization of Aspinwall-Morrison formula to general type hypersurfaces in some special cases.

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1. Introduction

In this paper, we discuss a generalization of the multiple cover formula for rational Gromov-Witten invariants of Calabi-Yau manifolds [AM], [M] to double cover maps of a line L on a degree k hypersurface $M^k_N$ in $\mathbb{P}^{N-1}$. Naively, for a given finite set of elements $\alpha_j \in H^*(M^k_N, \mathbb{Z})$, the rational Gromov-Witten invariant $(\mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \cdots \mathcal{O}_{\alpha_n})_{0,d}$ of $M^k_N$ counts the number of degree $d$ (possibly singular and reducible) rational curves on $M^k_N$ that intersect real sub-manifolds of $M^k_N$ that are Poincaré-dual to $\alpha_j$.

Recently, the mirror computation of rational Gromov-Witten invariants of $M^k_N$ with negative first Chern class $(k-N>0)$ was established in [CG, [IR], [J]]. Using the method presented in these articles, we can compute $(\mathcal{O}_{\alpha_{e-1}} \mathcal{O}_{\alpha_{e-2}} \cdots \mathcal{O}_{\alpha_{e-n}})_{0,d}$ where $e$ is the generator of $H^{1,1}(M^k_N, \mathbb{Z})$. Briefly, mirror computation of $M^k_N$ $(k > N)$ in [J] goes as follows. We start from the following ODE:

\[ (\partial_x)^{N-1} - k \cdot \exp(x) \cdot (k\partial_x + k - 1)(k\partial_x + k - 2) \cdots (k\partial_x + 1) \cdot w(x) = 0, \]

and construct the virtual Gauss-Manin system associated with (1):

\[ \partial_x \tilde{\psi}_{N-2-m}(x) = \tilde{\psi}_{N-1-m}(x) + \sum_{d=1}^{\infty} \exp(dx) \cdot \tilde{L}^{N,k \cdot d} \cdot \tilde{\psi}_{N-1-m-(N-k)d}(x), \]

where $m$ runs through all the integers and $\tilde{L}^{N,k \cdot d}$ is non-zero only if $0 \leq m \leq N - 1 + (k - N)d$. From the compatibility of (1) and (2), we can derive the recursive formulas that determine all the $\tilde{L}^{N,k \cdot d,e}$:

\[ \tilde{L}^{N,k \cdot d,e}_{m} = k \cdot \prod_{j=1}^{k-1} (jw + (k - j)) \cdot \prod_{j=1}^{k-1} \left( \prod_{l=2}^{d} (-1)^l \sum_{0 = i_0 < i_1 < \cdots < i_d} \right) \times \]

\[ \sum_{m=0}^{N - 1 + (k - N)d} \cdot \sum_{j_0=0}^{j_a} \cdots \sum_{j_d=0}^{j_a} \left( \prod_{n=1}^{d} \frac{i_n - 1 + (d - i_n - 1)z}{d} \right) \tilde{L}^{N,k \cdot j_a -(N-k)j_{a-1}} \]
With these data, we can construct the formulas that represent rational three point Gromov-Witten invariant \( \langle \mathcal{O}_e \mathcal{O}_{e,N-2-m} \mathcal{O}_{e,m-1-(k-N)d} \rangle_d \) in terms of \( \tilde{L}_m^N k,d \). These three point Gromov-Witten invariants are enough for reconstruction of all the rational Gromov-Witten invariants \( \langle \mathcal{O}_{e,1} \mathcal{O}_{e,2} \cdots \mathcal{O}_{e,m} \rangle_{0,d} \) [KM]. In particular, we obtain the following formula in the \( d = 2 \) case:

\[
(3) \quad \langle \mathcal{O}_e \mathcal{O}_{e,N-2-m} \mathcal{O}_{e,m-1-(k-N)d} \rangle_2 = k \cdot \left( \tilde{L}_m^N k,2 - \tilde{L}_{1+2(k-N)}^N k,1 - 2 \tilde{L}_{1+2(k-N)}^N k,1 \sum_{j=0}^{k-N} (\tilde{L}_{n-j}^N - \tilde{L}_{1+2(k-N)-j}^N k,1) \right).
\]

According to the results of this procedure, rational three point Gromov-Witten invariants can be rational numbers with large denominator if \( k > N \), in contrast to the Calabi-Yau case where rational three point Gromov-Witten invariants are always integers.

One of the reasons of this rationality (non-integrality) comes from the contributions of multiple cover maps to Gromov-Witten invariants. In the Calabi-Yau case \( (N = k) \), for any divisor \( m \) of \( d \) there are some contributions from degree \( m \) multiple cover maps \( \phi \) of a rational curve \( \mathbb{P}^1 \) onto a degree \( \frac{d}{m} \) rational curve \( C \hookrightarrow M^k_\beta \). The contributions from the multiple cover maps are expressed in terms of the virtual fundamental class of Gromov-Witten invariants. Let \( C \) be a general degree \( d \) rational curve in \( M^k_\beta \). Its normal bundle \( N_{C/M^k_\beta} \) is decomposed into a direct sum of line bundles as follows:

\[
N_{C/M^k_\beta} \simeq O_C(-1) \oplus O_C(-1) \oplus O_C^{\oplus(k-5)}.
\]

Let \( \phi : \mathbb{P}^1 \to C \) be a holomorphic map of degree \( m \). Since the pull-back \( \phi^*(N_{C/M^k_\beta}) \) is given by

\[
\phi^*(N_{C/M^k_\beta}) \simeq O_{\mathbb{P}^1}(-m) \oplus O_{\mathbb{P}^1}(-m) \oplus O_{\mathbb{P}^1}^{\oplus k-5},
\]

we obtain \( h^1(\phi^*(N_{C/M^k_\beta})) = 2m - 2 \). On the other hand, let \( \overline{M}_{0,0}(M,d) \) be the moduli space of 0-pointed stable maps of degree \( d \) from genus 0 curve to \( M \). Then the moduli space of \( \phi \) is the fiber space \( \pi : \overline{M}_{0,0}(C,m) \to \overline{M}_{0,0}(M^k, \frac{d}{m}) \), whose fibre \( \overline{M}_{0,0}(C,m) \) over \( C \) (fixed) has complex dimension \( 2m - 2 \). Then the push-forward of the virtual fundamental class \( \pi_*(c_{top}(\mathcal{H}^1(\phi^*(N_{C/M^k_\beta})))) \) can be computed only by intersection theory on the fiber \( \overline{M}_{0,0}(C,m) \), which turns out to be equal to \( \frac{1}{m} \). This depends on neither the structure of the base \( \overline{M}_{0,0}(M^k, \frac{d}{m}) \) nor the global structure of the fibration.

But when \( k < N \), the situation is more complicated than \( M^k_\beta \) because of negative first Chern class. Let us concentrate on the case of \( d = 2, m = 2 \) that we discuss in this paper. In this case, \( C \) is just a line \( L \) on the hypersurface \( M^k_\beta \). The moduli space \( \overline{M}_{0,0}(M^k_\beta, 1) \) is a sub-manifold of \( \overline{M}_{0,0}(\mathbb{P}^{N-1}, 1) \), while \( \overline{M}_{0,0}(\mathbb{P}^{N-1}, 1) \) is the Grassmannian \( G(2,N) \), the moduli space of rank 2 quotients of \( V = \mathcal{O}^N \). As will be shown later, for a generic line \( L \), \( N_{L/M^k_\beta} \) is decomposed into

\[
N_{L/M^k_\beta} \simeq O_L(-1)^{\oplus k-N+2} \oplus O_L^{\oplus 2N-k-5}.
\]
By pulling back it by the degree 2 map $\phi : \mathbb{P}^1 \to L$, we obtain,

$$\phi^* N_{L/M^k_N} \simeq O_{\mathbb{P}^1}(-2)^{k-N+2} \oplus O_{\mathbb{P}^1}\otimes^{\oplus 2N-k-5}.$$ 

Therefore, $h^1(\phi^*(N_{L/M^k_N})) = k-N+2$, which is strictly greater than two, the complex dimension of the fiber $\overline{\mathcal{M}}_{0,0}(L,2)$. Thus we need to know the global structure of the fibration $\pi$ in order to compute the multiple cover contribution to degree 2 rational Gromov-Witten invariants of $M^k_N$.

In order to estimate the contributions from double cover maps $\phi : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,0}(L,2)$, we first computed the number of conics, that intersect cycles Poâncaré dual to $e^a$, $e^b$ and $e^c$, on $M^k_N$ (whose normal bundle are of the same type) by using the method in [K2]. Then we found the following formula by comparing these integers with the results obtained from (3):

$$\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,2} = (\text{number of corresponding conics}) +$$

$$\int_{G(2,N)} c_{top}(S^k Q) \wedge \left[ \frac{c(S^k-1)}{1 - \frac{1}{2} c_1(Q)} \right]^{k-N} \wedge \sigma_{a-1} \wedge \sigma_{b-1} \wedge \sigma_{c-1},$$

where $Q$ is the universal rank 2 quotient bundle of $G(2,N)$, $\sigma_a$ is a Schubert cycle defined by $\sum_{a=0}^N \sigma_a := \frac{1}{\prod_{j=0}^{N-1} j!}$ and $[\ast]_{k-N}$ is the operation of picking up degree $2(k-N)$ part of Chern classes.

