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CONICS ON A GENERIC HYPERSURFACE

MASAOS JINZENJI, IKU NAKAMURA AND YASUKI SUZUKI

ABSTRACT. In this paper, we compute the contributions from double cover maps to genus 0 degree 2 Gromov-Witten invariants of general type projective hypersurfaces. Our results correspond to a generalization of Aspinwall-Morrison formula to general type hypersurfaces in some special cases.
MSC-class: 14H99, 14N35, 32G20

1. INTRODUCTION

In this paper, we discuss a generalization of the multiple cover formula for rational Gromov-Witten invariants of Calabi-Yau manifolds [AM], [M] to double cover maps of a line $L$ on a degree $k$ hypersurface $M^k_N$ in $\mathbb{P}^{N-1}$. Naively, for a given finite set of elements $\alpha_j \in H^*(M^k_N, \mathbb{Z})$, the rational Gromov-Witten invariant $\langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \cdots \mathcal{O}_{\alpha_n} \rangle_{0,d}$ of $M^k_N$ counts the number of degree $d$ (possibly singular and reducible) rational curves on $M^k_N$ that intersect real sub-manifolds of $M^k_N$ that are Poincaré-dual to $\alpha_j$.

Recently, the mirror computation of rational Gromov-Witten invariants of $M^k_N$ with negative first Chern class $(k-N>0)$ was established in [CG], [Iri], [J]. Using the method presented in these articles, we can compute $\langle \mathcal{O}_{c_1} \mathcal{O}_{c_2} \cdots \mathcal{O}_{c_m} \rangle_{0,d}$ where $c$ is the generator of $H^{1,1}(M^k_N, \mathbb{Z})$. Briefly, mirror computation of $M^k_N$ $(k > N)$ in [J] goes as follows. We start from the following ODE:

$$(1) \quad \left( (\partial_x)^{N-1} - k \cdot \exp(x) \cdot (k\partial_x + k - 1)(k\partial_x + k - 2) \cdots (k\partial_x + 1) \right) w(x) = 0,$$

and construct the virtual Gauss-Manin system associated with (1):

$$(2) \quad \partial_x \tilde{w}_{N-2-m}(x) = \tilde{w}_{N-1-m}(x) + \sum_{d=1}^{\infty} \exp(dx) \cdot \tilde{L}_{m}^{N,k,d} \cdot \tilde{w}_{N-1-m-(N-k)d}(x),$$

where $m$ runs through all the integers and $\tilde{L}_{m}^{N,k,d}$ is non-zero only if $0 \leq m \leq N-1+(k-N)d$. From the compatibility of (1) and (2), we can derive the recursive formulas that determine all the $\tilde{L}_{m}^{N,k,d}$:

$$\sum_{n=0}^{k-1} \tilde{L}_{m}^{N,k,1}w^n = k \cdot \prod_{j=1}^{k-1} (jw + (k-j)),$$

$$\sum_{m=0}^{N-1+(k-N)d} \tilde{L}_{m}^{N,k,d} \cdot \tilde{w}_{N-1-m} = \sum_{l=2}^{d} (-1)^{l-1} \sum_{0=i_0 < \cdots < j_l = d} \times$$

$$\sum_{j_l = 0}^{j_0} \cdots \sum_{j_2 = 0}^{j_1} \sum_{j_1 = 0}^{j_2} l \left( \frac{i_{j_l-1} + (d-i_{j_l-1})z}{d} j_{j_l} - j_{j_l-1} \right) \tilde{L}_{m}^{N,k,1} \tilde{w}_{N-1-(N-k)d}.$$
With these data, we can construct the formulas that represent rational three point Gromov-Witten invariant \( \langle \mathcal{O}_c, \mathcal{O}_{e,N-2-m}, \mathcal{O}_{e,m-1-(k-N)d} \rangle_d \) in terms of \( \tilde{L}^{N,k,d}_m \). These three point Gromov-Witten invariants are enough for reconstruction of all the rational Gromov-Witten invariants \( \langle \mathcal{O}_{e_1, \ldots, e_m} \rangle_0, d \) [KM]. In particular, we obtain the following formula in the \( d = 2 \) case:

\[
(3) \quad \langle \mathcal{O}_c, \mathcal{O}_{e,N-2-m}, \mathcal{O}_{e,m-1-(k-N)d} \rangle_2 = k \cdot \left( \tilde{L}^{N,k,2}_m - \tilde{L}^{N,2}_1 + (k-N) 2 \tilde{L}^{N,k,1}_m - \sum_{j=0}^{k-N} (\tilde{L}^{N,k,1}_n - \tilde{L}^{N,k,1}_{n+j}) \right).
\]

According to the results of this procedure, rational three point Gromov-Witten invariants can be rational numbers with large denominator if \( k > N \), in contrast to the Calabi-Yau case where rational three point Gromov-Witten invariants are always integers.

One of the reasons of this rationality (non-integrality) comes from the contributions of multiple cover maps to Gromov-Witten invariants. In the Calabi-Yau case \( (N = k) \), for any divisor \( m \) of \( d \) there are some contributions from degree \( m \) multiple cover maps \( \phi \) of a rational curve \( \mathbb{P}^1 \) onto a degree \( \frac{d}{m} \) rational curve \( C \hookrightarrow M^k_0 \). The contributions from the multiple cover maps are expressed in terms of the virtual fundamental class of Gromov-Witten invariants. Let \( C \) be a general degree \( d \) rational curve in \( M^k_0 \). Its normal bundle \( N_{C/M^k_0} \) is decomposed into a direct sum of line bundles as follows:

\[
N_{C/M^k_0} \simeq \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1) \oplus \mathcal{O}_C^\oplus(k-5).
\]

Let \( \phi : \mathbb{P}^1 \to C \) be a holomorphic map of degree \( m \). Since the pull-back \( \phi^*(N_{C/M^k_0}) \) is given by

\[
\phi^*(N_{C/M^k_0}) \simeq \mathcal{O}_{\mathbb{P}^1}(-m) \oplus \mathcal{O}_{\mathbb{P}^1}(-m) \oplus \mathcal{O}_{\mathbb{P}^1}^\oplus(k-5),
\]

we obtain \( h^1(\phi^*(N_{C/M^k_0})) = 2m - 2 \). On the other hand, let \( \overline{M}_{0,0}(M,d) \) be the moduli space of 0-pointed stable maps of degree \( d \) from genus 0 curve to \( M \). Then the moduli space of \( \phi \) is the fiber space \( \pi : \overline{M}_{0,0}(C,m) \to \overline{M}_{0,0}(M^k, \frac{d}{m}) \), whose fibre \( \overline{M}_{0,0}(C,m) \) over \( C \) (fixed) has complex dimension \( 2m - 2 \). Then the push-forward of the virtual fundamental class \( \pi_*([c_{0,0}(H^1(\phi^*(N_{C/M^k_0}))))] \) can be computed only by intersection theory on the fiber \( \overline{M}_{0,0}(C,m) \), which turns out to be equal to \( \frac{1}{m^k} \). This depends on neither the structure of the base \( \overline{M}_{0,0}(M^k, \frac{d}{m}) \) nor the global structure of the fibration.

But when \( k < N \), the situation is more complicated than \( M^k_0 \) because of negative first Chern class. Let us concentrate on the case of \( d = 2, m = 2 \) that we discuss in this paper. In this case, \( C \) is just a line \( L \) on the hypersurface \( M^k_0 \). The moduli space \( \overline{M}_{0,0}(M^k_0,1) \) is a sub-manifold of \( \overline{M}_{0,0}(P^{N-1},1) \), while \( \overline{M}_{0,0}(P^{N-1},1) \) is the Grassmannian \( G(2,N) \), the moduli space of rank 2 quotients of \( V = \mathbb{C}^N \). As will be shown later, for a generic line \( L, N_{L/M^k_0} \) is decomposed into

\[
N_{L/M^k_0} \simeq \mathcal{O}_L(-1)^{\oplus k-N+2} \oplus \mathcal{O}_L^\oplus(2N-k-5).
\]
By pulling back it by the degree 2 map $\phi : \mathbb{P}^1 \to L$, we obtain,

$$\phi^* N_{L/M_k^k} \simeq O_{\mathbb{P}^1}(-2) \oplus O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}$$

Therefore, $h^1(\phi^* (N_{L/M_k^k})) = k - N + 2$, which is strictly greater than two, the complex dimension of the fiber $\mathbb{M}_{0,0}(L, 2)$. Thus we need to know the global structure of the fibration $\pi$ in order to compute the multiple cover contribution to degree 2 rational Gromov-Witten invariants of $M_k^k$.

