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# BOUNDARY BEHAVIOR OF QUOTIENTS OF MARTIN KERNELS

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ABSTRACT. Given two intersecting domains, we investigate the boundary behavior of the quotient of Martin kernels of each domain. To this end, we give a characterization of the minimal thinness for a difference of two subdomains in terms of Martin kernels of each domain. As a consequence of our main theorem (Theorem 2.1), we obtain the boundary growth of the Martin kernel of a Lipschitz domain, which corresponds to earlier results for the boundary decay of the Green function for a Lipschitz domain investigated by Burdzy, Carroll and Gardiner.

## 1. INTRODUCTION

One of the purposes of this paper is to show the boundary growth of the Martin kernel of a Lipschitz domain. This is motivated by earlier works due to Burdzy, Carroll and Gardiner. We write  $\mathbf{0}$  for the origin of  $\mathbb{R}^n$  ( $n \geq 2$ ) to distinct from  $0 \in \mathbb{R}$ , and denote  $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$  and  $e = (\mathbf{0}', 1)$ . Suppose that  $\phi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$  satisfies  $\phi(\mathbf{0}') = 0$  and the Lipschitz property: there is a positive constant  $L$  such that

$$|\phi(x') - \phi(y')| \leq L|x' - y'| \quad \text{for } x', y' \in \mathbb{R}^{n-1}.$$

We put  $\Omega_\phi = \{(x', x_n) : x_n > \phi(x')\}$  and set

$$(1.1) \quad I^+ = \int_{\{|x'| < 1\}} \frac{\max\{\phi(x'), 0\}}{|x'|^n} dx',$$

$$(1.2) \quad I^- = \int_{\{|x'| < 1\}} \frac{\max\{-\phi(x'), 0\}}{|x'|^n} dx'.$$

In [2, 3], Burdzy obtained a result on the angular derivative problem of analytic functions in a Lipschitz domain. The key step was to show the relationship between the convergence of integrals above and the boundary behavior of the Green function  $G_{\Omega_\phi}$  for  $\Omega_\phi$ . Burdzy's approach was based on probabilistic methods and the minimal fine topology. Analytic proofs were given by Carroll [4, 5] and Gardiner [7].

**Theorem A.** *Suppose that  $I^+$  and  $I^-$  are as in (1.1) and (1.2). The following statements hold.*

(i) *If  $I^+ < \infty$  and  $I^- = \infty$ , then*

$$\lim_{t \rightarrow 0^+} \frac{G_{\Omega_\phi}(te, e)}{t} = \infty.$$

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(ii) If  $I^+ = \infty$  and  $I^- < \infty$ , then

$$\lim_{t \rightarrow 0^+} \frac{G_{\Omega_\phi}(te, e)}{t} = 0.$$

(iii) If  $I^+ < \infty$  and  $I^- < \infty$ , then the limit of  $G_{\Omega_\phi}(te, e)/t$ , as  $t \rightarrow 0^+$ , exists and

$$0 < \lim_{t \rightarrow 0^+} \frac{G_{\Omega_\phi}(te, e)}{t} < \infty.$$

In view of the boundary Harnack principle, Theorem A shows *the boundary decay* of positive harmonic functions on  $\Omega_\phi$  vanishing continuously on a part of the boundary of  $\Omega_\phi$  near the origin. We are now interested in a relationship between the convergence of the integrals  $I^+, I^-$  and *the boundary growth* of positive harmonic functions on  $\Omega_\phi$  with singularity at the origin. In view of the Fatou-Näim-Doob theorem, it is enough to investigate it for the Martin kernel of  $\Omega_\phi$  with pole at the origin. See the first paragraph of Section 2 for the definition of the Martin kernel.

**Theorem 1.1.** *Suppose that  $I^+$  and  $I^-$  are as in (1.1) and (1.2). The following statements hold.*

(i) If  $I^+ < \infty$  and  $I^- = \infty$ , then

$$\lim_{t \rightarrow 0^+} t^{n-1} K_{\Omega_\phi}(te, \mathbf{0}) = 0.$$

(ii) If  $I^+ = \infty$  and  $I^- < \infty$ , then

$$\lim_{t \rightarrow 0^+} t^{n-1} K_{\Omega_\phi}(te, \mathbf{0}) = \infty.$$

(iii) If  $I^+ < \infty$  and  $I^- < \infty$ , then the limit of  $t^{n-1} K_{\Omega_\phi}(te, \mathbf{0})$ , as  $t \rightarrow 0^+$ , exists and

$$0 < \lim_{t \rightarrow 0^+} t^{n-1} K_{\Omega_\phi}(te, \mathbf{0}) < \infty.$$

When  $I^+ = \infty$  and  $I^- = \infty$ , the limit of  $t^{n-1} K_{\Omega_\phi}(te, \mathbf{0})$  may take any values 0, positive and finite, or  $\infty$ , as the following simple example shows.

**Example 1.2.** To simplify the notation, we write  $\mathbb{R}_{1^+}^{n-1} = \{x' \in \mathbb{R}^{n-1} : x_1 \geq 0\}$  and  $\mathbb{R}_{1^-}^{n-1} = \{x' \in \mathbb{R}^{n-1} : x_1 \leq 0\}$  in this example.

