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OPTIMAL LONG TERM INVESTMENT MODEL WITH MEMORY

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Abstract. We consider an investment model with memory in which the prices of n risky assets are driven by an $\mathbf{R}^n$-valued Gaussian process with stationary increments that is different from Brownian motion. The driving process consists of n independent components, and each component is characterized by two parameters describing the memory. For the model, we explicitly solve the problem of maximizing the expected growth rate as well as that of maximizing the probability of overperforming a given benchmark.

1. Introduction

Let $X^{x,\pi}(t)$ be the investor's wealth at time t, who follows a strategy π starting from initial wealth x. Among various optimal long term investment problems, we are concerned with the following one:

$$\sup_{\pi\in\mathcal{A}} \limsup_{T\to\infty} \frac{1}{\alpha T} \log E\left[(X^{x,\pi}(T))^{\alpha}\right], \tag{1.1}$$

where $\mathcal{A}$ is the class of admissible strategies and $\alpha \in (-\infty, 1) \setminus \{0\}$. This is the problem of maximizing the long term expected growth rate, and may be seen as an infinite horizon risk-sensitive stochastic control problem. In Bielecki and Pliska [4], this problem is investigated for a factor model which itself is introduced in the same paper. See, e.g., Fleming and Sheu [6, 7], Kuroda and Nagai [12] and Nagai and Peng [15] where the same problem is studied under various settings.

We are also interested in the following problem:

$$\sup_{\pi\in\mathcal{A}} \limsup_{T\to\infty} \frac{1}{T} \log P\left[X^{x,\pi}(T) \geq e^{cT}\right], \tag{1.2}$$

where c is a given benchmark. This is the problem of maximizing the probability of overperforming the target c, and may be seen as a large deviations stochastic control problem. This problem is studied in Pham [16, 17], and an important dual relation between the two problems (1.1) and (1.2) is exploited. See Hata and Iida [8] and Hata and Sekine [9] for related work.

In this paper, we consider the two problems above for an investment model with memory consisting of n risky assets and one riskless asset. The prices of the risky assets are driven by an n-dimensional Gaussian process $(Y(t))_{t\geq 0}$ with stationary increments, which is different from Brownian motion. Denoting by $Y_j(t)$ the jth component of $Y(t)$, the process $(Y(t))_{t\in\mathbf{R}}$ with extended time domain $\mathbf{R}$ is described

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by the following continuous-time AR(∞)-type equation:

$$\frac{dY_j(t)}{dt} = -\int_{-\infty}^{t} p_j e^{-q_j(t-s)} \frac{dY_j(s)}{ds} ds + \frac{dW_j(t)}{dt}, \ Y_j(0) = 0, \ j = 1, \dots, n, \tag{1.3}$$

where

$$p_j \in [0, \infty), \qquad q_j \in (0, \infty), \qquad j = 1, \dots, n, \tag{1.4}$$

$(W(t))_{t\in\mathbf{R}}$ is an n-dimensional standard Brownian motion with components $W_j(t)$, and $dY_j(t)/dt$ and $dW_j(t)/dt$ are the derivatives of $Y_j(t)$ and $W_j(t)$, respectively, in the random distribution sense. For $j = 1, \dots, n$, the parameters p_j and q_j describe the memory of $Y_j(t)$. We refer to Anh and Inoue [1] and Anh *et al.* [2] where, in the case $n = 1$, models based on similar equations are considered. The adoption of such a noise with memory is motivated by an analysis of real market data; see Anh *et al.* [3]. In this paper as well as Inoue *et al.* [10], the estimation of the parameters p_j and q_j is discussed.

For the above model with memory, we start with the classical optimal investment problem over a finite planning horizon $[0, T]$ where the goal is to maximize the expected utility of terminal wealth $\alpha^{-1}E[(X^{x,\pi}(T))^\alpha]$ over the class of admissible strategies π. The optimal strategy $\hat{\pi}_T(t)$ is expressed explicitly in terms of the past prices of the risky assets.

We give a solution to the problem (1.1) by verifying that a candidate for the optimal strategy suggested by $\hat{\pi}_T$ in the finite horizon problem above is actually optimal. Using the results thus obtained as well as Pham's dual relation stated above, we solve the large deviations control problem (1.2) for the model. These results have the virtue that they are explicit.

In the arguments, existence of solutions of relevant Riccati equations (Lemmas 3.1 and 4.7) plays a key role. The result of Nagai and Peng [15] on the asymptotic behavior of solutions to Riccati equations is also an essential ingredient.

In Section 2, we describe the model and formulate the problems. In Section 3, we derive the solution to the finite horizon investment problem. Section 4 is the main body of this paper, in which we solve the infinite horizon optimization problem (1.1). In Section 5, we investigate the large deviation probability control problem (1.2) using the results in Section 4. Appendix A provides a necessary result on a Cameron–Martin type formula. In Appendix B, we summarize the results on the asymptotic behavior of solutions of Riccati equations that we use.

2. The model and problems

In this section, we describe the model and formulate the optimal investment problems that we consider in this paper.

Let $(\Omega, \mathcal{F}, P)$ be the underlying probability space. Let the process $(W(t))_{t\in\mathbf{R}}$ be an $n \geq 1$ dimensional standard Brownian motion with components $W_j(t)$. For $j = 1, \dots, n$, let p_j and q_j be constants satisfying (1.4). Denoting by $Y_j(t)$ the jth component of $Y(t)$, the solution $(Y(t))_{t\in\mathbf{R}}$ to (1.3) in the sense of [1] is given by

$$Y_j(t) = W_j(t) - \int_0^t \left[\int_{-\infty}^{s} p_j e^{-(q_j+p_j)(s-u)} dW_j(u)\right] ds, \quad t \in \mathbf{R} \tag{2.1}$$

(see Examples 2.2 and 2.4 in [1]). The process $(Y(t))_{t\in\mathbf{R}}$ is an n-dimensional centered Gaussian process with stationary increments. Since $W_1,\dots,W_n$ are independent, so are the components $Y_1,\dots,Y_n$ of Y. If $p_j=0$, then Y_j is reduced to the one-dimensional Brownian motion W_j, i.e., $Y_j(t)=W_j(t)$ for $t\geq 0$.

We put

$$D_j(t-s)=\frac{1}{t-s}E\left[(Y_j(t)-Y_j(s))^2\right],\quad t>s. \tag{2.2}$$

Then $D_j(t)$ is given by

$$D_j(t)=\frac{q_j^2}{(p_j+q_j)^2}+\frac{p_j(2q_j+p_j)}{(p_j+q_j)^3}\cdot\frac{(1-e^{-(p_j+q_j)t})}{t},\qquad t>0. \tag{2.3}$$

The function $D_j(t)$ is decreasing, and satisfies

$$\lim_{t\to 0+}D_j(t)=1,\qquad \lim_{t\to\infty}D_j(t)=\left(\frac{q_j}{p_j+q_j}\right)^2.$$

In the case $p_j=0$, i.e., $Y_j=W_j$, we have $D_j(t)=1$ for $t>0$. The equality (2.2) with (2.3) is useful in estimating the values of the parameters p_j and q_j from given data, whence in fitting the model to them. See [3] and [10] for details.

Let $(\mathcal{F}_t)_{t\geq 0}$ be the P-augmentation of the filtration $(\sigma(Y(s),0\leq s\leq t))_{t\geq 0}$ generated by the process $(Y(t))_{t\geq 0}$. We take $(\mathcal{F}_t)_{t\geq 0}$ as the underlying information structure of $(\Omega,\mathcal{F},P)$. Notice that the equality (2.1) does not give a semimartingale representation of $Y(t)$ since $W(t)$ is not $\mathcal{F}_t$-adapted. However, $Y(t)$ does have the following semimartingale representation:

$$Y_j(t)=B_j(t)-\int_0^t\left[\int_0^s l_j(s,u)dB_j(u)\right]ds,\quad t\geq 0,\quad j=1,\dots,n, \tag{2.4}$$

where the deterministic kernels $l_j(t,s)$ are given by

$$l_j(t,s)=e^{-(p_j+q_j)(t-s)}l_j(s),\quad 0\leq s\leq t,\quad j=1,\dots,n$$

with

$$l_j(s)=p_j\left[1-\frac{2p_jq_j}{(2q_j+p_j)^2e^{2q_js}-p_j^2}\right],\quad s\geq 0,\quad j=1,\dots,n, \tag{2.5}$$

and the n-dimensional process $(B(t))_{t\geq 0}$ with components $B_j(t)$ is a standard Brownian motion satisfying $\sigma(Y(s):0\leq s\leq t)=\sigma(B(s):0\leq s\leq t)$ for $t\geq 0$. We refer to [2, 10] for details. The Brownian motion $(B(t))_{t\geq 0}$ is the so-called *innovation process* associated with $(Y(t))_{t\geq 0}$.