In order to estimate the contributions from double cover maps $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ to $(\mathcal{O}_{\mathbb{P}^2} \mathcal{O}_{\mathbb{P}^2} \mathcal{O}_{\mathbb{P}^2})_{0,2}$, we first computed the number of conics, that intersect cycles Poâncaré dual to $e^a, e^b$ and $\mathcal{O}_{\mathbb{P}^2}$, on $M_k^k$ (whose normal bundle are of the same type) by using the method in [K2]. Then we found the following formula by comparing these integers with the results obtained from (3):

$$\langle \mathcal{O}_{\mathbb{P}^2} \mathcal{O}_{\mathbb{P}^2} \mathcal{O}_{\mathbb{P}^2} \rangle_{0,2} = \text{(number of corresponding conics)} +$$

$$\int_{G(2, N)} c_{\text{top}}(S^k Q) \wedge \left[ \frac{c(S^k Q)}{1 - \frac{1}{2}c_1(Q)} \right]_{k-N} \wedge \sigma_{a-1} \wedge \sigma_{b-1} \wedge \sigma_{c-1},$$

where $Q$ is the universal rank 2 quotient bundle of $G(2, N)$, $\sigma_a$ is a Schubert cycle defined by $\sum_{a=0}^{1} \sigma_a := \frac{1}{\pi \sqrt{\mathbb{V}}} \text{ and } \#_{k-N}$ is the operation of picking up degree $2(k-N)$ part of Chern classes.

On the other hand, we have the following formula which directly follows from the definition of the virtual fundamental class of $\mathbb{M}_{0,0}(M_k^k, 2)$:

$$\langle \mathcal{O}_{\mathbb{P}^2} \mathcal{O}_{\mathbb{P}^2} \mathcal{O}_{\mathbb{P}^2} \rangle_{0,2} = \text{(number of corresponding conics)} +$$

$$8 \int_{G(2, N)} c_{\text{top}}(S^k Q) \wedge \left[ \pi_*(c_{\text{top}}(H^1(\phi^* N_{L/M_k^k}))) \right]_{k-N} \wedge \sigma_{a-1} \wedge \sigma_{b-1} \wedge \sigma_{c-1},$$

where $\pi : \mathbb{M}_{0,0}(L, 2) \to \mathbb{M}_{0,0}(M_k^k, 1)$ is the natural projection. Here, the factor 8 comes from the divisor axiom of Gromov-Witten invariants.

In this paper, we prove the following formula

$$\pi_*(c_{\text{top}}(H^1(\phi^* N_{L/M_k^k}))) = \frac{1}{8} \left[ \frac{c(S^k Q)}{1 - \frac{1}{2}c_1(Q)} \right]_{k-N}.$$ 

By combining (5) with (6), we can derive the formula (4) immediately.

From (4), we see that $\langle \mathcal{O}_{\mathbb{P}^2} \mathcal{O}_{\mathbb{P}^2} \mathcal{O}_{\mathbb{P}^2} \rangle_{0,2}$ of $M_k^k$ is a rational number with denominator at most $2^{k-N}$. Therefore rationality (non-integrality) of the Gromov-Witten invariant $(\mathcal{O}_{\mathbb{P}^2} \mathcal{O}_{\mathbb{P}^2} \mathcal{O}_{\mathbb{P}^2})_{0,2}$ is caused by the effect of multiple cover map in this case.

We note here that the total moduli space of double cover maps of lines is isomorphic to $\mathbb{P}(S^2 Q)$ over $G := \mathbb{M}_{0,0}(M_k^k, 1) \to G(2, N)$, which is an algebraic $\mathbb{Q}$-stack $\mathbb{P}(S^2 Q)^{\text{stack}}$ (in the sense of Mumford). As a consequence, the union of all $H^1(\phi^* N_{\mathbb{P}(S^2 Q)^{\text{stack}}})$ turns out to be a coherent sheaf on $\mathbb{P}(S^2 Q)^{\text{stack}}$ with fractional Chern class in (6), as was suggested in [BT]. See [V, Section 9].

We also did some numerical experiments on degree 3 Gromov-Witten invariants of $M_k^k$ by using the results of [ES]. For $k - N > 0$, there is a new contribution from multiple cover maps to nodal conics in $M_k^k$ that did not appear in the Calabi-Yau case. Therefore, multiple cover map contributions are far more complicated than Calabi-Yau, and we leave general analysis on this problem to future works.
This paper is organized as follows. In Section 1, we analyze characteristics of moduli space of lines in $M^k_N$ and derive $N^{L/M}_k \simeq O_L(-1)^{2k-N+2} \oplus O_L^{2N-k-5}$. In Section 2, we study the moduli space $\mathcal{M}_{0,0}(\mathbb{P}^1, 2)$ from the point of view of stability and identify it with $\mathbb{P}^2$ and show that the moduli space $\mathcal{M}_{0,0}(\mathbb{P}^1, 2)$ is isomorphic to $\mathbb{P}(S^2 Q)$ over $G$. In section 4, we describe $H^1(\mathfrak{g} N_{L/M_k})$ as a coherent sheaf over $\mathbb{P}(S^2 Q)^{stack}$. In section 5, we derive the main theorem (6) of this paper by using Segre-Witten classes. In section 6, we mention some generalization to degree 3 Gromov-Witten invariants.

2. Lines on a hypersurface

Let $M$ be a generic hypersurface of degree $k$ of the projective space $\mathbb{P}^{N-1} = \mathbb{P}(V)$. We assume $2N - 5 \geq k \geq N - 2 \geq 2$ throughout this note. In this note we count the number of rational curves of virtual degree two, namely rational curves which doubly cover lines on $M$.

Let $\mathcal{P} = \mathbb{P}(V)$ be the projective space parameterizing all one-dimensional quotients of $V$, which is usually denoted by $\mathcal{P}(V)$ in the standard notation in algebraic geometry. In this notation let $W$ be a subspace of $V$. Then $\mathcal{P}(W)$ is naturally a linear subspace of $\mathcal{P}(V)$ of dimension $\dim W - 1$.

Let $G(2, V)$ be the Grassmann variety of lines in $\mathcal{P}(V)$, the scheme parameterizing all lines of $\mathcal{P} = \mathbb{P}(V)$. This is also the universal scheme parameterizing all one-dimensional quotient linear spaces of $V$. Let $W$ be a two dimensional quotient linear space, $\psi \in G(2, V)$, namely $\psi : \mathcal{P}(W) \to \mathbb{P}(V)$ the natural immersion and $i_\psi : V \to W$ the quotient homomorphism. The space $W$ is denoted by $W(\psi)$ when necessary.

There exists the universal bundle $Q_{G(2, V)}$ over $G(2, V)$ and a homomorphism $i_{\text{univ}}^* : O_{G(2, V)} \otimes V \to Q_{G(2, V)}$ whose fiber $i_{\psi}^* : V \to Q_{G(2, V), \psi}$ is the quotient $i_\psi^* : V \to W(\psi)$ of $V$ corresponding to $\psi$.

2.1. Existence of a line on $M$. Let $L = \mathcal{P}(W)$ be a line of $\mathcal{P}$, equivalently $W \in G(2, V)$. Then the condition $L \subset M$ imposes at most $k + 1$ conditions on $W$, while the number of moduli of lines of $\mathcal{P}$ equals $\dim G(2, V) = 2N - 4$. Hence we infer

Lemma 2.2. If $2N \geq k + 5$, then there exists at least a line on $M$.

See also [Katz p.152]. Let $G$ be the subscheme of $G(2, V)$ parameterizing all lines of $\mathcal{P}(V)$ lying on $M$. $Q = (Q_{G(2, V)})_G$ the restriction of $Q_{G(2, V)}$ to $G$. By Lemma 2.2, $G$ is nonempty. Let $i^* : O_G \otimes V \to Q$ be the restriction of $i_{\text{univ}}^*$ to $G$. Let $P = \mathcal{P}(Q)$ and $\pi : P \to G$ the natural projection. Then $\pi$ is the universal line of $M$ over $G$, to be more exact, the universal family over $G$ of lines lying on $M$. In other words, the natural epimorphism $i : O_G \otimes V \to Q$ induces a morphism $i : P \to \mathcal{P}_G(V) := G \times \mathcal{P}(V)$, which is a closed immersion into $\mathcal{P}_G(V)$, thus $P$ is a subscheme of $\mathcal{P}_G(V)$ such that $\pi = (p_1)_p$. Let $L_\psi = \mathcal{P}(Q_\psi)$. Note that

$$L_\psi = P_\psi := \pi^{-1}(\psi) \subset \mathcal{P}(Q_\psi) \subset \{\psi\} \times \mathcal{P}(V) \subset \mathcal{P}(V).$$
2.3. The normal bundle $N_{L/M}$. The argument of this section is standard and well known. Let $P = P(V)$, $L = P(W)$ and $i_W^*: V \to W \in G$. Let us recall the following exact sequence:

$$0 \longrightarrow O_P \longrightarrow O_P(1) \otimes V^\vee \xrightarrow{D} T_P \longrightarrow 0$$

where the homomorphism $D$ is defined by

$$D(a \otimes v^\vee) := aD_{(v^\vee)} \quad (a \in O_P(1))$$

$$(D_{v^\vee}F)(u^\vee) := \left(\frac{d}{dt}F(u^\vee + tv^\vee)\right)_{t=0}$$

for a homogeneous polynomial $F \in S(V)$ and $u^\vee, v^\vee \in V^\vee$. We note $H^0(O_P(1)) \otimes V^\vee = V \otimes V^\vee = \text{End}(V, V)$ and that the image of $H^0(O_P)$ in $\text{End}(V, V)$ is $\text{Cid}_V$. We also have the following exact sequences:

$$0 \longrightarrow T_L \longrightarrow (T_P)_L \longrightarrow N_{L/P} \longrightarrow 0$$

$$0 \longrightarrow O_L \longrightarrow O_L(1) \otimes V^\vee \xrightarrow{D_L} (T_P)_L \longrightarrow 0.$$