On the other hand, we have the following formula which directly follows from the definition of the virtual fundamental class of $\overline{\mathcal{M}}_{0,0}(M^k_N,2)$:

$$\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,2} = (\text{number of corresponding conics}) +$$

$$8 \int_{G(2,N)} c_{top}(S^k Q) \wedge \pi_*(c_{top}(H^1(\phi^* N_{L/M^k_N})))^{k-N} \wedge \sigma_{a-1} \wedge \sigma_{b-1} \wedge \sigma_{c-1},$$

where $\pi : \overline{\mathcal{M}}_{0,0}(L,2) \to \overline{\mathcal{M}}_{0,0}(M^k_N,1)$ is the natural projection. Here, the factor 8 comes from the divisor axiom of Gromov-Witten invariants.

In this paper, we prove the following formula

$$\pi_*(c_{top}(H^1(\phi^* N_{L/M^k_N}))) = \frac{1}{8} \left[ \frac{c(S^k-1)}{1 - \frac{1}{2} c_1(Q)} \right]^{k-N}.$$ 

By combining (5) with (6), we can derive the formula (4) immediately.

From (4), we see that $\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,2}$ of $M^k_N$ is a rational number with denominator at most $2^{k-N}$. Therefore rationality (non-integrality) of the Gromov-Witten invariant $\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,2}$ is caused by the effect of multiple cover map in this case.

We note here that the total moduli space of double cover maps of lines is isomorphic to $\mathbb{P}(S^2 Q)$ over $G := \overline{\mathcal{M}}_{0,0}(M^k_N,1) \hookrightarrow G(2,N)$, which is an algebraic $\mathbb{Q}$-stack $\mathbb{P}(S^2 Q)^{stack}$ (in the sense of Mumford). As a consequence, the union of all $H^1(\phi^* N_{L/M^k_N})$ turns out to be a coherent sheaf on $\mathbb{P}(S^2 Q)^{stack}$ with fractional Chern class in (6), as was suggested in [BT]. See [V, Section 9].

We also did some numerical experiments on degree 3 Gromov-Witten invariants of $M^k_N$ by using the results of [ES]. For $k-N > 0$, there is a new contribution from multiple cover maps to nodal conics in $M^k_N$ that did not appear in the Calabi-Yau case. Therefore, multiple cover map contributions are far more complicated than Calabi-Yau, and we leave general analysis on this problem to future works.
This paper is organized as follows. In Section 1, we analyze characteristics of moduli space of lines in $M^k_N$ and derive $N_{L/M} \simeq O_L(-1)^{g_k+2-N} \oplus O_L^{g_2N-k-5}$. In Section 2, we study the moduli space $M_{0,0}(\mathbb{P}^1, 2)$ from the point of view of stability and identify it with $\mathbb{P}^2$ and show that the moduli space $M_{0,0}(\mathbb{P}^1, 2)$ is isomorphic to $\mathbb{P}(S^2 Q)$ over $G$. In section 4, we describe $H^1(\phi^* N_{L/M})$ as an coherent sheaf over $\mathbb{P}(S^2 Q)^{stack}$. In section 5, we derive the main theorem (6) of this paper by using Segre-Witten classes. In Section 6, we mention some generalization to degree 3 Gromov-Witten invariants.

2. LINES ON A HYPERSURFACE

Let $M$ be a generic hypersurface of degree $k$ of the projective space $\mathbb{P}^{N-1} = \mathbb{P}(V)$. We assume $2N - 5 \geq k \geq N - 2 \geq 2$ throughout this note. In this note we count the number of rational curves of virtual degree two, namely rational curves which doubly cover lines on $M$.

Let $\mathbb{P} = \mathbb{P}(V)$ be the projective space parameterizing all one-dimensional quotients of $V$, which is usually denoted by $\mathbb{P}(V)$ in the standard notation in algebraic geometry. In this notation let $W$ be a subspace of $V$. Then $\mathbb{P}(W)$ is naturally a linear subspace of $\mathbb{P}(V)$ of dimension $\dim W - 1$.

Let $G(2, V)$ be the Grassmann variety of lines in $\mathbb{P}(V)$, the scheme parameterizing all lines of $\mathbb{P} = \mathbb{P}(V)$. This is also the universal scheme parameterizing all one-dimensional quotients linear spaces of $V$. Let $W$ be a two dimensional quotient linear space, $\psi \in G(2, V)$, namely $\psi : \mathbb{P}(W) \to \mathbb{P}(V)$ the natural immersion and $i^* : W \to V$ the quotient homomorphism. The space $W$ is denoted by $W(\psi)$ when necessary.

There exists the universal bundle $Q_{G(2, V)}$ over $G(2, V)$ and a homomorphism $i^{univ*} : O_{G(2, V)} \otimes V \to Q_{G(2, V)}$ whose fiber $i^{univ*}_\psi : V \to Q_{G(2, V)}$ is the quotient $i^*_\psi : V \to W(\psi)$ of $V$ corresponding to $\psi$.

2.1. Existence of a line on $M$. Let $L = \mathbb{P}(W)$ be a line of $\mathbb{P}$, equivalently $W \in G(2, V)$. Then the condition $L \subseteq M$ imposes at most $k + 1$ conditions on $W$, while the number of moduli of lines of $\mathbb{P}$ equals $\dim G(2, V) = 2N - 4$. Hence we infer

Lemma 2.2. If $2N \geq k + 5$, then there exists at least a line on $M$.

See also [Katz,p.152]. Let $G$ be the subscheme of $G(2, V)$ parameterizing all lines of $\mathbb{P}(V)$ lying on $M$, $Q = (Q_{G(2, V)})_G$ the restriction of $Q_{G(2, V)}$ to $G$. By Lemma 2.2, $G$ is nonempty. Let $i^* : O_G \otimes V \to Q$ be the restriction of $i^{univ*}$ to $G$. Let $P = \mathbb{P}(Q)$ and $\pi : P \to G$ the natural projection. Then $\pi$ is the universal line of $M$ over $G$, to be more exact, the universal family over $G$ of lines lying on $M$. In other words, the natural epimorphism $i^* : O_G \otimes V \to Q$ induces a morphism $i : P \to \mathbb{P}_G(V) := G \times \mathbb{P}(V)$, which is a closed immersion into $\mathbb{P}_G(V)$, thus $P$ is a subscheme of $\mathbb{P}_G(V)$ such that $\pi = (p_1)_p$. Let $L_\psi = \mathbb{P}(Q_\psi)$. Note that

$L_\psi = P_\psi := \pi^{-1}(\psi) \simeq \mathbb{P}(Q_\psi) \cap \{\psi\} \times \mathbb{P}(V) \simeq \mathbb{P}(V)$. 
2.3. The normal bundle $N_{L/M}$. The argument of this section is standard and well known. Let $P = P(V)$, $L = P(W)$ and $i^*_W : V \to W \in G$. Let us recall the following exact sequence:

$$0 \to O_P \to O_P(1) \otimes V^\vee \xrightarrow{D} T_P \to 0$$

where the homomorphism $D$ is defined by

$$D(a \otimes v^\vee) := a D_{(v^\vee)} \quad (a \in O_P(1))$$

$$(D_{v^\vee} F)(u^\vee) := \left( \frac{d}{dt} F(u^\vee + tv^\vee) \right)_{t=0}$$

for a homogeneous polynomial $F \in S(V)$ and $u^\vee, v^\vee \in V^\vee$. We note $H^0(O_P(1)) \otimes V^\vee = V \otimes V^\vee = \text{End}(V, V)$ and that the image of $H^0(O_P)$ in $\text{End}(V, V)$ is $\text{Id}_V$. We also have the following exact sequences:

$$0 \to T_L \to (T_P)_L \to N_{L/P} \to 0$$

$$0 \to O_L \to O_L(1) \otimes V^\vee \xrightarrow{D_L} (T_P)_L \to 0.$$

**Lemma 2.4.** Let $L = P(W)$. Then

$$N_{L/P} \simeq O_L(1) \otimes (V^\vee/W^\vee), \quad H^0(N_{L/P}) \simeq W \otimes (V^\vee/W^\vee).$$

**Proof.** The assertion is clear from the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
0 & \to & O_L & \to & O_L(1) \otimes W^\vee & \xrightarrow{(D_L)_{W^\vee}} & T_L & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & O_L & \to & O_L(1) \otimes V^\vee & \xrightarrow{D_L} & (T_P)_L & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & O_L(1) \otimes (V^\vee/W^\vee) & \to & N_{L/P} & \to & 0
\end{array}
$$

The second assertion is clear from $H^0(L, O_L(1)) = W$.

