DRAPEAU THEOREM FOR DIFFERENTIAL SYSTEMS

KAZUHIRO SHIBUYA AND KEIZO YAMAGUCHI

Abstract. Generalizing the theorem for Goursat flags, we will characterize those flags which are obtained by "Rank 1 Prolongation" from the space of 1 jets for 1 independent and m dependent variables.

1. Introduction

This paper is concerned with the Drapeau theorem for differential systems. By a differential system (R, D), we mean a distribution D on a manifold R, i.e., D is a subbundle of the tangent bundle $T(R)$. The *derived system* ∂D of D is defined, in terms of sections, by

$$\partial\mathcal{D} = \mathcal{D} + [\mathcal{D}, \mathcal{D}].$$

where $\mathcal{D} = \Gamma(D)$ denotes the space of sections of D. In general ∂D is obtained as a subsheaf of the tangent sheaf of R (for the precise argument, see e.g.[Y1], [BCG3]). Moreover higher derived systems $\partial^i D$ are defined successively by

$$\partial^i D = \partial(\partial^{i-1} D),$$

where we put $\partial^0 D = D$ by convention. In this paper, a differential system (R, D) is called regular if $\partial^i D$ are subbundles of $T(M)$ for every $i \geq 1$.

We say that (R, D) is an m-flag of length k, if (R, D) is regular and has a derived length k, i.e., $\partial^k D = T(R)$;

$$D \subset \partial D \subset \cdots \subset \partial^{k-2} D \subset \partial^{k-1} D \subset \partial^k D = T(R),$$

such that $\operatorname{rank} D = m+1$ and $\operatorname{rank}\ \partial^i D = \operatorname{rank} \partial^{i-1} D + m$ for $i = 1, \ldots, k$. In particular $\dim R = (k+1)m + 1$.

Especially (R, D) is called a *Goursat flag* (un drapeau de Goursat) of length k when $m = 1$. Historically, by Engel, Goursat and Cartan, it is known that a Goursat flag (R, D) of length k is locally isomorphic, at a generic point, to the canonical system $(J^k(M, 1), C^k)$ on the k-jet spaces of 1 independent and 1 dependent variable (for the definition of the canonical system $(J^k(M, 1), C^k)$, see §2). The characterization of the canonical (contact) systems on jet spaces was given by R. Bryant in [B] for the first order systems and in [Y1] and [Y2] for higher order systems for n independent and m dependent variables. However, it was first explicitly exhibited by A.Giaro, A. Kumpera and C. Ruiz in [GKR] that a Goursat flag of length 3 has singularities and the research of singularities of Goursat flags of length k ($k \geq 3$) began as in [M1]. To this situation, R. Montgomery and M. Zhitomirskii constructed the "Monster Goursat manifold" by successive applications of the "Cartan prolongation of rank 2 distributions [BH]" to a surface and showed that every germ of a Goursat flag (R, D) of length k appears in this "Monster Goursat manifold" in [MZ] , by first exhibitting the following Sandwich Lemma for (R, D);

$$\begin{array}{ccccccc} & D & \subset & \partial D & \subset\cdots\subset & \partial^{k-2}D & \subset\partial^{k-1}D\subset \partial^k D = T(R) \\ & \cup & & \cup & & \cup & \\ \mathrm{Ch}\,(D)\subset & \mathrm{Ch}\,(\partial D) & \subset & \mathrm{Ch}\,(\partial^2 D) & \subset\cdots\subset & \mathrm{Ch}\,(\partial^{k-1}D) & \end{array}$$

where $\mathrm{Ch}\,(\partial^i D)$ is the Cauchy characteristic system of $\partial^i D$ and $\mathrm{Ch}\,(\partial^i D)$ is a subbundle of $\partial^{i-1}D$ of corank 1 for $i = 1, \ldots, k-1$. Here the *Cauchy Characteristic System* $\mathrm{Ch}\,(C)$ of a differential system (R, C) is defined by

$$\mathrm{Ch}\,(C)(x) = \{X \in C(x) \mid X \rfloor d\omega_i \equiv 0 \quad (\mathrm{mod}\ \ \omega_1, \ldots, \omega_s) \quad \text{for } i = 1, \ldots, s\},$$

where $C = \{\,\omega_1 = \cdots = \omega_s = 0\,\}$ is defined locally by defining 1-forms $\{\omega_1, \ldots, \omega_s\}$. Moreover, after [MZ], P.Mormul defined the notion of a *special* m*- flag* of length k for $m \geq 2$ to characterize those m-flags which are obtained by successive applications of the "generalized Cartan prolongation" to the space of 1-jets of 1 independent and m dependent variables.

The main purpose of this paper is first to clarify the procedure of "Rank 1 Prolongation" of an arbitrary differential system (R, D) of rank $m+1$, and to give good criteria for an m-flag of length k to be special,i.e., to be locally isomorphic to the k-th Rank 1 Prolongation $(P^k(M), C^k)$ of a manifold M of dimension $m+1$. More precisely we will show for an m-flag of length k for $m \geq 3$;

Corollary 5.8. *An m-flag (R, D) of length k for $m \geq 3$ is locally isomorphic to $(P^k(M), C^k)$ if and only if $\partial^{k-1}D$ is of Cartan rank 1, and, moreover for $m \geq 4$, if and only if $\partial^{k-1}D$ is of Engel rank 1.*

Here, the *Cartan rank* of (R, C) is the smallest integer ρ such that there exist 1-forms $\{\pi^1, \ldots, \pi^\rho\}$, which are independent modulo $\{\omega_1, \ldots, \omega_s\}$ and satisfy

$$d\alpha \wedge \pi^1 \wedge \cdots \wedge \pi^\rho \equiv 0 \quad (\mathrm{mod}\ \ \omega_1, \ldots, \omega_s) \qquad \text{for } \forall\alpha \in \mathcal{C}^\perp = \Gamma(C^\perp),$$

where $C = \{\omega_1 = \cdots = \omega_s = 0\ \}$. Furthermore the *Engel (half) rank* of (R, C) is the smallest integer ρ such that

$$(d\alpha)^{\rho+1} \equiv 0 \quad (\mathrm{mod}\ \ \omega_1, \ldots, \omega_s) \qquad \text{for } \forall\alpha \in \mathcal{C}^\perp,$$

Moreover we will show for an m-flag of length k for $m \geq 2$,

Corollary 6.3. *An m-flag (R, D) of length k is locally isomorphic to $(P^k(M), C^k)$ if and only if there exists a completely integrable subbundle F of $\partial^{k-1}D$ of corank 1.*

For this purpose, we will first review the geometric construction of jet spaces in §2 and clarify the procedure of Rank 1 Prolongation in §3. In §4, we will analyze the notion of a special m-flag of length k and reestablish the local characterization of $(P^k(M), C^k)$ by utilizing the Realization Lemma [Y1]. In §5 and §6, we will show the above criteria (the Drapeau Theorem) for an m-flag of length k.

2. Geometric construction of Jet Spaces

In this section, we will briefly recall the geometric construction of jet bundles in general, following [Y1] and [Y2], which is our basis for the later considerations.

Let M be a manifold of dimension $m+n$. Fixing the number n, we form the space of n-dimensional *contact elements* to M, i.e., the Grassmann bundle $J(M,n)$ over M consisting of n-dimensional subspaces of tangent spaces to M. Namely, $J(M,n)$ is defined by

$$J(M,n) = \bigcup_{x\in M} J_x, \qquad J_x = \mathrm{Gr}(T_x(M),n),$$

where $\mathrm{Gr}(T_x(M),n)$ denotes the Grassmann manifold of n-dimensional subspaces in $T_x(M)$. Let $\pi : J(M,n) \to M$ be the bundle projection. The *canonical system* C on $J(M,n)$ is, by definition, the differential system of codimension m on $J(M,n)$ defined by

$$C(u) = \pi_*^{-1}(u) = \{v \in T_u(J(M,n)) \mid \pi_*(v) \in u\} \subset T_u(J(M,n)) \xrightarrow{\pi_*} T_x(M),$$

where $\pi(u) = x$ for $u \in J(M,n)$.

Let us describe C in terms of a canonical coordinate system in $J(M,n)$. Let $u_o \in J(M,n)$. Let $(x_1,\ldots,x_n,z^1,\ldots,z^m)$ be a coordinate system on a neighborhood U' of $x_o = \pi(u_o)$ such that $dx_1,\ldots,dx_n$ are linearly independent when restricted to $u_o \subset T_{x_o}(M)$. We put $U = \{\ u \in \pi^{-1}(U') | dx_1|_u,\ldots,dx_n|_u$ are linearly independent $\}$. Then U is a neighborhood of u_o in $J(M,n)$. Here $dz^\alpha|_u$ is a linear combination of $dx_i|_u$'s,i.e., $dz^\alpha|_u = \sum_{i=1}^n p_i^\alpha(u) dx_i|_u$. Thus, there exist unique functions p_i^α on U such that C is defined on U by the following 1-forms;

$$\varpi^\alpha = dz^\alpha - \sum_{i=1}^n p_i^\alpha dx_i \qquad (\alpha = 1,\ldots,m),$$

where we identify z^α and x_i on U' with their lifts on U. The system of functions $(x_i, z^\alpha, p_i^\alpha)$ $(\alpha = 1,\ldots,m, i = 1,\ldots,n)$ on U is called a *canonical coordinate system* of $J(M,n)$ subordinate to (x_i, z^α).