The equality (2.4) gives a representation of Y_j in terms of B_j. On the other hand, it holds that

$$\int_0^t l_j(t,s)dB_j(s)=\int_0^t k_j(t,s)dY_j(s),\quad t\geq 0,\quad j=1,\dots,n, \tag{2.6}$$

with

$$k_j(t,s)=p_j(2q_j+p_j)\frac{(2q_j+p_j)e^{q_js}-p_je^{-q_js}}{(2q_j+p_j)^2e^{q_jt}-p_j^2e^{-q_jt}},\quad 0\leq s\leq t,\quad j=1,\dots,n,$$

whence, conversely, we have the following representation of B_j in terms of Y_j (see [2, Example 5.3]):

$$B_j(t) = Y_j(t) + \int_0^t \left[\int_0^s k_j(s,u) dY_j(u) \right] ds, \quad t \geq 0, \quad j = 1, \dots, n.$$

We consider an investment model consisting of n risky assets and one riskless asset. We denote by $S_i(t)$ the price of the ith risky asset and put $S(t) = (S_1(t), \dots, S_n(t))^*$, where A^* stands for the transpose of a matrix A. We suppose that the process $(S(t))_{t \geq 0}$ is governed by

$$dS_i(t) = S_i(t) \left[\mu_i(t) dt + \sum_{j=1}^n \sigma_{ij}(t) dY_j(t) \right], \quad S_i(0) = s_i, \quad i = 1, \dots, n, \tag{2.7}$$

where $s_i \in (0, \infty)$, $Y(t)$ is the $\mathbf{R}^n$-valued process with stationary increments defined above, $\mu_i : [0, \infty) \to \mathbf{R}$ and $\sigma_{ij} : [0, \infty) \to \mathbf{R}$ are continuous, deterministic functions such that $\sigma(t) = (\sigma_{ij}(t))_{1 \leq i,j \leq n}$ is nonsingular for $t \geq 0$. The dynamics of the price $S_0(t)$ of the riskless asset is given by

$$dS_0(t) = r(t) S_0(t) dt, \qquad S_0(0) = 1,$$

where $r : [0, \infty) \to [0, \infty)$ is a continuous, deterministic function. More explicitly, $S_0(t) = \exp[\int_0^t r(s) ds]$ for $t \geq 0$, and, in view of (2.4), we have

$$S_i(t) = s_i \exp \left[\int_0^t \sum_{j=1}^n \sigma_{ij}(s) dY_j(s) + \int_0^t \left(\mu_i(s) - \frac{1}{2} \sum_{j=1}^n \sigma_{ij}^2(s) \right) ds \right], \tag{2.8}$$
$$t \geq 0, \quad i = 1, \dots, n.$$

The $2n$ parameters p_j and q_j describe the memory of $Y(t)$, whence that of $S(t)$.

For $T \in (0, \infty)$, we put

$$\mathcal{A}_T = \left\{ \pi = (\pi(t))_{0 \leq t \leq T} : \begin{array}{c} \pi \text{ is an } \mathbf{R}^n\text{-valued, progressively measurable} \\ \text{process satisfying } \int_0^T \|\pi(t)\|^2 dt < \infty \text{ a.s.} \end{array} \right\},$$

where $\|v\|$ denotes the Euclidean norm of $v \in \mathbf{R}^n$. The class $\mathcal{A}_T$ is that of admissible strategies on the finite horizon $[0, T]$. The corresponding class of admissible strategies on the infinite horizon $[0, \infty)$ is defined by

$$\mathcal{A} = \left\{ (\pi(t))_{t \geq 0} : (\pi(t))_{0 \leq t \leq T} \in \mathcal{A}_T \text{ for every } T \in (0, \infty) \right\}.$$

Let $x \in (0, \infty)$ and let π be an admissible strategy, i.e., π is in $\mathcal{A}$ or $\mathcal{A}_T$ for some $T > 0$. Denoting by $\pi_i(t)$ the ith component of $\pi(t)$, we consider the wealth process $X^{x,\pi}(t)$ of an investor with initial wealth x who invests, at time t, $\pi_i(t) X^{x,\pi}(t)$ dollars in the ith risky asset for $i = 1, \dots, n$ and $[1 - \sum_{i=1}^n \pi_i(t)] X^{x,\pi}(t)$ dollars in the riskless asset. Thus $\pi_i(t)$ denotes the proportion of the capital invested in the ith risky asset at time t.

We define an $\mathbf{R}^n$-valued deterministic function $\eta(t) = (\eta_1(t), \dots, \eta_n(t))^*$ by

$$\eta(t) = \sigma^{-1}(t) \left[\mu(t) - r(t) \mathbf{1} \right], \qquad t \geq 0, \tag{2.9}$$

where $\mathbf{1} = (1, \dots, 1)^*$. We also define an $\mathbf{R}^n$-valued Gaussian process $\xi(t) = (\xi_1(t), \dots, \xi_n(t))^*$ by

$$\xi_i(t) = \int_0^t l_j(t,s) dB_j(s), \quad t \geq 0, \quad j = 1, \dots, n. \tag{2.10}$$

By the self-financing condition, $X^{x,\pi}(t)$ evolves according to $X^{x,\pi}(0) = x$ and

$$\begin{aligned}\frac{dX^{x,\pi}(t)}{X^{x,\pi}(t)} &= \left[1 - \sum\nolimits_{i=1}^n \pi_i(t)\right] \frac{dS_0(t)}{S_0(t)} + \sum\nolimits_{i=1}^n \pi_i(t) \frac{dS_i(t)}{S_i(t)} \\ &= \left[1 - \sum\nolimits_{i=1}^n \pi_i(t)\right] r(t)dt + \sum\nolimits_{i=1}^n \pi_i(t) \left[\mu_i(t)dt + \sum\nolimits_{j=1}^n \sigma_{ij}(t)dY_j(t)\right] \\ &= r(t)dt + \pi(t)^*\sigma(t)\theta(t)dt + \pi(t)^*\sigma(t)dB(t),\end{aligned}$$

where $\theta(t)$ is an $\mathbf{R}^n$-valued Gaussian process defined by

$$\theta(t) = \eta(t) - \xi(t), \qquad t \geq 0. \tag{2.11}$$

Applying the Itô formula, we find that, for $t \geq 0$,

$$\begin{aligned}X^{x,\pi}(t) = x \exp \Bigg[\int_0^t \left\{ r(s) + \pi^*(s)\sigma(s)\theta(s) - \frac{1}{2}\|\sigma^*(s)\pi(s)\|^2 \right\} ds \\ + \int_0^t \pi^*(s)\sigma(s)dB(s)\Bigg].\end{aligned} \tag{2.12}$$

From (2.6) and (2.10), we have

$$\xi_j(t) = \int_0^t k_j(t,s)dY_j(s), \quad t \geq 0, \quad j = 1, \ldots, n.$$

This and (2.8) imply that we can express $\xi(t)$ and $\theta(t)$ explicitly in terms of the past prices $S(u)$, $0 \leq u \leq t$, of the risky assets.

For $T \in (0,\infty)$ and $\alpha \in (-\infty, 1) \setminus \{0\}$, we consider the following classical optimal investment problem over the finite horizon $[0,T]$:

$$V(T) = \sup_{\pi \in \mathcal{A}_T} \frac{1}{\alpha} E\left[(X^{x,\pi}(T))^\alpha\right]. \tag{2.13}$$

Starting with the solution to this problem, we explicitly solve the infinite horizon risk-sensitive control problem

$$J(\alpha) = \sup_{\pi \in \mathcal{A}} J(\alpha, \pi) \tag{2.14}$$

with

$$J(\alpha,\pi) = \limsup_{T\to\infty} \frac{1}{\alpha T} \log E\left[(X^{x,\pi}(T))^\alpha\right]. \tag{2.15}$$

This result in turn enables us to apply the duality relation in Pham [16, 17] to solve the following large deviation probability control problem:

$$I(c) = \sup_{\pi\in\mathcal{A}} \limsup_{T\to\infty} \frac{1}{T} \log P\left[X^{x,\pi}(T) \geq e^{cT}\right], \qquad c \in \mathbf{R}. \tag{2.16}$$

3. Finite time horizon

In this section, we consider the finite horizon optimization problem (2.13).

Let $\alpha \in (-\infty, 1) \setminus \{0\}$ and let β be the conjugate exponent of α defined by $(1/\alpha) + (1/\beta) = 1$, that is, $\beta = \alpha/(\alpha - 1)$. If $-\infty < \alpha < 0$ (resp. $0 < \alpha < 1$), then $0 < \beta < 1$ (resp. $-\infty < \beta < 0$).

For $j = 1, \ldots, n$ and $T \in (0,\infty)$, we consider the one-dimensional backward Riccati equation

$$\dot{R}_j(t) - l_j^2(t)R_j^2(t) + 2b_j(t)R_j(t) + \beta(1-\beta) = 0, \quad 0 \leq t \leq T, \quad R_j(T) = 0, \tag{3.1}$$

where $l_j(t)$ is as in (2.5) and $b_j(t)$ is defined by

$$b_j(t) = -(p_j + q_j) + \beta l_j(t), \qquad (t \geq 0,\ j = 1, \dots, n). \tag{3.2}$$

If $p_j = 0$, then (3.1) is linear, whence it has a unique solution $R_j(t) = R_j(t;T)$. If $p_j > 0$ and $\alpha < 0$, then $\beta(1-\beta) > 0$, so that, by the well-known result on Riccati equations (see, e.g., Fleming and Rishel [5, Theorem 5.2] and Liptser and Shiryayev [13, Theorem 10.2]), (3.1) has a unique nonnegative solution $R_j(t) = R_j(t;T)$.

When $p_j > 0$ and $0 < \alpha < 1$, we have the following key result.