Since $T_L \simeq O_L(2)$, there follow exact sequences

$$0 \to H^0(T_L) \to H^0((T_P)_L) \to H^0(N_{L/P}) \to 0$$

$$0 \to H^0(O_L) \to H^0(O_L(1)) \otimes V^\vee \xrightarrow{H^0(D_L)} H^0((T_P)_L) \to 0.$$

We also note

$$H^0(T_L) = \text{LieAut}^0(L) = \text{End}(W, W)/\text{center} = \text{End}(W, W)/\text{Id}_W.$$

Since $H^0(O_L(1)) = W$, we see

$$H^0((T_P)_L) = W \otimes V^\vee/\text{Im} H^0(O_L) = \text{Hom}(V, W)/\text{Id}_W.$$

Hence we again see

$$H^0(N_{L/P}) = (\text{Hom}(V, W)/\text{Id}_W)^*/(\text{Hom}(W, W)/\text{Id}_W)$$

$$= W \otimes (V^\vee/W^\vee) = \text{Hom}(V/W, W).$$

For any line $L = P(W)$ of $P$ the following sequence is exact:

$$0 \to N_{L/M} \to N_{L/P} \to (N_{M/P})_L(\simeq O_L(k)) \to 0.$$
Hence so is the following sequence as well:

$$
0 \longrightarrow H^0(N_{L/M}) \longrightarrow H^0(N_{L/P}) \xrightarrow{H^0(D_L)} H^0(O_L(k)) \xrightarrow{H^0(D_L)} H^1(N_{L/M}) \longrightarrow 0.
$$

Hence we have

**Lemma 2.5.** The following is exact:

(8) \hspace{1cm} 0 \to H^0(N_{L/M}) \to W \otimes (V^\vee/W^\vee) \xrightarrow{H^0(D_L)} S^kW \to H^1(N_{L/M}) \to 0.

**Corollary 2.6.** \hspace{0.5cm} \text{dim } G \geq 2N - k - 5, equality holding if \( H^1(N_{L/M}) = 0 \).

**Proof.** As is well-known, \( \text{dim } G \geq h^0(N_{L/M}) - h^1(N_{L/M}) \). Note \( \text{dim } W \otimes (V^\vee/W^\vee) = 2(N - 2) \) and \( \text{dim } S^kW = k + 1 \). Hence the corollary follows from Lemma 2.5. \( \square \)

**Lemma 2.7.** For a generic line \( L \) on a generic hypersurface \( M \) of degree \( k \)

(i) \( N_{L/M} \cong O^\text{gen}_{L} \oplus O_L(-1)^{\text{gen}_L} \), where \( a = 2N - k - 5 \) and \( b = k - N + 2 \),

(ii) \( \text{Coker } H^0(D^\vee_L) \cong S^{k-1}W/(V^\vee/W^\vee) \) where \( D^\vee_L := D_L \oplus O_L(-1) \).

**Proof.** Let \( M \) be a generic hypersurface of degree \( k \) and \( L \) a generic line on \( M \). Without loss of generality we may assume that \( W^\vee \) is generated by \( e_1^\vee \) and \( e_2^\vee \) in other words, \( \psi : L \to \mathbb{P} \) is given by

\[
\psi : [s : t] \to [x_1, \ldots, x_N] = [s, t, 0, \ldots, 0].
\]

Then \( F \), the polynomial of degree \( k \) defining \( M \), is written as

\[
F = x_3F_3 + x_4F_4 + \cdots + x_NF_N
\]

for some polynomials \( F_j \) of degree \( k - 1 \). Let \( f_j = \psi^*F_j = F_j(s, t, 0, \ldots, 0) \).

Now we consider the exact sequence

$$
0 \longrightarrow H^0(N_{L/M}(-1)) \longrightarrow H^0(N_{L/P}(-1)) \xrightarrow{H^0(D^\vee_L)} H^0(O_L(k - 1)) \xrightarrow{H^0(D^\vee_L)} H^1(N_{L/M}(-1)) \longrightarrow 0,
$$

where we note \( H^0(N_{L/P}(-1)) = V^\vee/W^\vee \). Hence the following is exact:

(9) \hspace{1cm} 0 \longrightarrow H^0(N_{L/M}(-1)) \longrightarrow V^\vee/W^\vee \xrightarrow{H^0(D^\vee_L)} S^{k-1}W \xrightarrow{H^0(D^\vee_L)} H^1(N_{L/M}(-1)) \longrightarrow 0

where \( H^0(D^\vee_L) \) is given by \( H^0(D^\vee_L)(e_j^\vee) = f_j \) (\( j = 3, 4, \cdots, N \)).

A generic choice of \( F \) implies a generic choice of degree \( k - 1 \) polynomials \( f_j \) \( (j = 3, 4, \cdots, N) \) in \( s \) and \( t \). By the assumptions

\[
\text{dim } S^{k-1}W = k \geq N - 2 = \text{dim } V^\vee/W^\vee,
\]

\[
\text{dim } W \otimes V^\vee/W^\vee = 2(N - 2) \geq k + 1 = \text{dim } S^kW,
\]

the generic choice of \( F \) implies that we can choose \( f_j \in S^{k-1}W \) (\( j = 3, 4, \cdots, N \)) (and fix once for all) such that

(iii) \( f_j \) (\( j = 3, 4, \cdots, N \)) are linearly independent,

(iv) \( Wf_3 + Wf_4 + \cdots + Wf_N = S^kW \).
Hence $H^0(D_L^7)$ is injective by (iii). It follows that $H^0(N_{L/M}(-1)) = 0$. Hence (ii) is clear. Next we consider $H^0(D_L)$. By (iv), we see
\[
S^k W = W \cdot H^0(D_L)(V^*/W^*) = H^0(D_L)(W \oplus V^*/W^*),
\]
whence $H^0(D_L)$ is surjective. It follows that $H^1(N_{L/M}) = 0$. Hence $N_{L/M} \simeq O^a_L \oplus O_L^{b(-1)^{gb}}$ for some $a$ and $b$. Since $a + b = \text{rank}(N_{L/M}) = N - 3$ and $-b = \deg(N_{L/M}) = N - 2 - k$, we have (i).

2.8. Lines on a quintic hypersurface in $\mathbb{P}^4$. See [Katz, Appendix A] for the subsequent examples. Let $N = 5$ and $k = 5$. Hence $M$ is a hypersurface of degree 5 in $\mathbb{P}^4$, a Calabi-Yau 3-fold. Let
\[
F = x_4 x_1^4 + x_5 x_2^4 + x_3^5 + x_4^5 + x_5^5.
\]
First we note that $M = \{F = 0\}$ is nonsingular. Let $L = \{x_3 = x_4 = x_5 = 0\} = \{s, t, 0, 0, 0\}$. In this case $f_3 = 0$, $f_4 = s^4$ and $f_5 = t^4$. In the exact sequence (1) we see $H^0(N_{L/M}(-1)) = \text{Ker} H^0(D_L^7) = C(\mathcal{O}_L^1)$ and $H^1(N_{L/M}(-1)) = \text{Coker} H^0(D_L^7)$ is 3-dimensional. Hence $N_{L/M} = O_L(1) \oplus O_L(-3)$.

We summarize the above. If $\dim \text{Ker} H^0(D_L^7) = 1$ and if $M$ is nonsingular, then $N_{L/M} = O_L(1) \oplus O_L(-3)$. Hence $H^0(N_{L/M}) = \text{Ker} H^0(D_L) = W \oplus \text{Ker} H^0(D_L^7)$ is 2-dimensional. Therefore we can choose $f_3 = 0$ and a linearly independent pair $f_4$ and $f_5 \in S^1 W$ so that $W f_4 + W f_5$ is 4-dimensional. The choice $f_4 = s^4$ and $f_5 = t^4$ satisfies the conditions. This enables us to find a nonsingular hypersurface $M$ as above. However if we choose $f_3 = 0$, $f_4 = s^4$ and $f_5 = s^2 t$, then $W f_4 + W f_5$ is 3-dimensional. Hence $M$ is singular.

Next in the same manner we find $L$ on a nonsingular hypersurface $M$ with $N_{L/M} = O_L \oplus O_L(-2)$ or $N_{L/M} = O_L(-1)^{gb}$. Let
\[
F = x_3 x_1^4 + x_4 x_2^4 + x_5 x_3^4 + x_6^5 + x_4^5 + x_5^5.
\]
Then we have $f_3 = s^4$, $f_4 = s^3 t$ and $f_5 = t^4$. Since $W f_3 + W f_4 + W f_5$ is 5-dimensional, $H^0(N_{L/M}(-1)) = \text{Ker} H^0(D_L^7) = 0$, $H^0(N_{L/M}) = \text{Ker} H^0(D_L) = C(\mathcal{O}_L^1 \otimes \mathcal{O}_L(-1))$. We see also that $\dim H^1(N_{L/M}) = \dim \text{Coker} H^0(D_L^7) = 1$ and $N_{L/M} = O_L \oplus O_L(-2)$. The hypersurface $M = \{F = 0\}$ is easily shown to be nonsingular.

If $F = x_3 x_1^4 + x_4 x_2^4 + x_5 x_3^4 + x_6^5 + x_4^5 + x_5^5$ and $M = \{F = 0\}$, then $N_{L/M} = O_L(-1)^{gb}$.

