Extending the Picard-Fuchs system of local mirror symmetry

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Abstract

We propose an extended set of differential operators for local mirror symmetry. If X is Calabi-Yau such that $\dim H_4(X, \mathbb{Z}) = 0$, then we show that our operators fully describe mirror symmetry. In the process, a conjecture for intersection theory for such X is uncovered. We also find new operators on several examples of type $X = K_S$ through similar techniques. In addition, open string PF systems are considered.

1 Introduction.

For some time now, mirror symmetry has been successfully used to make enumerative predictions on certain Calabi-Yau manifolds. While mirror symmetry for compact Calabi-Yau's has been extensively studied, local mirror symmetry is relatively new, and a complete formulation does not yet exist.

The first unified treatment of local mirror symmetry was written down in [5], in the case that the space looks like $X = K_S$, where S is a Fano surface and K_S is its canonical bundle. Very recently [17], the work of [5] was formulated more mathematically. With the ideas of [17], we are able to determine all information relevant for mirror symmetry directly from the Picard-Fuchs equations of the mirror. These techniques are limited to the case that X satisfies $b_2(X) = b_4(X)$.

The aim of this paper is to further the program of local mirror symmetry. We propose a new set of differential operators, whose solutions contain the usual local mirror symmetry solutions as a subset. In the event that the space in question contains no 4 cycle, the new operators completely solve the problem for an arbitrary number of Kähler parameters. For the more traditional

mirror symmetry constructions of [5], our methods still complete missing data; however, a general formulation here is lacking.

The key point in the construction of the extended Picard-Fuchs system is the determination of triple intersection numbers of Kähler classes for open Calabi-Yau manifolds. Up to now, a natural definition of triple intersection numbers of Kähler classes on open Calabi-Yau manifolds is not known, but in this paper, we search for triple intersection numbers that are "natural" from the point of view of mirror symmetry in the sense of the following conjecture:

Conjecture 1 *Consider the A-model on a Calabi-Yau threefold X. Let u_i be the logarithm of the B-model complex deformation parameter z_i obtained from the toric construction of the mirror Calabi-Yau manifold $\hat{X}$. Then the B-model Yukawa coupling $C_{u_i u_j u_k}$ of $\hat{X}$ obtained from the A-model Yukawa coupling of X is a rational function in $z_i = \exp(u_i)$. Moreover, its denominator includes the divisor of the defining equation of the discriminant locus of $\hat{X}$.*

Since the triple intersection number is just the constant term of the B-model Yukawa coupling $C_{u^i u^j u^k}$, the above conjecture imposes constraints on these triple intersection numbers. In this paper, we regard these triple intersection numbers as intersection numbers of some minimal compactification $\overline{X}$ of X. With these intersection numbers in hand, we can construct a quantum cohomology ring of $\overline{X}$ and an associated Gauss-Manin system. In the examples treated in this paper, this quantum cohomology ring satisfies Poincare duality as a compact 3-fold, even if X is an open Calabi-Yau 3-fold. With this Gauss-Manin system, we can write down differential equations for $\psi_0(t^i)$, the function associated to the identity element of the quantum cohomology ring. Then we can rewrite these differential equations by using the mirror map $t_i = t_i(z_*)$. Our assertion in this paper is that the differential equations so obtained are the extension of the Picard-Fuchs operators obtained from the standard toric construction of $\hat{X}$. Moreover, the extended Picard-Fuchs system has all the properties that the Picard-Fuchs system associated to a compact Calabi-Yau 3-fold should have: a unique triple log solution, Yukawa couplings, etc.

One problem in the construction is that in some cases, the constraints obtained from the above conjecture are not strong enough to determine all the triple intersection numbers. In other words, we have some real moduli parameters in the triple intersection numbers. Yet, we can still construct the extended Picard-Fuchs system for each value of the moduli parameters, and these systems have all the properties desired for a Picard-Fuchs system associated to a compact Calabi-Yau 3-fold. In the case that $b_4(X) = b_6(X) = 0$, we find unique triple intersection numbers compatible with the above conjecture by considering the change of the prepotential under flops. Hence, our construction of an extended Picard-Fuchs system has no ambiguity in this situation.

Here is the organization of the paper. Section 2 spells out the generalities of the Gauss-Manin system and intersection theory for open Calabi-Yau manifolds. In Section 3, we thoroughly consider $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$, giving a PF operator for mirror symmetry and a geometric view of the meaning of the operator. Section 4 is the generalization, which spells out a conjecture on how to deal with all X such that $\dim H_4(X, \mathbb{Z}) = 0$. This is subsequently applied to several cases and shown to produce the expected results. Section 5 explores the application of our techniques to open string theory on $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$, while Sections 6 and 7 work through examples of type K_S. Some of the results for our more unwieldy examples are collected in the appendices.

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2 The Main Strategy: Overview of the Gauss-Manin System.

Suppose that we have obtained "natural" classical triple intersection numbers $\int_{\overline{X}} k_a \wedge k_b \wedge k_c$ for an open Calabi-Yau 3-fold X under the assumption of Conjecture 1, and that we know the instanton part of the prepotential for X. Let us denote this instanton part by $\mathcal{F}^{inst}(t_*)$, where t_a is the Kähler deformation parameter associated to the Kähler form k_a $(a = 1, \cdots, h^{1,1}(X))$. With this data, we can construct an A-model Yukawa coupling for X:

$$Y_{abc}(t_*) = \int_{\overline{X}} k_a \wedge k_b \wedge k_c + \frac{\partial^3 \mathcal{F}^{inst}(t_*)}{\partial t_a \partial t_b \partial t_c}. \tag{2.1}$$

Using the classical intersection numbers $\int_{\overline{X}} k_a \wedge k_b \wedge k_c$, we can construct a basis m_α $(\alpha = 1, \cdots, h^{1,1}(X))$ of $H^4(\overline{X}, \mathbb{Z})$ that has the following property:

$$\eta_{a\alpha} := \int_{\overline{X}} k_a \wedge m_\alpha = \delta_{a\alpha}. \tag{2.2}$$

With this setup, we obtain a (virtually compact) quantum cohomology ring of $\overline{X}$,

$$\begin{aligned} k_a \circ 1 &= k_a, \\ k_a \circ k_b &= \sum_{c,\gamma} Y_{abc}(t_*)\eta^{c\gamma} m_\gamma = \sum_\gamma Y_{ab\gamma}(t_*) m_\gamma, \\ k_a \circ m_\alpha &= Y_{a\alpha 0} v = \delta_{a\alpha} v, \\ k_a \circ v &= 0, \end{aligned} \tag{2.3}$$

where we used standard property of quantum cohomology ring:

$$Y_{a\alpha 0} = \eta_{a\alpha}. \tag{2.4}$$

Here v is the volume form of $\overline{X}$ and we use the subscript 0 to denote the identity 1 of $H^*(\overline{X}, \mathbb{Z})$. Then we consider the associated Gauss-Manin system:

$$\begin{aligned} \partial_a \psi_0 &= \psi_a, \\ \partial_a \psi_b &= \sum_{c,\gamma} Y_{abc}(t_*)\eta^{c\gamma}\psi_\gamma = \sum_\gamma Y_{ab\gamma}(t_*)\psi_\gamma, \\ \partial_a \psi_\alpha &= Y_{a\alpha 0}\psi_v = \delta_{a\alpha}\psi_v, \\ \partial_a \psi_v &= 0. \end{aligned} \tag{2.5}$$

Next, we consider the inverse matrix $(Y_a^{-1}(t_*))^{bc}$ of $(Y_a(t_*))_{bc} := Y_{abc}(t_*)$. From (2.5), we obtain,

$$\psi_\alpha = \sum_b (Y_a^{-1}(t_*))^{\alpha b} \partial_a \partial_b \psi_0. \tag{2.6}$$

Since, ψ_α is unique for each α, we have to impose integrability conditions:

$$\sum_c (Y_a^{-1}(t_*))^{\alpha c} \partial_a \partial_c \psi_0 = \sum_c (Y_b^{-1}(t_*))^{\alpha c} \partial_b \partial_c \psi_0 \ \ (a \neq b) \tag{2.7}$$

for any $a,\ b \in \{1, 2, \cdots, h^{1,1}(X)\}$. We have another integrability condition from the third line of (2.5):

$$\begin{aligned} &\partial_a \big(\sum_c (Y_a^{-1}(t_*))^{ac} \partial_a \partial_c \psi_0 \big) = \partial_b \big(\sum_c (Y_b^{-1}(t_*))^{bc} \partial_b \partial_c \psi_0 \big), \ \ (a \neq b) \\ &\partial_b \big(\sum_c (Y_a^{-1}(t_*))^{ac} \partial_a \partial_c \psi_0 \big) = 0, \ \ (a \neq b). \end{aligned} \tag{2.8}$$

for any $a,\ b \in \{1, 2, \cdots, h^{1,1}(X)\}$. Finally, we can derive differential equations from the fourth line of (2.5):

$$\partial_a^2 \big(\sum_c (Y_a^{-1}(t_*))^{ac} \partial_a \partial_c \psi_0 \big) = 0. \tag{2.9}$$

Our strategy in this paper is to translate the equations (2.7), (2.8) and (2.9) in terms of the mirror map

$$t_a = t_a(z_*) \tag{2.10}$$

into the differential equations of the complex deformation parameters z_a of the mirror manifold $\hat{X}$. Of course, in the one parameter case, the equations (2.7) and (2.8) become trivial, and the only nontrivial equation is

$$\partial_t^2 \left(\frac{1}{Y_{ttt}} \right) \partial_t^2 \psi_0(t) = 0, \tag{2.11}$$

which is well-known from the literature.

In the following, we will explicitly compute (2.7), (2.8) and (2.9) in many examples, and we find that these equations are highly degenerate. Therefore, in this paper, we will choose the minimal independent set of equations for an extended Picard-Fuchs system.

By construction, the Picard-Fuchs system so obtained has a solution space given by

$$\Big(1, t_1, \ldots, t_{h^{1,1}(X)}, \frac{\partial \mathcal{F}}{\partial t_1}, \ldots, \frac{\partial \mathcal{F}}{\partial t_{h^{1,1}(X)}}, 2\mathcal{F} - \sum_{a=1}^{h^{1,1}(X)} t_a \frac{\partial \mathcal{F}}{\partial t_a} \Big). \tag{2.12}$$

And this is exactly the data one would like from mirror symmetry.

3 Mirror symmetry for local $\mathbb{P}^1$.

3.1 A Picard-Fuchs operator for local $\mathbb{P}^1$.

Before diving into the details of Gauss-Manin systems and the like, we will first take a simple-minded look at a familiar example, namely $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$. We will see that in trying to apply the techniques of local mirror symmetry to this basic case, we are inevitably led to introduce the generalized intersection theory explained in the introduction. In fact, this is the example that originally motivated the investigations of this paper.

Recall the symplectic quotient definition of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$:

$$X = \{(w_1, \ldots, w_4) \in \mathbb{C}^4 - Z : |w_1|^2 + |w_2|^2 - |w_3|^2 - |w_4|^2 = r\}/S^1. \tag{3.13}$$

Above, $Z = \{w_1 = w_2 = 0\}$, $r \in \mathbb{R}^+$ and

$$S^1 : (w_1, \ldots, w_4) \to (e^{i\theta} w_1, e^{i\theta} w_2, e^{-i\theta} w_3, e^{-i\theta} w_4).$$

We can naively employ the methods of [17] to produce a Picard-Fuchs operator associated to the mirror Calabi-Yau $\hat{X}$ of X. The family $\hat{X}$ is described as [15]

$$\hat{X}_z = \{(u, v, y_1, y_2) \in \mathbb{C}^2 \times (\mathbb{C}^*)^2 : uv + 1 + y_1 + y_2 + z y_1 y_2^{-1} = 0\}. \tag{3.14}$$

Then [17] provides a recipe for dealing with non-compact period integrals for such an $\hat{X}$. They are defined by

$$\Pi_\Gamma(z) = \int_\Gamma \frac{du dv dy_1 dy_2/(y_1 y_2)}{uv + 1 + y_1 + y_2 + z y_1 y_2^{-1}}$$

for $\Gamma \in H_4(\mathbb{C}^2 \times (\mathbb{C}^*)^2 - \hat{X}, \mathbb{Z})$. As usual, we utilize the GKZ formalism in order to exhibit an differential operator which annihilates the Π_Γ. One finds

$$\mathcal{D} = (1 - z)\theta^2, \quad \theta = z\frac{d}{dz} \tag{3.15}$$

as the relevant PF operator.

This is a puzzling situation. Clearly, the solutions of $\mathcal{D}f = 0$ are given by $\{1, \log z\}$. This is sensible, because noncompact PF systems always have a constant solution [5], and the mirror map is trivial in this case, leading to a $\log z$ solution. However, there is no double logarithmic solution, because $\mathcal{D}$ is only of order 2. Hence, we have no function $\mathcal{F}$ with which to count holomorphic curves on X! But, since X contains exactly 1 holomorphic curve, we know that the sought after function should be of the form

$$\mathcal{F}(z) = K\frac{(\log z)^3}{6} + \sum_{n>0} \frac{z^n}{n^3}. \tag{3.16}$$

Here K is a classical triple intersection number for $\mathbb{P}^1 \hookrightarrow X$.

Also, notice that the leading factor of $1 - z$ in front of $\mathcal{D}$, while naturally appearing through the techniques of the GKZ formalism, is auxiliary to the solution set of $\mathcal{D}$.

At this point, we can gain a bit of insight from the compact case. Recall [6] that in the event of a compact Calabi-Yau X with one Kähler parameter, there is always a flat coordinate t in which the Picard-Fuchs operator for the mirror family is given as

$$\mathcal{D}_{compact}(t) = \partial_t^2 \Big(\frac{1}{Y}\Big) \partial_t^2, \tag{3.17}$$

which is the same as the formula (2.11). This is reminiscent of our situation (3.15), upon making the identification $t = \log z$.

If one surrendered to the impulse of emulating the above compact expression, one would be compelled to work with the following modified differential operator:

$$\mathcal{D} \longrightarrow \mathcal{D}' = \theta^2 \mathcal{D}.$$

Rewrite this as

$$\theta^2 (1-z)\theta^2 = \theta^2 \Big(\frac{1}{1-z}\Big)^{-1} \theta^2.$$

By comparison with (3.17), it is natural to identify the Yukawa coupling Y of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ as

$$Y = \frac{1}{1-z}. \tag{3.18}$$

And indeed, the condition that $Y = \theta^3 \mathcal{F}$, which follows from the form of $\mathcal{D}'$, yields the expected function $\mathcal{F}$ (3.16). Then the resultant period vector is

$$\Pi' = (1, \log z, \theta\mathcal{F}, 2\mathcal{F} - (\log z)\theta\mathcal{F}),$$

which is the period vector encountered when dealing with compact Calabi-Yaus. Hence, we have found a 'cure' for mirror symmetry on $X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$; the new operator $\theta^2\mathcal{D}$ reproduces all relevant data to describe mirror symmetry for X.

We can also view this $\theta^2\mathcal{D}$ from the vantage of the Frobenius method. The geometry of X is determined by the set of vertices $\{\nu_0, \ldots, \nu_3\} = \{(0,0), (1,0), (0,1), (1,1)\}$, together with a choice of triangulation of the resulting toric graph. These give rise to the lattice vector $l = (1, 1, -1, -1)$, and identify z as the correct variable on the complex moduli space of $\hat{X}$.

Then the solutions of our extended PF operator $\theta^2\mathcal{D}$ can be generated, via the Frobenius method, from the function $\omega_0(z, \rho) = \sum_{n \geq 0} c(n, \rho) z^{n+\rho}$, with

$$c(n, \rho) = (\Gamma(1 + n + \rho)^2 \Gamma(1 - n - \rho)^2)^{-1}.$$

It is a simple matter to verify that

$$\Pi' = (\omega_0(z, 0),\ \partial_\rho \omega_0(z, \rho)|_{\rho=0},\ \partial_\rho^2 \omega_0(z, \rho)|_{\rho=0},\ \partial_\rho^3 \omega_0(z, \rho)|_{\rho=0}).$$

Clearly this had to be the case, since the extension of the original $\mathcal{D}$ on the left by each factor of θ adds one more Frobenius-generated solution.

There is, however, one additional subtlety here; the constant K from eqn. (3.16) was not determined. It is expected from physics that such a K ought to be ambiguous. However, there is

a unique choice which is compatible with the solutions of $\theta^2\mathcal{D}$, which turns out to be $K = 1$. This is the same as was used in the physical considerations of [27]. Later, we will see that the choice more natural for generalization is $K = 1/2$.

Next, we will take up the question of the geometric meaning of the solutions of this new operator.

3.2 PF extensions and Riemann surfaces.

In physics literature [2] [17], a frequently used technique of local mirror symmetry is to consider periods on a Riemann surface $\Sigma \hookrightarrow \hat{X}$, rather than periods of the full mirror geometry $\hat{X}$. In this section, we will review evidence in favor of this approach.

Looking back at the mirror geometry (3.14), this can be rewritten as

$$\hat{X}_z = \{uv + 1 + y_1 + y_2 + zy_1y_2^{-1} = uv + f(z, y_1, y_2) = 0\},$$

which is a hypersurface in $\mathbb{C}^2 \times (\mathbb{C}^*)^2$. Notice that there is an imbedded Riemann surface in this space, defined as

$$\Sigma_z = \{(y_1, y_2) \in (\mathbb{C}^*)^2 : f(z, y_1, y_2) = 0\}. \tag{3.19}$$

In fact, this statement applies not only to the local $\mathbb{P}^1$ case, but to all toric local mirror symmetry constructions [2]. The only difference is that there may be more complex moduli involved; in such cases, we can simply set $z = (z_1, \ldots, z_n)$, where now each z_i is a complex structure modulus.

Now take $(a_0, \ldots, a_3)$ as homogeneous coordinates on the moduli spaces of $\hat{X}$ and Σ, i.e.

$$\hat{X}_a = \{uv + a_0 + a_1y_1 + a_2y_2 + a_3zy_1y_2^{-1} = uv + f(a, y_1, y_2) = 0\}.$$

Recall that the GKZ operators are differential operators $\{\mathcal{L}_i\}$ in the variables a, such that

$$\mathcal{L}_i \int_\Gamma \frac{dudvdy_1dy_2/(y_1y_2)}{uv + f(a, y_1, y_2)} = 0, \quad \forall i.$$

We can recover the PF operators from the GKZ operators via a canonical reduction on the homogeneous moduli space.