$(J(M,n),C)$ is the (geometric) 1-jet space and especially, in case $m=1$, is the so-called contact manifold. Let M, $\hat{M}$ be manifolds of dimension $m+n$ and $\varphi : M \to \hat{M}$ be a diffeomorphism. Then φ induces the isomorphism $\varphi_* : (J(M,n),C) \to (J(\hat{M},n),\hat{C})$, i.e., the differential map $\varphi_* : J(M,n) \to J(\hat{M},n)$ is a diffeomorphism sending C onto $\hat{C}$. The reason why the case $m=1$ is special is explained by the following theorem of Bäcklund.

Theorem(Bäcklund) *Let M and $\hat{M}$ be manifolds of dimension $m+n$. Assume $m \geq 2$. Then, for an isomorphism $\Phi : (J(M,n),C) \to (J(\hat{M},n),\hat{C})$, there exists a diffeomorphism $\varphi : M \to \hat{M}$ such that $\Phi = \varphi_*$.*

The essential part of this theorem is to show that $F = \mathrm{Ker}\,\pi_*$ is the covariant system of $(J(M,n),C)$ for $m \geq 2$. Namely an isomorphism Φ sends F onto $\hat{F} = \mathrm{Ker}\,\hat{\pi}_*$ for $m \geq 2$. For the proof, we refer the reader to Theorem 1.4 in [Y2].

In case $m = 1$, it is a well known fact that the group of isomorphisms of $(J(M,n),C)$, i.e., the group of contact transformations, is larger than the group of diffeomorphisms of M. Therefore, when we consider the geometric 2-jet spaces, the situation differs according to whether the number m of dependent variables is 1 or greater.

(1) Case $m=1$. We should start from a contact manifold (J,C) of dimension $2n+1$, which is locally a space of 1-jet for one dependent variable by Darboux's theorem. Then we can construct the geometric second order jet space $(L(J),E)$ as follows: We consider the Lagrange-Grassmann bundle $L(J)$ over J consisting of all n-dimensional integral elements of (J,C);

$$L(J)=\bigcup_{u\in J} L_u \subset J(J,n),$$

where L_u is the Grassmann manifolds of all Lagrangian (or Legendrian) subspaces of the symplectic vector space $(C(u),d\varpi)$. Here ϖ is a local contact form on J. Namely, $v\in J(J,n)$ is an integral element if and only if $v\subset C(u)$ and $d\varpi|_v=0$, where $u=\pi(v)$. Let $\pi: L(J)\to J$ be the projection. Then the canonical system E on $L(J)$ is defined by

$$E(v)=\pi_*^{-1}(v)\subset T_v(L(J)) \xrightarrow{\pi_*} T_u(J),$$

where $\pi(v)=u$ for $v\in L(J)$. We have $\partial E=\pi_*^{-1}(C)$ and $\mathrm{Ch}\,(C)=\{0\}$ (cf.[Y1]). Hence we get $\mathrm{Ch}\,(\partial E)=\mathrm{Ker}\,\pi_*$, which implies the Bäcklund theorem for $(L(J),E)$ (cf. [Y1]).

Now we put

$$(J^2(M,n),C^2)=(L(J(M,n)),E),$$

where M is a manifold of dimension $n+1$.

(2) Case $m\geq 2$. Since $F=\mathrm{Ker}\,\pi_*$ is a covariant system of $(J(M,n),C)$, we define $J^2(M,n)\subset J(J(M,n),n)$ by

$$J^2(M,n)=\{n\text{-dim. } \textit{integral elements} \text{ of } (J(M,n),C),\ \text{transversal to } F\},$$

C^2 is defined as the restriction to $J^2(M,n)$ of the canonical system on $J(J(M,n),n)$.

Now the higher order (geometric) jet spaces $(J^{k+1}(M,n),C^{k+1})$ for $k\geq 2$ are defined (simultaneously for all m) by induction on k. Namely, for $k\geq 2$, we define $J^{k+1}(M,n)\subset J(J^k(M,n),n)$ and C^{k+1} inductively as follows:

$$J^{k+1}(M,n)=\{n\text{-dim. } \textit{integral elements} \text{ of } (J^k(M,n),C^k),\ \text{transversal to Ker } (\pi^k_{k-1})_*\},$$

where $\pi^k_{k-1}: J^k(M,n)\to J^{k-1}(M,n)$ is the projection. Here we have

$$\mathrm{Ker}\ (\pi^k_{k-1})_*=\mathrm{Ch}\,(\partial C^k),$$

and C^{k+1} is defined as the restriction to $J^{k+1}(M,n)$ of the canonical system on $J(J^k(M,n),n)$. Then we have ([Y1],[Y2])

$$\begin{array}{ccccccc}
 & C^k & \subset\cdots\subset & \partial^{k-2}C^k & \subset\partial^{k-1}C^k\subset & \partial^k C^k = T(J^k(M,n)) \\
 & \cup & & \cup & \cup & \\
\{0\}=\mathrm{Ch}\,(C^k)\subset & \mathrm{Ch}\,(\partial C^k)\subset\cdots\subset & & \mathrm{Ch}\,(\partial^{k-1}C^k)\subset & F &
\end{array}$$

where $\mathrm{Ch}\,(\partial^{i+1}C^k)$ is a subbundle of $\partial^i C^k$ of corank n for $i=0,\ldots,k-2$ and, when $m\geq 2$, F is a subbundle of $\partial^{k-1}C^k$ of corank n. The transversality conditions are expressed as

$$C^k\cap F=\mathrm{Ch}\,(\partial C^k)\quad \text{for } m\geqq 2,\qquad C^k\cap \mathrm{Ch}\,(\partial^{k-1}C^k)=\mathrm{Ch}\,(\partial C^k)\quad \text{for } m=1$$

By the above diagram together with the rank condition, Jet spaces $(J^k(M,n), C^k)$ can be characterized as higher order contact manifolds as in [Y1] and [Y2].

Here we observe that, if we drop the transversality condition in our definition of $J^k(M,n)$ and collect all n-dimensional integral elements, we may have some singularities in $J^k(M,n)$ in general. However, since every 2-form vanishes on 1-dimensional subspaces, in case $n=1$, the integrability condition for $v \in J(J^{k-1}(M,1),1)$ reduces to $v \subset C^{k-1}(u)$ for $u = \pi^k_{k-1}(v)$. Hence, in this case, we can safely drop the transversality condition in the above construction as in the next section, which constitutes the key construction for the Drapeau theorem in later considerations.

3. Rank 1 Prolongation

Let (R, D) be a differential system, i.e., R is a manifold of dimension $s+m+1$ and D is a subbundle of $T(R)$ of rank $m+1$. Starting from (R, D), we define $(P(R), \widehat{D})$ as follows (cf. [BH]):

$$P(R) = \bigcup_{x \in R} P_x \subset J(R, 1),$$

where

$$P_x = \{\text{1-dim. integral elements of } (R, D)\} = \{u \subset D(x) \mid \text{1-dim. subspaces}\} \cong \mathbb{P}^m.$$

Let $p : P(R) \to R$ be the projection. We define the canonical system $\widehat{D}$ on $P(R)$ by

$$\widehat{D}(u) = p_*^{-1}(u) = \{v \in T_u(P(R)) \mid p_*(v) \in u\} \subset T_u(P(R)) \xrightarrow{p_*} T_x(R),$$

where $p(u) = x$ for $u \in P(R)$.

We call $(P(R), \widehat{D})$ the *prolongation of rank 1* (or *Rank* 1 *Prolongation* for short) of (R, D). Then $P(R)$ is a manifold of dimension $2m+s+1$ and $\widehat{D}$ is a differential system of rank $m+1$. In case $(R, D) = (M, T(M))$, we have $(P(M), \widehat{D}) = (J(M,1), C)$. Moreover

$$J^2(M,1) \subset P(J(M,1)) \subset J(J(M,1),1)$$

As for the prolongation of rank 1, we have

Proposition 3.1. *Let (R, D) be a differential system of rank $m+1$ and let $(P(R), \widehat{D})$ be the prolongation of rank 1 of (R, D). Then $\widehat{D}$ is of rank $m+1$, $\partial\widehat{D} = p_*^{-1}(D)$ and* $\mathrm{Ch}\,(\widehat{D})$ *is trivial. Moreover, if* $\mathrm{Ch}\,(D)$ *is trivial, then* $\mathrm{Ch}\,(\partial\widehat{D})$ *is a subbundle of $\widehat{D}$ of corank* 1.

