CONICS ON A GENERIC HYPERSURFACE

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ABSTRACT. In this paper, we compute the contributions from double cover maps
to genus 0 degree 2 Gromov-Witten invariants of general type projective hypersur-
faces. Our results correspond to a generalization of Aspinwall-Morrison formula
to general type hypersurfaces in some special cases.
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1. INTRODUCTION

In this paper, we discuss a generalization of the multiple cover formula for rational
Gromov-Witten invariants of Calabi-Yau manifolds [AM], [M] to double cover maps
of a line \( L \) on a degree \( k \) hypersurface \( M^k_N \) in \( P^{N-1} \). Naively, for a given finite set of
elements \( \alpha_j \in H^*(M^k_N, \mathbb{Z}) \), the rational Gromov-Witten invariant \( \langle \mathcal{O}_{e_1} \mathcal{O}_{e_2} \cdots \mathcal{O}_{e_m} \rangle_{0,d} \)
of \( M^k_N \) counts the number of degree \( d \) (possibly singular and reducible) rational
curves on \( M^k_N \) that intersect real sub-manifolds of \( M^k_N \) that are Poincaré-dual to \( \alpha_j \).

Recently, the mirror computation of rational Gromov-Witten invariants of \( M^k_N \)
with negative first chern class \( (k-N>0) \) was established in [CG], [Iri], [J]. Using the
method presented in these articles, we can compute \( \langle \mathcal{O}_{e_1} \mathcal{O}_{e_2} \cdots \mathcal{O}_{e_m} \rangle_{0,d} \) where \( e \)
is the generator of \( H^{1,1}(M^k_N, \mathbb{Z}) \). Briefly, mirror computation of \( M^k_N \) \( (k>N) \) in [J]
goes as follows. We start from the following ODE:

\[
(\partial_x)^{N-1} - k \cdot \exp(x) \cdot (k \partial_x + k - 1)(k \partial_x + k - 2) \cdots (k \partial_x + 1) w(x) = 0,
\]

and construct the virtual Gauss-Manin system associated with (1):

\[
\partial_x \tilde{\psi}_{N-2-m}(x) = \tilde{\psi}_{N-1-m}(x) + \sum_{d=1}^{\infty} \exp(dx) \cdot \tilde{L}^{N,k,d} \cdot \tilde{\psi}_{N-1-m-(N-k)d}(x),
\]

where \( m \) runs through all the integers and \( \tilde{L}^{N,k,d} \) is non-zero only if \( 0 \leq m \leq N-1+(k-N)d \). From the compatibility of (1) and (2), we can derive the recursive
formulas that determine all the \( \tilde{L}^{N,k,d} \):

\[
\sum_{n=0}^{k-1} \tilde{L}^{N,k,1}_{1j} w^m = k \cdot \prod_{j=1}^{k-1} (jw + (k-j)),
\]

\[
\sum_{m=0}^{N-1+(k-N)d} \tilde{L}^{N,k,d}_{1m} = \sum_{l=2}^{d} (-1)^l \sum_{0 \leq i_0 < \cdots < i_d = d} \times
\]

\[
\sum_{j_0=0}^{N-1+(k-N)d} \cdots \sum_{j_d=0}^{N-1+(k-N)d} \prod_{n=1}^{l} \left( \frac{i_{n-1} + (d-i_{n-1})z}{d} j_n - j_{n-1} \right) \tilde{L}^{N,k,i_0-i_{n-1}} j_{n+1} + (N-k)j_{n-1}.
\]

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With these data, we can construct the formulas that represent rational three point Gromov-Witten invariant $\langle \mathcal{O}_e \mathcal{O}_{e_N-2-m} \mathcal{O}_{e_{m-1-(k-N)d}} \rangle_d$ in terms of $\tilde{L}_m^{N,k,d}$. These three point Gromov-Witten invariants are enough for reconstruction of all the rational Gromov-Witten invariants $\langle \mathcal{O}_{e_1} \mathcal{O}_{e_2} \cdots \mathcal{O}_{e_{m+n}} \rangle_{d,0}$ [KM]. In particular, we obtain the following formula in the $d = 2$ case:

$$
(3) \quad \langle \mathcal{O}_e \mathcal{O}_{e_N-2-m} \mathcal{O}_{e_{m-1-(k-N)2}} \rangle_2 = k \cdot \left( \tilde{L}_m^{N,k,2} - \tilde{L}_1^{N,k,2} - 2 \tilde{L}_1^{N,k,1} \left( \sum_{j=0}^{k-N} \left( \tilde{L}_m^{N,k,1} - \tilde{L}_1^{N,k,1} \right) \right) \right).
$$

According to the results of this procedure, rational three point Gromov-Witten invariants can be rational numbers with large denominator if $k > N$, in contrast to the Calabi-Yau case where rational three point Gromov-Witten invariants are always integers.

One of the reasons of this rationality (non-integrality) comes from the contributions of multiple cover maps to Gromov-Witten invariants. In the Calabi-Yau case ($N = k$), for any divisor $m$ of $d$ there are some contributions from degree $m$ multiple cover maps $\phi$ of a rational curve $\mathbf{P}^1$ onto a degree $\frac{d}{m}$ rational curve $C \hookrightarrow M_k^N$. The contributions from the multiple cover maps are expressed in terms of the virtual fundamental class of Gromov-Witten invariants. Let $C$ be a general degree $d$ rational curve in $M_k^N$. Its normal bundle $N_{C/M_k^N}$ is decomposed into a direct sum of line bundles as follows:

$$
N_{C/M_k^N} \simeq \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1) \oplus \mathcal{O}_C^{\oplus (k-5)}.
$$

Let $\phi : \mathbf{P}^1 \to C$ be a holomorphic map of degree $m$. Since the pull-back $\phi^*(N_{C/M_k^N})$ is given by

$$
\phi^*(N_{C/M_k^N}) \simeq \mathcal{O}_{\mathbf{P}^1}(-m) \oplus \mathcal{O}_{\mathbf{P}^1}(-m) \oplus \mathcal{O}_{\mathbf{P}^1}^{\oplus (k-5)},
$$

we obtain $h^1(\phi^*(N_{C/M_k^N})) = 2m - 2$. On the other hand, let $\overline{M}_{0,0}(M,d)$ be the moduli space of 0-pointed stable maps of degree $d$ from genus 0 curve to $M$. Then the moduli space of $\phi$ is the fiber space $\pi : \overline{M}_{0,0}(C,m) \to \overline{M}_{0,0}(M_k^N, \frac{d}{m})$, whose fibre $\overline{M}_{0,0}(C,m)$ over $C$ (fixed) has complex dimension $2m - 2$. Then the push-forward of the virtual fundamental class $\pi_*\mathcal{O}_{\overline{M}_{0,0}(C,m)}(H^1(\phi^*N_{C/M_k^N}))$ can be computed only by intersection theory on the fiber $\overline{M}_{0,0}(C,m)$, which turns out to be equal to $\frac{1}{m^2}$. This depends on neither the structure of the base $\overline{M}_{0,0}(M_k^N, \frac{d}{m})$ nor the global structure of the fibration.

But when $k < N$, the situation is more complicated than $M_k^N$ because of negative first Chern class. Let us concentrate on the case of $d = 2, m = 2$ that we discuss in this paper. In this case, $C$ is just a line $L$ on the hypersurface $M_k^N$. The moduli space $\overline{M}_{0,0}(M_k^{N,1})$ is a sub-manifold of $\overline{M}_{0,0}(P^{N-1},1)$, while $\overline{M}_{0,0}(P^{N-1},1)$ is the Grassmannian $G(2,N)$, the moduli space of rank 2 quotients of $V = \mathbb{C}^N$. As will be shown later, for a generic line $L$, $N_{L/M_k^N}$ is decomposed into

$$
N_{L/M_k^N} \simeq \mathcal{O}_L(-1)^{\oplus k-N+2} \oplus \mathcal{O}_L^{\oplus 2N-k-5}.
$$
By pulling back it by the degree 2 map $\phi : \mathbb{P}^1 \to L$, we obtain,
$$\phi^* N_{L/M^k_N} \simeq O_{\mathbb{P}^1}(-2)^{2(k-N+2)} \oplus O_{\mathbb{P}^1}^{2N-k-5}.$$ 
Therefore, $h^1(\phi^*(N_{L/M^k_N})) = k - N + 2$, which is strictly greater than two, the complex dimension of the fiber $\mathcal{M}_{0,0}(L, 2)$. Thus we need to know the global structure of the fibration $\pi$ in order to compute the multiple cover contribution to degree 2 rational Gromov-Witten invariants of $M^k_N$.

In order to estimate the contributions from double cover maps $\phi : \mathbb{P}^1 \to \mathcal{O}_{C^e_i} \mathcal{O}_{C^e_i} \mathcal{O}_{C^e_i}$, we first computed the number of conics, that intersect cycles Poâncaré dual to $e^a, e^b$ and $e^c$, on $M^k_N$ (whose normal bundle are of the same type) by using the method in [K2]. Then we found the following formula by comparing these integers with the results obtained from (3):

$$\langle \mathcal{O}_{C^e_i} \mathcal{O}_{C^e_i} \mathcal{O}_{C^e_i} \rangle_{0,2} = \text{(number of corresponding conics)} + \int_{G(2,N)} c_{\text{top}}(S^k Q) \wedge \left[ \frac{c(S^k Q)}{1 - 2c_1(Q)} \right]_{k-N} \wedge \sigma_a \wedge \sigma_b \wedge \sigma_c,$$

where $Q$ is the universal rank 2 quotient bundle of $G(2,N)$, $\sigma_a$ is a Schubert cycle defined by $\sum_{a=0}^2 \sigma_a := \frac{1}{|\text{PGL}_2|}$ and $[\ast]_{k-N}$ is the operation of picking up degree $2(k - N)$ part of Chern classes.