With these things in mind, we note the general

Proposition 1 *The GKZ operators associated to the geometry $\hat{X}$ are the same as those associated to Σ.*

Proof. Notice that Σ is a complex dimension 1 noncompact Calabi-Yau manifold. In particular, it makes sense to define the period integrals of Σ as

$$\Pi_\gamma^\Sigma(a) = \int_\gamma \frac{dy_1dy_2/(y_1y_2)}{f(a, y_1, y_2)}$$

with $\gamma \in H_2((\mathbb{C}^*)^2 - \Sigma, \mathbb{Z})$. Let $\mathcal{L}$ be a GKZ operator on the moduli space of Σ_a, so that

$$\mathcal{L}\Pi_\gamma^\Sigma(a) = 0.$$

Recall that the period integrals of $\hat{X}_a$ are given as

$$\Pi_\Gamma^{\hat{X}}(a) = \int_\Gamma \frac{dudvdy_1dy_2/(y_1y_2)}{uv + f(a, y_1, y_2)} \tag{3.20}$$

for $\Gamma \in H_4(\mathbb{C}^2 \times (\mathbb{C}^*)^2 - \hat{X}, \mathbb{Z})$. Then it is clear that we must also have

$$\mathcal{L}\Pi_\Gamma^{\hat{X}}(a) = 0,$$

because the additive factor of uv in the period integrals of $\hat{X}$ is independent of a. Clearly the converse of this statement is also true, so the proposition follows. □

Note that, as pointed out in [17], there us a subtlety in terms of the scaling properties of the period integrals on $\hat{X}_a$ and Σ_a. Further, this scaling difference implies that the PF operators we derive from the above $\mathcal{L}$ will be different on Σ_z and $\hat{X}_z$. In the following, we will ignore this point, and carry on as though the period integrals on Σ_z actually reproduce the same PF operators.

As the geometry of Σ is far simpler than that of $\hat{X}$, this proposition will greatly aid the search for a geometric interpretation of the formal procedure $\mathcal{D} \to \theta^2\mathcal{D}$ on Picard-Fuchs operators. We will explore this in the next section.

3.3 Geometric interpretation through the Riemann surface.

First, we will give a brief description of what "adding extra period integrals" means (which we are doing, by raising the power of the PF operator) in the context of the space $\hat{X}$. This follows the lead of e.g. [15],[4].

Recall that mirror symmetry between the spaces X and $\hat{X}$ means, in particular, that

$$\dim H^{1,1}(X) = \dim H^{2,1}(\hat{X}).$$

Hence, for every 2 cycle of X, we can expect a mirror 3 cycle of $\hat{X}$. Let $\dim H_3(\hat{X}, \mathbb{Z}) = n$, and take $\Gamma_i, \Gamma_j \in H_3(\hat{X}, \mathbb{Z})$ with Poincaré duals $\alpha_i, \alpha_j \in H^3(\hat{X}, \mathbb{Z})$. Then there is a symplectic structure on $H_3(\hat{X}, \mathbb{Z})$, defined by the intersection pairing

$$(\Gamma_i, \Gamma_j) = \int_{\hat{X}} \alpha_i \wedge \alpha_j.$$

In the compact case, we can find a basis $\{\Phi_1, \ldots, \Phi_{n/2}, \Psi_1, \ldots, \Psi_{n/2}\}$ for $H_3(\hat{X}, \mathbb{Z})$ satisfying

$$(\Phi_i, \Phi_j) = (\Psi_i, \Psi_j) = 0 \ , \quad (\Phi_i, \Psi_j) = \delta_{ij}.$$

However, there is no such nice construction for the noncompact case. In fact, we can explicitly exhibit this failure in the example we are considering, $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$. First, rewrite the equation for the mirror $\hat{X}$:

$$\hat{X}_z = \{\tilde{u}v + z + \tilde{y}_1 + \tilde{y}_2 + \tilde{y}_1\tilde{y}_2 = 0\}$$

where we have taken $\tilde{y}_i = y_i^{-1}$ and $\tilde{u} = u/y_1y_2$. Set $z = 1 - a$, where $a \in \mathbb{R}^+$. Then

$$\hat{X}_z = \{\tilde{u}v + 1 + \tilde{y}_1 + \tilde{y}_2 + \tilde{y}_1\tilde{y}_2 = a\},$$

and we can identify a 3 cycle

$$\Gamma_1 = \hat{X}_z \cap \{\tilde{u} = \bar{v}, \tilde{y}_2 = \bar{\tilde{y}}_1\} = \{v\bar{v} + (1 + \tilde{y}_1)\overline{(1 + \tilde{y}_1)} = a\}.$$

It is easy to verify that this cycle has no symplectic dual in $H_3(\hat{X}, \mathbb{Z})$.

Of course, there is a noncompact symplectic dual for Γ_1. Let $p \in \Gamma_1$, and take $\tilde{\Gamma}_1 = (N_{\Gamma_1/\hat{X}})_p$. Then $\tilde{\Gamma}_1$ intersects Γ_1 in a point, and could be thought of as a dual; however, integrals over $\tilde{\Gamma}_1$ may not be well defined. Instead take

$$\tilde{\Gamma}_1^\lambda = \{v \in (N_{\Gamma_1/\hat{X}})_p : |v| \leq \lambda\}, \quad \lambda \in \mathbb{R}^+.$$

Set

$$\Omega_{\hat{X}} = \mathrm{Res}_{\{uv+f(z,y_1,y_2)=0\}} \Big(\frac{dudvdy_1dy_2/(y_1y_2)}{uv + f(z, y_1, y_2)}\Big).$$

Then the proposal of [15] is that we should also consider integrals of the form

$$\int_{\tilde{\Gamma}_1^\lambda} \Omega_{\hat{X}}$$

as periods of the noncompact geometry $\hat{X}$. Mathematically, this means that the definition of noncompact period integrals of $\hat{X}$ are to be taken as all $\int_\Gamma \Omega_{\hat{X}}$ for $\Gamma \in H_3(\hat{X}, \mathbb{Z}) \oplus (H_3(\hat{X}, \mathbb{Z}))_c$. Here, the subscript c indicates compactly supported homology.

In view of Proposition 1, one can make the following

Definition 1 *Let $\hat{X}$ be the noncompact Calabi-Yau hypersurface*

$$\hat{X} = \{(u, v, y_1, y_2) \in \mathbb{C}^2 \times (\mathbb{C}^*)^2 : uv + f(z, y_1, y_2) = 0\}$$

and Σ the imbedded Riemann surface $\Sigma_z = \hat{X} \cap \{u = v = 0\}$. Then the period integrals of $\hat{X}$ *are defined to be*

$$\Pi_\gamma(z) = \int_\gamma \mathrm{Res}_{f=0} \Big(\frac{dy_1dy_2/(y_1y_2)}{f(z, y_1, y_2)}\Big)$$

for $\gamma \in H_1(\Sigma, \mathbb{Z}) \oplus (H_1(\Sigma, \mathbb{Z}))_c$.

In the next subsection, we will apply this definition to $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$, and see that the explicit evaluation of the period integrals on Σ gives the same answer as the extended PF operator of section 3.1.

3.4 Period integrals for local $\mathbb{P}^1$.

Before describing the cycles of Σ and computing their associated integrals, we will need to make use of the following proposition. This will give a $(1,0)$ form $\alpha \in H_1(\log(\Sigma),\mathbb{Z})$, which can be integrated over lines in Σ. Note that, as was the case of the previous proposition, the validity is for all Riemann surfaces appearing as mirrors of the toric local mirror symmetry construction.

Some of the arguments below can be found in [22].

Proposition 2 *Let Σ be as above, and choose $\gamma \in H_1(\Sigma,\mathbb{Z}) \oplus (H_1(\Sigma,\mathbb{Z}))_c$. Then*

$$\int_\gamma \mathrm{Res}_{f=0}\Big(\frac{dy_1dy_2/(y_1y_2)}{f(z,y_1,y_2)}\Big) = -\int_\gamma \log y_2 \frac{dy_1}{y_1} = -\int_\gamma \log y_1 \frac{dy_2}{y_2}.$$

Proof. Let $T(\gamma)$ be a tubular neighborhood of γ in Σ. One easily sees that the PF operators of

$$\int_{T(\gamma)} \frac{dy_1dy_2/(y_1y_2)}{f(z,y_1,y_2)} \quad \text{and} \quad \int_{T(\gamma)} \log(f)\, dy_1\, dy_2/(y_1\, y_2)$$

are in fact the same. Then, if we assume that Definition 1 gives an equivalence between PF solutions and period integrals, we have

$$\int_\gamma \mathrm{Res}_{f=0}\Big(\frac{dy_1dy_2/(y_1y_2)}{f(z,y_1,y_2)}\Big) = \int_\gamma \mathrm{Res}_{f=0}\Big(\log(f)dy_1dy_2/(y_1y_2)\Big) =$$

$$-\int_\gamma \mathrm{Res}_{f=0}\Big(d(\log(f))\log y_2\frac{dy_1}{y_1}\Big) = -\int_\gamma \mathrm{Res}_{f=0}\Big(\frac{df}{f}\log y_2\frac{dy_1}{y_1}\Big) =$$

$$-\int_\gamma \log y_2 \frac{dy_1}{y_1}.$$

From the first to the second line, we have integrated by parts, and then the residue around $f=0$ was taken from the second to third line.

This argument applies equally well upon exchanging y_1 and y_2. □

With this proposition in hand, we can take up the task of working out period integrals on local $\mathbb{P}^1$. Recall that the original, unmodified PF operator for this space was found to be

$$\mathcal{D} = (1-z)\theta^2,$$

with solution set $\{1, \log z\}$. We will at first content ourselves with finding cycles $\gamma_0, \gamma_1 \in H_1(\Sigma,\mathbb{Z})$ whose period integrals reproduce these solutions.

To this end, note that the defining equation

$$\Sigma = \{(y_1,y_2) \in (\mathbb{C}^*)^2 : 1 + y_1 + y_2 + zy_1y_2^{-1} = 0\} \tag{3.21}$$

can be solved in 2 ways:

$$y_1 = \frac{-1-y_2}{1+zy_2^{-1}}, \quad y_2^{\pm} = \frac{-1-y_1 \pm \sqrt{(1+y_1)^2 - 4zy_1}}{2}.$$

Now, since y_1 and y_2 are $\mathbb{C}^*$ variables, we can define three homology elements from these equations:

$$\gamma_0 = \Big\{(y_1, y_2) \in (\mathbb{C}^*)^2 : y_1 = \frac{-1 - y_2}{1 + zy_2^{-1}}\ ,\quad |y_2| = \epsilon\Big\},$$

$$\tau_\pm = \Big\{(y_1, y_2) \in (\mathbb{C}^*)^2 : y_2^\pm = \frac{-1 - y_1 \pm \sqrt{(1 + y_1)^2 - 4zy_1}}{2}\ ,\quad |y_1| = \epsilon\Big\}.$$

Then, two of these must be responsible for the solution set $\{1, \log z\}$. To motivate the correct choice of cycles, let us first look closely at the mirror construction that originally provided equation (3.21).

Starting with the description $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1 =$

$$\{(w_1, \ldots, w_4) \in \mathbb{C}^4 - Z : |w_1|^2 + |w_2|^2 - |w_3|^2 - |w_4|^2 = r\}/S^1,$$

from [2], the mirror geometry can be characterized as

$$\{uv + x_1 + \cdots + x_4 = 0 : x_1x_2x_3^{-1}x_4^{-1} = z,\ x_i = 1 \text{ for some } i\}.$$

Here $u, v \in \mathbb{C}$ and $x_i \in \mathbb{C}^*$. Also, the x_i obey $|x_i| = \exp(-|w_i|^2)$, and $|z| = e^{-r}$. To arrive at the form (3.21), we set $x_3 = 1$ and solved for x_1 in the constraint $x_1x_2x_3^{-1}x_4^{-1} = z$. Finally, the identification $y_1 = x_4,\ y_2 = x_2$ was made.

The equation for local $\mathbb{P}^1$ indicates that $[w_1, w_2]$ can be taken as homogeneous coordinates on the $\mathbb{P}^1$. In the locus where $w_3 = w_4 = 0$, we thus have a bound $0 \le |w_2|^2 \le r$. Then $|y_2| = \exp(-|w_2|^2)$ implies that $1 \ge |y_2| \ge e^{-r} = |z|$; taking these considerations together, we may accurately label the following figure:

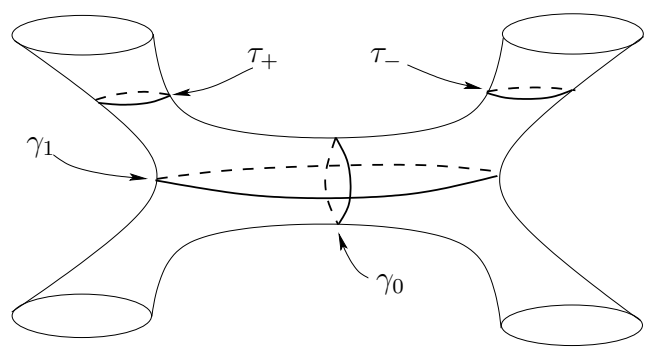

Figure 1: 1-cycles on Σ. $\{\gamma_0, \tau_+, \tau_-\}$ is a basis for $H_1(\Sigma, \mathbb{Z})$.

Proposition 3 *Let $\gamma_0,\ \tau_\pm$ be as described above, and set*

$$\gamma_1 = [\tau_+] + [\tau_-]$$

with the sum taken in $H_1(\Sigma, \mathbb{Z})$. Then

$$\int_{\gamma_0} \log y_1 \frac{dy_2}{y_2} = 1, \qquad \int_{\gamma_1} \log y_2 \frac{dy_1}{y_1} = \log z$$

when appropriately normalized.

Proof. The first integral is trivial:

$$\int_{\gamma_0} \log y_1 \frac{dy_2}{y_2} = \int_{|y_2|=\epsilon} \log\left(\frac{-1-y_2}{1+zy_2^{-1}}\right)\frac{dy_2}{y_2} =$$

$$\int_{|y_2|=\epsilon} \left(i\pi + \log\left(\frac{1+y_2}{1+zy_2^{-1}}\right)\right)\frac{dy_2}{y_2}$$

and this is a constant. Of course, the branch cut of log must be taken to lie off the negative real axis.

For the second,

$$\int_{\gamma_1} \log y_2 \frac{dy_1}{y_1} = \int_{\tau_+} \log y_2^+ \frac{dy_1}{y_1} + \int_{\tau_-} \log y_2^- \frac{dy_1}{y_1} =$$

$$\int_{|y_1|=\epsilon} \log(y_2^+ y_2^-)\frac{dy_1}{y_1} = \int_{|y_1|=\epsilon} \log(zy_1)\frac{dy_1}{y_1} = const + 2\pi i \log z.$$

□

Next, the existence of a new period integral, based on Definition 1, will be demonstrated.

3.5 A period integral from $(H_1(\Sigma,\mathbb{Z}))_c$.

Since the constraint $1 \geq |y_2| \geq |z|$ only applies in regions with $z_3 = z_4 = 0$, outside of this locus, it is sensible to define a path on Σ as follows. Let $\lambda < |z|$ be real, and take a smooth increasing function $\sigma : [0,1] :\to \Sigma$ such that $\sigma(0) = \lambda, \sigma(1) = z$. Then

$$\gamma_2(\lambda) = \left\{(y_1,y_2) \in (\mathbb{C}^*)^2 : y_1 = \frac{-1-y_2}{1+zy_2^{-1}}, \quad y_2 = \sigma[0,1], \quad y_1 \neq 1\right\}$$

defines an element of $(H_1(\Sigma,\mathbb{Z}))_c$.

Proposition 4 *Let Σ, γ_2 be defined as above. Then*

$$\theta\mathcal{F} = \int_{\gamma_2} \log y_1 \frac{dy_2}{y_2},$$

where $\theta = z\frac{d}{dz}$ and $\theta\mathcal{F}$ is the double logarithmic solution of the extended PF operator

$$\mathcal{D}' = \theta^2(1-z)\theta^2$$

associated to the mirror of the local model $\mathcal{O}(-1)\oplus\mathcal{O}(-1) \to \mathbb{P}^1$.

Proof. The computation is straightforward:

$$\int_{\gamma_2} \log y_1 \frac{dy_2}{y_2} = \int_{\lambda}^{z} \log\left(\frac{-1-y_2}{1+zy_2^{-1}}\right)\frac{dy_2}{y_2} =$$

$$\int_\lambda^z \Big(i\pi + \sum_{n>0} \frac{(-y_2)^n}{n} - \sum_{n>0} \frac{(-zy_2^{-1})^n}{n}\Big)\frac{dy_2}{y_2} =$$

$$const + (\lambda - \text{dependent}) + \sum_{n>0} \frac{(-z)^n}{n^2} = \big(\theta\mathcal{F}\big)(-z).$$

In order to achieve the result, we should have z rather than $-z$ in the above. However, this is accounted for by the fact that $\arg(y_2)$ is not determined in local mirror symmetry, and hence we are free to use the variable $y_2' = e^{i\pi}y_2$ in place of y_2. □

Notice that, with this definition of period integrals, the logarithmic terms of $\theta\mathcal{F}$ are not uniquely determined, as they depend on λ. It is for this reason that λ dependent terms are disregarded in the calculation.

It may seem that the choice of γ_2 is artificial, since we could have equally well chosen an increasing function $\sigma' : [0,1] \to \Sigma$ with $\sigma'(0) = 1, \sigma'(1) = \lambda > 1$. However, it is easy to show that this is equivalent; if

$$\gamma_2'(\lambda) = \Big\{(y_1, y_2) \in (\mathbb{C}^*)^2 : y_1 = \frac{-1-y_2}{1+zy_2^{-1}}, \quad y_2 = \sigma'[0,1], \quad y_1 \neq 1\Big\},$$

then

$$\lim_{\lambda\to\infty} \Big(\theta \int_{\gamma_2'} \log y_1 \frac{dy_2}{y_2}\Big) = \sum_{n>0} \frac{(-z)^n}{n},$$

so the two approaches are interchangeable.

3.6 Discussion.

So far, we have considered only one example: $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$. However, we have learned a great deal already. Firstly, the PF system cannot always be relied on to provide all the information we need for local mirror symmetry. Yet, one might be tempted to hope that, in general, we can recover the missing data through a simple modification of the known PF systems. Let us consider our new operator $\theta^2\mathcal{D}$ from one final perspective.