Lemma 3.1. *We assume $p_j > 0$ and $0 < \alpha < 1$. Then, for every $T > 0$, the equation* (3.1) *has a unique solution $R_j(t) = R_j(t;T)$ such that $R_j(t) \geq b_j(t)/l_j^2(t)$ for $t \in [0,T]$.*

Proof. We put, for $t \geq 0$,

$$a_1(t) = l_j^2(t), \quad a_2(t) = b_j(t), \quad a_3 = \beta(1-\beta), \tag{3.3}$$

and consider the transform

$$P(t) = R_j(t) - \frac{a_2(t)}{a_1(t)}.$$

Then the equation for $P(t)$ is given by

$$\dot{P}(t) - a_1(t)P^2(t) + a_4(t) = 0, \qquad 0 \leq t \leq T, \tag{3.4}$$

where

$$a_4(t) = \frac{a_2^2(t) + a_1(t)a_3}{a_1(t)} + \frac{d}{dt}\left(\frac{a_2(t)}{a_1(t)}\right).$$

Since $dl_j(t)/dt > 0$ and $\beta < 0$, we have

$$\frac{d}{dt}\left(\frac{a_2(t)}{a_1(t)}\right) = \frac{2(p_j + q_j) - \beta l_j(t)}{l_j(t)^3} \cdot \frac{dl_j}{dt}(t) > 0.$$

Moreover, since $0 \leq l_j(t) \leq p_j$, it holds that

$$\begin{aligned} &a_2^2(t) + a_1(t)a_3 \\ &\quad = (1-\beta)\left[(p_j+q_j)^2 - \{(p_j+q_j) - l_j(t)\}^2\right] + [(p_j+q_j) - l_j(t)]^2 > 0. \end{aligned}$$

Thus $a_4(t) > 0$, so that (3.3) has a unique nonnegative solution $P(t) = P(t;T)$. The desired solution to (3.1) is given by $R_j(t) = P(t) + [a_2(t)/a_1(t)]$. □

In what follows, we write $R_j(t) = R_j(t;T)$ for the unique solution to (3.1) in the sense above. We define the $n \times n$ diagonal matrix $R(t)$ by

$$R(t) = \mathrm{diag}(R_1(t), \dots, R_n(t)).$$

Recall $\eta(t) = (\eta_1(t), \dots, \eta_n(t))^*$ from (2.9). For $j = 1, \dots, n$, let $v_j(t) = v_j(t;T)$ be the solution to the following one-dimensional linear equation:

$$\begin{aligned} &\dot{v}_j(t) + [b_j(t) - l_j^2(t)R_j(t;T)]v_j(t) + \beta(1-\beta)\eta_j(t) - R_j(t;T)\rho_j(t) = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leq t \leq T, \\ &v_j(T) = 0, \end{aligned} \tag{3.5}$$

where

$$\rho_j(t) = -\beta l_j(t)\eta_j(t), \qquad t \geq 0, \quad j = 1, \dots, n. \tag{3.6}$$

We define $v(t) = v(t;T)$ and $\rho(t) = \rho(t;T)$ by

$$v(t;T) = (v_1(t;T), \ldots, v_n(t;T))^*, \qquad \rho(t;T) = (\rho_1(t;T), \ldots, \rho_n(t;T))^*.$$

Recall $\theta(t)$ and $B(t)$ from Section 2. We define a real-valued processes $Z(t)$ by

$$Z(t) = \exp\left[-\int_0^t \theta^*(s)dB(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\right], \qquad t \geq 0.$$

Since $\theta(t)$ is a continuous Gaussian process, the process $Z(t)$ is a P-martingale (see, e.g., Example 5 in [13, Section 6.2]). We define the process $(\Gamma(t))_{0\leq t\leq T}$ by

$$\Gamma(t) = E\left[Z^\beta(T) \middle| \mathcal{F}_t\right], \qquad 0 \leq t \leq T.$$

We put

$$g_j(t;T) = v_j^2(t;T)l_j^2(t) + 2\rho_j(t)v_j(t;T) - l_j^2(t)j(t;T) - \beta(1-\beta)\eta_j^2(t), \quad (t,T) \in \Delta,\ j = 1, \ldots, n, \tag{3.7}$$

where Δ is the domain defined by

$$\Delta = \{(t,T) : 0 < T < \infty,\ 0 \leq t \leq T\}. \tag{3.8}$$

We put

$$l(t) = \mathrm{diag}(l_1(t), \ldots, l_n(t)).$$

The next proposition gives a stochastic integral representation of $\Gamma(t)$.

Proposition 3.2. *For $t \in [0,T]$, $\Gamma(t)$ is finite a.s. and we have $\Gamma(t) = \Gamma(0) + \int_0^t \psi^*(s)dB(s)$ with*

$$\Gamma(0) = \exp\left[\frac{1}{2}\int_0^T \sum_{j=1}^n g_j(s;T)ds\right] \tag{3.9}$$

and

$$\psi(t) = \Gamma(t)[-\beta\theta(t) - l(t)R(t;T)\xi(t) + l(t)v(t;T)], \qquad 0 \leq t \leq T. \tag{3.10}$$

Proof. We define an $\mathbf{R}$-valued P-martingale $K(t)$ by

$$K(t) = \exp\left[-\beta\int_0^t \theta^*(s)dB(s) - \frac{\beta^2}{2}\int_0^t \|\theta(s)\|^2 ds\right], \qquad t \geq 0,$$

and the probability measure $\bar{P}$ by $d\bar{P}/dP = K(T)$. Recall $\xi(t)$ from Section 2. By Bayes' rule, we have, for $t \in [0,T]$,

$$\begin{aligned}
\Gamma(t) &= E\left[K(T)\exp\left\{-\frac{1}{2}\beta(1-\beta)\int_0^T \|\theta(s)\|^2 ds\right\}\middle| \mathcal{F}_t\right] \\
&= K(t)\bar{E}\left[\exp\left\{-\frac{1}{2}\beta(1-\beta)\int_0^T \|\theta(s)\|^2 ds\right\}\middle| \mathcal{F}_t\right] \\
&= Z^\beta(t)\exp\left\{-\frac{1}{2}\beta(1-\beta)\int_t^T \|\eta(s)\|^2 ds\right\} \\
&\qquad \times \bar{E}\left[\exp\left\{-\frac{1}{2}\beta(1-\beta)\int_t^T \left(\|\xi(s)\|^2 - 2\eta^*(s)\xi(s)\right)ds\right\}\middle| \mathcal{F}_t\right],
\end{aligned}$$

where $\bar{E}$ stands for the expectation with respect to $\bar{P}$.

Let $(\bar{B}(t))_{0\le t\le T}$ be the $\mathbf{R}^n$-valued standard $\bar{P}$-Brownian motion defined by

$$\bar{B}(t) = B(t) + \int_0^t \beta\theta(s)ds, \qquad 0 \le t \le T.$$

Then

$$\begin{aligned} d\xi(t) &= -(p+q)\xi(t) + l(t)dB(t) \\ &= [-\beta l(t)\eta(t) + \{-(p+q) + \beta l(t)\}\xi(t)]\, dt + l(t)d\bar{B}(t) \\ &= [\rho(t) + b(t)\xi(t)]\, dt + l(t)d\bar{B}(t), \end{aligned}$$

where

$$b(t) = \mathrm{diag}(b_1(t), \ldots, b_n(t)), \quad p = \mathrm{diag}(p_1, \ldots, p_n), \quad q = \mathrm{diag}(q_1, \ldots, q_n).$$

Since $R(t)$ satisfies the matrix Riccati equation

$$\begin{aligned} &\dot{R}(t) - R(t)l^2(t)R(t) + b(t)R(t) + R(t)b(t) + \beta(1-\beta)I_n = 0, \quad 0 \le t \le T, \\ &R(T) = 0, \end{aligned}$$

with $n \times n$ unit matrix I_n, it follows from Theorem A.1 in Appendix A (Cameron–Martin type formula) that $\Gamma(t)$ is equal to

$$Z^\beta(t) \exp\left[\sum_{j=1}^n v_j(t)\xi_j(t) - \frac{1}{2}\sum_{j=1}^n \xi_j^2(t)R_j(t) + \frac{1}{2}\int_t^T \sum_{j=1}^n g_j(s;T)ds\right].$$

In particular, we see that $\Gamma(0)$ is given by (3.9).

By the expression of $\Gamma(t)$ above and the Itô formula, we have

$$d\log\Gamma(t) = \psi^*(t)dB(t) + [\cdots]dt$$

with $\psi(t)$ as in (3.10), so that the P-martingale $(\Gamma(t))_{0\le t\le T}$ is given by

$$\Gamma(t) = \Gamma(0)\exp\left[\int_0^t \psi^*(s)dB(s) - \frac{1}{2}\int_0^t \|\psi(s)\|^2 ds\right], \qquad 0 \le t \le T.$$

Thus the proposition follows. □

For $T \in (0, \infty)$, we define the admissible strategy $\hat{\pi}_T = (\hat{\pi}_T(t))_{0\le t\le T} \in \mathcal{A}_T$ by

$$\hat{\pi}_T(t) = (\sigma^*)^{-1}(t)\left[(1-\beta)\theta(t) - l(t)R(t;T)\xi(t) + l(t)v(t;T)\right], \quad 0 \le t \le T. \tag{3.11}$$

Since $\theta(t)$ and $\xi(t)$ are expressed in terms of the past prices $S(u)$, $u \in [0, t]$, of the risky assets (see Section 2), so is $\hat{\pi}_T(t)$.

Here is the solution to the finite time horizon problem (2.13).