(i) If  $\phi(x')$  is equal to  $x_1/2$  on  $\mathbb{R}_{1^+}^{n-1}$  and  $x_1$  on  $\mathbb{R}_{1^-}^{n-1}$ , then

$$\lim_{t \rightarrow 0^+} t^{n-1} K_{\Omega_\phi}(te, \mathbf{0}) = 0.$$

(ii) If  $\phi(x')$  is equal to  $x_1$  on  $\mathbb{R}_{1^+}^{n-1}$  and  $x_1$  on  $\mathbb{R}_{1^-}^{n-1}$ , then the limit of  $t^{n-1} K_{\Omega_\phi}(te, \mathbf{0})$ , as  $t \rightarrow 0^+$ , exists and

$$0 < \lim_{t \rightarrow 0^+} t^{n-1} K_{\Omega_\phi}(te, \mathbf{0}) < \infty.$$

(iii) If  $\phi(x')$  is equal to  $x_1$  on  $\mathbb{R}_{1^+}^{n-1}$  and  $x_1/2$  on  $\mathbb{R}_{1^-}^{n-1}$ , then

$$\lim_{t \rightarrow 0^+} t^{n-1} K_{\Omega_\phi}(te, \mathbf{0}) = \infty.$$

It is easy to check that  $I^+ = \infty$  and  $I^- = \infty$ . The value of the limit in each case follows from [9, Theorems 1 and 2].

Let  $\mathbb{R}_+^n = \{(x', x_n) : x_n > 0\}$ . As we will state in Section 5, the convergence of the integrals  $I^+$  and  $I^-$  is connected with the minimal thinness of the sets  $\mathbb{R}_+^n \setminus \Omega_\phi$  and  $\Omega_\phi \setminus \mathbb{R}_+^n$ . See Section 2 for the definition of minimal thinness. Since  $K_{\mathbb{R}_+^n}(te, \mathbf{0}) = t^{1-n}$ , Theorem 1.1 may be interpreted as the relationship between the minimal thinness of the sets  $\mathbb{R}_+^n \setminus \Omega_\phi$ ,  $\Omega_\phi \setminus \mathbb{R}_+^n$  and the boundary behavior of the quotient of Martin kernels of  $\Omega_\phi$  and  $\mathbb{R}_+^n$ . So, given two intersecting domains  $\Phi$  and  $\Psi$ , it is valuable for us to investigate a relationship between the minimal thinness of the differences  $\Phi \setminus \Psi$ ,  $\Psi \setminus \Phi$  and the boundary behavior of the quotient of Martin kernels of  $\Phi$  and  $\Psi$  (Theorem 2.1).

## 2. STATEMENT FOR GENERAL DOMAINS

Let  $\Omega$  be a Greenian domain in  $\mathbb{R}^n$  with  $n \geq 2$ . Here a Greenian domain means a domain possessing the Green function  $G_\Omega$  for the Laplace operator. Let  $x_0$  be a reference point in  $\Omega$ . The Martin kernel of  $\Omega$  is defined for  $(x, y) \in (\Omega \times \Omega) \setminus \{(x_0, x_0)\}$  by

$$K_\Omega(x, y) = \frac{G_\Omega(x, y)}{G_\Omega(x_0, y)}.$$

Now, let  $\{y_j\}$  be a sequence in  $\Omega$  with no limit point in  $\Omega$ . We observe that if  $j_0$  is sufficiently large, then  $\{K_\Omega(\cdot, y_j)\}_{j \geq j_0}$  is a uniformly bounded sequence of positive harmonic functions on a relatively compact open subset of  $\Omega$ . Therefore the Harnack principle shows that there exists a subsequence  $\{K_\Omega(\cdot, y_{j_k})\}$  converging to a positive harmonic function on  $\Omega$ . The Martin boundary  $\Delta(\Omega)$  of  $\Omega$  is defined as an ideal boundary consisting of all positive harmonic functions on  $\Omega$  that can be obtained as the limit of  $\{K_\Omega(\cdot, y_j)\}$  for some sequence  $\{y_j\}$  in  $\Omega$  with no limit point in  $\Omega$ . The set  $\Omega \cup \Delta(\Omega)$  (equipped with a suitable metric) is called a Martin compactification of  $\Omega$ . See [1, Section 8.1] for details. In the sequel, we write  $K_\Omega(\cdot, \xi)$  for the positive harmonic function on  $\Omega$  corresponding to  $\xi \in \Delta(\Omega)$ . We say that  $\xi \in \Delta(\Omega)$  is *minimal* if every positive harmonic function on  $\Omega$  less than or equal to the corresponding Martin kernel  $K_\Omega(\cdot, \xi)$  coincides with a constant multiple of  $K_\Omega(\cdot, \xi)$ . By  $\Delta_1(\Omega)$ , we denote the collection of all minimal Martin boundary points in  $\Delta(\Omega)$ .