The incompleteness of the PF system on $X = \mathcal{O}(-1)\oplus\mathcal{O}(-1) \to \mathbb{P}^1$ is supposed to emerge from the noncompactness of the space; that is, as a result of the facts that $b_4 = b_6 = 0$, where b_i denotes the ith Betti number. On a compact space, the PF system will have one regular solution, b_2(resp. b_4) logarithmic (resp. double logarithmic) solutions, and one ($= b_6$) triple logarithmic solution. On X, our two usual PF solutions can be summarized by a cohomology-valued hypergeometric series [12][16][17]

$$\omega(z, J) = \omega_0 + \omega_1(z)J.$$

Here $\omega_0 = 1$, and $\omega_1 = \log z$ is the mirror map. Also, J is the cohomology element dual to $\mathbb{P}^1 \hookrightarrow X$. Solutions of the PF system $\mathcal{D}f = 0$ are recovered as $\omega_i(z) = \frac{d^i\omega(z,J)}{dJ^i}|_{J=0}$.

It is clear that by instead working with $\theta^2\mathcal{D}f = 0$, we have a larger cohomology-valued series as a generating function of solutions:

$$\omega(z, J) = \omega_0 + \omega_1(z)J + \omega_2(z)J^2 + \omega_3(z)J^3.$$

From this perspective, the addition of θ^2 on the left-hand side of the operator $\mathcal{D}$ is a sort of compactification of the model, in that it represents the addition of a 4-cycle and a 6-cycle for X.

We will now generalize these ideas to situations with no 4-cycle, and an arbitrary number of Kähler classes.

4 Mirror Symmetry for Toric Trees.

In this section, we will show that if X is a noncompact Calabi-Yau manifold such that $\dim H_4(X,\mathbb{Z}) = 0$, then we may apply our methods to obtain a complete system of differential equations which fully determines the prepotential. The Yukawa couplings take on a central role in the following.

4.1 Ordinary Picard-Fuchs systems.

Our first interest will be to take a look at the PF systems one would arrive at through use of existing local mirror symmetry techniques. We wish to understand exactly how much information one might recover through these systems alone, in order to determine an appropriate 'fix'.

Let us clarify what we are exploring here. Let $\{l^1,\ldots,l^n\} \subset \mathbb{Z}^m$ be a choice of basis for the secondary fan of a noncompact toric Calabi-Yau threefold X satisfying $\dim H_4(X,\mathbb{Z}) = 0$. Consider the generating function

$$\omega(z,\rho) = \sum_{n>0} c(n,\rho) z^{n+\rho} \tag{4.22}$$

where

$$c(n,\rho)^{-1} = \prod_i \Gamma\big(1 + \sum_k l_i^k (n_k + \rho_k)\big). \tag{4.23}$$

Here we are using the convention that $l^k = (l_1^k,\ldots,l_m^k)$. Then we want to look at the functions

$$\Pi_{ij} = \big(\partial_{\rho_i}\partial_{\rho_j}\omega(z,\rho)\big)|\rho = 0. \tag{4.24}$$

Our interest in this subsection, then, is to ascertain how much information we can find by looking at the Π_{ij}. In so doing, we will gain a better understanding of what to do in order to remedy mirror symmetry in this situation.

Example 1 Consider the space

$$X_1 = \{-2|w_1|^2 + |w_2|^2 + |w_3|^2 = Re(t),\ |w_1|^2 - |w_2|^2 + |w_4|^2 - |w_5|^2 = Re(s)\}/(S^1)^2$$

where $(w_1,\ldots,w_5) \in \mathbb{C}^5 - (\{w_2 = w_3 = 0\} \cup \{w_1 = w_4 = 0\} \cup \{w_4 = w_5 = 0\})$. This contains two curves C_t, C_s with respective normal bundles $\mathcal{O} \oplus \mathcal{O}(-2)$ and $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ in X_1. We have that $b_2 = 2$ and $b_4 = 0$.

From the work [23], we can draw a planar trivalent graph for X_1 corresponding to the torus weights. Using the rules of that paper, the resulting graph looks like Through the GKZ formalism,

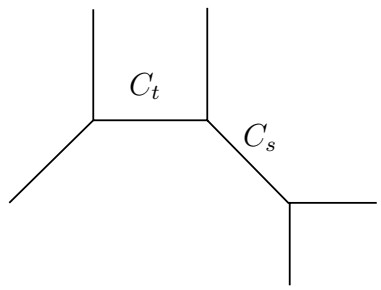

Figure 2: Toric diagram for X_1.

we have the PF operators associated to X_1:

$$\begin{aligned}
\mathcal{D}_1 &= \theta_1(\theta_1-\theta_2) - z_1(2\theta_1-\theta_2)(2\theta_1-\theta_2+1) \\
\mathcal{D}_2 &= (2\theta_1-\theta_2)\theta_2 - z_2(\theta_1-\theta_2)\theta_2 \\
\mathcal{D}_3 &= \theta_1\theta_2 - z_1z_2(2\theta_1-\theta_2)\theta_2.
\end{aligned} \tag{4.25}$$

Let $\omega(z,\rho)$ be the generating function for solutions of this, with logarithmic solutions t, s.

Set $\Pi_{ij} = \partial_{\rho_i}\partial_{\rho_j}\omega|_{\rho=0}$. Then $\{\mathcal{D}_1, \mathcal{D}_2\}$ also has a double logarithmic solution, given as the linear combination

$$W(z) = \frac{1}{2}\Pi_{11} + \Pi_{12} + \Pi_{22}.$$

We are interested in the relationship between this double logarithmic solution and the prepotential $\mathcal{F}_1$ for X. From physics [9][18], the instanton part of the prepotential is

$$\mathcal{F}_1^{inst} = \sum_{n>0} \frac{e^{ns}}{n^3} + \frac{e^{n(s+t)}}{n^3} - \frac{e^{nt}}{n^3}. \tag{4.26}$$

Naturally, this is the expression gotten after use of the inverse mirror map.

Let us first take a look at the Π_{ij}'s, after the insertion of the mirror map:

$$\Pi_{11}(s,t) = 0,$$

$$\Pi_{12}(s,t) = \sum_{n>0} \frac{e^{n(t+s)}}{n^2} - \frac{e^{ns}}{n^2},$$

$$\Pi_{22}(s,t) = 2\sum_{n>0} \frac{e^{ns}}{n^2}.$$

We have neglected the logarithmic terms of each function. Notice that there is no linear combination of Π_{ij}'s that we can take to reproduce the term $\sum_{n>0} e^{nt}/n^2$.

From these expressions,

$$W(s,t) = \sum_{n>0} \frac{e^{n(s+t)}}{n^2} + \frac{e^{ns}}{n^2} = \frac{\partial \mathcal{F}_1^{inst}}{\partial s}.$$

Apparently, the PF system cannot 'see' the curve with normal bundle $\mathcal{O} \oplus \mathcal{O}(-2)$.

Example 2 Next, consider the space

$$X_2 = \{|w_1|^2 + |w_2|^2 - |w_3|^2 - |w_4|^2 = Re(s_1),\ -|w_1|^2 - |w_3|^2 + |w_4|^2 + |w_5|^2 = Re(s_2)\}/(S^1)^2,$$

with $(w_1, \ldots, w_5) \in \mathbb{C}^5 - (\{w_1 = w_2 = 0\} \cup \{w_4 = w_5 = 0\} \cup \{w_2 = w_5 = 0\})$. We again have $b_2 = 2, b_4 = 0$, and now $\mathcal{N}_{C_i/X_2} \cong O(-1) \oplus \mathcal{O}(-1)$ for each i. Notice that we can flop from X_2 to X_1: if $l^1 = (1, 1, -1, -1, 0)$ and $l^2 = (-1, 0, -1, 1, 1)$, then the combinations $l^1 + l^2, -l^2$ give the secondary fan for X_1. The planar trivalent toric graph for X_2 is given in figure 3. We have the

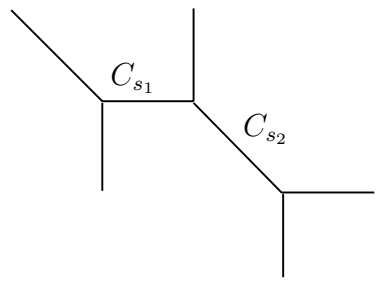

Figure 3: Toric diagram for X_2.

PF system from the mirror manifold:

$$\begin{aligned}
\mathcal{D}_1 &= (\theta_1 - \theta_2)\theta_1 - z_1(-\theta_1 - \theta_2)(-\theta_1 + \theta_2), \\
\mathcal{D}_2 &= (\theta_2 - \theta_1)\theta_2 - z_2(-\theta_2 - \theta_1)(-\theta_2 + \theta_1), \\
\mathcal{D}_3 &= \theta_1\theta_2 - z_1 z_2(\theta_1 + \theta_2 + 1)(\theta_1 + \theta_2).
\end{aligned} \tag{4.27}$$

Let s_i be the logarithmic solutions. Using the same conventions as example 1, we find

$$\Pi_{11}(s_1, s_2) = \sum_{n>0} \frac{e^{ns_1}}{n^2} - \frac{e^{ns_2}}{n^2},$$

$$\Pi_{12}(s_1, s_2) = 0,$$

with $\Pi_{22} = -\Pi_{11}$. Again, these expressions already include the mirror map.

Let's take a look at the prepotential:

$$\mathcal{F}_2^{inst} = \sum_{n\geq 0} \frac{e^{ns_1}}{n^3} + \frac{e^{ns_2}}{n^3} - \frac{e^{n(s_1+s_2)}}{n^3}.$$

Then we see that

$$\Pi_{11} = \frac{\partial \mathcal{F}_2^{inst}}{\partial s_1} - \frac{\partial \mathcal{F}_2^{inst}}{\partial s_2}.$$

Hence, we could recover a bit of information by using the basic extended system $\{\theta_1\mathcal{D}_1, \mathcal{D}_2\}$, which has Π_{11} as a solution. However, the cross term corresponding to $C_{s_1+s_2}$ cannot be detected from the Π_{ij}'s.

Our work with example 1 suggests the reason for the problem with the $C_{s_1+s_2}$ curve. To exhibit this, recall that the lattice vectors $\{l^1, l^2\}$ for this geometry are

$$\begin{pmatrix} l^1 \\ l^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & -1 & 0 \\ 0 & -1 & -1 & 1 & 1 \end{pmatrix}$$

Each vector represents a curve C_{s_i} in X_2. Then $C_{s_1+s_2}$ is determined by the single vector

$$l^1 + l^2 = \begin{pmatrix} 1 & 0 & -2 & 0 & 1 \end{pmatrix}.$$

This curve satisfies $\mathcal{N}_{C_{s_1+s_2}/X_2} \cong \mathcal{O} \oplus \mathcal{O}(-2)$. Hence, we do not expect that we can retrieve its information from the PF system.

We have performed similar computations for 3 and 4 parameter cases, all of which support this general principle. This leads us to make the

Conjecture 2 *Let X be a noncompact toric Calabi-Yau threefold with $\dim H_4(X, \mathbb{Z}) = 0$, and say $\{l^1, \dots l^m\}$ define X via symplectic quotient. Let $\omega = \sum_{n>0} c(n, \rho) z^{n+\rho}$ be the generating function for*

$$\Pi_{ij}^{inst} = \sum_n \left(\partial_{\rho_i} \partial_{\rho_j} c(n, \rho)\right)|_{\rho=0} z^n \tag{4.28}$$

with

$$c(n, \rho)^{-1} = \prod_i \Gamma\big(1 + \sum_k l_i^k (n_k + \rho_k)\big). \tag{4.29}$$

If $\mathcal{F}$ is the prepotential, and $s_i = \partial_{\rho_i} \omega|_{\rho=0}$ for each i such that

$$l^i = \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & \dots & 0 \end{pmatrix}$$

(up to a permutation of the columns of l^i), then there are rational numbers $m_{ij} \in \mathbb{Q}$ such that

$$\sum_{i,j} m_{ij} \Pi_{ij}^{inst} = \sum_i (-1)^{i-1} \frac{\partial \mathcal{F}^{inst}}{\partial s_i}.$$

Here, $\mathcal{F}^{inst}$ is the instanton part of the prepotential. We use the notation Π_{ij}^{inst} to distinguish these functions from the usual derivatives of ω (i.e. $\Pi_{ij} = \partial_{\rho_i} \partial_{\rho_j} \omega|_{\rho=0}$).

This conjecture is equivalent to the statement that although we cannot detect curves with normal bundle $\mathcal{O} \oplus \mathcal{O}(-2)$ via the Π_{ij}, we can exhibit all curves with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ using these functions.

We will now exploit this idea, and in so doing discover a way to construct a new system of differential operators which provides *all* mirror symmetry data for this class of examples.

4.2 Two building blocks of solutions.

Assume X is a noncompact Calabi-Yau threefold such that $\dim H_4(X,\mathbb{Z}) = 0$, and that every two cycle $C \hookrightarrow X$ has normal bundle $\mathcal{O} \oplus \mathcal{O}(-2)$ or $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. We will refer to these as t and s curves, respectively, in the following.

Then, as any such space X is obtained by gluing s and t curves together in some way, it is reasonable to expect that we can solve all these models by extension from the two basic one parameter cases

$$X_s = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \longrightarrow \mathbb{P}^1,$$
$$X_t = \mathcal{O} \oplus \mathcal{O}(-2) \longrightarrow \mathbb{P}^1.$$

We have already exhibited the solution on X_s. We will now modify this slightly to allow extension to the general cases, and subsequently demonstrate a similar solution on X_t.

Recall, from section 2.1, the differential operator for X_s:

$$\tilde{\mathcal{D}}_1 = \theta_s^2(1 - z_s)\theta_s^2.$$

As before, $\theta_s = z_s d/dz_s$, and $\tilde{Y}_1 = 1/(1 - z_s)$ is the Yukawa coupling. Note that this expression for $\tilde{Y}_1$ implies a classical triple intersection number 1 for $\mathbb{P}^1 \hookrightarrow X_s$.

We will now need to make a slightly different choice of Yukawa coupling on X_s. Recall [28][13] that, in the context of the toric flop $s \to -s$, the natural value for the triple intersection number is $1/2$. There is a simple proof for this, which we give now. From section 2.1, we had the prepotential on local $\mathbb{P}^1$ with arbitrary triple intersection number (eqn.(3.16)):

$$\mathcal{F}(z) = K\frac{(\log z)^3}{6} + \sum_{n>0} \frac{z^n}{n^3}.$$

A flop of the $\mathbb{P}^1$ on X_s is the same as a change of variables $z \longrightarrow 1/z$ in the prepotential:

$$\mathcal{F}^{flop}(z) = K\frac{(-\log z)^3}{6} + \sum_{n>0} \frac{z^{-n}}{n^3}.$$

Taking the difference of these,

$$\mathcal{F}(z) - \mathcal{F}^{flop}(z) = -\frac{1}{3}K(\log z)^3.$$

We have ignored the terms including $\sqrt{-1}$, since the Yukawa coupling is insensitive to them. Then according to Witten [28], we are supposed to find

$$\mathcal{F}(z) - \mathcal{F}^{flop}(z) = -\frac{1}{6}(\log z)^3,$$

and hence $K = 1/2$ is uniquely determined.

This means that we should really be using

$$Y_1 = \frac{1}{2} + \frac{z_s}{1 - z_s}$$

for the Yukawa coupling. We obtain the following differential operator describing mirror symmetry for X_s :

$$\mathcal{D}_1 = \theta_s^2\Big(\frac{2(1-z_s)}{1+z_s}\Big)\theta_s^2.$$

Next, let's turn to X_t. Note that the naturally occurring PF operator on the mirror to $\mathcal{O}\oplus\mathcal{O}(-2)\longrightarrow\mathbb{P}^1$, which is

$$\mathcal{D}_2' = \theta_t^2 - z_t(2\theta_t)(2\theta_t+1),$$

has no curve information, since there is no double logarithmic solution. Moreover, the second Frobenius derivative of the generating function of solutions has no instanton part.

Yet, in view of the solution on X_s, we can easily exhibit a Picard-Fuchs operator for X_t; it is given by

$$\mathcal{D}_2 = \partial_t^2(1/Y_2)\partial_t^2.$$

Here

$$t(z) = \log(z_t) + 2\sum_{n>0}\frac{(2n-1)!}{n!^2}z_t^n$$

is the mirror map for X_t, and Y_2 is the Yukawa coupling on X_t, which in these coordinates is

$$Y_2 = -\frac{1+e^t}{2(1-e^t)}.$$

The overall negative has no effect on the solution space of the differential operator $\mathcal{D}_2$, but it is taken so that $\mathbb{P}^1\hookrightarrow X_t$ has a classical triple self-intersection number of $-1/2$. We make this choice in analogy with the case $\mathcal{O}(-3)\to\mathbb{P}^2$, which has intersection number $-1/3$ [7].

Then, as in the compact case, it follows automatically that the solutions Π_t of $\mathcal{D}_2\Pi_t = 0$ are given as

$$\Pi_t = \Big(1, t, \frac{\partial\mathcal{F}}{\partial t}, t\frac{\partial\mathcal{F}}{\partial t} - 2\mathcal{F}\Big)$$

where $\mathcal{F}$ is a holomorphic function in t such that

$$\frac{\partial^3\mathcal{F}}{\partial t^3} = Y_2.$$

Then it is a simple matter to write down and explicit differential operator on the mirror of X_t, by a change of coordinates for $\mathcal{D}_2$. We find

$$\mathcal{D}_2 = \theta_t^4 - z_t(2\theta_t+2)(2\theta_t+1)^2\theta_t + (z_t)^2(2\theta_t+4)(2\theta_t+3)(2\theta_t+1)2\theta_t, \quad (\theta_t := z_t\frac{d}{dz_t}). \tag{4.30}$$

The solutions of (4.30) are generated by the Fröbenius function:

$$w(z_t,\rho) := \sum_{n=0}^{\infty}\frac{1}{\Gamma(1-2n-2\rho)(\Gamma(1+n+\rho))^2}(1+\sum_{j=1}^{n}\frac{\rho}{j+\rho})z_t^{n+\rho}.$$

We can easily check that the vector space:

$$\langle\ 1,\ t,\ \frac{\partial \mathcal{F}}{\partial t},\ 2\mathcal{F} - t\frac{\partial \mathcal{F}}{\partial t}\ \rangle_{\mathbb{C}},$$

is equal to the vector space:

$$\langle\ w(z_t, 0),\ \partial_\rho w(z_t, 0),\ \partial_\rho^2 w(z_t, 0),\ \partial_\rho^3 w(z_t, 0)\ \rangle_{\mathbb{C}}.$$

Hence, we have demonstrated the existence of mirror symmetry for both X_s and X_t, in terms of solutions of new differential operators. It should be noted that $\mathcal{D}_1, \mathcal{D}_2$ cannot be derived from any GKZ system on these spaces.

