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CONICS ON A GENERIC HYPERSURFACE

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ABSTRACT. In this paper, we compute the contributions from double cover maps to genus 0 degree 2 Gromov-Witten invariants of general type projective hypersurfaces. Our results correspond to a generalization of Aspinwall-Morrison formula to general type hypersurfaces in some special cases.

MSC-class: 14H99, 14N35, 32G20

1. INTRODUCTION

In this paper, we discuss a generalization of the multiple cover formula for rational Gromov-Witten invariants of Calabi-Yau manifolds [AM], [M] to double cover maps of a line L on a degree k hypersurface M_N^k in \mathbf{P}^{N-1} . Navely, for a given finite set of elements $\alpha_j \in H^*(M_N^k, \mathbf{Z})$, the rational Gromov-Witten invariant $\langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \cdots \mathcal{O}_{\alpha_n} \rangle_{0,d}$ of M_N^k counts the number of degree d (possibly singular and reducible) rational curves on M_N^k that intersect real sub-manifolds of M_N^k that are Poincare-dual to α_j .

Recently, the mirror computation of rational Gromov-Witten invariants of M_N^k with negative first chern class ($k-N > 0$) was established in [CG], [Iri], [J]. Using the method presented in these articles, we can compute $\langle \mathcal{O}_{e^{m_1}} \mathcal{O}_{e^{m_2}} \cdots \mathcal{O}_{e^{m_n}} \rangle_{0,d}$ where e is the generator of $H^{1,1}(M_N^k, \mathbf{Z})$. Briefly, mirror computation of M_N^k ($k > N$) in [J] goes as follows. We start from the following ODE:

$$(1) \quad \left((\partial_x)^{N-1} - k \cdot \exp(x) \cdot (k\partial_x + k-1)(k\partial_x + k-2) \cdots (k\partial_x + 1) \right) w(x) = 0,$$

and construct the virtual Gauss-Manin system associated with (1):

$$(2) \quad \partial_x \tilde{\psi}_{N-2-m}(x) = \tilde{\psi}_{N-1-m}(x) + \sum_{d=1}^{\infty} \exp(dx) \cdot \tilde{L}_m^{N,k,d} \cdot \tilde{\psi}_{N-1-m-(N-k)d}(x),$$

where m runs through all the integers and $\tilde{L}_m^{N,k,d}$ is non-zero only if $0 \leq m \leq N-1+(k-N)d$. From the compatibility of (1) and (2), we can derive the recursive formulas that determine all the $\tilde{L}_m^{N,k,d}$ s:

$$\begin{aligned} \sum_{n=0}^{k-1} \tilde{L}_n^{N,k,1} w^n &= k \cdot \prod_{j=1}^{k-1} (jw + (k-j)), \\ \sum_{m=0}^{N-1+(k-N)d} \tilde{L}_m^{N,k,d} z^m &= \sum_{l=2}^d (-1)^l \sum_{0=i_0 < \cdots < i_l=d} \times \\ &\times \sum_{j_l=0}^{N-1+(k-N)d} \cdots \sum_{j_2=0}^{j_3} \sum_{j_1=0}^{j_2} \prod_{n=1}^l \left(\left(\frac{i_{n-1} + (d-i_{n-1})z}{d} \right)^{j_n-j_{n-1}} \cdot \tilde{L}_{j_n+(N-k)i_{n-1}}^{N,k,i_n-i_{n-1}} \right). \end{aligned}$$

With these data, we can construct the formulas that represent rational three point Gromov-Witten invariant $\langle \mathcal{O}_e \mathcal{O}_{e^{N-2-m}} \mathcal{O}_{e^{m-1-(k-N)d}} \rangle_d$ in terms of $\tilde{L}_m^{N,k,d}$. These three point Gromov-Witten invariants are enough for reconstruction of all the rational Gromov-Witten invariants $\langle \mathcal{O}_{e^{m_1}} \mathcal{O}_{e^{m_2}} \cdots \mathcal{O}_{e^{m_n}} \rangle_{0,d}$ [KM]. In particular, we obtain the following formula in the $d = 2$ case:

$$(3) \quad \langle \mathcal{O}_e \mathcal{O}_{e^{N-2-m}} \mathcal{O}_{e^{m-1-(k-N)2}} \rangle_2 = k \cdot \left(\tilde{L}_n^{N,k,2} - \tilde{L}_{1+2(k-N)}^{N,k,2} - 2\tilde{L}_{1+(k-N)}^{N,k,1} \left(\sum_{j=0}^{k-N} (\tilde{L}_{n-j}^{N,k,1} - \tilde{L}_{1+2(k-N)-j}^{N,k,1}) \right) \right).$$

According to the results of this procedure, rational three point Gromov-Witten invariants can be rational numbers with large denominator if $k > N$, in contrast to the Calabi-Yau case where rational three point Gromov-Witten invariants are always integers.

One of the reasons of this rationality (non-integrality) comes from the contributions of multiple cover maps to Gromov-Witten invariants. In the Calabi-Yau case ($N = k$), for any divisor m of d there are some contributions from degree m multiple cover maps ϕ of a rational curve \mathbf{P}^1 onto a degree $\frac{d}{m}$ rational curve $C \hookrightarrow M_k^k$. The contributions from the multiple cover maps are expressed in terms of the virtual fundamental class of Gromov-Witten invariants. Let C be a general degree d rational curve in M_k^k . Its normal bundle N_{C/M_k^k} is decomposed into a direct sum of line bundles as follows:

$$N_{C/M_k^k} \simeq \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1) \oplus \mathcal{O}_C^{\oplus(k-5)}.$$

Let $\phi : \mathbf{P}^1 \rightarrow C$ be a holomorphic map of degree m . Since the pull-back $\phi^*(N_{C/M_k^k})$ is given by

$$\phi^*(N_{C/M_k^k}) \simeq \mathcal{O}_{\mathbf{P}^1}(-m) \oplus \mathcal{O}_{\mathbf{P}^1}(-m) \oplus \mathcal{O}_{\mathbf{P}^1}^{\oplus(k-5)},$$

we obtain $h^1(\phi^*(N_{C/M_k^k})) = 2m - 2$. On the other hand, let $\overline{M}_{0,0}(M, d)$ be the moduli space of 0-pointed stable maps of degree d from genus 0 curve to M . Then the moduli space of ϕ is the fiber space $\pi : \overline{M}_{0,0}(C, m) \rightarrow \overline{M}_{0,0}(M_k^k, \frac{d}{m})$, whose fibre $\overline{M}_{0,0}(C, m)$ over C (fixed) has complex dimension $2m - 2$. Then the push-forward of the virtual fundamental class $\pi_*(c_{top}(H^1(\phi^*N_{C/M_k^k})))$ can be computed only by intersection theory on the fiber $\overline{M}_{0,0}(C, m)$, which turns out to be equal to $\frac{1}{d^3}$. This depends on neither the structure of the base $\overline{M}_{0,0}(M_k^k, \frac{d}{m})$ nor the global structure of the fibration.

But when $k < N$, the situation is more complicated than M_k^k because of negative first Chern class. Let us concentrate on the case of $d = 2, m = 2$ that we discuss in this paper. In this case, C is just a line L on the hypersurface M_N^k . The moduli space $\overline{M}_{0,0}(M_N^k, 1)$ is a sub-manifold of $\overline{M}_{0,0}(\mathbf{P}^{N-1}, 1)$, while $\overline{M}_{0,0}(\mathbf{P}^{N-1}, 1)$ is the Grassmannian $G(2, N)$, the moduli space of rank 2 quotients of $V = \mathbf{C}^N$. As will be shown later, for a generic line L , N_{L/M_N^k} is decomposed into

$$N_{L/M_N^k} \simeq \mathcal{O}_L(-1)^{\oplus k-N+2} \oplus \mathcal{O}_L^{\oplus 2N-k-5}.$$

By pulling back it by the degree 2 map $\phi : \mathbf{P}^1 \rightarrow L$, we obtain,

$$\phi^* N_{L/M_N^k} \simeq O_{\mathbf{P}^1}(-2)^{\oplus k-N+2} \oplus O_{\mathbf{P}^1}^{\oplus 2N-k-5}.$$

Therefore, $h^1(\phi^*(N_{L/M_N^k})) = k-N+2$, which is strictly greater than two, the complex dimension of the fiber $\overline{M}_{0,0}(L, 2)$. Thus we need to know the global structure of the fibration π in order to compute the multiple cover contribution to degree 2 rational Gromov-Witten invariants of M_N^k .