Proof. Let $s+m+1$ be the dimension of R. For $x \in R$, let $\{\varpi^\beta,\ \theta^\alpha\}$ $(\alpha = 1, \cdots, m+1\ ,\ \beta = 1, \cdots, s)$ be a coframe on a neighborhood U of x such that

$$D = \{\varpi^1 = \cdots = \varpi^s = 0\}.$$

$p^{-1}(U)$ is covered by $m+1$ open sets $\hat{U}_i = \{v \in p^{-1}(U) \mid \theta^i|_v \neq 0\}$ in $P(R)$:

$$p^{-1}(U) = \hat{U}_1 \cup \cdots \cup \hat{U}_{m+1}.$$

For $v \in \hat{U}_i$, v is a 1-dimensional subspace of $T_x(R), x = p(v)$. Hence, restricting θ^α to v, we have

$$\theta^\alpha|_v = p^\alpha_i(v)\theta^i|_v \quad \text{for} \quad \alpha = 1, \cdots, \check{i}, \cdots, m+1$$

where $\check{}$ over a symbol means that symbol is deleted. These p_i^α $(\alpha = 1, \cdots, \check{i}, \cdots, m+1)$ constitute a fiber coordinate on $\hat{U}_i$.

Now we put

$$\pi_i^\alpha = \theta^\alpha - p_i^\alpha \theta^i \quad \text{for} \quad \alpha = 1, \cdots, \check{i}, \cdots, m+1.$$

Then we have

$$\widehat{D} = \{p^*\varpi^1 = \cdots = p^*\varpi^s = \pi_i^\alpha = 0 \quad (\alpha = 1, \cdots, \check{i}, \cdots, m+1)\}.$$

Since $d\varpi^\beta, d\theta^\alpha$ are 2-forms on M, $d\varpi^\beta|_u = 0, d\theta^\alpha|_u = 0$ for $u \in P(R)$. These imply that

$$d\varpi^\beta \equiv d\theta^\alpha \equiv 0 \quad (\text{mod} \quad \varpi^1, \ldots, \varpi^s, \pi_i^\alpha \quad (\alpha = 1, \cdots, \check{i}, \cdots, m+1)),$$

where we write $\varpi^\beta, \theta^\alpha$ instead of $p^*\varpi^\beta, p^*\theta^\alpha$, respectively.

Thus the structure equation for $\widehat{D}$ reads

$$\begin{cases} d\varpi^\beta & \equiv 0 \qquad\qquad (\text{mod} \quad \varpi^1, \ldots, \varpi^s, \pi_i^\alpha \quad (\alpha = 1, \cdots, \check{i}, \cdots, m+1)) \\ d\pi_i^\alpha & \equiv \theta^i \wedge dp_i^\alpha \quad (\text{mod} \quad \varpi^1, \ldots, \varpi^s, \pi_i^\alpha \quad (\alpha = 1, \cdots, \check{i}, \cdots, m+1)) \end{cases}$$

Therefore

$$\partial\widehat{D} = \{\varpi^1 = \cdots = \varpi^s = 0\},$$

$$\mathrm{Ch}\,(\widehat{D}) = \{\varpi^1 = \cdots = \varpi^s = \pi_i^\alpha = \theta^i = dp_i^\alpha = 0 \quad (\alpha = 1, \cdots, \check{i}, \cdots, m+1)\}$$

These imply that $\partial\widehat{D} = p_*^{-1}(D)$ and $\mathrm{Ch}\,(\widehat{D})$ is trivial.

Moreover, if $Ch(D)$ is trivial, it follows that

$$\mathrm{Ch}\,(\partial\widehat{D}) = \mathrm{Ch}\,(p_*^{-1}(D)) = p_*^{-1}(\mathrm{Ch}\,(D)) = \mathrm{Ker}\ \ p_*$$

Then, by the very definition of canonical system $\widehat{D}$, it follows that $\mathrm{Ch}\,(\partial\widehat{D})$ is a subbundle of $\widehat{D}$ of corank 1. □

This proposition implies that, starting from any differential system (R, D), we can repeat the procedure of Rank 1 Prolongation. Let $(P^1(R), D^1)$ be the prolongation of rank 1 of (R, D). Then $(P^k(R), D^k)$ is defined inductively as the prolongation of rank 1 of $(P^{k-1}(R), D^{k-1})$, which is called *k-th prolongation of rank 1* of (R, D). Moreover, starting from a manifold M of dimension $m+1$, we put

$$(P^k(M), C^k) = (P(P^{k-1}(M)), \widehat{C}^{k-1})$$

where $(P^1(M), C^1) = (J(M, 1), C)$. When $m = 1$, $(P^k(M), C^k)$ are called "monster Goursat manifolds" in [MZ].

Here we observe that the above proposition also implies

Proposition 3.2. *Let (R, D) be an m-flag of length* 1*, i.e.,* $\dim R = 2m+1$, $\mathrm{rank}\, D = m+1$ *and* $\partial D = T(R)$. *Then the k-th prolongation $(P^k(R), D^k)$ of rank 1 of (R, D) is an m-flag of length $k+1$. Namely, D^k satisfies* $\mathrm{rank}\, D^k = m+1$, $\mathrm{rank}\ \partial^{i+1}D^k = \mathrm{rank}\ \partial^i D^k + m$ *for $i = 0, \ldots, k$ and $\partial^{k+1}D^k = T(P^k(R))$.*

Schematically we have the following diagram;

$$
\begin{array}{ccccccccccc}
D^k & \subset & \partial D^k & \subset & \cdots\cdots\cdots & \subset & \partial^{k-1}D^k & \subset & \partial^k D^k & \subset & \partial^{k+1}D^k = T(P^k(R)) \\
 & & \downarrow{\scriptstyle p^k_*} & & & & \downarrow{\scriptstyle p^k_*} & & \downarrow{\scriptstyle p^k_*} & & \downarrow{\scriptstyle p^k_*} \\
 & & D^{k-1} & \subset & \cdots\cdots\cdots & \subset & \partial^{k-2}D^{k-1} & \subset & \partial^{k-1}D^{k-1} & \subset & \partial^k D^{k-1} = T(P^{k-1}(R)) \\
 & & & & \downarrow{\scriptstyle p^{k-1}_*} & & \downarrow{\scriptstyle p^{k-1}_*} & & \downarrow{\scriptstyle p^{k-1}_*} & & \downarrow{\scriptstyle p^{k-1}_*} \\
 & & & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & & & & & & \vdots & & & \\
 & & & & & & & \vdots & & & \\
 & & & & & & \vdots & & \vdots & & \vdots \\
 & & & & & & \downarrow{\scriptstyle p^2_*} & & \downarrow{\scriptstyle p^2_*} & & \downarrow{\scriptstyle p^2_*} \\
 & & & & & & D^1 & \subset & \partial D^1 & \subset & \partial^2 D^1 = T(P^1(R)) \\
 & & & & & & & & \downarrow{\scriptstyle p^1_*} & & \downarrow{\scriptstyle p^1_*} \\
 & & & & & & & & D & \subset & \partial D = T(R)
\end{array}
$$

where $p^i : P^i(R) \to P^{i-1}(R)$ is the projection. Here we note

$$\partial^k D^k = (p^k_0)^{-1}_*(D),$$

where $p^k_0 : P^k(R) \to R$ is the projection.

4. Special m-Flags of length k

An m- flag (R, D) $(m \geq 2)$ of length k is called a *special m-flag* if there exists a completely integrable subbundle F of $\partial^{k-1}D$ of corank 1, which contains $\mathrm{Ch}\,(\partial^{k-1}D)$, and $\mathrm{Ch}\,(\partial^i D)$ is a subbundle of $\partial^{i-1}D$ of corank 1 for $i = 1, \ldots, k-1$, such that $\mathrm{Ch}\,(D)$ is trivial, i.e., if the following diagram holds for (R, D);

$$
\begin{array}{ccccccccccc}
 & & D & \subset & \partial D & \subset\cdots\subset & \partial^{k-2}D & \subset & \partial^{k-1}D & \subset & \partial^k D = T(R) \\
 & & \cup & & \cup & & \cup & & \cup & & \\
\{0\} = \mathrm{Ch}\,(D) & \subset & \mathrm{Ch}\,(\partial D) & \subset & \mathrm{Ch}\,(\partial^2 D) & \subset\cdots\subset & \mathrm{Ch}\,(\partial^{k-1}D) & \subset & F & &
\end{array}
$$

where $\mathrm{rank}\,\partial^i D = \mathrm{rank}\,\partial^{i-1}D + m$ for $i = 1, \ldots, k$ and $\mathrm{rank}\,D = m + 1$.

First, by repeated use of Rank 1 prolongations starting from a manifold M of dimension $m + 1$, we obtain by Proposition 3.1,

Proposition 4.1. *$(P^k(M), C^k)$ is a special m-flag of length k.*

Conversely, by utilizing the following Realization Lemma, we will show that every special m- flag of length k is locally isomorphic to $(P^k(M), C^k)$.