With these at hand, we can propose a general prescription for local mirror symmetry in absence of a 4 cycle.

4.3 Mirror Symmetry when $\dim H_4(X, \mathbb{Z}) = 0$.

We can use the results of the previous section to find a general solution for such spaces, as follows. From the considerations of [18],we see that if X a noncompact toric Calabi-Yau threefold with $\dim H_4(X, \mathbb{Z}) = 0$, then for each $C \in H_2(X, \mathbb{Z})$, we have

$$\mathcal{N}_{C/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1) \text{ or}$$

$$\mathcal{N}_{C/X} \cong \mathcal{O} \oplus \mathcal{O}(-2).$$

This is also apparent from the vectors which span the secondary fan. We will choose X such that $\{C_{s_1}, \ldots, C_{s_m}, C_{t_1}, \ldots, C_{t_n}\}$ is a basis of $H_2(X, \mathbb{Z})$, where $\mathcal{N}_{C_{s_i}/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, $\mathcal{N}_{C_{t_j}/X} \cong \mathcal{O} \oplus \mathcal{O}(-2)\ \forall i, j$. Also, let $u = (s_1, \ldots, s_m, t_1, \ldots, t_n)$.

From the topological vertex formalism, the authors of [18] were able to determine the instanton part of the prepotential for the class of examples we're considering. Explicitly,

$$\mathcal{F}^{inst} = \sum_{C_s} \sum_{k>0} \frac{e^{ks}}{k^3} - \sum_{C_t} \sum_{k>0} \frac{e^{kt}}{k^3}.$$

Here, the sum over C_s represents the sum over all curves $C_s \hookrightarrow X$ such that $N_{C_s/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, and similarly for the sum over C_t.

As explained in the introduction, our problem reduces to that of defining a consistent (triple) intersection theory on X. Thanks to the simple structure of X, together with our preliminary choice of intersection numbers for $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \longrightarrow \mathbb{P}^1$ and $\mathcal{O} \oplus \mathcal{O}(-2) \longrightarrow \mathbb{P}^1$, there is in fact a unique choice. We will first give the general definition, and afterward explain its significance through an example.

To give the prescription for intersection theory for the general case, we will only consider X with toric diagram as in the following figure (4). That is, only two curves in X are allowed to intersect at any point. Hence, we exclude cases where three curves meet at one point in X, etc. With this restriction, we can introduce an ordering on the curves in X:

$$C_{u_1} < C_{u_2} < C_{u_3} < \ldots$$

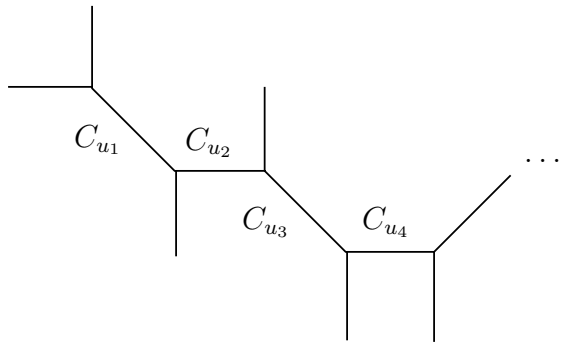

Figure 4: X containing a string of curves.

Define a function

$$sgn : H_2(X, \mathbb{Z}) \longrightarrow \{1, -1\}$$

so that $sgn(C) = 1$ if $\mathcal{N}_{C/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, and $sgn(C) = -1$ otherwise.

With these conventions, we can now state our conjecture on intersection theory.

Definition 2 *Let X be a noncompact toric Calabi-Yau threefold such that* $\dim H_4(X, \mathbb{Z}) = 0$, *and suppose $\mathcal{N}_C \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ or $\mathcal{N}_C \cong \mathcal{O} \oplus \mathcal{O}(-2)$ $\forall$ $C \in H_2(X, \mathbb{Z})$. Then the* classical intersection numbers *for X are given by*

$$K_{abc} = \frac{1}{2} \sum_{C \notin \mathcal{A}} sgn\Big([C_{abc}] + [C]\Big), \tag{4.31}$$

where the sum is taken in homology, and

$$[C_{abc}] = [C_{u_a}] + \sum_{C_{u_a} < C_\alpha < C_{u_b}} [C_\alpha] + [C_{u_b}] + \sum_{C_{u_b} < C_\beta < C_{u_c}} [C_\beta] + [C_{u_c}].$$

The sum is taken away from the set

$$\mathcal{A} = \{[C_{u_a}], [C_{u_b}], \ldots, [C_{u_a} + C_{u_{a+1}}], \ldots\}.$$

This formula can be most simply understood as follows. The curve of minimum volume containing all three curves C_{u_a}, C_{u_b} and C_{u_c} can be represented by the homology class

$$[C_{abc}] = [C_{u_a}] + \sum_{C_{u_a} < C_\alpha < C_{u_b}} [C_\alpha] + [C_{u_b}] + \sum_{C_{u_b} < C_\beta < C_{u_c}} [C_\beta] + [C_{u_c}].$$

Then each term of the sum $[C_{abc}] + [C]$ corresponds to a curve in X containing C_{abc}.

For example, consider the case $a = b = c$. Then both of the sums collapse, we are left with only $sgn([C_{u_a}] + [C])$. The sum will contribute $\pm 1/2$ for each curve containing C_{u_a}, depending on the normal bundle of that curve.

Let us now apply this definition to a concrete case. Consider again the instanton part of the prepotential from example 1 above, equation (4.26):

$$\mathcal{F}^{inst} = \sum_{n>0} \frac{e^{ns}}{n^3} + \frac{e^{n(s+t)}}{n^3} - \frac{e^{nt}}{n^3}.$$

Then e.g.

$$\frac{\partial^3 \mathcal{F}^{inst}}{\partial s^3} = \sum_{n>0} \left(e^{ns} + e^{n(s+t)} \right).$$

Both the s curve and the $s+t$ curve have normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ (this can be seen from the toric diagram, or directly from the vectors defining the secondary fan). Thus, each curve should have an intersection number equal to $1/2$, which implies

$$K_{sss} = \frac{1}{2} + \frac{1}{2} = 1.$$

By applying similar reasoning, we obtain the other intersection numbers

$$K_{tss} = K_{tts} = 1/2, \; K_{ttt} = 0.$$

This intersection theory is also compatible with the flop $X_1 \longrightarrow X_2$ given above. We have also verified that this prescription gives the simplest possible form for the extended Picard-Fuchs system on three parameter models of this type.

In fact, using the results of [18], we can argue that thus definition is the right one in general. In [18], it was shown that the Gopakumar-Vafa invariants for the spaces of interest are invariant under flops; this was done by considering a 3 parameter flop on the strip. The classical intersection numbers that preserve the polynomial part of the prepotential turn to be the ones given above, for the same 3 parameter models of [18]. Hence, this definition is the unique one in order to have a theory that transforms sensibly under flops.

4.4 Extended Picard-Fuchs System for X_1 and X_2

In this subsection, we derive an extended PF system under the assumption of the conjecture given in the previous subsection. First, we look at the example X_1. In this case, we start from four A-model Yukawa couplings:

$$\begin{aligned}
Y_{ttt} &:= -\frac{e^t}{1-e^t} + \frac{e^{s+t}}{1-e^{s+t}}, \\
Y_{tts} &:= \frac{1}{2} + \frac{e^{s+t}}{1-e^{s+t}}, \\
Y_{tss} &:= \frac{1}{2} + \frac{e^{s+t}}{1-e^{s+t}} \\
Y_{sss} &:= 1 + \frac{e^s}{1-e^s} + \frac{e^{s+t}}{1-e^{s+t}}.
\end{aligned} \tag{4.32}$$

The constant part of each Yukawa coupling is given by the conjecture, and the instanton (non-constant) part was taken from [18]. We repeat here the PF operators given by the standard toric construction of the mirror manifold $\hat{X}_1$:

$$\begin{aligned}
\mathcal{D}_1 &= \theta_1(\theta_1-\theta_2) - z_1(2\theta_1-\theta_2)(2\theta_1-\theta_2+1), \\
\mathcal{D}_2 &= (2\theta_1-\theta_2)\theta_2 - z_2(\theta_1-\theta_2)\theta_2, \\
\mathcal{D}_3 &= \theta_1\theta_2 - z_1z_2(2\theta_1-\theta_2)\theta_2.
\end{aligned} \tag{4.33}$$

By solving (4.33), we obtain mirror maps $s = s(z_1, z_2)$ and $t = t(z_1, z_2)$. In particular, the Jacobian of these mirror maps are written in terms of simple functions, as follows:

$$\frac{\partial t}{\partial u_1} = \frac{1}{\sqrt{1-4z_1}}, \quad \frac{\partial t}{\partial u_2} := 0, \quad \frac{\partial s}{\partial u_1} = \frac{1}{2}\frac{-1+4z_1+\sqrt{1-4z_1}}{-1+4z_1}, \quad \frac{\partial s}{\partial u_2} = 1, \tag{4.34}$$

where $u_i = \log(z_i)$. With this data, we can compute the B model Yukawa couplings in u^i coordinates, and they turn out to be rational functions in z_i whose denominators are given by the divisor of the defining equation of discriminant locus of $\hat{X}_1$:

$$dis(\hat{X}_1) = (1-z_2+z_1z_2^2)(1-4z_1). \tag{4.35}$$

The explicit results are given as follows:

$$\begin{aligned}
Y_{111} &= -\frac{1}{2}\frac{z_1(-4z_1+5-7z_2+12z_1z_2+2z_2^2-5z_1z_2^2+4z_1^2z_2^2)}{(1-z_2+z_1z_2^2)(-1+4z_1)^2}, \\
Y_{112} &= -\frac{1}{2}\frac{(1-2z_1-z_2+4z_1z_2-z_1z_2^2+2z_1^2z_2^2)}{(1-z_2+z_1z_2^2)(-1+4z_1)}, \\
Y_{122} &= -\frac{1}{2}\frac{(-1+z_2+z_1z_2^2)}{(1-z_2+z_1z_2^2)}, \\
Y_{222} &= \frac{1-z_1z_2^2}{1-z_2+z_1z_2^2}.
\end{aligned} \tag{4.36}$$

These results show that the conjecture given in the previous section is compatible with Conjecture 1 in Section 1. Therefore, we can construct an extended PF system by using the strategy outlined in Section 2. For brevity, we introduce here the following notation:

$$M_a^\alpha(t_*) := \sum_b (Y_a^{-1}(t_*))^{\alpha b}\partial_a\partial_b\psi_0. \tag{4.37}$$

In the case of X_1, we have two integrability conditions given in (2.7):

$$M_1^1(t,s) = M_2^1(t,s), \quad M_1^2(t,s) = M_2^2(t,s), \tag{4.38}$$

where we use the subscript 1 and 2 for t and s. By explicit computation, these two conditions turn out to be the same, and translated into a differential equation in z_i variables by using (4.34) and (4.36), we obtain:

$$\big((1+z_2+z_1z_2^2)\mathcal{D}_1 + \mathcal{D}_2 + z_2\mathcal{D}_3\big)\psi_0 = 0. \tag{4.39}$$

Next, we consider the second integrability condition given in (2.8):

$$\partial_1 M_1^1(t,s) = \partial_2 M_2^2(t,s), \;\; \partial_2 M_1^1(t,s) = 0, \;\; \partial_1 M_2^2(t,s) = 0. \tag{4.40}$$

By explicit computation, we found that the second and the third conditions are translated into one rational differential equation:

$$\theta_2 \mathcal{D}_1 \psi_0 = 0. \tag{4.41}$$

The first condition is also translated into a rational differential equation but the result is very complicated. Now, we assert that (4.39) and (4.41) are a minimal set of extended PF operators for X_1. The reason is the following. Let us consider the large radius limit of (4.39) and (4.41);

$$(\theta_1^2 + \theta_1\theta_2 - \theta_2^2)\psi_0 = 0, \;\; (\theta_1^2\theta_2 - \theta_1\theta_2^2)\psi_0 = 0. \tag{4.42}$$

These conditions are equivalent to the relations of the classical cohomology ring of $\overline{X}_1$:

$$k_t^2 + k_t k_s - k_s^2 = 0, \;\; k_t^2 k_s - k_t k_s^2 = 0, \tag{4.43}$$

which reproduces the conjectured triple intersection numbers, up to an overall scaling. From this fact, we can see that (4.39) and (4.41) give us a complete set of relations for the classical cohomology ring of $\overline{X}_1$ at the large radius limit. Since the PF equations are nothing but the non-commutative version of the relations of the quantum cohomology ring of $\overline{X}_1$, which reduce to relations of classical cohomology at the large radius limit, [14], we can propose the following set of differential operators as an extended PF system:

$$\begin{aligned} \tilde{\mathcal{D}}_1 &= (1 + z_2 + z_1 z_2^2)\mathcal{D}_1 + \mathcal{D}_2 + z_2 \mathcal{D}_3, \\ \tilde{\mathcal{D}}_2 &= \theta_2 \mathcal{D}_1. \end{aligned} \tag{4.44}$$

We checked that the solution space of (4.44) is given by,

$$\langle\, 1,\; t,\; s,\; \frac{\partial \mathcal{F}}{\partial t},\; \frac{\partial \mathcal{F}}{\partial s},\; 2\mathcal{F} - t\frac{\partial \mathcal{F}}{\partial t} - s\frac{\partial \mathcal{F}}{\partial s} \,\rangle_{\mathbb{C}}. \tag{4.45}$$

Of course, we can derive the B-model Yukawa couplings (4.36) by using (4.44) as the starting point. An explicit example of this kind of computation will be given in Section 6 of this paper.

We can also construct an extended PF system of X_2 in the same way as X_1. Here, we briefly present the data of this construction. The starting point is the A-model Yukawa couplings:

$$\begin{aligned} Y_{111} &:= \frac{e^{s_1}}{1-e^{s_1}} - \frac{e^{s_1+s_2}}{1-e^{s_1+s_2}}, \\ Y_{112} &:= -\frac{1}{2} - \frac{e^{s_1+s_2}}{1-e^{s_1+s_2}} \\ Y_{122} &:= -\frac{1}{2} - \frac{e^{s_1+s_2}}{1-e^{s_1+s_2}}, \\ Y_{222} &:= \frac{e^{s_2}}{1-e^{s_2}} - \frac{e^{s_1+s_2}}{1-e^{s_1+s_2}}, \end{aligned} \tag{4.46}$$

and the ordinary PF operators:

$$\begin{aligned}
\mathcal{D}_1 &:= \theta_1^2 - \theta_1\theta_2 - z_1(\theta_1+\theta_2)(\theta_1-\theta_2), \\
\mathcal{D}_2 &:= \theta_2^2 - \theta_1\theta_2 - z_2(\theta_2+\theta_1)(\theta_2-\theta_1), \\
\mathcal{D}_3 &:= \theta_1\theta_2 - z_1z_2(\theta_1+\theta_2+1)(\theta_1+\theta_2).
\end{aligned} \tag{4.47}$$

Let us introduce the logarithm of the B model coordinates z_i.

$$u_1 = \log(z_1), \quad u_2 = \log(z_2). \tag{4.48}$$

By solving (4.47), we obtain the mirror maps $s_1 = s_1(z_1, z_2)$ and $s_2 = s_2(z_1, z_2)$ and their Jacobian:

$$\begin{aligned}
\frac{\partial s_1}{\partial u_1} &= \frac{1}{2}\frac{(-\sqrt{1-4z_1z_2}-1+4z_1z_2)}{(4z_1z_2-1)}, \quad \frac{\partial s_1}{\partial u_2} = -\frac{1}{2}\frac{(-1+4z_1z_2+\sqrt{1-4z_1z_2})}{(4z_1z_2-1)}, \\
\frac{\partial s_2}{\partial u_1} &= -\frac{1}{2}\frac{(-1+4z_1z_2+\sqrt{1-4z_1z_2})}{(4z_1z_2-1)}, \quad \frac{\partial s_2}{\partial u_2} = \frac{1}{2}\frac{(-\sqrt{1-4z_1z_2}-1+4z_1z_2)}{(4z_1z_2-1)}.
\end{aligned} \tag{4.49}$$

With this data, we can compute the B model Yukawa couplings in u^i:

$$\begin{aligned}
Y_{111} &= \frac{1}{2}z_1\frac{(5z_1z_2-2-12z_1z_2^2+7z_2+4z_2^3z_1-5z_2^2-4z_1^2z_2^2)}{(z_2-1+z_1)(4z_1z_2-1)^2}, \\
Y_{112} &= \frac{1}{2}\frac{(1-z_1-z_2-z_1z_2-z_2z_1^2+z_1z_2^2+4z_1^2z_2^2+4z_2^2z_1^3-4z_2^3z_1^2)}{(z_2-1+z_1)(4z_1z_2-1)^2}, \\
Y_{122} &= \frac{1}{2}\frac{(1-z_2-z_1-z_2z_1-z_1z_2^2+z_2z_1^2+4z_2^2z_1^2+4z_1^2z_2^3-4z_1^3z_2^2)}{(z_1-1+z_2)(4z_2z_1-1)^2}, \\
Y_{222} &= \frac{1}{2}z_2\frac{(5z_2z_1-2-12z_2z_1^2+7z_1+4z_1^3z_2-5z_1^2-4z_2^2z_1^2)}{(z_1-1+z_2)(4z_2z_1-1)^2},
\end{aligned} \tag{4.50}$$

and they turn out to be rational functions in z_i whose denominators are divisors of defining equation of discriminant locus of $\hat{X}_2$:

$$dis(\hat{X}_2) = (1-z_1-z_2)(1-4z_1z_2). \tag{4.51}$$

The derivation of the extended PF system by using the recipe in Section 2 proceeds in the same way as X_1. In this case, we only have to consider

$$M_1^1(s_1,s_2) = M_2^1(s_1,s_2), \quad M_1^2(s_1,s_2) = M_2^2(s_1,s_2), \tag{4.52}$$

and

$$\partial_1 M_1^1(s_1,s_2) = \partial_2 M_2^2(s_1,s_2), \quad \partial_2 M_1^1(s_1,s_2) = 0, \quad \partial_1 M_2^2(s_1,s_2) = 0. \tag{4.53}$$

(4.52) gives us one differential equation for ψ_0 with rational function coefficients in z_i:

$$\big(\mathcal{D}_1 + \mathcal{D}_2 + (1+z_1+z_2)\mathcal{D}_3\big)\psi_0 = 0. \tag{4.54}$$

As for (4.53), the second and the third conditions give us a differential equations for ψ_0:

$$(\theta_1 - \theta_2)\mathcal{D}_3\psi_0 = 0, \tag{4.55}$$

and the first one gives us a complicated rational differential equation. For the same reasoning as X_1, we can propose an extended PF system for X_2 as follows:

$$\begin{aligned} \tilde{\mathcal{D}}_1 &:= \mathcal{D}_1 + \mathcal{D}_2 + (1 + z_1 + z_2)\mathcal{D}_3, \\ \tilde{\mathcal{D}}_2 &:= (\theta_1 - \theta_2)\mathcal{D}_3. \end{aligned} \tag{4.56}$$

We have also constructed an extended PF system for a three parameter space X_3, in order to further test the conjecture made in the previous section. Specifically, X_3 satisfies $\dim H_2(X, \mathbb{Z}) = 3$, $\dim H_4(X, \mathbb{Z}) = 0$, and is defined by the following vectors:

$$\begin{pmatrix} l^1 \\ l^2 \\ l^3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}. \tag{4.57}$$

The results are collected in Appendix A.