2.9. Lines on a generic hypersurface $M^8$ of $\mathbb{P}^6$. Let $N = 7$ and $k = 8$. In view of Lemma 2.2 there exists a line $L$ on any generic hypersurface $M$ of degree 8 in $\mathbb{P}(V) = \mathbb{P}^6$. In view of Lemma 2.7, $a = 1$, $b = 3$ and $N_{L/M} \simeq O_L \oplus O_L(-1)^{gb}$. For example let $L : x_j = 0$ ($j \geq 3$) and we take
\[
F_3 = 8 x_1^4, F_4 = 8 x_2^4 x_1, F_5 = 8 x_2^4 x_2, F_6 = 8 x_3^2 x_2^5, F_7 = 8 x_5^2,
\]
\[
F = x_2 F_3 + x_4 F_4 + x_5 F_5 + x_6 F_6 + x_7 F_7 + x_8 + x_9 + x_0 + x_7^2.
\]
and let $M = M^8 \setminus F = 0$. We see that $M$ is nonsingular near $L$ and has at most isolated singularities. However it is still unclear to us whether $M = M^8$ is
nonsingular everywhere. The space $H^0(N_{L/M})$ is spanned by $te^s - se^t$, hence an
infinitesimal deformation $L_{e}$ of $L$ is given by

$$[s, t] \mapsto [s, t, \varepsilon s, 0, 0, 0]$$

which yields $F_{|L_{e}} = \varepsilon^8 (s^8 + t^8) \equiv 0 \mod \varepsilon^8$. Since $H^1(N_{L/M}) = 0$, this
infinitesimal deformation is integrable and $G (:= \text{the moduli of lines of $P^6$ contained in $M$})$

We note that $M$ also contains 8 lines

$$L' := L'_{e} : \varepsilon^s x_1 - x_2 = x_3 + \varepsilon^s x_4 = x_j = 0 \quad (j \geq 5),$$

with $N_{L/M} = O_{L'}(-1)^{\oplus 3} \oplus O_{L'}(-6)$ where $\varepsilon^8 = -1$.

3. Stability

**Definition 3.1.** Suppose that a reductive algebraic group $G$ acts on a vector space $V$. Let $v \in V$, $v \neq 0$.

1. the vector $v$ is said to be semistable if there exists a $G$-invariant homogeneous polynomial $F$ on $V$ such that $F(v) \neq 0$,
2. the vector $v$ is said to be stable if $p$ has a closed $G$-orbit in $X_{ss}$ and the stabilizer subgroup of $v$ in $G$ is finite.

Let $\pi : V \setminus \{0\} \to P(V^\vee)$ be the natural surjection. Then $v \in V$ is semistable (resp. stable) if and only if $\pi(v)$ is semistable (resp. stable).

3.2. Grassmann variety. Let $V$ be an $N$-dimensional vector space, and $G(r, N)$
the Grassmann variety parameterizing all $r$-dimensional quotient spaces of $V$. Here
is a natural way of understanding $G(r, N)$ via GIT-stability. Let $U$ be an $r$-
dimensional vector space, $X = \text{Hom}(V, U)$ and $\pi : X \setminus \{0\} \to P(X^\vee)$ the natural
map. Then $SL(U)$ acts on $X$ from the left by:

$$(g \cdot \phi^*)(v) = g \cdot (\phi^*(v)) \quad \text{for } \phi^* \in X, v \in V.$$ 

We see that for $\phi^* \in X$

$$\phi^* \text{ is } SL(U)-\text{stable } \iff \text{rank } \phi^* = r,$$

$$\phi^* \text{ is } SL(U)-\text{semistable } \iff \phi^* \text{ is } SL(U)-\text{stable.}$$

In fact, if rank $\phi^* = r - 1$, then there is a one-parameter torus $T$ of $SL(U)$ such
that the closure of the orbit $T \cdot \phi$ contains the zero vector as the following simple
example ($r = 2$) shows

$$\lim_{t \to 0} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \lim_{t \to 0} \begin{pmatrix} ta_{11} & ta_{12} & \cdots & ta_{1N} \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$ 

Let $X_s$ be the set of all (semi)stable points and $P_s$ the image of $X_s$ by $\pi$. It is, as
we saw above, just the set of all $\phi \in X$ with rank $\phi^* = r$. Therefore the GIT-orbit
space $P_s/SL(U)$ is the orbit space $P_s/SL(U)$ by the free action, the Grassmann
variety $G(r, N)$.
3.3. Moduli of double coverings of $\mathbb{P}^1$ (1). Let $W$ and $U$ be a pair of two dimensional vector spaces, $X = \text{Hom}(W, S^2 U)$, and $\pi : X \setminus \{0\} \to \mathbb{P}(X^*)$ the natural morphism. Note that $\text{SL}(U)$ acts on $S^2 U$ from the left via the natural action: $\sigma(u_1 u_2) = \sigma(u_1) \sigma(u_2)$ for $u_1, u_2 \in U$. Thus $\text{SL}(U)$ acts on $X$ from the left in the same manner in the subsection 3.2.

Lemma 3.4. Let $\phi^* \in X$.

(i) $\phi^*$ is unstable iff $\phi^*(w)$ has a double root for any $w \in W$.
(ii) $\phi^*$ is semistable iff $\phi^*(w)$ has no double roots for some nonzero $w \in W$.
(iii) $\phi^*$ is stable iff $\phi^*(W)$ is a base-point free linear subsystem of $S^2 U$ on $\mathbb{P}(U)$.

Proof. We note that $\phi^*$ is unstable iff there is a suitable basis $s$ and $t$ of $U$ such that $\phi^*(w) = a(w)s^2$ for any $w \in W$ since a torus orbit $T \cdot \phi^*$ contains the zero vector. This proves (i). This also proves (ii). Next we prove (iii). If $\phi^*(W)$ has a base point, then it is clear that $\phi^*$ is not stable. If $\phi^*$ is semistable and it is not stable, then we choose a basis $s, t$ of $U$ and a basis $w_1, w_2$ of $W$ such that $\phi^*(w_1) = st$. If $\phi(w_2) = as^2 + bst$, then $\phi^*$ is not stable. This proves the lemma. □

Theorem 3.5. Let $X_{ss}$ be the Zariski open subset of $X$ consisting of all semistable points of $X$, $\pi(X_{ss})$ the image of $X_{ss}$ by $\pi$, and $Y := \pi(X_{ss})/\text{SL}(U)$. Then $Y \simeq \mathbb{P}^2$.

Proof. First consider a simplest case. We choose a basis $s, t$ of $U$. Let $w_1$ and $w_2$ be a basis of $W$, $T$ the subgroup of $\text{SL}(U)$ of diagonal matrices and $X' = \{ \phi^* \in X; \phi^*(w_1) = 2st \}$. Let $Z' = \text{SL}(U) \cdot X'$.

We note that $Z'$ is an $\text{SL}(U)$-invariant subset of $X_{ss}$. We prove $\pi(Z')/\text{SL}(U) \simeq \mathbb{C}^2$. Let $\phi^*$ and $\psi^*$ be points of $X'$. Let $\phi^*(w_2) = As^2 + 2bst + Ct^2$ and $\psi^*(w_2) = As^2 + 2bst + Ct^2$. Then it is easy to check

$$g \cdot \phi^* = \psi^* \text{ for } \exists g \in \text{SL}(U) \iff g \cdot \phi^* = \psi^* \text{ for } \exists g \in T$$

$$\iff A = au^2, B = b, C = u^{-2}c \text{ for } \exists u \neq 0.$$ 

Therefore each equivalence class of $\pi(Z)/\text{SL}(U)$ is represented by the pair $(AC, B)$, which proves $\pi(Z)/\text{SL}(U) \simeq \mathbb{C}^2$.