Theorem 3.3. *For $T \in (0, \infty)$, the strategy $\hat{\pi}_T$ is the unique solution to the problem (2.13), i.e., $\hat{\pi}_T$ is the unique admissible strategy that attains the supremum in (2.13). The value function $V(T)$ in (2.13) is given by*

$$V(T) = \frac{1}{\alpha}[xS_0(T)]^\alpha \exp\left[\frac{(1-\alpha)}{2}\sum_{j=1}^n \int_0^T g_j(t;T)dt\right]. \tag{3.12}$$

Proof. Let the process $\psi(t)$ be as in (3.10). Then, by Theorem 7.6 in Karatzas and Shreve [11, Chapter 3], the unique optimal strategy $(\pi_T(t))_{0\le t\le T}$ is given by

$$\pi_T(t) = (\sigma^*)^{-1}(t)\left[\Gamma^{-1}(t)\psi(t) + \theta(t)\right], \qquad 0 \le t \le T.$$

However, by (3.10), we have $\pi_T(t) = \hat{\pi}_T(t)$, whence the first assertion follows. By the same theorem in [11], we also see that $V(T) = \alpha^{-1}[xS_0(T)]^{\alpha}\Gamma^{1-\alpha}(0)$. This and (3.9) yield (3.12). □

Remark 3.4. From Theorem 7.6 in [11, Chapter 3], we also find that

$$X^{x,\hat{\pi}_T}(t) = x\frac{S_0(t)\Gamma(t)}{Z(t)\Gamma(0)}, \qquad 0 \le t \le T.$$

4. Infinite time horizon

In this section, we consider the infinite horizon problem (2.11).

Recall $\eta(t)$ from (2.9). Throughout this section we assume the following two conditions:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T r(t)dt = \bar{r} \quad \text{for some } \bar{r} \in \mathbf{R}; \tag{4.1}$$

$$\lim_{T\to\infty} \eta(t) = \bar{\eta} \quad \text{for some } \bar{\eta} = (\bar{\eta}_1, \dots, \bar{\eta}_n)^* \in \mathbf{R}^n. \tag{4.2}$$

Let α and β be as in Section 3. Notice that $(1-\alpha)(1-\beta) = 1$. Recall $b_j(t)$ from (3.2). For $j = 1, \dots, n$, we define a negative constant $\bar{b}_j$ by

$$\bar{b}_j = -(1-\beta)p_j - q_j.$$

Then $\lim_{t\to\infty} b_j(t) = \bar{b}_j$. We consider the equation

$$p_j^2 x^2 - 2\bar{b}_j x - \beta(1-\beta) = 0. \tag{4.3}$$

In view of

$$\bar{b}_j^2 + \beta(1-\beta)p_j^2 = (1-\beta)[(p_j+q_j)^2 - q_j^2] + q_j^2 \ge q_j^2 > 0,$$

we may write $\bar{R}_j$ for the larger (resp. unique) solution of (4.3) when $p_j > 0$ (resp. $p_j = 0$). We define a positive constant K_j by

$$K_j = \sqrt{\bar{b}_j^2 + \beta(1-\beta)p_j^2}.$$

Then it holds that $\bar{b}_j - p_j^2\bar{R}_j = -K_j < 0$.

In what follows, we write $R_j(t) = R_j(t;T)$ for the unique solution to the Riccati equation (3.1) in the sense of Section 3. Recall the domain Δ from (3.8). We need the next lemma when we apply Theorem B.2 in Appendix B to $R_j(t;T)$.

Lemma 4.1. *Let $j \in \{1, \dots, n\}$. We assume $0 < \alpha < 1$ and $p_j > 0$. Then $R_j(t;T)$ is bounded in Δ.*

Proof. Let $a_1(t)$, $a_2(t)$ and a_3 be as in (3.3). Since $b_j(t)/l_j(t)^2$ is bounded from below in $t > 0$, so is $R_j(t;T)$ in Δ. To show that $R_j(t;T)$ is bounded from above in Δ, we consider the solution $M(t) = M(t;T)$ to the linear equation

$$\dot{M}(t) + 2[a_2(t) - \bar{R}_j a_1(t)]M(t) + a_3 + a_1(t)\bar{R}_j^2 = 0. \quad M(T) = 0.$$

Then, since $M(T) - R(T) = 0$ and

$$[\dot{M}(t) - \dot{R}_j(t)] + 2[a_2(t) - \bar{R}_j a_1(t)][M(t) - R_j(t)] = -a_1(t)\left[R_j(t) - \bar{R}_j\right]^2 \le 0,$$

we see that $R(t;T) \le M(t;T)$ in Δ. However, $\lim_{t\to\infty}\left[a_2(t) - \bar{R}a_1(t)\right] = \bar{b}_j - \bar{R}_j\bar{p}_j^2 < 0$, so that $M(t;T)$ is bounded from above in Δ, whence so is $R_j(t;T)$. □

The next proposition provides the necessary results on the asymptotic behavior of $R_j(t;T)$.

Proposition 4.2. *For $j = 1, \dots, n$, we have the following:*

$$R_j(t;T) \text{ is bounded in } \Delta;$$
$$\lim_{T-t\to\infty,\ t\to\infty} R_j(t;T) = \bar{R}_j;$$
$$\lim_{T\to\infty} \sup_{\delta T \le t \le (1-\epsilon)T} |R_j(t;T) - \bar{R}_j| = 0 \quad \text{for } \delta, \epsilon \in (0,\infty) \text{ such that } \delta + \epsilon < 1.$$

Proof. Since

$$|l_j(t) - p_j| \le \frac{p_j^2}{2(p_j + q_j)} e^{-2q_j t}, \qquad t \ge 0,$$

the function $l_j(t)$ converges to p_j exponentially fast as $t \to \infty$. Hence the coefficients in the equations (3.1) also converge to those in (4.3) exponentially fast. If $p_j > 0$ and $-\infty < \alpha < 0$ (resp. $0 < \alpha < 1$), the desired assertions follow from Theorem B.1 in Appendix B due to Nagai and Peng [15] (resp. Lemma 4.1 and Theorem B.2). If $p_j = 0$, then $l_j(t) = 0$ and $b_j(t) = -q_j < 0$ for $t \ge 0$, so that the assertions follow from Theorem B.3. □

Let $v_j(t) = v_j(t;T)$ be as in Section 3. Define $\bar{v}_j$ by

$$\left(\bar{b}_j - p_j^2 \bar{R}_j\right) \bar{v}_j + \beta(1-\beta)\bar{\eta}_j - \bar{R}_j \bar{\rho}_j = 0, \tag{4.4}$$

where

$$\bar{\rho}_j = -\beta p_j \bar{\eta}_j, \qquad j = 1, \dots, n.$$

Recall $\rho_j(t)$ from (3.6). It holds that $\lim_{t\to\infty} \rho_j(t) = \bar{\rho}_j$.

Proposition 4.3. *For $j = 1, \dots, n$, we have the following:*

$$v_j(t;T) \text{ is bounded in } \Delta;$$
$$\lim_{T-t\to\infty,\ t\to\infty} v_j(t;T) = \bar{v}_j;$$
$$\lim_{T\to\infty} \sup_{\delta T \le t \le (1-\epsilon)T} |v_j(t;T) - \bar{v}_j| = 0 \quad \text{for } \delta, \epsilon \in (0,\infty) \text{ such that } \delta + \epsilon < 1.$$

Proof. The coefficients in (3.5) converge to those in (4.4), and it holds that

$$\lim_{T-t\to\infty,\, t\to\infty} [b_j(t) - l_j^2(t) R_j(t;T)] = \bar{b}_j - p_j^2 \bar{R}_j = -K_j < 0,$$

Thus the proposition follows from Theorem B.3 in Appendix B. □

Recall the value function $V(T)$ from (2.13) and $g_j(t;T)$ from (3.7). We put

$$\bar{g}_j = \bar{v}_j^2 p_j^2 + 2\bar{\rho}_j \bar{v}_j - p_j^2 \bar{R}_j - \beta(1-\beta)\bar{\eta}_j^2, \qquad j = 1, \dots, n.$$

Proposition 4.4. *For every $\alpha \in (-\infty, 1) \setminus \{0\}$, the limit*

$$\tilde{J}(\alpha) = \lim_{T\to\infty} \frac{1}{\alpha T} \log\left[\alpha V(T)\right]$$

exists and is given by

$$\tilde{J}(\alpha) = \bar{r} + \frac{1-\alpha}{2\alpha} \sum_{j=1}^n \bar{g}_j.$$

Proof. By Propositions 4.2 and 4.3, we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T g_j(t;T)dt = \bar{g}_j, \qquad j=1,\dots,n.$$

This and (3.12) yield

$$\begin{aligned}\frac{1}{\alpha T}\log\left[\alpha V(T)\right] &= \frac{\log x}{T}+\frac{1}{T}\int_0^T r(t)dt+\frac{1-\alpha}{2\alpha}\sum_{j=1}^n\frac{1}{T}\int_0^T g_j(t;T)dt\\ &\to \bar{r}+\frac{1-\alpha}{2\alpha}\sum_{j=1}^n\bar{g}_j, \qquad T\to\infty,\end{aligned}$$

as desired. ☐

The next proposition represents $\tilde{J}(\alpha)$ explicitly.