The notion of minimal thinness was introduced by Naïm [11], using a regularized reduced function. Let  $u$  be a positive superharmonic function on  $\Omega$  and let  $E$  be a subset of  $\Omega$ . A reduced function of  $u$  relative to  $E$  on  $\Omega$  is defined by

$$\Omega R_u^E(x) = \inf\{v(x)\},$$

where the infimum is taken over all positive superharmonic functions  $v$  on  $\Omega$  such that  $v \geq u$  on  $E$ . By  $\Omega \widehat{R}_u^E$ , we denote the lower semicontinuous regularization of  $\Omega R_u^E$ . Observe that  $\Omega \widehat{R}_u^E \leq u$  in general. Let  $\xi \in \Delta_1(\Delta)$ . A set  $E$  is said to be *minimally thin* at  $\xi$  with respect to  $\Omega$  if

$$\Omega \widehat{R}_{K_\Omega(\cdot, \xi)}^E(z) < K_\Omega(z, \xi) \quad \text{for some } z \in \Omega.$$

Minimal thinness enables us to equip the minimal fine topology in the Martin compactification of  $\Omega$ . Roughly speaking, the minimal fine topology is the collection of subsets  $W$  of the Martin compactification such that  $\Omega \setminus W$  is minimally thin at every point of  $W \cap \Delta_1(\Omega)$ . See [1, Definition 9.2.3] for the precise definition. Let  $U$

be a minimal fine neighborhood of  $\xi \in \Delta_1(\Omega)$ . We say that a function  $f$  on  $U$  has *minimal fine limit*  $l$  at  $\xi$  with respect to  $\Omega$  if there is a subset  $E$  on  $\Omega$ , minimally thin at  $\xi$  with respect to  $\Omega$ , such that  $f(x) \rightarrow l$  as  $x \rightarrow \xi$  along  $U \setminus E$ , and then we write

$$\text{mf-}\lim_{\Omega} f(x) = l.$$

We note from the definition that a function is not necessarily defined on the whole of a domain when we consider the minimal fine limit.

The following theorem is our main result.

**Theorem 2.1.** *Suppose that  $\Phi$  and  $\Psi$  are Greenian domains in  $\mathbb{R}^n$  such that  $\Phi \cap \Psi$  is a non-empty domain. Let  $\xi \in \Delta_1(\Phi)$ , where  $\xi$  is in the closure of  $\Phi \cap \Psi$  in the Martin compactification of  $\Phi$ . Let  $\zeta \in \Delta_1(\Psi)$ , where  $\zeta$  is in the closure of  $\Phi \cap \Psi$  in the Martin compactification of  $\Psi$ . If  $\Phi \setminus \Psi$  is minimally thin at  $\xi$  with respect to  $\Phi$ , then  $K_\Psi(\cdot, \zeta)/K_\Phi(\cdot, \xi)$  has a finite minimal fine limit at  $\xi$  with respect to  $\Phi$ . Furthermore, the following statements hold.*

(i) *If  $\Psi \setminus \Phi$  is not minimally thin at  $\zeta$  with respect to  $\Psi$ , then*

$$\text{mf-}\lim_{\Phi} \frac{K_\Psi(x, \zeta)}{K_\Phi(x, \xi)} = 0.$$

(ii) *If  $\Psi \setminus \Phi$  is minimally thin at  $\zeta$  with respect to  $\Psi$ , where  $\zeta$  is the point such that*

$$(2.1) \quad K_\Psi(\cdot, \zeta) - {}^\Psi R_{K_\Psi(\cdot, \zeta)}^{\Psi \setminus \Phi} = \alpha (K_\Phi(\cdot, \xi) - {}^\Phi R_{K_\Phi(\cdot, \xi)}^{\Phi \setminus \Psi}) \quad \text{on } \Phi \cap \Psi$$

*for some positive constant  $\alpha$ , then*

$$0 < \text{mf-}\lim_{\Phi} \frac{K_\Psi(x, \zeta)}{K_\Phi(x, \xi)} < \infty.$$

(iii) *If  $\Psi \setminus \Phi$  is minimally thin at  $\zeta$  with respect to  $\Psi$ , where  $\zeta$  is a point such that (2.1) is not satisfied, then*

$$\text{mf-}\lim_{\Phi} \frac{K_\Psi(x, \zeta)}{K_\Phi(x, \xi)} = 0.$$

For Lipschitz domains  $\Phi$  and  $\Psi$ , Theorem 2.1 can be restated as the corollary below. We note from [8] that each Euclidean boundary point of a Lipschitz domain has a unique Martin boundary point and it is minimal. So, we identify a Martin boundary point with a Euclidean boundary point. Let  $\Omega$  be a Lipschitz domain in  $\mathbb{R}^n$  and let  $c > 1$ . We define a non-tangential region at  $y \in \partial\Omega$  (the Euclidean boundary of  $\Omega$ ) by

$$\Gamma_c(y) = \{x \in \Omega : |x - y| < c \text{dist}(x, \partial\Omega)\}.$$

Note that this region makes sense if  $c$  is sufficiently large. We say that a function  $f$  on  $\Omega$  has non-tangential limit  $l$  at  $y$  if, for each  $c$  sufficiently large,  $f(x)$  has limit  $l$  as  $x \rightarrow y$  along  $\Gamma_c(y)$ . Then we write

$$\text{nt-}\lim_{\Omega} f(x) = l.$$

**Corollary 2.2.** *Suppose that  $\Phi$  and  $\Psi$  are Lipschitz domains in  $\mathbb{R}^n$  such that  $\Phi \cap \Psi$  is also a Lipschitz domain. Let  $y \in \partial\Phi \cap \partial\Psi$ , and suppose that  $\Phi \setminus \Psi$  is minimally thin at  $y$  with respect to  $\Phi$ . The following statements hold.*

