CONICS ON A GENERIC HYPERSURFACE

MASAIO JINZENJI, IKU NAKAMURA AND YASUKI SUZUKI

ABSTRACT. In this paper, we compute the contributions from double cover maps to genus 0 degree 2 Gromov-Witten invariants of general type projective hypersurfaces. Our results correspond to a generalization of Aspinwall-Morrison formula to general type hypersurfaces in some special cases.
MSC-class: 14H99, 14N35, 32G20

1. INTRODUCTION

In this paper, we discuss a generalization of the multiple cover formula for rational Gromov-Witten invariants of Calabi-Yau manifolds [AM], [M] to double cover maps of a line L on a degree k hypersurface $M_N^k$ in $\mathbb{P}^{N-1}$. Naïvely, for a given finite set of elements $\alpha_j \in H^*(M_N^k, \mathbb{Z})$, the rational Gromov-Witten invariant $\langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \cdots \mathcal{O}_{\alpha_s} \rangle_{0,d}$ of $M_N^k$ counts the number of degree $d$ (possibly singular and reducible) rational curves on $M_N^k$ that intersect real sub-manifolds of $M_N^k$ that are Poincaré-dual to $\alpha_j$.

Recently, the mirror computation of rational Gromov-Witten invariants of $M_N^k$ with negative first Chern class $(k-N > 0)$ was established in [CG], [IR], [J]. Using the method presented in these articles, we can compute $\langle \mathcal{O}_{e_1} \mathcal{O}_{e_2} \cdots \mathcal{O}_{e_m} \rangle_{0,d}$ where $e$ is the generator of $H^{1,1}(M_N^k, \mathbb{Z})$. Briefly, mirror computation of $M_N^k$ ($k > N$) in [J] goes as follows. We start from the following ODE:

$$\left( (\partial_x)^N - k \cdot \exp(x) \cdot (k\partial_x + k - 1)(k\partial_x + k - 2) \cdots (k\partial_x + 1) \right) w(x) = 0,$$

and construct the virtual Gauss-Manin system associated with (1):

$$\partial_x \tilde{\psi}_{N-2-m}(x) = \tilde{\psi}_{N-1-m}(x) + \sum_{d=1}^{\infty} \exp(dx) \cdot \tilde{L}_{N,k,d} \cdot \tilde{\psi}_{N-1-m-(N-k)d}(x),$$

where $m$ runs through all the integers and $\tilde{L}_{N,k,d}$ is non-zero only if $0 \leq m \leq N - 1 + (k-N)d$. From the compatibility of (1) and (2), we can derive the recursive formulas that determine all the $\tilde{L}_{N,k,d}$:

$$\sum_{n=0}^{k-1} \tilde{L}_{N,k,1} w_n = k \cdot \prod_{j=1}^{k-1} (jw + (k - j)),$$

$$\sum_{m=0}^{N-1+(k-N)d} \tilde{L}_{N,k,d} w_m = \sum_{l=2}^{d} (-1)^{l-1} \sum_{0 \leq \ell_0 < \cdots < \ell_l = d} \times$$

$$\sum_{j_0=0}^{\ell_0} \cdots \sum_{j_l=0}^{\ell_l} \sum_{n=1}^{l} \left( \frac{\ell_n - 1 + (d - \ell_n - 1)z}{d} \cdot \tilde{L}_{N,k,j_0 + (N-k)j_{n-1}} \right).$$
With these data, we can construct the formulas that represent rational three point Gromov-Witten invariant \( \langle O_0, O_{e,N-2-m}, O_{e,m-1-(k-N)d} \rangle_d \) in terms of \( \bar{L}^N_{m,k,d} \). These three point Gromov-Witten invariants are enough for reconstruction of all the rational Gromov-Witten invariants \( \langle O_{e,m_1} O_{e,m_2} \cdots O_{e,m_n} \rangle_0 \) [KM]. In particular, we obtain the following formula in the \( d = 2 \) case:

\[
\langle O_0, O_{e,N-2-m}, O_{e,m-1-(k-N)d} \rangle_2 = k \cdot \left( \bar{L}^N_{m,k,2} - \bar{L}^N_{1+2(k-N)} - 2 \bar{L}^N_{1+i(k-N)} \sum_{j=0}^{k-N} (\bar{L}^N_{m-j} - \bar{L}^N_{1+i(k-N)-j}) \right).
\]

According to the results of this procedure, rational three point Gromov-Witten invariants can be rational numbers with large denominator if \( k > N \), in contrast to the Calabi-Yau case where rational three point Gromov-Witten invariants are always integers.

One of the reasons of this rationality (non-integrality) comes from the contributions of multiple cover maps to Gromov-Witten invariants. In the Calabi-Yau case \( (N = k) \), for any divisor \( m \) of \( d \) there are some contributions from degree \( m \) multiple cover maps \( \phi \) of a rational curve \( \mathbb{P}^1 \) onto a degree \( \frac{d}{m} \) rational curve \( C \hookrightarrow M^k \). The contributions from the multiple cover maps are expressed in terms of the virtual fundamental class of Gromov-Witten invariants. Let \( C \) be a general degree \( d \) rational curve in \( M^k \). Its normal bundle \( N_{C/M^k} \) is decomposed into a direct sum of line bundles as follows:

\[
N_{C/M^k} \simeq O_C(-1) \oplus O_C(-1) \oplus O_C^{\oplus(k-5)}.
\]

Let \( \phi : \mathbb{P}^1 \to C \) be a holomorphic map of degree \( m \). Since the pull-back \( \phi^*(N_{C/M^k}) \) is given by

\[
\phi^*(N_{C/M^k}) \simeq O_{\mathbb{P}^1}(-m) \oplus O_{\mathbb{P}^1}(-m) \oplus O_{\mathbb{P}^1}^{\oplus k-5},
\]

we obtain \( h^1(\phi^*(N_{C/M^k})) = 2m - 2 \). On the other hand, let \( \bar{M}_{0,0}(M,d) \) be the moduli space of 0-pointed stable maps of degree \( d \) from genus 0 curve to \( M \). Then the moduli space of \( \phi \) is the fiber space \( \pi : \bar{M}_{0,0}(C,m) \to \bar{M}_{0,0}(M_k, \frac{d}{m}) \), whose fibre \( \bar{M}_{0,0}(C,m) \) over \( C \) (fixed) has complex dimension \( 2m - 2 \). Then the push-forward of the virtual fundamental class \( \pi_*c_{top}(H^1(\phi^*N_{C/M^k}))) \) can be computed only by intersection theory on the fiber \( \bar{M}_{0,0}(C,m) \), which turns out to be equal to \( \frac{1}{2} \). This depends on neither the structure of the base \( \bar{M}_{0,0}(M_k, \frac{d}{m}) \) nor the global structure of the fibration.

But when \( k < N \), the situation is more complicated than \( M^k \) because of negative first Chern class. Let us concentrate on the case of \( d = 2, m = 2 \) that we discuss in this paper. In this case, \( C \) is just a line \( L \) on the hypersurface \( M^k \). The moduli space \( \bar{M}_{0,0}(M^k,1) \) is a sub-manifold of \( \bar{M}_{0,0}(P^{N-1},1) \), while \( \bar{M}_{0,0}(P^{N-1},1) \) is the Grassmannian \( G(2,N) \), the moduli space of rank 2 quotients of \( V = C^N \). As will be shown later, for a generic line \( L, N_{L/M^k} \) is decomposed into

\[
N_{L/M^k} \simeq O_L(-1)^{\oplus k-N+2} \oplus O_L^{\oplus 2N-k-5}.
\]
By pulling back it by the degree 2 map $\phi : \mathbb{P}^1 \to L$, we obtain,
\[
\phi^* N_{L/M^k_N} \simeq O_{\mathbb{P}^1}((-2)^{g_k - N + 2} \oplus O_{\mathbb{P}^1}^{g_k - N - 1}.
\]
Therefore, $h^1(\phi^*(N_{L/M^k_N})) = k - N + 2$, which is strictly greater than two, the complex dimension of the fiber $\overline{M}_{0,0}(L, 2)$. Thus we need to know the global structure of the fibration $\pi$ in order to compute the multiple cover contribution to degree 2 rational Gromov-Witten invariants of $M^k_N$.

In order to estimate the contributions from double cover maps $\phi : \mathbb{P}^1 \to L$ to $\langle O_c \cup_c O_c \cup_c \rangle_{0,2}$, we first computed the number of conics, that intersect cycles Poicaré dual to $e^a, e^b$ and $e^c$, on $M^k_N$ (whose normal bundle are of the same type) by using the method in [K2]. Then we found the following formula by comparing these integers with the results obtained from (3):

\[
\langle O_c \cup_c O_c \cup_c \rangle_{0,2} = \text{(number of corresponding conics)} + \int_{G(2,N)} c_{top}(S_k^2 Q) \wedge \left[ c_1(Q)^{g_k - 1} \right]_{k-N} \wedge \sigma_{a-1} \wedge \sigma_{b-1} \wedge \sigma_{c-1},
\]
where $Q$ is the universal rank 2 quotient bundle of $G(2,N)$, $\sigma_a$ is a Schubert cycle defined by $\sum_{a=0}^{\infty} \sigma_a := \frac{1}{\pi_a^N}$ and $[\ast]_{k-N}$ is the operation of picking up degree $2(k - N)$ part of Chern classes.