5 Adding open strings to $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$.

So far, we have been able to demonstrate the existence of new differential operators which determine mirror symmetry for noncompact toric Calabi-Yau manifolds which have no 4 cycle. One may also wonder what the applications are to the PF system derived for open mirror symmetry [24][10]. We will see that again, some modification of open string PF operators is necessary.

5.1 Review of open string geometry.

Recall [2][10] that open string mirror symmetry is a local isomorphism of moduli spaces $(\mathcal{X}, \mathcal{L})$ and $(\hat{\mathcal{X}}, \mathcal{C})$; X and $\hat{X}$ are Calabi-Yau manifolds which are mirror in the usual sense, and $L \subset X$ is Lagrangian, while $C \subset \hat{X}$ is holomorphic. In the case at hand, $(\mathcal{X}, \mathcal{L})$ will be given by

$$X_r = \{(w_1, \ldots, w_4) \in \mathbb{C}^4 - Z : |w_1|^2 + |w_2|^2 - |w_3|^2 - |w_4|^2 = r\}/S^1,$$

together with either

$$L_{r,c} = X_r \cap \{|w_2|^2 - |w_4|^2 = c,\ |w_3|^2 - |w_4|^2 = 0,\ \sum_i \arg(w_i) = 0\} \tag{5.58}$$

or

$$L'_{r,c} = X_r \cap \{|w_2|^2 - |w_4|^2 = 0,\ |w_3|^2 - |w_4|^2 = c,\ \sum_i \arg(w_i) = 0\}. \tag{5.59}$$

There are then local moduli space isomorphisms $(\mathcal{X}, \mathcal{L}) \cong (\hat{\mathcal{X}}, \mathcal{C})$, $(\mathcal{X}, \mathcal{L}') \cong (\hat{\mathcal{X}}, \mathcal{C}')$, where

$$\hat{X}_{z_1} = \{(u, v, y_1, y_2) \in \mathbb{C}^2 \times (\mathbb{C}^*)^2 : uv + 1 + y_1 + y_2 + z_1 y_1 y_2^{-1} = 0\},$$

$$C_{z_1,z_2} = \hat{X}_{z_1} \cap \{y_2^{-1}y_1 = z_2,\ y_1 = 1\}, \tag{5.60}$$

$$C'_{z_1,z_2} = \hat{X}_{z_1} \cap \{y_2^{-1}y_1 = 1,\ y_1 = z_2\}. \tag{5.61}$$

The detailed derivation of these spaces is given in [2]. One mathematical implication of open string mirror symmetry is that the geometry of $(\hat{X}, C)$ should determine the genus 0 open Gromov-Witten invariants of (X, L). This means that there should be functions defined on the moduli space $(\hat{\mathcal{X}}, \mathcal{C})$ which count holomorphic maps $f : D = \{z \in \mathbb{C} : |z| \leq 1\} \to X$ such that $f(\partial D) \subset L$. Furthermore, in many cases such functions can be derived from an open string Picard-Fuchs system on $(\hat{X}, C)$. However, for the case at hand, it will be shown that the same sort of modification of the PF system, proposed for ordinary closed mirror symmetry, is also necessary in the open string setting.

We turn to the PF system on the moduli spaces $(\hat{\mathcal{X}}, \mathcal{C})$ and $(\hat{\mathcal{X}}, \mathcal{C}')$.

5.2 Moduli space and Picard-Fuchs system for $(\hat{\mathcal{X}}, \mathcal{C})$.

From [10], we can define "open period integrals" on $(\hat{\mathcal{X}}, \mathcal{C})$ by

$$\Pi_\Gamma(z_1, z_2) = \int_\Gamma \frac{dudvdy_1dy_2/(y_1y_2)}{(uv + 1 + y_1 + y_2 + z_1y_1y_2^{-1})(y_1 - z_2y_2)(y_1 - 1)}, \tag{5.62}$$

$\Gamma \in H_4(\mathbb{C}^2 \times (\mathbb{C}^*)^2 - \hat{X}, \hat{X} - C, \mathbb{Z})$. Then the derivations in [10] lead to a Picard-Fuchs system for $(\hat{\mathcal{X}}, \mathcal{C})$, given as

$$\mathcal{D}_1 = \theta_1(\theta_1 + \theta_2) - z_1\theta_1(\theta_1 + \theta_2), \tag{5.63}$$

$$\mathcal{D}_2 = \theta_2(\theta_1 + \theta_2) - z_2\theta_2(\theta_1 + \theta_2),$$

$$\theta_i = z_i\frac{d}{dz_i};$$

these operators satisfy $\mathcal{D}_i\Pi_\Gamma(z_1, z_2) = 0$. This is exactly the noncompact PF system of the vectors $l^1 = (1, 1, -1, -1, 0, 0)$, $l^2 = (0, 1, 0, -1, 1, -1)$, and agrees with the results of [24].

As was shown in [5], the solution space of $\{\mathcal{D}_1, \mathcal{D}_2\}$ can be obtained, using the Frobenius method, from a function

$$\omega_0(z, \rho) = \sum_{n\geq 0} c(n, \rho)z_1^{n_1+\rho_1}z_2^{n_2+\rho_2},$$

where

$$c(n, \rho)^{-1} = \Gamma(1 + n_1 + \rho_1)\Gamma(1 + n_1 + \rho_1 + n_2 + \rho_2)\Gamma(1 - n_1 - \rho_1)*$$
$$\Gamma(1 - n_1 - \rho_1 - n_2 - \rho_2)\Gamma(1 + n_2 + \rho_2)\Gamma(1 - n_2 - \rho_2).$$

According to [25], the solutions are expected to be

$$(1, t_1, t_2, W_1, W_2, \dots).$$

t_1 and t_2 give the open string mirror map, and this is trivial for the present example, so we have $t_i(z) = log(z_i)$. Also, the W_i count discs on (X, L).

Upon looking at the equations of (X, L), we can make the following geometric observation about a map $f : D \to X$ with $f(\partial D) \subset L$. In the region where $w_3 = w_4 = 0$, L will intersect the $\mathbb{P}^1$ of X; hence, such an f must obey $f(D) \subset \mathbb{P}^1$. Then the natural interpretation of the variable z_2 is as a parameter controlling the size of a holomorphic disc $D \hookrightarrow X$. It is therefore expected that one of the double logarithmic solutions of (5.63) will look like

$$W_1(z_2) = \sum_{n>0} \frac{z_2^n}{n^2}, \tag{5.64}$$

where the log terms have been disregarded due to ambiguity [2]. And indeed, it is the case that

$$W_1(z_2) = (\partial^2_{\rho_2}\omega_0)|_{\rho=0}.$$

The problem, though, is that $(\partial^2_{\rho_2}\omega_0)|_{\rho=0}$ is not a solution of (5.63). The easiest way to see this is to note that W_1 is independent of z_1, and (5.63) reduces to $\mathcal{D}_2 = (1 - z_2)\theta_2^2$ if $z_1 = 0$.

The minimal resolution of this issue, which continues in the spirit of raising the power of PF operators, is to instead work with the system

$$\{\mathcal{D}_1, \theta_2\mathcal{D}_2\}. \tag{5.65}$$

W_1 is indeed a solution of these higher order operators.

A lingering difficulty, even after such a PF extension, is that (as will be shown below) there is expected to be another disc counting function

$$W_2(z_1, z_2) = \sum_{n>0} \frac{(z_1/z_2)^n}{n^2},$$

which measures the size of the "other disc"; that is, since L splits $\mathbb{P}^1$ into two discs, there should be a function corresponding to each disc. Functions with essential singularities such as W_2 are not allowed to be Picard-Fuchs solutions.

One way around this is to instead use the vectors $l^1, -l^2$ to write down the noncompact PF system. Then it is easy to see that the resulting extended system

$$\{\tilde{\mathcal{D}}_1, \theta_2\tilde{\mathcal{D}}_2\}$$

will have solutions

$$\tilde{W}_1(z_2) = \sum_{n>0} \frac{z_2^n}{n^2}, \qquad \tilde{W}_2(z_1, z_2) = \sum_{n>0} \frac{(z_1 z_2)^n}{n^2}. \tag{5.66}$$

These are the same as found in [26].

Similarly, we can perform calculations on the family $(\hat{X}, C')$. This moduli space is given by vectors $k^1 = l^1,\ k^2 = (0, 0, 1, -1, 1, -1)$, and the open string PF system we arrive at is

$$\mathcal{D}_1' = \theta_1^2 - z_1(\theta_1 - \theta_2)(\theta_1 + \theta_2),$$

$$\mathcal{D}_2' = (\theta_2 - \theta_1)\theta_2 - z_2(\theta_1 + \theta_2)\theta_2.$$

Again, analogously to the above, let ω_0' be the generator of solutions of $\{\mathcal{D}_1', \mathcal{D}_2'\}$. Then there is a disc counting function

$$\partial^2_{\rho_2}\omega_0'|_{\rho=0} = W'(z_1, z_2) = \sum_{n_2>n_1\geq 0} \frac{(-1)^{n_1}(n_1+n_2-1)!}{(n_1!)^2(n_2-n_1)!n_2} z_1^{n_1} z_2^{n_2} \tag{5.67}$$

which agrees with the result of [2]. Yet, once again we have the problem of this not being a solution of the given PF system; the same modification gives the system

$$\{\mathcal{D}_1', \theta_2\mathcal{D}_2'\}.$$

W' is indeed among the solutions of this.

The moral of this discussion is that the open string PF system, constructed in [24], is also incomplete in certain cases. Though the geometric meaning of raising the power of operators is less clear this time, we find that the same techniques are effective in open and closed string calculations. Moreover, the extension $\mathcal{D}_2 \to \theta_2\mathcal{D}_2$ (rather than $\mathcal{D}_1 \to \theta_1\mathcal{D}_1$) is the natural one. This follows because z_2 is the open string variable, and $(\partial^2_{\rho_2}\omega_0)|_{\rho=0}$ counts discs; hence, we must assure that the second partial derivative in ρ_2 is a solution of the system.

Next, we will give an open string period integral definition for these degenerate situations.

5.3 Period integrals for $(\hat{\mathcal{X}}, \mathcal{C})$.

So far, it has been seen that the open PF system found in [24], which was later shown to be derived from a set of period integrals [10], does not always give the disc-counting functions one is interested in. Since local mirror symmetry on $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$yields an incomplete PF system, it is not so surprising that open strings on this same space should exhibit a similar failing.

Hence, we need a definition of open string period integrals. For motivation, let's review some geometric facts about $(\hat{\mathcal{X}}, \mathcal{C})$. Let y be a local coordinate on Σ. Then following [2], we can think of the curve C as

$$C_{z_1,z_2} = \hat{X}_{z_1} \cap \{v = 0, y = z_2\} = \mathbb{C} \times \{z_2 \in \Sigma\}.$$

Then the coordinate on $C \cong \mathbb{C}$ is u, and z_2 parameterizes a family of curves in Σ.

Earlier, it was noted that the problem of period integrals was reducible to that of integrals on Σ. Here it is beneficial to make the same simplification. Notice that, when projected to Σ, the family of curves $\{C_{z_1,y} | y \in [z, z_2]\}$ becomes a real curve connecting z to z_2. Hence, the sensible extension of Definition 1 to open strings is

Definition 3 *Let $\hat{X}, \Sigma$ be as given in Definition 1, and C as above. Choose $z, z_2 \in \Sigma$ and $\hat{\gamma} \in H_1(\Sigma, \{z, z_2\}, \mathbb{Z})$. Then the* open period integrals *of $(\hat{\mathcal{X}}, \mathcal{C})$ are defined to be*

$$W(z_1, z_2) = \int_{\hat{\gamma}} \mathrm{Res}_{f=0}\Big(\frac{dy_1 dy_2/(y_1 y_2)}{f(z_1, y_1, y_2)}\Big).$$

For the purposes of the definition, z is considered to be fixed on Σ, and z_2 is taken as a parameter. In the local $\mathbb{P}^1$ example, the relevant curves $\hat{\gamma}$ are shown in Figure 5. We now move on to the

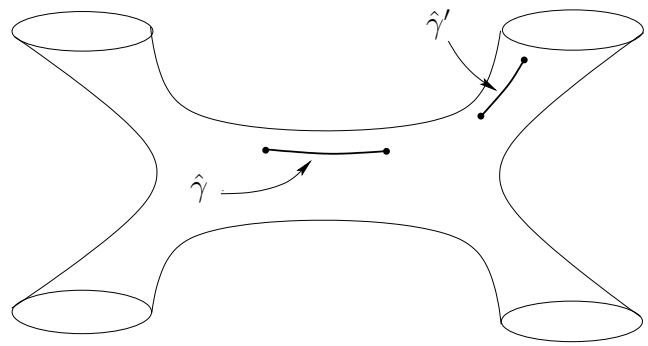

Figure 5: Real curves defining open string periods on Σ.

evaluation of these, and show agreement with the solutions of the proposed extended open string PF system of the last section.

There are two integrals, associated to the curves of the figure, and their calculation proceeds as follows.

$$W(z_1, z_2) = \int_{\hat{\gamma}} \log y_1 \frac{dy_2}{y_2} = \int_z^{z_2} \log\Big(\frac{-1-y_2}{1+z_1 y_2^{-1}}\Big)\frac{dy_2}{y_2} = \sum_{n>0} \frac{(-z_2)^n}{n^2} - \sum_{n>0} \frac{(-z_2^{-1}z_1)^n}{n^2},$$

which is what we saw in the previous section from the PF system (after the rotation $y_2 \to e^{i\pi} y_2$). If one is so inclined, the functions of (5.66) can be reproduced in the same way; this simply amounts to a change of the coordinate choices made in the mirror construction.

We also find

$$W'(z_1, z_2) = \int_{\hat{\gamma}'} \log y_2 \frac{dy_1}{y_1} = \int_z^{z_2} \log\Big(\frac{-1-y_1-\sqrt{(1+y_1)^2-4z_1y_1}}{2}\Big)\frac{dy_1}{y_1}.$$

This integral is more difficult to directly evaluate, but as in [2] we can simply note that

$$z_2 \frac{d}{dz_2} W' = \log\Big(\frac{-1-z_2-\sqrt{(1+z_2)^2-4z_1z_2}}{2}\Big).$$

After Taylor expanding about $z_1 = 0$ and integrating the result in z_2, this matches (5.67).

6 More general local geometries.

We have come a long way toward a more complete picture of the differential equations governing local mirror symmetry. However, we have yet to test these ideas in the domain of applicability of [5]; namely, local Calabi-Yau manifolds K_S, where S is a Fano surface and K_S is its canonical bundle. The system of differential operators as given in [5] cannot be complete, since it is not possible to fix the prepotential simply by analyzing the solution set of the system. We will find that our techniques still apply, though a unified treatment for all examples is less clear.

6.1 One 1-parameter example.

The simplest, though rather trivial, example is $K_{\mathbb{P}^2} = \mathcal{O}(-3) \to \mathbb{P}^2$. This can be defined as a symplectic quotient

$$K_{\mathbb{P}^2} = \{(w_1, \ldots, w_4) \in \mathbb{C}^4 - Z : |w_1|^2 + |w_2|^2 + |w_3|^2 - 3|w_4|^2 = r\}/S^1,$$

where $Z = \{w_1 = w_2 = w_3 = 0\}$, $r \in \mathbb{R}^+$ and the S^1 action is given as

$$(w_1, \ldots, w_4) \to (e^{i\theta} w_1, e^{i\theta} w_2, e^{i\theta} w_3, e^{-3i\theta} w_4).$$

The original paper [5] associates a Picard-Fuchs system here, which is ultimately derivable from the period integrals of the mirror

$$\hat{X}_z = \{(u, v, y_1, y_2) \in \mathbb{C}^2 \times (\mathbb{C}^*)^2 : uv + 1 + y_1 + y_2 + z y_1^{-1} y_2^{-1} = 0\}.$$

Again, from [17]:

$$\Pi_\Gamma(z) = \int_\Gamma \frac{du dv dy_1 dy_2/(y_1 y_2)}{uv + 1 + y_1 + y_2 + z y_1^{-1} y_2^{-1}}$$

for $\Gamma \in H_4(\mathbb{C}^2 \times (\mathbb{C}^*)^2 - \hat{X}, \mathbb{Z})$ are the period integrals. Then we immediately recover the well known PF operator

$$\mathcal{D} = \theta^3 + z(3\theta)(3\theta + 1)(3\theta + 2), \quad \theta = z\frac{d}{dz},$$

whose solution space is generated by a function $\omega_0(z, \rho) = \sum_{n \geq 0} c(n, \rho) z^{n+\rho}$. Here, the coefficients can be written

$$c(n, \rho) = (\Gamma(1 - 3n - 3\rho)\Gamma(1 + n + \rho)^3)^{-1}.$$

If we write the solutions in the variable $t = \partial_\rho \omega_0|_{\rho=0}$, we get

$$\Pi = (1, t, \partial\mathcal{F}/\partial t)$$

Naturally, this implies that in the t variable, it must be the case that

$$\mathcal{D} = \partial_t \Big(\frac{\partial^3 \mathcal{F}}{\partial t^3}\Big)^{-1} \partial_t^2.$$

Then, we can again give a 'compactified' operator

$$\partial_t \mathcal{D}$$

which possesses a completed set of solutions. It is actually equivalent to just work with

$$\theta\mathcal{D} = \theta(\theta^3 + z(3\theta)(3\theta + 1)(3\theta + 2))$$

on account of the invertibility of the Jacobian. And this is our new, extended PF operator.