(i) If  $\Psi \setminus \Phi$  is not minimally thin at  $y$  with respect to  $\Psi$ , then

$$\text{nt-}\lim_{\Phi \cap \Psi} \frac{K_{\Psi}(x, y)}{K_{\Phi}(x, y)} = 0.$$

(ii) If  $\Psi \setminus \Phi$  is minimally thin at  $y$  with respect to  $\Psi$ , then the non-tangential limit of  $K_{\Psi}(\cdot, y)/K_{\Phi}(\cdot, y)$  at  $y$  with respect to  $\Phi \cap \Psi$  exists and

$$0 < \text{nt-}\lim_{\Phi \cap \Psi} \frac{K_{\Psi}(x, y)}{K_{\Phi}(x, y)} < \infty.$$

*Remark 2.3.* If  $\Phi \setminus \Psi$  is not minimally thin at  $y$  with respect to  $\Phi$  and  $\Psi \setminus \Phi$  is not minimally thin at  $y$  with respect to  $\Psi$ , then the non-tangential limit of  $K_{\Psi}(\cdot, y)/K_{\Phi}(\cdot, y)$  may take any values 0, positive and finite, or  $\infty$ . See Example 1.2.

### 3. CHARACTERIZATION OF THE MINIMAL THINNESS FOR A DIFFERENCE OF TWO SUBDOMAINS

Naïm [11, Théorème 11] gave a characterization of the minimal thinness for a difference of two subdomains in terms of Green functions for each domain, which played an important role in the proof of Theorem A. In order to prove Theorem 2.1, we need a new characterization of the minimal thinness for a difference.

**Lemma 3.1.** *Suppose that  $\Omega$  is a Greenian domain in  $\mathbb{R}^n$  and that  $D$  is a subdomain of  $\Omega$ . Let  $\xi \in \Delta_1(\Omega)$ , where  $\xi$  is in the closure of  $D$  in the Martin compactification of  $\Omega$ . The following statements are equivalent:*

- (i)  $\Omega \setminus D$  is minimally thin at  $\xi$  with respect to  $\Omega$ ;
- (ii) there exists  $\eta \in \Delta_1(D)$  such that

$$(3.1) \quad \text{mf-}\lim_D \frac{K_{\Omega}(x, \xi)}{K_D(x, \eta)} > 0.$$

Furthermore, the point  $\eta \in \Delta_1(D)$  in (ii) is uniquely determined and the corresponding Martin kernel is represented as

$$K_D(\cdot, \eta) = \alpha(K_{\Omega}(\cdot, \xi) - {}^{\Omega}R_{K_{\Omega}(\cdot, \xi)}^{\Omega \setminus D}) \quad \text{on } D$$

for some positive constant  $\alpha$ .

*Remark 3.2.* We note in Lemma 3.1 that the minimal fine limit in (3.1) exists and satisfies that

$$(3.2) \quad \text{mf-}\lim_D \frac{K_{\Omega}(x, \xi)}{K_D(x, \eta)} = \mu_{K_{\Omega}(\cdot, \xi)}^D(\{\eta\}) = \inf_{x \in D} \frac{K_{\Omega}(x, \xi)}{K_D(x, \eta)} = \liminf_{x \rightarrow \eta} \frac{K_{\Omega}(x, \xi)}{K_D(x, \eta)} < \infty,$$

where  $\mu_{K_{\Omega}(\cdot, \xi)}^D$  is the measure on  $\Delta(D)$  associated with  $K_{\Omega}(\cdot, \xi)$  in the Martin representation. See [1, Theorems 9.2.6 and 9.3.3]. Thus the minimal thinness of  $\Omega \setminus D$  can be also characterized in terms of any of quantities in (3.2) instead of the minimal fine limit.

For the proof of Lemma 3.1, we need the following lemmas. Lemma 3.3 can be deduced from [1, Theorems 9.2.6 and 9.3.3]. Lemma 3.4 is due to Naïm [11, Théorème 15] (cf. [1, Theorem 9.5.5]).

**Lemma 3.3.** *Let  $E$  be a subset of a Greenian domain  $\Omega$  in  $\mathbb{R}^n$  and let  $\xi \in \Delta_1(\Omega)$ . The following statements are equivalent:*

- (i)  $E$  is minimally thin at  $\xi$  with respect to  $\Omega$ ;
- (ii) there exists a positive superharmonic function  $u$  on  $\Omega$  such that

$$\inf_{x \in \Omega} \frac{u(x)}{K_{\Omega}(x, \xi)} < \inf_{x \in E} \frac{u(x)}{K_{\Omega}(x, \xi)}.$$

**Lemma 3.4.** *Suppose that  $\Omega$  is a Greenian domain in  $\mathbb{R}^n$  and that  $D$  is a subdomain of  $\Omega$ . Let  $\xi \in \Delta_1(\Omega)$ , where  $\xi$  is in the closure of  $D$  in the Martin compactification of  $\Omega$ . Assume that  $\Omega \setminus D$  is minimally thin at  $\xi$  with respect to  $\Omega$ , and let  $\eta \in \Delta_1(D)$  be the point such that*

$$K_D(\cdot, \eta) = \alpha(K_{\Omega}(\cdot, \xi) - {}^{\Omega}R_{K_{\Omega}(\cdot, \xi)}^{\Omega \setminus D}) \quad \text{on } D$$

for some positive constant  $\alpha$ . The following statements for a subset  $E$  of  $D$  are equivalent:

- (i)  $E$  is minimally thin at  $\eta$  with respect to  $D$ ;
- (ii)  $E$  is minimally thin at  $\xi$  with respect to  $\Omega$ .