On the other hand, we have the following formula which directly follows from the definition of the virtual fundamental class of $\overline{M}_{0,0}(M^k_N, 2)$:

\[
\langle O_c \cup_c O_c \cup_c \rangle_{0,2} = \text{(number of corresponding conics)} + 8 \int_{G(2,N)} c_{top}(S_k^2 Q) \wedge \left[ \pi_*(c_{top}(H^1(\phi^* N_{L/M^k_N}))) \right]_{k-N} \wedge \sigma_{a-1} \wedge \sigma_{b-1} \wedge \sigma_{c-1}.
\]
where $\pi : \overline{M}_{0,0}(L, 2) \to \overline{M}_{0,0}(M^k_N, 1)$ is the natural projection. Here, the factor 8 comes from the divisor axiom of Gromov-Witten invariants.

In this paper, we prove the following formula

\[
\pi_*(c_{top}(H^1(\phi^* N_{L/M^k_N}))) = \frac{1}{8} \left[ \frac{c_1(Q)^{g_k - 1}}{1 - c_1(Q)} \right]_{k-N}.
\]

By combining (5) with (6), we can derive the formula (4) immediately.

From (4), we see that $\langle O_c \cup_c O_c \cup_c \rangle_{0,2}$ of $M^k_N$ is a rational number with denominator at most $2^{k-N}$. Therefore rationality (non-integrality) of the Gromov-Witten invariant $(O_c \cup_c O_c \cup_c)_{0,2}$ is caused by the effect of multiple cover map in this case.

We note here that the total moduli space of double cover maps of lines is isomorphic to $\mathbb{P}(S^2 Q)$ over $G := \overline{M}_{0,0}(M^k_N, 1) \leftrightarrow G(2,N)$, which is an algebraic $Q$-stack $\mathbb{P}(S^2 Q)^{stack}$ (in the sense of Mumford). As a consequence, the union of all $H^1(\phi^* N_{C_{/M^k_N}})$ turns out to be a coherent sheaf on $\mathbb{P}(S^2 Q)^{stack}$ with fractional Chern class in (6), as was suggested in [BT]. See [V, Section 9].

We also did some numerical experiments on degree 3 Gromov-Witten invariants of $M^k_N$ by using the results of [ES]. For $k - N > 0$, there is a new contribution from multiple cover maps to nodal conics in $M^k_N$ that did not appear in the Calabi-Yau case. Therefore, multiple cover map contributions are far more complicated than Calabi-Yau, and we leave general analysis on this problem to future works.
This paper is organized as follows. In Section 1, we analyze characteristics of moduli space of lines in \( M^k_N \) and derive \( N_{L/M^k_N} \simeq O_L(-1)^{g(k-N+2)} \circ O_L^{g2N-k-5} \). In Section 2, we study the moduli space \( \mathcal{M}_{0,0}(\mathbb{P}^1, 2) \) from the point of view of stability and identify it with \( \mathbb{P}^2 \) and show that the moduli space \( \mathcal{M}_{0,0}(\mathbb{P}^1, 2) \) is isomorphic to \( \mathbb{P}(S^2 Q) \) over \( G \). In section 4, we describe \( H^1(\phi^* N_{L/M^k_N}) \) as an coherent sheaf over \( \mathbb{P}(S^2 Q)^{\text{stack}} \). In section 5, we derive the main theorem (6) of this paper by using Segre-Witten classes. In Section 6, we mention some generalization to degree 3 Gromov-Witten invariants.

2. Lines on a hypersurface

Let \( M \) be a generic hypersurface of degree \( k \) of the projective space \( \mathbb{P}^N = \mathbb{P}(V) \). We assume \( 2N - 5 \geq k \geq N - 2 \geq 2 \) throughout this note. In this note we count the number of rational curves of virtual degree two, namely rational curves which doubly cover lines on \( M \).

Let \( \mathcal{P} = \mathbb{P}(V) \) be the projective space parameterizing all one-dimensional quotients of \( V \), which is usually denoted by \( \mathcal{P}(V) \) in the standard notation in algebraic geometry. In this notation let \( W \) be a subspace of \( V \). Then \( \mathcal{P}(W) \) is naturally a linear subspace of \( \mathcal{P}(V) \) of dimension \( \text{dim } W - 1 \).

Let \( G(2, V) \) be the Grassmann variety of lines in \( \mathcal{P}(V) \), the scheme parameterizing all lines of \( \mathcal{P} = \mathbb{P}(V) \). This is also the universal scheme parameterizing all one-dimensional quotient linear spaces of \( V \). Let \( W \) be a two dimensional quotient linear space, \( \psi \in G(2, V) \), namely \( \psi : \mathcal{P}(W) \to \mathcal{P}(V) \) the natural immersion and \( i_\psi : V \to W \) the quotient homomorphism. The space \( W \) is denoted by \( W(\psi) \) when necessary.

There exists the universal bundle \( Q_{G(2,V)} \) over \( G(2, V) \) and a homomorphism \( i_{\text{univ}}^* : O_{G(2,V)} \otimes V \to Q_{G(2,V)} \) whose fiber \( i_{\psi}^* : V \to Q_{G(2,V)} \) is the quotient \( i_\psi : V \to W(\psi) \) of \( V \) corresponding to \( \psi \).

2.1. Existence of a line on \( M \). Let \( L = \mathcal{P}(W) \) be a line of \( \mathcal{P} \), equivalently \( W \in G(2, V) \). Then the condition \( L \subset M \) imposes at most \( k + 1 \) conditions on \( W \), while the number of moduli of lines of \( \mathcal{P} \) equals \( \text{dim } G(2, V) = 2N - 4 \). Hence we infer

**Lemma 2.2.** If \( 2N \geq k + 5 \), then there exists at least a line on \( M \).

See also [Katz,p.152]. Let \( G \) be the subscheme of \( G(2, V) \) parameterizing all lines of \( \mathcal{P}(V) \) lying on \( M \), \( Q = (Q_{G(2,V)})_G \) the restriction of \( Q_{G(2,V)} \) to \( G \). By Lemma 2.2, \( G \) is nonempty. Let \( i^* : O_G \otimes V \to Q \) be the restriction of \( i_{\text{univ}}^* \) to \( G \). Let \( P = \mathcal{P}(Q) \) and \( \pi : P \to G \) the natural projection. Then \( \pi \) is the universal line of \( M \) over \( G \), to be more exact, the universal family over \( G \) of lines lying on \( M \). In other words, the natural epimorphism \( i^* : O_G \otimes V \to Q \) induces a morphism \( i : P \to \mathcal{P}_G(V) := G \times \mathcal{P}(V) \), which is a closed immersion into \( \mathcal{P}_G(V) \), thus \( P \) is a subscheme of \( \mathcal{P}_G(V) \) such that \( \pi = (p_1)p \). Let \( L_\psi = \mathcal{P}(Q_\psi) \). Note that

\[
L_\psi = P_\psi := \pi^{-1}(\psi) \simeq \mathcal{P}(Q_\psi) \times \{\psi\} \times \mathcal{P}(V) \simeq \mathcal{P}(V).
\]
2.3. The normal bundle $N_{L/M}$. The argument of this section is standard and well known. Let $P = P(V)$, $L = P(W)$ and $i_W: V \to W \in G$. Let us recall the following exact sequence:

$$
0 \to O_P \to O_P(1) \otimes V^\vee \xrightarrow{D} T_P \to 0
$$

where the homomorphism $D$ is defined by

$$
D(a \otimes v^\vee) := aD_{(v^\vee)} \quad (a \in O_P(1))
$$

$$(D_{v^\vee}F)(u^\vee) := \frac{d}{dt}F(u^\vee + tv^\vee)|_{t=0}
$$

for a homogeneous polynomial $F \in S(V)$ and $u^\vee, v^\vee \in V^\vee$. We note $H^0(O_P(1)) \otimes V^\vee = V \otimes V^\vee = \text{End}(V, V)$ and that the image of $H^0(O_P)$ in $\text{End}(V, V)$ is $\text{Cid}_V$.