6.2 Completing mirror symmetry for Hirzebruch surfaces.

One parameter spaces of type K_S have already been exhausted, by the $K_{\mathbb{P}^2}$ case. We will now turn to the two parameter spaces, namely the canonical bundle over the Hirzebruch surfaces F_0, F_1, F_2. As is well-known [5], the instanton part of the double log solution of the standard PF system is given by a linear combination of the $\frac{\partial \mathcal{F}_{inst.}}{\partial t_a}$. This fact tells us that the standard PF system of K_S already includes the information coming from $Y_a^{-1}(t_*)$ in (2.6). Therefore, we can take a short cut in the process of constructing an extended PF system on K_S. Examples of this explicit construction will be given in the next subsection.

The symplectic quotient description is given by $K_{F_n} =$

$$\{-2|w_1|^2 + |w_2|^2 + |w_3|^2 = r_1^n,\ (-2+n)|w_1|^2 - n|w_2| + |w_4|^2 + |w_5|^2 = r_2^n\}/(S^1)^2$$

where $(w_1, \ldots, w_5) \in \mathbb{C}^5 - Z_n$. That is, the vectors in the secondary fan are

$$\begin{pmatrix} l_n^1 \\ l_n^2 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -2+n & -n & 0 & 1 & 1 \end{pmatrix}.$$

The methods of [5] lead to PF operators:

K_{F_0} :

$$\mathcal{D}_1^0 = \theta_1^2 - z_1(2\theta_1 + 2\theta_2)(2\theta_1 + 2\theta_2 + 1),$$
$$\mathcal{D}_2^0 = \theta_2^2 - z_2(2\theta_1 + 2\theta_2)(2\theta_1 + 2\theta_2 + 1)$$

K_{F_1} :

$$\mathcal{D}_1^1 = \theta_1(\theta_1 - \theta_2) - z_1(2\theta_1 + \theta_2)(2\theta_1 + \theta_2 + 1),$$
$$\mathcal{D}_2^1 = \theta_2^2 - z_2(2\theta_1 + \theta_2)(\theta_1 - \theta_2)$$

K_{F_2} :

$$\mathcal{D}_1^2 = \theta_1(\theta_1 - 2\theta_2) - z_1 2\theta_1(2\theta_1 + 1),$$
$$\mathcal{D}_2^2 = \theta_2^2 - z_2(2\theta_2 - \theta_1)(2\theta_2 - \theta_1 + 1).$$

For each respective system, we let t_1^n, t_2^n be the logarithmic solutions. Each case comes equipped with a single double log solution W_n. If ω^n is the generating function of solutions on K_{F_n} and $\Pi_{ij}^n = \partial_{\rho_i}\partial_{\rho_j}\omega^n|_{\rho=0}$, then we can write these as

$$W_0 = \Pi_{12}^0,\ W_1 = \Pi_{11}^1 + 2\Pi_{12}^1,\ W_2 = \Pi_{11}^2 + \Pi_{12}^2.$$

Taking $\mathcal{F}_n$ for the prepotential on K_{F_n}, we have the following equalities:

$$W_n = 2\frac{\partial \mathcal{F}_n}{\partial t_1^n} + (2-n)\frac{\partial \mathcal{F}_n}{\partial t_2^n}.$$

By a comparison of power series, we can demonstrate that the Π_{ij}^n contain all the information necessary to derive the (instanton part of) the prepotential on K_{F_n}. We find

$$\begin{pmatrix} \Pi_{11}^n \\ \Pi_{12}^n \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 2 & -3n+2 \end{pmatrix} \begin{pmatrix} \partial \mathcal{F}_n/\partial t_1^n \\ \partial \mathcal{F}_n/\partial t_2^n \end{pmatrix}. \tag{6.68}$$

The equality above holds at the level of instanton parts of $\mathcal{F}_n$. We will now investigate the classical terms of these prepotentials.

6.3 B-model Yukawa Coupling of K_{F_n}

First, we give here the discriminant locus of K_{F_n} where the corresponding mirror hypersurface $\hat{K}_{F_n}$ becomes singular:

$$\begin{aligned} dis(\hat{\mathbf{K}}_{\mathbf{F}_0}) &= 1-8(z_1+z_2)+16(z_1-z_2)^2, && (6.69)\\ dis(\hat{\mathbf{K}}_{\mathbf{F}_1}) &= (1-4z_1)^2-z_2+36z_1z_2-27z_1z_2^2, && (6.70)\\ dis(\hat{\mathbf{K}}_{\mathbf{F}_2}) &= (1-4z_1)^2-64z_1^2z_2. && (6.71)\end{aligned}$$

With these results, M. Naka determined the B-model Yukawa couplings of K_{F_n} with respect to $u=\log(z_1), v=\log(z_2)$ variables by assuming compatibility with the instanton expansion given by the double-log solution, and that they should be written in terms of simple rational functions multiplied by $1/dis(\hat{\mathbf{K}}_{\mathbf{F}_n})$ [21]:

$n=0:$

$$Y_{uuu}=\frac{-4z_1^2+4z_2^2-4z_1-2z_2+\frac{1}{4}}{dis(\hat{\mathbf{K}}_{\mathbf{F}_0})},\quad Y_{uuv}=\frac{4z_1^2-4z_2^2+2z_2-\frac{1}{4}}{dis(\hat{\mathbf{K}}_{\mathbf{F}_0})},$$
$$Y_{uvv}=\frac{-4z_1^2+4z_2^2+2z_1-\frac{1}{4}}{dis(\hat{\mathbf{K}}_{\mathbf{F}_0})},\quad Y_{vvv}=\frac{4z_1^2-4z_2^2-2z_1-4z_2+\frac{1}{4}}{dis(\hat{\mathbf{K}}_{\mathbf{F}_0})},$$

$n=1:$

$$Y_{uuu}=$$
$$-\frac{((-162x+9)z_1z_2^2+(96x-4)z_1^2+(216x-14)z_1z_2+(5-48x)z_1-6x(z_2-1))}{dis(\hat{\mathbf{K}}_{\mathbf{F}_1})},$$
$$Y_{uuv}=$$
$$-\frac{((324x-18)z_1z_2^2+(8-192x)z_1^2+(25-432x)z_1z_2+(96x-6)z_1+(12x-1)(z_2-1))}{dis(\hat{\mathbf{K}}_{\mathbf{F}_1})},$$
$$Y_{uvv}=$$
$$-\frac{((36-648x)z_1z_2^2+(384x-16)z_1^2+(864x-44)z_1z_2+(8-192x)z_1-(24x-1)(z_2-1))}{dis(\hat{\mathbf{K}}_{\mathbf{F}_1})},$$
$$Y_{vvv}=$$
$$-\frac{((1296x-72)z_1z_2^2+(32-768x)z_1^2+(76-1728x)z_1z_2+(384x-16)z_1+(48x-2)(z_2-1)-z_2)}{dis(\hat{\mathbf{K}}_{\mathbf{F}_1})},$$

$n=2:$

$$Y_{uuu}=\frac{-1}{dis(\hat{\mathbf{K}}_{\mathbf{F}_2})},\quad Y_{uuv}=\frac{2z_1-\frac{1}{2}}{dis(\hat{\mathbf{K}}_{\mathbf{F}_2})},$$
$$Y_{uvv}=\frac{-z_2(8z_1-1)}{dis(\hat{\mathbf{K}}_{\mathbf{F}_2})(1-4z_2)},\quad Y_{vvv}=\frac{-z_2(24z_1z_2+2z_1-2z_2-\frac{1}{2})}{dis(\hat{\mathbf{K}}_{\mathbf{F}_2})(1-4z_2)^2}. \qquad (6.72)$$

In the $n=1$ case, there exists a moduli parameter x that leaves the instanton part of Y_{ijk} invariant.[1] In other words, we cannot determine the value of x from the compatibility of the instanton numbers. The aim of this section is to give a derivation of these Yukawa couplings by using an extended set Picard-Fuchs operators of local F_n as the starting point. First, we notice that (6.72) tells us of the existence of a natural classical triple intersection theory on K_{F_n} compatible with the instanton expansion:

$n=0:$

$$\langle k_u k_u k_u\rangle = \frac{1}{4},\ \ \langle k_u k_u k_v\rangle = -\frac{1}{4},\ \ \langle k_u k_v k_v\rangle = -\frac{1}{4},\ \ \langle k_v k_v k_v\rangle = \frac{1}{4},$$

$n=1:$

$$\langle k_u k_u k_u\rangle = -6x,\ \ \langle k_u k_u k_v\rangle = -1+12x,\ \ \langle k_u k_v k_v\rangle = -24x+1,\ \ \langle k_v k_v k_v\rangle = -2+48x,$$

$n=2:$

$$\langle k_u k_u k_u\rangle = -1,\ \ \langle k_u k_u k_v\rangle = -\frac{1}{2},\ \ \langle k_u k_v k_v\rangle = 0,\ \ \langle k_v k_v k_v\rangle = 0. \tag{6.73}$$

In (6.73), we denote the classical triple intersection numbers of Kähler forms $k_u,\ k_v$ by $\langle k_u k_u k_u\rangle$, etc. Therefore, we have to reproduce (6.73) from the information obtained from some extended PF system. The key idea of constructing such an extended system becomes more clear upon looking at the triple log series obtained from the generating hypergeometric series of the solution of the PF system:

$$w(u,v,r_1,r_2) := \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{\prod_{j=0}^{4}\Gamma(1+l_j^1(m+r_1)+l_j^2(n+r_2))}\exp((m+r_1)u+(n+r_2)v). \tag{6.74}$$

It is well-known that $w(u,v,r_1,r_2)|_{(r1,r2)=(0,0)}=1$ is the trivial solution of the PF system, and that the log solutions $\partial_{r_i} w(u,v,r_1,r_2)|_{(r1,r2)=(0,0)}$ gives us the mirror map t^i. Then we consider the relation between the triple log series $W_{ijk}(u,v) := \partial_{r_i}\partial_{r_j}\partial_{r_k} w(u,v,r_1,r_2)|_{(r1,r2)=(0,0)}$ and the prepotential $\mathcal{F}(t^1,t^2)$ of K_{F_n}. Surprisingly, the classical intersection number (6.73) is determined from the following assumption:

$$\begin{aligned}
&\frac{1}{6}\langle k_u k_u k_u\rangle\cdot W_{111}(u(t_*),v(t_*)) + \frac{1}{2}\langle k_u k_u k_v\rangle\cdot W_{112}(u(t_*),v(t_*))\\
&+\frac{1}{2}\langle k_u k_v k_v\rangle\cdot W_{122}(u(t_*),v(t_*)) + \frac{1}{6}\langle k_v k_v k_v\rangle\cdot W_{222}(u(t_*),v(t_*))\\
&= t_1\frac{\partial\mathcal{F}(t_*)}{\partial t_1} + t_2\frac{\partial\mathcal{F}(t_*)}{\partial t_2} - 2\mathcal{F}(t_*).
\end{aligned} \tag{6.75}$$

In the $n=0,2$ cases, we can determine classical intersection number (6.73) uniquely from the instanton expansion of the r.h.s. of (6.75), but in the $n=1$ case, we have one moduli parameter

[1] In the $n=0$ case, we also have one moduli parameter if we don't assume symmetry between u and v.

x that leaves the instanton expansion invariant. Therefore, this situation is the same as in Naka's result. With these intersection numbers, we can construct the set of relations of the classical cohomology ring of "compactified" K_{F_n} (which we denote by $\overline{K}_{F_n}$) as follows:

$$
\begin{aligned}
&n=0: \\
&\qquad k_u^2+k_v^2=0, \quad k_u k_v^2 - k_u^2 k_v = 0, \\
&n=1: \\
&\qquad (24x-1)(k_u^2-k_u k_v)-(18x-1)k_v^2=0, \quad 2k_u k_v^2+k_v^3=0, \quad (x\neq \frac{1}{24}) \\
&\qquad k_v^2=0, \quad 2k_u^3-k_u^2k_v=0, \quad (x=\frac{1}{24}) \\
&n=2: \\
&\qquad k_v^2=0, \quad k_u^3-2k_u^2k_v=0.
\end{aligned}
\tag{6.76}
$$

With this set up, we can construct an extended Picard-Fuchs system of $\overline{K}_{F_n}$ that has the same principle part as (6.76), and is constructed from the linear combination of $\mathcal{D}_i, \quad \theta_1\mathcal{D}_i, \ \theta_2\mathcal{D}_i$: $(i=1,2)$:

$$
\begin{aligned}
&n=0: \\
&\tilde{\mathcal{D}}_1^0=\mathcal{D}_1^0+\mathcal{D}_2^0, \quad \tilde{\mathcal{D}}_2^0=\theta_1\mathcal{D}_2^0-\theta_2\mathcal{D}_1^0, \\
&n=1: \\
&\tilde{\mathcal{D}}_1^1=(24x-1)\mathcal{D}_1^1-(18x-1)\mathcal{D}_2^1, \ \tilde{\mathcal{D}}_2^1=(2\theta_1+\theta_2)\mathcal{D}_2^1, \ (x\neq\frac{1}{24}) \\
&\tilde{\mathcal{D}}_1^1=\mathcal{D}_2^1, \ \tilde{\mathcal{D}}_2^1=(2\theta_1+\theta_2)\mathcal{D}_1^1, \ (x=\frac{1}{24}) \\
&n=2: \\
&\tilde{\mathcal{D}}_1^2=\mathcal{D}_2^2, \ \tilde{\mathcal{D}}_2^2=\theta_1\mathcal{D}_1^2.
\end{aligned}
\tag{6.77}
$$

In the remaining part of this section, we briefly discuss the derivation of the Yukawa coupling of F_0 in (6.72) by using (6.77) as the starting point. The other cases can be done in exactly the same way as in this computation. First, we use the standard definition of the B-model Yukawa coupling of mirror symmetry for a compact Calabi-Yau 3-fold:

$$
Y_{ijk}=\int_{\overline{\mathbf{K}}_{F_0}} \Omega\wedge\partial_i\partial_j\partial_k\Omega. \tag{6.78}
$$

In the case of $\overline{K}_{F_n}$, the existence of a global holomomorphic three form Ω is not guaranteed, but we proceed here by assuming the existence of such an Ω. We also apply the standard results obtained from Kodaira-Spencer theory on a compact Calabi-Yau 3-fold to the computation on $\overline{K}_{F_0}$. It is easy to show the following formula by application of this machinery:

$$
\int_M \Omega\wedge\partial_i\partial_j\partial_k\partial_l\Omega=\frac{1}{2}(\partial_i Y_{jkl}+\partial_j Y_{ikl}+\partial_k Y_{ijl}+\partial_l Y_{ijk}). \tag{6.79}
$$

Next, we derive two relations among different Yukawa couplings obtained from $\theta_1(\mathcal{D}_1^0+\mathcal{D}_2^0)\Omega = \theta_2(\mathcal{D}_1^0+\mathcal{D}_2^0)\Omega = 0$:

$$\begin{aligned} &Y_{uuu}+Y_{uvv}-4(z_1+z_2)(Y_{uuu}+2Y_{uuv}+Y_{uvv})=0,\\ &Y_{uuv}+Y_{vvv}-4(z_1+z_2)(Y_{uuv}+2Y_{uvv}+Y_{vvv})=0. \end{aligned} \tag{6.80}$$

We can also derive another relation from $(\theta_1\mathcal{D}_2^0-\theta_2\mathcal{D}_1^0)\Omega=0$:

$$Y_{uuv}-4z_1(Y_{uuv}+2Y_{uvv}+Y_{vvv}) = Y_{uvv}-4z_2(Y_{uuu}+2Y_{uuv}+Y_{uvv}). \tag{6.81}$$

Since (6.80) and (6.81) are linear relations, we can easily solve them and obtain,

$$\begin{aligned} Y_{uuu} &= (1-16z_1-8z_2-16z_1^2+16z_2^2)S(u,v),\\ Y_{uuv} &= (-1+8z_2+16z_1^2-16z_2^2)S(u,v),\\ Y_{uuv} &= (-1+8z_1+16z_2^2-16z_1^2)S(u,v),\\ Y_{vvv} &= (1-16z_2-8z_1-16z_2^2+16z_1^2)S(u,v). \end{aligned} \tag{6.82}$$

where $S(u,v)$ is an unknown function at this stage. Then we can derive differential equations of $S(u,v)$ from the relations $((\theta_1)^2-(\theta_2)^2)\mathcal{D}_2^0\Omega = ((\theta_1)^2-(\theta_2)^2)\mathcal{D}_1^0\Omega = 0$ and (6.79) by substituting the r.h.s. of (6.82). These operations result in the following differential equations of $S(u,v)$:

$$\begin{aligned} \frac{\partial_u S(u,v)}{S(u,v)} &= \frac{8z_1-32z_1^2+32z_1z_2}{1-8z_1-8z_2+16z_1^2-32z_1z_2+16z_2^2},\\ \frac{\partial_v S(u,v)}{S(u,v)} &= \frac{8z_2-32z_2^2+32z_1z_2}{1-8z_1-8z_2+16z_1^2-32z_1z_2+16z_2^2}. \end{aligned} \tag{6.83}$$

We can immediately solve the above equations and obtain,

$$S(u,v) = (const.)\cdot\frac{1}{1-8z_1-8z_2+16z_1^2-32z_1z_2+16z_2^2}. \tag{6.84}$$

Finally, the classical intersection numbers in (6.73) tell us that $(const.)=\frac{1}{4}$.

7 The del Pezzo surface K_{dP_2}.

We can also look to a three parameter model, in order to determine what we might expect in more general situations. The examples of the two parameter case might cause one to hope that, for every local geometry of the form K_S, we can extend the original PF system to give a complete description of mirror symmetry from the B model geometry alone. Here we will demonstrate that this is indeed the case for K_{dP_2}. However, it is no longer necessary to use a higher order system for three and higher parameter cases; we can find a complete set of solutions by "forgetting" about some of the originally proposed local mirror symmetry operators.

The symplectic quotient description of K_{dP_2} may be written as

$$\{(w_1,\ldots,w_6)\in\mathbb{C}^6-Z : \sum_{k=1}^{6} l_k^i|w_k|^2 = r_i,\ i=1,2,3\}/(S^1)^3$$

with the vectors

$$\begin{pmatrix} l^1 \\ l^2 \\ l^3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & -1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & -1 & 1 & 0 \end{pmatrix}.$$

Note that $b_2 = 3$ and $b_4 = 1$.

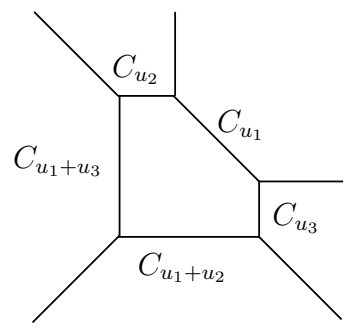

Figure 6: The del Pezzo dP_2. Each curve C_{u_i} corresponds to a vector l^i.