We say that a property holds quasi-everywhere if it holds apart from a polar set. The following lemma is elementary. For the convenience sake of the reader, we give a proof.

**Lemma 3.5.** *Let  $D$  be a Greenian domain in  $\mathbb{R}^n$  and let  $\zeta \in \Delta_1(D)$ . Then  $K_D(\cdot, \zeta)$  vanishes quasi-everywhere on  $\partial D$ .*

*Proof.* Let  $V$  be a Martin topology (closed) neighborhood of  $\zeta$  with respect to  $D$ . Then  $V \cap D$  is not minimally thin at  $\zeta$  with respect to  $D$ . Therefore we have from [1, Theorem 6.9.1] that

$$K_D(x, \zeta) = {}^D R_{K_D(\cdot, \zeta)}^{V \cap D}(x) = H_{K_D(\cdot, \zeta) \mathcal{X}_{\partial V \cap D}}^{D \setminus V}(x) \quad \text{for } x \in D \setminus V,$$

where  $H_{K_D(\cdot, \zeta) \mathcal{X}_{\partial V \cap D}}^{D \setminus V}$  denotes the Perron-Wiener-Brelot solution of the Dirichlet problem in  $D \setminus V$  with the boundary function  $K_D(\cdot, \zeta)$  on  $\partial(V \cap D) \cap D$  and 0 on  $\partial D$ . Since  $V$  is arbitrary, we obtain the lemma.  $\square$

Let  $\Omega$  be a domain in  $\mathbb{R}^n$  and let  $D$  be a subdomain of  $\Omega$ . If  $h$  is a positive harmonic function on  $D$  which vanishes quasi-everywhere on  $\partial D \cap \Omega$  and is bounded near each point of  $\partial D \cap \Omega$ , then we see from [1, Theorem 5.2.1] that  $h$  has a subharmonic extension  $h^*$  to  $\Omega$  which is valued 0 quasi-everywhere on  $\partial D \cap \Omega$  and everywhere on  $\Omega \setminus \overline{D}$ . In what follows, we use the mark  $*$ , like as  $h^*$ , to denote such a subharmonic extension.

Let us prove Lemma 3.1.

*Proof of Lemma 3.1.* By [11, Théorème 12] (cf. [1, Theorem 9.5.5]), we can easily show that (i) implies (ii). In fact,  $f := K_{\Omega}(\cdot, \xi) - {}^{\Omega}R_{K_{\Omega}(\cdot, \xi)}^{\Omega \setminus D}$  is a minimal harmonic function on  $D$ , and so there exists  $\eta \in \Delta_1(D)$  such that  $K_D(\cdot, \eta) = f/f(x_0)$  on  $D$ . Hence we obtain that

$$\inf_{x \in D} \frac{K_{\Omega}(x, \xi)}{K_D(x, \eta)} \geq f(x_0) > 0,$$

and thus (3.1) follows from (3.2).

We next show that (ii) implies (i). We may assume that  $\Omega \setminus D$  is non-polar. Let  $\eta \in \Delta_1(D)$  be a point such that

$$\alpha := \text{mf-}\lim_D \frac{K_\Omega(x, \xi)}{K_D(x, \eta)} > 0.$$

By (3.2), we have  $K_D(\cdot, \eta) \leq \alpha^{-1}K_\Omega(\cdot, \xi)$  on  $D$ . This shows that  $K_D(\cdot, \eta)$  is bounded near each point of  $\partial D \cap \Omega$ . Also,  $K_D(\cdot, \eta)$  vanishes quasi-everywhere on  $\partial D \cap \Omega$  by Lemma 3.5. Thus  $K_D^*(\cdot, \eta)$  is well-defined as a subharmonic function on  $\Omega$  and is dominated by  $\alpha^{-1}K_\Omega(\cdot, \xi)$  on  $\Omega$ . Let  $u = \alpha^{-1}K_\Omega(\cdot, \xi) - K_D^*(\cdot, \eta)$ . Then  $u$  is superharmonic on  $\Omega$ . Since  $\Omega \setminus D$  is non-polar, there is a point in  $\Omega \setminus D$  at which  $u$  is positive. Therefore the minimum principle yields that  $u$  is positive on  $\Omega$ . Also, we have that

$$\begin{aligned} \inf_{x \in \Omega} \frac{u(x)}{K_\Omega(x, \xi)} &= \alpha^{-1} - \sup_{x \in D} \frac{K_D(x, \eta)}{K_\Omega(x, \xi)} < \alpha^{-1}, \\ \inf_{x \in \Omega \setminus (D \cup F)} \frac{u(x)}{K_\Omega(x, \xi)} &= \alpha^{-1} - \sup_{x \in \Omega \setminus (D \cup F)} \frac{K_D^*(x, \eta)}{K_\Omega(x, \xi)} = \alpha^{-1}, \end{aligned}$$

where  $F$  is a polar set in  $\partial D \cap \Omega$  such that  $K_D^*(\cdot, \eta) > 0$  on  $F$ . Hence it follows from Lemma 3.3 that  $\Omega \setminus (D \cup F)$  is minimally thin at  $\xi$  with respect to  $\Omega$ , and so is  $\Omega \setminus D$ .