Realization Lemma ([Y1;p.122]) *Let R and M be manifolds. Assume that the quadruple (R, D, p, M) satisfies the following conditions:*

(i) p is a map of R into M of constant rank.

(ii) D is a differential system on R such that $F = \operatorname{Ker} p_*$ is a subbundle of D of corank n.

Then there exists a unique map ψ of R into $J(M, n)$ satifying $p = \pi \cdot \psi$ and $D = \psi_*^{-1}(C)$. Furthermore, let u be any point of R. Then ψ is in fact defined by

$$\psi(u) = p_*(D(u)) \quad \text{as a point of } Gr(T_x(M), n), \quad x = \pi(u),$$

and satisfies $\operatorname{Ker}(\psi_*)_u = F(u) \cap \operatorname{Ch}(D)(u)$.

Theorem 4.2. *A special m-flag (R, D) of length k is locally isomorphic to $(P^k(M), C^k)$, where M is a manifold of dimension $m + 1$. Especially F is unique for (R, D).*

Proof. Let (R, D) be a special m-flag of length k. Matters being of local nature, we may assume that the leaf space $M = R/F$ of the foliation F defined on R is a manifold of dimension $2m + 1$ so that $p : R \to M$ is a submersion and $\operatorname{Ker} p_* = F$. Putting $p = \psi^0$, we will define maps $\psi^i : R \to P^i(M)$ such that $\operatorname{Ker} \psi^i_* = \operatorname{Ch}(\partial^{k-i} D)$ for $i = 1, \ldots, k$ as follows; First, Realization Lemma for the quadruple $(R, \partial^{k-1} D, p, M)$ gives us the map ψ^1 of R into $P^1(M) = J(M, 1)$ such that $(\psi^1_*)^{-1}(C^1) = \partial^{k-1} D$ and $\operatorname{Ker}(\psi^1)_* = \operatorname{Ch}(\partial^{k-1} D)$. By dimension count, we see that ψ^1 is locally a submersion of R onto $P^1(M)$. $\psi^j : R \to P^j(M)$ such that $\operatorname{Ker} \psi^j_* = \operatorname{Ch}(\partial^{k-j} D)$ being defined for $j = 1, \ldots, i-1$, Realization Lemma for $(R, \partial^{k-i} D, \psi^{i-1}, P^{i-1}(M))$ gives us the map ψ^i of R into $P^i(M)$ such that $(\psi^i_*)^{-1}(C^i) = \partial^{k-i} D$ and $\operatorname{Ker}(\psi^i)_* = \operatorname{Ch}(\partial^{k-i} D)$. Thus, for $i = k$, we obtain the map ψ^k of R into $P^k(M)$ such that $(\psi^k_*)^{-1}(C^k) = D$ and $\operatorname{Ker}(\psi^k)_* = \operatorname{Ch}(D) = \{0\}$. Then, by dimension count, ψ^k is a local isomorphism of (R, D) onto $(P^k(M), C^k)$.

For the uniqueness of F, we first observe that, for a special m-flag (R, D) of length 1, ψ^1 is an isomophism of (R, D) onto $(J(M, 1), C)$. In this case, the uniqueness of F follows from Proposition 1.3 in [Y2], which gives the characterization of the covariant system F. For a special m-flag (R, D) of length k $(k \geq 2)$, we consider, locally, the leaf space $\bar{J} = R/\operatorname{Ch}(\partial^{k-1} D)$ by $\operatorname{Ch}(\partial^{k-1} D)$. Let $\bar{p} : R \to \bar{J}$ be the projection. On $\bar{J}$, we have differential systems $\bar{D} = \partial^{k-1} D/\operatorname{Ch}(\partial^{k-1} D)$ and $\bar{F} = F/\operatorname{Ch}(\partial^{k-1} D)$ such that $\bar{F}$ is a completely integrable subbundle of $\bar{D}$ of corank 1 and $\operatorname{Ch}(\bar{D})$ is trivial, i.e., $(\bar{J}, \bar{D})$ is a special m-flag of length 1. Then the uniqueness of $F = \bar{p}_*^{-1}(\bar{F})$ follows from that of $\bar{F}$. This completes the proof of Theorem. □

Remark 4.3. After [MZ], P.Mormul first defined the notion of special m-flags of length k for $m \geq 2$ in a slightly different form in [M2] (cf. Theorem 6.2), generalizing the works of [KR] or [PR]. The above theorem was first observed by him in Remark 3 [M2].

In view of Theorem 4.2, our task is to characterize the special m-flags among m-flags of length k, which will be accomplished in the following sections.

5. Main Theorem ($m \geq 3$)

Let (R, D) be an m-flag of length 1, i.e., R is a manifold of dimension $2m + 1$ such that $\operatorname{rank} D = m+1$ and $\partial D = T(R)$. By definition, (R, D) is a special m-flag ($m \geq 2$) if there exists a completely integrable subbundle F of D of corank 1 and $\mathrm{Ch}\,(D)$ is trivial. Then, by Relization Lemma, (R, D) is locally isomorphic to $(P^1(M), C^1) = (J(M, 1), C)$, where $M = R/F$ is (locally) the leaf space of the foliatin F on R. In case $m = 1$, it is easy to see that a 1-flag of length 1 is a contact manifold of dimension 3. 2-flags of length 1 have peculiar aspects and were extensively studied in [C] (cf. §6). Now we start with the following characterization of special m-flags of length 1 for $m \geq 3$.

Proposition 5.1. *An m-flag (R, D) of length 1 for $m \geq 3$ is a special m-flag if and only if D is of Cartan rank 1.*

Here, the *Cartan rank* of (R, D) is the smallest integer ρ such that there exist 1-forms $\{\pi^1, \ldots, \pi^\rho\}$, which are independent modulo $\{\eta^1, \ldots, \eta^m\}$ and satisfy

$$d\alpha \wedge \pi^1 \wedge \cdots \wedge \pi^\rho \equiv 0 \pmod{\eta^1, \ldots, \eta^m} \qquad \text{for } \forall \alpha \in \mathcal{D}^\perp = \Gamma(D^\perp),$$

where $D = \{\eta^1 = \cdots = \eta^m = 0\ \}$.

Proof of Proposition 5.1. First, assume that (R, D) is special. Then we can take local defining 1-forms $\{\eta^1, \ldots, \eta^m, \omega\}$, which are independent at each point, such that

$$D = \{\eta^1 = \cdots = \eta^m = 0\}, \quad F = \{\eta^1 = \cdots = \eta^m = \omega = 0\}.$$

Since F is completely integrable, $d\eta^\beta \equiv 0 \pmod{\eta^1, \ldots, \eta^m, \omega}$ for $\beta = 1, \ldots, m$. Hence there exist 1-forms $\{\varpi^1, \ldots, \varpi^m\}$ such that

$$d\eta^\beta \equiv \omega \wedge \varpi^\beta \pmod{\eta^1, \ldots, \eta^m} \qquad \text{for } \beta = 1, \ldots, m.$$

This implies that D is of Cartan rank 1.

Conversely assume that the Cartan rank of (R, D) is 1. Let us take local defining 1-forms $\{\eta^1, \ldots, \eta^m\}$ of D as above;

$$D = \{\eta^1 = \cdots = \eta^m = 0\}.$$

Since the Cartan rank of D is 1, there exists 1-form ω, which is independent modulo $\{\eta^1, \ldots, \eta^m\}$ such that

$$\omega \wedge d\eta^\beta \equiv 0 \pmod{\eta^1, \ldots, \eta^m} \qquad \text{for } \beta = 1, \ldots, m.$$

Hence there exist 1-forms $\{\varpi^1, \ldots, \varpi^m\}$ such that

$$d\eta^\beta \equiv \omega \wedge \varpi^\beta \pmod{\eta^1, \ldots, \eta^m} \qquad \text{for } \beta = 1, \ldots, m.$$

Then, from $\operatorname{rank}\ \partial D = \operatorname{rank}\ D + m$, it follows that $\{\eta^1, \ldots, \eta^m, \omega, \varpi^1, \ldots, \varpi^m\}$ are linearly independent. Taking exterior derivative of both sides of the above mod equality, we get

$$0 \equiv d\omega \wedge \varpi^\beta \pmod{\eta^1, \ldots, \eta^m, \omega} \qquad \text{for } \beta = 1, \ldots, m.$$

Hence, from $m \geq 3$, we obtain $d\omega \equiv 0 \pmod{\eta^1, \ldots, \eta^m, \omega}$. Putting

$$F = \{\eta^1 = \cdots = \eta^m = \omega = 0\},$$

we have

$$d\eta^\beta \equiv d\omega \equiv 0 \pmod{\eta^1, \ldots, \eta^m, \omega} \qquad \text{for } \beta = 1, \ldots, m.$$