**Lemma 2.4.** Let $L = P(W)$. Then

$$N_{L/P} \simeq O_L(1) \otimes (V^\vee/W^\vee), \; H^0(N_{L/P}) \simeq W \otimes (V^\vee/W^\vee).$$

**Proof.** The assertion is clear from the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & O_L & \longrightarrow & O_L(1) \otimes W^\vee & \xrightarrow{(D_L)_{W^\vee}} & T_L & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow_{\text{id} \otimes v^\vee} & & \downarrow & & \\
0 & \longrightarrow & O_L & \longrightarrow & O_L(1) \otimes V^\vee & \xrightarrow{D_L} & (T_P)_L & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & O_L(1) \otimes (V^\vee/W^\vee) & \longrightarrow & N_{L/P} & \longrightarrow & 0
\end{array}$$

The second assertion is clear from $H^0(L, O_L(1)) = W$. \qed

Since $T_L \simeq O_L(2)$, there follow exact sequences

$$0 \longrightarrow H^0(T_L) \longrightarrow H^0((T_P)_L) \longrightarrow H^0(N_{L/P}) \longrightarrow 0$$

$$0 \longrightarrow H^0(O_L) \longrightarrow H^0(O_L(1)) \otimes V^\vee \xrightarrow{H^0(D_L)} H^0((T_P)_L) \longrightarrow 0.$$

We also note

$$H^0(T_L) = \text{LieAut}^0(L) = \text{End}(W, W)/\text{center} = \text{End}(W, W)/\text{Cid}_W.$$

Since $H^0(O_L(1)) = W$, we see

$$H^0((T_P)_L) = W \otimes V^\vee/\text{Im} H^0(O_L) = \text{Hom}(V, W)/\text{Cid}_W.$$

Hence we again see

$$H^0(N_{L/P}) = \frac{\text{Hom}(V, W)/\text{Cid}_W}{\text{Hom}(W, W)/\text{Cid}_W} = W \otimes (V^\vee/W^\vee) = \text{Hom}(V/W, W).$$

For any line $L = P(W)$ of $P$ the following sequence is exact:

$$0 \rightarrow N_{L/M} \rightarrow N_{L/P} \rightarrow (N_{M/P})_L \simeq O_L(k) \rightarrow 0.$$
Hence so is the following sequence as well:

\[ 0 \rightarrow H^0(N_{L/M}) \rightarrow H^0(N_{L/P}) \xrightarrow{H^0(D_L)} H^0(O_L(k)) \rightarrow 0. \]

Hence we have

**Lemma 2.5.** The following is exact:

\[ (8) \quad 0 \rightarrow H^0(N_{L/M}) \rightarrow W \otimes (V^\vee/W^\vee) \xrightarrow{H^0(D_L)} S^k W \rightarrow H^1(N_{L/M}) \rightarrow 0. \]

**Corollary 2.6.** \( \dim G \geq 2N - k - 5 \), equality holding if \( H^1(N_{L/M}) = 0. \)

**Proof.** As is well known, \( \dim G \geq h^0(N_{L/M}) - h^1(N_{L/M}) \). Note \( \dim W \otimes (V^\vee/W^\vee) = 2(N - 2) \) and \( \dim S^k W = k + 1 \). Hence the corollary follows from Lemma 2.5. \( \square \)

**Lemma 2.7.** For a generic line \( L \) on a generic hypersurface \( M \) of degree \( k \)

(i) \( N_{L/M} \cong O_L^a \oplus O_L(-1)^b \), where \( a = 2N - k - 5 \) and \( b = k - N + 2 \),

(ii) \( \text{Coker} H^0(D_L) \cong S^{k-1} W/(V^\vee/W^\vee) \) where \( D_L := D_L \oplus O_L(-1). \)

**Proof.** Let \( M \) be a generic hypersurface of degree \( k \) and \( L \) a generic line on \( M \). Without loss of generality we may assume that \( W^\vee \) is generated by \( e_1^\vee \) and \( e_2^\vee \). In other words, \( \psi : L \rightarrow P \) is given by

\[ \psi : [s : t] \rightarrow [x_1, \ldots, x_N] = [s, t, 0, \ldots, 0]. \]

Then \( F \), the polynomial of degree \( k \) defining \( M \), is written as

\[ F = x_3 f_3 + x_4 f_4 + \cdots + x_N f_N \]

for some polynomials \( f_j \) of degree \( k - 1 \). Let \( f_j = \psi^* f_j = f_j(s, t, 0, \ldots, 0). \)

Now we consider the exact sequence

\[ 0 \rightarrow H^0(N_{L/M}(-1)) \rightarrow H^0(N_{L/P}(-1)) \xrightarrow{H^0(D_L^-)} H^0(O_L(k - 1)) \rightarrow 0. \]

where we note \( H^0(N_{L/P}(-1)) = V^\vee/W^\vee \). Hence the following is exact:

\[ (9) \quad 0 \rightarrow H^0(N_{L/M}(-1)) \rightarrow V^\vee/W^\vee \xrightarrow{H^0(D_L^-)} S^{k-1} W \rightarrow H^1(N_{L/M}(-1)) \rightarrow 0. \]

where \( H^0(D_L^-) \) is given by \( H^0(D_L^-)(e_j^\vee) = f_j \) \((j = 3, 4, \cdots, N)\). A generic choice of \( F \) implies a generic choice of degree \( k - 1 \) polynomials \( f_j \) \((j = 3, 4, \cdots, N)\) in \( s \) and \( t \). By the assumptions

\[ \dim S^{k-1} W = k \geq N - 2 = \dim V^\vee/W^\vee, \]

\[ \dim W \otimes V^\vee/W^\vee = 2(N - 2) \geq k + 1 = \dim S^k W, \]

the generic choice of \( F \) implies that we can choose \( f_j \in S^{k-1} W \) \((j = 3, 4, \cdots, N)\) (and fix once for all) such that

(iii) \( f_j \) \((j = 3, 4, \cdots, N)\) are linearly independent,

(iv) \( W f_3 + W f_4 + \cdots + W f_N = S^k W. \)
Hence $H^0(D^*_{\mathcal{L}})$ is injective by (iii). It follows that $H^0(N_{L/M}(-1)) = 0$. Hence (ii) is clear. Next we consider $H^0(D_L)$. By (iv), we see

$$S^kW = W \cdot H^0(D^*_L)(V^*/W^*).$$

whence $H^0(D_L)$ is surjective. It follows that $H^1(N_{L/M}) = 0$. Hence $N_{L/M} \cong O_L^{b-a} \oplus O_L(-1)^{b+b}$ for some $a$ and $b$. Since $a + b = \text{rank}(N_{L/M}) = N - 3$ and $b = \text{deg}(N_{L/M}) = N - 2 - k$, we have (i). \hfill \Box

2.8. Lines on a quintic hypersurface in $\mathbb{P}^4$. See [Katz, Appendix A] for the subsequent examples. Let $N = 5$ and $k = 5$. Hence $M$ is a hypersurface of degree 5 in $\mathbb{P}^4$, a Calabi-Yau 3-fold. Let

$$F = x_4x_1^4 + x_5x_2^3 + x_3^3 + x_4^2 + x_5^2.$$

First we note that $M = \{F = 0\}$ is nonsingular. Let $L = \{x_3 = x_4 = x_5 = 0\}$. In this case $f_3 = 0$, $f_4 = s^4$ and $f_5 = t^4$. In the exact sequence (1) we see $H^0(N_{L/M}(-1)) = \text{Ker} H^0(D_{\mathcal{L}}) = C_{\mathcal{L}}^0$ and $H^1(N_{L/M}(-1)) = \text{Coker} H^0(D_{\mathcal{L}})$ is 3-dimensional. Hence $N_{L/M} = O_L(1) \oplus O_L(-3)$. We summarize the above. If $\text{dim Ker} H^0(D^*_L) = 1$ and if $M$ is nonsingular, then $N_{L/M} = O_L(1) \oplus O_L(-3)$. Hence $H^0(N_{L/M}) = \text{Ker} H^0(D_L) = W \oplus \text{Ker} H^0(D^*_L)$ is 2-dimensional. Therefore we can choose $f_3 = 0$ and a linearly independent pair $f_4$ and $f_5$ in $S^4W$ so that $W f_4 + W f_5$ is 4-dimensional. The choice $f_4 = s^4$ and $f_5 = t^4$ satisfies the conditions. This enables us to find a nonsingular hypersurface $M$ as above. However if we choose $f_3 = 0$, $f_4 = s^4$ and $f_5 = s^2 t$, then $W f_4 + W f_5$ is 3-dimensional. Hence $M$ is singular.