We finally show the uniqueness of  $\eta \in \Delta_1(D)$ . We suppose to the contrary that there exists  $\zeta \in \Delta_1(D)$  such that  $K_D(\cdot, \zeta) \leq \beta K_\Omega(\cdot, \xi)$  on  $D$  and  $K_D(\cdot, \zeta)$  is different from  $K_D(\cdot, \eta) := \gamma(K_\Omega(\cdot, \xi) - \Omega R_{K_\Omega(\cdot, \xi)}^{\Omega \setminus D})$ , where  $\beta$  and  $\gamma$  are some positive constants. We may assume that  $\beta$  is the smallest number satisfying  $K_D(\cdot, \zeta) \leq \beta K_\Omega(\cdot, \xi)$  on  $D$ . Since  $\xi \in \Delta_1(\Omega)$ , it follows that  $\beta K_\Omega(\cdot, \xi)$  is the least harmonic majorant of  $K_D^*(\cdot, \zeta)$  on  $\Omega$ . Let  $W$  be a Martin topology neighborhood of  $\zeta$  with respect to  $D$  such that  $\eta$  is apart from  $W$ . Then  $W \cap D$  is minimally thin at  $\eta$  with respect to  $D$ . Thus the minimal thinness of  $\Omega \setminus D$  at  $\xi$  with respect to  $\Omega$ , together with Lemma 3.4, yields that  $W \cap D$  is minimally thin at  $\xi$  with respect to  $\Omega$ .

On the other hand, since  $W \cap D$  is not minimally thin at  $\zeta$  with respect to  $D$ , we have that

$$K_D(\cdot, \zeta) = {}^D \widehat{R}_{K_D(\cdot, \zeta)}^{W \cap D} \leq \beta {}^D \widehat{R}_{K_\Omega(\cdot, \xi)}^{W \cap D} \leq \beta {}^\Omega \widehat{R}_{K_\Omega(\cdot, \xi)}^{W \cap D} \quad \text{on } D.$$

Since  $\beta K_\Omega(\cdot, \xi)$  is the least one among superharmonic functions  $u$  on  $\Omega$  satisfying  $K_D^*(\cdot, \zeta) \leq u$  on  $\Omega$ , we have  ${}^\Omega \widehat{R}_{K_\Omega(\cdot, \xi)}^{W \cap D} = K_\Omega(\cdot, \xi)$  on  $\Omega$ , so that  $W \cap D$  is not minimally thin at  $\xi$  with respect to  $\Omega$ . Thus we obtain a contradiction, and hence the uniqueness of  $\eta \in \Delta_1(D)$  is established. The proof of Lemma 3.1 is complete.  $\square$

#### 4. PROOFS OF THEOREM 2.1 AND COROLLARY 2.2

In this section, we give proofs of Theorem 2.1 and Corollary 2.2.

*Proof of Theorem 2.1.* In order to prove the first assertion, we assume that  $\Phi \setminus (\Phi \cap \Psi)$  is minimally thin at  $\xi$  with respect to  $\Phi$ . Let  $\eta \in \Delta_1(\Phi \cap \Psi)$  be the point such that  $K_{\Phi \cap \Psi}(\cdot, \eta) = \alpha(K_\Phi(\cdot, \xi) - \Phi R_{K_\Phi(\cdot, \xi)}^{\Phi \setminus \Psi})$  on  $\Phi \cap \Psi$  for some positive constant  $\alpha$ . Then we have by Lemma 3.1 with  $D := \Phi \cap \Psi$  and  $\Omega := \Phi$  and its remark that the minimal



fine limit of  $K_\Phi(\cdot, \xi)/K_{\Phi \cap \Psi}(\cdot, \eta)$  at  $\eta$  with respect to  $\Phi \cap \Psi$  exists and

$$(4.1) \quad 0 < \text{mf} - \lim_{\Phi \cap \Psi} \frac{K_\Phi(x, \xi)}{K_{\Phi \cap \Psi}(x, \eta)} < \infty.$$

It also follows from [1, Theorem 9.3.3] that  $K_\Psi(\cdot, \zeta)/K_{\Phi \cap \Psi}(\cdot, \eta)$  has a finite minimal fine limit at  $\eta$  with respect to  $\Phi \cap \Psi$ . The minimal thinness of  $\Phi \setminus (\Phi \cap \Psi)$  at  $\xi$  with respect to  $\Phi$ , together with Lemma 3.4 with  $D := \Phi \cap \Psi$  and  $\Omega := \Phi$ , concludes that  $K_\Psi(\cdot, \zeta)/K_\Phi(\cdot, \xi)$  has a finite minimal fine limit at  $\xi$  with respect to  $\Phi$ .