Thus F is completely integrable. Moreover

$$\mathrm{Ch}\,(D) = \{\eta^1 = \cdots = \eta^m = \omega = \varpi^1 = \cdots = \varpi^m = 0\}$$

implies $\mathrm{Ch}\,(D)$ is a subbundle of F of corank m. Namely $\mathrm{Ch}\,(D)$ is trivial. This completes the proof of Proposition. $\square$

Remark 5.2. As a characterization of 1-jet spaces, Bryant's normal form theorem is well known ([B], [BCG3]). This theorem in 1 independent variable case says that an m-flag (R, D) of length 1 for $m \geq 3$ is a special m-flag if and only if D is of Engel (half-) rank 1 and $\mathrm{Ch}\,(D)$ is trivial. Here the Engel rank of (R, D) is the smallest integer ρ such that

$$(d\alpha)^{\rho+1} \equiv 0 \pmod{\eta^1, \ldots, \eta^m} \qquad \text{for } \forall \alpha \in \mathcal{D}^{\perp},$$

where $D = \{\eta^1 = \cdots = \eta^m = 0\ \}$. Here we observe that we cannot replace the Cartan rank 1 condition in the above Proposition by the Engel rank 1 condition when $m = 3$, as the following example shows; Let $(y^1, y^2, y^3, x^0, x^1, x^2, x^3)$ be a coordinate system of R. Let us take a coframe $\{\eta^1, \eta^2, \eta^3, \theta^i, (i = 0, 1, 2, , 3)\}$as follows;

$$\eta^1 = dy^1 + x^2 dx^3,\ \eta^2 = dy^2 + x^3 dx^1,\ \eta^3 = dy^3 + x^1 dx^2, \theta^i = dx^i.$$

Then, for $D = \{\eta^1 = \eta^2 = \eta^3 = 0\}$, we have

$$\begin{cases} d\eta^1 & \equiv \theta^2 \wedge \theta^3 \pmod{\eta^1, \eta^2, \eta^3}, \\ d\eta^2 & \equiv \theta^3 \wedge \theta^1 \pmod{\eta^1, \eta^2, \eta^3}, \\ d\eta^3 & \equiv \theta^1 \wedge \theta^2 \pmod{\eta^1, \eta^2, \eta^3}. \end{cases}$$

Thus (R, D) is a 3-flag of length 1 such that (R, D) is of Engel rank 1 and has non-trivial $\mathrm{Ch}\,(D)$.

However we can replace the Cartan rank 1 condition in the above Proposition by the Engel rank 1 condition when $m \geq 4$, as the following Lemma implies.

Lemma 5.3. *Let V be a vector space over $\mathbb{R}$. Let $\omega_1, \ldots, \omega_r \in \wedge^2 V$ be 2-forms such that $\{\omega_1, \ldots, \omega_r\}$ are linearly independent and $\omega_i \wedge \omega_j = 0$ for $1 \leq i \leq j \leq r$. Then*

(1) *In case $r = 2$. There exist vectors $v_0, v_1, v_2 \in V$, which are linearly independent, such that*

$$\omega_1 = v_0 \wedge v_1, \qquad \omega_2 = v_0 \wedge v_2.$$

(2) *In case $r = 3$. Either of the followings holds*

(*i*) *There exist vectors $v_1, v_2, v_3 \in V$, which are linearly independent, such that*

$$\omega_1 = v_2 \wedge v_3, \quad \omega_2 = v_3 \wedge v_1, \quad \omega_3 = \pm v_1 \wedge v_2.$$

(*ii*) *There exist vectors $v_0, v_1, v_2, v_3 \in V$, which are linearly independent, such that*

$$\omega_1 = v_0 \wedge v_1, \quad \omega_2 = v_0 \wedge v_2, \quad \omega_3 = v_0 \wedge v_3.$$

(3) *In case $r \geq 4$. There exist vectors $v_0, \ldots, v_r \in V$, which are linearly independent, such that*

$$\omega_1 = v_0 \wedge v_1, \quad \omega_2 = v_0 \wedge v_2, \quad \ldots, \quad \omega_r = v_0 \wedge v_r.$$

In case $m = 1$, every Goursat flag of length k ($k \geq 2$) is a special 1-flag, i.e., the Sandwich Lemma holds automatically ([MZ]). By contrast, we need some condition for an m-flag of length 2 ($m \geq 2$) to be special as the following example shows.

Example 5.4. Let R be a manifold of dimension $3m+1$ ($m \geq 2$), and let $(x^\alpha, y^\beta, z^\beta)$ ($\alpha = 0, 1, \cdots, m,\ \beta = 1, \cdots, m$) be a coordinate system on R. For a fixed $a \in \{0, 1, \cdots, m-2\}$, let us take a coframe $\{\eta^1, \ldots, \eta^m, \zeta^1, \ldots, \zeta^m, \theta^0, \ldots, \theta^m\}$ as follows;

$$\begin{array}{lll} \theta^\alpha = dx^\alpha & , \ \eta^\gamma = dz^\gamma + y^\gamma dx^0 - \frac{1}{2}(x^0)^2 dx^\gamma & (\gamma = 1, \cdots, m-a-1) \\ \zeta^\beta = dy^\beta + x^0 dx^\beta & , \ \eta^\delta = dz^\delta + y^{\delta-1} dx^{\delta-1} & (\delta = m-a, \cdots, m) \end{array}$$

We consider $D = \{\eta^1 = \cdots = \eta^m = \zeta^1 = \cdots = \zeta^m = 0\}$. Then we have

$$\left\{ \begin{array}{lll} d\eta^\beta & \equiv 0 & (\mathrm{mod}\ \ \eta^1, \ldots, \eta^m, \zeta^1, \ldots, \zeta^m) \quad \text{for } \beta = 1, \ldots, m, \\ d\zeta^\beta & \equiv \theta^0 \wedge \theta^\beta & (\mathrm{mod}\ \ \eta^1, \ldots, \eta^m, \zeta^1, \ldots, \zeta^m) \quad \text{for } \beta = 1, \ldots, m. \end{array} \right.$$

$$\left\{ \begin{array}{lll} d\eta^\gamma & \equiv \zeta^\gamma \wedge \theta^0 & (\mathrm{mod}\ \ \eta^1, \ldots, \eta^m) \quad \text{for } \gamma = 1, \ldots, m-a-1, \\ d\eta^\delta & \equiv \zeta^{\delta-1} \wedge \theta^{\delta-1} & (\mathrm{mod}\ \ \eta^1, \ldots, \eta^m) \quad \text{for } \delta = m-a, \ldots, m. \end{array} \right.$$

Hence we get

$$\partial D = \{\eta^1 = \cdots = \eta^m = 0\} \quad , \quad \partial^2 D = T(R)$$

$$\mathrm{Ch}\,(\partial D) = \{\eta^1 = \cdots = \eta^m = \zeta^1 = \cdots = \zeta^{m-1} = \theta^0 = \theta^{m-a-1} = \cdots = \theta^{m-1} = 0\}$$

Thus, (R, D) is an m-flag of length 2, but $Ch(\partial D)$ is not a subbundle of D. Moreover $\mathrm{rank}\,\mathrm{Ch}\,(\partial D)$ is $m-a$.

In order to get good control over $\mathrm{Ch}\,(\partial D)$, we prepare the following proposition, which gives us the Sandwich Lemma for $m \geq 3$.

Proposition 5.5. *Let (R, D) be a regular differential system such that* $\mathrm{rank}\,\partial^2 D = \mathrm{rank}\,\partial D + m$ *and* $\mathrm{rank}\,\partial D = \mathrm{rank}\,D + m$. *Assume $m \geq 3$ and the Cartan rank of ∂D is* 1*, then* $\mathrm{Ch}\,(\partial D)$ *is a subbundle of D of corank* 1*. Moreover the Cartan rank of D is* 1

In view of Lemma 5.3, we can replace the Cartan rank 1 condition by the Engel rank 1 condition when $m \geq 4$ (cf. Remark 5.6).