Now we prove the lemma. Let $\phi^* \in X_{ss}, \phi_j = \phi^*(w_j)$ and $\phi_0 = -(\phi_j + \phi_2)$. Let

$$\phi_0 = r_1s^2 + 2r_2st + r_3t^2,$$
$$\phi_1 = p_1s^2 + 2p_2st + p_3t^2,$$
$$\phi_2 = q_1s^2 + 2q_2st + q_3t^2,$$

and we define

$$D_1 = p_2^2 - p_1p_3, \quad D_2 = q_2^2 - q_1q_3,$$
$$D_0 = v_2^2 - r_1r_3 = D_1 + D_2 + 2p_2q_2 - (p_1q_3 + p_3q_1).$$

To show the lemma, we prove the more precise isomorphism

$$\pi(X_{ss})/\text{SL}(U) = \text{Proj} \mathbb{C}[D_0, D_1, D_2]$$

For this purpose we define $Y_j = \pi(\{ \phi^* \in X_{ss}; \phi_j \text{ has no double roots} \})/\text{SL}(U)$. It suffices to prove $Y_1 = \text{Spec} \mathbb{C}[D_0, D_1, D_2]$ by reducing it to the first simplest case.
Let \( \phi^* \in Y_1 \). Let \( \alpha \) and \( \beta \) be the roots of \( \phi_1 = 0 \). By the assumption \( \phi_1 \) has no double roots, hence \( \alpha \neq \beta \). Let

\[
    u = \frac{1}{\gamma}(s - \alpha t), \quad v = \frac{1}{\gamma}(s - \beta t), \quad g = \frac{1}{\gamma} \left( \frac{1}{1 - \alpha}, -\frac{1}{1 - \beta} \right)
\]

where \( \gamma = \sqrt{\alpha - \beta} \). Note that \( g \in \text{SL}(U) \). Hence we see

\[
    (\phi_1(s, t), \phi_2(s, t)) \equiv (p_1^4 uv, A_1 u^2 + 2B_1 uv + C_1 v^2)
\]

where

\[
    A_1 = q_1^2 \alpha^2 + 2q_2 \alpha + q_3,
\]

\[
    -B_1 = q_1 \alpha \beta + q_2 (\alpha + \beta) + q_3,
\]

\[
    C_1 = q_1^2 \beta^2 + 2q_2 \beta + q_3.
\]

Thus we see

\[
    (\phi_1(s, t), \phi_2(s, t)) \equiv (2st, As^2 + 2Bst + Ct^2)
\]

where

\[
    A = \frac{2A_1}{p_1^4}, \quad B = \frac{2B_1}{p_1^4}, \quad C = \frac{2C_1}{p_1^4}, \quad p_1^4 = 4D_1,
\]

\[
    AC = B^2 - \frac{D_2}{D_1}, \quad B = \frac{D_0 - D_1 - D_2}{2D_1}.
\]

Therefore by the first half of the proof

\[
    Y_1 \simeq \text{Spec } \mathbb{C}[AC, B] = \text{Spec } \mathbb{C}[\frac{D_0}{D_1}, \frac{D_2}{D_1}].
\]

This completes the proof of the lemma. \( \square \)

**Corollary 3.6.** Let \( Y^s = \pi(X_x)/\text{SL}(U) \). Then \( Y \setminus Y^s \) is a conic of \( Y \) defined by

\[
    Y \setminus Y^s : D_0^2 + D_1^2 + D_2^2 - 2D_0D_1 - 2D_1D_2 - 2D_2D_0 = 0.
\]

**Proof.** In view of Theorem 3.5, \( Y_1 \simeq \text{Spec } \mathbb{C}[AC, B] \). The complement of \( Y_0 \) in \( Y_1 \) is then the curve defined by \( AC = 0 \), which is easily identified with the above conic. \( \square \)

**Corollary 3.7.** Let \( X^0 \) be the Zariski open subset of \( X \) consisting of all semistable points \( \phi^* \) of \( X \) with \( \text{rank } \phi^* = 2 \), and let \( Y^0 := \pi(X^0)/\text{SL}(U) \). Then \( Y^0 \simeq \pi(X^0)/\text{SL}(U) \simeq Y \simeq \mathbb{P}^2 \).

**Proof.** It suffices to compare \( Y_1 \) and \( Y^0 \cap Y_1 \). As in the proof of Theorem 3.5 we let \( X' = \{ \phi^* \in X; \phi^*(w_1) = 2st \} \). Let \( Z = \text{SL}(U) \cdot X' \) and \( Z^0 = \text{SL}(U) \cdot (X' \cap X^0) \).

Then with the notation in Theorem 3.5, we recall \( X' = \{ \phi^* \in X; \phi^*(w_1) = 2st, \phi^*(w_2) = As^2 + 2Bst + Ct^2 \} \), \( \pi(Z)/\text{SL}(U) \simeq \text{Spec } \mathbb{C}[AC, B] \) where

\[
    X' \cap X^0 = \{ \phi^* \in X'; A \neq 0 \text{ or } C \neq 0 \}.
\]

In the same manner as before we see \( \pi(Z^0)/\text{SL}(U) \simeq \text{Spec } \mathbb{C}[AC, B] \), whence \( \pi(Z^0)/\text{SL}(U) = \pi(Z)/\text{SL}(U) \). This proves \( Y^0 \cap Y_1 = Y_1 \). This completes the proof of the corollary. \( \square \)
3.8. Moduli of double coverings of $\mathbb{P}(W)$ (2). There is an alternative way of understanding $\pi(X_{X_2})/\pi SL(U) \simeq \mathbb{P}^2$ by using the isomorphism $S^2\mathbb{P}^1 \simeq \mathbb{P}^2$. We use the following convention to denote a point of $\mathbb{P}(U) = U^\vee \setminus \{0\}/G_m$: $(u : v) = us^\vee + vt^\vee \in U^\vee$ where $s^\vee$ and $t^\vee$ are a basis dual to $s$ and $t$. In what follows we fix a basis $w_1$ and $w_2$ of $W$. Let $P := (a_1 : a_2)$ and $Q := (b_1 : b_2)$ be a pair of points of $\mathbb{P}(W) \simeq \mathbb{P}^1$. If $P \neq Q$, there is a double covering $\phi : \mathbb{P}(U) \to \mathbb{P}(W)$ ramifying at $P$ and $Q$, unique up to isomorphism once we fix the base $w_1$ and $w_2$:

$$\frac{b_2 w_1 - b_1 w_2}{a_2 w_1 - a_1 w_2} = \left(\frac{t}{s}\right)^2.$$

Thus $\phi$ is given explicitly by

$$\phi_1 := \phi^*(w_1) = b_1 s^2 - a_1 t^2, \quad \phi_2 := \phi^*(w_2) = b_2 s^2 - a_2 t^2, \quad \phi_0 = -(\phi_1 + \phi_2)$$

for which we have

$$D_1 = a_1 b_1, \quad D_2 = a_2 b_2, \quad D_0 = (a_1 + a_2)(b_1 + b_2).$$

The isomorphism $S^2\mathbb{P}^1 \simeq \mathbb{P}^2$ is given by $(P, Q) \mapsto (D_0, D_1, D_2)$. This shows

**Corollary 3.9.** We have a natural isomorphism: $Y \simeq \mathbb{P}(S^2W)$.

4. The virtual normal bundle of a double covering

4.1. The case $N = 7$ and $k = 8$ revisited. We revisit the example in the subsection 2.9. Let $N = 7$ and $k = 8$. Let $L : x_j = 0$ ($j \geq 3$) and we take

$$F_3 = 8x_1^7, F_4 = 8x_1^6x_2, F_5 = 8x_1^5x_2^2, F_6 = 8x_1^4x_2^3, F_7 = 8x_2^7,$$

$$F = x_3F_3 + x_4F_4 + x_5F_5 + x_6F_6 + x_7F_7 + x_3^8 + x_4^8 + x_5^8 + x_6^8 + x_7^8.$$

and let $M = M_8^0 : F = 0$. We often denote $L$ also by $\mathbb{P}(W)$ with $W$ a two dimensional vector space for later convenience. Since $H^0(D_L^-)$ is injective and $H^0(D_L)$ is surjective, we have $N_{L/M} \simeq O_L \oplus O_L(1)^{83}$. Hence $H^1(N_{L/M}(-1)) = H^1(O_L(-2)^{83})$ is 3-dimensional. As we see easily, this follows also from the fact that $\text{Coker} H^0(D_L^-)$ is freely generated by $x_1^7x_2^3, x_1^6x_2^2$ and $x_1^5x_2^3$.

Let $\phi^* = (\phi_1, \phi_2) \in X^0$. Then $\text{Ker} H^0(\phi^*D_L)$ is generated by a single element $\phi_2\phi_3^\vee - \phi_1\phi_4^\vee$, while $\text{Coker} H^0(\phi^*D_L)$ is generated by $S^2U \cdot \phi_1^2\phi_2$, $S^2U \cdot \phi_1^3\phi_2^4$ and $S^2U \cdot \phi_1\phi_2^8$. To be more precise, we see

$$\text{Coker} H^0(\phi^*D_L) = \{\phi_1^2\phi_2, \phi_1^3\phi_2^4, \phi_1\phi_2^8\} \oplus S^2U/\{\phi_1, \phi_2\}.$$

In fact, this is proved as follows: first we consider the case where $\phi_1$ and $\phi_2$ has no common zeroes. In this case $\phi^*$ gives rise to a double covering $\phi : \mathbb{P}(U) \to \mathbb{P}(W)$ ($\equiv L$), which we denote by $L_\phi$ for brevity. By pulling back by $\phi^*$ the normal sequence $0 \to N_{L/M} \to N_{L/P} \to O_L(k) \to 0$ ($k = 8$) for the line $L$ we infer an exact sequence

$$0 \to \phi^*N_{L/M} \to \phi^*N_{L/P} \phi^*D_L \to \phi^*O_L(k) \to 0,$$

which yields an exact sequence

$$0 \to \text{H}^0(\phi^*N_{L/M}) \to \text{H}^0(\phi^*N_{L/P} / (V^\vee/W^\vee)) \to \text{H}^0(\phi^*D_L) \to \text{H}^0(\phi^*O_L(2k)) \to 0.$$
Let \( \eta = q_3\varepsilon_3^\gamma + \cdots + q_7\varepsilon_7^\gamma \in \text{Ker } H^0(\phi^*D_L), q_j \in S^2 U. \) Then we have

\[
\phi_1^2(q_3\varepsilon_3^\gamma + q_4\varepsilon_4^\gamma + q_5\varepsilon_5^2 + q_6\varepsilon_6^\gamma) = -q_7\varepsilon_7^\gamma.
\]