Next in the same manner we find $L$ on a nonsingular hypersurface $M$ with $N_{L/M} = O_L \oplus O_L(-2)$ or $N_{L/M} = O_L(-1)^{\oplus 2}$. Let

$$F = x_3x_1^4 + x_4x_2^3 + x_5x_2^2 + x_3^3 + x_4^2 + x_5^2.$$

Then we have $f_3 = s^4$, $f_4 = s^3 t$ and $f_5 = t^4$. Since $W f_3 + W f_4 + W f_5$ is 5-dimensional, $H^0(N_{L/M}(-1)) = \text{Ker} H^0(D_L) = 0$, $H^0(N_{L/M}) = \text{Ker} H^0(D_L) = C(\mathcal{L}^0 - s \mathcal{L}^0)$. We see also that $\text{dim} H^1(N_{L/M}) = \text{dim Coker} H^0(D_L) = 1$ and $N_{L/M} = O_L \oplus O_L(-2)$. The hypersurface $M = \{F = 0\}$ is easily shown to be nonsingular.

If $F = x_3x_1^4 + x_4x_2^3 + x_5x_2^2 + x_3^3 + x_4^2 + x_5^2$ and $M = \{F = 0\}$, then $N_{L/M} = O_L(-1)^{\oplus 2}$. 2.9. Lines on a generic hypersurface $M^8$ of $\mathbb{P}^6$. Let $N = 7$ and $k = 8$. In view of Lemma 2.2 there exists a line $L$ on any generic hypersurface $M$ of degree 8 in $\mathbb{P}(V) = \mathbb{P}^6$. In view of Lemma 2.7, $a = 1$, $b = 3$ and $N_{L/M} \cong O_L \oplus O_L(-1)^{\oplus 3}$. For example let $L : x_j = 0$ ($j \geq 3$) and we take

$$F_3 = 8x_1^2, \quad F_4 = 8x_1^2 x_2, \quad F_5 = 8x_1^2 x_2^2, \quad F_6 = 8x_1^2 x_2^3, \quad F_7 = 8x_1^2,$$

$$F = x_3F_3 + x_4F_4 + x_5F_5 + x_6F_6 + x_7F_7 + x_3^8 + x_4^8 + x_5^8.$$

and let $M = M^8 : F = 0$. We see that $M$ is nonsingular near $L$ and has at most isolated singularities. However it is still unclear to us whether $M = M^8$ is
nonsingular everywhere. The space $H^0(N_{L/M})$ is spanned by $te_N^*$, hence an
infinitesimal deformation $L_e$ of $L$ is given by

$$\delta t \mapsto \delta t, \varepsilon, -\delta t, 0, 0, 0$$

which yields $F|L_e = \varepsilon (s^8 t^8) \equiv 0 \mod \varepsilon ^8$. Since $H^1(N_{L/M}) = 0$, this
infinitesimal deformation is integrable and $G := \text{moduli of lines of } \mathbf{P}^6 \text{ contained in } M$ is
nonsingular and one dimensional at the point $[L]$.

We note that $M$ also contains 8 lines

$$L^i := L_{e_8} : \varepsilon_8 x_1 - x_2 = x_3 + \varepsilon_8 x_4 = x_5 = 0 \quad (j \geq 5),$$

with $N_{L/M} = O_L(1)^{\oplus 3} \oplus O_L(-6)$ where $\varepsilon_8 = -1$.

3. Stability

**Definition 3.1.** Suppose that a reductive algebraic group $G$ acts on a vector space $V$. Let $v \in V$, $v \neq 0$.

1. the vector $v$ is said to be semi-stable if there exists a $G$-invariant homogeneous polynomial $F$ on $V$ such that $F(v) \neq 0$.

2. the vector $v$ is said to be stable if $p$ has a closed $G$-orbit in $X_{ss}$ and the stabilizer subgroup of $v$ in $G$ is finite.

Let $\pi : V \setminus \{0\} \to \mathbf{P}(V^\vee)$ be the natural surjection. Then $v \in V$ is semi-stable (resp.
stable) if and only if $\pi(v)$ is semi-stable (resp. stable).

3.2. Grassmann variety. Let $V$ be an $N$-dimensional vector space, and $G(r, N)$
the Grassmann variety parameterizing all $r$-dimensional quotient spaces of $V$. Here
is a natural way of understanding $G(r, N)$ via GIT-stability. Let $U$ be an $r$-
dimensional vector space, $X = \text{Hom}(V, U)$ and $\pi : X \setminus \{0\} \to \mathbf{P}(X^\vee)$ the natural
map. Then $\text{SL}(U)$ acts on $X$ from the left by:

$$(g \cdot \phi^*)(v) = g \cdot (\phi^*(v)) \quad \text{for } \phi^* \in X, \ v \in V.$$ 

We see that for $\phi^* \in X$

$\phi^*$ is $\text{SL}(U)$-stable $\iff$ rank $\phi^* = r$,

$\phi^*$ is $\text{SL}(U)$-semi-stable $\iff$ $\phi^*$ is $\text{SL}(U)$-stable.

In fact, if rank $\phi^* = r - 1$, then there is a one-parameter torus $T$ of $\text{SL}(U)$ such
that the closure of the orbit $T \cdot \phi$ contains the zero vector as the following simple
example ($r = 2$) shows

$$\lim_{t \to 0} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \lim_{t \to 0} \begin{pmatrix} ta_{11} & ta_{12} & \cdots & ta_{1N} \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$ 

Let $X_s$ be the set of all (semi)stable points and $P_s$ the image of $X_s$ by $\pi$. It is, as
we saw above, just the set of all $\phi \in X$ with rank $\phi^* = r$. Therefore the GIT-orbit
space $P_s/\text{SL}(U)$ is the orbit space $P_s/\text{SL}(U)$ by the free action, the Grassmann
variety $G(r, N)$. 
3.3. Moduli of double coverings of $\mathbf{P}^1$ (1). Let $W$ and $U$ be a pair of two dimensional vector spaces, $X = \text{Hom}(W,S^2U)$, and $\pi : X \setminus \{0\} \to \mathbf{P}(X^\vee)$ the natural morphism. Note that $\text{SL}(U)$ acts on $S^2U$ from the left via the natural action: $\sigma(u_1u_2) = \sigma(u_1)\sigma(u_2)$ for $\forall u_1, u_2 \in U$. Thus $\text{SL}(U)$ acts on $X$ from the left in the same manner in the subsection 3.2.

**Lemma 3.4.** Let $\phi^* \in X$.

(i) $\phi^*$ is unstable iff $\phi^*(w)$ has a double root for any $w \in W$.

(ii) $\phi^*$ is semistable iff $\phi^*(w)$ has no double roots for some nonzero $w \in W$.

(iii) $\phi^*$ is stable iff $\phi^*(W)$ is a base-point free linear subsystem of $S^2U$ on $\mathbf{P}(U)$.

**Proof.** We note that $\phi^*$ is unstable iff there is a suitable basis $s$ and $t$ of $U$ such that $\phi^*(w) = a(w)s^2$ for any $w \in W$ since a torus orbit $T \cdot \phi^*$ contains the zero vector. This proves (i). This also proves (ii). Next we prove (iii). If $\phi^*(W)$ has a base point, then it is clear that $\phi^*$ is not stable. If $\phi^*$ is semistable and it is not stable, then we choose a basis $s$, $t$ of $U$ and a basis $w_1$, $w_2$ of $W$ such that $\phi^*(w_1) = st$. If $\phi(w_2) = as^2 + bst$, then $\phi^*$ is not stable. This proves the lemma. □

**Theorem 3.5.** Let $X_{ss}$ be the Zariski open subset of $X$ consisting of all semistable points of $X$, $\pi(X_{ss})$ the image of $X_{ss}$ by $\pi$, and $Y := \pi(X_{ss})/\text{SL}(U)$. Then $Y \simeq \mathbf{P}^2$.

**Proof.** First consider a simplest case. We choose a basis $s$, $t$ of $U$. Let $w_1$ and $w_2$ be a basis of $W$, $T$ the subgroup of $\text{SL}(U)$ of diagonal matrices and $X^t = \{\phi^* \in X; \phi^*(w_1) = 2st\}$. Let $Z^t = \text{SL}(U) \cdot X^t$.