To prove (i), we assume in addition that  $\Psi \setminus (\Phi \cap \Psi)$  is not minimally thin at  $\zeta$  with respect to  $\Psi$ . Then Lemma 3.1 with  $D := \Phi \cap \Psi$  and  $\Omega := \Psi$  shows that for any  $\eta \in \Delta_1(\Phi \cap \Psi)$ , the minimal fine limit in (3.1) is zero. Therefore we have that

$$\text{mf} - \lim_{\Phi \cap \Psi} \frac{K_\Psi(x, \zeta)}{K_{\Phi \cap \Psi}(x, \eta)} = 0.$$

Hence (i) follows from (4.1) and Lemma 3.4 with  $D := \Phi \cap \Psi$  and  $\Omega := \Phi$ .

To prove (ii), we assume in addition that  $\Psi \setminus (\Phi \cap \Psi)$  is minimally thin at  $\zeta$  with respect to  $\Psi$ , where  $\zeta$  is the point in  $\Delta_1(\Psi)$  such that (2.1) is satisfied. We note from (2.1) that  $K_{\Phi \cap \Psi}(\cdot, \eta)$  is also written as  $\beta(K_\Psi(\cdot, \zeta) - {}^\Psi R_{K_\Psi(\cdot, \zeta)}^{\Psi \setminus \Phi})$  on  $\Phi \cap \Psi$  for some positive constant  $\beta$ . Then we have by Lemma 3.1 with  $D := \Phi \cap \Psi$  and  $\Omega := \Psi$  and its remark that the minimal fine limit of  $K_\Psi(\cdot, \zeta)/K_{\Phi \cap \Psi}(\cdot, \eta)$  at  $\eta$  with respect to  $\Phi \cap \Psi$  exists and

$$0 < \text{mf} - \lim_{\Phi \cap \Psi} \frac{K_\Psi(x, \zeta)}{K_{\Phi \cap \Psi}(x, \eta)} < \infty.$$

Therefore (ii) follows from (4.1) and Lemma 3.4 with  $D := \Phi \cap \Psi$  and  $\Omega := \Psi$ .

To prove (iii), we assume in addition that  $\Psi \setminus (\Phi \cap \Psi)$  is minimally thin at  $\zeta$  with respect to  $\Psi$ , where  $\zeta$  is a point in  $\Delta_1(\Psi)$  such that (2.1) is not satisfied. Then the normalization  $K_{\Phi \cap \Psi}(\cdot, \omega)$  of  $K_\Psi(\cdot, \zeta) - {}^\Psi R_{K_\Psi(\cdot, \zeta)}^{\Psi \setminus \Phi}$  at a reference point is a minimal Martin kernel of  $\Phi \cap \Psi$ , but is different from  $K_{\Phi \cap \Psi}(\cdot, \eta)$ . We note from the uniqueness in Lemma 3.1 that for only  $\omega \in \Delta_1(\Phi \cap \Psi)$ ,  $K_\Psi(\cdot, \zeta)/K_{\Phi \cap \Psi}(\cdot, \omega)$  has a positive minimal fine limit at  $\omega$  with respect to  $\Phi \cap \Psi$ . Therefore we have that

$$\text{mf} - \lim_{\Phi \cap \Psi} \frac{K_\Psi(x, \zeta)}{K_{\Phi \cap \Psi}(x, \eta)} = 0.$$

Hence (iii) follows from (4.1) and Lemma 3.4 with  $D := \Phi \cap \Psi$  and  $\Omega := \Phi$ . Thus Theorem 2.1 is established.  $\square$

*Proof of Corollary 2.2.* Let  $y \in \partial\Phi \cap \partial\Psi$ . We first show (i). By Theorem 2.1 (i) and Lemma 3.4, we have that  $K_\Psi(\cdot, y)/K_\Phi(\cdot, y)$  has minimal fine limit 0 at  $y$  with respect to  $\Phi \cap \Psi$ . Since the non-tangential region  $\Gamma_c(y)$  is not minimally thin at  $y$  with respect to  $\Phi \cap \Psi$  (cf. [8, Section 5]), the existence of the minimal fine limit of  $K_\Psi(\cdot, y)/K_\Phi(\cdot, y)$  with respect to  $\Phi \cap \Psi$  implies the existence of the non-tangential limit with respect to  $\Phi \cap \Psi$ , and the both values coincide. Hence (i) follows.

We next show (ii). We observe that  $K_\Phi(\cdot, y)$  and  $K_\Psi(\cdot, y)$  satisfies (2.1) on  $\Phi \cap \Psi$ , since  $K_\Phi(\cdot, y) - {}^\Phi R_{K_\Phi(\cdot, y)}^{\Phi \setminus \Psi}$  and  $K_\Psi(\cdot, y) - {}^\Psi R_{K_\Psi(\cdot, y)}^{\Psi \setminus \Phi}$  are minimal harmonic functions on  $\Phi \cap \Psi$  with pole at  $y$ . Therefore (ii) follows from Theorem 2.1 (ii).  $\square$

## 5. PROOF OF THEOREM 1.1

In order to prove Theorem 1.1, we collect lemmas on relationships between the convergence of the integrals  $I^+, I^-$  in (1.1), (1.2) and the minimal thinness of the differences  $\Omega_\phi \setminus \mathbb{R}_+^n, \mathbb{R}_+^n \setminus \Omega_\phi$ . See [7, Lemma 1 and Proof of Theorem 1] for Lemma 5.1 and [6, Theorem 4.2] for Lemma 5.2.