Since \( \phi_1 \) and \( \phi_2 \) are mutually prime and \( q_j \) is of degree two, we have \( q_7 = 0 \) and

\[
\phi_1^2(q_3\varepsilon_3^\gamma + q_4\varepsilon_4^\gamma + q_5\varepsilon_5^2) = -q_6\varepsilon_6^\gamma.
\]

Hence \( q_6 = 0 \) and similarly we infer also \( q_7 = 0. \) Thus we have \( q_3\phi_1 + q_4\phi_2 = 0. \) This proves that \( \text{Ker } H^0(\phi^*D_L) \) is generated by \( \phi_2e_2^\gamma - \phi_1e_1^\gamma. \)

Next we prove that \( \text{Coker } H^0(\phi^*D_L) \) is generated by \( \phi^* \) \( \text{Coker } H^0(D_L^\gamma) \) over \( S^2 U, \) in fact over \( S^2 U/\phi^*(W). \) Without loss of generality we may assume that \( \phi_1 = 2st \) and \( \phi_2 = \lambda s^2 + 2nu t + t^2 \) for some \( \lambda \neq 0 \) and \( \nu \in \mathbb{C}. \) Let \( \phi^*W = \{ \phi_1, \phi_2 \}. \) Then one checks \( U \cdot \phi^*W = S^2 U, \) and hence \( S^2 U \cdot \phi^*W = S^4 U, S^{2m-2} U \cdot \phi^*W = S^{2m} U \) for \( m \geq 2. \) It follows \( S^2 U \cdot \phi^*(S^{m-1} W) = S^{2m} U \) for \( m \geq 1. \) In fact, by the induction on \( m \)

\[
S^2 U \cdot \phi^*(S^{m} W) = S^2 U \cdot \phi^*(S^{m-1} W) = S^2 U \cdot \phi^*(S^{m-1} W)
\]

\[
= S^{2m} U.
\]

Therefore \( H^0(OL(k)) = S^{16} U = S^2 U \cdot \phi^*(S^7 W). \) Hence

\[
\text{Coker } H^0(\phi^*D_L) = S^{16} U / \text{Im } H^0(\phi^*D_L)
\]

\[
= S^{2m} U / \phi^*(S^7 W) / S^2 U \cdot \phi^*(\text{Im } H^0(D_L^\gamma))
\]

\[
= (S^2 U / \phi^*(W)) \cdot \phi^*(S^7 W / \text{Im } H^0(D_L^\gamma)).
\]

because \( \text{Coker } H^0(D_L^\gamma) = S^7 W / \text{Im } H^0(D_L^\gamma) \) and \( \text{W} \cdot S^7 W \subset W \cdot \text{Im } H^0(D_L^\gamma) = S^7 W \) by the choice of \( L. \) This proves that \( \text{Coker } H^0(\phi^*D_L) \) is generated by \( \phi^* \) \( \text{Coker } H^0(D_L^\gamma) \) over \( S^2 U / \phi^*(W). \) It follows \( \text{Coker } H^0(\phi^*D_L) = (\phi^* \text{Coker } H^0(D_L^\gamma)) \tilde{\otimes} (S^2 U / \phi^*W). \)

Finally we consider the case where \( \phi_1 \) and \( \phi_2 \) have a common zero. In this case we may assume \( \phi_1 = 2st \) and \( \phi_2 = 2nu t + t^2. \) In this case \( L_\phi \) is a chain of two rational curves \( C_\phi \) and \( \phi'' \) where \( C_\phi \) is the proper transform of \( \text{P}(U), \) where the double covering map from \( L_\phi \) to \( \text{P}(W) \) is the union of the isomorphisms \( \phi' \) and \( \phi'' \), say, \( \phi = \phi' \cup \phi''. \) Let \( \psi_1 = 2s \) and \( \psi_2 = 2nu + t. \) Then \( \phi' \) is induced by the homomorphism \( (\phi')^* \in \text{Hom}(W, U) \) such that \( (\phi')^*(w_j) = \psi_j. \) On the other hand let \( U_{\phi} = CL + Ct, \phi_1' = 2t \) and \( \phi_2' = \lambda + 2t \) where we note \( \psi_j'' \) is the linear part of \( \phi_j \) in \( t \) with \( s = 1. \) Then \( C_\phi = \text{P}(U_{\phi}) \) and \( \phi'' \) is induced by the homomorphism \( (\phi'')^* \in \text{Hom}(W, U_{\phi}) \) such that \( (\phi'')^*(w_j) = \psi_j'' \). Furthermore the pull back by \( \phi^* \) of the normal sequence for \( L \)

\[
0 \rightarrow \phi^*N_{L/M} \rightarrow \phi^*N_{L/P} \rightarrow \phi^*D_L \rightarrow \phi^*O_L(k) \rightarrow 0,
\]
yields exact sequences with natural vertical homomorphisms:

\[
0 \rightarrow \phi^* N_{L/M} \rightarrow (\phi')^* N_{L/M} \oplus (\phi'')^* N_{L/M} \rightarrow C \rightarrow 0
\]

This yields the following long exact sequences:

\[
0 \rightarrow H^0((\phi')^* N_{L/M}) \rightarrow U \otimes V^\vee/W^\vee \xrightarrow{H^0((\phi')^* D_L)} S^k U \\
0 \rightarrow H^1((\phi')^* N_{L/M}) \rightarrow H^0((\phi'')^* N_{L/M}) \rightarrow 0 \\
0 \rightarrow H^0((\phi'')^* N_{L/M}) \rightarrow U'' \otimes V^\vee/W^\vee \xrightarrow{H^0((\phi'')^* D_L)} S^k U''_{\phi} \\
0 \rightarrow H^1((\phi'')^* N_{L/M}) \rightarrow 0
\]

whence \( H^1((\phi')^* N_{L/M}) = H^1((\phi'')^* N_{L/M}) = 0 \), and both \( H^0((\phi')^* N_{L/M}) \) and \( H^0((\phi'')^* N_{L/M}) \) are one-dimensional. Let \( U^t \) be the subspace of \( U \) consisting of elements vanishing at \( C_{\phi} \cap C_{\phi}' \), namely the subspace spanned by \( t \). Then the restriction of \( H^0((\phi')^* D_L) \) to \( U^t \otimes V^\vee/W^\vee \) equals \( t \cdot H^0((\phi')^* D_L^-) \). Hence

\[
\text{Coker } H^0((\phi')^* D_L) \simeq t \cdot S^2 U/t \cdot \text{Im } H^0((\phi')^* D_L^-) \oplus \text{Coker } H^0((\phi'')^* D_L) \\
\simeq S^2 U/t \cdot \text{Im } H^0((\phi')^* D_L^-) \simeq \text{Coker } H^0((\phi')^* D_L^-).
\]

One could understand the above isomorphism as

\[
\text{Coker } H^0((\phi')^* D_L) = \text{Coker } (\phi) \cdot H^0(D_L^-) \oplus (S^2 U/\phi^* W).
\]

Thus \( H^0(\phi^* N_{L/M}) \) is one-dimensional, while \( H^1(\phi^* N_{L/M}) \) is 3-dimensional. This is immediately generalized into the following

**Lemma 4.2.** For any \( \phi^* \in X^0 \) we have

\[
\text{Ker } H^0(\phi^* D_L) = \phi^* \text{Ker } H^0(D_L), \\
\text{Coker } H^0(\phi^* D_L) = (\phi^* \text{Coker } H^0(D_L^-)) \oplus (S^2 U/\phi^* W).
\]

**Lemma 4.3.** We define a line bundle \( L_0 \) (resp. \( L_1 \)) on \( Y \) (resp. \( P(S^2 W) \)) by the assignment:

\[
X^0 \ni \phi^* \mapsto \phi^* \text{Ker } H^0(D_L) \ (\text{resp. } \phi^* \text{Coker } H^0(D_L^-)).
\]

Then \( L_k \simeq O_{P(S^2 W)} \).

**Proof.** We know that \( \phi^* \text{Ker } H^0(D_L) \) is generated by \( \phi_2 c_2^\gamma - \phi_1 c_1^\gamma \). By the SL(2)-variable change of \( s \) and \( t \), \( \phi_j \) is transformed into a new quadratic polynomial, which is however the same as the first \( \phi_j \). This shows the generator is unchanged, whence \( L_0 \simeq O_{P(S^2 W)} \). The proof for \( L_1 \) is the same. \( \square \)
Lemma 4.4. We define a coherent sheaf \( L \) on the stack \( Y \) (\( \simeq \mathbb{P}(S^2W) \)) (See Remark below) by the assignment:

\[
X^0 \ni \phi^* \mapsto S^2U/\phi^*W.
\]

Then \( L^2 \simeq O_{\mathbb{P}(S^2W)}(-1) \).