We also have the following exact sequences:

$$
0 \to T_L \to (T_P)_L \to N_{L/P} \to 0
$$

$$
0 \to O_L \to O_L(1) \otimes V^\vee \xrightarrow{D_L} (T_P)_L \to 0
$$

Lemma 2.4. Let $L = P(W)$. Then

$$
N_{L/P} \simeq O_L(1) \otimes (V^\vee/W^\vee), \quad H^0(N_{L/P}) \simeq W \otimes (V^\vee/W^\vee).
$$

Proof. The assertion is clear from the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccc}
0 & \to & O_L & \to & O_L(1) \otimes W^\vee & \xrightarrow{(D_L)_{W^\vee}} & T_L & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & O_L & \to & O_L(1) \otimes V^\vee & \xrightarrow{D_L} & (T_P)_L & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & O_L(1) \otimes (V^\vee/W^\vee) & \to & N_{L/P} & \to & 0
\end{array}
$$

The second assertion is clear from $H^0(L, O_L(1)) = W$. \hfill \Box

Since $T_L \simeq O_L(2)$, there follow exact sequences

$$
0 \to H^0(T_L) \to H^0((T_P)_L) \to H^0(N_{L/P}) \to 0
$$

$$
0 \to H^0(O_L) \to H^0(O_L(1)) \otimes V^\vee \xrightarrow{H^0(D_L)} H^0((T_P)_L) \to 0.
$$

We also note

$$
H^0(T_L) = \text{Lie Aut}^0(L) = \text{End}(W, W)/\text{center} = \text{End}(W, W)/\text{Cid}_W.
$$

Since $H^0(O_L(1)) = W$, we see

$$
H^0((T_P)_L) = W \otimes V^\vee/\text{Im} H^0(O_L) = \text{Hom}(V, W)/\text{Cid}_W^*.
$$

Hence we again see

$$
H^0((N_{L/P})) = (\text{Hom}(V, W)/\text{Cid}_W^*)/(\text{Hom}(W, W)/\text{Cid}_W)
$$

$$
= W \otimes (V^\vee/W^\vee) = \text{Hom}(V/W, W).
$$

For any line $L = P(W)$ of $P$ the following sequence is exact:

$$
0 \to N_{L/M} \to N_{L/P} \to (N_{M/P})_L(\simeq O_L(k)) \to 0.
$$
Hence so is the following sequence as well:

\[
\begin{align*}
0 \longrightarrow & H^0(N_{L/M}) \longrightarrow H^0(N_{L/P}) \xrightarrow{H^0(D_L)} H^0(O_L(k)) \\
\quad & \longrightarrow H^1(N_{L/M}) \longrightarrow 0.
\end{align*}
\]

Hence we have

**Lemma 2.5.** The following is exact:

\[
(8) \quad 0 \rightarrow H^0(N_{L/M}) \rightarrow W \otimes (V^\vee/W^\vee) \xrightarrow{H^0(D_L)} S^kW \rightarrow H^1(N_{L/M}) \rightarrow 0.
\]

**Corollary 2.6.** \(\dim G \geq 2N - k - 5\), equality holding if \(H^1(N_{L/M}) = 0\).

**Proof.** As is well-known, \(\dim G \geq h^0(N_{L/M}) - h^1(N_{L/M})\). Note \(\dim W \otimes (V^\vee/W^\vee) = 2(N - 2)\) and \(\dim S^kW = k + 1\). Hence the corollary follows from Lemma 2.5. \(\square\)

**Lemma 2.7.** For a generic line \(L\) on a generic hypersurface \(M\) of degree \(k\)

(i) \(N_{L/M} \cong O_L^{a_0} \oplus O_L(-1)^{b_b}\), where \(a = 2N - k - 5\) and \(b = k - N + 2\);

(ii) \(\text{Coker } H^0(D_L^-) \cong S^{k-1}W/(V^\vee/W^\vee)\) where \(D_L^- := D_L \oplus O_L(-1)\).

**Proof.** Let \(M\) be a generic hypersurface of degree \(k\) and \(L\) a generic line on \(M\). Without loss of generality we may assume that \(W^\vee\) is generated by \(e_1^y\) and \(e_2^y\), in other words, \(\psi : L \rightarrow \mathbb{P}\) is given by

\[
\psi : [s : t] \rightarrow [x_1, \ldots, x_N] = [s, t, 0, \ldots, 0].
\]

Then \(F\), the polynomial of degree \(k\) defining \(M\), is written as

\[
F = x_3F_3 + x_4F_4 + \cdots + x_NF_N
\]

for some polynomials \(F_j\) of degree \(k - 1\). Let \(f_j = \psi^*F_j = F_j(s, t, 0, \ldots, 0)\).

Now we consider the exact sequence

\[
\begin{align*}
0 \longrightarrow & H^0(N_{L/M}(-1)) \longrightarrow H^0(N_{L/P}(-1)) \xrightarrow{H^0(D_L^-)} H^0(O_L(k - 1)) \\
\quad & \longrightarrow H^1(N_{L/M}(-1)) \longrightarrow 0,
\end{align*}
\]

where we note \(H^0(N_{L/P}(-1)) = V^\vee/W^\vee\). Hence the following is exact:

\[
(9) \quad 0 \longrightarrow H^0(N_{L/M}(-1)) \longrightarrow V^\vee/W^\vee \xrightarrow{H^0(D_L^-)} S^{k-1}W \\
\quad \longrightarrow H^1(N_{L/M}(-1)) \longrightarrow 0
\]

where \(H^0(D_L^-)\) is given by \(H^0(D_L^-)(e_j^y) = f_j\) \((j = 3, 4, \ldots, N)\).

A generic choice of \(F\) implies a generic choice of degree \(k - 1\) polynomials \(f_j\) \((j = 3, 4, \ldots, N)\) in \(s\) and \(t\). By the assumptions

\[
\dim S^{k-1}W = k \geq N - 2 = \dim V^\vee/W^\vee,
\]

\[
\dim W \otimes V^\vee/W^\vee = 2(N - 2) \geq k + 1 = \dim S^kW,
\]

the generic choice of \(F\) implies that we can choose \(f_j \in S^{k-1}W\) \((j = 3, 4, \ldots, N)\) (and fix once for all) such that

(iii) \(f_j\) \((j = 3, 4, \ldots, N)\) are linearly independent,

(iv) \(Wf_3 + Wf_4 + \cdots + Wf_N = S^kW\).
Hence $H^0(D_L^*)$ is injective by (iii). It follows that $H^0(N_{L/M}(-1)) = 0$. Hence (ii) is clear. Next we consider $H^0(D_L)$. By (iv), we see

$$S^kW = W - H^0(D_L^*)(V/W) = H^0(D_L)(W \oplus V/W),$$

whence $H^0(D_L)$ is surjective. It follows that $H^1(N_{L/M}) = 0$. Hence $N_{L/M} \simeq O_L^{ba} \oplus O_L(-1)^{gb}$ for some $a$ and $b$. Since $a + b = \text{rank}(N_{L/M}) = N - 3$ and $-b = \text{deg}(N_{L/M}) = N - 2 - k$, we have (i).

\[\square\]

2.8. Lines on a quintic hypersurface in $\mathbb{P}^4$. See [Katz, Appendix A] for the subsequent examples. Let $N = 5$ and $k = 5$. Hence $M$ is a hypersurface of degree 5 in $\mathbb{P}^4$, a Calabi-Yau 3-fold. Let

$$F = x_4x_1^3 + x_5x_2^2 + x_3x_2^5 + x_4x_5^2 + x_5^3.$$

First we note that $M = \{F = 0\}$ is nonsingular. Let $L = \{x_3 = x_4 = x_5 = 0\} = \{[s, t, 0, 0, 0]\}$. In this case $f_3 = 0$, $f_4 = s^4$ and $f_5 = t^4$. In the exact sequence (1) we see $H^0(N_{L/M}(-1)) = \text{Ker} H^0(D_L^*) = Cc_3^{(5)}$ and $H^1(N_{L/M}(-1)) = \text{Coker} H^0(D_L^*)$ is 3-dimensional. Hence $N_{L/M} = O_L(1) \oplus O_L(-3)$.

We summarize the above. If $\dim \text{Ker} H^0(D_L^*) = 1$ and if $M$ is nonsingular, then $N_{L/M} = O_L(1) \oplus O_L(-3)$. Hence $H^0(N_{L/M}) = \text{Ker} H^0(D_L) = W \oplus \text{Ker} H^0(D_L^*)$ is 2-dimensional. Therefore we can choose $f_3 = 0$ and a linearly independent pair $f_4$ and $f_5 \in S^4W$ so that $Wf_4 + Wf_5$ is 4-dimensional. The choice $f_4 = s^4$ and $f_5 = t^4$ satisfies the conditions. This enables us to find a nonsingular hypersurface $M$ as above. However if we choose $f_3 = 0$, $f_4 = s^4$ and $f_5 = s^3t$, then $Wf_4 + Wf_5$ is 3-dimensional. Hence $M$ is singular.

Next in the same manner we find $L$ on a nonsingular hypersurface $M$ with $N_{L/M} = O_L \oplus O_L(-2)$ or $N_{L/M} = O_L(-1)^{gb}$. Let

$$F = x_3x_1^4 + x_4x_1^2x_2 + x_5x_2^3 + x_3^5 + x_4^5 + x_5^5.$$  

Then we have $f_3 = s^4$, $f_4 = s^3t$ and $f_5 = t^4$. Since $Wf_3 + Wf_4 + Wf_5$ is 5-dimensional, $H^0(N_{L/M}(-1)) = \text{Ker} H^0(D_L^*) = 0$, $H^0(N_{L/M}) = \text{Ker} H^0(D_L) = C(c_3^{(2)} - sc_1^{(2)})$. We see also that $\dim H^1(N_{L/M}) = \dim \text{Coker} H^0(D_L) = 1$ and $N_{L/M} = O_L \oplus O_L(-2)$. The hypersurface $M = \{F = 0\}$ is easily shown to be nonsingular.

If $F = x_3x_1^4 + x_4x_1^2x_2 + x_5x_2^3 + x_3^5 + x_4^5 + x_5^5$ and $M = \{F = 0\}$, then $N_{L/M} = O_L(-1)^{gb}$.