Proposition 4.5. *For $\alpha\in(-\infty,1)\setminus\{0\}$, we have*

$$\begin{aligned}\tilde{J}(\alpha) &= \bar{r}+\sum_{j=1}^n\frac{(p_j+q_j)^2\bar{\eta}_j{}^2}{2\left[(1-\alpha)(p_j+q_j)^2+p_j(p_j+2q_j)\alpha\right]}\\ &\quad+\sum_{j=1}^n\frac{(p_j+q_j)-q_j\alpha-(1-\alpha)^{1/2}\left[(1-\alpha)(p_j+q_j)^2+p_j(p_j+2q_j)\alpha\right]^{1/2}}{2\alpha}.\end{aligned}$$

Proof. We have

$$\begin{aligned}\beta p_j^2(p_j\bar{R}_j)^2-2\beta p_j^2\bar{R}_jK_j &= \beta p_j^2\bar{R}_j(p_j^2\bar{R}_j-2K_j)=\beta p_j^2\bar{R}_j(\bar{b}_j-K_j)\\ &= -\beta p_j^2\beta(1-\beta).\end{aligned}$$

and $\bar{v}_j=\beta\bar{\eta}_j(1-\beta+p_j\bar{R}_j)/K_j$. Thus

$$\begin{aligned}&\bar{v}_j^2p_j^2+2\bar{\rho}_j\bar{v}_j-\beta(1-\beta)\bar{\eta}_j^2\\ &\quad=\frac{\beta\bar{\eta}_j^2}{K_j^2}\left[\beta p_j^2(1-\beta+p_j\bar{R}_j)^2-2\beta p_j(1-\beta+p_j\bar{R}_j)K_j-(1-\beta)K_j^2\right]\\ &\quad=\frac{\beta(1-\beta)\bar{\eta}_j^2}{K_j^2}\left[\beta p_j^2-2\beta^2p_j^2-K_j^2+2\beta p_j(p_j^2\bar{R}_j-K_j)\right]\\ &\quad=\frac{\beta(1-\beta)\bar{\eta}_j^2}{K_j^2}\left[\beta p_j^2-2\beta^2p_j^2-\{\bar{b}_j^2+\beta(1-\beta)p_j^2\}+2\beta p_j\bar{b}_j\right]\\ &\quad=-\frac{\beta(1-\beta)\bar{\eta}_j^2}{K_j^2}[\bar{b}_j-\beta p_j]^2=\frac{\alpha}{(\alpha-1)^2}\frac{(p_j+q_j)^2}{K_j^2}\bar{\eta}_j^2.\end{aligned}$$

From this and $p_j^2\bar{R}_j=\bar{b}_j+K_j$, we get

$$\bar{g}_j=\frac{\alpha}{(1-\alpha)^2}\frac{(p_j+q_j)^2}{K_j^2}\bar{\eta}_j^2-(\bar{b}_j+K_j).$$

Since $(1-\alpha)(1-\beta)=1$, it follows that

$$(1-\alpha)\bar{b}_j=(1-\alpha)[(\beta-1)p_j-q_j]=(\alpha-1)q_j-p_j=q_j\alpha-(p_j+q_j).$$

Also,

$$K_j^2 = (p_j+q_j)^2 - \beta p_j(p_j+q_j) = (1-\alpha)(p_j+q_j)^2 + \frac{\alpha}{1-\alpha} p_j(p_j+q_j).$$

Combining, we obtain the desired assertion. □

For $j = 1, \ldots, n$, we put

$$\alpha_j^* = \begin{cases} -\infty & \text{if} \quad 0 \le p_j \le 2q_j, \\ -3 - \dfrac{8q_j}{p_j - 2q_j} & \text{if} \quad 2q_j < p_j < \infty \end{cases}$$

and define $\alpha^* \in [-\infty, -3)$ by

$$\alpha^* = \max(\alpha_1^*, \ldots, \alpha_n^*).$$

Lemma 4.6. *If $\alpha^* < \alpha < 0$, then $p_j^2 \bar{R}_j^2 < (1-\beta)^2$ for all $j \in \{1, \ldots, n\}$.*

Proof. We may assume that $p_j \neq 0$. We put $f(x) = p_j^2 x^2 - 2\bar{b}_j x - \beta(1-\beta)$. Then the inequality $p_j^2 \bar{R}_j^2 < (1-\beta)^2$ holds if and only if $f((1-\beta)/|p_j|) > 0$. By elementary arguments, we obtain the lemma. □

In what follows, we assume

$$\alpha \in (\alpha^*, 0) \cup (0, 1). \tag{4.5}$$

For $j = 1, \ldots, n$ and $T \in (0, \infty)$, we consider the one-dimensional backward Riccati equation

$$\dot{U}_j(t) - l_j^2(t) U_j^2(t) + 2d_j(t) U_j(t) + Q_j = 0, \quad 0 \le t \le T, \quad U_j(T) = 0, \tag{4.6}$$

where

$$Q_j = \beta \left[1 - (1-\alpha)^2 p_j^2 \bar{R}_j^2\right], \qquad j = 1, \ldots, n$$

and

$$d_j(t) = b_j(t) - \alpha p_j \bar{R}_j l_j(t), \qquad t \ge 0, \; j = 1, \ldots, n.$$

If $p_j = 0$, then (4.6) is linear, whence it has a unique solution $U_j(t) = U_j(t; T)$. If $p_j > 0$ and $\alpha^* < \alpha < 0$, then $Q_j > 0$, so that, by Lemma 4.6, (4.6) has a unique nonnegative solution $U_j(t) = U_j(t; T)$.

When $p_j > 0$ and $0 < \alpha < 1$, we have the following key result.

Lemma 4.7. *We assume $p_j > 0$ and $0 < \alpha < 1$. Then, for every $T > 0$, the equation* (4.1) *has a unique solution $U_j(t) = U_j(t; T)$ such that $U_j(t; T) \ge (1-\alpha) R_j(t; T)$ for $t \in [0, T]$.*

Proof. We consider the transform

$$V(t) = \frac{U_j(t)}{1-\alpha} - R_j(t).$$

Then, we see that the equation for $V(t)$ is given by

$$\begin{aligned} \dot{V}(t) - (1-\alpha) l_j^2(t) V^2(t) - 2\left[(1-\alpha) l_j^2(t) R_j(t) - d_j(t)\right] V(t) \\ + \alpha [l_j(t) R_j(t) - p_j \bar{R}_j]^2 = 0, \quad 0 \le t \le T, \end{aligned}$$

with $V(T) = 0$, which has a unique nonnegative solution. Thus the lemma follows. □

In what follows, we write $U_j(t) = U_j(t;T)$ for the unique solution to (3.1) in the sense above.

We put

$$\bar{d}_j = \bar{b}_j - \alpha p_j^2 \bar{R}_j, \qquad j = 1, \dots, n.$$

Then $d_j(t) \to \bar{d}_j$ as $t \to \infty$ exponentially fast. Since

$$\begin{aligned} \bar{d}_j^2 + p_j^2 Q_j &= (\bar{b}_j - \alpha p_j^2 \bar{R}_j)^2 + p_j^2 \beta \left[1 - (1-\alpha)^2 p_j^2 \bar{R}_j^2\right] \\ &= \bar{b}_j^2 - 2\alpha \bar{b}_j(\bar{b}_j + K_j) + \alpha^2 (\bar{b}_j + K_j)^2 + p_j^2 \beta - \alpha(\alpha-1)(\bar{b}_j + K_j)^2 \\ &= (1-\alpha)\bar{b}_j^2 + \alpha K_j^2 + p_j^2 \beta = \bar{b}_j^2 + p_j^2 \beta(1-\beta), \end{aligned}$$

we see that

$$\sqrt{\bar{d}_j^2 + p_j^2 Q_j} = K_j > 0. \tag{4.7}$$

Let $\bar{U}_j$ be the larger (resp. unique) solution of the following equation when $p_j > 0$ (resp. $p_j = 0$):

$$p_j^2 x^2 - 2\bar{d}_j x - Q_j = 0.$$

From (4.7), it follows that $\bar{d}_j - p_j^2 \bar{U}_j = -K_j$.

We need the next lemma when we apply Theorem B.2 in Appendix B to $U_j(t;T)$.

Lemma 4.8. *Let $j \in \{1, \dots, n\}$. We assume $0 < \alpha < 1$ and $p_j > 0$. Then $U_j(t;T)$ is bounded in Δ.*

Proof. Since $R_j(t;T)$ is bounded from below in Δ, so is $U_j(t;T)$. Let $N(t) = N(t;T)$ be the solution to the linear equation

$$\dot{N}(t) + 2[d_2(t) - \bar{U}_j l_j^2(t)]N(t) + Q_j + l_j^2(t)\bar{U}_j^2 = 0, \quad N(T) = 0.$$

Then, as in the proof of Lemma 4.1, we see that $U_j(t;T) \le N(t;T)$ and that $N(t;T)$ is bounded from above in Δ. Hence so is $U_j(t;T)$. □

The next proposition gives the necessary results on the asymptotic behavior of $U_j(t;T)$.

Proposition 4.9. *For $j = 1, \dots, n$, we have the following:*

$$U_j(t;T) \text{ is bounded in } \Delta;$$

$$\lim_{T-t\to\infty,\ t\to\infty} U_j(t;T) = \bar{U}_j;$$

$$\lim_{T\to\infty} \sup_{\delta T \le t \le (1-\epsilon)T} |U_j(t;T) - \bar{U}_j| = 0 \quad \text{for } \delta, \epsilon \in (0,\infty) \text{ such that } \delta + \epsilon < 1.$$

The proof is similar to that of Proposition 4.2, and so we omit it.

For $j = 1, \dots, n$, let $m_j(t) = m_j(t;T)$ be the solution to the following one-dimensional linear equation:

$$\begin{aligned} &\dot{m}_j(t) + [d_j(t) - l_j^2(t) U_j(t;T)] m_j(t) - h_j(t) - U_j(t;T)\gamma_j(t) = 0, \quad 0 \le t \le T, \\ &m_j(T) = 0, \end{aligned}$$

where, for $t \ge 0$ and $j = 1, \dots, n$, we define

$$h_j(t) = \alpha(\alpha-1)p_j^2 \bar{R}_j \bar{v}_j - \beta \eta_j(t), \qquad \gamma_j(t) = \rho_j(t) + \alpha p_j l_j(t) \bar{v}_j.$$

We put

$$m(t;T) = (m_1(t;T), \ldots, m_n(t;T))^*, \qquad \gamma(t;T) = (\gamma_1(t;T), \ldots, \gamma_n(t;T))^*,$$
$$h(t;T) = (h_1(t;T), \ldots, h_n(t;T))^*.$$

Define $\bar{m}_j$ by

$$\left(\bar{d}_j - p_j^2 \bar{U}_j\right) \bar{m}_j - \bar{h}_j - \bar{U}_j \bar{\gamma}_j = 0,$$

where the constants $\bar{h}_j$ and $\bar{\gamma}_j$ are defined by

$$\bar{h}_j = \alpha(\alpha-1)p_j^2 \bar{R}_j \bar{v}_j - \beta \bar{\eta}_j, \qquad \bar{\gamma}_j = \bar{\rho}_j + \alpha p_j^2 \bar{v}_j.$$

We have $\lim_{t\to\infty} h_j(t) = \bar{h}_j$ and $\lim_{t\to\infty} \gamma_j(t) = \bar{\gamma}_j$.