On the other hand, we have the following formula which directly follows from the definition of the virtual fundamental class of $\overline{\mathcal{M}}_{0,0}(M^k_N, 2)$:

$$\langle \mathcal{O}_{C^e_i} \mathcal{O}_{C^e_i} \mathcal{O}_{C^e_i} \rangle_{0,2} = \text{(number of corresponding conics)} + \int_{G(2,N)} c_{\text{top}}(S^k Q) \wedge \left[ \pi_*(c_{\text{top}}(H^1(\phi^* N_{L/M^k_N}))) \right]_{k-N} \wedge \sigma_a \wedge \sigma_b \wedge \sigma_c,$$

where $\pi : \overline{\mathcal{M}}_{0,0}(L, 2) \to \overline{\mathcal{M}}_{0,0}(M^k_N, 1)$ is the natural projection. Here, the factor 8 comes from the divisor axiom of Gromov-Witten invariants.

In this paper, we prove the following formula

$$\pi_* (c_{\text{top}}(H^1(\phi^* N_{L/M^k_N}))) = \frac{1}{8} \left[ \frac{c(S^k Q)}{1 - 2c_1(Q)} \right]_{k-N}.$$

By combining (5) with (6), we can derive the formula (4) immediately.

From (4), we see that $\langle \mathcal{O}_{C^e_i} \mathcal{O}_{C^e_i} \mathcal{O}_{C^e_i} \rangle_{0,2}$ of $M^k_N$ is a rational number with denominator at most $2^{k-N}$. Therefore rationality (non-integrality) of the Gromov-Witten invariant $\langle \mathcal{O}_{C^e_i} \mathcal{O}_{C^e_i} \mathcal{O}_{C^e_i} \rangle_{0,2}$ is caused by the effect of multiple cover map in this case.

We note here that the total moduli space of double cover maps of lines is isomorphic to $\mathbb{P}(S^2 Q)$ over $G := \overline{\mathcal{M}}_{0,0}(M^k_N, 1) \hookrightarrow G(2,N)$, which is an algebraic $\mathbb{Q}$-stack $\mathbb{P}(S^2 Q)^{\text{stack}}$ (in the sense of Mumford). As a consequence, the union of all $H^1(\phi^* N_{L/M^k_N})$ turns out to be a coherent sheaf on $\mathbb{P}(S^2 Q)^{\text{stack}}$ with fractional Chern class in (6), as was suggested in [BT]. See [V, Section 9].

We also did some numerical experiments on degree 3 Gromov-Witten invariants of $M^k_N$ by using the results of [ES]. For $k - N > 0$, there is a new contribution from multiple cover maps to nodal conics in $M^k_N$ that did not appear in the Calabi-Yau case. Therefore, multiple cover map contributions are far more complicated than Calabi-Yau, and we leave general analysis on this problem to future works.
This paper is organized as follows. In Section 1, we analyze characteristics of moduli space of lines in $M^k_N$ and derive $N_{L/M^k_N} \simeq O_L(-1)^{2k-N+2} \oplus O_L^{2N-k-5}$. In Section 2, we study the moduli space $M_{0,0}(P^1, 2)$ from the point of view of stability and identify it with $P^2$ and show that the moduli space $M_{0,0}(P^1, 2)$ is isomorphic to $P(S^2 Q)$ over $G$. In section 4, we describe $H^1(\phi^*N_{L/M^k_N})$ as a coherent sheaf over $P(S^2 Q)_{\text{stack}}$. In section 5, we derive the main theorem (6) of this paper by using Segre-Witten classes. In Section 6, we mention some generalization to degree 3 Gromov-Witten invariants.

2. LINES ON A HYPERSURFACE

Let $M$ be a generic hypersurface of degree $k$ of the projective space $P^{N-1} = P(V)$. We assume $2N - 5 \geq k \geq N - 2 \geq 2$ throughout this note. In this note we count the number of rational curves of virtual degree two, namely rational curves which doubly cover lines on $M$.

Let $P = P(V)$ be the projective space parameterizing all one-dimensional quotients of $V$, which is usually denoted by $P(V)$ in the standard notation in algebraic geometry. In this notation let $W$ be a subspace of $V$. Then $P(W)$ is naturally a linear subspace of $P(V)$ of dimension $\dim W - 1$.

Let $G(2, V)$ be the Grassmann variety of lines in $P(V)$, the scheme parameterizing all lines of $P = P(V)$. This is also the universal scheme parameterizing all one-dimensional quotient linear spaces of $V$. Let $W$ be a two dimensional quotient linear space, $\psi \in G(2, V)$, namely $\psi : P(W) \to P(V)$ the natural immersion and $i_\psi^* : V \to W$ the quotient homomorphism. The space $W$ is denoted by $W(\psi)$ when necessary.

There exists the universal bundle $Q_{G(2, V)}$ over $G(2, V)$ and a homomorphism $i^{\text{univ}}^* : O_{G(2, V)} \otimes V \to Q_{G(2, V)}$ whose fiber $i^{\text{univ}}_{\psi}^* : V \to Q_{G(2, V), \psi}$ is the quotient $i_{\psi}^* : V \to W(\psi)$ of $V$ corresponding to $\psi$.

2.1. Existence of a line on $M$. Let $L = P(W)$ be a line of $P$, equivalently $W \in G(2, V)$. Then the condition $L \subset M$ imposes at most $k + 1$ conditions on $W$, while the number of moduli of lines of $P$ equals $\dim G(2, V) = 2N - 4$. Hence we infer

Lemma 2.2. If $2N \geq k + 5$, then there exists at least a line on $M$.

See also [Katz.p.152]. Let $G$ be the subscheme of $G(2, V)$ parameterizing all lines of $P(V)$ lying on $M$, $Q = (Q_{G(2, V)})|_{G}$ the restriction of $Q_{G(2, V)}$ to $G$. By Lemma 2.2, $G$ is nonempty. Let $i^* : O_G \otimes V \to Q$ be the restriction of $i^{\text{univ}}^*$ to $G$. Let $P = P(Q)$ and $\pi : P \to G$ the natural projection. Then $\pi$ is the universal line of $M$ over $G$, to be more exact, the universal family over $G$ of lines lying on $M$. In other words, the natural epimorphism $i^* : O_G \otimes V \to Q$ induces a morphism $i : P \to P_G(V) := G \times P(V)$, which is a closed immersion into $P_G(V)$, thus $P$ is a subscheme of $P_G(V)$ such that $\pi = (p_1)_p$. Let $L_\psi = P(\phi_\psi)$.

Note that

$L_\psi = P_\psi := \pi^{-1}(\psi) \simeq P(Q_\psi) \subset \{\psi\} \times P(V) \simeq P(V)$.
2.3. The normal bundle $N_{L/M}$. The argument of this section is standard and well known. Let $P = P(V)$, $L = P(W)$ and $i_W^*: V \to W \in G$. Let us recall the following exact sequence:

$$0 \longrightarrow O_P \longrightarrow O_P(1) \otimes V^\vee \xrightarrow{D} T_P \longrightarrow 0$$

where the homomorphism $D$ is defined by

$$D(a \otimes v^\vee) = aD_{(v)} \quad (a \in O_P(1))$$

$$(D_{v^\vee}F)(u^\vee) = \frac{d}{dt}F(u^\vee + tv^\vee)|_{t=0}$$

for a homogeneous polynomial $F \in S(V)$ and $u^\vee, v^\vee \in V^\vee$. We note $H^0(O_P(1)) \otimes V^\vee = V \otimes V^\vee = \text{End}(V, V)$ and that the image of $H^0(O_P)$ in $\text{End}(V, V)$ is $\text{Clid}_V$. We also have the following exact sequences:

$$0 \longrightarrow T_L \longrightarrow (T_P)_L \longrightarrow N_{L/P} \longrightarrow 0$$

$$0 \longrightarrow O_L \longrightarrow O_L(1) \otimes V^\vee \xrightarrow{D_L} (T_P)_L \longrightarrow 0.$$ 

**Lemma 2.4.** Let $L = P(W)$. Then

$$N_{L/P} \simeq O_L(1) \otimes (V^\vee/W^\vee), \quad H^0(N_{L/P}) \simeq W \otimes (V^\vee/W^\vee).$$

**Proof.** The assertion is clear from the following commutative diagram with exact rows and columns:

$$\begin{array}{cccccc}
0 & \longrightarrow & O_L & \longrightarrow & O_L(1) \otimes W^\vee & \xrightarrow{(D_L)_W^\vee} & T_L & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & O_L & \longrightarrow & O_L(1) \otimes V^\vee & \xrightarrow{D_L} & (T_P)_L & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & O_L(1) \otimes (V^\vee/W^\vee) & \longrightarrow & N_{L/P} & \longrightarrow & 0
\end{array}$$