We note that $Z^t$ is an $\text{SL}(U)$-invariant subset of $X_{ss}$. We prove $\pi(Z^t)/\text{SL}(U) \simeq \mathbf{C}^2$. Let $\phi^*$ and $\psi^*$ be points of $X^t$. Let $\phi^*(w_2) = As^2 + 2Bst + Ct^2$ and $\psi^*(w_2) = as^2 + 2bst + ct^2$. Then it is easy to check

$$g \cdot \phi^* = \psi^*$$

for $\exists g \in \text{SL}(U) \iff g \cdot \phi^* = \psi^* \text{ for } \exists g \in T$

$$\iff A = au^2, \ B = b, \ C = u^{-2}c \text{ for } \exists u \neq 0.$$ Therefore each equivalence class of $\pi(Z)/\text{SL}(U)$ is represented by the pair $(A, B)$, which proves $\pi(Z)/\text{SL}(U) \simeq \mathbf{C}^2$.

Now we prove the lemma. Let $\phi^* \in X_{ss}$, $\phi_j = \phi^*(w_j)$ and $\phi_0 = -(\phi_1 + \phi_2)$. Let

$$\phi_0 = r_1s^2 + 2r_2st + r_3t^2,$$

$$\phi_1 = p_1s^2 + 2p_2st + p_3t^2,$$

$$\phi_2 = q_1s^2 + 2q_2st + q_3t^2,$$

and we define

$$D_1 = p_2^2 - p_1p_3, \ D_2 = q_2^2 - q_1q_3,$$

$$D_0 = r_2^2 + r_1r_3 = D_1 + D_2 + 2p_2q_2 - (p_1q_3 + p_3q_1).$$

To show the lemma, we prove the more precise isomorphism

$$\pi(X_{ss})/\text{SL}(U) \cong \text{Proj} \mathbf{C}[D_0, D_1, D_2]$$

For this purpose we define $Y_j = \pi(\{\phi^* \in X_{ss}; \phi_j \text{ has no double roots}\})/\text{SL}(U)$. It suffices to prove $Y_1 = \text{Spec} \mathbf{C}[D_1^a, D_1^b]$ by reducing it to the first simplest case.
Let $\phi^* \in Y_1$. Let $\alpha$ and $\beta$ be the roots of $\phi_1 = 0$. By the assumption $\phi_1$ has no double roots, hence $\alpha \neq \beta$. Let
\[
u = \frac{1}{\gamma} (s - \alpha t), \quad v = \frac{1}{\gamma} (s - \beta t), \quad g = \frac{1}{\gamma} \left( \frac{1 - \alpha}{1 - \beta} \right)
\]
where $\gamma = \sqrt{\alpha - \beta}$. Note that $g \in SL(U)$. Hence we see
\[(\phi_1(s,t), \phi_2(s,t)) \equiv (p_1\gamma^4 uv, A_1u^2 + 2B_1uv + C_1v^2)\]
where
\[A_1 = q_1^2 \beta^2 + 2q_2 \beta + q_3,\]
\[-B_1 = q_1 \alpha + q_2 (\alpha + \beta) + q_3,\]
\[C_1 = q_1^2 \alpha^2 + 2q_2 \alpha + q_3.
\]
Thus we see
\[(\phi_1(s,t), \phi_2(s,t)) \equiv (2st, As^2 + 2Bst + Ct^2)\]
where
\[A = \frac{2A_1}{p_1^2}, \quad B = \frac{2B_1}{p_1^2}, \quad C = \frac{2C_1}{p_1^4}, \quad p_1 = 4D_1,
\]
\[AC = B^2 - \frac{D_1}{D_1}, \quad B = \frac{D_0 - D_1 - D_2}{2D_1}.
\]
Therefore by the first half of the proof
\[Y_1 \simeq \text{Spec } \mathbb{C}[AC, B] = \text{Spec } \mathbb{C}^{D_0, D_1, D_2}.
\]
This completes the proof of the lemma. \qed

**Corollary 3.6.** Let $Y^s = \pi(X_s)/SL(U)$. Then $Y \setminus Y^s$ is a conic of $Y$ defined by
\[Y \setminus Y^s : D_0^2 + D_1^2 + D_2^2 - 2D_0D_1 - 2D_1D_2 - 2D_2D_0 = 0.
\]

**Proof.** In view of Theorem 3.5, $Y_1 \simeq \text{Spec } \mathbb{C}[AC, B]$. The complement of $Y_s$ in $Y_1$ is then the curve defined by $AC = 0$, which is easily identified with the above conic. \qed

**Corollary 3.7.** Let $X^0$ be the Zariski open subset of $X$ consisting of all semistable points $\phi^*$ of $X$ with rank $\phi^* = 2$, and let $Y^0 := \pi(X^0)/SL(U)$. Then $Y^0 \simeq \pi(X^0)/SL(U) \simeq Y \simeq \mathbb{P}^2$.

**Proof.** It suffices to compare $Y_1$ and $Y^0 \cap Y_1$. As in the proof of Theorem 3.5 we let $X' = \{ \phi^* \in X; \phi^*(w_1) = 2st \}$. Let $Z = SL(U) \cdot X'$ and $Z^0 = SL(U) \cdot (X' \cap X^0)$.

Then with the notation in Theorem 3.5, we recall $X' = \{ \phi^* \in X; \phi^*(w_1) = 2st, \phi^*(w_2) = As^2 + 2Bst + Ct^2 \}, \pi(Z)/SL(U) \simeq \text{Spec } \mathbb{C}[AC, B]$ where
\[X' \cap X^0 = \{ \phi^* \in X'; A \neq 0 \text{ or } C \neq 0 \} \]
In the same manner as before we see $\pi(Z^0)/SL(U) \simeq \text{Spec } \mathbb{C}[AC, B]$, whence $\pi(Z^0)/SL(U) = \pi(Z)/SL(U)$. This proves $Y^0 \cap Y_1 = Y_1$. This completes the proof of the corollary. \qed
3.8. **Moduli of double coverings of** $\mathbb{P}(W)$ (2). There is an alternative way of understanding $\pi(X_2)/\text{SL}(U) \simeq \mathbb{P}^2$ by using the isomorphism $S^2\mathbb{P}^1 \simeq \mathbb{P}^2$. We use the following convention to denote a point of $\mathbb{P}(U) = U^\vee \setminus \{0\}/G_m$: $(u : v) = u^s v^t + v^t \in U^\vee$ where $s^\vee$ and $t^\vee$ are a basis dual to $s$ and $t$. In what follows we fix a basis $w_1$ and $w_2$ of $W$. Let $P := (a_1 : a_2)$ and $Q := (b_1 : b_2)$ be a pair of points of $\mathbb{P}(W) \simeq \mathbb{P}^1$. If $P \neq Q$, there is a double covering $\phi : \mathbb{P}(U) \to \mathbb{P}(W)$ ramifying at $P$ and $Q$, unique up to isomorphism once we fix the base $w_1$ and $w_2$:

$$\frac{b_2 w_1 - b_1 w_2}{a_2 w_1 - a_1 w_2} = \left(\frac{s}{t}\right)^2.$$ 

Thus $\phi$ is given explicitly by

$$\phi_1 := \phi^*(w_1) = b_1 s^2 - a_1 t^2, \quad \phi_2 := \phi^*(w_2) = b_2 s^2 - a_2 t^2, \quad \phi_0 = -(\phi_1 + \phi_2)$$

for which we have

$$D_1 = a_1 b_1, \quad D_2 = a_2 b_2, \quad D_0 = (a_1 + a_2)(b_1 + b_2).$$

The isomorphism $S^2\mathbb{P}^1 \simeq \mathbb{P}^2$ is given by $(P, Q) \mapsto (D_0, D_1, D_2)$. This shows

**Corollary 3.9.** We have a natural isomorphism: $Y \simeq \mathbb{P}(S^2W)$. 

4. **The virtual normal bundle of a double covering**

4.1. **The case** $N = 7$ and $k = 8$ revisited. We revisit the example in the subsection 2.9. Let $N = 7$ and $k = 8$. Let $L : x_j = 0$ ($j \geq 3$) and we take

$$F_3 = 8x_1^7, F_4 = 8x_2^6x_2, F_5 = 8x_1^4x_3^2, F_6 = 8x_1^2x_2^2, F_7 = 8x_7^2,$$

$$F = x_3F_3 + x_4F_4 + x_5F_5 + x_6F_6 + x_7F_7 + x_8^4 + x_8^4 + x_8^4 + x_8^4 + x_8^4.$$

and let $M = M_8^8 : F = 0$. We often denote $L$ also by $\mathbb{P}(W)$ with $W$ a two dimensional vector space for later convenience. Since $H^0(D_L^-)$ is injective and $H^0(D_L)$ is surjective, we have $N_{L/M} \simeq O_L \oplus O_L(-1)^{\mathbb{G}_m}$. Hence $H^1(N_{L/M}(-1)) = H^1(O_L(-2)^{\mathbb{G}_m})$ is 3-dimensional. As we see easily, this follows also from the fact that $\text{Coker}H^0(D_L^-)$ is freely generated by $x_1^4x_2^4, x_1^4x_2^4$ and $x_1^4x_2^4$.