Proof. The GIT-quotient \( Y^0 \) is covered with the images of \( X_j^0 \):

\[
X_1^0 = \{(\phi_1, \phi_2) \in X^0; \phi_1 = 2st, \phi_2 = \lambda s^2 + 2\nu st + t^2, \lambda, \nu \in \mathbb{C} \},
\]
\[
X_2^0 = \{(\phi_1, \phi_2) \in X^0; \phi_1 = ps^2 + 2qst + t^2, \phi_2 = 2st, p, q \in \mathbb{C} \}.
\]

It is clear that the natural image of \( X_j^0 \) in \( Y \) is \( Y_j \). The map \( \phi \) given by \( \phi^* = (\phi_1, \phi_2) \in Y_1 \) has natural \( \mathbb{Z}_2 \) involution generated by,

\[
r : (\sqrt{\lambda}s + t, \sqrt{\lambda}s - t) \mapsto (\sqrt{\lambda}s + t, - (\sqrt{\lambda}s - t)).
\]

Since

\[
2st = \frac{1}{2\sqrt{\lambda}}((\sqrt{\lambda}s + t)^2 - (\sqrt{\lambda}s - t)^2),
\]
\[
\lambda s^2 + 2\nu st + t^2 = \frac{\nu}{2\sqrt{\lambda}}((\sqrt{\lambda}s + t)^2 - (\sqrt{\lambda}s - t)^2) + \frac{1}{2}((\sqrt{\lambda}s + t)^2 + (\sqrt{\lambda}s - t)^2),
\]

it is clear that,

\[
r^*(\phi_1) = \phi_1, \ r^*(\phi_2) = \phi_2, \ r^*(\lambda s^2 - \nu t^2) = -(\lambda s^2 - \nu t^2).
\]

Therefore, we can decompose \( S^2U \) into \( \langle \lambda s^2 - \nu t^2 \rangle_{\mathbb{C}} \oplus \langle \phi_1, \phi_2 \rangle_{\mathbb{C}} \) with respect to eigenvalue of \( r^* \) and take \( \lambda s^2 - \nu t^2 \) as canonical generator of \( S^2U/\phi^*W \). Similarly \( S^2U/\phi^*W \) is generated by \( ps^2 - \nu t^2 \) on \( Y_2 \). The problem is therefore to write \( \lambda s^2 - \nu t^2 \) as an \( \Gamma(O_{Y_1}O_{Y_2}) \)-multiple of \( pu^2 + v^2 \) when we write \( \phi_2 = 2uv \) by a variable change in \( GL(2) \). The following variable change \( (s, t) \mapsto (u, v) \) is in \( GL(2) \):

\[
s = \frac{\sqrt{2}\alpha}{(\beta - \alpha)^2}(2u - \frac{\beta - \alpha}{2\alpha}v), \quad t = \frac{\sqrt{2}\alpha}{(\beta - \alpha)^2}(2\beta u - \frac{\beta - \alpha}{2\alpha}v),
\]

where \( \alpha, \beta \) are roots of the equation \( \lambda s^2 + 2\nu st + t^2 = 0 \). Under this coordinate change, \( \phi_1 \) and \( \phi_2 \) are rewritten as follows:

\[
\phi_1 = \frac{\lambda}{(\nu^2 - \lambda)^2}u^2 + 2\frac{\nu}{\nu^2 - \lambda}uv + v^2 = pu^2 + 2quv + v^2, \quad \phi_2 = 2uv.
\]

Then we have

\[
pu^2 + v^2 = -\frac{2}{\beta - \alpha}(\lambda s^2 - \nu t^2) = -\frac{1}{\sqrt{\nu^2 - \lambda}}(\lambda s^2 - \nu t^2) = -\sqrt{\frac{D_1}{D_2}}(\lambda s^2 - \nu t^2).
\]

Similarly by computing the effect on \( S^2U/\phi^*W \) by the variable change from \( X_1^0 \) into \( X_0^0 \), we see that \( L^2 \) is isomorphic to \( O_{\mathbb{P}(S^2W)}(-1) \). This completes the proof. \( \square \)

Remark 4.5. We remark that the space \( X \) must be regared as a \( \mathbb{Q} \)-stack \( \mathcal{Y} \)-stack as follows: First we define \( \phi_0 = -\phi_1 - \phi_2 \). For each atlas \( X_\alpha^0 \) we define an atlas \( \mathcal{Y}_\alpha^0 \)-stack
\((\alpha = 0, 1, 2)\) by

\[
Y^{stack}_{0} = \{(\phi_0, \phi_1, \phi_2, \pm \psi_0) \in X^0 \times S^2 U; \phi_0 = 2st, \phi_1 = \alpha s^2 + 2bst + t^2, \\
\psi_0 = \alpha s^2 - t^2 a, b \in C\},
\]

\[
Y^{stack}_{1} = \{(\phi_0, \phi_1, \phi_2, \pm \psi_1) \in X^0 \times S^2 U; \phi_1 = 2st, \phi_2 = \lambda s^2 + 2\nu st + t^2, \\
\psi_1 = \lambda s^2 - t^2 \nu, \lambda, \nu \in C\},
\]

\[
Y^{stack}_{2} = \{(\phi_0, \phi_1, \phi_2, \pm \psi_2) \in X^0 \times S^2 U; \phi_0 = \psi_1 = ps^2 + 2qst + t^2, \phi_2 = 2st, \\
\psi_2 = ps^2 - t^2 p, q \in C\}.
\]

Since \(L^2 \simeq O_{\mathbb{P}(S^2 W)}(-1)\) we have \(c_1(L) = -\frac{1}{2} c_1(O_{\mathbb{P}(S^2 W)}(1))\) in the Chow ring

\[A(Y^{stack})_Q \simeq A(X)_Q = A(\mathbb{P}(S^2 W)_Q).\]

5. Proof of the main theorem

**Theorem 5.1.**

\[
\pi_*(c_{top}(H^1)) = \frac{1}{8} \left[ \frac{c(S^{k-1}Q)}{1 - \frac{1}{2} c_1(Q)} \right]_{k-N},
\]

where \(\pi\) is the natural projection from \(\tilde{M}_{0,0}(L, 2)\) to \(G\) and \([\pi]\) is the operation of picking up the degree \((k - N)\) part of Chern classes.

**Proof.** From now on we denote the coherent sheaf \(L\) in Lemma 4.4 by \(O_{\mathbb{P}}(-\frac{1}{2})\). In view of the results from the previous section, what remains is to evaluate the top chern class of \((S^{k-1}Q/(\mathcal{V}^\vee \otimes O_G)/\mathcal{Q}^\vee)) \otimes O_{\mathbb{P}}(-\frac{1}{2})\) on \(P(S^2 Q)\). Since double cover maps parametrized by \(P(S^2 Q)\) have natural \(\mathbb{Z}_2\) involution \(r\) given in the previous section, we have to multiply the result of integration on \(P(S^2 Q)\) by the factor \(\frac{1}{2}\) [BT], [FP]. With this set-up, let \(\pi' : P(S^2 Q) \to G\) be the natural projection. Then what we have to compute is \(\pi_*(c_{top}(H^1)) = \frac{1}{2} \pi_*'(c_{top}(H^1)) = \frac{1}{2} \pi_*'(c_{top}(S^{k-1}Q/(\mathcal{V}^\vee \otimes O_G)/\mathcal{Q}^\vee)) \otimes O_{\mathbb{P}}(-\frac{1}{2}))\). Let \(\varepsilon\) be \(c_1(O_{\mathbb{P}}(1))\). Then we obtain,

\[
\frac{1}{2} \pi_*'(c_{top}(S^{k-1}Q/(\mathcal{V}^\vee \otimes O_G)/\mathcal{Q}^\vee)) \otimes O_{\mathbb{P}}(-\frac{1}{2})) = \frac{1}{2} \sum_{j=0}^{k-N+2} c_{k-N+2-j}(S^{k-1}Q \oplus \mathcal{Q}^\vee) \cdot \pi_*(\varepsilon^j) \cdot (\varepsilon^{-\frac{1}{2}})^j
\]

\[
= \frac{1}{8} \sum_{j=0}^{k-N} c_{k-N-j}(S^{k-1}Q \oplus \mathcal{Q}^\vee) \cdot s_j(S^2 Q) \cdot (\varepsilon^{-\frac{1}{2}})^j
\]

\[
= \frac{1}{8} \left[ \frac{c(S^{k-1}Q) \cdot c(Q^\vee)}{1 - \frac{1}{2} c_1(S^2 Q) + \frac{1}{2} c_2(S^2 Q) - \frac{1}{8} c_3(S^2 Q)} \right]_{k-N},
\]

where \(s_j(S^2 Q)\) is the \(j\)-th Segre class of \(S^2 Q\). But if we decompose \(c(Q)\) into \((1 + \alpha)(1 + \beta)\), we can easily see,

\[
\frac{c(Q^\vee)}{1 - \frac{1}{2} c_1(S^2 Q) + \frac{1}{2} c_2(S^2 Q) - \frac{1}{8} c_3(S^2 Q)} = \frac{(1 - \alpha)(1 - \beta)}{(1 - \alpha)(1 - \frac{1}{2}(\alpha + \beta))(1 - \beta)} = \frac{1}{1 - \frac{1}{2} c_1(Q)}.
\]
Finally, by combining the above theorem with the divisor axiom of Gromov-Witten invariants, we can prove the decomposition formula of degree 2 rational Gromov-Witten invariants of $M_k^r$ found from numerical experiments.