The second assertion is clear from $H^0(L, O_L(1)) = W$. \hfill \Box

Since $T_L \simeq O_L(2)$, there follow exact sequences

$$0 \longrightarrow H^0(T_L) \longrightarrow H^0((T_P)_L) \longrightarrow H^0(N_{L/P}) \longrightarrow 0$$

$$0 \longrightarrow H^0(O_L) \longrightarrow H^0(O_L(1)) \otimes V^\vee \xrightarrow{H^0(D_L)} H^0((T_P)_L) \longrightarrow 0.$$ 

We also note

$$H^0(T_L) = \text{LieAut}^0(L) = \text{End}(W, W)/\text{center} = \text{End}(W, W)/\text{Clid}_W.$$ 

Since $H^0(O_L(1)) = W$, we see

$$H^0((T_P)_L) = W \otimes V^\vee/\text{Im} \ H^0(O_L) = \text{Hom}(V, W)/\text{Clid}_W.$$ 

Hence we again see

$$H^0(N_{L/P}) = (\text{Hom}(V, W)/\text{Clid}_W)/(\text{Hom}(W, W)/\text{Clid}_W)$$

$$= W \otimes (V^\vee/W^\vee) = \text{Hom}(V/W, W).$$

For any line $L = P(W)$ of $P$ the following sequence is exact:

$$0 \rightarrow N_{L/M} \rightarrow N_{L/P} \rightarrow (N_{M/P})_L (\simeq O_L(k)) \rightarrow 0.$$ (7)
Hence so is the following sequence as well:

\[
\begin{array}{c}
0 \rightarrow H^0(N_{L/M}) \rightarrow H^0(N_{L/P}) \xrightarrow{h^0(D_L)} H^0(O_L(k)) \\
\rightarrow H^1(N_{L/M}) \rightarrow 0.
\end{array}
\]

Hence we have

**Lemma 2.5.** The following is exact:

\[(8) \quad 0 \rightarrow H^0(N_{L/M}) \rightarrow W \otimes (V^\vee/W^\vee) \xrightarrow{h^0(D_L)} S^kW \rightarrow H^1(N_{L/M}) \rightarrow 0.
\]

**Corollary 2.6.** \(\dim G \geq 2N - k - 5\), equality holding if \(H^1(N_{L/M}) = 0\).

**Proof.** As is well-known, \(\dim G \geq h^0(N_{L/M}) - h^1(N_{L/M})\). Note \(\dim W \otimes (V^\vee/W^\vee) = 2(N - 2)\) and \(\dim S^kW = k + 1\). Hence the corollary follows from Lemma 2.5. \(\square\)

**Lemma 2.7.** For a generic line \(L\) on a generic hypersurface \(M\) of degree \(k\)

(i) \(N_{L/M} \simeq O_L^{ab} \oplus O_L(-1)^b\), where \(a = 2N - k - 5\) and \(b = k - N + 2\),

(ii) \(\text{Coker} H^0(D_L) \simeq S^{k-1}W/(V^\vee/W^\vee)\) where \(D_L \simeq D_L \oplus O_L(-1)\).

**Proof.** Let \(M\) be a generic hypersurface of degree \(k\) and \(L\) a generic line on \(M\). Without loss of generality we may assume that \(W^\vee\) is generated by \(e^1, e^2, \ldots \), in other words, \(\psi : L \rightarrow P\) is given by

\[\psi : [s : t] \rightarrow [x_1, \ldots, x_N] = [s, t, 0, \ldots, 0].\]

Then \(F\), the polynomial of degree \(k\) defining \(M\), is written as

\[F = x_3F_3 + x_4F_4 + \cdots + x_NF_N\]

for some polynomials \(F_j\) of degree \(k - 1\). Let \(f_j = \psi^*F_j = F_j(s, t, 0, \ldots, 0)\).

Now we consider the exact sequence

\[
\begin{array}{c}
0 \rightarrow H^0(N_{L/M}(-1)) \rightarrow H^0(N_{L/P}(-1)) \xrightarrow{h^0(D_L)} H^0(O_L(k-1)) \\
\rightarrow H^1(N_{L/M}(-1)) \rightarrow 0.
\end{array}
\]

where we note \(H^0(N_{L/P}(-1)) = V^\vee/W^\vee\). Hence the following is exact:

\[(9) \quad 0 \rightarrow H^0(N_{L/M}(-1)) \rightarrow V^\vee/W^\vee \xrightarrow{h^0(D_L)} S^{k-1}W \\
\rightarrow H^1(N_{L/M}(-1)) \rightarrow 0.
\]

where \(H^0(D_L)\) is given by \(H^0(D_L)(e^j) = f_j\) \((j = 3, 4, \ldots, N)\).

A generic choice of \(F\) implies a generic choice of degree \(k - 1\) polynomials \(f_j\) \((j = 3, 4, \ldots, N)\) in \(s\) and \(t\). By the assumptions

\[
\dim S^{k-1}W = k \geq N - 2 = \dim V^\vee/W^\vee,
\]

\[
\dim W \otimes V^\vee/W^\vee = 2(N - 2) \geq k + 1 = \dim S^kW,
\]

the generic choice of \(F\) implies that we can choose \(f_j \in S^{k-1}W\) \((j = 3, 4, \ldots, N)\)

and fix once for all) such that

(iii) \(f_j\) \((j = 3, 4, \ldots, N)\) are linearly independent,

(iv) \(Wf_3 + Wf_4 + \cdots + Wf_N = S^kW\).
Hence $H^0(D_L^7)$ is injective by (iii). It follows that $H^0(N_{L/M}(-1)) = 0$. Hence (ii) is clear. Next we consider $H^0(D_L)$. By (iv), we see

$$S^kW = W - H^0(D_L)(V^*/W^*) = H^0(D_L)(W \otimes V^*/W^*),$$

whence $H^0(D_L)$ is surjective. It follows that $H^1(N_{L/M}) = 0$. Hence $N_{L/M} \simeq O_L^{ba} \oplus O_L(-1)^{gb}$ for some $a$ and $b$. Since $a + b = \mathrm{rank}(N_{L/M}) = N - 3$ and $-b = \deg(N_{L/M}) = N - 2 - k$, we have (i). \hfill \Box

2.8. Lines on a quintic hypersurface in $\mathbb{P}^4$. See [Katz, Appendix A] for the subsequent examples. Let $N = 5$ and $k = 5$. Hence $M$ is a hypersurface of degree 5 in $\mathbb{P}^4$, a Calabi-Yau 3-fold. Let

$$F = x_4x_1^4 + x_5x_2^4 + x_3^5 + x_4^5 + x_5^5.$$

First we note that $M = \{F = 0\}$ is nonsingular. Let $L = \{x_3 = x_4 = x_5 = 0\} = \{[s,t,0,0,0]\}$. In this case $f_3 = 0$, $f_4 = s^4$ and $f_5 = t^4$. In the exact sequence (1) we see $H^0(N_{L/M}(-1)) = \mathrm{Ker}H^0(D_L^7) = Cc_3^0$ and $H^1(N_{L/M}(-1)) = \mathrm{Coker}H^0(D_L^7)$ is 3-dimensional. Hence $N_{L/M} = O_L(1) \oplus O_L(-3)$.

We summarize the above. If $\dim\mathrm{Ker}H^0(D_L^7) = 1$ and if $M$ is nonsingular, then $N_{L/M} = O_L(1) \oplus O_L(-3)$. Hence $H^0(N_{L/M}) = \mathrm{Ker}H^0(D_L) = W \otimes \mathrm{Ker}H^0(D_L^7)$ is 2-dimensional. Therefore we can choose $f_3 = 0$ and a linearly independent pair $f_4$ and $f_5 \in S^4W$ so that $Wf_4 + Wf_5$ is 4-dimensional. The choice $f_3 = s^4$ and $f_5 = t^4$ satisfies the conditions. This enables us to find a nonsingular hypersurface $M$ as above. However if we choose $f_3 = 0$, $f_4 = s^4$ and $f_5 = s^4t$, then $Wf_4 + Wf_5$ is 3-dimensional. Hence $M$ is singular.

Next in the same manner we find $L$ on a nonsingular hypersurface $M$ with $N_{L/M} = O_L \oplus O_L(-2)$ or $N_{L/M} = O_L(-1)^{gb}$. Let

$$F = x_3x_1^4 + x_4x_1^2x_2 + x_5x_2^4 + x_3^5 + x_4^5 + x_5^5.$$

Then we have $f_3 = s^4$, $f_4 = s^3t$ and $f_5 = t^4$. Since $Wf_3 + Wf_4 + Wf_5$ is 5-dimensional, $H^0(N_{L/M}(-1)) = \mathrm{Ker}H^0(D_L^7) = 0$, $H^0(N_{L/M}) = \mathrm{Ker}H^0(D_L) = C(c_3^0)$. We see also that $\dim H^1(N_{L/M}) = \dim \mathrm{Coker}H^0(D_L) = 1$ and $N_{L/M} = O_L \oplus O_L(-2)$. The hypersurface $M = \{F = 0\}$ is easily shown to be nonsingular.