In order to estimate the contributions from double cover maps $\phi : \mathbf{P}^1 \rightarrow L$ to $\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,2}$, we first computed the number of conics, that intersect cycles Poincaré dual to e^a, e^b and e^c , on M_N^k (whose normal bundle are of the same type) by using the method in [K2]. Then we found the following formula by comparing these integers with the results obtained from (3):

$$(4) \quad \langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,2} = (\text{number of corresponding conics}) + \int_{G(2,N)} c_{top}(S^k Q) \wedge \left[\frac{c(S^{k-1} Q)}{1 - \frac{1}{2}c_1(Q)} \right]_{k-N} \wedge \sigma_{a-1} \wedge \sigma_{b-1} \wedge \sigma_{c-1},$$

where Q is the universal rank 2 quotient bundle of $G(2, N)$, σ_a is a Schubert cycle defined by $\sum_{a=0}^{\infty} \sigma_a := \frac{1}{c(Q^{\vee})}$ and $[*]_{k-N}$ is the operation of picking up degree $2(k-N)$ part of Chern classes.

On the other hand, we have the following formula which directly follows from the definition of the virtual fundamental class of $\overline{M}_{0,0}(M_N^k, 2)$:

$$(5) \quad \langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,2} = (\text{number of corresponding conics}) + 8 \int_{G(2,N)} c_{top}(S^k Q) \wedge [\pi_*(c_{top}(H^1(\phi^* N_{L/M_N^k})))]_{k-N} \wedge \sigma_{a-1} \wedge \sigma_{b-1} \wedge \sigma_{c-1}.$$

where $\pi : \overline{M}_{0,0}(L, 2) \rightarrow \overline{M}_{0,0}(M_N^k, 1)$ is the natural projection. Here, the factor 8 comes from the divisor axiom of Gromov-Witten invariants.

In this paper, we prove the following formula

$$(6) \quad \pi_*(c_{top}(H^1(\phi^* N_{L/M_N^k}))) = \frac{1}{8} \left[\frac{c(S^{k-1} Q)}{1 - \frac{1}{2}c_1(Q)} \right]_{k-N}.$$

By combining (5) with (6), we can derive the formula (4) immediately.

From (4), we see that $\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,2}$ of M_N^k is a rational number with denominator at most 2^{k-N} . Therefore rationality (non-integrality) of the Gromov-Witten invariant $\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,2}$ is caused by the effect of multiple cover map in this case.

We note here that the total moduli space of double cover maps of lines is isomorphic to $\mathbf{P}(S^2 Q)$ over $G := \overline{M}_{0,0}(M_N^k, 1) \hookrightarrow G(2, N)$, which is an algebraic \mathbf{Q} -stack $\mathbf{P}(S^2 Q)^{stack}$ (in the sense of Mumford). As a consequence, the union of all $H^1(\phi^* N_{C/M_N^k})$ turns out to be a coherent sheaf on $\mathbf{P}(S^2 Q)^{stack}$ with fractional Chern class in (6), as was suggested in [BT]. See [V, Section 9].

We also did some numerical experiments on degree 3 Gromov-Witten invariants of M_N^k by using the results of [ES]. For $k-N > 0$, there is a new contribution from multiple cover maps to nodal conics in M_N^k that did not appear in the Calabi-Yau case. Therefore, multiple cover map contributions are far more complicated than Calabi-Yau, and we leave general analysis on this problem to future works.

This paper is organized as follows. In Section 1, we analyze characteristics of moduli space of lines in M_N^k and derive $N_{L/M_N^k} \simeq O_L(-1)^{\oplus k-N+2} \oplus O_L^{\oplus 2N-k-5}$. In Section 2, we study the moduli space $\overline{M}_{0,0}(\mathbf{P}^1, 2)$ from the point of view of stability and identify it with \mathbf{P}^2 and show that the moduli space $\overline{M}_{0,0}(\mathbf{P}^1, 2)$ is isomorphic to $\mathbf{P}(S^2Q)$ over G . In section 4, we describe $H^1(\phi^*N_{L/M_N^k})$ as a coherent sheaf over $\mathbf{P}(S^2Q)^{stack}$. In section 5, we derive the main theorem (6) of this paper by using Segre classes. In Section 6, we mention some generalization to degree 3 Gromov-Witten invariants.

2. LINES ON A HYPERSURFACE

Let M be a generic hypersurface of degree k of the projective space $\mathbf{P}^{N-1} = \mathbf{P}(V)$. We assume $2N - 5 \geq k \geq N - 2 \geq 2$ throughout this note. In this note we count the number of rational curves of virtual degree two, namely rational curves which doubly cover lines on M .

Let $\mathbf{P} = \mathbf{P}(V)$ be the projective space parameterizing all one-dimensional quotients of V , which is usually denoted by $\mathbf{P}(V)$ in the standard notation in algebraic geometry. In this notation let W be a subspace of V . Then $\mathbf{P}(W)$ is naturally a linear subspace of $\mathbf{P}(V)$ of dimension $\dim W - 1$.

Let $G(2, V)$ be the Grassmann variety of lines in $\mathbf{P}(V)$, the scheme parameterizing all lines of $\mathbf{P} = \mathbf{P}(V)$. This is also the universal scheme parameterizing all one-dimensional quotient linear spaces of V . Let W be a two dimensional quotient linear space, $\psi \in G(2, V)$, namely $\psi : \mathbf{P}(W) \rightarrow \mathbf{P}(V)$ the natural immersion and $i_\psi^* : V \rightarrow W$ the quotient homomorphism. The space W is denoted by $W(\psi)$ when necessary.

There exists the universal bundle $Q_{G(2,V)}$ over $G(2, V)$ and a homomorphism $i^{\text{univ}*} : O_{G(2,V)} \otimes V \rightarrow Q_{G(2,V)}$ whose fiber $i_\psi^{\text{univ}*} : V \rightarrow Q_{G(2,V),\psi}$ is the quotient $i_\psi^* : V \rightarrow W(\psi)$ of V corresponding to ψ .

2.1. Existence of a line on M . Let $L = \mathbf{P}(W)$ be a line of \mathbf{P} , equivalently $W \in G(2, V)$. Then the condition $L \subset M$ imposes at most $k + 1$ conditions on W , while the number of moduli of lines of \mathbf{P} equals $\dim G(2, V) = 2N - 4$. Hence we infer

Lemma 2.2. *If $2N \geq k + 5$, then there exists at least a line on M .*

See also [Katz,p.152]. Let G be the subscheme of $G(2, V)$ parameterizing all lines of $\mathbf{P}(V)$ lying on M , $Q = (Q_{G(2,V)})|_G$ the restriction of $Q_{G(2,V)}$ to G . By Lemma 2.2, G is nonempty. Let $i^* : O_G \otimes V \rightarrow Q$ be the restriction of $i^{\text{univ}*}$ to G . Let $P = \mathbf{P}(Q)$ and $\pi : P \rightarrow G$ the natural projection. Then π is the universal line of M over G , to be more exact, the universal family over G of lines lying on M . In other words, the natural epimorphism $i^* : O_G \otimes V \rightarrow Q$ induces a morphism $i : P \rightarrow \mathbf{P}_G(V) := G \times \mathbf{P}(V)$, which is a closed immersion into $\mathbf{P}_G(V)$, thus P is a subscheme of $\mathbf{P}_G(V)$ such that $\pi = (p_1)|_P$. Let $L_\psi = \mathbf{P}(Q_\psi)$. Note that

$$L_\psi = P_\psi := \pi^{-1}(\psi) \simeq \mathbf{P}(Q_\psi) \subset \{\psi\} \times \mathbf{P}(V) \simeq \mathbf{P}(V).$$

2.3. The normal bundle $N_{L/M}$. The argument of this section is standard and well known. Let $\mathbf{P} = \mathbf{P}(V)$, $L = \mathbf{P}(W)$ and $i_W^* : V \rightarrow W \in G$. Let us recall the following exact sequence:

$$0 \longrightarrow O_{\mathbf{P}} \longrightarrow O_{\mathbf{P}}(1) \otimes V^{\vee} \xrightarrow{D} T_{\mathbf{P}} \longrightarrow 0$$

where the homomorphism D is defined by

$$\begin{aligned} D(a \otimes v^{\vee}) &:= aD_{(v^{\vee})} \quad (a \in O_{\mathbf{P}}(1)) \\ (D_{v^{\vee}}F)(u^{\vee}) &:= \left(\frac{d}{dt}F(u^{\vee} + tv^{\vee})\right)_{t=0} \end{aligned}$$

for a homogeneous polynomial $F \in S(V)$ and $u^{\vee}, v^{\vee} \in V^{\vee}$. We note $H^0(O_{\mathbf{P}}(1)) \otimes V^{\vee} = V \otimes V^{\vee} = \text{End}(V, V)$ and that the image of $H^0(O_{\mathbf{P}})$ in $\text{End}(V, V)$ is $\mathbf{C} \text{id}_V$. We also have the following exact sequences:

$$\begin{aligned} 0 &\longrightarrow T_L \longrightarrow (T_{\mathbf{P}})_L \longrightarrow N_{L/\mathbf{P}} \longrightarrow 0 \\ 0 &\longrightarrow O_L \longrightarrow O_L(1) \otimes V^{\vee} \xrightarrow{D_L} (T_{\mathbf{P}})_L \longrightarrow 0. \end{aligned}$$