2.9. Lines on a generic hypersurface $M^8$ of $\mathbb{P}^6$. Let $N = 7$ and $k = 8$. In view of Lemma 2.2 there exists a line $L$ on any generic hypersurface $M$ of degree 8 in $\mathbb{P}(V) = \mathbb{P}^6$. In view of Lemma 2.7, $a = 1$, $b = 3$ and $N_{L/M} \simeq O_L \oplus O_L(-1)^{gb}$. For example let $L : x_j = 0$ $(j \geq 3)$ and we take

$$F = x_3x_1^4 + x_4x_1^2x_2 + x_5x_2^3 + x_3^5 + x_4^5 + x_5^5,$$

and let $M = M^8 : F = 0$. We see that $M$ is nonsingular near $L$ and has at most isolated singularities. However it is still unclear to us whether $M = M^8$ is
nonsingular everywhere. The space $H^0(N_{L/M})$ is spanned by $te^\epsilon - se^\epsilon$, hence an infinitesimal deformation $L_c$ of $L$ is given by

$$[s,t] \mapsto [s,t, \epsilon s, 0, 0, 0]$$

which yields $F|_{L_c} = \epsilon^8(s^8 + t^8) \equiv 0 \mod \epsilon^8$. Since $H^1(N_{L/M}) = 0$, this infinitesimal deformation is integrable and $G (:= \text{the moduli of lines of } P^6 \text{ contained in } M)$ is nonsingular and one dimensional at the point $[L]$.

We note that $M$ also contains 8 lines

$$L' := L_{c_8} : \epsilon s x_1 - x_2 = x_3 + \epsilon s x_4 = x_j = 0 \ (j \geq 5),$$

with $N_{L/M} = O_L(-1)^{\mathbb{R}^3} \oplus O_L(-6)$ where $\epsilon^8 = -1$.

3. Stability

**Definition 3.1.** Suppose that a reductive algebraic group $G$ acts on a vector space $V$. Let $v \in V$, $v \neq 0$.

1. the vector $v$ is said to be semi-stable if there exists a $G$-invariant homogeneous polynomial $F$ on $V$ such that $F(v) \neq 0$,
2. the vector $v$ is said to be stable if $p$ has a closed $G$-orbit in $X_{ss}$ and the stabilizer subgroup of $v$ in $G$ is finite.

Let $\pi : V \setminus \{0\} \to P(V^\vee)$ be the natural surjection. Then $v \in V$ is semi-stable (resp. stable) if and only if $\pi(v)$ is semi-stable (resp. stable).

3.2. Grassmann variety.** Let $V$ be an $N$-dimensional vector space, and $G(r,N)$ the Grassmann variety parameterizing all $r$-dimensional quotient spaces of $V$. Here is a natural way of understanding $G(r,N)$ via GIT-stability. Let $U$ be an $r$-dimensional vector space, $X = \text{Hom}(V,U)$ and $\pi : X \setminus \{0\} \to P(X^\vee)$ the natural map. Then $SL(U)$ acts on $X$ from the left by:

$$(g \cdot \phi^*)(v) = g \cdot (\phi^*(v)) \quad \text{for } \phi^* \in X, \ v \in V.$$  

We see that for $\phi^* \in X$ 

- $\phi^*$ is $SL(U)$-stable $\iff$ rank $\phi^* = r$,
- $\phi^*$ is $SL(U)$-semi-stable $\iff$ $\phi^*$ is $SL(U)$-stable.

In fact, if rank $\phi^* = r - 1$, then there is a one-parameter torus $T$ of $SL(U)$ such that the closure of the orbit $T \cdot \phi$ contains the zero vector as the following simple example $(r = 2)$ shows

$$\begin{aligned}
\lim_{t \to 0} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \\
\lim_{t \to 0} \begin{pmatrix} ta_{11} & ta_{12} & \cdots & ta_{1N} \\ 0 & 0 & \cdots & 0 \end{pmatrix}.
\end{aligned}$$

Let $X_s$ be the set of all (semi)stable points and $P_s$ the image of $X_s$ by $\pi$. It is, as we saw above, just the set of all $\phi \in X$ with rank $\phi^* = r$. Therefore the GIT-orbit space $P_s/SL(U)$ is the orbit space $P_s/SL(U)$ by the free action, the Grassmann variety $G(r,N)$. 

3.3. Moduli of double coverings of $\mathbb{P}^1$ (1). Let $W$ and $U$ be a pair of two dimensional vector spaces, $X = \text{Hom}(W, S^2 U)$, and $\pi : X \setminus \{0\} \to \mathbb{P}(X')$ the natural morphism. Note that $\text{SL}(U)$ acts on $S^2U$ from the left via the natural action: $\sigma(u_1 u_2) = \sigma(u_1) \sigma(u_2)$ for $\forall u_1, u_2 \in U$. Thus $\text{SL}(U)$ acts on $X$ from the left in the same manner in the subsection 3.2.

Lemma 3.4. Let $\phi^* \in X$.

(i) $\phi^*$ is unstable iff $\phi^*(w)$ has a double root for any $w \in W$.

(ii) $\phi^*$ is semistable iff $\phi^*(w)$ has no double roots for some nonzero $w \in W$.

(iii) $\phi^*$ is stable iff $\phi^*(W)$ is a base-point free linear subsystem of $S^2U$ on $\mathbb{P}(U)$.

Proof. We note that $\phi^*$ is unstable iff there is a suitable basis $s$ and $t$ of $U$ such that $\phi^*(w) = a(w)s^2$ for any $w \in W$ since a torus orbit $T \cdot \phi^*$ contains the zero vector. This proves (i). This also proves (ii). Next we prove (iii). If $\phi^*(W)$ has a base point, then it is clear that $\phi^*$ is not stable. If $\phi^*$ is semistable and it is not stable, then we choose a basis $s$, $t$ of $U$ and a basis $w_1, w_2$ of $W$ such that $\phi^*(w_1) = st$. If $\phi^*(w_2) = as^2 + bst$, then $\phi^*$ is not stable. This proves the lemma. □

Theorem 3.5. Let $X_{ss}$ be the Zariski open subset of $X$ consisting of all semistable points of $X$, $\pi(X_{ss})$ the image of $X_{ss}$ by $\pi$, and $Y := \pi(X_{ss})//\text{SL}(U)$. Then $Y \simeq \mathbb{P}^2$.

Proof. First consider a simplest case. We choose a basis $s$, $t$ of $U$. Let $w_1$ and $w_2$ be a basis of $W$, $T$ the subgroup of $\text{SL}(U)$ of diagonal matrices and $X^t = \{ \phi^* \in X; \phi^*(w_1) = 2st \}$. Let $Z' = \text{SL}(U) \cdot X'$.

We note that $Z'$ is an $\text{SL}(U)$-invariant subset of $X_{ss}$. We prove $\pi(Z')//\text{SL}(U) \simeq \mathbb{C}^2$. Let $\phi^*$ and $\psi^*$ be points of $X'$. Let $\phi^*(w_2) = As^2 + 2Bst + Ct^2$ and $\psi^*(w_2) = as^2 + bst + ct^2$. Then it is easy to check

$$g \cdot \phi^* = \psi^* \quad \text{for } \exists g \in \text{SL}(U) \iff g \cdot \phi^* = \psi^* \quad \text{for } \exists g \in T \iff \begin{array}{c} A = au^2, \quad B = b, \quad C = u^{-2}c \quad \text{for } \exists u \neq 0. \end{array}$$

Therefore each equivalence class of $\pi(Z')//\text{SL}(U)$ is represented by the pair $(AC, B)$, which proves $\pi(Z)//\text{SL}(U) \simeq \mathbb{C}^2$.

Now we prove the lemma. Let $\phi^* \in X_{ss}$, $\phi_j = \phi^*(w_j)$ and $\phi_0 = - (\phi_1 + \phi_2)$. Let

$$\begin{align*}
\phi_0 &= r_1 s^2 + 2r_2 st + r_3 t^2, \\
\phi_1 &= p_1 s^2 + 2p_2 st + p_3 t^2, \\
\phi_2 &= q_1 s^2 + 2q_2 st + q_3 t^2,
\end{align*}$$

and we define

$$D_1 = p_2^2 - p_1 p_3, \quad D_2 = q_2^2 - q_1 q_3,$$

$$D_0 = r_2^2 - r_1 r_3 = D_1 + D_2 + 2p_2 q_2 - (p_1 q_3 + p_3 q_1).$$

To show the lemma, we prove the more precise isomorphism

$$\pi(X_{ss})//\text{SL}(U) = \text{Proj } \mathbb{C}[D_0, D_1, D_2]$$

For this purpose we define $Y_j = \pi(\{ \phi^* \in X_{ss}; \phi_j \text{ has no double roots} \})//\text{SL}(U)$. It suffices to prove $Y_1 = \text{Spec } \mathbb{C}[D_0, D_1]$ by reducing it to the first simplest case.
Let $\phi^* \in Y_1$. Let $\alpha$ and $\beta$ be the roots of $\phi_1 = 0$. By the assumption $\phi_1$ has no double roots, hence $\alpha \neq \beta$. Let 

$$u = \frac{1}{\gamma} (s - \alpha t), \quad v = \frac{1}{\gamma} (s - \beta t), \quad g = \frac{1}{\gamma} \left( \frac{1}{\gamma} \begin{vmatrix} -\alpha \\ -\beta \end{vmatrix} \right)$$

where $\gamma = \sqrt{\alpha - \beta}$. Note that $g \in \text{SL}(U)$. Hence we see 

$$(\phi_1(s,t), \phi_2(s,t)) \equiv (p_1gamma^4uv, A_1u^2 + 2B_1uv + C_1v^2)$$

where 

$$A_1 = q_1^2\beta^2 + 2q_2\beta + q_3,$$

$$-B_1 = q_1\alpha\beta + q_2(\alpha + \beta) + q_3,$$

$$C_1 = q_1^2\alpha^2 + 2q_2\alpha + q_3.$$ 

Thus we see 

$$(\phi_1(s,t), \phi_2(s,t)) \equiv (2st, As^2 + 2Bst + Ct^2)$$

where 

$$A = \frac{2A_1}{p_1gamma^4}, \quad B = \frac{2B_1}{p_1gamma^4}, \quad C = \frac{2C_1}{p_1gamma^4}, \quad p_1gamma^4 = 4D_1,$$

$$AC = B^2 - \frac{D_2}{D_1}, \quad B = \frac{D_0 - D_1 - D_2}{2D_1}.$$ 

Therefore by the first half of the proof 

$$Y_1 \cong \text{Spec } \mathbb{C}[AC,B] = \text{Spec } \mathbb{C}[\frac{D_0}{D_1}, \frac{D_2}{D_1}].$$