If $F = x_3x_1^4 + x_4x_1^2x_2 + x_5x_2^4 + x_3^5 + x_4^5 + x_5^5$ and $M = \{F = 0\}$, then $N_{L/M} = O_L(-1)^{gb}$.

2.9. Lines on a generic hypersurface $M^8$ of $\mathbb{P}^6$. Let $N = 7$ and $k = 8$. In view of Lemma 2.2 there exists a line $L$ on any generic hypersurface $M$ of degree 8 in $\mathbb{P}(V) = \mathbb{P}^6$. In view of Lemma 2.7, $a = 1$, $b = 3$ and $N_{L/M} \simeq O_L \oplus O_L(-1)^{gb}$. For example let $L : x_j = 0 (j \geq 3)$ and we take

$$F_3 = 8x_1^4, \quad F_4 = 8x_1^3x_2, \quad F_5 = 8x_1^2x_2^2, \quad F_6 = 8x_1x_2^3, \quad F_7 = 8x_2^4,$$

$$F = x_2F_3 + x_4F_4 + x_5F_5 + x_6F_6 + x_7F_7 + x_8^8 + x_9^8 + x_1^8 + x_2^8,$$

and let $M = M^8 : F = 0$. We see that $M$ is nonsingular near $L$ and has at most isolated singularities. However it is still unclear to us whether $M = M^8$ is
nonsingular everywhere. The space $H^0(N_{L/M})$ is spanned by $te^\epsilon_s \cdot e^\epsilon_4$, hence an infinitesimal deformation $L_c$ of $L$ is given by

$$[s, t] \mapsto [s, t, \varepsilon s, 0, 0, 0]$$

which yields $F|_{L_c} = \varepsilon^8(s^8 + t^8) \equiv 0 \mod \varepsilon^8$. Since $H^1(N_{L/M}) = 0$, this infinitesimal deformation is integrable and $G :=$ the moduli of lines of $\mathbb{P}^6$ contained in $M$ is nonsingular and one dimensional at the point $[L]$.

We note that $M$ also contains 8 lines

$$L' := L_{c_8} : \varepsilon s x_1 - x_2 = x_3 + \varepsilon x_4 = x_j = 0 \quad (j \geq 5),$$

with $N_{L/M} = O_L(1)^{\mathbb{R}^3} \oplus O_L(-6)$ where $\varepsilon^8 = -1$.

3. Stability

**Definition 3.1.** Suppose that a reductive algebraic group $G$ acts on a vector space $V$. Let $v \in V$, $v \neq 0$.

1. the vector $v$ is said to be semistable if there exists a $G$-invariant homogeneous polynomial $F$ on $V$ such that $F(v) \neq 0$,
2. the vector $v$ is said to be stable if $p$ has a closed $G$-orbit in $X_{ss}$ and the stabilizer subgroup of $v$ in $G$ is finite.

Let $\pi : V \setminus \{0\} \to \mathbb{P}(V^\vee)$ be the natural surjection. Then $v \in V$ is semistable (resp. stable) if and only if $\pi(v)$ is semistable (resp. stable).

3.2. Grassmann variety. Let $V$ be an $N$-dimensional vector space, and $G(r, N)$ the Grassmann variety parameterizing all $r$-dimensional quotient spaces of $V$. Here is a natural way of understanding $G(r, N)$ via GIT-stability. Let $U$ be an $r$-dimensional vector space, $X = \text{Hom}(V, U)$ and $\pi : X \setminus \{0\} \to \mathbb{P}(X^\vee)$ the natural map. Then $\text{SL}(U)$ acts on $X$ from the left by:

$$(g \cdot \phi^*)(v) = g \cdot (\phi^*(v)) \quad \text{for } \phi^* \in X, \quad v \in V.$$

We see that for $\phi^* \in X$

$$\phi^* \text{ is } \text{SL}(U)\text{-stable} \iff \text{rank } \phi^* = r,$$

$$\phi^* \text{ is } \text{SL}(U)\text{-semistable} \iff \phi^* \text{ is } \text{SL}(U)\text{-stable}.$$

In fact, if $\text{rank } \phi^* = r - 1$, then there is a one-parameter torus $T$ of $\text{SL}(U)$ such that the closure of the orbit $T \cdot \phi$ contains the zero vector as the following simple example ($r = 2$) shows

$$\lim_{t \to 0} \left( \begin{array}{cc} t & 0 \\ 0 & t^{-1} \end{array} \right) \left( \begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1N} \\ 0 & 0 & \cdots & 0 \end{array} \right) = \lim_{t \to 0} \left( \begin{array}{cccc} ta_{11} & ta_{12} & \cdots & ta_{1N} \\ 0 & 0 & \cdots & 0 \end{array} \right).$$

Let $X_s$ be the set of all (semi)stable points and $\mathbb{P}_s$ the image of $X_s$ by $\pi$. It is, as we saw above, just the set of all $\phi \in X$ with $\text{rank } \phi^* = r$. Therefore the GIT-orbit space $\mathbb{P}_s//\text{SL}(U)$ is the orbit space $\mathbb{P}_s/\text{SL}(U)$ by the free action, the Grassmann variety $G(r, N)$.
3.3. Moduli of double coverings of $\mathbb{P}^1 (1)$. Let $W$ and $U$ be a pair of two dimensional vector spaces, $X = \text{Hom}(W, S^2 U)$, and $\pi : X \setminus \{0\} \to \mathbb{P}(X^*)$ the natural morphism. Note that $\text{SL}(U)$ acts on $S^2 U$ from the left via the natural action: $\sigma(u_1, u_2) = \sigma(u_1)\sigma(u_2)$ for $\forall u_1, u_2 \in U$. Thus $\text{SL}(U)$ acts on $X$ from the left in the same manner in the subsection 3.2.

**Lemma 3.4.** Let $\phi^* \in X$.

(i) $\phi^*$ is unstable iff $\phi^*(w)$ has a double root for any $w \in W$.

(ii) $\phi^*$ is semistable iff $\phi^*(w)$ has no double roots for some nonzero $w \in W$.

(iii) $\phi^*$ is stable iff $\phi^*(W)$ is a base-point free linear subsystem of $S^2 U$ on $\mathbb{P}(U)$.

**Proof.** We note that $\phi^*$ is unstable iff there is a suitable basis $s$ and $t$ of $U$ such that $\phi^*(w) = a(w)s^2$ for any $w \in W$ since a torus orbit $T \cdot \phi^*$ contains the zero vector. This proves (i). This also proves (ii). Next we prove (iii). If $\phi^*(W)$ has a base point, then it is clear that $\phi^*$ is not stable. If $\phi^*$ is semistable and it is not stable, then we choose a basis $s, t$ of $U$ and a basis $w_1, w_2$ of $W$ such that $\phi^*(w_1) = st$. If $\phi^*(w_2) = as^2 + bst$, then $\phi^*$ is not stable. This proves the lemma. \qed

**Theorem 3.5.** Let $X_{ss}$ be the Zariski open subset of $X$ consisting of all semistable points of $X$, $\pi(X_{ss})$ the image of $X_{ss}$ by $\pi$, and $Y := \pi(X_{ss})//\text{SL}(U)$. Then $Y \simeq \mathbb{P}^2$.

**Proof.** First consider a simplest case. We choose a basis $s, t$ of $U$. Let $w_1$ and $w_2$ be a basis of $W$, $T$ the subgroup of $\text{SL}(U)$ of diagonal matrices and $X^t = \{\phi^* \in X; \phi^*(w_1) = 2st\}$. Let $Z' = \text{SL}(U) \cdot X^t$.

We note that $Z'$ is an $\text{SL}(U)$-invariant subset of $X_{ss}$. We prove $\pi(Z')//\text{SL}(U) \simeq \mathbb{C}^2$. Let $\phi^*$ and $\psi^*$ be points of $X^t$. Let $\phi^*(w_2) = 2Bst + Ct^2$ and $\psi^*(w_2) = as^2 + 2bst + ct^2$. Then it is easy to check
\[
g \cdot \phi^* = \psi^* \quad \text{for} \quad \exists g \in \text{SL}(U) \iff g \cdot \phi^* = \psi^* \quad \text{for} \quad \exists g \in T
\]
\[
\iff A = au^2, \quad B = b, \quad C = u^{-2}c \quad \text{for} \quad \exists u \neq 0.
\]

Therefore each equivalence class of $\pi(Z)/\text{SL}(U)$ is represented by the pair $(AC, B)$, which proves $\pi(Z)//\text{SL}(U) \simeq \mathbb{C}^2$.