Lemma 2.4. *Let $L = \mathbf{P}(W)$. Then*

$$N_{L/\mathbf{P}} \simeq O_L(1) \otimes (V^{\vee}/W^{\vee}), \quad H^0(N_{L/\mathbf{P}}) \simeq W \otimes (V^{\vee}/W^{\vee}).$$

Proof. The assertion is clear from the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc} 0 & \longrightarrow & O_L & \longrightarrow & O_L(1) \otimes W^{\vee} & \xrightarrow{(D_L)|_{W^{\vee}}} & T_L & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \text{id} \otimes i^{\vee} & & \downarrow & & \downarrow \\ 0 & \longrightarrow & O_L & \longrightarrow & O_L(1) \otimes V^{\vee} & \xrightarrow{D_L} & (T_{\mathbf{P}})_L & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & O_L(1) \otimes (V^{\vee}/W^{\vee}) & \longrightarrow & N_{L/\mathbf{P}} & \longrightarrow & 0 \end{array}$$

The second assertion is clear from $H^0(L, O_L(1)) = W$. □

Since $T_L \simeq O_L(2)$, there follow exact sequences

$$\begin{aligned} 0 &\longrightarrow H^0(T_L) \longrightarrow H^0((T_{\mathbf{P}})_L) \longrightarrow H^0(N_{L/\mathbf{P}}) \longrightarrow 0 \\ 0 &\longrightarrow H^0(O_L) \longrightarrow H^0(O_L(1)) \otimes V^{\vee} \xrightarrow{H^0(D_L)} H^0((T_{\mathbf{P}})_L) \longrightarrow 0. \end{aligned}$$

We also note

$$H^0(T_L) = \text{Lie Aut}^0(L) = \text{End}(W, W) / \text{center} = \text{End}(W, W) / \mathbf{C} \text{id}_W.$$

Since $H^0(O_L(1)) = W$, we see

$$H^0((T_{\mathbf{P}})_L) = W \otimes V^{\vee} / \text{Im } H^0(O_L) = \text{Hom}(V, W) / \mathbf{C} i_W^*.$$

Hence we again see

$$\begin{aligned} H^0(N_{L/\mathbf{P}}) &= (\text{Hom}(V, W) / \mathbf{C} i_W^*) / (\text{Hom}(W, W) / \mathbf{C} \text{id}_W) \\ &= W \otimes (V^{\vee}/W^{\vee}) = \text{Hom}(V/W, W). \end{aligned}$$

For any line $L = \mathbf{P}(W)$ of \mathbf{P} the following sequence is exact:

$$(7) \quad 0 \rightarrow N_{L/M} \rightarrow N_{L/\mathbf{P}} \rightarrow (N_{M/\mathbf{P}})_L (\simeq O_L(k)) \rightarrow 0.$$

Hence $H^0(D_L^-)$ is injective by (iii). It follows that $H^0(N_{L/M}(-1)) = 0$. Hence (ii) is clear. Next we consider $H^0(D_L)$. By (iv), we see

$$S^k W = W \cdot H^0(D_L^-)(V^\vee/W^\vee) = H^0(D_L)(W \otimes V^\vee/W^\vee),$$

whence $H^0(D_L)$ is surjective. It follows that $H^1(N_{L/M}) = 0$. Hence $N_{L/M} \simeq O_L^{\oplus a} \oplus O_L(-1)^{\oplus b}$ for some a and b . Since $a + b = \text{rank}(N_{L/M}) = N - 3$ and $-b = \text{deg}(N_{L/M}) = N - 2 - k$, we have (i). \square

2.8. Lines on a quintic hypersurface in \mathbf{P}^4 . See [Katz, Appendix A] for the subsequent examples. Let $N = 5$ and $k = 5$. Hence M is a hypersurface of degree 5 in \mathbf{P}^4 , a Calabi-Yau 3-fold. Let

$$F = x_4 x_1^4 + x_5 x_2^4 + x_3^5 + x_4^5 + x_5^5.$$

First we note that $M = \{F = 0\}$ is nonsingular. Let $L = \{x_3 = x_4 = x_5 = 0\} = \{[s, t, 0, 0, 0]\}$. In this case $f_3 = 0$, $f_4 = s^4$ and $f_5 = t^4$. In the exact sequence (1) we see $H^0(N_{L/M}(-1)) = \text{Ker } H^0(D_L^-) = \mathbf{C}e_3^\vee$ and $H^1(N_{L/M}(-1)) = \text{Coker } H^0(D_L^-)$ is 3-dimensional. Hence $N_{L/M} = O_L(1) \oplus O_L(-3)$.

We summarize the above. If $\dim \text{Ker } H^0(D_L^-) = 1$ and if M is nonsingular, then $N_{L/M} = O_L(1) \oplus O_L(-3)$. Hence $H^0(N_{L/M}) = \text{Ker } H^0(D_L) = W \otimes \text{Ker } H^0(D_L^-)$ is 2-dimensional. Therefore we can choose $f_3 = 0$ and a linearly independent pair f_4 and $f_5 \in S^4 W$ so that $Wf_4 + Wf_5$ is 4-dimensional. The choice $f_4 = s^4$ and $f_5 = t^4$ satisfies the conditions. This enables us to find a nonsingular hypersurface M as above. However if we choose $f_3 = 0$, $f_4 = s^4$ and $f_5 = s^3 t$, then $Wf_4 + Wf_5$ is 3-dimensional. Hence M is singular.

Next in the same manner we find L on a nonsingular hypersurface M with $N_{L/M} = O_L \oplus O_L(-2)$ or $N_{L/M} = O_L(-1)^{\oplus 2}$. Let

$$F = x_3 x_1^4 + x_4 x_1^3 x_2 + x_5 x_2^4 + x_3^5 + x_4^5 + x_5^5.$$

Then we have $f_3 = s^4$, $f_4 = s^3 t$ and $f_5 = t^4$. Since $Wf_3 + Wf_4 + Wf_5$ is 5-dimensional, $H^0(N_{L/M}(-1)) = \text{Ker } H^0(D_L^-) = 0$, $H^0(N_{L/M}) = \text{Ker } H^0(D_L) = \mathbf{C}(te_3^\vee - se_4^\vee)$. We see also that $\dim H^1(N_{L/M}) = \dim \text{Coker } H^0(D_L) = 1$ and $N_{L/M} = O_L \oplus O_L(-2)$. The hypersurface $M = \{F = 0\}$ is easily shown to be nonsingular.

If $F = x_3 x_1^4 + x_4 x_1^2 x_2^2 + x_5 x_2^4 + x_3^5 + x_4^5 + x_5^5$ and $M = \{F = 0\}$, then $N_{L/M} = O_L(-1)^{\oplus 2}$.

2.9. Lines on a generic hypersurface M_7^8 of \mathbf{P}^6 . Let $N = 7$ and $k = 8$. In view of Lemma 2.2 there exists a line L on any generic hypersurface M of degree 8 in $\mathbf{P}(V) = \mathbf{P}^6$. In view of Lemma 2.7, $a = 1$, $b = 3$ and $N_{L/M} \simeq O_L \oplus O_L(-1)^{\oplus 3}$. For example let $L : x_j = 0$ ($j \geq 3$) and we take

$$\begin{aligned} F_3 &= 8x_1^7, F_4 = 8x_1^6 x_2, F_5 = 8x_1^4 x_2^3, F_6 = 8x_1^2 x_2^5, F_7 = 8x_2^7, \\ F &= x_3 F_3 + x_4 F_4 + x_5 F_5 + x_6 F_6 + x_7 F_7 + x_3^8 + x_4^8 + x_5^8 + x_6^8 + x_7^8. \end{aligned}$$

and let $M = M_7^8 : F = 0$. We see that M is nonsingular near L and has at most isolated singularities. However it is still unclear to us whether $M = M_7^8$ is

nonsingular everywhere. The space $H^0(N_{L/M})$ is spanned by $te_3^\vee - se_4^\vee$, hence an infinitesimal deformation L_ε of L is given by

$$[s, t] \mapsto [s, t, \varepsilon t, -\varepsilon s, 0, 0, 0]$$

which yields $F|_{L_\varepsilon} = \varepsilon^8(s^8 + t^8) \equiv 0 \pmod{\varepsilon^8}$. Since $H^1(N_{L/M}) = 0$, this infinitesimal deformation is integrable and G ($:=$ the moduli of lines of \mathbf{P}^6 contained in M) is nonsingular and one dimensional at the point $[L]$.

We note that M also contains 8 lines

$$L^j := L_{\varepsilon^8}^j : \varepsilon_8 x_1 - x_2 = x_3 + \varepsilon_8 x_4 = x_j = 0 \quad (j \geq 5),$$

with $N_{L^j/M} = O_{L^j}(1)^{\oplus 3} \oplus O_{L^j}(-6)$ where $\varepsilon_8^8 = -1$.