Proposition 4.10. *For $j = 1, \ldots, n$, the following hold:*

$$m_j(t;T) \text{ is bounded in } \Delta,$$
$$\lim_{T-t\to\infty,\ t\to\infty} m_j(t;T) = \bar{m}_j,$$
$$\lim_{T\to\infty} \sup_{\delta T \le t \le (1-\epsilon)T} |m_j(t;T) - \bar{m}_j| = 0 \quad \text{for } \delta, \epsilon \in (0,\infty) \text{ such that } \delta + \epsilon < 1.$$

The proof is similar to that of Proposition 4.3, and so we omit it.

With (3.11) in mind, we consider the admissible strategy $\hat{\pi} = (\hat{\pi}(t))_{t\ge 0} \in \mathcal{A}$ defined by

$$\hat{\pi}(t) = (\sigma^*)^{-1}(t)[(1-\beta)\theta(t) - p\bar{R}\xi(t) + p\bar{v}], \qquad t \ge 0, \tag{4.8}$$

where $p = \mathrm{diag}(p_1, \ldots, p_n)$ as before, and $\bar{R}$ is defined by

$$\bar{R} = \mathrm{diag}(\bar{R}_1, \ldots, \bar{R}_n).$$

The next theorem shows that $\hat{\pi}$ is the solution to the problem (2.12). Notice that $\hat{\pi}(t)$ can be expressed explicitly in terms of $S(u)$, $0 \le u \le t$.

Recall that we have assumed (4.5).

Theorem 4.11. *The strategy $\hat{\pi}$ is a solution to the problem* (2.14) *with a limit in* (2.15)*, i.e.,*

$$J(\alpha) = \lim_{T\to\infty} \frac{1}{\alpha T} \log E\left[(X^{x,\hat{\pi}}(T))^\alpha\right].$$

The optimal rate $J(\alpha)$ is given by

$$J(\alpha) = \bar{r} + \sum_{j=1}^n \frac{(p_j+q_j)^2 \bar{\eta}_j{}^2}{2\left[(1-\alpha)(p_j+q_j)^2 + p_j(p_j+2q_j)\alpha\right]}$$
$$+ \sum_{j=1}^n \frac{(p_j+q_j) - q_j\alpha - (1-\alpha)^{1/2}\left[(1-\alpha)(p_j+q_j)^2 + p_j(p_j+2q_j)\alpha\right]^{1/2}}{2\alpha}.$$

Proof. We put $X(t) = X^{x,\hat{\pi}}(t)$ for $t \ge 0$. We claim

$$\tilde{J}(\alpha) = \lim_{T\to\infty} \frac{1}{\alpha T} \log E[X(T)^\alpha]. \tag{4.9}$$

Since $J(\alpha) \le \tilde{J}(\alpha)$ by definition, this implies $J(\alpha) = \tilde{J}(\alpha)$. From this and Proposition 4.5, the theorem follows.

We complete the proof by proving (4.9). From (2.12), we have

$$X^{\alpha}(t)=[xS_0(t)]^{\alpha}L(t)\exp\left[\int_0^t N(s)ds\right], \qquad t\geq 0,$$

where, for $t\geq 0$,

$$L(t)=\exp\left[\int_0^t \alpha\{\sigma^*(s)\hat{\pi}(s)\}^* dB_s-\frac{1}{2}\int_0^t \alpha^2\|\sigma^*(s)\hat{\pi}(s)\|^2 ds\right],$$
$$N(t)=\alpha\{\sigma^*(t)\hat{\pi}(t)\}^*\left[\theta(t)+\frac{1}{2}(\alpha-1)\sigma^*(t)\hat{\pi}(t)\right].$$

We see that

$$N(t)=-\frac{1}{2}\xi^*(t)Q\xi(t)-h^*(t)\xi(t)+\frac{1}{2}\sum_{j=1}^{n}u_j(t),$$

where $Q=\mathrm{diag}(Q_1,\dots,Q_n)$ and

$$u_j(t)=\alpha(\alpha-1)\left[p_j^2\bar{v}_j^2-(1-\beta)^2\eta_j^2(t)\right].$$

We write $\bar{E}$ for the expectation with respect to the probability measure $\bar{P}$ defined by

$$\left.\frac{d\bar{P}}{dP}\right|_{\mathcal{F}_T}=L(T), \quad T>0.$$

Then

$$E[X^{\alpha}(T)]=[xS_0(T)]^{\alpha}\exp\left[\frac{1}{2}\sum_{j=1}^{n}\int_0^T u_j(t)dt\right]\phi(T)$$

with

$$\phi(T)=\bar{E}\left[\exp\left\{-\int_0^T\left(\frac{1}{2}\xi^*(t)Q\xi(t)+h^*(t)\xi(t)\right)dt\right\}\right].$$

Let $(\bar{B}(t))_{t\geq 0}$ be the standard $\bar{P}$-Brownian motion defined by

$$\bar{B}(t)=B(t)-\int_0^t \alpha\sigma^*(s)\hat{\pi}(s)ds, \qquad t\geq 0.$$

Then, the process $(\xi(t))_{t\geq 0}$ evolves according to

$$d\xi(t)=[\gamma(t)+d(t)\xi(t)]dt-l(t)d\bar{B}_t, \qquad t\geq 0,$$

where $d(t)=\mathrm{diag}(d_1(t),\dots,d_n(t))$. We define $U(t)=U(t;T)$ by $U(t;T)=\mathrm{diag}(U_1(t;T),\dots,U_n(t;T))$. Then $U(t)$ satisfies the matrix Riccati equation

$$\dot{U}(t)-U(t)l^2(t)U(t)+d(t)U(t)+U(t)d(t)+Q=0, \quad 0\leq t\leq T, \quad U(T)=0.$$

Therefore, it follows from Theorem A.1 that

$$\exp\left[\frac{1}{2}\sum_{j=1}^{n}\int_0^T u_j(t)dt\right]\phi(T)=\exp\left[\frac{1}{2}\sum_{j=1}^{n}\int_0^T f_j(t;T)dt\right]$$

with

$$f_j(t;T) = l_j^2(t)m_j^2(t;T) + 2\gamma_j(t)m_j(t) - l_j^2(t)U_j(t;T) + u_j(t), \qquad (t,T) \in \Delta.$$

From Propositions 4.9 and 4.10, we obtain

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f_j(t;T)dt = \bar{f}_j,$$

where

$$\bar{f}_j = p_j^2 \bar{m}_j^2 + 2\bar{\gamma}_j \bar{m}_j - p_j^2 \bar{U}_j + \alpha(\alpha-1)\left[p_j^2\bar{v}_j^2 - (1-\beta)^2\bar{\eta}_j^2\right].$$

Thus

$$\begin{aligned}\frac{1}{\alpha T}\log E[X^\alpha(T)] &= \frac{\log x}{T} + \frac{1}{T}\int_0^T r(t)d + \frac{1}{2\alpha T}\sum_{j=1}^n \frac{1}{T}\int_0^T f_j(t;T)dt \\ &\to \bar{r} + \frac{1}{2\alpha}\sum_{j=1}^n \bar{f}_j, \qquad T\to\infty.\end{aligned}$$

We have

$$p_j^2\bar{U}_j = \bar{d}_j + K_j = \bar{b}_j - \alpha p_j^2 \bar{R}_j + K_j = p_j^2\bar{R}_j - \alpha p_j^2 \bar{R}_j = (1-\alpha)p_j^2\bar{R}_j,$$

whence $\bar{U}_j = (1-\alpha)\bar{R}_j$ (this also holds when $p_j = 0$). Moreover, we have

$$\begin{aligned}\bar{h}_j + \bar{U}_j\bar{\gamma}_j &= \alpha(\alpha-1)p_j^2\bar{R}_j\bar{v}_j - \beta\bar{\eta}_j + (1-\alpha)\bar{R}_j[-\beta p_j\bar{\eta}_j + \alpha p_j^2\bar{v}_j] \\ &= \bar{\eta}_j(-\beta + \alpha p_j\bar{R}_j) = -(1-\alpha)\beta\bar{\eta}_j[(1-\beta) + p_j\bar{R}_j],\end{aligned}$$

so that

$$\bar{m}_j = \frac{(1-\alpha)}{K_j}\beta\bar{\eta}_j[(1-\beta) + p_j\bar{R}_j] = (1-\alpha)\bar{v}_j.$$

Using these equalities, we find that

$$\bar{f}_j = (1-\alpha)\left[\bar{v}_j^2 p_j^2 + 2\bar{\rho}_j\bar{v}_j - p_j^2\bar{R}_j - \beta(1-\beta)\bar{\eta}_j^2\right] = (1-\alpha)\bar{g}_j.$$

Thus we obtain (4.9) from Proposition 4.4. □

5. Large deviation probability control

In this section, we study the large deviation probability control problem (2.16). Throughout this section, we assume (4.1) and (4.2). We also assume that

(5.1) $\qquad$ either $\bar{\eta} \neq (0,\ldots,0)^*$ or $(p_1,\ldots,p_n) \neq (0,\ldots,0)$ holds.