Proof. Let x be any point of R. By the rank condition, there exist linearly independent 1-forms $\{\pi^i, \eta^\beta, \zeta^\beta (i = 1, \ldots, s, \beta = 1, \ldots, m)\}$ defined on a neighborhood U of x, where $s = \mathrm{corank}\,\partial^2 D$, such that

$$\begin{array}{rl} \partial^2 D & = \{\pi^1 = \cdots = \pi^s = 0\ \}, \\ \partial D & = \{\pi^1 = \cdots = \pi^s = \eta^1 = \cdots = \eta^m = 0\ \}, \\ D & = \{\pi^1 = \cdots = \pi^s = \eta^1 = \cdots = \eta^m = \zeta^1 = \cdots = \zeta^m = 0,\ \}. \end{array}$$

$$\left\{ \begin{array}{lll} d\pi^i \ \equiv 0, & d\eta^\beta \ \not\equiv 0 & (\mathrm{mod}\ \ \pi^1, \ldots, \pi^s, \eta^1, \ldots, \eta^m) \\ d\eta^\beta \ \equiv 0, & d\zeta^\beta \ \not\equiv 0 & (\mathrm{mod}\ \ \pi^1, \ldots, \pi^s, \eta^1, \ldots, \eta^m, \zeta^1, \ldots, \zeta^m) \end{array} \right.$$

Since the Cartan rank of ∂D is 1, there exist 1-forms $\{\omega, \varpi^1, \ldots, \varpi^m\}$ on a neighborhood $V \subset U$ of x such that

$$d\eta^\beta \equiv \omega \wedge \varpi^\beta \quad (\mathrm{mod}\ \ \pi^1, \ldots, \pi^s,\ \eta^1, \ldots, \eta^m)$$

From $\mathrm{rank}\,\partial^2 D = \mathrm{rank}\,\partial D + m$, it follows that $\{\pi^i, \eta^\beta, \omega, \varpi^\beta (i = 1, \ldots, s,\ \beta = 1, \ldots, m)\}$ are linearly independent at each $y \in V$. Then we have

$$\mathrm{Ch}\,(\partial D) = \{\pi^1 = \cdots = \pi^s = \eta^1 = \cdots = \eta^m = \omega = \varpi^1 = \cdots = \varpi^m = 0\},$$

Thus $\mathrm{Ch}\,(\partial D)$ is a subbubdle of ∂D of corank $m+1$.

Now the structure equation for D implies

$$\omega \wedge \varpi^\beta \equiv 0 \quad (\mathrm{mod}\ \ \pi^1, \ldots, \pi^s, \eta^1, \ldots, \eta^m, \zeta^1, \ldots, \zeta^m).$$

First of all, we claim: *There exists no open neighborhood $V' \subset V$ of x such that ω vanishes identically on V' modulo $\mathcal{D}^\perp$.* Assume the contrary, i.e., there exists V' such that $\omega_{V'} \equiv 0$ (mod $\pi^1, \ldots, \pi^s, \eta^1, \ldots, \eta^m, \zeta^1, \ldots, \zeta^m$). Then we may assume $\omega = \zeta^1$, so that

$$d\eta^\beta \equiv \zeta^1 \wedge \varpi^\beta \qquad (\mathrm{mod}\ \pi^1, \ldots, \pi^s, \eta^1, \ldots, \eta^m)$$

Taking the exterior derivative of both sides of this mod equation, we obtain

$$0 \equiv d\zeta^1 \wedge \varpi^\beta \qquad (\mathrm{mod}\ \pi^1, \ldots, \pi^s, \eta^1, \ldots, \eta^m, \zeta^1).$$

Since $\{\pi^i, \eta^\beta, \zeta^1, \varpi^\beta (i = 1, \ldots, s, \beta = 1, \ldots, m)\}$ are linearly independent and $m \geq 3$, we get

$$d\zeta^1 \equiv 0 \qquad (\mathrm{mod}\ \pi^1, \ldots, \pi^s, \eta^1, \ldots, \eta^m, \zeta^1),$$

which contradicts the structure equation for D.

Now we divide the proof according to the dependence of ω_x modulo $D^\perp(x)$.

(1) $\omega_x \not\equiv 0$ (mod $\pi^1, \ldots, \pi^s, \eta^1, \ldots, \eta^m, \zeta^1, \ldots, \zeta^m$).

From $\omega \wedge \varpi^\beta \equiv 0 \quad (\mathrm{mod}\ \mathcal{D}^\perp)$, we have

$$\varpi^\beta \equiv 0 \quad (\mathrm{mod}\ \pi^1, \ldots, \pi^s, \eta^1, \ldots, \eta^m, \zeta^1, \ldots, \zeta^m, \omega).$$

Hence we have

$$\varpi^\beta \equiv \sum_{\gamma=1}^{m} a^\beta_\gamma \zeta^\gamma \qquad (\mathrm{mod}\ \ \pi^1, \ldots, \pi^s, \eta^1, \ldots, \eta^m, \omega).$$

Since $\{\pi^i, \eta^\beta, \omega, \varpi^\beta (i = 1, \ldots, s, \beta = 1, \ldots, m)\}$ are linearly independent, it follows that $\det\big(a^\beta_\gamma(x)\big) \neq 0$. Therefore

$$\begin{aligned} \mathrm{Ch}\,(\partial D) &= \{\pi^1 = \cdots = \pi^s = \eta^1 = \cdots = \eta^m = \omega = \varpi^1 = \cdots = \varpi^m = 0\} \\ &= \{\pi^1 = \cdots = \pi^s = \eta^1 = \cdots = \eta^m = \zeta^1 = \cdots = \zeta^m = \omega = 0\} \subset D. \end{aligned}$$

Thus $\mathrm{Ch}\,(\partial D)$ is a completely integrable subbundle of D of corank 1 so that $d\zeta^\beta \equiv 0$ (mod $\pi^1, \ldots, \pi^s, \eta^1, \ldots, \eta^m, \zeta^1, \ldots, \zeta^m, \omega$). Hence we have

$$d\zeta^\beta \equiv \omega \wedge \theta^\beta \quad (\mathrm{mod}\ \ \pi^1, \ldots, \pi^s, \eta^1, \ldots, \eta^m, \zeta^1, \ldots, \zeta^m),$$

Since $\mathrm{rank}\,\partial D = \mathrm{rank}\, D + m$, $\{\pi^i, \eta^\beta, \zeta^\beta, \omega, \theta^\beta (i = 1, \ldots, s, \beta = 1, \ldots, m)\}$ are linearly independent and the Cartan rank of D is 1.

(2) $\omega_x \equiv 0 \quad (\mathrm{mod}\ \pi^1, \ldots, \pi^s, \eta^1, \ldots, \eta^m, \zeta^1, \ldots, \zeta^m)$.

Since $\{\pi^i, \eta^\beta, \omega, \varpi^\beta (i = 1, \ldots, s,\ \beta = 1, \ldots, m)\}$ are linearly independent, there exists $\beta_0 \in \{1, \ldots, m\}$ such that $\varpi^{\beta_0}_x \not\equiv 0$ (mod $D^\perp(x)$). We may shrink our neighborhood V of x so that $\varpi^{\beta_0}_y \not\equiv 0$ (mod $D^\perp(y)$) for each $y \in V$. Then, from $\omega \wedge \varpi^{\beta_0} \equiv 0 \quad (\mathrm{mod}\ \mathcal{D}^\perp)$, we have

$$\omega \equiv 0 \quad (\mathrm{mod}\ \pi^1, \ldots, \pi^s, \eta^1, \ldots, \eta^m, \zeta^1, \ldots, \zeta^m, \varpi^{\beta_0}).$$

Moreover we claim:

$$\varpi^\beta \wedge \varpi^{\beta_0} \equiv 0 \quad (\mathrm{mod}\ \pi^1, \ldots, \pi^s, \eta^1, \ldots, \eta^m, \zeta^1, \ldots, \zeta^m),$$

hold on V for each $\beta \in \{1, \ldots, m\}$.

In fact, for each $y \in V$, we consider the following two cases.
(a) $\omega_y \not\equiv 0 \pmod{\pi^1, \ldots, \pi^s, \eta^1, \ldots, \eta^m, \zeta^1, \ldots, \zeta^m}$.

From $\omega \wedge \varpi^\beta \equiv 0 \pmod{\mathcal{D}^\perp}$, we have $\varpi_y^\beta \equiv \lambda^\beta \omega_y \pmod{D^\perp(y)}$. Since $\lambda^{\beta_0} \neq 0$, we get $\omega_y \equiv \lambda \varpi_y^{\beta_0}$ for $\lambda \neq 0$. Hence $\varpi_y^\beta \wedge \varpi_y^{\beta_0} \equiv 0 \pmod{D^\perp(y)}$ as desired.

(b) $\omega_y \equiv 0 \pmod{\pi^1, \ldots, \pi^s, \eta^1, \ldots, \eta^m, \zeta^1, \ldots, \zeta^m}$.

Assume the contrary, i.e., there exists $\gamma \in \{1, \ldots, m\}$ such that $\varpi_y^\gamma \wedge \varpi_y^{\beta_0} \not\equiv 0 \pmod{D^\perp(y)}$ Then we may take a neighborhood $V_0 \subset V$ of y so that

$$\varpi_z^\gamma \wedge \varpi_z^{\beta_0} \not\equiv 0 \pmod{D^\perp(z)},$$

for each $z \in V_0$. However ω cannot vanish identically on V_0 as shown above. Hence there exists a point $z_0 \in V_0$ such that $\omega_{z_0} \not\equiv 0 \pmod{D^\perp(z_0)}$. Then, as in (a), we get $\varpi_{z_0}^\gamma \wedge \varpi_{z_0}^{\beta_0} \equiv 0 \pmod{D^\perp(z_0)}$, which is a contradiction.