3. STABILITY

Definition 3.1. Suppose that a reductive algebraic group G acts on a vector space V . Let $v \in V$, $v \neq 0$.

- (1) the vector v is said to be *semistable* if there exists a G -invariant homogeneous polynomial F on V such that $F(v) \neq 0$,
- (2) the vector v is said to be *stable* if p has a closed G -orbit in X_{ss} and the stabilizer subgroup of v in G is finite.

Let $\pi : V \setminus \{0\} \rightarrow \mathbf{P}(V^\vee)$ be the natural surjection. Then $v \in V$ is semistable (resp. stable) if and only if $\pi(v)$ is semistable (resp. stable).

3.2. Grassmann variety. Let V be an N -dimensional vector space, and $G(r, N)$ the Grassmann variety parameterizing all r -dimensional quotient spaces of V . Here is a natural way of understanding $G(r, N)$ via GIT-stability. Let U be an r -dimensional vector space, $X = \text{Hom}(V, U)$ and $\pi : X \setminus \{0\} \rightarrow \mathbf{P}(X^\vee)$ the natural map. Then $\text{SL}(U)$ acts on X from the left by:

$$(g \cdot \phi^*)(v) = g \cdot (\phi^*(v)) \quad \text{for } \phi^* \in X, v \in V.$$

We see that for $\phi^* \in X$

$$\begin{aligned} \phi^* \text{ is } \text{SL}(U)\text{-stable} &\iff \text{rank } \phi^* = r, \\ \phi^* \text{ is } \text{SL}(U)\text{-semistable} &\iff \phi^* \text{ is } \text{SL}(U)\text{-stable.} \end{aligned}$$

In fact, if $\text{rank } \phi^* = r - 1$, then there is a one-parameter torus T of $\text{SL}(U)$ such that the closure of the orbit $T \cdot \phi$ contains the zero vector as the following simple example ($r = 2$) shows

$$\lim_{t \rightarrow 0} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \lim_{t \rightarrow 0} \begin{pmatrix} ta_{11} & ta_{12} & \cdots & ta_{1N} \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Let X_s be the set of all (semi)stable points and \mathbf{P}_s the image of X_s by π . It is, as we saw above, just the set of all $\phi \in X$ with $\text{rank } \phi^* = r$. Therefore the GIT-orbit space $\mathbf{P}_s // \text{SL}(U)$ is the orbit space $\mathbf{P}_s / \text{SL}(U)$ by the free action, the Grassmann variety $G(r, N)$.

3.3. Moduli of double coverings of \mathbf{P}^1 (1). Let W and U be a pair of two dimensional vector spaces, $X = \text{Hom}(W, S^2U)$, and $\pi : X \setminus \{0\} \rightarrow \mathbf{P}(X^\vee)$ the natural morphism. Note that $\text{SL}(U)$ acts on S^2U from the left via the natural action: $\sigma(u_1u_2) = \sigma(u_1)\sigma(u_2)$ for $\forall u_1, u_2 \in U$. Thus $\text{SL}(U)$ acts on X from the left in the same manner in the subsection 3.2.

Lemma 3.4. *Let $\phi^* \in X$.*

- (i) ϕ^* is unstable iff $\phi^*(w)$ has a double root for any $w \in W$,
- (ii) ϕ^* is semistable iff $\phi^*(w)$ has no double roots for some nonzero $w \in W$,
- (iii) ϕ^* is stable iff $\phi^*(W)$ is a base-point free linear subsystem of S^2U on $\mathbf{P}(U)$.

Proof. We note that ϕ^* is unstable iff there is a suitable basis s and t of U such that $\phi^*(w) = a(w)s^2$ for any $w \in W$ since a torus orbit $T \cdot \phi^*$ contains the zero vector. This proves (i). This also proves (ii). Next we prove (iii). If $\phi^*(W)$ has a base point, then it is clear that ϕ^* is not stable. If ϕ^* is semistable and it is not stable, then we choose a basis s, t of U and a basis w_1, w_2 of W such that $\phi^*(w_1) = st$. If $\phi^*(w_2) = as^2 + bst$, then ϕ^* is not stable. This proves the lemma. \square

Theorem 3.5. *Let X_{ss} be the Zariski open subset of X consisting of all semistable points of X , $\pi(X_{ss})$ the image of X_{ss} by π , and $Y := \pi(X_{ss}) // \text{SL}(U)$. Then $Y \simeq \mathbf{P}^2$.*

Proof. First consider a simplest case. We choose a basis s, t of U . Let w_1 and w_2 be a basis of W , T the subgroup of $\text{SL}(U)$ of diagonal matrices and $X' = \{\phi^* \in X; \phi^*(w_1) = 2st\}$. Let $Z' = \text{SL}(U) \cdot X'$.

We note that Z' is an $\text{SL}(U)$ -invariant subset of X_{ss} . We prove $\pi(Z') // \text{SL}(U) \simeq \mathbf{C}^2$. Let ϕ^* and ψ^* be points of X' . Let $\phi^*(w_2) = As^2 + 2Bst + Ct^2$ and $\psi^*(w_2) = as^2 + 2bst + ct^2$. Then it is easy to check

$$\begin{aligned} g \cdot \phi^* = \psi^* \text{ for } \exists g \in \text{SL}(U) &\iff g \cdot \phi^* = \psi^* \text{ for } \exists g \in T \\ &\iff A = au^2, B = b, C = u^{-2}c \text{ for } \exists u \neq 0. \end{aligned}$$

Therefore each equivalence class of $\pi(Z) // \text{SL}(U)$ is represented by the pair (AC, B) , which proves $\pi(Z) // \text{SL}(U) \simeq \mathbf{C}^2$.

Now we prove the lemma. Let $\phi^* \in X_{ss}$, $\phi_j = \phi^*(w_j)$ and $\phi_0 = -(\phi_1 + \phi_2)$. Let

$$\begin{aligned} \phi_0 &= r_1s^2 + 2r_2st + r_3t^2, \\ \phi_1 &= p_1s^2 + 2p_2st + p_3t^2, \\ \phi_2 &= q_1s^2 + 2q_2st + q_3t^2, \end{aligned}$$

and we define

$$\begin{aligned} D_1 &= p_2^2 - p_1p_3, \quad D_2 = q_2^2 - q_1q_3, \\ D_0 &= r_2^2 - r_1r_3 = D_1 + D_2 + 2p_2q_2 - (p_1q_3 + p_3q_1). \end{aligned}$$

To show the lemma, we prove the more precise isomorphism

$$\pi(X_{ss}) // \text{SL}(U) = \text{Proj } \mathbf{C}[D_0, D_1, D_2]$$

For this purpose we define $Y_j = \pi(\{\phi^* \in X_{ss}; \phi_j \text{ has no double roots}\}) // \text{SL}(U)$. It suffices to prove $Y_1 = \text{Spec } \mathbf{C}[\frac{D_0}{D_1}, \frac{D_2}{D_1}]$ by reducing it to the first simplest case.

Let $\phi^* \in Y_1$. Let α and β be the roots of $\phi_1 = 0$. By the assumption ϕ_1 has no double roots, hence $\alpha \neq \beta$. Let

$$u = \frac{1}{\gamma}(s - \alpha t), \quad v = \frac{1}{\gamma}(s - \beta t), \quad g = \frac{1}{\gamma} \begin{pmatrix} 1 & -\alpha \\ 1 & -\beta \end{pmatrix}$$

where $\gamma = \sqrt{\alpha - \beta}$. Note that $g \in \mathrm{SL}(U)$. Hence we see

$$(\phi_1(s, t), \phi_2(s, t)) \equiv (p_1 \gamma^4 uv, A_1 u^2 + 2B_1 uv + C_1 v^2)$$

where

$$\begin{aligned} A_1 &= q_1^2 \beta^2 + 2q_2 \beta + q_3, \\ -B_1 &= q_1 \alpha \beta + q_2(\alpha + \beta) + q_3, \\ C_1 &= q_1^2 \alpha^2 + 2q_2 \alpha + q_3. \end{aligned}$$

Thus we see

$$(\phi_1(s, t), \phi_2(s, t)) \equiv (2st, As^2 + 2Bst + Ct^2)$$

where

$$\begin{aligned} A &= \frac{2A_1}{p_1 \gamma^4}, \quad B = \frac{2B_1}{p_1 \gamma^4}, \quad C = \frac{2C_1}{p_1 \gamma^4}, \quad p_1^2 \gamma^4 = 4D_1, \\ AC &= B^2 - \frac{D_2}{D_1}, \quad B = \frac{D_0 - D_1 - D_2}{2D_1}. \end{aligned}$$

Therefore by the first half of the proof

$$Y_1 \simeq \mathrm{Spec} \mathbf{C}[AC, B] = \mathrm{Spec} \mathbf{C}\left[\frac{D_0}{D_1}, \frac{D_2}{D_1}\right].$$

This completes the proof of the lemma. \square

Corollary 3.6. *Let $Y^s = \pi(X_s) // \mathrm{SL}(U)$. Then $Y \setminus Y^s$ is a conic of Y defined by*

$$Y \setminus Y^s : D_0^2 + D_1^2 + D_2^2 - 2D_0 D_1 - 2D_1 D_2 - 2D_2 D_0 = 0.$$

Proof. In view of Theorem 3.5, $Y_1 \simeq \mathrm{Spec} \mathbf{C}[AC, B]$. The complement of Y_s in Y_1 is then the curve defined by $AC = 0$, which is easily identified with the above conic. \square

Corollary 3.7. *Let X^0 be the Zariski open subset of X consisting of all semistable points ϕ^* of X with $\mathrm{rank} \phi^* = 2$, and let $Y^0 := \pi(X^0) // \mathrm{SL}(U)$. Then $Y^0 \simeq \pi(X^0) / \mathrm{SL}(U) \simeq Y \simeq \mathbf{P}^2$.*

Proof. It suffices to compare Y_1 and $Y^0 \cap Y_1$. As in the proof of Theorem 3.5 we let $X' = \{\phi^* \in X; \phi^*(w_1) = 2st\}$. Let $Z = \mathrm{SL}(U) \cdot X'$ and $Z^0 = \mathrm{SL}(U) \cdot (X' \cap X^0)$.