For $x \in (0,\infty)$ and $\pi \in \mathcal{A}$, let $L^{x,\pi}(T)$ be the growth rate defined by

$$L^{x,\pi}(T) = \frac{\log X^{x,\pi}(T)}{T}, \qquad T > 0.$$

Then we have

$$P\left[L^{x,\pi}(T) \geq c\right] = P\left[X^{x,\pi}(T) \geq e^{cT}\right].$$

Following Pham [16, 17], we consider the following optimal logarithmic moment generating function:

$$\Lambda(\alpha) = \sup_{\pi\in\mathcal{A}} \limsup_{T\to\infty} \frac{1}{T}\log E[\exp(\alpha T L^{x,\pi}(T))], \qquad 0 < \alpha < 1.$$

Since $\Lambda(\alpha) = \alpha J(\alpha)$ for $\alpha \in (0,1)$, it follows from Theorem 4.11 that

$$\Lambda(\alpha) = \bar{r}\alpha + \frac{1}{2}\sum_{j=1}^{n} F_j(\alpha) + \frac{1}{2}\sum_{j=1}^{n} G_j(\alpha), \qquad 0 < \alpha < 1,$$

where

$$F_j(\alpha) = \frac{(p_j+q_j)^2 \bar{\eta}_j{}^2 \alpha}{[(1-\alpha)(p_j+q_j)^2 + p_j(p_j+2q_j)\alpha]},$$
$$G_j(\alpha) = (p_j+q_j) - q_j\alpha - (1-\alpha)^{1/2}\left[(1-\alpha)(p_j+q_j)^2 + p_j(p_j+2q_j)\alpha\right]^{1/2}.$$

Proposition 5.1. *We have*

$$\frac{d\Lambda}{d\alpha}(0+) = \bar{r} + \frac{1}{4}\sum_{j=1}^{n}\frac{p_j^2}{p_j+q_j} + \frac{1}{2}\|\bar{\eta}\|^2, \qquad \lim_{\alpha\uparrow 1}\frac{d\Lambda}{d\alpha}(\alpha) = \infty.$$

Proof. Since $\dot{F}_j(\alpha)$ is equal to

$$\frac{(p_j+q_j)^2\bar{\eta}_j^2}{[(1-\alpha)(p_j+q_j)^2 + p_j(p_j+2q_j)\alpha]} + \frac{(p_j+q_j)^2\bar{\eta}_j^2 q_j^2\alpha}{[(1-\alpha)(p_j+q_j)^2 + p_j(p_j+2q_j)\alpha]^2},$$

we have $\dot{F}_j(0+) = \bar{\eta}_j^2$. If $p_j = 0$ and $\eta_j \neq 0$, then

$$\frac{dF_j}{d\alpha}(\alpha) \sim \bar{\eta}_j^2(1-\alpha)^{-2}, \qquad \alpha \uparrow 1.$$

On the other hand, since

$$\frac{dG_j}{d\alpha}(\alpha) = -q_j + \frac{(1-\alpha)^{-1/2}}{2}\left[(1-\alpha)(p_j+q_j)^2 + p_j(p_j+2q_j)\alpha\right]^{1/2} + \frac{q_j^2(1-\alpha)^{1/2}}{2\left[(1-\alpha)(p_j+q_j)^2 + p_j(p_j+2q_j)\alpha\right]^{1/2}},$$

we get $\dot{G}_j(0+) = p_j^2/[2(p_j+q_j)]$. If $p_j > 0$, then

$$\frac{dG_j}{d\alpha}(\alpha) \sim \frac{\sqrt{p_j(p_j+2q_j)}}{2}(1-\alpha)^{-1/2}, \qquad \alpha \uparrow 1.$$

Thus the proposition follows. □

Remark 5.2. From the proof of Proposition 5.1, we see that

$$\frac{d\Lambda}{d\alpha}(\alpha) \sim \frac{(1-\alpha)^{-1/2}}{4}\sum\nolimits_{j=1}^{n}\sqrt{p_j(p_j+2q_j)}, \qquad \alpha \uparrow 1$$

when $(p_1,\ldots,p_n) \neq (0,\ldots,0)$, i.e., the market has memory. Compare the following result: when $p_1 = \cdots = p_n = 0$, i.e., the market has no memory, we have

$$\frac{d\Lambda}{d\alpha}(\alpha) \sim \frac{1}{2}\|\bar{\eta}\|^2(1-\alpha)^{-2}, \qquad \alpha \uparrow 1$$

We put $\bar{c} = \dot{\Lambda}(0+)$, that is,

$$\bar{c} = \bar{r} + \frac{1}{4}\sum_{j=1}^{n}\frac{p_j^2}{p_j+q_j} + \frac{1}{2}\|\bar{\eta}\|^2.$$

For $\alpha \in (0,1)$, we write $\hat{\pi}(t;\alpha)$ for the optimal strategy $\hat{\pi}(t)$ in (4.8). Recall $I(c)$ from (2.16).

From Theorem 4.11, Proposition 5.1, and [16, Theorem 3.1], we immediately obtain the following solution to the problem (2.16).

Theorem 5.3. *We have*

$$I(c) = -\sup_{\alpha \in (0,1)} \left[\alpha c - \Lambda(\alpha) \right], \qquad c \in \mathbf{R}.$$

Moreover, for $c \geq \bar{c}$, the sequence of strategies

$$\hat{\pi}^n(t) = \hat{\pi}\left(t; \alpha\left(c + \tfrac{1}{n}\right)\right)$$

with $\alpha(d) \in (0,1)$ such that $\dot{\Lambda}(\alpha(d)) = d \in (\bar{c}, \infty)$, is nearly optimal in the sense that

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \log P\left[X^{x,\hat{\pi}^n}(T) \geq e^{cT}\right] = I(c), \qquad c \geq \bar{c}.$$

Remark 5.4. Though Theorem 3.1 in Pham [16] is formally stated for a model different from that in the present paper, the arguments in his proof are general enough to prove Theorem 5.3 in the same way.

In the rest of this section, we derive an optimal strategy for the problem (2.16), rather than a nearly optimal sequence, when $c < \bar{c}$. Recall $\theta(t)$ from (2.11). We define a strategy $\hat{\pi}_0 \in \mathcal{A}$ by

$$\hat{\pi}_0(t) = (\sigma^*)^{-1}(t)\theta(t), \quad t \geq 0.$$

From (2.12), we obtain

$$L^{x,\hat{\pi}_0}(T) = \frac{\log x}{T} + \frac{1}{T}\int_0^T r(t)dt + \frac{1}{2T}\int_0^T \|\theta(t)\|^2 dt + \frac{1}{T}\int_0^T \theta^*(t)dB(t).$$

Proposition 5.5. *The growth rate $L^{x,\hat{\pi}_0}(T)$ converges to $\bar{c}$, as $T \to \infty$, in prob.*

Proof. In this proof, we denote by C positive constants, which may not be necessarily equal.

For $j = 1, \ldots, n$, we consider the decomposition

$$\theta_j(t) = [\eta_j(t) - \bar{\eta}_j] + [\bar{\eta}_j - p_j K(t)] + N(t),$$

where

$$K(t) = \int_0^t e^{-(p_j+q_j)(t-s)} dB_j(s), \qquad N(t) = \int_0^t e^{-(p_j+q_j)(t-s)} f(s) dB_j(s)$$

with

$$f(s) = \frac{2p_j^2 q_j}{(2q_j + p_j)^2 e^{2q_j s} - p_j^2}.$$

The dynamics of the process $K(t)$ is given by

$$dK(t) = -(p_j + q_j)K(t)dt + dB_j(t),$$

so that $K(t)$ is a positively recurrent one-dimensional diffusion with the speed measure $m(dx) = 2e^{-(p_j+q_j)x^2}dx$. By the ergodic theorem (cf. Rogers and Williams [18, v.53]), we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T [\bar{\eta}_j - p_j K(t)]^2 dt = \int_{-\infty}^{\infty} (\bar{\eta}_j - p_j y)^2 \nu(dy) = \bar{\eta}_j^2 + \frac{p_j^2}{2(p_j + q_j)} \quad \text{a.s.,}$$

where $\nu(dy)$ is the Gaussian measure with mean 0 and variance $1/[2(p_j+q_j)]$.