**Lemma 5.1.** *The following statements hold.*

- (i)  $I^+ < \infty$  if and only if  $\mathbb{R}_+^n \setminus \Omega_\phi$  is minimally thin at  $\mathbf{0}$  with respect to  $\mathbb{R}_+^n$ .
- (ii) If  $I^+ < \infty$  and  $I^- = \infty$ , then  $\Omega_\phi \setminus \mathbb{R}_+^n$  is not minimally thin at  $\mathbf{0}$  with respect to  $\Omega_\phi$ .

**Lemma 5.2.** *Let  $\Omega$  be a Greenian domain in  $\mathbb{R}^n$  containing  $\mathbb{R}_+^n$ . Suppose that  $\Omega$  has a unique Martin boundary point at infinity and it is minimal. If  $\Omega \setminus \overline{\mathbb{R}_+^n}$  is minimally thin at  $\infty$  with respect to  $\mathbb{R}_-^n := \{(x', x_n) : x_n < 0\}$ , then  $\Omega \setminus \mathbb{R}_+^n$  is minimally thin at  $\infty$  with respect to  $\Omega$ .*

**Lemma 5.3.** *If  $I^- < \infty$ , then  $\Omega_\phi \setminus \mathbb{R}_+^n$  is minimally thin at  $\mathbf{0}$  with respect to  $\Omega_\phi \cup \mathbb{R}_+^n$ .*

*Proof.* By Lemma 5.1, we see that  $\Omega_\phi \setminus \overline{\mathbb{R}_+^n}$  is minimally thin at  $\mathbf{0}$  with respect to  $\mathbb{R}_-^n$ . Since minimal thinness is invariant under the inversion with respect to the unit sphere, it follows from Lemma 5.2 that  $\Omega_\phi \setminus \mathbb{R}_+^n$  is minimally thin at  $\mathbf{0}$  with respect to  $\Omega_\phi \cup \mathbb{R}_+^n$ .  $\square$

**Lemma 5.4.** *If  $I^+ < \infty$  and  $I^- < \infty$ , then  $\Omega_\phi \setminus \mathbb{R}_+^n$  is minimally thin at  $\mathbf{0}$  with respect to  $\Omega_\phi$ .*

*Proof.* We note from Lemma 5.3 that  $(\Omega_\phi \cup \mathbb{R}_+^n) \setminus \mathbb{R}_+^n$  is minimally thin at  $\mathbf{0}$  with respect to  $\Omega_\phi \cup \mathbb{R}_+^n$ . Therefore we see from Lemmas 3.4 and 5.1 that  $(\Omega_\phi \cup \mathbb{R}_+^n) \setminus \Omega_\phi$  is minimally thin at  $\mathbf{0}$  with respect to  $\Omega_\phi \cup \mathbb{R}_+^n$ . Applying Lemma 3.4 again, we obtain the lemma.  $\square$

Let us prove Theorem 1.1.

*Proof of Theorem 1.1.* We can easily obtain (i) and (iii) from Corollary 2.2 with  $\Phi := \mathbb{R}_+^n$  and  $\Psi := \Omega_\phi$  and Lemmas 5.1 and 5.4. We show (ii). Since  $(\Omega_\phi \cup \mathbb{R}_+^n) \setminus \mathbb{R}_+^n$  is minimally thin at  $\mathbf{0}$  with respect to  $\Omega_\phi \cup \mathbb{R}_+^n$  by Lemma 5.3, we have by Lemma 3.1 with  $D := \mathbb{R}_+^n$  and  $\Omega := \Omega_\phi \cup \mathbb{R}_+^n$  that  $K_{\Omega_\phi \cup \mathbb{R}_+^n}(\cdot, \mathbf{0})/K_{\mathbb{R}_+^n}(\cdot, \mathbf{0})$  has a positive minimal fine limit at  $\mathbf{0}$  with respect to  $\mathbb{R}_+^n$ . Therefore  $t^{n-1}K_{\Omega_\phi \cup \mathbb{R}_+^n}(te, \mathbf{0})$  has a positive limit as  $t \rightarrow 0+$ . Also, it follows from Lemmas 3.4 and 5.1 that  $(\Omega_\phi \cup \mathbb{R}_+^n) \setminus \Omega_\phi$  is not minimally thin at  $\mathbf{0}$  with respect to  $\Omega_\phi \cup \mathbb{R}_+^n$ . Therefore we have by Lemma 3.1 with  $D := \Omega_\phi$  and  $\Omega := \Omega_\phi \cup \mathbb{R}_+^n$  that  $K_{\Omega_\phi \cup \mathbb{R}_+^n}(\cdot, \mathbf{0})/K_{\Omega_\phi}(\cdot, \mathbf{0})$  has minimal fine limit 0 at  $\mathbf{0}$  with respect to  $\Omega_\phi$ , and so  $K_{\Omega_\phi}(te, \mathbf{0})/K_{\Omega_\phi \cup \mathbb{R}_+^n}(te, \mathbf{0})$  has limit  $\infty$  as  $t \rightarrow 0+$ . Thus we conclude that  $t^{n-1}K_{\Omega_\phi}(te, \mathbf{0})$  has limit  $\infty$  as  $t \rightarrow 0+$ .  $\square$

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