This completes the proof of the lemma. 

**Corollary 3.6.** Let $Y^* = \pi(X_*)//\text{SL}(U)$. Then $Y \setminus Y^*$ is a conic of $Y$ defined by 

$$Y \setminus Y^* : D_0^2 + D_1^2 + D_2^2 - 2D_0D_1 - 2D_1D_2 - 2D_2D_0 = 0.$$ 

**Proof.** In view of Theorem 3.5, $Y_1 \cong \text{Spec } \mathbb{C}[AC,B]$. The complement of $Y_*$ in $Y_1$ is then the curve defined by $AC = 0$, which is easily identified with the above conic. 

**Corollary 3.7.** Let $X^0$ be the Zariski open subset of $X$ consisting of all semistable points $\phi^*$ of $X$ with rank $\phi^* = 2$, and let $Y^0 := \pi(X^0)//\text{SL}(U)$. Then $Y^0 \cong \pi(X^0)/\text{SL}(U) \cong Y$ or $X^0 \cong \mathbb{P}^2$. 

**Proof.** It suffices to compare $Y_1$ and $Y^0 \cap Y_1$. As in the proof of Theorem 3.5 we let 

$$X' = \{ \phi^* \in X : \phi^*(w_1) = 2st \}. \quad \text{Let } Z = \text{SL}(U) \cdot X' \text{ and } Z^0 = \text{SL}(U) \cdot (X' \cap X^0).$$

Then with the notation in Theorem 3.5, we recall $X' = \{ \phi^* \in X : \phi^*(w_1) = 2st, \phi^*(w_2) = As^2 + 2Bst + Ct^2 \}, \pi(Z)//\text{SL}(U) \cong \text{Spec } \mathbb{C}[AC,B]$ where 

$$X' \cap X^0 = \{ \phi^* \in X' : A \neq 0 \text{ or } C \neq 0 \}.$$ 

In the same manner as before we see $\pi(Z^0)//\text{SL}(U) \cong \text{Spec } \mathbb{C}[AC,B]$, whence $\pi(Z^0)//\text{SL}(U) = \pi(Z)//\text{SL}(U)$. This proves $Y^0 \cap Y_1 = Y_1$. This completes the proof of the corollary. 

\[ \Box \]
3.8. Moduli of double coverings of $\mathbb{P}(W)$ (2). There is an alternative way of understanding $\pi(X_{s})//\text{SL}(U) \cong \mathbb{P}^{2}$ by using the isomorphism $S^{2}\mathbb{P}^{1} \cong \mathbb{P}^{2}$. We use the following convention to denote a point of $\mathbb{P}(U) = U^{\vee} \setminus \{0\}/\Gamma_{m}$: $(u : v) = us^{\vee} + tv^{\vee} \in U^{\vee}$ where $s^{\vee}$ and $t^{\vee}$ are a basis dual to $s$ and $t$. In what follows we fix a basis $w_{1}$ and $w_{2}$ of $W$. Let $P := (a_{1} : a_{2})$ and $Q := (b_{1} : b_{2})$ be a pair of points of $\mathbb{P}(W) \cong \mathbb{P}^{1}$. If $P \neq Q$, there is a double covering $\phi : \mathbb{P}(U) \rightarrow \mathbb{P}(W)$ ramifying at $P$ and $Q$, unique up to isomorphism once we fix the base $w_{1}$ and $w_{2}$:

$$
\frac{b_{2}w_{1} - b_{1}w_{2}}{a_{2}w_{1} - a_{1}w_{2}} = \left(\frac{t}{s}\right)^{2}.
$$

Thus $\phi$ is given explicitly by

$$
\phi_{1} := \phi^{*}(w_{1}) = b_{1}s^{2} - a_{1}t^{2}, \quad \phi_{2} := \phi^{*}(w_{2}) = b_{2}s^{2} - a_{2}t^{2}, \quad \phi_{0} = -(\phi_{1} + \phi_{2})
$$

for which we have

$$
D_{1} = a_{1}b_{1}, \quad D_{2} = a_{2}b_{2}, \quad D_{0} = (a_{1} + a_{2})(b_{1} + b_{2}).
$$

The isomorphism $S^{2}\mathbb{P}^{1} \cong \mathbb{P}^{2}$ is given by $(P, Q) \mapsto (D_{0}, D_{1}, D_{2})$. This shows

**Corollary 3.9.** We have a natural isomorphism: $Y \cong \mathbb{P}(S^{2}W)$.

4. The virtual normal bundle of a double covering

4.1. The case $N = 7$ and $k = 8$ revisited. We revisit the example in the subsection 2.9. Let $N = 7$ and $k = 8$. Let $L : x_{j} = 0$ ($j \geq 3$) and we take

$$
F_{3} = 8x_{1}^{7}, F_{4} = 8x_{2}^{6}x_{2}, F_{5} = 8x_{1}x_{2}^{3}, F_{6} = 8x_{1}x_{2}^{2}, F_{7} = 8x_{2}^{2},
$$

$$
F = x_{3}F_{3} + x_{4}F_{4} + x_{5}F_{5} + x_{6}F_{6} + x_{7}F_{7} + x_{8}^{3} + x_{9}^{2} + x_{10}^{2} + x_{11}^{2} + x_{12}^{2}.
$$

and let $M = M_{6}^{\phi} : F = 0$. We often denote $L$ also by $b_{0}(W)$ with $L$ a two dimensional vector space for later convenience. Since $H^{0}(D_{-}^{L})$ is injective and $H^{0}(D_{L})$ is surjective, we have $N_{L/M} \cong O_{L} \oplus O_{L}(-1)$. Hence $H^{1}(N_{L/M}(-1)) \cong H^{1}(O_{L}(-2))$ is 3-dimensional. As we see easily, this follows also from the fact that $\text{Coker} H^{0}(D_{L})$ is freely generated by $(x_{1}x_{2}^{2}, x_{1}^{2}x_{2}^{2})$ and $(x_{1}x_{2}^{3})^{2}$.

Let $\phi^{\ast} = (\phi_{1}, \phi_{2}) \in X^{0}$. Then $\text{Coker} H^{0}(\phi^{\ast}D_{L})$ is generated by a single element $\phi_{2}^{\ast}\phi_{3}^{\ast} - \phi_{1}^{\ast}\phi_{4}^{\ast}$, while $\text{Coker} H^{0}(\phi^{\ast}D_{L})$ is generated by $S^{2}U \cdot \phi_{1}^{\ast}\phi_{2}^{\ast}$, $S^{2}U \cdot \phi_{1}^{\ast}\phi_{2}^{\ast}$ and $S^{2}U \cdot \phi_{1}^{\ast}\phi_{2}^{\ast}$. To be more precise, we see

$$
\text{Coker} H^{0}(\phi^{\ast}D_{L}) = \{\phi_{1}^{\ast}\phi_{2}^{\ast}, \phi_{1}^{\ast}\phi_{2}^{\ast}, \phi_{3}^{\ast}\phi_{4}^{\ast}\} \otimes S^{2}U / \{\phi_{1}^{\ast}, \phi_{2}^{\ast}\}.
$$

In fact, this is proved as follows: first we consider the case where $\phi_{1}$ and $\phi_{2}$ has no common zeroes. In this case $\phi^{\ast}$ gives rise to a double covering $\phi : \mathbb{P}(U) \rightarrow \mathbb{P}(W)$ (= $L$), which we denote by $L_{\phi}$ for brevity. By pulling back by $\phi^{\ast}$ the normal sequence

$$
0 \rightarrow N_{L/M} \rightarrow N_{L/P} \rightarrow O_{L}(k) \rightarrow 0 \quad (k = 8)
$$

for the line $L$ we infer an exact sequence

$$
0 \rightarrow \phi^{\ast}N_{L/M} \rightarrow \phi^{\ast}N_{L/P} \xrightarrow{\phi^{\ast}D_{L}} \phi^{\ast}O_{L}(k) \rightarrow 0,
$$

which yields an exact sequence

$$
0 \rightarrow H^{0}(\phi^{\ast}N_{L/M}) \rightarrow S^{2}U \otimes (V^{\vee} / W^{\vee}) \xrightarrow{H^{0}(\phi^{\ast}D_{L})} H^{0}(O_{L}(2k)) \rightarrow 0.
$$
Let $\eta = q_3 e_3^V + \cdots + q_r e_r^V \in \text{Ker} H^0(\phi^* D_L)$, $q_j \in S^2 U$. Then we have

$$\phi_1^2(q_3 \phi_1^3 + q_4 \phi_1^4 \phi_2 + q_5 \phi_1^2 \phi_2^3 + q_6 \phi_2^5) = -q_7 \phi_2^5.$$ 

Since $\phi_1$ and $\phi_2$ are mutually prime and $q_j$ is of degree two, we have $q_7 = 0$ and

$$\phi_1^2(q_3 \phi_1^3 + q_4 \phi_1^4 \phi_2 + q_5 \phi_1^2 \phi_2^3) = -q_6 \phi_2^5.$$ 

Hence $q_6 = 0$ and similarly we infer also $q_7 = 0$. Thus we have $q_3 \phi_1 + q_4 \phi_2 = 0$.