Then with the notation in Theorem 3.5, we recall $X' = \{\phi^* \in X; \phi^*(w_1) = 2st, \phi^*(w_2) = As^2 + 2Bst + Ct^2\}$, $\pi(Z) // \mathrm{SL}(U) \simeq \mathrm{Spec} \mathbf{C}[AC, B]$ where

$$X' \cap X^0 = \{\phi^* \in X'; A \neq 0 \text{ or } C \neq 0\}.$$

In the same manner as before we see $\pi(Z^0) // \mathrm{SL}(U) \simeq \mathrm{Spec} \mathbf{C}[AC, B]$, whence $\pi(Z^0) // \mathrm{SL}(U) = \pi(Z) // \mathrm{SL}(U)$. This proves $Y^0 \cap Y_1 = Y_1$. This completes the proof of the corollary. \square

3.8. Moduli of double coverings of $\mathbf{P}(W)$ (2). There is an alternative way of understanding $\pi(X_{s,s})//\mathrm{SL}(U) \simeq \mathbf{P}^2$ by using the isomorphism $S^2\mathbf{P}^1 \simeq \mathbf{P}^2$. We use the following convention to denote a point of $\mathbf{P}(U) = U^\vee \setminus \{0\}/\mathbf{G}_m$: $(u : v) = us^\vee + vt^\vee \in U^\vee$ where s^\vee and t^\vee are a basis dual to s and t . In what follows we fix a basis w_1 and w_2 of W . Let $P := (a_1 : a_2)$ and $Q := (b_1 : b_2)$ be a pair of points of $\mathbf{P}(W) \simeq \mathbf{P}^1$. If $P \neq Q$, there is a double covering $\phi : \mathbf{P}(U) \rightarrow \mathbf{P}(W)$ ramifying at P and Q , unique up to isomorphism once we fix the base w_1 and w_2 :

$$\frac{b_2 w_1 - b_1 w_2}{a_2 w_1 - a_1 w_2} = \left(\frac{t}{s}\right)^2.$$

Thus ϕ is given explicitly by

$$\phi_1 := \phi^*(w_1) = b_1 s^2 - a_1 t^2, \quad \phi_2 := \phi^*(w_2) = b_2 s^2 - a_2 t^2, \quad \phi_0 = -(\phi_1 + \phi_2)$$

for which we have

$$D_1 = a_1 b_1, \quad D_2 = a_2 b_2, \quad D_0 = (a_1 + a_2)(b_1 + b_2).$$

The isomorphism $S^2\mathbf{P}^1 \simeq \mathbf{P}^2$ is given by $(P, Q) \mapsto (D_0, D_1, D_2)$. This shows

Corollary 3.9. *We have a natural isomorphism: $Y \simeq \mathbf{P}(S^2W)$.*

4. THE VIRTUAL NORMAL BUNDLE OF A DOUBLE COVERING

4.1. The case $N = 7$ and $k = 8$ revisited. We revisit the example in the subsection 2.9. Let $N = 7$ and $k = 8$. Let $L : x_j = 0$ ($j \geq 3$) and we take

$$\begin{aligned} F_3 &= 8x_1^7, F_4 = 8x_1^6x_2, F_5 = 8x_1^4x_2^3, F_6 = 8x_1^2x_2^5, F_7 = 8x_2^7, \\ F &= x_3F_3 + x_4F_4 + x_5F_5 + x_6F_6 + x_7F_7 + x_3^8 + x_4^8 + x_5^8 + x_6^8 + x_7^8. \end{aligned}$$

and let $M = M_8^5 : F = 0$. We often denote L also by $\mathbf{P}(W)$ with W a two dimensional vector space for later convenience. Since $H^0(D_L^-)$ is injective and $H^0(D_L)$ is surjective, we have $N_{L/M} \simeq O_L \oplus O_L(-1)^{\oplus 3}$. Hence $H^1(N_{L/M}(-1)) = H^1(O_L(-2)^{\oplus 3})$ is 3-dimensional. As we see easily, this follows also from the fact that $\mathrm{Coker} H^0(D_L^-)$ is freely generated by $x_1^5x_2^2$, $x_1^3x_2^4$ and $x_1x_2^6$.

Let $\phi^* = (\phi_1, \phi_2) \in X^0$. Then $\mathrm{Ker} H^0(\phi^*D_L)$ is generated by a single element $\phi_2e_3^\vee - \phi_1e_4^\vee$, while $\mathrm{Coker} H^0(\phi^*D_L)$ is generated by $S^2U \cdot \phi_1^5\phi_2^2$, $S^2U \cdot \phi_1^3\phi_2^4$ and $S^2U \cdot \phi_1\phi_2^6$. To be more precise, we see

$$\mathrm{Coker} H^0(\phi^*D_L) = \{\phi_1^5\phi_2^2, \phi_1^3\phi_2^4, \phi_1\phi_2^6\} \otimes S^2U/\{\phi_1, \phi_2\}.$$

In fact, this is proved as follows: first we consider the case where ϕ_1 and ϕ_2 has no common zeroes. In this case ϕ^* gives rise to a double covering $\phi : \mathbf{P}(U) \rightarrow \mathbf{P}(W)$ ($= L$), which we denote by L_ϕ for brevity. By pulling back by ϕ^* the normal sequence $0 \rightarrow N_{L/M} \rightarrow N_{L/\mathbf{P}} \rightarrow O_L(k) \rightarrow 0$ ($k = 8$) for the line L we infer an exact sequence

$$0 \rightarrow \phi^*N_{L/M} \rightarrow \phi^*N_{L/\mathbf{P}} \xrightarrow{\phi^*D_L} \phi^*O_L(k) \rightarrow 0,$$

which yields an exact sequence

$$\begin{aligned} 0 &\longrightarrow H^0(\phi^*N_{L/M}) \longrightarrow S^2U \otimes (V^\vee/W^\vee) \xrightarrow{H^0(\phi^*D_L)} H^0(O_{L_\phi}(2k)) \\ &\longrightarrow H^1(\phi^*N_{L/M}) \longrightarrow 0. \end{aligned}$$

Let $\eta = q_3 e_3^\vee + \cdots + q_7 e_7^\vee \in \text{Ker } H^0(\phi^* D_L)$, $q_j \in S^2 U$. Then we have

$$\phi_1^2(q_3 \phi_1^5 + q_4 \phi_1^4 \phi_2 + q_5 \phi_1^2 \phi_2^3 + q_6 \phi_2^5) = -q_7 \phi_2^7.$$

Since ϕ_1 and ϕ_2 are mutually prime and q_j is of degree two, we have $q_7 = 0$ and

$$\phi_1^2(q_3 \phi_1^3 + q_4 \phi_1^2 \phi_2 + q_5 \phi_2^3) = -q_6 \phi_2^5,$$

Hence $q_6 = 0$ and similarly we infer also $q_5 = 0$. Thus we have $q_3 \phi_1 + q_4 \phi_2 = 0$. This proves that $\text{Ker } H^0(\phi^* D_L)$ is generated by $\phi_2 e_3^\vee - \phi_1 e_4^\vee$.