Now we prove the lemma. Let $\phi^* \in X_{ss}$, $\phi_j = \phi^*(w_j)$ and $\phi_0 = - (\phi_1 + \phi_2)$. Let
\[
\phi_0 = r_1 s^2 + 2r_2 st + r_3 t^2,
\]
\[
\phi_1 = p_1 s^2 + 2p_2 st + p_3 t^2,
\]
\[
\phi_2 = q_1 s^2 + 2q_2 st + q_3 t^2,
\]
and we define
\[
D_1 = p_2^2 - p_1 p_3, \quad D_2 = q_2^2 - q_1 q_3,
\]
\[
D_0 = r_2^2 - r_1 r_3 = D_1 + D_2 + 2p_2 q_2 - (p_1 q_3 + p_3 q_1).
\]

To show the lemma, we prove the more precise isomorphism
\[
\pi(X_{ss})//\text{SL}(U) = \text{Proj} \mathbb{C}[D_0, D_1, D_2]
\]

For this purpose we define $Y_j = \pi(\{\phi^* \in X_{ss}; \phi_j \text{ has no double roots}\})//\text{SL}(U)$. It suffices to prove $Y_1 = \text{Spec} \mathbb{C}[\frac{D_0}{D_1^2}, \frac{D_2}{D_1^2}]$ by reducing it to the first simplest case.
Let \( \phi^* \in Y_1 \). Let \( \alpha \) and \( \beta \) be the roots of \( \phi_1 = 0 \). By the assumption \( \phi_1 \) has no double roots, hence \( \alpha \neq \beta \). Let
\[
    u = \frac{1}{\gamma} (s - \alpha t), \quad v = \frac{1}{\gamma} (s - \beta t), \quad g = \frac{1}{\gamma} \left( \frac{1}{1} - \alpha \right)
\]
where \( \gamma = \sqrt{\alpha - \beta} \). Note that \( g \in \text{SL}(U) \). Hence we see
\[
    (\phi_1(s, t), \phi_2(s, t)) \equiv (p_1 \alpha^4 uv, A_1 u^2 + 2B_1 uv + C_1 v^2)
\]
where
\[
    A_1 = q_1^2 \beta^2 + 2q_2 \beta + q_3,
    -B_1 = q_1 \alpha \beta + q_2 (\alpha + \beta) + q_3,
    C_1 = q_1^2 \alpha^2 + 2q_2 \alpha + q_3.
\]
Thus we see
\[
    (\phi_1(s, t), \phi_2(s, t)) \equiv (2st, As^2 + 2Bst + Ct^2)
\]
where
\[
    A = \frac{2A_1}{p_1^4 \gamma^4}, \quad B = \frac{2B_1}{p_1^4 \gamma^4}, \quad C = \frac{2C_1}{p_1^4 \gamma^4}, \quad p_1^4 \gamma^4 = 4D_1,
    AC = B_1^2 - \frac{D_2}{D_1}, \quad B = \frac{D_0 - D_1 - D_2}{2D_1}.
\]
Therefore by the first half of the proof
\[
    Y_1 \simeq \text{Spec } \mathbb{C}[AC, B] = \text{Spec } \mathbb{C}\left[\frac{D_0}{D_1}, \frac{D_2}{D_1}\right].
\]
This completes the proof of the lemma. \( \Box \)

**Corollary 3.6.** Let \( Y^s = \pi(X_\alpha)/\text{SL}(U) \). Then \( Y \setminus Y^s \) is a conic of \( Y \) defined by
\[
    Y \setminus Y^s : D_0^2 + D_1^2 + D_2^2 - 2D_0 D_1 - 2D_1 D_2 - 2D_2 D_0 = 0.
\]

**Proof.** In view of Theorem 3.5, \( Y_1 \simeq \text{Spec } \mathbb{C}[AC, B] \). The complement of \( Y_\alpha \) in \( Y_1 \) is then the curve defined by \( AC = 0 \), which is easily identified with the above conic. \( \Box \)

**Corollary 3.7.** Let \( X^0 \) be the Zariski open subset of \( X \) consisting of all semistable points \( \phi^* \) of \( X \) with rank \( \phi^* = 2 \), and let \( Y^0 = \pi(X^0)/\text{SL}(U) \). Then \( Y^0 \simeq \pi(X^0)/\text{SL}(U) \simeq Y \simeq \mathbb{P}^2 \).

**Proof.** It suffices to compare \( Y_1 \) and \( Y^0 \cap Y_1 \). As in the proof of Theorem 3.5 we let \( X' = \{ \phi^* \in X; \phi^*(w_1) = 2st \} \). Let \( Z = \text{SL}(U) \cdot X' \) and \( Z^0 = \text{SL}(U) \cdot (X' \cap X^0) \).

Then with the notation in Theorem 3.5, we recall \( X' = \{ \phi^* \in X; \phi^*(w_1) = 2st, \phi^*(w_2) = As^2 + 2Bst + Ct^2 \}, \pi(Z)/\text{SL}(U) \simeq \text{Spec } \mathbb{C}[AC, B] \) where
\[
    X' \cap X^0 = \{ \phi^* \in X'; A \neq 0 \text{ or } C \neq 0 \}.
\]

In the same manner as before we see \( \pi(Z^0)/\text{SL}(U) \simeq \text{Spec } \mathbb{C}[AC, B] \), whence \( \pi(Z^0)/\text{SL}(U) = \pi(Z)/\text{SL}(U) \). This proves \( Y^0 \cap Y_1 = Y_1 \). This completes the proof of the corollary. \( \Box \)
3.8. Moduli of double coverings of $\mathbf{P}(W)$ (2). There is an alternative way of understanding $\pi(X_{ss})//\mathbf{SL}(U) \simeq \mathbf{P}^2$ by using the isomorphism $\mathbf{S}^2\mathbf{P}^1 \simeq \mathbf{P}^2$. We use the following convention to denote a point of $\mathbf{P}(U) = U^\vee \setminus \{0\}/G_m$: $(u : v) = us^\alpha + vt^\beta \in U^\vee$ where $s^\alpha$ and $t^\beta$ are a basis dual to $s$ and $t$. In what follows we fix a basis $w_1$ and $w_2$ of $W$. Let $P := (a_1 : a_2)$ and $Q := (b_1 : b_2)$ be a pair of points of $\mathbf{P}(W) \simeq \mathbf{P}^1$. If $P \neq Q$, there is a double covering $\phi : \mathbf{P}(U) \to \mathbf{P}(W)$ ramifying at $P$ and $Q$, unique up to isomorphism once we fix the base $w_1$ and $w_2$:

$$\frac{b_2w_1 - b_1w_2}{a_2w_1 - a_1w_2} = \frac{(t)^2}{s^2}.$$ 

Thus $\phi$ is given explicitly by

$$\phi_1 := \phi^s(w_1) = b_1s^2 - a_1t^2, \quad \phi_2 := \phi^t(w_2) = b_2s^2 - a_2t^2, \quad \phi_0 = -(\phi_1 + \phi_2)$$

for which we have

$$D_1 = a_1b_1, \quad D_2 = a_2b_2, \quad D_0 = (a_1 + a_2)(b_1 + b_2).$$

The isomorphism $\mathbf{S}^2\mathbf{P}^1 \simeq \mathbf{P}^2$ is given by $(P, Q) \mapsto (D_0, D_1, D_2)$. This shows

**Corollary 3.9.** We have a natural isomorphism: $Y \simeq \mathbf{P}(\mathbf{S}^2W)$.

4. The virtual normal bundle of a double covering

4.1. The case $N = 7$ and $k = 8$ revisited. We revisit the example in the subsection 2.9. Let $N = 7$ and $k = 8$. Let $L : x_j = 0$ ($j \geq 3$) and we take

$$F_3 = 8x_1^7, F_4 = 8x_1^6x_2, F_5 = 8x_1^5x_3^2, F_6 = 8x_1^3x_2^2, F_7 = 8x_1^2,$$

$$F = x_3F_3 + x_4F_4 + x_5F_5 + x_6F_6 + x_7F_7 + x_8^3 + x_9^2 + x_6^3 + x_7^2.$$ 

and let $M = M_{\phi}^0 : F = 0$. We often denote $L$ also by $\mathbf{P}(W)$ with $W$ a two dimensional vector space for later convenience. Since $H^0(D_L^0)$ is injective and $H^0(D_L)$ is surjective, we have $N_{L/M} \simeq O_L \oplus O_L(-1)_{\mathbf{g}^3}$. Hence $H^1(N_{L/M}(-1)) = H^1(O_L(-2)_{\mathbf{g}^3})$ is 3-dimensional. As we see easily, this follows also from the fact that $\text{Coker} H^0(D_L)$ is freely generated by $x_1^3x_2^2, x_1^4x_2$ and $x_1x_2^3$.