This proves that $\text{Ker} H^0(\phi^* D_L)$ is generated by $\phi_2 e_2^V - \phi_1 e_1^V$.

Next we prove that $\text{Coker} H^0(\phi^* D_L)$ is generated by $\phi^* \text{Coker} H^0(D_L^-)$ over $S^2 U$ in fact over $S^2 U/\phi^*(W)$. Without loss of generality we may assume that $\phi_1 = 2st$ and $\phi_2 = \lambda s^2 + 2\nu st + t^2$ for some $\lambda \neq 0$ and $\nu \in \mathbb{C}$. Let $\phi^* W = \{ \phi_1, \phi_2 \}$. Then one checks $U \cdot \phi^* W = S^2 U$, and hence $S^2 U \cdot \phi^* W = S^2 U$, $S^{2m-2} U \cdot \phi^* W = S^{2m} U$ for $m \geq 2$. It follows $S^2 U \cdot \phi^* (S^{m-1} W) = S^{2m} U$ for $m \geq 1$. In fact, by the induction on $m$

$$S^2 U \cdot \phi^* (S^{m-1} W) = S^2 U \cdot \phi^* (W) \cdot \phi^* (S^{m-1} W) = S^2 U \cdot \phi^* (S^{m-1} W)$$

$$= S^2 U \cdot \phi^* (S^{m-1} W)$$

$$= S^2 U \cdot \phi^* (S^{m-1} W)$$

Therefore $H^0(OL_2(2k)) = S^1 U = S^2 U \cdot \phi^*(S^W W)$. Hence

$$\text{Coker} H^0(\phi^* D_L) = S^1 U / \text{Im} H^0(\phi^* D_L^-)$$

$$= S^2 U \cdot \phi^*(S W) / S^2 U \cdot \phi^*(\text{Im} H^0(D_L^-))$$

$$= (S^2 U / \phi^*(W)) \cdot \phi^*(S^W / \text{Im} H^0(D_L^-)).$$

Because $\text{Coker} H^0(D_L^-) = S^W W / \text{Im} H^0(D_L^-)$ and $W \cdot S^W W \subset W \cdot \text{Im} H^0(D_L^-) = S^W W$ by the choice of $L$, this proves that $\text{Coker} H^0(\phi^* D_L)$ is generated by $\phi^* \text{Coker} H^0(D_L^-)$ over $S^2 U/\phi^*(W)$. It follows $\text{Coker} H^0(\phi^* D_L) = (\phi^* \text{Coker} H^0(D_L^-)) \oplus (S^2 U/\phi^*(W))$. 

Finally we consider the case where $\phi_1$ and $\phi_2$ has a common zero. In this case we may assume $\phi_1 = 2st$ and $\phi_2 = 2\nu st + t^2$. In this case $L_\phi$ is a chain of two rational curves $C^\prime_0$ and $C^\prime_\phi$ where $C_\phi$ is the proper transform of $\text{P}(U)$, where the double covering map from $L_\phi$ to $\text{P}(W)$ is the union of the isomorphisms $\phi'$ and $\phi''$, say, $\phi = \phi' \cup \phi''$. Let $\psi_1 = 2s$ and $\psi_2 = 2\nu s + t$. Then $\phi'$ is induced by the homomorphism $(\phi')^* \in \text{Hom}(W, U)$ such that $(\phi')^*(w_j) = \psi_j$. On the other hand let $U_\phi = C_\lambda + C_t$, $\psi'_1 = 2t$ and $\psi'_2 = \lambda + 2t$ where we note $\psi'_2$ is the linear part of $\phi_j$ in $t$ with $s = 1$. Then $C_\phi = \text{P}(U_\phi)$ and $\phi''$ is induced by the homomorphism $(\phi'')^* \in \text{Hom}(W, U''_\phi)$ such that $(\phi'')^*(w_j) = \psi''_j$. Furthermore the pull back by $\phi^*$ of the normal sequence for $L$

$$0 \to \phi^* N_{L/M} \to \phi^* N_{L/P} \xrightarrow{\phi^* D_L} \phi^* O_L(k) \to 0,$$
yields exact sequences with natural vertical homomorphisms:

\[
\begin{array}{cccccccc}
0 & \longrightarrow & \phi^*N_{L/M} & \longrightarrow & (\phi')^*N_{L/M} \oplus (\phi'')^*N_{L/M} & \longrightarrow & C & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \phi^*N_{L/P} & \longrightarrow & (\phi')^*N_{L/P} \oplus (\phi'')^*N_{L/P} & \longrightarrow & V^\vee /W^\vee & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \phi^*O_L(k) & \longrightarrow & O_{\mathcal{C}_\wedge}(k) \oplus O_{\mathcal{C}_\wedge}(k) & \longrightarrow & C & \longrightarrow & 0.
\end{array}
\]

This yields the following long exact sequences:

\[
\begin{array}{cccccccc}
0 & \longrightarrow & H^0((\phi')^*N_{L/M}) & \longrightarrow & U \otimes V^\vee /W^\vee & \overset{H^0((\phi')^*D_L)}{\longrightarrow} & S^kU \\
& \longrightarrow & H^1((\phi')^*N_{L/M}) & \longrightarrow & 0 \\
0 & \longrightarrow & H^0((\phi'')^*N_{L/M}) & \longrightarrow & U'' \otimes V^\vee /W^\vee & \overset{H^0((\phi'')^*D_L)}{\longrightarrow} & S^kU'' \\
& \longrightarrow & H^1((\phi'')^*N_{L/M}) & \longrightarrow & 0
\end{array}
\]

whence \( H^1((\phi')^*N_{L/M}) = H^1((\phi'')^*N_{L/M}) = 0 \), and both \( H^0((\phi')^*N_{L/M}) \) and \( H^0((\phi'')^*N_{L/M}) \) are one-dimensional. Let \( U^t \) be the subspace of \( U \) consisting of elements vanishing at \( C'_\phi \cap C''_\phi \), namely the subspace spanned by \( t \). Then the restriction of \( H^0((\phi')^*D_L) \) to \( U^t \otimes V^\vee /W^\vee \) equals \( t \cdot H^0((\phi')^*D_L^-) \). Hence

\[
\text{Coker } H^0(\phi^*D_L) \simeq t \cdot S^2U/t \cdot \text{Im } H^0((\phi')^*D_L^-) \oplus \text{Coker } H^0((\phi'')^*D_L^-) \simeq S^2U/t \cdot \text{Im } H^0((\phi')^*D_L^-) \simeq \text{Coker } H^0((\phi')^*D_L^-).
\]

One could understand the above isomorphism as

\[
\text{Coker } H^0(\phi^*D_L) = \text{Coker } (\phi^*H^0(D_L^-)) \oplus (S^2U/\phi^*W).
\]

Thus \( H^0(\phi^*N_{L/M}) \) is one-dimensional, while \( H^1(\phi^*N_{L/M}) \) is 3-dimensional. This is immediately generalized into the following

**Lemma 4.2.** For any \( \phi^* \in X^0 \) we have

\[
\begin{align*}
\text{Ker } H^0(\phi^*D_L) &= \phi^* \text{Ker } H^0(D_L), \\
\text{Coker } H^0(\phi^*D_L) &= (\phi^* \text{Coker } H^0(D_L^-)) \oplus (S^2U/\phi^*W).
\end{align*}
\]

**Lemma 4.3.** We define a line bundle \( L_0 \) (resp. \( L_1 \)) on \( Y \) (\( \cong \mathbb{P}(S^2W) \)) by the assignment:

\[
X^0 \ni \phi^* \mapsto \phi^* \text{Ker } H^0(D_L) \text{ (resp. } \phi^* \text{Coker } H^0(D_L^-)).
\]

Then \( L_k \cong O_{\mathbb{P}(S^2W)} \).