Next we prove that $\text{Coker } H^0(\phi^* D_L)$ is generated by $\phi^* \text{Coker } H^0(D_L^-)$ over $S^2 U$, in fact over $S^2 U / \phi^*(W)$. Without loss of generality we may assume that $\phi_1 = 2st$ and $\phi_2 = \lambda s^2 + 2\nu st + t^2$ for some $\lambda \neq 0$ and $\nu \in \mathbf{C}$. Let $\phi^* W = \{\phi_1, \phi_2\}$. Then one checks $U \cdot \phi^* W = S^3 U$, and hence $S^2 U \cdot \phi^* W = S^4 U$, $S^{2m-2} U \cdot \phi^* W = S^{2m} U$ for $m \geq 2$. It follows $S^2 U \cdot \phi^*(S^{m-1} W) = S^{2m} U$ for $m \geq 1$. In fact, by the induction on m

$$\begin{aligned} S^2 U \cdot \phi^*(S^m W) &= S^2 U \cdot \phi^*(W) \cdot \phi^*(S^{m-1} W) \\ &= S^4 U \cdot \phi^*(S^{m-1} W) \\ &= S^2 U \cdot (S^2 U \cdot \phi^*(S^{m-1} W)) \\ &= S^2 U \cdot S^{2m} U = S^{2m+2} U. \end{aligned}$$

Therefore $H^0(O_{L_\phi}(2k)) = S^{16} U = S^2 U \cdot \phi^*(S^7 W)$. Hence

$$\begin{aligned} \text{Coker } H^0(\phi^* D_L) &= S^{16} U / \text{Im } H^0(\phi^* D_L) \\ &= S^2 U \cdot \phi^*(S^7 W) / S^2 U \cdot \phi^*(\text{Im } H^0(D_L^-)) \\ &= (S^2 U / \phi^*(W)) \cdot \phi^*(S^7 W / \text{Im } H^0(D_L^-)). \end{aligned}$$

because $\text{Coker } H^0(D_L^-) = S^7 W / \text{Im } H^0(D_L^-)$ and $W \cdot S^7 W \subset W \cdot \text{Im } H^0(D_L^-) = S^8 W$ by the choice of L . This proves that $\text{Coker } H^0(\phi^* D_L)$ is generated by $\phi^* \text{Coker } H^0(D_L^-)$ over $S^2 U / \phi^*(W)$. It follows $\text{Coker } H^0(\phi^* D_L) = (\phi^* \text{Coker } H^0(D_L^-)) \otimes (S^2 U / \phi^* W)$.

Finally we consider the case where ϕ_1 and ϕ_2 has a common zero. In this case we may assume $\phi_1 = 2st$ and $\phi_2 = 2\nu st + t^2$. In this case L_ϕ is a chain of two rational curves C'_ϕ and C''_ϕ where C_ϕ is the proper transform of $\mathbf{P}(U)$, where the double covering map from L_ϕ to $\mathbf{P}(W)$ is the union of the isomorphisms ϕ' and ϕ'' , say, $\phi = \phi' \cup \phi''$. Let $\psi_1 = 2s$ and $\psi_2 = 2\nu s + t$. Then ϕ' is induced by the homomorphism $(\phi')^* \in \text{Hom}(W, U)$ such that $(\phi')^*(w_j) = \psi_j$. On the other hand let $U''_\phi = \mathbf{C}\lambda + \mathbf{C}t$, $\psi''_1 = 2t$ and $\psi''_2 = \lambda + 2\nu t$ where we note ψ''_j is the linear part of ϕ_j in t with $s = 1$. Then $C''_\phi = \mathbf{P}(U''_\phi)$ and ϕ'' is induced by the homomorphism $(\phi'')^* \in \text{Hom}(W, U''_\phi)$ such that $(\phi'')^*(w_j) = \psi''_j$. Furthermore the pull back by ϕ^* of the normal sequence for L

$$0 \rightarrow \phi^* N_{L/M} \rightarrow \phi^* N_{L/\mathbf{P}} \xrightarrow{\phi^* D_L} \phi^* O_L(k) \rightarrow 0,$$

yields exact sequences with natural vertical homomorphisms:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \phi^* N_{L/M} & \longrightarrow & (\phi')^* N_{L/M} \oplus (\phi'')^* N_{L/M} & \longrightarrow & \mathbf{C} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \phi^* N_{L/\mathbf{P}} & \longrightarrow & (\phi')^* N_{L/\mathbf{P}} \oplus (\phi'')^* N_{L/\mathbf{P}} & \longrightarrow & V^\vee/W^\vee & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \phi^* O_L(k) & \longrightarrow & O_{C'_\phi}(k) \oplus O_{C''_\phi}(k) & \longrightarrow & \mathbf{C} & \longrightarrow & 0.
\end{array}$$

This yields the following long exact sequences:

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^0((\phi')^* N_{L/M}) & \longrightarrow & U \otimes V^\vee/W^\vee & \xrightarrow{H^0((\phi')^* D_L)} & S^k U \\
& & \longrightarrow & H^1((\phi')^* N_{L/M}) & \longrightarrow & 0 & \\
0 & \longrightarrow & H^0((\phi'')^* N_{L/M}) & \longrightarrow & U''_\phi \otimes V^\vee/W^\vee & \xrightarrow{H^0((\phi'')^* D_L)} & S^k U''_\phi \\
& & \longrightarrow & H^1((\phi'')^* N_{L/M}) & \longrightarrow & 0 &
\end{array}$$

whence $H^1((\phi')^* N_{L/M}) = H^1((\phi'')^* N_{L/M}) = 0$, and both $H^0((\phi')^* N_{L/M})$ and $H^0((\phi'')^* N_{L/M})$ are one-dimensional. Let U' be the subspace of U consisting of elements vanishing at $C'_\phi \cap C''_\phi$, namely the subspace spanned by t . Then the restriction of $H^0((\phi')^* D_L)$ to $U' \otimes V^\vee/W^\vee$ equals $t \cdot H^0((\phi')^* D_L^-)$. Hence

$$\begin{aligned}
\text{Coker } H^0(\phi^* D_L) &\simeq t \cdot S^7 U/t \cdot \text{Im } H^0((\phi')^* D_L^-) \oplus \text{Coker } H^0((\phi'')^* D_L) \\
&\simeq S^7 U/t \cdot \text{Im } H^0((\phi')^* D_L^-) \simeq \text{Coker } H^0((\phi')^* D_L^-).
\end{aligned}$$

One could understand the above isomorphism as

$$\text{Coker } H^0(\phi^* D_L) = \text{Coker}(\phi)^* H^0(D_L^-) \otimes (S^2 U/\phi^* W).$$

Thus $H^0(\phi^* N_{L/M})$ is one-dimensional, while $H^1(\phi^* N_{L/M})$ is 3-dimensional.

This is immediately generalized into the following

Lemma 4.2. *For any $\phi^* \in X^0$ we have*

$$\begin{aligned}
\text{Ker } H^0(\phi^* D_L) &= \phi^* \text{Ker } H^0(D_L), \\
\text{Coker } H^0(\phi^* D_L) &= (\phi^* \text{Coker } H^0(D_L^-)) \otimes (S^2 U/\phi^* W).
\end{aligned}$$

Lemma 4.3. *We define a line bundle \mathbf{L}_0 (resp. \mathbf{L}_1) on $Y (\simeq \mathbf{P}(S^2 W))$ by the assignment:*

$$X^0 \ni \phi^* \mapsto \phi^* \text{Ker } H^0(D_L) \text{ (resp. } \phi^* \text{Coker } H^0(D_L^-)).$$

Then $\mathbf{L}_k \simeq O_{\mathbf{P}(S^2 W)}$.

Proof. We know that $\phi^* \text{Ker } H^0(D_L)$ is generated by $\phi_2 e_3^\vee - \phi_1 e_4^\vee$. By the $\text{SL}(2)$ -variable change of s and t , ϕ_j is transformed into a new quadratic polynomial, which is however the same as the first ϕ_j . This shows the generator is unchanged, whence $\mathbf{L}_0 \simeq O_{\mathbf{P}(S^2 W)}$. The proof for \mathbf{L}_1 is the same. \square

Lemma 4.4. *We define a coherent sheaf \mathbf{L} on the stack $Y (\simeq \mathbf{P}(S^2W))$ (See Remark below) by the assignment:*

$$X^0 \ni \phi^* \mapsto S^2U/\phi^*W.$$

Then $\mathbf{L}^2 \simeq O_{\mathbf{P}(S^2W)}(-1)$.