Let $\phi^* = (\phi_1, \phi_2) \in X^0$. Then $\text{Coker} H^0(\phi^*D_L)$ is generated by a single element $\phi_2x_3 - \phi_1x_4$, while $\text{Coker} H^0(\phi^*D_L)$ is generated by $S^2U \cdot \phi_1^3\phi_2^4$, $S^2U \cdot \phi_2 \phi_1^4$ and $S^2U \cdot \phi_1 \phi_2^4$. To be more precise, we see

$$\text{Coker} H^0(\phi^*D_L) = \{\phi_1^3\phi_2^4, \phi_2 \phi_1^4, \phi_1 \phi_2^4\} \subset S^2U \setminus \{\phi_1, \phi_2\}.$$ 

In fact, this is proved as follows: first we consider the case where $\phi_1$ and $\phi_2$ has no common zeroes. In this case $\phi^*$ gives rise to a double covering $\phi : \mathbf{P}(U) \to \mathbf{P}(W)$ ($= L$), which we denote by $L_\phi$ for brevity. By pulling back by $\phi^*$ the normal sequence

$$0 \to N_{L/M} \to N_{L/P} \to O_L(k) \to 0 \quad (k = 8)$$

for the line $L$ we infer an exact sequence

$$0 \to \phi^*N_{L/M} \to \phi^*N_{L/P} \to \phi^*O_L(k) \to 0,$$

which yields an exact sequence

$$0 \to H^0(\phi^*N_{L/M}) \to S^2U \otimes (V^\vee / W^\vee) \xrightarrow{H^0(\phi^*D_L)} H^0(O_L(2k)) \to 0.$$
Let $\eta = q_3 e_3^\nu + \cdots + q_7 e_7^\nu \in \text{Ker } H^0(\phi^* D_L)$, $q_j \in S^2 U$. Then we have

$$\phi_1^2(q_3 \phi_1^5 + q_4 \phi_1^4 \phi_2 + q_5 \phi_1^2 \phi_2^3 + q_6 \phi_2^5) = -q_7 \phi_2^5.$$  

Since $\phi_1$ and $\phi_2$ are mutually prime and $q_j$ is of degree two, we have $q_7 = 0$ and

$$\phi_1^2(q_3 \phi_1^5 + q_4 \phi_1^4 \phi_2 + q_5 \phi_1^2 \phi_2^3) = -q_6 \phi_2^5.$$  

Hence $q_6 = 0$ and similarly we infer also $q_5 = 0$. Thus we have $q_3 \phi_1 + q_4 \phi_2 = 0$.

This proves that $\text{Ker } H^0(\phi^* D_L)$ is generated by $\phi_2 e_2^\nu - \phi_1 e_2^\nu$.

Next we prove that $\text{Coker } H^0(\phi^* D_L)$ is generated by $\phi^* \text{Coker } H^0(D_L^-)$ over $S^2 U$, in fact over $S^2 U/\phi^*(W)$. Without loss of generality we may assume that $\phi_1 = 2st$ and $\phi_2 = \lambda s^2 + 2\nu st + t^2$ for some $\lambda \neq 0$ and $\nu \in \mathbb{C}$. Let $\phi^* W = \{\phi_1, \phi_2\}$. Then one checks $U \cdot \phi^* W = S^2 U$, and hence $S^2 U \cdot \phi^* W = S^4 U$, $S^2 m^{-2} U \cdot \phi^* W = S^2 m U$ for $m \geq 2$. It follows $S^2 U \cdot \phi^* (S^m - 1 W) = S^2 m U$ for $m \geq 1$. In fact, by the induction on $m$

$$S^2 U \cdot \phi^* (S^m W) = S^2 U \cdot \phi^* (W) \cdot \phi^* (S^{m-1} W) = S^2 U \cdot \phi^* (S^{m-1} W) = S^2 U \cdot (S^2 U \cdot \phi^* (S^{m-1} W)) = S^2 U \cdot S^{2 m-2} U = S^2 U \cdot S^{2 m} U = S^2 U \cdot S^{2 m+2} U.$$  

Therefore $H^0(\text{O}_{L^2}(2k)) = S^1 U = S^2 U \cdot \phi^* (S^2 W)$. Hence

$$\text{Coker } H^0(\phi^* D_L) = S^1 U \cap \text{Im } H^0(\phi^* D_L) = S^2 U \cdot \phi^* (S^2 W) \cdot \phi^* (\text{Im } H^0(D_L^-)) = (S^2 U / \phi^* (W)) \cdot \phi^* (S^2 W / \text{Im } H^0(D_L^-)).$$  

because $\text{Coker } H^0(D_L^-) = S^7 W / \text{Im } H^0(D_L^-)$ and $W : S^7 W \subset W \cdot \text{Im } H^0(D_L^-) = S^6 W$ by the choice of $L$. This proves that $\text{Coker } H^0(\phi^* D_L)$ is generated by $\phi^* \text{Coker } H^0(D_L^-)$ over $S^2 U/\phi^*(W)$. It follows $\text{Coker } H^0(\phi^* D_L) = (\phi^* \text{Coker } H^0(D_L^-)) \cap (S^2 U / \phi^* W)$.

Finally we consider the case where $\phi_1$ and $\phi_2$ has a common zero. In this case we may assume $\phi_1 = 2st$ and $\phi_2 = 2\nu st + t^2$. In this case $L_\phi$ is a chain of two rational curves $C'_\phi$ and $C''_\phi$ where $C_\phi$ is the proper transform of $\text{P}(U)$, where the double covering map from $L_\phi$ to $\text{P}(W)$ is the union of the isomorphisms $\phi'$ and $\phi''$, say, $\phi = \phi' \cup \phi''$. Let $\psi_1 = 2s$ and $\psi_2 = 2\nu s + t$. Then $\phi'$ is induced by the homomorphism $(\phi')^* \in \text{Hom}(W, U)$ such that $(\phi')^*(w_j) = \psi_j$. On the other hand let $U^U_\phi = C \lambda + Ct$, $\psi_1' = 2t$ and $\psi_2' = \lambda + 2t$ where we note $\psi_j'$ is the linear part of $\phi_j$ in $t$ with $s = 1$. Then $C''_\phi = \text{P}(U^U_\phi)$ and $\phi''$ is induced by the homomorphism $(\phi'')^* \in \text{Hom}(W, U^U_\phi)$ such that $(\phi'')^*(w_j) = \psi_j''$. Furthermore the pull back by $\phi^*$ of the normal sequence for $L$

$$0 \to \phi^* N_{L/M} \to \phi^* N_{L/P} \to \phi^* O_L(k) \to 0,$$
yields exact sequences with natural vertical homomorphisms:
\[
\begin{array}{c}
0 \\[-0.5ex] \downarrow \\[-0.5ex] \downarrow \\[-0.5ex] \downarrow
\end{array}
\xrightarrow{} \phi^*N_{L/M} \xrightarrow{} (\phi')^*N_{L/M} \oplus (\phi'')^*N_{L/M} \xrightarrow{} C \xrightarrow{} 0
\]
\[
\begin{array}{c}
0 \\[-0.5ex] \downarrow \\[-0.5ex] \downarrow \\[-0.5ex] \downarrow
\end{array}
\xrightarrow{} \phi^*N_{L/P} \xrightarrow{} (\phi')^*N_{L/P} \oplus (\phi'')^*N_{L/P} \xrightarrow{} V^\vee /W^\vee \xrightarrow{} 0
\]
\[
\begin{array}{c}
0 \\[-0.5ex] \downarrow \\[-0.5ex] \downarrow \\[-0.5ex] \downarrow
\end{array}
\xrightarrow{} \phi^*O_L(k) \xrightarrow{} O_{C_L}(k) \oplus O_{C_P}(k) \xrightarrow{} C \xrightarrow{} 0.
\]

This yields the following long exact sequences:
\[
\begin{array}{c}
0 \\ \xrightarrow{} H^0((\phi')^*N_{L/M}) \\ \xrightarrow{} H^1((\phi')^*N_{L/M}) \\ \xrightarrow{} 0
\end{array}
\xrightarrow{H^0((\phi')^*D_L)} S^kU
\]
\[
\begin{array}{c}
0 \\ \xrightarrow{} H^0((\phi'')^*N_{L/M}) \\ \xrightarrow{} H^1((\phi'')^*N_{L/M}) \\ \xrightarrow{} 0
\end{array}
\xrightarrow{H^0((\phi''')^*D_L)} S^kU''
\]

whence \(H^1((\phi')^*N_{L/M}) = H^1((\phi'')^*N_{L/M}) = 0\), and both \(H^0((\phi')^*N_{L/M})\) and \(H^0((\phi'')^*N_{L/M})\) are one-dimensional. Let \(U^t\) be the subspace of \(U\) consisting of elements vanishing at \(C'_\phi \cap C''_\phi\), namely the subspace spanned by \(t\). Then the restriction of \(H^0((\phi')^*D_L)\) to \(U^t \otimes V^\vee /W^\vee\) equals \(t \cdot H^0((\phi')^*D_L^-)\). Hence
\[
\text{Coker } H^0((\phi')^*D_L) \simeq t \cdot S^2 U/t \cdot \text{Im } H^0((\phi')^*D_L^-) \oplus \text{Coker } H^0((\phi'')^*D_L) \\
\simeq S^2 U/t \cdot \text{Im } H^0((\phi')^*D_L^-) \simeq \text{Coker } H^0((\phi')^*D_L^-).
\]

One could understand the above isomorphism as
\[
\text{Coker } H^0((\phi')^*D_L) = \text{Coker } (\phi')^*H^0(D_L^-) \oplus (S^2 U/\phi^*W).
\]

Thus \(H^0(\phi^*N_{L/M})\) is one-dimensional, while \(H^1(\phi^*N_{L/M})\) is 3-dimensional. This is immediately generalized into the following

\textbf{Lemma 4.2.} For any \(\phi^* \in X^0\) we have
\[
\text{Ker } H^0((\phi')^*D_L) = \phi^* \text{Ker } H^0(D_L),
\]
\[
\text{Coker } H^0((\phi')^*D_L) = (\phi^* \text{Coker } H^0(D_L^-)) \oplus (S^2 U/\phi^*W).
\]

\textbf{Lemma 4.3.} We define a line bundle \(L_0\) (resp. \(L_1\)) on \(Y \simeq \mathbb{P}(S^2 W)\) by the assignment:
\[
X^0 \ni \phi^* \mapsto \phi^* \text{Ker } H^0(D_L) \quad (\text{resp. } \phi^* \text{Coker } H^0(D_L^-)).
\]

Then \(L_k \simeq O_{\mathbb{P}(S^2 W)}\).