**Proof.** We know that \( \phi^* \text{Ker } H^0(D_L) \) is generated by \( \phi_2 c'_\wedge - \phi_1 c'_\vee \). By the SL(2)-variable change of \( s \) and \( t \), \( \phi_j \) is transformed into a new quadratic polynomial, which is however the same as the first \( \phi_j \). This shows the generator is unchanged, whence \( L_0 \cong O_{\mathbb{P}(S^2W)} \). The proof for \( L_1 \) is the same. \( \square \)
Lemma 4.4. We define a coherent sheaf \( L \) on the stack \( Y (\simeq \mathbf{P}(S^2 W)) \) (See Remark below) by the assignment:

\[
X^0 \ni \phi^* \mapsto S^2 U/\phi^* W.
\]

Then \( L^2 \simeq O_{\mathbf{P}(S^2 W)}(-1) \).

Proof. The GIT-quotient \( Y^0 \) is covered with the images of \( X'_j \):

\[
X'_1 = \{ (\phi_1, \phi_2) \in X^0; \phi_1 = 2st, \ \phi_2 = \lambda s^2 + 2\nu st + t^2, \ \lambda, \nu \in \mathbb{C} \},
\]

\[
X'_2 = \{ (\phi_1, \phi_2) \in X^0; \phi_1 = ps^2 + 2qst + t^2, \ \phi_2 = 2st, \ p, q \in \mathbb{C} \}.
\]

It is clear that the natural image of \( X'_j \) in \( Y \) is \( Y_j \). The map \( \phi \) given by \( \phi^* = (\phi_1, \phi_2) \in Y \) has natural \( \mathbb{Z}_2 \) involution generated by,

\[
r : (\sqrt{\lambda}s + t, \sqrt{\lambda}s - t) \mapsto (\sqrt{\lambda}s + t, -(\sqrt{\lambda}s - t)).
\]

Since

\[
2st = \frac{1}{2\sqrt{\lambda}}((\sqrt{\lambda}s + t)^2 - (\sqrt{\lambda}s - t)^2),
\]

\[
\lambda s^2 + 2s \nu t + t^2 = \frac{\nu}{2\sqrt{\lambda}}((\sqrt{\lambda}s + t)^2 - (\sqrt{\lambda}s - t)^2) + \frac{1}{2}((\sqrt{\lambda}s + t)^2 + (\sqrt{\lambda}s - t)^2),
\]

it is clear that,

\[
r^*(\phi_1) = \phi_1, \ r^*(\phi_2) = \phi_2, \ r^*(\lambda s^2 - t^2) = -(\lambda s^2 - t^2).
\]

Therefore, we can decompose \( S^2 U \) into \( \langle \lambda s^2 - t^2 \rangle_{\mathbb{C}} \oplus \langle \phi_1, \phi_2 \rangle_{\mathbb{C}} \) with respect to eigenvalue of \( r^* \) and take \( \lambda s^2 - t^2 \) as canonical generator of \( S^2 U/\phi^* W \). Similarly \( S^2 U/\phi^* W \) is generated by \( ps^2 - t^2 \) on \( Y_2 \). The problem is therefore to write \( \lambda s^2 - t^2 \) as an \( \Gamma(O_Y) \)-multiple of \( pu^2 - v^2 \) when we write \( \phi_2 = 2uv \) by a variable change in \( GL(2) \). The following variable change \( (s, t) \mapsto (u, v) \) is in \( GL(2) \):

\[
s = \frac{\sqrt{2\alpha}}{(\beta - \alpha)^2} (2u - \frac{(\beta - \alpha)^2}{2\alpha} v), \quad t = \frac{\sqrt{2\alpha}}{(\beta - \alpha)^2} (2\beta u - \frac{(\beta - \alpha)^2}{2} v),
\]

where \( \alpha, \beta \) are roots of the equation \( \lambda s^2 + 2s \nu t + t^2 = 0 \). Under this coordinate change, \( \phi_1 \) and \( \phi_2 \) are rewritten as follows:

\[
\phi_1 = \frac{\lambda}{(\nu^2 - \lambda)^2} u^2 + 2\frac{\nu}{\nu^2 - \lambda} uv + v^2 = pu^2 + 2quv + v^2, \ \phi_2 = 2uv.
\]

Then we have

\[
pu^2 - v^2 = -\frac{2}{\beta - \alpha} (\lambda s^2 - t^2) = -\frac{1}{\sqrt{\nu^2 - \lambda}} (\lambda s^2 - t^2) = -\sqrt{\frac{D_1}{D_2}} (\lambda s^2 - t^2).
\]

Similarly by computing the effect on \( S^2 U/\phi^* W \) by the variable change from \( X'_1 \) into \( X'_0 \), we see that \( L^2 \) is isomorphic to \( O_{\mathbf{P}(S^2 W)}(-1) \). This completes the proof. \( \square \)

Remark 4.5. We remark that the space \( X \) must be regard as a \( \mathbb{Q} \)-stack \( Y^{stack} \) as follows: First we define \( \phi_0 = -\phi_1 - \phi_2 \). For each atlas \( X^i \) we define an atlas \( Y^{stack} \).
\( (\alpha = 0, 1, 2) \) by
\[
Y_{0}^{\text{stack}} = \{(\phi_0, \phi_1, \phi_2, \pm \psi_0) \in X^0 \times S^2 U ; \phi_0 = 2st, \phi_1 = as^2 + 2bst + t^2, \\
\psi_0 = as^2 - t^2 a, b \in C\},
\]
\[
Y_{1}^{\text{stack}} = \{(\phi_0, \phi_1, \phi_2, \pm \psi_1) \in X^0 \times S^2 U ; \phi_1 = 2st, \phi_2 = \lambda s^2 + 2\nu st + t^2, \\
\psi_1 = \lambda s^2 - t^2 \lambda, \nu \in C\},
\]
\[
Y_{2}^{\text{stack}} = \{(\phi_0, \phi_1, \phi_2, \pm \psi_2) \in X^0 \times S^2 U ; \phi_1 = ps^2 + 2qst + t^2, \phi_2 = 2st, \\
\psi_2 = ps^2 - t^2 p, q \in C\}.
\]

Since \( L^2 \cong O_{P(S^2W)}(-1) \) we have \( c_1(L) = -\frac{1}{2}c_1(O_{P(S^2W)}(1)) \) in the Chow ring \( A(Y_{\text{stack}}^0) = A(X)_{Q} = A(P(S^2W))_{Q} \).

5. Proof of the main theorem

**Theorem 5.1.**

\[
\pi_*(c_{\text{top}}(H^1)) = \frac{1}{8} \left[ \frac{c(S^{k-1}Q)}{1 - \frac{1}{2}c_1(Q)} \right]_{k-N},
\]

where \( \pi \) is the natural projection from \( \tilde{M}_{0,0}(L, 2) \) to \( G \) and \([*]_{k-N} \) is the operation of picking up the degree \( 2(k-N) \) part of Chern classes.

**Proof.** From now on we denote the coherent sheaf \( L \) in Lemma 4.4 by \( O_{P}(-\frac{1}{2}) \). In view of the results from the previous section, what remains is to evaluate the top chern class of \((S^{k-1}Q)/(V^\vee \otimes O_{G})/Q^\vee) \otimes O_{P}(-\frac{1}{2}) \) on \( P(S^2Q) \). Since double cover maps parametrized by \( P(S^2Q) \) have natural \( \mathbb{Z}_2 \) involution \( r \) given in the previous section, we have to multiply the result of integration on \( P(S^2Q) \) by the factor \( \frac{1}{2} \) [BT], [FP]. With this set-up, let \( \pi' : P(S^2Q) \to G \) be the natural projection. Then what we have to compute is \( \pi_*(c_{\text{top}}(H^1)) = \frac{1}{2} \pi'_*(c_{\text{top}}(H^1)) = \frac{1}{2} \pi'_*(c_{\text{top}}((S^{k-1}Q)/(V^\vee \otimes O_{G})/Q^\vee)) \otimes O_{P}(-\frac{1}{2})) \). Let \( z \) be \( c_1(O_{P}(1)) \). Then we obtain,

\[
\frac{1}{2} \pi'_*(c_{\text{top}}((S^{k-1}Q)/(V^\vee \otimes O_{G})/Q^\vee)) \otimes O_{P}(-\frac{1}{2}))
\]
\[
= \frac{1}{2} \sum_{j=0}^{k-N+2} c_{k-N+2-j}(S^{k-1}Q \oplus Q^\vee) \cdot \pi_*(z^j) \cdot (-\frac{1}{2})^j
\]
\[
= \frac{1}{8} \sum_{j=0}^{k-N} c_{k-N-j}(S^{k-1}Q \oplus Q^\vee) \cdot s_j(S^2Q) \cdot (-\frac{1}{2})^j
\]
\[
= \frac{1}{8} \left[ \frac{c(S^{k-1}Q) \cdot c(Q^\vee)}{1 - \frac{1}{2}c_1(S^2Q) + \frac{1}{4}c_2(S^2Q) - \frac{1}{8}c_3(S^2Q)} \right]_{k-N},
\]

where \( s_j(S^2Q) \) is the \( j \)-th Segre class of \( S^2Q \). But if we decompose \( c(Q) \) into \( (1 + \alpha)(1 + \beta) \), we can easily see,

\[
\frac{c(Q^\vee)}{1 - \frac{1}{2}c_1(S^2Q) + \frac{1}{4}c_2(S^2Q) - \frac{1}{8}c_3(S^2Q)} = \frac{(1-\alpha)(1-\beta)}{(1-\alpha)(1-\frac{1}{2}(\alpha+\beta))(1-\beta)} \cdot \frac{1}{1 - \frac{1}{2}c_1(Q)}.
\]
Finally, by combining the above theorem with the divisor axiom of Gromov-Witten invariants, we can prove the decomposition formula of degree 2 rational Gromov-Witten invariants of $M^{k}_{N}$ found from numerical experiments.