**Corollary 5.2.**

\[ \langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,2} = \langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,2} + S(\pi_*(c_{top}(H^1)) O_{e^a} O_{e^b} O_{e^c})_{0,1}, \]

where $\langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,2}$ is the number of conics that intersect cycles Poincaré dual to $e^a$, $e^b$ and $e^c$. We also denote by $\langle \pi_*(c_{top}(H^1)) O_{e^a} O_{e^b} O_{e^c} \rangle_{0,1}$ the integral:

\[ \int_{G(2,\mathbb{V})} c_{top}(S^3 Q) \wedge \pi_*(c_{top}(H^1)) \wedge \sigma_{a-1} \wedge \sigma_{b-1} \wedge \sigma_{c-1}. \]

6. GENERALIZATION TO TWISTED CUBICS

In this section, we present a decomposition formula of degree 3 rational Gromov-Witten invariants found from numerical experiments using the results of [ES].

**Conjecture 6.1.** If $k - N = 1$, we have the following equality:

\[ \pi_*(c_{top}(H^1)) = \frac{1}{27} \left( \left( \frac{1}{24} (27k^2 - 55k + 26)k(k - 1) + \frac{2}{9} \right) c_1(Q)^2 + \left( \frac{7}{6} (k + 1)k(k - 1) + \frac{1}{9} \right) c_2(Q) \right). \]

where $\pi : \overline{M}_{0,9}(L, 3) \to \overline{M}_{0,9}(M_k^r, 1)$ is the natural projection.

In the $k - N > 1$ case, we have not found the explicit formula, because in the $d = 3$ case, we have another contribution from multiple cover maps of type $(2+1) \to (1+1)$. Here multiple cover map of type $(2+1) \to (1+1)$ is the map from nodal curve $\mathbb{P}^1 \cup \mathbb{P}^1$ to nodal conic $L_1 \sqcup L_2 \subset M_k^r$, that maps the first (resp. the second) $\mathbb{P}^1$ to $L_1$ (resp. $L_2$) by two to one (resp. one to one). In the $k - N = 1$ case, we have also determined the contributions from multiple cover maps of $(2+1) \to (1+1)$ to nodal conics.

**Corollary 6.2.** If $k - N = 1$, $\langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,3}$ is decomposed into the following contributions:

\[ \langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,3} = \langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,3} + \frac{1}{k} \left( \frac{9}{4} \langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,1} \langle O_{e^a} \rangle_{0,1} + \frac{3}{2} \langle O_{e^a} O_{e^b} O_{e^c+2} \rangle_{0,1} \langle O_{e^a} \rangle_{0,1} + \frac{3}{2} \langle O_{e^a} O_{e^b+2} O_{e^c} \rangle_{0,1} \langle O_{e^a} \rangle_{0,1} \right) \]

where $\langle O_{e^a} O_{e^b} O_{e^c} \rangle_{0,3-3}$ is the number of twisted cubics that intersect cycles Poincaré dual to $e^a$, $e^b$ and $e^c$.

**Proof.** In the $k - N = 1$ case, dimension of moduli space of multiple cover maps of $(2+1) \to (1+1)$ to nodal conics is given by $N - 6 + N - 6 - (N - 4) + 2 = N - 6$, hence the rank of $H^1$ is given by $N - 6 - (N - 5 - 3) = 2$. On the other hand, dimension of moduli space of $d = 2$ multiple cover maps of $\mathbb{P}^1 \to \mathbb{P}^1$ is 2, the degree of the form of $\pi_*(c_{top}(H^1))$ equals to $2 - 2 = 0$, where $\pi$ is the projection map.
that projects out the fiber locally isomorphic to the moduli space of \( d = 2 \) multiple cover maps. This situation is exactly the same as the Calabi-Yau case. Therefore, we can use the well-known result by Aspinwall and Morrison, that says for \( n \)-point rational Gromov-Witten invariants for Calabi-Yau manifold, \( \hat{\pi}_*(c_{\text{top}}(H^1)) \) for degree \( d \) multiple cover map is given by,

\[
\hat{\pi}_*(c_{\text{top}}(H^1)) = \frac{1}{d^{N-1}}.
\]

With this formula, we add up all the combinatorial possibility of insertion of external operator \( \mathcal{O}_{e^w}, \mathcal{O}_{e^b} \) and \( \mathcal{O}_{e^e} \),

\[
\frac{1}{k} \left( (\hat{\pi}_*(c_{\text{top}}(H^1)))\mathcal{O}_{e^w} \mathcal{O}_{e^b} \mathcal{O}_{e^e} \mathcal{O}_{e^x} \otimes_0 \mathcal{O}_{e^{N-x}} \right)_{0,1} + \left( \hat{\pi}_*(c_{\text{top}}(H^1)))\mathcal{O}_{e^w} \mathcal{O}_{e^b} \mathcal{O}_{e^{x+2}} \otimes_0 \mathcal{O}_{e^{N-x-1}} \right)_{0,1} + \left( \hat{\pi}_*(c_{\text{top}}(H^1)))\mathcal{O}_{e^b} \mathcal{O}_{e^w} \mathcal{O}_{e^{x+2}} \otimes_0 \mathcal{O}_{e^{N-x-1}} \right)_{0,1} + \left( \hat{\pi}_*(c_{\text{top}}(H^1)))\mathcal{O}_{e^w} \mathcal{O}_{e^b} \mathcal{O}_{e^{x+2}} \otimes_0 \mathcal{O}_{e^{N-x-1}} \mathcal{\pi}_*(c_{\text{top}}(H^1))) \mathcal{O}_{e^e} \otimes_0 \mathcal{O}_{e^{N-x}} \mathcal{\pi}_*(c_{\text{top}}(H^1))) \mathcal{O}_{e^e} \otimes_0 \mathcal{O}_{e^{N-x}} \hat{\pi}_*(c_{\text{top}}(H^1))) \mathcal{O}_{e^e} \otimes_0 \mathcal{O}_{e^{N-x}} \right)_{0,1} + \left( \hat{\pi}_*(c_{\text{top}}(H^1)))\mathcal{O}_{e^b} \mathcal{O}_{e^w} \mathcal{O}_{e^{x+2}} \otimes_0 \mathcal{O}_{e^{N-x-1}} \mathcal{\pi}_*(c_{\text{top}}(H^1))) \mathcal{O}_{e^e} \otimes_0 \mathcal{O}_{e^{N-x}} \hat{\pi}_*(c_{\text{top}}(H^1))) \mathcal{O}_{e^e} \otimes_0 \mathcal{O}_{e^{N-x}} \right)_{0,1} + \left( \hat{\pi}_*(c_{\text{top}}(H^1)))\mathcal{O}_{e^w} \mathcal{O}_{e^b} \mathcal{O}_{e^{x+2}} \otimes_0 \mathcal{O}_{e^{N-x-1}} \mathcal{\pi}_*(c_{\text{top}}(H^1))) \mathcal{O}_{e^e} \otimes_0 \mathcal{O}_{e^{N-x}} \hat{\pi}_*(c_{\text{top}}(H^1))) \mathcal{O}_{e^e} \otimes_0 \mathcal{O}_{e^{N-x}} \right)_{0,1} + \left( \hat{\pi}_*(c_{\text{top}}(H^1)))\mathcal{O}_{e^w} \mathcal{O}_{e^b} \mathcal{O}_{e^{x+2}} \otimes_0 \mathcal{O}_{e^{N-x-1}} \mathcal{\pi}_*(c_{\text{top}}(H^1))) \mathcal{O}_{e^e} \otimes_0 \mathcal{O}_{e^{N-x}} \hat{\pi}_*(c_{\text{top}}(H^1))) \mathcal{O}_{e^e} \otimes_0 \mathcal{O}_{e^{N-x}} \right)_{0,1} + \left( \hat{\pi}_*(c_{\text{top}}(H^1)))\mathcal{O}_{e^w} \mathcal{O}_{e^b} \mathcal{O}_{e^{x+2}} \otimes_0 \mathcal{O}_{e^{N-x-1}} \mathcal{\pi}_*(c_{\text{top}}(H^1))) \mathcal{O}_{e^e} \otimes_0 \mathcal{O}_{e^{N-x}} \hat{\pi}_*(c_{\text{top}}(H^1))) \mathcal{O}_{e^e} \otimes_0 \mathcal{O}_{e^{N-x}} \right)_{0,1}
\]

The last expression is nothing but the formula we want. □

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