\textbf{Proof.} We know that \(\phi^* \text{Ker } H^0(D_L)\) is generated by \(\phi_2 e_3 - \phi_1 e_4\). By the \(\text{SL}(2)\)-variable change of \(s\) and \(t\), \(\phi_j\) is transformed into a new quadratic polynomial, which is however the same as the first \(\phi_j\). This shows the generator is unchanged, whence \(L_0 \simeq O_{\mathbb{P}(S^2 W)}\). The proof for \(L_1\) is the same. \(\square\)
Lemma 4.4. We define a coherent sheaf $\mathbf{L}$ on the stack $Y \simeq \mathbf{P}(S^2W)$ (See Remark below) by the assignment:

$$X^0 \ni \phi^* \mapsto S^2U/\phi^*W.$$ 

Then $\mathbf{L}^2 \simeq O_{\mathbf{P}(S^2W)}(-1)$.

Proof. The GIT-quotient $Y^0$ is covered with the images of $X^0_j$:

$$X^0_j = \{(\phi_1, \phi_2) \in X^0; \phi_1 = 2st, \phi_2 = 2\nu st + t^2, \lambda, \nu \in \mathbb{C}\},$$

$$X^0_j = \{(\phi_1, \phi_2) \in X^0; \phi_1 = \nu s^2 + 2qst + t^2, \phi_2 = 2st, p, q \in \mathbb{C}\}.$$ 

It is clear that the natural image of $X^0_j$ in $Y$ is $Y_j$. The map $\phi$ given by $\phi^* = (\phi_1, \phi_2) \in Y_j$ has natural $\mathbb{Z}_2$ involution generated by,

$$r: (\sqrt{\lambda}s + t, \sqrt{\lambda}s - t) \mapsto (\sqrt{\lambda}s + t, -(\sqrt{\lambda}s - t)).$$ 

Since

$$2st = \frac{1}{2\sqrt{\lambda}}((\sqrt{\lambda}s + t)^2 - (\sqrt{\lambda}s - t)^2),$$

$$\lambda s^2 + 2\nu st + t^2 = \frac{\nu}{2\sqrt{\lambda}}((\sqrt{\lambda}s + t)^2 - (\sqrt{\lambda}s - t)^2) + \frac{1}{2}((\sqrt{\lambda}s + t)^2 + (\sqrt{\lambda}s - t)^2),$$

it is clear that,

$$r^*(\phi_1) = \phi_1, \quad r^*(\phi_2) = \phi_2, \quad r^*(\lambda s^2 - t^2) = -(\lambda s^2 - t^2).$$ 

Therefore, we can decompose $S^2U$ into $\langle \lambda s^2 - t^2 \rangle_C \oplus \langle \phi_1, \phi_2 \rangle_C$ with respect to eigenvalue of $r^*$ and take $\lambda s^2 - t^2$ as canonical generator of $S^2U/\phi^*W$. Similarly $S^2U/\phi^*W$ is generated by $ps^2 - t^2$ on $Y_2$. The problem is therefore to write $\lambda s^2 - t^2$ as an $\Gamma(O_{Y_1};\mathcal{L})$-multiple of $pu^2 - v^2$ when we write $\phi_2 = 2uv$ by a variable change in $GL(2)$. The following variable change $(s, t) \mapsto (u, v)$ is in $GL(2)$:

$$s = \frac{\sqrt{2}\alpha}{(\beta - \alpha)^2}(2u - \frac{(\beta - \alpha)^2}{2\alpha}v), \quad t = \frac{\sqrt{2}\alpha}{(\beta - \alpha)^2}(2\beta u - \frac{(\beta - \alpha)^2}{2}v),$$

where $\alpha, \beta$ are roots of the equation $\lambda s^2 + 2\nu st + t^2 = 0$. Under this coordinate change, $\phi_1$ and $\phi_2$ are rewritten as follows:

$$\phi_1 = \frac{\lambda}{(\nu^2 - \lambda)^2}u^2 + 2\frac{\nu}{\nu^2 - \lambda}uv + v^2 = pu^2 + 2quv + v^2, \quad \phi_2 = 2uv.$$ 

Then we have

$$pu^2 - v^2 = -\frac{2}{\beta - \alpha}(\lambda s^2 - t^2) = -\frac{1}{\sqrt{\nu^2 - \lambda}}(\lambda s^2 - t^2) = -\frac{D_1}{D_2}(\lambda s^2 - t^2).$$ 

Similarly by computing the effect on $S^2U/\phi^*W$ by the variable change from $X^0_j$ into $X^0_0$, we see that $\mathbf{L}^2$ is isomorphic to $O_{\mathbf{P}(S^2W)}(-1)$. This completes the proof. $\square$

Remark 4.5. We remark that the space $X$ must be regarded as a $\mathbb{Q}$-stack $Y^{\text{stack}}$ as follows: First we define $\phi_0 = -\phi_1 - \phi_2$. For each atlas $X^0_{\alpha}$ we define an atlas $Y^\text{stack}_{\alpha}$.
(α = 0, 1, 2) by
\[ Y_0^{\text{stack}} = \{ (φ_0, φ_1, φ_2, ±ψ_0) ∈ X^0 × S^2U; φ_0 = 2st, φ_1 = as^2 + 2bst + t^2, \]
\[ ψ_0 = as^2 - t^2 a, b ∈ C \}, \]
\[ Y_1^{\text{stack}} = \{ (φ_0, φ_1, φ_2, ±ψ_1) ∈ X^0 × S^2U; φ_1 = 2st, φ_2 = λs^2 + 2νst + t^2, \]
\[ ψ_1 = λs^2 - t^2 λ, ν ∈ C \}, \]
\[ Y_2^{\text{stack}} = \{ (φ_0, φ_1, φ_2, ±ψ_2) ∈ X^0 × S^2U; φ_1 = ps^2 + 2qst + t^2, φ_2 = 2st, \]
\[ ψ_2 = ps^2 - t^2 p, q ∈ C \}. \]

Since \( L^2 ≃ O_{\mathbb{P}(S^2W)}(-1) \) we have \( c_1(L) = -\frac{1}{2} c_1(O_{\mathbb{P}(S^2W)}(1)) \) in the Chow ring \( A(Y^{\text{stack}})_Q = A(X)_Q = A(\mathbb{P}(S^2W))_Q \).

5. Proof of the main theorem

Theorem 5.1.

\[ \pi_*(c_{\text{top}}(H^1)) = \frac{1}{8} \left[ \frac{c(S^{k-1}Q)}{1 - \frac{1}{2}c_1(Q)} \right]_{k-N}, \]

where \( \pi \) is the natural projection from \( \tilde{M}_{0,0}(L, 2) \) to \( G \) and \([*]_{k-N}\) is the operation of picking up the degree \( 2(k - N) \) part of Chern classes.

Proof. From now on we denote the coherent sheaf \( L \) in Lemma 4.4 by \( O_{\mathbb{P}}(-\frac{1}{2}) \). In view of the results from the previous section, what remains is to evaluate the top chern class of \( (S^{k-1}Q/((V^\vee \otimes O_G)/(Q^\vee)) \otimes O_{\mathbb{P}}(-\frac{1}{2}) \) on \( \mathbb{P}(S^2Q) \). Since double cover maps parametrized by \( \mathbb{P}(S^2Q) \) have natural \( \mathbb{Z}_2 \) involution \( r \) given in the previous section, we have to multiply the result of integration on \( \mathbb{P}(S^2Q) \) by the factor \( \frac{1}{2} \) [BT], [FP]. With this set-up, let \( \pi': \mathbb{P}(S^2Q) → G \) be the natural projection. Then we have to compute is \( \pi_*(c_{\text{top}}(H^1)) = \frac{1}{2} \pi'_*(c_{\text{top}}(H^1)) = \frac{1}{2} \pi'_*(c_{\text{top}}((S^{k-1}Q/((V^\vee \otimes O_G)/(Q^\vee)) \otimes O_{\mathbb{P}}(-\frac{1}{2}))) \). Let \( z \) be \( c_1(O_{\mathbb{P}}(1)). \) Then we obtain,

\[ \frac{1}{2} \pi'_*(c_{\text{top}}((S^{k-1}Q/((V^\vee \otimes O_G)/(Q^\vee)) \otimes O_{\mathbb{P}}(-\frac{1}{2}))) = \frac{1}{2} \sum_{j=0}^{k-N+2} c_{k-N+2-j}(S^{k-1}Q \oplus Q^\vee) \cdot \pi_*(z^j) \cdot (-\frac{1}{2})^j = \frac{1}{8} \sum_{j=0}^{k-N} c_{k-N-j}(S^{k-1}Q \oplus Q^\vee) \cdot s_j(S^2Q) \cdot (-\frac{1}{2})^j = \frac{1}{8} \left[ \frac{c(S^{k-1}Q) \cdot c(Q^\vee)}{1 - \frac{1}{2}c_1(S^2Q) + \frac{1}{4}c_2(S^2Q) - \frac{1}{8}c_3(S^2Q)} \right]_{k-N}, \]

where \( s_j(S^2Q) \) is the \( j \)-th Segre class of \( S^2Q \). But if we decompose \( c(Q) \) into \( (1 + α)(1 + β) \), we can easily see,

\[ \frac{c(Q^\vee)}{1 - \frac{1}{2}c_1(S^2Q) + \frac{1}{4}c_2(S^2Q) - \frac{1}{8}c_3(S^2Q)} = \frac{(1 - α)(1 - β)}{(1 - α)(1 - \frac{1}{2}(α + β))(1 - β)} = \frac{1}{1 - \frac{1}{2}c_1(Q)}. \]
Finally, by combining the above theorem with the divisor axiom of Gromov-Witten invariants, we can prove the decomposition formula of degree 2 rational Gromov-Witten invariants of $M_k^k$ found from numerical experiments.