This comes with PF operators

$$\begin{aligned}
\mathcal{D}_1 &= (\theta_1 - \theta_2)(\theta_1 - \theta_3) - z_1(\theta_1 + \theta_2 + \theta_3)(\theta_1 - \theta_2 - \theta_3), \\
\mathcal{D}_2 &= (\theta_2 + \theta_3 - \theta_1)\theta_2 - z_2(\theta_1 + \theta_2 + \theta_3)(\theta_2 - \theta_1), \\
\mathcal{D}_3 &= (\theta_2 + \theta_3 - \theta_1)\theta_3 - z_3(\theta_1 + \theta_2 + \theta_3)(\theta_3 - \theta_1), \\
\mathcal{D}_4 &= \theta_2(\theta_1 - \theta_3) - z_1 z_2(\theta_1 + \theta_2 + \theta_3 + 1)(\theta_1 + \theta_2 + \theta_3), \\
\mathcal{D}_5 &= \theta_3(\theta_1 - \theta_2) - z_1 z_3(\theta_1 + \theta_2 + \theta_3 + 1)(\theta_1 + \theta_2 + \theta_3).
\end{aligned} \tag{7.85}$$

Let $\mathcal{F}$, t_1, t_2, t_3 and the Π_{ij}'s be as before. Then [5] provides a single double logarithmic solution W, corresponding to the four cycle in the base of the A model geometry, which satisfies

$$W = \Pi_{11} + \Pi_{12} + \Pi_{13} + \Pi_{23} = \frac{\partial \mathcal{F}}{\partial t_1} + \frac{\partial \mathcal{F}}{\partial t_2} + \frac{\partial \mathcal{F}}{\partial t_3}.$$

The naive approach in this case suggests that we need to add two 4 cycles to the A model geometry. Using as motivation the notion that each double logarithmic solution of the system should correspond to a 4 cycle on the mirror, we are led to work with a system consisting of 3 of the original 5 PF differential operators. For reasons to be discussed shortly, we will use the set

$$\begin{aligned}
\tilde{\mathcal{D}}_1 &= (6x + 2y - 1)(\mathcal{D}_1 - \mathcal{D}_2 - \mathcal{D}_3) - x(\mathcal{D}_2 + \mathcal{D}_3), \\
\tilde{\mathcal{D}}_2 &= (5x + y - 1)(\mathcal{D}_2 + \mathcal{D}_3 + \mathcal{D}_4 + \mathcal{D}_5) + (x + y)(\mathcal{D}_4 + \mathcal{D}_5), \\
\tilde{\mathcal{D}}_3 &= (x - y - 1)(\mathcal{D}_2 - \mathcal{D}_3 + \mathcal{D}_4 - \mathcal{D}_5) + (x - y)(\mathcal{D}_4 - \mathcal{D}_5),
\end{aligned} \tag{7.86}$$

where x and y are real free parameters. If we set $(x, y) = (0, 0)$, we can easily show that a basis of

the solution space with double logarithmic singularities is three dimensional, and is provided by

$$\Pi_{12} = \frac{\partial \mathcal{F}}{\partial t_2},$$
$$\Pi_{13} = \frac{\partial \mathcal{F}}{\partial t_3},$$
$$\Pi_{11} + \Pi_{23} = \frac{\partial \mathcal{F}}{\partial t_1}. \tag{7.87}$$

Hence, we can recover the full prepotential from solutions of the new PF system alone. We expect that this phenomenon will continue to hold true in all cases of the type K_S.

The extended PF system of K_{dP_2} (7.86) is derived in the same way as K_{F_n}. Let us introduce the logarithm of the standard B-model coordinates z_i:

$$u_1 = \log(z_1), \quad u_2 = \log(z_2), \quad u_3 = \log(z_3), \tag{7.88}$$

and consider the generating function of solutions of the PF system (7.85):

$$\begin{aligned} w(u_1, u_2, u_3, r_1, r_2, r_3) := \quad & \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} \frac{1}{\prod_{j=0}^{5}\Gamma(1 + l_j^1(n_1+r_1) + l_j^2(n_2+r_2) + l_j^3(n_3+r_3))} \times \\ & \times \exp((n_1+r_1)u_1 + (n_2+r_2)u_2 + (n_3+r_3)u_3). \end{aligned} \tag{7.89}$$

The mirror map of K_{dP_2} is given by

$$t_i(u_*) := \partial_{r_i} w(u_1, u_2, u_3, r_1, r_2, r_3)|_{r_i=0}. \tag{7.90}$$

From the triple log function

$$W_{ijk}(u_1, u_2, u_3) := \partial_{r_i}\partial_{r_2}\partial_{r_3} w(u_1, u_2, u_3, r_1, r_2, r_3)|_{r_i=0}, \tag{7.91}$$

obtained from (7.89), we can find the classical triple intersection number $\langle k_{u_i} k_{u_j} k_{u_m}\rangle$ of Kähler forms k_{u_i} by assuming the following relation:

$$\begin{aligned} & \frac{1}{6}\langle k_{u_1}k_{u_1}k_{u_1}\rangle \cdot W_{111}(u_1(t_*), u_2(t_*), u_3(t_*)) + \frac{1}{6}\langle k_{u_2}k_{u_2}k_{u_2}\rangle \cdot W_{222}(u_1(t_*), u_2(t_*), u_3(t_*)) + \\ & \frac{1}{6}\langle k_{u_3}k_{u_3}k_{u_3}\rangle \cdot W_{333}(u_1(t_*), u_2(t_*), u_3(t_*)) + \\ & \frac{1}{2}\langle k_{u_1}k_{u_1}k_{u_2}\rangle \cdot W_{112}(u_1(t_*), u_2(t_*), u_3(t_*)) + \frac{1}{2}\langle k_{u_1}k_{u_2}k_{u_2}\rangle \cdot W_{122}(u_1(t_*), u_2(t_*), u_3(t_*)) + \\ & \frac{1}{2}\langle k_{u_2}k_{u_2}k_{u_3}\rangle \cdot W_{223}(u_1(t_*), u_2(t_*), u_3(t_*)) + \frac{1}{2}\langle k_{u_2}k_{u_3}k_{u_3}\rangle \cdot W_{233}(u_1(t_*), u_2(t_*), u_3(t_*)) + \\ & \frac{1}{2}\langle k_{u_1}k_{u_1}k_{u_3}\rangle \cdot W_{113}(u_1(t_*), u_2(t_*), u_3(t_*)) + \frac{1}{2}\langle k_{u_1}k_{u_3}k_{u_3}\rangle \cdot W_{133}(u_1(t_*), u_2(t_*), u_3(t_*)) + \\ & \langle k_{u_1}k_{u_2}k_{u_3}\rangle \cdot W_{123}(u_1(t_*), u_2(t_*), u_3(t_*)) \\ & = t_1\frac{\partial \mathcal{F}(t_*)}{\partial t_1} + t_2\frac{\partial \mathcal{F}(t_*)}{\partial t_2} + t_3\frac{\partial \mathcal{F}(t_*)}{\partial t_3} - 2\mathcal{F}(t_*), \end{aligned} \tag{7.92}$$

where we used instanton expansion part of $\mathcal{F}(t_1, t_2, t_3)$ read off from double log solution of (7.85). Taking the symmetry between k_{u_2} and k_{u_3} into account, we found the following classical intersection numbers with two free parameters[2] x and y:

$$\begin{aligned}
&\langle k_{u_1}k_{u_1}k_{u_1}\rangle = -1+6x+2y, \quad \langle k_{u_2}k_{u_2}k_{u_2}\rangle = -y, \quad \langle k_{u_3}k_{u_3}k_{u_3}\rangle = -y, \quad \langle k_{u_1}k_{u_1}k_{u_2}\rangle = -3x-y,\\
&\langle k_{u_1}k_{u_2}k_{u_2}\rangle = x+y, \quad \langle k_{u_2}k_{u_2}k_{u_3}\rangle = -x, \quad \langle k_{u_2}k_{u_3}k_{u_3}\rangle = -x, \quad \langle k_{u_1}k_{u_1}k_{u_3}\rangle = -3x-y,\\
&\langle k_{u_1}k_{u_3}k_{u_3}\rangle = x+y, \quad \langle k_{u_1}k_{u_2}k_{u_3}\rangle = -1+2x.
\end{aligned} \tag{7.93}$$

With this data, we can construct a complete set of relations of k_{u^i} that reproduce (7.93) as follows:

$$\begin{aligned}
R_1 &= (6x+2y-1)(p_1-p_2-p_3) - x(p_2+p_3),\\
R_2 &= (5x+y-1)(p_2+p_3+p_4+p_5) + (x+y)(p_4+p_5),\\
R_3 &= (x-y-1)(p_2-p_3+p_4-p_5) + (x-y)(p_4-p_5),
\end{aligned} \tag{7.94}$$

where

$$\begin{aligned}
&p_1 = (k_{u_1}-k_{u_2})(k_{u_1}-k_{u_3}),\ p_2 = k_{u_2}(k_{u_2}+k_{u_3}-k_{u_1}),\ p_3 = k_{u_3}(k_{u_2}+k_{u_3}-k_{u_1}),\\
&p_4 = k_{u_2}(k_{u_1}-k_{u_3}),\ p_5 = k_{u_3}(k_{u_1}-k_{u_2}).
\end{aligned} \tag{7.95}$$

The extended PF system (7.86) is obtained from linear combinations of the $\mathcal{D}_i$'s that reduce to (7.94) at the large radius limit. Of course, we can compute the B-model Yukawa coupling of K_{dP_2} by using (7.86) as the starting point in the same way as F_0, and we collect the resulting Yukawa couplings in Appendix B.

8 Conclusion.

Through a variety of examples, we have seen the emergence of a new set of differential operators for local mirror symmetry. In this sense, we may view this as a next step towards a complete treatment of the program initiated in [5]. However, at present, we lack an understanding of how to choose classical intersection theory. Is there, in fact, a canonical way to associate triple intersection numbers on K_S, as we saw in the $\dim H_4(X,\mathbb{Z}) = 0$ cases? Also, although the methods presented here produce results consistent with physical expectations, it is unsatisfying to be without a better geometric understanding of the extended PF systems. It would be interesting to find compact spaces whose period integrals agree with the solutions of the extended PF systems. We leave these questions for future work.

[2]In this case, we have four moduli parameters unless we assume symmetry between k_{u_2} and k_{u_3}.

Appendix A : An extended Picard-Fuchs system for X_3.

Here, we apply our conjecture on intersection theory for X satisfying $\dim H_4(X,\mathbb{Z}) = 0$ to a three parameter example, in order to more fully explore its applicability. We will work with $X_3 =$

$$\{(w_1,\ldots,w_6) \in \mathbb{C}^6 - Z : \sum_{k=1}^{6} l_k^i |w_k|^2 = r_i,\ i = 1,2,3\}/(S^1)^3$$

where

$$\begin{pmatrix} l^1 \\ l^2 \\ l^3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}. \tag{8.96}$$

The toric graph is provided in figure 7. Then X_3 has no 4 cycle, and for each curve C_{s_i} corresponding to the secondary fan vector l^i, we have that $\mathcal{N}_{C_{s_i}/X_3} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. By making use of the

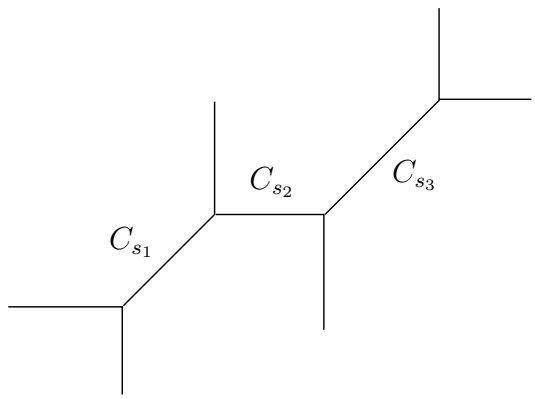

Figure 7: Toric graph for X_3.

instanton parts given in [18] and our conjecture, we immediately have the Yukawa couplings:
A-model Yukawa Couplings of X_3 (w.r.t. s_i)

$$\begin{aligned}
Y_{111} &= \frac{1}{2} + \frac{e^{s_1}}{1-e^{s_1}} - \frac{e^{s_1+s_2}}{1-e^{s_1+s_2}} + \frac{e^{s_1+s_2+s_3}}{1-e^{s_1+s_2+s_3}}, \\
Y_{113} &= \frac{1}{2} + \frac{e^{s_1+s_2+s_3}}{1-e^{s_1+s_2+s_3}}, \\
Y_{113} &= \frac{1}{2} + \frac{e^{s_1+s_2+s_3}}{1-e^{s_1+s_2+s_3}}, \\
Y_{333} &= \frac{1}{2} + \frac{e^{s_3}}{1-e^{s_3}} - \frac{e^{s_3+s_2}}{1-e^{s_3+s_2}} + \frac{e^{s_1+s_2+s_3}}{1-e^{s_1+s_2+s_3}},
\end{aligned}$$

$$
\begin{aligned}
Y_{112} &= -\frac{e^{s_1+s_2}}{1-e^{s_1+s_2}} + \frac{e^{s_1+s_2+s_3}}{1-e^{s_1+s_2+s_3}}, \\
Y_{233} &= -\frac{e^{s_3+s_2}}{1-e^{s_3+s_2}} + \frac{e^{s_1+s_2+s_3}}{1-e^{s_1+s_2+s_3}}, \\
Y_{122} &= -\frac{e^{s_1+s_2}}{1-e^{s_1+s_2}} + \frac{e^{s_1+s_2+s_3}}{1-e^{s_1+s_2+s_3}}, \\
Y_{223} &= -\frac{e^{s_3+s_2}}{1-e^{s_3+s_2}} + \frac{e^{s_1+s_2+s_3}}{1-e^{s_1+s_2+s_3}}, \\
Y_{222} &= \frac{e^{s_2}}{1-e^{s_2}} - \frac{e^{s_1+s_2}}{1-e^{s_1+s_2}} - \frac{e^{s_3+s_2}}{1-e^{s_3+s_2}} + \frac{e^{s_1+s_2+s_3}}{1-e^{s_1+s_2+s_3}}, \\
Y_{123} &= \frac{1}{2} + \frac{e^{s_1+s_2+s_3}}{1-e^{s_1+s_2+s_3}}.
\end{aligned} \tag{8.97}
$$

Once we have these, the rest of the calculations are routine. We list the results here.

Ordinary Picard-Fuchs system of X_3

$$
\begin{aligned}
&\mathcal{D}_1 = \theta_1(\theta_1-\theta_2-\theta_3) - z_1(\theta_1-\theta_2)(\theta_1+\theta_2-\theta_3), \\
&\mathcal{D}_2 = (\theta_2-\theta_1)(\theta_2-\theta_3) - z_2(\theta_2+\theta_3-\theta_1)(\theta_2+\theta_1-\theta_3), \\
&\mathcal{D}_3 = \theta_3(\theta_3-\theta_2-\theta_1) - z_3(\theta_3-\theta_2)(\theta_3+\theta_2-\theta_1), \\
&\mathcal{D}_4 = \theta_1(\theta_1-\theta_3) - z_1z_2(\theta_1+\theta_2-\theta_3+1)(\theta_1+\theta_2-\theta_3), \\
&\mathcal{D}_5 = \theta_1\theta_3 - z_1z_3(\theta_2-\theta_1)(\theta_2-\theta_3), \\
&\mathcal{D}_6 = \theta_3(\theta_3-\theta_1) - z_2z_3(\theta_3+\theta_2-\theta_1+1)(\theta_3+\theta_2-\theta_1).
\end{aligned} \tag{8.98}
$$

Jacobian of the Mirror Map

$$
\begin{aligned}
&u_1 = \log(z_1), \quad u_2 = \log(z_2), \quad u_3 = \log(z_3). \\
&\frac{\partial s_1}{\partial u_1} = \frac{1}{2}\frac{-\sqrt{1-4z_1z_2}-1+4z_1z_2}{4z_1z_2-1}, \\
&\frac{\partial s_1}{\partial u_2} = \frac{1}{2}\frac{4z_1z_2\sqrt{1-4z_2z_3}+\sqrt{1-4z_1z_2}-\sqrt{1-4z_2z_3}-4z_2z_3\sqrt{1-4z_1z_2}}{(4z_1z_2-1)(4z_2z_3-1)}, \\
&\frac{\partial s_1}{\partial u_3} = \frac{1}{2}\frac{-1+4z_2z_3+\sqrt{1-4z_2z_3}}{4z_2z_3-1}, \\
&\frac{\partial s_2}{\partial u_1} = -\frac{1}{2}\frac{-1+4z_1z_2+\sqrt{1-4z_1z_2}}{4z_1z_2-1},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial s_2}{\partial u_2} &= -\frac{1}{2}\frac{-\sqrt{1-4z_1z_2}+4z_1z_2\sqrt{1-4z_2z_3}-\sqrt{1-4z_2z_3}+4z_2z_3\sqrt{1-4z_1z_2}}{(4z_1z_2-1)(4z_2z_3-1)},\\
\frac{\partial s_2}{\partial u_3} &= -\frac{1}{2}\frac{-1+4z_2z_3+\sqrt{1-4z_2z_3}}{4z_2z_3-1},\\
\frac{\partial s_3}{\partial u_1} &= \frac{1}{2}\frac{-1+4z_1z_2+\sqrt{1-4z_1z_2}}{4z_1z_2-1},\\
\frac{\partial s_3}{\partial u_2} &= \frac{1}{2}\frac{\sqrt{1-4z_2z_3}+4z_2z_3\sqrt{1-4z_1z_2}-4z_1z_2\sqrt{1-4z_2z_3}-\sqrt{1-4z_1z_2}}{(4z_1z_2-1)(4z_2z_3-1)},\\
\frac{\partial s_3}{\partial u_3} &= \frac{1}{2}\frac{-\sqrt{1-4z_2z_3}-1+4z_2z_3}{4z_2z_3-1}.
\end{aligned}
\tag{8.99}
$$