Since $\{\pi^i, \eta^\beta, \omega, \varpi^\beta (i = 1, \ldots, s,\ \beta = 1, \ldots, m)\}$ are linearly independent, we obtain

$$\begin{aligned} \mathrm{Ch}\,(\partial D) &= \{\pi^1 = \cdots = \pi^s = \eta^1 = \cdots = \eta^m = \omega = \varpi^1 = \cdots = \varpi^m = 0\} \\ &= \{\pi^1 = \cdots = \pi^s = \eta^1 = \cdots = \eta^m = \zeta^1 = \cdots = \zeta^m = \varpi^{\beta_0} = 0\} \subset D. \end{aligned}$$

Thus $\mathrm{Ch}\,(\partial D)$ is a completely integrable subbundle of D of corank 1. Moreover, as in (1), the Cartan rank of D is 1. This completes the proof of Proposition. □

Remark 5.6. We cannot replace the Cartan rank 1 condition in the above Proposition by the Engel rank 1 condition when $m = 3$, as the following example shows; Let $(z^1, z^2.z^3, y^1, y^2.y^3, x^0, x^1, x^2, x^3)$ be a coordinate system of R. Let us take a coframe $\{\eta^1, \eta^2, \eta^3, \zeta^1, \zeta^2, \zeta^3, \theta^0, \theta^1, \theta^2, \theta^3\}$ as follows;

$$\begin{aligned} &\eta^1 = dz^1 + y^1 dx^0, \quad \eta^2 = dz^2 + y^2 dy^1,\ \eta^3 = dz^3 + x^0 dy^2, \quad \theta^0 = dx^0,\ \theta^1 = dx^1, \\ &\zeta^1 = dy^1 - x^1 dx^0, \quad \zeta^2 = dy^2 - x^2 dx^0,\ \zeta^3 = dy^3 - x^3 dx^0, \quad \theta^2 = dx^2,\ \theta^3 = dx^3. \end{aligned}$$

We consider $D = \{\eta^1 = \eta^2 = \eta^3 = \zeta^1 = \zeta^2 = \zeta^3 = 0\}$. Then we have

$$\begin{cases} d\eta^\beta & \equiv 0 \qquad\quad \pmod{\eta^1, \eta^2, \eta^3, \zeta^1, \zeta^2, \zeta^3} \quad \text{for } \beta = 1, 2, 3, \\ d\zeta^\beta & \equiv \theta^0 \wedge \theta^\beta \quad \pmod{\eta^1, \eta^2, \eta^3, \zeta^1, \zeta^2, \zeta^3} \quad \text{for } \beta = 1, 2, 3. \end{cases}$$

$$\begin{cases} d\eta^1 & \equiv \zeta^1 \wedge \theta^0 \qquad\qquad\qquad\quad \pmod{\eta^1, \eta^2, \eta^3}, \\ d\eta^2 & \equiv (\zeta^2 + x^2\theta^0) \wedge (\zeta^1 + x^1\theta^0) \quad \pmod{\eta^1, \eta^2, \eta^3}, \\ d\eta^3 & \equiv \theta^0 \wedge \zeta^2 \qquad\qquad\qquad\quad \pmod{\eta^1, \eta^2, \eta^3}. \end{cases}$$

Hence we get

$$\begin{gathered} \partial D = \{\eta^1 = \eta^2 = \eta^3 = 0\} \quad , \quad \partial^2 D = T(R), \\ \mathrm{Ch}\,(\partial D) = \{\eta^1 = \eta^2 = \eta^3 = \zeta^1 = \zeta^2 = \theta^0 = 0\}. \end{gathered}$$

Thus, (R, D) is an 3-flag of length 2 such that the Engel rank of ∂D is 1, but $Ch(\partial D)$ is not a subbundle of D.

However, by Lemma 5.3, we can replace the Cartan rank 1 condition in the above Proposition by the Engel rank 1 condition when $m \geq 4$.

By utilizing the above proposition repeatedly, we obtain

Theorem 5.7. *An m-flag (R, D) of length k for $m \geq 3$ is a special m-flag if and only if $\partial^{k-1}D$ is of Cartan rank 1. Moreover, an m-flag (R, D) of length k for $m \geq 4$ is a special m-flag if and only if $\partial^{k-1}D$ is of Engel rank 1.*

Proof. Only if part follows from the existence of the completely integrable subbundle F of $\partial^{k-1}D$ of corank 1 for the special m-flag as in the proof of Proposition 5.1.

For the if part, first, the proof of Proposition 5.1 shows the existence of a completely integrable subbundle F of $\partial^{k-1}D$, which contains $\mathrm{Ch}\,(\partial^{k-1}D)$. By repeated application of the previous propositon, we obtain that $\mathrm{Ch}\,(\partial^{i+1}D)$ is a subbundle of $\partial^i D$ of corank 1 for $i = 0, \ldots, k-2$. Thus we are left to show that $\mathrm{rank}\, D = \mathrm{rank}\, \mathrm{Ch}\,(D) + m + 1$.

Let us take defining 1-forms of D, ∂D and $\mathrm{Ch}\,(\partial D)$ such that

$$\begin{aligned} \partial D \quad &= \{\pi^1 = \cdots = \pi^s = 0\}\ ,\ D = \{\pi^1 = \cdots = \pi^s = \zeta^1 = \cdots = \zeta^m = 0\}, \\ \mathrm{Ch}\,(\partial D) \quad &= \{\pi^1 = \cdots = \pi^s = \zeta^1 = \cdots = \zeta^m = \omega = 0\}. \end{aligned}$$

where s is the corank of ∂D. Since $\mathrm{Ch}\,(\partial D)$ is completely integrable, we have

$$d\zeta^\alpha \equiv 0 \pmod{\pi^1, \ldots, \pi^s,\ \zeta^1, \ldots, \zeta^m, \omega}, \qquad \text{for } \alpha = 1, \ldots, m.$$

Therefore, there exist 1-forms $\{\theta^1, \ldots, \theta^m\}$ such that

$$\begin{cases} d\pi^i & \equiv 0, \qquad\quad \pmod{\pi^1, \ldots, \pi^s, \zeta^1, \ldots, \zeta^m} \quad \text{for } i = 1, \ldots, s, \\ d\zeta^\alpha & \equiv \omega \wedge \theta^\alpha, \quad \pmod{\pi^1, \ldots, \pi^s, \zeta^1, \ldots, \zeta^m} \quad \text{for } \alpha = 1, \ldots, m. \end{cases}$$

Then, from rank $\partial D =$ rank $D + m$, it follows that $\{\pi^i,\ \zeta^\alpha, \omega, \theta^\alpha (i = 1, \ldots, s, \alpha = 1, \ldots, m)\}$ are linealy independent. Hence

$$\mathrm{Ch}\,(D) = \{\pi^1 = \cdots = \pi^s = \zeta^1 = \cdots = \zeta^m = \omega = \theta^1 = \cdots = \theta^m = 0\}.$$

Thus rank $D =$ rank $\mathrm{Ch}\,(D) + m + 1$. This completes the proof of Theorem. □

Hence, by Theorem 4.2, we obtain the Drapeau Theorem for $m \geq 3$

Corollary 5.8. *Let M be a manifold of dimension $m+1$. An m-flag (R, D) of length k for $m \geq 3$ is locally isomorphic to $(P^k(M), C^k)$ if and only if $\partial^{k-1}D$ is of Cartan rank 1, and, moreover for $m \geq 4$, if and only if $\partial^{k-1}D$ is of Engel rank 1.*

6. Integrable subbundle of corank 1

Let (R, D) be a 2-flag of length 1. Then it can be shown ([C]) that there exists a local coframe $\{\eta^1, \eta^2, \theta^0, \theta^1, \theta^2\}$ such that $D = \{\eta^1 = \eta^2 = 0\}$,

$$\begin{cases} d\eta^1 & \equiv \theta^0 \wedge \theta^1 \qquad \pmod{\eta^1, \eta^2}, \\ d\eta^2 & \equiv \theta^0 \wedge \theta^2 \qquad \pmod{\eta^1, \eta^2} \end{cases}$$

Thus the Cartan rank of (R, D) is always 1 and we have the covariant system $F = \{\eta^1 = \eta^2 = \theta^0 = 0\}$ of D of corank 1 (cf. [Y3]). As is well known, F is not necessarily completely integrable.