Let $\phi^* = (\phi_1, \phi_2) \in X^0$. Then $\text{Coker}H^0(\phi^*D_L)$ is generated by a single element $\phi_2 x_3^8 + \phi_1 x_4^8$, while $\text{Coker}H^0(\phi^*D_L)$ is generated by $S^2U \cdot \phi_1 \phi_2^3$, $S^2U \cdot \phi_1 \phi_2^3$ and $S^2U \cdot \phi_1 \phi_2^3$. To be more precise, we see

$$\text{Coker}H^0(\phi^*D_L) = \{\phi_1 \phi_2^3, \phi_1 \phi_2^3, \phi_1 \phi_2^3\} \otimes S^2U / \{\phi_1, \phi_2\}.$$ 

In fact, this is proved as follows: first we consider the case where $\phi_1$ and $\phi_2$ has no common zeroes. In this case $\phi^*$ gives rise to a double covering $\phi : \mathbb{P}(U) \to \mathbb{P}(W)$ ($= L$), which we denote by $L_\phi$ for brevity. By pulling back by $\phi^*$ the normal sequence $0 \to N_{L/M} \to N_{L/P} \to O_L(k) \to 0$ ($k = 8$) for the line $L$ we infer an exact sequence

$$0 \to \phi^*N_{L/M} \to \phi^*N_{L/P} \xrightarrow{\phi^*D_L} \phi^*O_L(k) \to 0,$$

which yields an exact sequence

$$0 \longrightarrow H^0(\phi^*N_{L/M}) \longrightarrow S^2U \otimes (V^\vee/W^\vee) \xrightarrow{H^0(\phi^*D_L)} H^0(O_L(2k)) \longrightarrow 0.$$
Let $\eta = q_3 e^\eta_3 + \cdots + q_7 e^\eta_7 \in \text{Ker } H^0(\phi^* D_L)$, $q_j \in S^2 U$. Then we have

$$\phi_1^2(q_3 \phi_1^2 + q_4 \phi_1^2 \phi_2 + q_5 \phi_1^2 \phi_2^2 + q_6 \phi_2^2) = -q_7 \phi_2^2.$$  

Since $\phi_1$ and $\phi_2$ are mutually prime and $q_j$ is of degree two, we have $q_7 = 0$ and

$$\phi_1^2(q_3 \phi_1^2 + q_4 \phi_1^2 \phi_2 + q_5 \phi_1^2 \phi_2^2) = -q_6 \phi_2^2.$$  

Hence $q_6 = 0$ and similarly we infer also $q_7 = 0$. Thus we have $q_3 \phi_1 + q_4 \phi_2 = 0$. This proves that $\text{Ker } H^0(\phi^* D_L)$ is generated by $\phi_2 e^\eta_2 - \phi_1 e^\eta_1$.

Next we prove that $\text{Coker } H^0(\phi^* D_L)$ is generated by $\phi^* \text{Coker } H^0(D_L^-)$ over $S^2 U$, in fact over $S^2 U / \phi^*(W)$. Without loss of generality we may assume that $\phi_1 = 2st$ and $\phi_2 = \lambda s^2 + 2\nu st + t^2$ for some $\lambda \neq 0$ and $\nu \in \mathbb{C}$. Let $\phi^* W = \{ \phi_1, \phi_2 \}$. Then one checks $U \cdot \phi^* W = S^2 U$, and hence $S^2 U \cdot \phi^* W = S^4 U$, $S^2 m - 2 U \cdot \phi^* W = S^2 m U$ for $m \geq 2$. It follows $S^2 U \cdot \phi^*(S^{m-1} W) = S^2 m U$ for $m \geq 1$. In fact, by the induction on $m$

$$S^2 U \cdot \phi^*(S^{m} W) = S^2 U \cdot \phi^*(W) \cdot \phi^*(S^{m-1} W)$$

$$= S^4 U \cdot \phi^*(S^{m-1} W)$$

$$= S^2 U \cdot (S^2 U \cdot \phi^*(S^{m-1} W))$$

$$= S^2 U \cdot S^{2m} U = S^{2m+2} U.$$  

Therefore $H^0(OL_2(2k)) = S^16 U = S^2 U \cdot \phi^*(S^7 W)$. Hence

$$\text{Coker } H^0(\phi^* D_L) = S^16 U / \text{Im } H^0(\phi^* D_L)$$

$$= S^2 U \cdot \phi^*(S^7 W) / S^2 U \cdot \phi^*(\text{Im } H^0(D_L^-))$$

$$= (S^2 U / \phi^*(W)) \cdot \phi^*(S^7 W / \text{Im } H^0(D_L^-)).$$

because $\text{Coker } H^0(D_L^-) = S^7 W / \text{Im } H^0(D_L^-)$ and $W \cdot S^7 W \subset W \cdot \text{Im } H^0(D_L^-) = S^8 W$ by the choice of $L$. This proves that $\text{Coker } H^0(\phi^* D_L)$ is generated by $\phi^* \text{Coker } H^0(D_L^-)$ over $S^2 U / \phi^*(W)$. It follows $\text{Coker } H^0(\phi^* D_L) = (\phi^* \text{Coker } H^0(D_L^-)) \otimes (S^2 U / \phi^* W)$.

Finally we consider the case where $\phi_1$ and $\phi_2$ has a common zero. In this case we may assume $\phi_1 = 2st$ and $\phi_2 = 2\nu st + t^2$. In this case $L_\phi$ is a chain of two rational curves $C_\phi$ and $C^\circ_\phi$ where $C_\phi$ is the proper transform of $\mathbb{P}(U)$, where the double covering map from $L_\phi$ to $\mathbb{P}(W)$ is the union of the isomorphisms $\phi^i$ and $\phi^\mu$, say, $\phi = \phi^i \cup \phi^\mu$. Let $\psi_1 = 2s$ and $\psi_2 = 2\nu s + t$. Then $\phi^i$ is induced by the homomorphism $(\phi^i)^* \in \text{Hom}(W, U)$ such that $(\phi^i)^*(w_j) = \psi_j$. On the other hand let $U^\mu_\phi = C\phi + C\mu$, $\psi_1^\mu = 2t$ and $\psi_2^\mu = \lambda + 2t$ where we note $\psi_j^\mu$ is the linear part of $\phi_j$ in $t$ with $s = 1$. Then $C_\phi = \mathbb{P}(U^\mu_\phi)$ and $\phi^\mu$ is induced by the homomorphism $(\phi^\mu)^* \in \text{Hom}(W, U^\mu_\phi)$ such that $(\phi^\mu)^*(w_j) = \psi_j^\mu$. Furthermore the pull back by $\phi^\mu$ of the normal sequence for $L$

$$0 \to \phi^\mu N_{L/M} \to \phi^\mu N_{L/P} \xrightarrow{\phi^* D_L} \phi^\mu O_L(k) \to 0,$$
yields exact sequences with natural vertical homomorphisms:

\[
\begin{array}{c}
0 \longrightarrow \phi^* N_{L/M} \longrightarrow (\phi')^* N_{L/M} \oplus (\phi'')^* N_{L/M} \longrightarrow C \longrightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow \phi^* N_{L/P} \longrightarrow (\phi')^* N_{L/P} \oplus (\phi'')^* N_{L/P} \longrightarrow V^\vee /W^\vee \longrightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow \phi^* O_L(k) \longrightarrow O_{C_{\phi}}(k) \oplus O_{C_{\psi}}(k) \longrightarrow C \longrightarrow 0.
\end{array}
\]

This yields the following long exact sequences:

\[
\begin{array}{c}
0 \longrightarrow H^0((\phi')^* N_{L/M}) \longrightarrow U \otimes V^\vee /W^\vee \xrightarrow{H^0((\phi')^* D_L)} S^k U \\
\longrightarrow H^1((\phi')^* N_{L/M}) \longrightarrow 0 \\
0 \longrightarrow H^0((\phi'')^* N_{L/M}) \longrightarrow U''_\phi \otimes V^\vee /W^\vee \xrightarrow{H^0((\phi'')^* D_L)} S^k U''_\phi \\
\longrightarrow H^1((\phi'')^* N_{L/M}) \longrightarrow 0
\end{array}
\]

whence \( H^1((\phi')^* N_{L/M}) = H^1((\phi'')^* N_{L/M}) = 0 \), and both \( H^0((\phi')^* N_{L/M}) \) and \( H^0((\phi'')^* N_{L/M}) \) are one-dimensional. Let \( U^t \) be the subspace of \( U \) consisting of elements vanishing at \( C_{\phi} \cap C_{\psi} \), namely the subspace spanned by \( t \). Then the restriction of \( H^0((\phi')^* D_L) \) to \( U^t \otimes V^\vee /W^\vee \) equals \( t \cdot H^0((\phi')^* D_L) \). Hence

\[
\text{Coker } H^0(\phi^* D_L) \simeq t \cdot S^2 U/t \cdot \text{Im } H^0((\phi')^* D_L) \oplus \text{Coker } H^0((\phi'')^* D_L) \\
\simeq S^2 U/t \cdot \text{Im } H^0((\phi')^* D_L) \simeq \text{Coker } H^0((\phi')^* D_L).
\]

One could understand the above isomorphism as

\[
\text{Coker } H^0(\phi^* D_L) = \text{Coker } (\phi^* H^0(D_L)) \otimes (S^2 U/\phi^* W).
\]

Thus \( H^0(\phi^* N_{L/M}) \) is one-dimensional, while \( H^1(\phi^* N_{L/M}) \) is 3-dimensional. This is immediately generalized into the following

**Lemma 4.2.** For any \( \phi^* \in X^0 \) we have

\[
\begin{align*}
\text{Ker } H^0(\phi^* D_L) &= \phi^* \text{Ker } H^0(D_L), \\
\text{Coker } H^0(\phi^* D_L) &= (\phi^* \text{Coker } H^0(D_L^-)) \oplus (S^2 U/\phi^* W).
\end{align*}
\]

**Lemma 4.3.** We define a line bundle \( L_0 \) (resp. \( L_1 \)) on \( Y \) (\( \simeq \mathbf{P}(S^2 W) \)) by the assignment:

\[
X^0 \ni \phi^* \mapsto \phi^* \text{Ker } H^0(D_L) \ (\text{resp. } \phi^* \text{Coker } H^0(D_L^-)).
\]

Then \( L_k \simeq O_{\mathbf{P}(S^2 W)} \).