Corollary 5.2. \[ \langle \mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e} \rangle_{0, 2} = \langle \mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e} \rangle_{0, 2-2} + 8(\pi_{*}(c_{top}(H^{1}))\mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e})_{0, 1}, \]
where $\langle \mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e} \rangle_{0, 2-2}$ is the number of conics that intersect cycles Poincaré dual to $e^a$, $e^b$ and $e^c$. We also denote by $\langle \pi_{*}(c_{top}(H^{1}))\mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e} \rangle_{0, 1}$ the integral:
\[ \int_{G(2, 3)} c_{top}(S^{k} Q) \wedge \pi_{*}(c_{top}(H^{1})) \wedge \sigma_{a-1} \wedge \sigma_{b-1} \wedge \sigma_{c-1}. \]

6. GENERALIZATION TO TWISTED CUBICS

In this section, we present a decomposition formula of degree 3 rational Gromov-Witten invariants found from numerical experiments using the results of [ES].

Conjecture 6.1. If $k - N = 1$, we have the following equality:
\[ \pi_{*}(c_{top}(H^{1})) = \frac{1}{27} \left( \frac{1}{24} (27k^{2} - 55k + 26)k(k - 1) + \frac{2}{9}c_{1}(Q)^{2} + \left( \frac{7}{6} (k + 1)k(k - 1) + \frac{1}{9} \right)c_{2}(Q) \right). \]
where $\pi : \overline{\mathcal{M}}_{0, 0}(L, 3) \to \overline{\mathcal{M}}_{0, 0}(M^{k}_{N}, 1)$ is the natural projection.

In the $k-N > 1$ case, we have not found the explicit formula, because in the $d = 3$ case, we have another contribution from multiple cover maps of type $(2+1) \to (1+1)$. Here multiple cover map of type $(2+1) \to (1+1)$ is the map from nodal curve $\mathbb{P}^{1} \cup \mathbb{P}^{1}$ to nodal conic $L_{1} \cup L_{2} \subset M^{k}_{N}$, that maps the first (resp. the second) $\mathbb{P}^{1}$ to $L_{1}$ (resp. $L_{2}$) by two to one (resp. one to one). In the $k-N = 1$ case, we have also determined the contributions from multiple cover maps of $(2+1) \to (1+1)$ to nodal conics.

Corollary 6.2. If $k - N = 1$, $\langle \mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e} \rangle_{0, 3}$ is decomposed into the following contributions:
\[ \langle \mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e} \rangle_{0, 3} = \langle \mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e} \rangle_{0, 3-3} + \frac{1}{k} \langle \mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e} \rangle_{0, 1} \langle \mathcal{O}_{e}, N-e \rangle_{0, 1} \]
\[ + \frac{3}{2} \langle \mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e} \rangle_{0, 1} \langle \mathcal{O}_{e}, N-e-i \rangle_{0, 1} + \frac{3}{2} \langle \mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e} \rangle_{0, 1} \langle \mathcal{O}_{e}, N-e-i \rangle_{0, 1} \]
\[ + \frac{3}{2} \langle \mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e} \rangle_{0, 1} \mathcal{O}_{e}, N-e-i \rangle_{0, 1} \]
\[ + 27(\pi_{*}(c_{top}(H^{1}))\mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e})_{0, 1}, \]
where $\langle \mathcal{O}_{e}, \mathcal{O}_{e}, \mathcal{O}_{e} \rangle_{0, 3-3}$ is the number of twisted cubics that intersect cycles Poincaré dual to $e^a$, $e^b$ and $e^c$.

Proof. In the $k-N = 1$ case, dimension of moduli space of multiple cover maps of $(2+1) \to (1+1)$ to nodal conics is given by $N-6+(N-6)-(N-4)+2=N-6$, hence the rank of $H^{1}$ is given by $N-6-(N-5-3)=2$. On the other hand, dimension of moduli space of $d=2$ multiple cover maps of $\mathbb{P}^{1} \to \mathbb{P}^{1}$ is 2, the degree of the form of $\pi_{*}(c_{top}(H^{1}))$ equals to $2-2=0$, where $\pi$ is the projection map.
that projects out the fiber locally isomorphic to the moduli space of \(d = 2\) multiple cover maps. This situation is exactly the same as the Calabi-Yau case. Therefore, we can use the well-known result by Aspinwall and Morrison, that says for \(n\)-point rational Gromov-Witten invariants for Calabi-Yau manifold, \(\tilde{\pi}_*(c_{top}(H^1))\) for degree \(d\) multiple cover map is given by,

\[
\tilde{\pi}_*(c_{top}(H^1)) = \frac{1}{d^{n-1}}.
\]

With this formula, we add up all the combinatorial possibility of insertion of external operator \(\mathcal{O}_{e^s}, \mathcal{O}_{e^h}\) and \(\mathcal{O}_{e^v}\),

\[
\begin{align*}
&\frac{1}{k} \left( \langle \tilde{\pi}_*(c_{top}(H^1)) \mathcal{O}_{e^s} \mathcal{O}_{e^h} \mathcal{O}_{e^v} \rangle_{0,1} \langle \mathcal{O}_{e^{N-s}} \rangle_{0,1} \\
&+ \langle \tilde{\pi}_*(c_{top}(H^1)) \mathcal{O}_{e^h} \mathcal{O}_{e^v} \mathcal{O}_{e^{s+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-s-t}} \mathcal{O}_{e^v} \rangle_{0,1} \\
&+ \langle \tilde{\pi}_*(c_{top}(H^1)) \mathcal{O}_{e^v} \mathcal{O}_{e^s} \mathcal{O}_{e^{h+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-s-t}} \mathcal{O}_{e^h} \rangle_{0,1} \\
&+ \langle \mathcal{O}_{e^s} \mathcal{O}_{e^h} \mathcal{O}_{e^{s+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-s-t}} \tilde{\pi}_*(c_{top}(H^1)) \mathcal{O}_{e^v} \rangle_{0,1} \\
&+ \langle \mathcal{O}_{e^h} \mathcal{O}_{e^v} \mathcal{O}_{e^{s+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-s-t}} \tilde{\pi}_*(c_{top}(H^1)) \mathcal{O}_{e^h} \rangle_{0,1} \\
&+ \langle \mathcal{O}_{e^v} \mathcal{O}_{e^s} \mathcal{O}_{e^{h+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-s-t}} \tilde{\pi}_*(c_{top}(H^1)) \mathcal{O}_{e^v} \rangle_{0,1} \\
&+ \langle \mathcal{O}_{e^s} \mathcal{O}_{e^h} \mathcal{O}_{e^{v+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-s-t}} \mathcal{O}_{e^s} \rangle_{0,1} \right) \\
= \frac{1}{k} \left( 2 \langle \mathcal{O}_{e^s} \mathcal{O}_{e^h} \mathcal{O}_{e^{v+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-s-t}} \mathcal{O}_{e^v} \rangle_{0,1} + \langle \mathcal{O}_{e^h} \mathcal{O}_{e^v} \mathcal{O}_{e^{s+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-s-t}} \mathcal{O}_{e^v} \rangle_{0,1} + \langle \mathcal{O}_{e^v} \mathcal{O}_{e^s} \mathcal{O}_{e^{h+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-s-t}} \mathcal{O}_{e^h} \rangle_{0,1} + \frac{1}{2} \langle \mathcal{O}_{e^s} \mathcal{O}_{e^h} \mathcal{O}_{e^{s+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-s-t}} \mathcal{O}_{e^v} \rangle_{0,1} \right) \\
&+ \frac{1}{2} \langle \mathcal{O}_{e^h} \mathcal{O}_{e^v} \mathcal{O}_{e^{s+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-s-t}} \mathcal{O}_{e^h} \rangle_{0,1} + \frac{1}{2} \langle \mathcal{O}_{e^v} \mathcal{O}_{e^s} \mathcal{O}_{e^{h+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-s-t}} \mathcal{O}_{e^v} \rangle_{0,1} + \frac{1}{4} \langle \mathcal{O}_{e^s} \mathcal{O}_{e^h} \mathcal{O}_{e^{v+2}} \rangle_{0,1} \langle \mathcal{O}_{e^{N-s-t}} \mathcal{O}_{e^s} \rangle_{0,1}. \end{align*}
\]

The last expression is nothing but the formula we want. \(\square\)

Masao Jinzenji\(^\dagger\), Iku Nakamura\(^\star\), Yasuki Suzuki

*Division of Mathematics, Graduate School of Science, Hokkaido University*
*Kita-ku, Sapporo, 060-0810, Japan*
e-mail address: \(\dagger\) jin@math.sci.hokudai.ac.jp, \(\star\) nakamura@math.sci.hokudai.ac.jp

**References**


[Iri] H. Iritani, Quantum D-modules and generalized mirror transformations, math.DG/0411111.