As for a 2-flag of length 2, we observe that, in Example 5.4, putting $m = 2$, we obtain the folllowing structure equation for $D = \{\eta^1 = \eta^2 = \zeta^1 = \zeta^2 = 0\}$, where

$$\begin{aligned} &\eta^1 = dz^1 + y^1 dx^0 - \tfrac{1}{2}(x^0)^2 dx^1,\ \eta^2 = dz^2 + y^1 dx^1 \\ &\zeta^1 = dy^1 + x^0 dx^1,\ \ \zeta^2 = dy^2 + x^0 dx^2,\ \ \theta^0 = dx^0,\ \ \theta^1 = dx^1,\ \theta^2 = dx^2, \end{aligned}$$

$$\begin{cases} d\eta^\beta & \equiv 0 \qquad\qquad (\text{mod} \quad \eta^1, \eta^2, \zeta^1, \zeta^2) \quad \text{for } \beta = 1, 2, \\ d\zeta^\beta & \equiv \theta^0 \wedge \theta^\beta \quad (\text{mod} \quad \eta^1, \eta^2, \zeta^1, \zeta^2) \quad \text{for } \beta = 1, 2. \end{cases}$$

$$\begin{cases} d\eta^1 & \equiv \zeta^1 \wedge \theta^0 \quad (\text{mod} \quad \eta^1, \eta^2), \\ d\eta^2 & \equiv \zeta^1 \wedge \theta^1 \quad (\text{mod} \quad \eta^1, \eta^2). \end{cases}$$

Thus $\partial D = \{\eta^1 = \eta^2 = 0\}$ and the Cartan rank of ∂D is 1, whereas $Ch(\partial D)$ is not a subbundle of D. This shows that the statement of Proposition 5.5 is false for $m = 2$.

To cover the case $m = 2$, we strengthen the hypothesis of Proposition 5.5 as in the following.

Proposition 6.1. *Let (R, D) be a regular differential system such that* $\operatorname{rank} \partial^2 D = \operatorname{rank} \partial D + m$ *and* $\operatorname{rank} \partial D = \operatorname{rank} D + m$. *Assume that there exists a completely integrable subbubdle F of ∂D of corank* 1, *then* $\operatorname{Ch}(\partial D)$ *is a subbundle of D of corank* 1.

Proof. Let x be any point of R. By the rank condition, there exist linearly independent 1-forms $\{\pi^i, \eta^\beta, \zeta^\beta (i = 1, \ldots, s, \beta = 1, \ldots, m)\}$ defined on a neighborhood U of x, where $s = \operatorname{corank} \partial^2 D$, such that

$$\begin{aligned} \partial^2 D &= \{\pi^1 = \cdots = \pi^s = 0\ \}, \\ \partial D &= \{\pi^1 = \cdots = \pi^s = \eta^1 = \cdots = \eta^m = 0\ \}, \\ D &= \{\pi^1 = \cdots = \pi^s = \eta^1 = \cdots = \eta^m = \zeta^1 = \cdots = \zeta^m = 0,\ \}. \end{aligned}$$

$$\begin{cases} d\pi^i & \equiv 0, \quad d\eta^\beta \not\equiv 0 \quad (\text{mod} \quad \pi^1, \ldots, \pi^s, \eta^1, \ldots, \eta^m) \\ d\eta^\beta & \equiv 0, \quad d\zeta^\beta \not\equiv 0 \quad (\text{mod} \quad \pi^1, \ldots, \pi^s, \eta^1, \ldots, \eta^m, \zeta^1, \ldots, \zeta^m) \end{cases}$$

Moreover, since F is a subbundle of ∂D of corank 1, there exists 1-form ω such that $\{\pi^1, \ldots, \pi^s, \eta^1, \ldots, \eta^m, \omega\}$ are linearly independent and

$$F = \{\pi^1 = \cdots = \pi^s = \eta^1 = \cdots = \eta^m = \omega = 0\ \}$$

Since F is completely integrable, we have $d\eta^\beta \equiv 0 \ (\text{mod} \quad \pi^1, \ldots, \pi^s,\ \eta^1, \ldots, \eta^m, \omega)$. Hence there exist 1-forms $\{\varpi^1, \ldots, \varpi^m\}$ on a neighborhood $V \subset U$ of x such that

$$d\eta^\beta \equiv \omega \wedge \varpi^\beta \quad (\text{mod} \quad \pi^1, \ldots, \pi^s,\ \eta^1, \ldots, \eta^m)$$

From $\operatorname{rank} \partial^2 D = \operatorname{rank} \partial D + m$, it follows that $\{\pi^i, \eta^\beta, \omega, \varpi^\beta (i = 1, \ldots, s,\ \beta = 1, \ldots, m)\}$ are linearly independent at each $y \in V$. Then we have

$$\operatorname{Ch}(\partial D) = \{\pi^1 = \cdots = \pi^s = \eta^1 = \cdots = \eta^m = \omega = \varpi^1 = \cdots = \varpi^m = 0\} \subset F,$$

Thus $\operatorname{Ch}(\partial D)$ is a subbubdle of F of corank m.

Now the structure equation for D implies

$$\omega \wedge \varpi^\beta \equiv 0 \quad (\text{mod} \quad \pi^1, \ldots, \pi^s, \eta^1, \ldots, \eta^m, \zeta^1, \ldots, \zeta^m).$$

First of all, we claim: *There exists no open neighborhood $V' \subset V$ of x such that ω vanishes identically on V' modulo $\mathcal{D}^\perp$.* Assume the contrary, i.e., there exists V' such that $\omega_{V'} \equiv 0 \ (\text{mod} \quad \pi^1, \ldots, \pi^s, \eta^1, \ldots, \eta^m, \zeta^1, \ldots, \zeta^m)$. Then we may assume $\omega = \zeta^1$ on V'. Since F is completely integrable and $F = \{\pi^1 = \cdots = \pi^s = \eta^1 = \cdots = \eta^m = \zeta^1 = 0\ \}$, we get

$$d\zeta^1 \equiv 0 \qquad (\text{mod}\ \pi^1, \ldots, \pi^s, \eta^1, \ldots, \eta^m, \zeta^1),$$

which contradicts the structure equation for D.

The rest of the proof is quite similar to that of Proposition 5.5, hence is omitted. $\square$

By utilizing the above proposition repeatedly, we obtain

Theorem 6.2. *An m-flag (R, D) of length k is a special m-flag if and only if there exists a completely integrable subbundle F of $\partial^{k-1}D$ of corank 1. Moreover, F is unique for (R, D).*

Proof. Only if part is trivial. For the if part, by repeated application of the above Proposition, we obtain that $F \supset \mathrm{Ch}\,(\partial^{k-1}D)$ and $\mathrm{Ch}\,(\partial^{i+1}D)$ is a subbundle of $\partial^i D$ of corank 1 for $i = 0, \ldots, k-2$. Thus we are left to show that $\operatorname{rank} D = \operatorname{rank}\mathrm{Ch}\,(D)+m+1$, but the proof is the same as in Theorem 5.7. The uniqueness of F follows from Theorem 4.2. $\square$

Hence, by Theorem 4.2, we obtain the following Drapeau Theorem for $m \geq 2$.

Corollary 6.3. *Let M be a manifold of dimension $m + 1$. An m-flag (R, D) of length k is locally isomorphic to $(P^k(M), C^k)$ if and only if there exists a completely integrable subbundle F of $\partial^{k-1}D$ of corank 1.*

References

[B] Bryant, R. : Some aspect of the local and global theory of Pfaffian systems, Thesis, University of North Carolina, Chapel Hill, 1979

[BCG3] Bryant, R., Chern, S., Gardner, R., Goldschmidt, H. and Griffiths, P. : *Exterior Differential Systems*, MSRI Publ. vol. **18**, Springer Verlag, Berlin 1991

[BH] Bryant, R. and Hsu, L.: Rigidity of integral curves of rank 2 distributions, Invent. Math. **144** (1993), 435-461

[C] Cartan, E. : Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre, Ann. École Normale, **27** (1910), 109-192

[GKR] Giaro, A., Kumpera, A. et Ruiz, C.: Sur la lecture correcte d'un resultat d'Elie Cartan, C.R.Acad.Sc.Paris, **287**, Sér.A(1978), 241-244.

[KR] Kumpera, A. and Rubin, J.L.: Multi-flag systems and ordinary differential equations, Nagoya Math.J. **166**(2002), 1-27.

[MZ] Montgomery, R. and Zhitomirskii, M.: Geometric approach to Goursat flags, Ann.Inst. H.Poincaré-AN **18**(2001), 459-493.

[M1] Mormul,P.: Goursat Flags:Classification of codimension-one singularities, J.of Dynamical and Control Systems,**6**,No. 3(2000), 311-330

[M2] Mormul, P.: Multi-dimentional Cartan prolongation and special k-flags, Banach center Publ.vol.**65**(2004), 157-178.

[PR] Pasillas-Lépine, W. and Respondek, W. : Contact systems and corank one involutive subdistributions, Acta Appl. Math. **69** (2001), 105-128

[Y1] Yamaguchi, K.: Contact geometry of higher order, Japan.J.Math **8**(1982), 109-176.

[Y2] Yamaguchi, K.: Geometrization of jet bundles, Hokkaido Math. J. **12** (1983), 27-40.

[Y3] Yamaguchi, K.: G_2-geometry of overdetermined systems of second order, Trends in Mathematics (Analysis and Geometry in Several Complex Variables) (1999), Birkhäuser, Boston, 289-314.

Department of Mathematics, Graduate School of Science, Hokkaido University, Sapporo 060-0810, Japan