B-model Yukawa Couplings of X_3 (w.r.t u_i)

$$
\begin{aligned}
Y_{111} &= (8z_2^3z_1^4z_3^2-10z_3^2z_1^3z_2^2+3z_1^2z_2z_3^2+8z_3z_1^3z_2^2-24z_3z_1^2z_2^2-2z_3z_2z_1^2+12z_3z_1z_2-z_3-\\
&\quad z_3z_1-8z_1^2z_2^3+16z_1^2z_2^2+10z_1z_2^2-z_2-12z_1z_2+1+z_1)/(\Delta(4z_1z_2-1)^2),\\
Y_{113} &= (2z_3^2z_1^3z_2^2-z_1^2z_2z_3^2+4z_3z_2z_1^2-4z_3z_1z_2-z_3z_1+z_3-2z_1z_2^2-2z_2z_1^2+2z_1z_2+z_2+z_1-1)\\
&\quad /(\Delta(4z_1z_2-1)),\\
Y_{133} &= (2z_1^2z_2^2z_3^3-2z_3z_2^2-z_1^2z_2z_3^2+4z_3^2z_2z_1-4z_3z_1z_2+z_2+2z_2z_3-2z_3^2z_2-z_3z_1+z_1+z_3-1)\\
&\quad /(\Delta(4z_2z_3-1)),\\
Y_{333} &= (8z_3^4z_2^3z_1^2-8z_3^2z_2^3-10z_1^2z_2^2z_3^3+8z_3^3z_2^2z_1-24z_3^2z_1z_2^2+16z_3^2z_2^2+10z_3z_2^2+\\
&\quad 3z_1^2z_2z_3^2-2z_3^2z_2z_1+12z_3z_1z_2-z_2-12z_2z_3+1+z_3-z_1-z_3z_1)/(\Delta(4z_2z_3-1)^2),\\
Y_{112} &= -2z_2z_1(2z_1^2z_2z_3^2-z_3^2z_1-8z_3z_2z_1^2+4z_3z_1z_2+4z_3z_1-2z_3+4z_2z_1^2-2z_2-3z_1+2)\\
&\quad /(\Delta(4z_1z_2-1)^2),\\
Y_{233} &= -2z_2z_3(2z_1^2z_2z_3^2-8z_3^2z_2z_1+4z_3^2z_2+4z_3z_1z_2+4z_3z_1-3z_3-z_3z_1^2-2z_2-2z_1+2)\\
&\quad /(\Delta(4z_2z_3-1)^2),\\
Y_{122} &= -2(4z_3^3z_2^2z_1^3-3z_3^3z_2z_1^2-16z_3^2z_1^3z_2^2+4z_3^2z_1^4z_2^2+12z_1^2z_2z_3^2-3z_3^2z_2z_1^3+\\
&\quad z_1^2z_3^2-z_3^2-4z_2^2z_3z_1+16z_3z_1^2z_2^2-z_2z_3+4z_3z_2z_1^3-12z_3z_2z_1^2+2z_3z_1+z_3-3z_3z_1^2-\\
&\quad 2z_1+2z_1^2+3z_1z_2-4z_1^2z_2^2)z_2/(\Delta(4z_2z_3-1)(4z_1z_2-1)^2),\\
Y_{223} &= -2(4z_3^4z_2^2z_1^2+4z_3^3z_2^2z_1^3-16z_1^2z_2^2z_3^3-3z_3^3z_2z_1^2+4z_3^3z_2z_1+16z_3^2z_1z_2^2-4z_3^2z_2^2-\\
&\quad 3z_3^2z_2z_1^3+12z_1^2z_2z_3^2-12z_3^2z_2z_1+z_1^2z_3^2-3z_3^2z_1+2z_3^2-4z_2^2z_3z_1+3z_2z_3+2z_3z_1-2z_3-\\
&\quad z_1z_2-z_1^2+z_1)z_2/(\Delta(4z_2z_3-1)^2(4z_1z_2-1)),
\end{aligned}
$$

$$
\begin{aligned}
Y_{222} &= 2(1+32z_3z_1z_2-28z_3z_2z_1^2-32z_3z_1^2z_2^2+32z_1^2z_2z_3^2-32z_3^2z_1^3z_2^2-28z_3^2z_2z_1-4z_3^2z_2z_1^3- \\
&\quad 32z_3^2z_1z_2^2-4z_3^3z_2z_1^2-32z_1^2z_2^2z_3^3+8z_3^4z_2^2z_1^2+8z_3^2z_1^4z_2^2+96z_3^2z_2^2z_1^2-2z_1-2z_3+ \\
&\quad 8z_1^2z_2^2-4z_1z_2+4z_3z_1+8z_3^2z_2^2+z_1^2z_3^2-2z_3^2z_1-4z_2z_3+z_1^2+z_3^2-2z_3z_1^2+4z_1^3z_2+ \\
&\quad 4z_3^3z_2)z_2/(\Delta(4z_2z_3-1)^2(4z_1z_2-1)^2), \\
\\
Y_{123} &= -(2z_1^2z_2^2z_3^3-16z_3^2z_2^2z_1^2+8z_3^2z_1z_2^2+2z_3^2z_1^3z_2^2+4z_3^2z_2z_1-z_1^2z_2z_3^2-2z_3^2z_2-2z_3z_2^2+ \\
&\quad 8z_3z_1^2z_2^2-8z_3z_1z_2+4z_3z_2z_1^2+2z_2z_3-z_3z_1+z_3-1+2z_1z_2-2z_1z_2^2+z_1-2z_2z_1^2z_2)/ \\
&\quad (\Delta(4z_2z_3-1)(4z_1z_2-1)). \qquad (8.100)
\end{aligned}
$$

Discriminant of X_3

$$
\Delta = 2(-z_1^2z_2z_3^2+2z_3z_1z_2+z_3z_1-z_3-z_2-z_1+1). \qquad (8.101)
$$

Extended Picard-Fuchs system of X_3

$$
\begin{aligned}
\tilde{\mathcal{D}}_1 &= a_1(z_1,z_2,z_3)\cdot\mathcal{D}_1+a_2(z_1,z_2,z_3)\cdot\mathcal{D}_2+a_3(z_1,z_2,z_3)\cdot\mathcal{D}_3+a_5(z_1,z_2,z_3)\cdot\mathcal{D}_5- \\
&\quad 2(1+z_1+z_2+z_3-2z_1z_2z_3+z_1z_3+z_1^2z_2z_3^2)\mathcal{D}_6 \\
&= (-\theta_2^2-2\theta_2\theta_3-\theta_1^2+2\theta_2\theta_1+3\theta_3^2)z_1^2z_2z_3^2+(((-4\theta_1^2+4\theta_2^2+8\theta_1\theta_3+4\theta_2\theta_3-8\theta_3^2)z_1+ \\
&\quad (4\theta_1\theta_3-4\theta_3^2+4\theta_2\theta_3)z_1^2)z_2+(-2\theta_2\theta_3+2\theta_2\theta_1-2\theta_1\theta_3-\theta_1^2-\theta_2^2+3\theta_3^2)z_1-\theta_2^2- \\
&\quad 2\theta_2\theta_3-2\theta_1\theta_3+3\theta_3^2+\theta_1^2)z_3+((-4\theta_1^2+4\theta_2^2+4\theta_1\theta_3-4\theta_2\theta_3)z_1+\theta_1^2+\theta_3^2-2\theta_2\theta_1+ \\
&\quad \theta_2^2-2\theta_2\theta_3)z_2+(-\theta_1^2+\theta_3^2+2\theta_2\theta_1-\theta_2^2-2\theta_1\theta_3)z_1+\theta_1^2+\theta_3^2-\theta_2^2-2\theta_1\theta_3- \\
&\quad 2(1+z_1+z_2+z_3-2z_1z_2z_3+z_1z_3+z_1^2z_2z_3^2)\mathcal{D}_6, \\
\tilde{\mathcal{D}}_2 &= \text{interchange between } z_1 \text{ and } z_3 \text{ of } \tilde{\mathcal{D}}_1, \\
\tilde{\mathcal{D}}_3 &= (1+z_3)\mathcal{D}_1+(1+z_1)\mathcal{D}_3+(1-z_1^2z_3^2)\mathcal{D}_2+(1-z_1z_3)\mathcal{D}_5, \qquad (8.102)
\end{aligned}
$$

where $a_j(z_1,z_2,z_3)$ are rational functions in z_i whose denominators are given by Δ.

Appendix B : B-model Yukawa couplings of K_{dP_2}

In this Appendix, we present the B-model Yukawa couplings of K_{dP_2} computed from the extended PF system (7.86), and the assumption of the existence of Kodaira-Spencer theory:

$$
Y_{ijk} = \int_{\overline{K}_{dP_2}} \Omega \wedge \frac{\partial^3\Omega}{\partial u_i \partial u_j \partial u_k}. \qquad (8.103)
$$

$$
\begin{aligned}
Y_{222} &= -(16z_3^3yz_1^2 + ((-27y-18)z_1z_2^2 + 16yz_1^3 - 16yz_1^2 + \\
&((-24y-8)z_1^2 + (4+36y)z_1)z_2 - 8z_1y)z_3^2 + (((-24y-14)z_1^2 + \\
&(22+36y)z_1)z_2^2 - 8yz_1^2 + ((64y+12)z_1^2 + (-2-46y)z_1 + (-32y-8)z_1^3 - \\
&y-1)z_2 + 8z_1y + y)z_3 + (12+16y)z_1^2z_2^3 + ((16y+8)z_1^3 + (-16y-4)z_1^2 + \\
&(-3-8y)z_1)z_2^2 + (y + (2-8y)z_1^2 + 1 + (-3+8y)z_1)z_2 + z_1y - y)/\Delta,
\end{aligned}
$$

$$
\begin{aligned}
Y_{223} &= -(16z_3^3xz_1^2 + ((-27x+9)z_1z_2^2 + 16xz_1^3 - 16xz_1^2 + ((-24x+8)z_1^2 + \\
&(36x-8)z_1)z_2 - 8z_1x)z_3^2 + (((36x-11)z_1 + (-24x+6)z_1^2)z_2^2 - 8xz_1^2 + \\
&((-16+64x)z_1^2 + (-32x+8)z_1^3 + (10-46x)z_1 - x)z_2 + 8z_1x + x)z_3 + (16x-8)z_1^2z_2^3 + \\
&((-16x+8)z_1^2 + (16x-8)z_1^3 + (-8x+2)z_1)z_2^2 + ((8x-2)z_1 + (-8x+2)z_1^2 + x)z_2 + \\
&z_1x - x)/\Delta,
\end{aligned}
$$

$$
\begin{aligned}
Y_{233} &= -((16x-8)z_1^2z_3^3 + ((-27x+9)z_1z_2^2 + (16x-8)z_1^3 + (-16x+8)z_1^2 + \\
&((36x-11)z_1 + (-24x+6)z_1^2)z_2 + (-8x+2)z_1)z_3^2 + (((-24x+8)z_1^2 + \\
&(36x-8)z_1)z_2^2 + (-8x+2)z_1^2 + ((-16+64x)z_1^2 + (-32x+8)z_1^3 + (10-46x)z_1 - x)z_2 + \\
&(8x-2)z_1 + x)z_3 + 16z_2^3xz_1^2 + (-16xz_1^2 + 16xz_1^3 - 8z_1x)z_2^2 + \\
&(8z_1x - 8xz_1^2 + x)z_2 + z_1x - x)/\Delta,
\end{aligned}
$$

$$
\begin{aligned}
Y_{333} &= -((12+16y)z_1^2z_3^3 + ((-27y-18)z_1z_2^2 + (-16y-4)z_1^2 + ((-24y-14)z_1^2 + \\
&(22+36y)z_1)z_2 + (16y+8)z_1^3 + (-3-8y)z_1)z_3^2 + ((2-8y)z_1^2 + ((-24y-8)z_1^2 + \\
&(4+36y)z_1)z_2^2 + (-3+8y)z_1 + 1 + ((64y+12)z_1^2 + (-2-46y)z_1 + (-32y-8)z_1^3 - \\
&y-1)z_2 + y)z_3 + 16z_2^3yz_1^2 + (16yz_1^3 - 16yz_1^2 - 8z_1y)z_2^2 + z_1y + \\
&(-8yz_1^2 + y + 8z_1y)z_2 - y)/\Delta,
\end{aligned}
$$

$$
\begin{aligned}
Y_{122} &= ((16x+16y)z_1^2z_3^3 + ((16x+16y)z_1^3 + (-27y-9-27x)z_1z_2^2 + \\
&((-24x-24y)z_1^2 + (8+36y+36x)z_1)z_2 + (-16x-16y)z_1^2 + (-8x-8y)z_1)z_3^2 + \\
&((8y+8x)z_1 + (-8x-8y)z_1^2 + ((-8-24x-24y)z_1^2 + (36y+36x+14)z_1)z_2^2 + \\
&((-32x-32y)z_1^3 + (-46y-46x-12)z_1 - x - y + (64y+8+64x)z_1^2)z_2 + x + y)z_3 + \\
&(4+16y+16x)z_1^2z_2^3 + ((16x+16y)z_1^3 + (-16x-16y)z_1^2 + (-8x-8y-5)z_1)z_2^2 + \\
&(y+x)z_1 + ((8y+4+8x)z_1 + x + y + (-8y-8x-4)z_1^2)z_2 - x - y)/\Delta,
\end{aligned}
$$

$$
\begin{aligned}
Y_{133} &= ((4+16y+16x)z_1^2z_3^3 + ((16x+16y)z_1^3 + (-27y-9-27x)z_1z_2^2 + ((-8-24x-24y)z_1^2 + \\
&(36y+36x+14)z_1)z_2 + (-16x-16y)z_1^2 + (-8x-8y-5)z_1)z_3^2 + ((8y+4+8x)z_1 + \\
&(-8y-8x-4)z_1^2 + ((-24x-24y)z_1^2 + (8+36y+36x)z_1)z_2^2 + ((-32x-32y)z_1^3 + \\
&(-46y-46x-12)z_1 - x - y + (64y+8+64x)z_1^2)z_2 + x + y)z_3 + (16x+16y)z_1^2z_2^3 + \\
&((16x+16y)z_1^3 + (-16x-16y)z_1^2 + (-8x-8y)z_1)z_2^2 + (y+x)z_1 + ((8y+8x)z_1 + \\
&x + y + (-8x-8y)z_1^2)z_2 - x - y)/\Delta,
\end{aligned}
$$

$$
\begin{aligned}
Y_{112} \;=\;& -((48x+16y-8)z_1^2z_3^3+((48x+16y-8)z_1^3+(-27y-81x+9)z_1z_2^2+\\
&((-24y-72x+14)z_1^2+(36y+108x-11)z_1)z_2+(8-16y-48x)z_1^2+\\
&(-8y+2-24x)z_1)z_3^2+((8y+24x-2)z_1+(-8y+2-24x)z_1^2+((6-72x-24y)z_1^2+\\
&(36y+108x-14)z_1)z_2^2+((-32y-96x+16)z_1^3+(-138x+16-46y)z_1-3x-y+\\
&(64y+192x-28)z_1^2)z_2+3x+y)z_3+(48x-4+16y)z_1^2z_2^3+((48x+16y-8)z_1^3+\\
&(4-48x-16y)z_1^2+(-8y+5-24x)z_1)z_2^2+(y+3x)z_1+((24x+8y-5)z_1+3x+y+\\
&(-24x-8y+6)z_1^2)z_2-3x-y)/\Delta,
\end{aligned}
$$

$$
\begin{aligned}
Y_{113} \;=\;& -((48x-4+16y)z_1^2z_3^3+((48x+16y-8)z_1^3+(-27y-81x+9)z_1z_2^2+\\
&((6-72x-24y)z_1^2+(36y+108x-14)z_1)z_2+(4-48x-16y)z_1^2+\\
&(-8y+5-24x)z_1)z_3^2+((24x+8y-5)z_1+(-24x-8y+6)z_1^2+\\
&((-24y-72x+14)z_1^2+(36y+108x-11)z_1)z_2^2+((-32y-96x+16)z_1^3+\\
&(-138x+16-46y)z_1-3x-y+(64y+192x-28)z_1^2)z_2+3x+y)z_3+\\
&(48x+16y-8)z_1^2z_2^3+((48x+16y-8)z_1^3+(8-16y-48x)z_1^2+\\
&(-8y+2-24x)z_1)z_2^2+(y+3x)z_1+((8y+24x-2)z_1+3x+y+\\
&(-8y+2-24x)z_1^2)z_2-3x-y)/\Delta,
\end{aligned}
$$

$$
\begin{aligned}
Y_{123} \;=\;& ((-8+32x)z_1^2z_3^3+((-32x+8)z_1^2+(6-16x)z_1+((72x-25)z_1+\\
&(14-48x)z_1^2)z_2+(-8+32x)z_1^3+(18-54x)z_1z_2^2)z_3^2+((16x-6)z_1+\\
&(1+(-40+128x)z_1^2+(16-64x)z_1^3-2x+(33-92x)z_1)z_2+(6-16x)z_1^2+2x-1+\\
&((72x-25)z_1+(14-48x)z_1^2)z_2^2)z_3+(-8+32x)z_1^2z_2^3+\\
&((-32x+8)z_1^2+(-8+32x)z_1^3+(6-16x)z_1)z_2^2+(-1+2x)z_1+\\
&((6-16x)z_1^2-1+2x+(16x-6)z_1)z_2+1-2x)/\Delta,
\end{aligned}
$$

$$
\begin{aligned}
Y_{111} \;=\;& ((-12+96x+32y)z_1^2z_3^3+((8-32y-96x)z_1^2+(-16+96x+32y)z_1^3+\\
&(-162x+18-54y)z_1z_2^2+(-48x-16y+3)z_1+((20-48y-144x)z_1^2+\\
&(216x+72y-22)z_1)z_2)z_3^2+((-6x+1+(-192x+32-64y)z_1^3-2y+\\
&(128y-48+384x)z_1^2+(-92y+26-276x)z_1)z_2+(8-16y-48x)z_1^2+2y+\\
&((20-48y-144x)z_1^2+(216x+72y-22)z_1)z_2^2+6x+(48x+16y-2)z_1-1)z_3+\\
&(-12+96x+32y)z_1^2z_2^3+((8-32y-96x)z_1^2+(-16+96x+32y)z_1^3+\\
&(-48x-16y+3)z_1)z_2^2+1+(2y-2+6x)z_1+(6x+2y+(8-16y-48x)z_1^2-1+\\
&(48x+16y-2)z_1)z_2-6x-2y)/\Delta. \qquad (8.104)
\end{aligned}
$$

$$
\begin{aligned}
\Delta \;=\;& -\big((-16z_2^2+32z_3z_2-16z_3^2)z_1^3+(24z_3^2z_2-16z_3^3-16z_2^3-64z_3z_2+24z_3z_2^2+16z_2^2+\\
&8z_3+16z_3^2+8z_2)z_1^2+(8z_3^2-1+27z_2^2z_3^2-36z_3z_2^2-8z_2+8z_2^2-8z_3-36z_3^2z_2+46z_3z_2)z_1+\\
&1-z_2+z_3z_2-z_3\big). \qquad (8.105)
\end{aligned}
$$

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