**Proof.** We know that \( \phi^* \text{Ker } H^0(D_L) \) is generated by \( \phi_2 e_{\phi}^\vee - \phi_1 e_{\psi}^\vee \). By the SL(2)-variable change of \( s \) and \( t \), \( \phi_j \) is transformed into a new quadratic polynomial, which is however the same as the first \( \phi_j \). This shows the generator is unchanged, whence \( L_0 \simeq O_{\mathbf{P}(S^2 W)} \). The proof for \( L_1 \) is the same. \( \square \)
Lemma 4.4. We define a coherent sheaf $L$ on the stack $Y$ ($\simeq \mathbb{P}(S^2 W)$) (See Remark below) by the assignment:

$$X^0 \ni \phi^* \mapsto S^2 U/\phi^* W.$$ 

Then $L^2 \simeq O_{\mathbb{P}(S^2 W)}(-1)$.

Proof. The GIT-quotient $Y^0$ is covered with the images of $X'_j$: 

$$X'_1 = \{ (\phi_1, \phi_2) \in X^0; \phi_1 = 2st, \phi_2 = \lambda s^2 + 2\nu st + t^2, \lambda, \nu \in \mathbb{C} \},$$

$$X'_2 = \{ (\phi_1, \phi_2) \in X^0; \phi_1 = ps^2 + 2qst + t^2, \phi_2 = 2st, p, q \in \mathbb{C} \}.$$

It is clear that the natural image of $X'_j$ in $Y$ is $Y_j$. The map $\phi$ given by $\phi^* = (\phi_1, \phi_2) \in Y_1$ has natural $\mathbb{Z}_2$ involution generated by,

$$r: (\sqrt{\lambda}s + t, \sqrt{\lambda}s - t) \rightarrow (\sqrt{\lambda}s + t, - (\sqrt{\lambda}s - t)).$$

Since

$$2st = \frac{1}{2\sqrt{\lambda}}((\sqrt{\lambda}s + t)^2 - (\sqrt{\lambda}s - t)^2),$$

$$\lambda s^2 + 2\nu st + t^2 = \frac{\nu}{2\sqrt{\lambda}}((\sqrt{\lambda}s + t)^2 - (\sqrt{\lambda}s - t)^2) + \frac{1}{2}((\sqrt{\lambda}s + t)^2 + (\sqrt{\lambda}s - t)^2),$$

it is clear that,

$$r^*(\phi_1) = \phi_1, \quad r^*(\phi_2) = \phi_2, \quad r^*(\lambda s^2 - t^2) = -(\lambda s^2 - t^2).$$

Therefore, we can decompose $S^2 U$ into $\langle \lambda s^2 - t^2 \rangle \odot \langle \phi_1, \phi_2 \rangle \mathbb{C}$ with respect to eigenvalue of $r^*$ and take $\lambda s^2 - t^2$ as canonical generator of $S^2 U/\phi^* W$. Similarly $S^2 U/\phi^* W$ is generated by $ps^2 - t^2$ on $Y_2$. The problem is therefore to write $\lambda s^2 - t^2$ as an $\Gamma(O_{Y_1})$-multiple of $pu^2 + v^2$ when we write $\phi_2 = 2uv$ by a variable change in $GL(2)$. The following variable change $(s, t) \mapsto (u, v)$ is in $GL(2)$:

$$s = \frac{\sqrt{2\alpha}}{(\beta - \alpha)^2} (2u - \frac{(\beta - \alpha)^2}{2\alpha}v), \quad t = \frac{\sqrt{2\alpha}}{(\beta - \alpha)^2} (2\beta u - \frac{(\beta - \alpha)^2}{2\alpha}v),$$

where $\alpha, \beta$ are roots of the equation $\lambda s^2 + 2\nu st + t^2 = 0$. Under this coordinate change, $\phi_1$ and $\phi_2$ are rewritten as follows:

$$\phi_1 = \frac{\lambda}{(\nu^2 - \lambda)^2} u^2 + 2\frac{\nu}{\nu^2 - \lambda} uv + v^2 = pu^2 + 2quv + v^2, \quad \phi_2 = 2uv.$$

Then we have

$$pu^2 - v^2 = -\frac{2}{\beta - \alpha} (\lambda s^2 - t^2) = -\frac{1}{\nu^2 - \lambda} (\lambda s^2 - t^2) = -\sqrt{\frac{D_1}{D_2}} (\lambda s^2 - t^2).$$

Similarly by computing the effect on $S^2 U/\phi^* W$ by the variable change from $X'_1$ into $X'_0$, we see that $L^2$ is isomorphic to $O_{\mathbb{P}(S^2 W)}(-1)$. This completes the proof. \( \square \)

Remark 4.5. We remark that the space $X$ must be regarded as a $\mathbb{Q}$-stack $Y^{\text{stack}}$ as follows: First we define $\phi_0 = -\phi_1 - \phi_2$. For each atlas $X'_\alpha$, we define an atlas $Y'_\alpha$ of
(α = 0, 1, 2) by
\[ Y_0^{stack} = \{(ϕ_0, ϕ_1, ϕ_2, ±ϕ_3) ∈ X^0 × S^2 U; ϕ_0 = 2st, \ ϕ_1 = as^2 + 2bst + t^2, \]
\[ ϕ_3 = as^2 - t^2, \ a, b ∈ C, \]
\[ Y_1^{stack} = \{(ϕ_0, ϕ_1, ϕ_2, ±ϕ_3) ∈ X^0 × S^2 U; ϕ_1 = 2st, \ ϕ_2 = λs^2 + 2νst + t^2, \]
\[ ϕ_3 = λs^2 - t^2, \ λ, ν ∈ C, \]
\[ Y_2^{stack} = \{(ϕ_0, ϕ_1, ϕ_2, ±ϕ_3) ∈ X^0 × S^2 U; ϕ_1 = ps^2 + 2qst + t^2, \ ϕ_2 = 2st, \]
\[ ϕ_3 = ps^2 - t^2, p, q ∈ C. \]

Since \( L^2 ≃ O_{P(S^2 W)}(-1) \) we have \( c_1(L) = -\frac{1}{2}c_1(O_{P(S^2 W)}(1)) \) in the Chow ring \( A(Y^{stack})_Q = A(X)_Q = A(P(S^2 W))_Q. \)

5. Proof of the main theorem

**Theorem 5.1.**

\[
π_∗(c_{top}(H^1)) = \frac{1}{8} \left[ \frac{c(S^{k-1} Q)}{1 - \frac{1}{2}c_1(Q)} \right]_{k-N},
\]

where \( π \) is the natural projection from \( \tilde{M}_{0,0}(L, 2) \) to \( G \) and \( [s]_{k-N} \) is the operation of picking up the degree \( 2(k-N) \) part of Chern classes.