Proof. The GIT-quotient Y^0 is covered with the images of X'_j :

$$X'_1 = \{(\phi_1, \phi_2) \in X^0; \phi_1 = 2st, \phi_2 = \lambda s^2 + 2\nu st + t^2, \lambda, \nu \in \mathbf{C}\},$$

$$X'_2 = \{(\phi_1, \phi_2) \in X^0; \phi_1 = ps^2 + 2qst + t^2, \phi_2 = 2st, p, q \in \mathbf{C}\}.$$

It is clear that the natural image of X'_j in Y is Y_j . The map ϕ given by $\phi^* = (\phi_1, \phi_2) \in Y_1$ has natural \mathbf{Z}_2 involution generated by,

$$r : (\sqrt{\lambda}s + t, \sqrt{\lambda}s - t) \rightarrow (\sqrt{\lambda}s + t, -(\sqrt{\lambda}s - t)).$$

Since

$$2st = \frac{1}{2\sqrt{\lambda}}((\sqrt{\lambda}s + t)^2 - (\sqrt{\lambda}s - t)^2),$$

$$\lambda s^2 + 2\nu st + t^2 = \frac{\nu}{2\sqrt{\lambda}}((\sqrt{\lambda}s + t)^2 - (\sqrt{\lambda}s - t)^2) + \frac{1}{2}((\sqrt{\lambda}s + t)^2 + (\sqrt{\lambda}s - t)^2),$$

it is clear that,

$$r^*(\phi_1) = \phi_1, \quad r^*(\phi_2) = \phi_2, \quad r^*(\lambda s^2 - t^2) = -(\lambda s^2 - t^2).$$

Therefore, we can decompose S^2U into $\langle \lambda s^2 - t^2 \rangle_{\mathbf{C}} \oplus \langle \phi_1, \phi_2 \rangle_{\mathbf{C}}$ with respect to eigenvalue of r^* and take $\lambda s^2 - t^2$ as canonical generator of S^2U/ϕ^*W . Similarly S^2U/ϕ^*W is generated by $ps^2 - t^2$ on Y_2 . The problem is therefore to write $\lambda s^2 - t^2$ as an $\Gamma(O_{Y_1 \cap Y_2})$ -multiple of $pu^2 - v^2$ when we write $\phi_2 = 2uv$ by a variable change in $\mathrm{GL}(2)$. The following variable change $(s, t) \mapsto (u, v)$ is in $\mathrm{GL}(2)$:

$$s = \frac{\sqrt{2\alpha}}{(\beta - \alpha)^2} \left(2u - \frac{(\beta - \alpha)^2}{2\alpha} v \right), \quad t = \frac{\sqrt{2\alpha}}{(\beta - \alpha)^2} \left(2\beta u - \frac{(\beta - \alpha)^2}{2} v \right),$$

where α, β are roots of the equation $\lambda s^2 + 2\nu st + t^2 = 0$. Under this coordinate change, ϕ_1 and ϕ_2 is rewritten as follows:

$$\phi_1 = \frac{\lambda}{(\nu^2 - \lambda)^2} u^2 + 2 \frac{\nu}{\nu^2 - \lambda} uv + v^2 =: pu^2 + 2quv + v^2, \quad \phi_2 = 2uv.$$

Then we have

$$pu^2 - v^2 = -\frac{2}{\beta - \alpha} (\lambda s^2 - t^2) = -\frac{1}{\sqrt{\nu^2 - \lambda}} (\lambda s^2 - t^2) = -\sqrt{\frac{D_1}{D_2}} (\lambda s^2 - t^2).$$

Similarly by computing the effect on S^2U/ϕ^*W by the variable change from X'_1 into X'_2 , we see that \mathbf{L}^2 is isomorphic to $O_{\mathbf{P}(S^2W)}(-1)$. This completes the proof. \square

Remark 4.5. We remark that the space X must be regarded as a \mathbf{Q} -stack Y^{stack} as follows: First we define $\phi_0 = -\phi_1 - \phi_2$. For each atlas X'_α we define an atlas Y_α^{stack}

($\alpha = 0, 1, 2$) by

$$\begin{aligned} Y_0^{stack} &= \{(\phi_0, \phi_1, \phi_2, \pm\psi_0) \in X^0 \times S^2U; \phi_0 = 2st, \phi_1 = as^2 + 2bst + t^2, \\ &\quad \psi_0 = as^2 - t^2 \ a, b \in \mathbf{C}\}, \\ Y_1^{stack} &= \{(\phi_0, \phi_1, \phi_2, \pm\psi_1) \in X^0 \times S^2U; \phi_1 = 2st, \phi_2 = \lambda s^2 + 2\nu st + t^2, \\ &\quad \psi_1 = \lambda s^2 - t^2 \ \lambda, \nu \in \mathbf{C}\}, \\ Y_2^{stack} &= \{(\phi_0, \phi_1, \phi_2, \pm\psi_2) \in X^0 \times S^2U; \phi_1 = ps^2 + 2qst + t^2, \phi_2 = 2st, \\ &\quad \psi_2 = ps^2 - t^2, p, q \in \mathbf{C}\}. \end{aligned}$$

Since $\mathbf{L}^2 \simeq O_{\mathbf{P}(S^2W)}(-1)$ we have $c_1(\mathbf{L}) = -\frac{1}{2}c_1(O_{\mathbf{P}(S^2W)}(1))$ in the Chow ring $A(Y^{stack})_{\mathbf{Q}} = A(X)_{\mathbf{Q}} = A(\mathbf{P}(S^2W))_{\mathbf{Q}}$.

5. PROOF OF THE MAIN THEOREM

Theorem 5.1.

$$\pi_*(c_{top}(H^1)) = \frac{1}{8} \left[\frac{c(S^{k-1}Q)}{1 - \frac{1}{2}c_1(Q)} \right]_{k-N},$$

where π is the natural projection from $\bar{M}_{0,0}(L, 2)$ to G and $[*]_{k-N}$ is the operation of picking up the degree $2(k-N)$ part of Chern classes.

Proof. From now on we denote the coherent sheaf \mathbf{L} in Lemma 4.4 by $O_{\mathbf{P}}(-\frac{1}{2})$. In view of the results from the previous section, what remains is to evaluate the top chern class of $(S^{k-1}Q/((V^\vee \otimes O_G)/Q^\vee)) \otimes O_{\mathbf{P}}(-\frac{1}{2})$ on $\mathbf{P}(S^2Q)$. Since double cover maps parametrized by $\mathbf{P}(S^2Q)$ have natural \mathbf{Z}_2 involution r given in the previous section, we have to multiply the result of integration on $\mathbf{P}(S^2Q)$ by the factor $\frac{1}{2}$ [BT], [FP]. With this set-up, let $\pi' : \mathbf{P}(S^2Q) \rightarrow G$ be the natural projection. Then what we have to compute is $\pi_*(c_{top}(H^1)) = \frac{1}{2}\pi'_*(c_{top}(H^1)) = \frac{1}{2}\pi'_*(c_{top}((S^{k-1}Q/((V^\vee \otimes O_G)/Q^\vee)) \otimes O_{\mathbf{P}}(-\frac{1}{2})))$. Let z be $c_1(O_{\mathbf{P}}(1))$. Then we obtain,

$$\begin{aligned} &\frac{1}{2}\pi'_*(c_{top}((S^{k-1}Q/((V^\vee \otimes O_G)/Q^\vee)) \otimes O_{\mathbf{P}}(-\frac{1}{2}))) \\ &= \frac{1}{2} \sum_{j=0}^{k-N+2} c_{k-N+2-j}(S^{k-1}Q \oplus Q^\vee) \cdot \pi_*(z^j) \cdot (-\frac{1}{2})^j \\ &= \frac{1}{8} \sum_{j=0}^{k-N} c_{k-N-j}(S^{k-1}Q \oplus Q^\vee) \cdot s_j(S^2Q) \cdot (-\frac{1}{2})^j \\ &= \frac{1}{8} \left[\frac{c(S^{k-1}Q) \cdot c(Q^\vee)}{1 - \frac{1}{2}c_1(S^2Q) + \frac{1}{4}c_2(S^2Q) - \frac{1}{8}c_3(S^2Q)} \right]_{k-N}, \end{aligned}$$

where $s_j(S^2Q)$ is the j -th Segre class of S^2Q . But if we decompose $c(Q)$ into $(1 + \alpha)(1 + \beta)$, we can easily see,

$$\begin{aligned} \frac{c(Q^\vee)}{1 - \frac{1}{2}c_1(S^2Q) + \frac{1}{4}c_2(S^2Q) - \frac{1}{8}c_3(S^2Q)} &= \frac{(1 - \alpha)(1 - \beta)}{(1 - \alpha)(1 - \frac{1}{2}(\alpha + \beta))(1 - \beta)} \\ &= \frac{1}{1 - \frac{1}{2}c_1(Q)}. \end{aligned}$$

□

Finally, by combining the above theorem with the divisor axiom of Gromov-Witten invariants, we can prove the decomposition formula of degree 2 rational Gromov-Witten invariants of M_N^k found from numerical experiments.

Corollary 5.2.

$$\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,2} = \langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,2 \rightarrow 2} + 8 \langle \pi_*(c_{top}(H^1)) \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,1},$$

where $\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,2 \rightarrow 2}$ is the number of conics that intersect cycles Poincaré dual to e^a , e^b and e^c . We also denote by $\langle \pi_*(c_{top}(H^1)) \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,1}$ the integral:

$$\int_{G(2,V)} c_{top}(S^k Q) \wedge \pi_*(c_{top}(H^1)) \wedge \sigma_{a-1} \wedge \sigma_{b-1} \wedge \sigma_{c-1}.$$

6. GENERALIZATION TO TWISTED CUBICS

In this section, we present a decomposition formula of degree 3 rational Gromov-Witten invariants found from numerical experiments using the results of [ES].