**Corollary 5.2.**

\[ \langle \mathcal{O}_{c, c, c}, \mathcal{O}_{c, c, c} \rangle_{0, 2} = \langle \mathcal{O}_{c, c, c}, \mathcal{O}_{c, c, c} \rangle_{0, 2} \to 2 + 8(\pi_i(\pi_k(H^1))) \langle \mathcal{O}_{c, c, c}, \mathcal{O}_{c, c, c} \rangle_{0, 1}, \]

where $\langle \mathcal{O}_{c, c, c}, \mathcal{O}_{c, c, c} \rangle_{0, 2} \to 2$ is the number of conics that intersect cycles Poincaré dual to $e^a$, $e^b$ and $e^c$. We also denote by $\langle \pi_i(\pi_k(H^1))) \langle \mathcal{O}_{c, c, c}, \mathcal{O}_{c, c, c} \rangle_{0, 1}$ the integral:

\[ \int_{G(2,\mathbb{V})} c_{top}(S^k Q) \wedge \pi_i(\pi_k(H^1))) \wedge \sigma_{a-1} \wedge \sigma_{b-1} \wedge \sigma_{c-1}. \]

6. GENERALIZATION TO TWISTED CUBICS

In this section, we present a decomposition formula of degree 3 rational Gromov-Witten invariants found from numerical experiments using the results of [ES].

**Conjecture 6.1.** If $k - N = 1$, we have the following equality:

\[ \pi_i(c_{top}(H^1)) = \]

\[ \frac{1}{27} \left( \frac{1}{24} (27k^2 - 55k + 26)k(k - 1) + \frac{2}{9} c_1(Q)^2 + \left( \frac{7}{6} (k + 1)k(k - 1) + \frac{1}{9} c_2(Q) \right) \right). \]

where $\pi : \overline{M}_{0,0}(L, 3) \to \overline{M}_{0,0}(M_k^k, 1)$ is the natural projection.

In the $k - N > 1$ case, we have not found the explicit formula, because in the $d = 3$ case, we have another contribution from multiple cover maps of type $(2+1) \to (1+1)$. Here multiple cover map of type $(2+1) \to (1+1)$ is the map from nodal curve $\mathbf{P}^1 \vee \mathbf{P}^1$ to nodal conic $L_1 \vee L_2 \subset M_k^k$, that maps the first (resp. the second) $\mathbf{P}^1$ to $L_1$ (resp. $L_2$) by two to one (resp. one to one). In the $k - N = 1$ case, we have also determined the contributions from multiple cover maps of $(2+1) \to (1+1)$ to nodal conics.

**Corollary 6.2.** If $k - N = 1$, $\langle \mathcal{O}_{c, c, c}, \mathcal{O}_{c, c, c} \rangle_{0, 3}$ is decomposed into the following contributions:

\[ \langle \mathcal{O}_{c, c, c}, \mathcal{O}_{c, c, c} \rangle_{0, 3} = \langle \mathcal{O}_{c, c, c}, \mathcal{O}_{c, c, c} \rangle_{0, 3} + \frac{1}{4} \langle \mathcal{O}_{c, c, c}, \mathcal{O}_{c, c, c} \rangle_{0, 1} \langle \mathcal{O}_{c, c, c} \rangle_{0, 1} \]

\[ + \frac{3}{2} \langle \mathcal{O}_{c, c, c}, \mathcal{O}_{c, c, c} \rangle_{0, 1} \langle \mathcal{O}_{c, c, c} \rangle_{0, 1} + \frac{3}{2} \langle \mathcal{O}_{c, c, c}, \mathcal{O}_{c, c, c} \rangle_{0, 1} \langle \mathcal{O}_{c, c, c} \rangle_{0, 1} \]

\[ + \frac{27(\pi_i(\pi_k(H^1))) \langle \mathcal{O}_{c, c, c}, \mathcal{O}_{c, c, c} \rangle_{0, 1}. \]

where $\langle \mathcal{O}_{c, c, c}, \mathcal{O}_{c, c, c} \rangle_{0, 3} \to 3$ is the number of twisted cubics that intersect cycles Poincaré dual to $e^a$, $e^b$ and $e^c$.

**Proof.** In the $k - N = 1$ case, dimension of moduli space of multiple cover maps of $(2+1) \to (1+1)$ to nodal conics is given by $N - 6 + N - 6 - (N - 4) + 2 = N - 6$, hence the rank of $H^1$ is given by $N - 6 - (N - 5) = 2$. On the other hand, dimension of moduli space of $d = 2$ multiple cover maps of $\mathbf{P}^1 \to \mathbf{P}^1$ is 2, the degree of the form of $\pi_i(c_{top}(H^1))$ equals to $2 - 2 = 0$, where $\pi$ is the projection map.
that projects out the fiber locally isomorphic to the moduli space of \( d = 2 \) multiple cover maps. This situation is exactly the same as the Calabi-Yau case. Therefore, we can use the well-known result by Aspinwall and Morrison, that says for \( n \)-point rational Gromov-Witten invariants for Calabi-Yau manifold, \( \tilde{\pi}_* (c_{top}(H^1)) \) for degree \( d \) multiple cover map is given by,

\[
\tilde{\pi}_* (c_{top}(H^1)) = \frac{1}{d^{n-1}}.
\]

With this formula, we add up all the combinatorial possibility of insertion of external operator \( \mathcal{O}_{c^1}, \mathcal{O}_{c^2} \) and \( \mathcal{O}_{c^3} \),

\[
\frac{1}{k} \left( (\tilde{\pi}_* (c_{top}(H^1)))_0 (\mathcal{O}_{c^1 c^2 c^3})_{0,1} (\mathcal{O}_{c^3})_{0,1} \right) + (\tilde{\pi}_* (c_{top}(H^1)))_0 (\mathcal{O}_{c^1 c^2 c^3})_{0,1} (\mathcal{O}_{c^3})_{0,1}
\]

\[
+ (\tilde{\pi}_* (c_{top}(H^1)))_0 (\mathcal{O}_{c^1 c^2 c^3})_{0,1} (\mathcal{O}_{c^3})_{0,1} + (\tilde{\pi}_* (c_{top}(H^1)))_0 (\mathcal{O}_{c^1 c^2 c^3})_{0,1} (\mathcal{O}_{c^3})_{0,1}
\]

\[
+ (\tilde{\pi}_* (c_{top}(H^1)))_0 (\mathcal{O}_{c^1 c^2 c^3})_{0,1} (\mathcal{O}_{c^3})_{0,1} + (\tilde{\pi}_* (c_{top}(H^1)))_0 (\mathcal{O}_{c^1 c^2 c^3})_{0,1} (\mathcal{O}_{c^3})_{0,1}
\]

\[
+ (\tilde{\pi}_* (c_{top}(H^1)))_0 (\mathcal{O}_{c^1 c^2 c^3})_{0,1} (\mathcal{O}_{c^3})_{0,1} + (\tilde{\pi}_* (c_{top}(H^1)))_0 (\mathcal{O}_{c^1 c^2 c^3})_{0,1} (\mathcal{O}_{c^3})_{0,1}
\]

\[
= \frac{1}{k} \left( 2 (\mathcal{O}_{c^1 c^2 c^3})_{0,1} (\mathcal{O}_{c^3})_{0,1} \right) + (\mathcal{O}_{c^1 c^2 c^3})_{0,1} (\mathcal{O}_{c^3})_{0,1} + (\mathcal{O}_{c^1 c^2 c^3})_{0,1} (\mathcal{O}_{c^3})_{0,1}
\]

\[
+ (\mathcal{O}_{c^1 c^2 c^3})_{0,1} (\mathcal{O}_{c^3})_{0,1} + (\mathcal{O}_{c^1 c^2 c^3})_{0,1} (\mathcal{O}_{c^3})_{0,1}
\]

\[
+ \frac{1}{2} (\mathcal{O}_{c^1 c^2 c^3})_{0,1} (\mathcal{O}_{c^3})_{0,1} + \frac{1}{2} (\mathcal{O}_{c^1 c^2 c^3})_{0,1} (\mathcal{O}_{c^3})_{0,1}
\]

\[
+ \frac{1}{4} (\mathcal{O}_{c^1 c^2 c^3})_{0,1} (\mathcal{O}_{c^3})_{0,1}.
\]

The last expression is nothing but the formula we want. \( \square \)

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