**Proof.** From now on we denote the coherent sheaf \( L \) in Lemma 4.4 by \( O_{P}(-\frac{1}{2}) \). In view of the results from the previous section, what remains is to evaluate the top chern class of \( (S^{k-1} Q / ((V^Y ⊕ O_G) / Q^Y)) \) or \( O_{P}(S^2 Q) \). Since double cover maps parametrized by \( P(S^2 Q) \) have natural \( Z_2 \) involution \( r \) given in the previous section, we have to multiply the result of integration on \( P(S^2 Q) \) by the factor \( \frac{1}{2} \) [BT], [FP]. With this set-up, let \( π^r : P(S^2 Q) → G \) be the natural projection. Then what we have to compute is \( π_∗(c_{top}(H^1)) = \frac{1}{8}π_∗(c_{top}(H^1)) = \frac{1}{4}π_∗(c_{top}(S^{k-1} Q / ((V^Y ⊕ O_G) / Q^Y)) \) or \( O_{P}(-\frac{1}{2})) \). Let \( z \) be \( c_1(O_{P}(1)) \). Then we obtain,

\[
\frac{1}{2}π_∗(c_{top}(S^{k-1} Q / ((V^Y ⊕ O_G) / Q^Y)) \) or \( O_{P}(-\frac{1}{2}))
\]

\[
= \frac{1}{2} \sum_{j=0}^{k-N+2} c_{k-N+2-j}(S^{k-1} Q ⊕ Q^Y) ∗ π_∗(z^j) ∗ (-\frac{1}{2})^j
\]

\[
= \frac{1}{8} \sum_{j=0}^{k-N} c_{k-N-j}(S^{k-1} Q ⊕ Q^Y) ∗ s_j(S^2 Q) ∗ (-\frac{1}{2})^j
\]

\[
= \frac{1}{8} \left[ \frac{c(S^{k-1} Q) ∗ c(Q^Y)}{1 - \frac{1}{2}c_1(S^2 Q) + \frac{1}{3}c_2(S^2 Q) - \frac{1}{5}c_3(S^2 Q)} \right]_{k-N},
\]

where \( s_j(S^2 Q) \) is the \( j \)-th Segre class of \( S^2 Q \). But if we decompose \( c(Q) \) into \( (1 + α)(1 + β) \), we can easily see,

\[
\frac{c(Q^Y)}{1 - \frac{1}{2}c_1(S^2 Q) + \frac{1}{3}c_2(S^2 Q) - \frac{1}{5}c_3(S^2 Q)} = \frac{(1 - α)(1 - β)}{(1 - α)(1 - \frac{1}{2}(α + β))(1 - β)}
\]

\[
= \frac{1}{1 - \frac{1}{2}c_1(Q)}.
\]
Finally, by combining the above theorem with the divisor axiom of Gromov-Witten invariants, we can prove the decomposition formula of degree 2 rational Gromov-Witten invariants of $M^k_N$ found from numerical experiments.

**Corollary 5.2.**
$$\langle O_{c^a}O_{c^b}O_{c^c} \rangle_{0,2} = \langle O_{c^a}O_{c^b}O_{c^c} \rangle_{0,2-2} + 8(\pi_* (c_{\text{top}}(H^1))O_{c^a}O_{c^b}O_{c^c})_{0,1},$$
where $\langle O_{c^a}O_{c^b}O_{c^c} \rangle_{0,2-2}$ is the number of conics that intersect cycles Poincaré dual to $e^a$, $e^b$ and $e^c$. We also denote by $\langle \pi_* (c_{\text{top}}(H^1))O_{c^a}O_{c^b}O_{c^c} \rangle_{0,1}$ the integral:
$$\int_{G(2,V)} c_{\text{top}}(S^k Q) \wedge \pi_* (c_{\text{top}}(H^1)) \wedge \sigma_{a-1} \wedge \sigma_{b-1} \wedge \sigma_{c-1}.$$  

6. GENERALIZATION TO TWISTED CUBICS

In this section, we present a decomposition formula of degree 3 rational Gromov-Witten invariants found from numerical experiments using the results of [ES].

**Conjecture 6.1.** If $k - N = 1$, we have the following equality:
$$\pi_* (c_{\text{top}}(H^1)) =$$
$$\frac{1}{27} \left( \frac{1}{24}(27k^2 - 55k + 26)k(k - 1) + \frac{2}{9}c_1(Q)^2 + \left( \frac{7}{6}(k + 1)k(k - 1) + \frac{1}{9}\right)c_2(Q) \right).$$
where $\pi : \overline{M}_{0,0}(L, 3) \to \overline{M}_{0,0}(M^k_N, 1)$ is the natural projection.

In the $k - N > 1$ case, we have not found the explicit formula, because in the $d = 3$ case, we have another contribution from multiple cover maps of type $(2+1) \to (1+1)$. Here multiple cover map of type $(2+1) \to (1+1)$ is the map from nodal curve $\mathbb{P}^1 \vee \mathbb{P}^1$ to nodal conic $L_1 \vee L_2 \subset M^k_N$, that maps the first (resp. the second) $\mathbb{P}^1$ to $L_1$ (resp. $L_2$) by two to one (resp. one to one). In the $k - N = 1$ case, we have also determined the contributions from multiple cover maps of $(2+1) \to (1+1)$ nodal conics.

**Corollary 6.2.** If $k - N = 1$, $\langle O_{c^a}O_{c^b}O_{c^c} \rangle_{0,3}$ is decomposed into the following contributions:
$$\langle O_{c^a}O_{c^b}O_{c^c} \rangle_{0,3} = \langle O_{c^a}O_{c^b}O_{c^c} \rangle_{0,3-3} + \frac{1}{k} \langle O_{c^a}O_{c^b}O_{c^c} \rangle_{0,1} \langle O_{c^a}O_{c^b}O_{c^c} \rangle_{0,1} + \frac{3}{2} \langle O_{c^a}O_{c^b}O_{c^c} \rangle_{0,1} \langle O_{c^a}O_{c^b}O_{c^c} \rangle_{0,1} + \frac{3}{2} \langle O_{c^a}O_{c^b}O_{c^c} \rangle_{0,1} \langle O_{c^a}O_{c^b}O_{c^c} \rangle_{0,1} + 27(\pi_* (c_{\text{top}}(H^1))O_{c^a}O_{c^b}O_{c^c})_{0,1},$$
where $\langle O_{c^a}O_{c^b}O_{c^c} \rangle_{0,3-3}$ is the number of twisted cubics that intersect cycles Poincaré dual to $e^a$, $e^b$ and $e^c$.

**Proof.** In the $k - N = 1$ case, dimension of moduli space of multiple cover maps of $(2+1) \to (1+1)$ to nodal conics is given by $N - 6 + N - 6 - (N - 4) + 2 = N - 6$, hence the rank of $H^1$ is given by $N - 6 = (N - 5 - 3) = 2$. On the other hand, dimension of moduli space of $d = 2$ multiple cover maps of $\mathbb{P}^1 \to \mathbb{P}^1$ is $2$, the degree of the form of $\pi_* (c_{\text{top}}(H^1))$ equals to $2 - 2 = 0$, where $\pi$ is the projection map.
that projects out the fiber locally isomorphic to the moduli space of \( d = 2 \) multiple cover maps. This situation is exactly the same as the Calabi-Yau case. Therefore, we can use the well-known result by Aspinwall and Morrison, that says for \( n \)-point rational Gromov-Witten invariants for Calabi-Yau manifold, \( \tilde{\pi}_*(\mathcal{O}_{c_{top}}(H^1)) \) for degree \( d \) multiple cover map is given by,

\[
\tilde{\pi}_*(\mathcal{O}_{c_{top}}(H^1)) = \frac{1}{d^{1/n}}.
\]

With this formula, we add up all the combinatorial possibility of insertion of external operator \( \mathcal{O}_{c_v}, \mathcal{O}_{c_b} \) and \( \mathcal{O}_{c_\ell} \),

\[
\frac{1}{k}((\tilde{\pi}_*(\mathcal{O}_{c_{top}}(H^1)))\mathcal{O}_{c_v}\mathcal{O}_{c_b}\mathcal{O}_{c_\ell})_{0,1}(C_{c_N-\ell})_{0,1}
\]

\[
+ (\tilde{\pi}_*(\mathcal{O}_{c_{top}}(H^1)))\mathcal{O}_{c_v}\mathcal{O}_{c_\ell}C_{c_b}(c_\ell)_{0,1}(C_{c_N-\ell})_{0,1}
\]

\[
+ (\tilde{\pi}_*(\mathcal{O}_{c_{top}}(H^1)))\mathcal{O}_{c_b}\mathcal{O}_{c_\ell}(c_{\ell})_{0,1}(C_{c_N-\ell})_{0,1}
\]

\[
+ (\tilde{\pi}_*(\mathcal{O}_{c_{top}}(H^1)))\mathcal{O}_{c_\ell}C_{c_b}(c_b)_{0,1}(C_{c_N-\ell})_{0,1}
\]

\[
+ (\mathcal{O}_{c_b}C_{c_v}C_{c_\ell})_{0,1}(C_{c_N-\ell})_{0,1}
\]

\[
+ (\mathcal{O}_{c_\ell}C_{c_v}C_{c_b})_{0,1}(C_{c_N-\ell})_{0,1}
\]

\[
+ \frac{1}{2}(\mathcal{O}_{c_v}C_{c_\ell}C_{c_b})_{0,1}(C_{c_N-\ell})_{0,1} + \frac{1}{2}(\mathcal{O}_{c_b}C_{c_v}C_{c_\ell})_{0,1}(C_{c_N-\ell})_{0,1}
\]

\[
+ \frac{1}{4}(\mathcal{O}_{c_v}C_{c_b}C_{c_\ell})_{0,1}(C_{c_N-\ell})_{0,1}.
\]

The last expression is nothing but the formula we want. 

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