Conjecture 6.1. *If $k - N = 1$, we have the following equality:*

$$\begin{aligned} \pi_*(c_{top}(H^1)) = \\ \frac{1}{27} \left(\left(\frac{1}{24} (27k^2 - 55k + 26) k(k-1) + \frac{2}{9} \right) c_1(Q)^2 + \left(\frac{7}{6} (k+1)k(k-1) + \frac{1}{9} \right) c_2(Q) \right). \end{aligned}$$

where $\pi : \overline{M}_{0,0}(L, 3) \rightarrow \overline{M}_{0,0}(M_N^k, 1)$ is the natural projection.

In the $k - N > 1$ case, we have not found the explicit formula, because in the $d = 3$ case, we have another contribution from multiple cover maps of type $(2+1) \rightarrow (1+1)$. Here multiple cover map of type $(2+1) \rightarrow (1+1)$ is the map from nodal curve $\mathbf{P}^1 \vee \mathbf{P}^1$ to nodal conic $L_1 \vee L_2 \subset M_N^k$, that maps the first (resp. the second) \mathbf{P}^1 to L_1 (resp. L_2) by two to one (resp. one to one). In the $k - N = 1$ case, we have also determined the contributions from multiple cover maps of $(2+1) \rightarrow (1+1)$ to nodal conics.

Corollary 6.2. *If $k - N = 1$, $\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,3}$ is decomposed into the following contributions:*

$$\begin{aligned} \langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,3} = & \langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,3 \rightarrow 3} + \frac{1}{k} \left(\frac{9}{4} \langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \mathcal{O}_{e^3} \rangle_{0,1} \langle \mathcal{O}_{e^{N-5}} \rangle_{0,1} \right. \\ & + \frac{3}{2} \langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^{c+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-c-4}} \mathcal{O}_{e^c} \rangle_{0,1} + \frac{3}{2} \langle \mathcal{O}_{e^b} \mathcal{O}_{e^c} \mathcal{O}_{e^{a+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-a-4}} \mathcal{O}_{e^a} \rangle_{0,1} \\ & \left. + \frac{3}{2} \langle \mathcal{O}_{e^c} \mathcal{O}_{e^a} \mathcal{O}_{e^{b+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-b-4}} \mathcal{O}_{e^b} \rangle_{0,1} \right) \\ & + 27 \langle \pi_*(c_{top}(H^1)) \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,1}, \end{aligned}$$

where $\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,3 \rightarrow 3}$ is the number of twisted cubics that intersect cycles Poincaré dual to e^a , e^b and e^c .

Proof. In the $k - N = 1$ case, dimension of moduli space of multiple cover maps of $(2+1) \rightarrow (1+1)$ to nodal conics is given by $N - 6 + N - 6 - (N - 4) + 2 = N - 6$, hence the rank of H^1 is given by $N - 6 - (N - 5 - 3) = 2$. On the other hand, dimension of moduli space of $d = 2$ multiple cover maps of $\mathbf{P}^1 \rightarrow \mathbf{P}^1$ is 2, the degree of the form of $\tilde{\pi}_*(c_{top}(H^1))$ equals to $2 - 2 = 0$, where $\tilde{\pi}$ is the projection map

that projects out the fiber locally isomorphic to the moduli space of $d = 2$ multiple cover maps. This situation is exactly the same as the Calabi-Yau case. Therefore, we can use the well-known result by Aspinwall and Morrison, that says for n -point rational Gromov-Witten invariants for Calabi-Yau manifold, $\tilde{\pi}_*(c_{top}(H^1))$ for degree d multiple cover map is given by,

$$\tilde{\pi}_*(c_{top}(H^1)) = \frac{1}{d^{3-n}}.$$

With this formula, we add up all the combinatorial possibility of insertion of external operator \mathcal{O}_{e^a} , \mathcal{O}_{e^b} and \mathcal{O}_{e^c} ,

$$\begin{aligned} & \frac{1}{k} (\langle \tilde{\pi}_*(c_{top}(H^1)) \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \mathcal{O}_{e^3} \rangle_{0,1} \langle \mathcal{O}_{e^{N-5}} \rangle_{0,1} \\ & + \langle \tilde{\pi}_*(c_{top}(H^1)) \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^{c+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-c-4}} \mathcal{O}_{e^c} \rangle_{0,1} \\ & + \langle \tilde{\pi}_*(c_{top}(H^1)) \mathcal{O}_{e^b} \mathcal{O}_{e^c} \mathcal{O}_{e^{a+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-a-4}} \mathcal{O}_{e^a} \rangle_{0,1} \\ & + \langle \tilde{\pi}_*(c_{top}(H^1)) \mathcal{O}_{e^c} \mathcal{O}_{e^a} \mathcal{O}_{e^{b+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-b-4}} \mathcal{O}_{e^b} \rangle_{0,1} \\ & + \langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^{c+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-c-4}} \tilde{\pi}_*(c_{top}(H^1)) \mathcal{O}_{e^c} \rangle_{0,1} \\ & + \langle \mathcal{O}_{e^b} \mathcal{O}_{e^c} \mathcal{O}_{e^{a+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-a-4}} \tilde{\pi}_*(c_{top}(H^1)) \mathcal{O}_{e^a} \rangle_{0,1} \\ & + \langle \mathcal{O}_{e^c} \mathcal{O}_{e^a} \mathcal{O}_{e^{b+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-b-4}} \tilde{\pi}_*(c_{top}(H^1)) \mathcal{O}_{e^b} \rangle_{0,1} \\ & + \langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \mathcal{O}_{e^3} \rangle_{0,1} \langle \mathcal{O}_{e^{N-5}} \tilde{\pi}_*(c_{top}(H^1)) \rangle_{0,1}) \\ & = \frac{1}{k} (2 \langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \mathcal{O}_{e^3} \rangle_{0,1} \langle \mathcal{O}_{e^{N-5}} \rangle_{0,1} \\ & + \langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^{c+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-c-4}} \mathcal{O}_{e^c} \rangle_{0,1} + \langle \mathcal{O}_{e^b} \mathcal{O}_{e^c} \mathcal{O}_{e^{a+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-a-4}} \mathcal{O}_{e^a} \rangle_{0,1} \\ & + \langle \mathcal{O}_{e^c} \mathcal{O}_{e^a} \mathcal{O}_{e^{b+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-b-4}} \mathcal{O}_{e^b} \rangle_{0,1} \\ & + \frac{1}{2} \langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^{c+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-c-4}} \mathcal{O}_{e^c} \rangle_{0,1} + \frac{1}{2} \langle \mathcal{O}_{e^b} \mathcal{O}_{e^c} \mathcal{O}_{e^{a+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-a-4}} \tilde{\pi}_* \mathcal{O}_{e^a} \rangle_{0,1} \\ & + \frac{1}{2} \langle \mathcal{O}_{e^c} \mathcal{O}_{e^a} \mathcal{O}_{e^{b+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-b-4}} \mathcal{O}_{e^b} \rangle_{0,1} \\ & + \frac{1}{4} \langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \mathcal{O}_{e^3} \rangle_{0,1} \langle \mathcal{O}_{e^{N-5}} \rangle_{0,1}). \end{aligned}$$

The last expression is nothing but the formula we want. \square

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REFERENCES

- [AM] P.S. Aspinwall, D.R. Morrison, *Topological Field Theory and Rational Curves* Commun.Math.Phys. 151 (1993) 245-262
- [BT] A. Bertram, M. Thaddeus, *On the quantum cohomology of a symmetric product of an algebraic curve*, math.AG/9803026.

- [CG] T. Coates, A. B. Givental, *Quantum Riemann-Roch, Lefschetz and Serre*, math.AG/0110142.
- [ES] G. Ellingsrud, S. A. Stromme, *Bott's Formula and enumerative geometry*, alg-geom/9411005.
- [FP] B. Fantechi, R. Pandharipande, *Stable maps and branch divisors*, math.AG/9905104.
- [Iri] H. Iritani, *Quantum D-modules and generalized mirror transformations*, math.DG/0411111.
- [J] M. Jinzenji, *Coordinate Change of Gauss-Manin System and Generalized Mirror Transformation*, math.AG/0310212.
- [K1] S. Katz, *On the finiteness of rational curves on quintic threefolds*, *Compositio Math.* **60** (1986) 151-162.
- [K2] S. Katz, *Rational curves on Calabi-Yau manifolds: verifying predictions of Mirror Symmetry*, alg-geom/9301006.
- [KM] M. Kontsevich, Yu. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, *Commun.Math.Phys.* 164 (1994) 525-562.
- [M] Yu. Manin. *Generating functions in algebraic geometry and sums over trees*, *The Moduli Space of Curves*, ed. by R. Dijkgraaf, C. Faber, G. van der Geer, *Progress in Math.* vol. 129, Birkhäuser, 1995, 401.
- [V] A. Vistoli. *Intersection theory on algebraic stacks and on their moduli spaces*, *Invent. Math.* 97 (1989) 613-670.