We have $E[K^2(t)] \leq C$ for $t \geq 0$. Since $0 \leq f(s) \leq Ce^{-2q_j s}$, it follows that

$$E\left[N^2(t)\right] \leq C\int_0^t e^{-2q_j(t+s)}ds \leq Ce^{-2q_j t}, \qquad t \geq 0.$$

Therefore, for example,

$$E\left[\frac{1}{T}\int_0^T |(\bar{\eta}_j - p_j K(t))N(t)|dt\right] \leq \frac{C}{T}\int_0^T E[N^2(t)]^{1/2}dt \to 0, \qquad T \to \infty.$$

Similarly,

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\int_0^T [\eta_j(t) - \bar{\eta}_j]^2 dt &= \lim_{T\to\infty} E\left[\frac{1}{T}\int_0^T (\eta_j(t) - \bar{\eta}_j)(\bar{\eta}_j - p_j K(t))dt\right] \\ &= \lim_{T\to\infty} E\left[\frac{1}{T}\int_0^T N^2(t)dt\right] = \lim_{T\to\infty} E\left[\frac{1}{T}\int_0^T (\eta_j(t) - \bar{\eta}_j)N(t)dt\right] = 0.\end{aligned}$$

Combining,

$$\frac{1}{T}\int_0^T \theta_j^2(t)dt \to \bar{\eta}_j^2 + \frac{p_j^2}{2(p_j+q_j)}, \quad T \to \infty, \quad \text{in prob.}$$

For $j = 1, \ldots, n$, we have

$$E\left[\theta_j^2(t)\right] \leq 2\eta_j^2(t) + 2E\left[\xi_j^2(t)\right] \leq C\left[1 + \int_0^t l_j^2(t,s)ds\right] \leq C, \quad t \geq 0,$$

so that $(1/T)\int_0^T \theta^*(t)dB(t)$ converges to zero, as $T \to \infty$, in the $L^2(\Omega)$-norm, whence in prob. Thus the proposition follows. □

Theorem 5.6. *For $c < \bar{c}$, $\hat{\pi}_0$ is an optimal strategy for (2.16) with a limit, i.e.,*

$$\lim_{T\to\infty}\frac{1}{T}\log P\left[X^{x,\hat{\pi}_0}(T) \geq e^{cT}\right] = I(c), \qquad c < \bar{c}.$$

Proof. Proposition 5.5 implies, for $c < \bar{c}$, $\lim_{T\to\infty} P\left[L^{x,\hat{\pi}_0}(T) \geq c\right] = 1$, so that

$$\lim_{T\to\infty}\frac{1}{T}\log P\left[L^{x,\hat{\pi}_0}(T) \geq c\right] = 0 \geq \sup_{\pi\in\mathcal{A}}\lim_{T\to\infty}\frac{1}{T}\log P\left[L^{x,\pi}(T) \geq c\right], \qquad c < \bar{c}.$$

Thus $\hat{\pi}_0$ is optimal if $c < \bar{c}$. □

Remark 5.7. From Theorem 10.1 in Karatzas and Shreve [11, Chapter 3], $\hat{\pi}_0$ is the log-optimal (or growth optimal) strategy in the sense that

$$\sup_{\pi\in\mathcal{A}}\limsup_{T\to\infty}\frac{1}{T}\log X^{x,\pi}(T) = \limsup_{T\to\infty}\frac{1}{T}\log X^{x,\hat{\pi}_0}(T) \quad \text{a.s.}$$

We also see that $\lim_{\alpha\downarrow 0}\hat{\pi}(t;\alpha) = \hat{\pi}_0(t)$ a.s. for $t \geq 0$.

Appendix A. A Cameron–Martin type formula

In this appendix, we derive a generalization of the Cameron–Martin formula that we need in the proofs of Proposition 3.2 and Theorem 4.11. We refer to Myers [14] for earlier work.

Let $T \in (0, \infty)$ and let $(\Omega, \mathcal{F}, P)$ be the underlying probability space with filtration $(\mathcal{F}_t)_{0 \le t \le T}$ satisfying the usual conditions. We assume that the dynamics of the $\mathbf{R}^n$-valued process $\xi(t)$ is given by the stochastic differential equation

$$d\xi(t) = [a(t) + b(t)\xi(t)]dt + c(t)dB(t), \qquad 0 \le t \le T,$$

where $B(t)$ is an $\mathbf{R}^n$-valued standard Brownian motion and $a : [0, T] \to \mathbf{R}^n$ and $b, c : [0, T] \to \mathbf{R}^{n\times n}$ are deterministic, bounded measurable functions.

Let $Q : [0, T] \to \mathbf{R}^{n\times n}$ be a deterministic, symmetric bounded measurable function, where we say that $A : [0, T] \to \mathbf{R}^{n\times n}$ is symmetric if $A(t)$ is a symmetric matrix for all $t \in [0, T]$. We consider the following backward Riccati equation for a symmetric function $R : [0, T] \to \mathbf{R}^{n\times n}$:

$$\begin{aligned} &\dot{R}(t) - R(t)c(t)c^*(t)R(t) + b(t)R(t) + R(t)b^*(t) + Q(t) = 0, \quad 0 \le t \le T, \\ &R(T) = 0. \end{aligned} \tag{A.1}$$

We also consider the following linear equation for $v : [0, T] \to \mathbf{R}^n$:

$$\begin{aligned} &\dot{v}(t) + [b(t) - c(t)c^*(t)R(t)]^* v(t) - h(t) - R(t)a(t) = 0, \quad 0 \le t \le T, \\ &v(T) = 0, \end{aligned} \tag{A.2}$$

where $h : [0, T] \to \mathbf{R}^n$ is a given deterministic, bounded measurable function.

Theorem A.1. *We assume that there exists a bounded symmetric function $R : [0, T] \to \mathbf{R}^{n\times n}$ that satisfies the Riccati equation* (A.1)*. Let $v : [0, T] \to \mathbf{R}$ be the solution to the equation* (A.2)*. Then we have, for $t \in [0, T]$,*

$$\begin{aligned} &E\left[\exp\left\{-\int_t^T \left(\frac{1}{2}\xi^*(s)Q(s)\xi(s) + h^*(s)\xi(s)\right) ds\right\}\middle| \mathcal{F}_t\right] \\ &= \exp\left[v^*(t)\xi(t) - \frac{1}{2}\xi^*(t)R(t)\xi(t)\right. \\ &\qquad\qquad \left. + \frac{1}{2}\int_t^T \{v^*(s)c(s)c^*(s)v(s) + 2a^*(s)v(s) - \mathrm{tr}(c(s)c^*(s)R(s))\} ds\right]. \end{aligned}$$

Proof. We put $K(t) = c^*(t)[v(t) - R(t)\xi(t)]$ for $t \in [0,T]$. Then, by the Itô formula,

$$d\left[v^*(t)\xi(t) - \frac{1}{2}\xi^*(t)R(t)\xi(t)\right] - \left[K^*(t)dB(t) - \frac{1}{2}\|K(t)\|^2 dt\right]$$

$$= \Bigg[-\frac{1}{2}\xi^*(t)\{\dot{R}(t) + R(t)b(t) + b^*(t)R(t) - R(t)c(t)c^*(t)R(t)\}\xi(t)$$

$$+ \{\dot{v}(t) + (b(t) - c(t)c^*(t)R(t))^* v(t) - R(t)a(t)\}\xi(t)$$

$$+ \frac{1}{2}\{v^*(t)c(t)c^*(t)v(t) + 2a^*(t)v(t) - \operatorname{tr}(c(t)c^*(t)R(t))\}\Bigg] dt$$

$$= \Bigg[\frac{1}{2}\xi^*(t)Q(t)\xi(t) + h^*(t)\xi(t)$$

$$+ \frac{1}{2}\{v^*(t)c(t)c^*(t)v(t) + 2a^*(t)v(t) - \operatorname{tr}(c(t)c^*(t)R(t))\}\Bigg] dt.$$

Therefore, $\int_t^T K^*(s)dB_s - \frac{1}{2}\int_t^T \|K(s)\|^2 ds$ is equal to

$$\frac{1}{2}\xi^*(t)R(t)\xi(t) - v^*(t)\xi(t) - \int_t^T \left(\frac{1}{2}\xi^*(s)Q(s)\xi(s) + h^*(s)\xi(s)\right) ds$$

$$- \frac{1}{2}\int_t^T \{v^*(s)c(s)c^*(s)v(s) + 2a^*(s)v(s) - \operatorname{tr}(c(s)c^*(s)R(s))\}\, ds.$$

Since $K(t)$ is a continuous Gaussian process, the process $M(t)$ defined by $M(t) = \exp\{\int_0^t K^*(s)dB(s) - \frac{1}{2}\int_0^t \|K(s)\|^2 ds\}$ is a martingale (cf. Example 5 in Liptser and Shiryayev [13, Section 6.2]). Hence

$$E\left[\exp\left\{\int_t^T K^*(s)dB(s) - \frac{1}{2}\int_t^T \|K(s)\|^2 ds\right\}\middle| \mathcal{F}_t\right] = 1.$$

Combining, we obtain the theorem. □

Appendix B. Asymptotics for solutions to Riccati equations

In this appendix, we summarize the results on the asymptotics for solutions to Riccati or linear equations in the form that we need in this paper. Let the domain Δ be as in (3.8).

For $T \in (0,\infty)$ and constants a_1, a_2 and a_3, we consider the following conditions:

(B.1) $a_i(\cdot) \in C([0,\infty) \to \mathbf{R})$ for $i = 1,2,3$;

(B.2) $a_1(t) \geq 0$ for $t \geq 0$;

(B.3) for $i = 1,2,3$, $a_i(t)$ converges to $\bar{a}_i$ exponentially fast as $t \to \infty$;

(B.4) $\bar{a}_1 > 0$ and $\bar{a_2}^2 + \bar{a}_1\bar{a}_3 > 0$;

(B.5) $a_3(t) \geq 0$ for $t \geq 0$.

The conditions (B.5) and (B.3) imply $\bar{a}_3 \geq 0$.

We are concerned with the following one-dimensional backward Riccati equation:

(B.6) $\dot{R}(t) - a_1(t)R^2(t) + 2a_2(t)R(t) + a_3(t) = 0, \quad 0 \leq t \leq T, \quad R(T) = 0.$

Under (B.4), let $\bar{R}$ be the larger solution of the quadratic equation

$$\bar{a}_1\bar{R}^2 - 2\bar{a}_2\bar{R} - \bar{a}_3 = 0.$$

Theorem B.1 (Nagai and Peng [15], Section 5)**.** *We assume* (B.1)–(B.5)*. Then, for* $T \in (0, \infty)$*, the equation* (B.6) *has a unique nonnegative solution* $R(t) = R(t;T)$*, and it satisfies the following:*

$$R(t;T) \text{ is bounded in } \Delta, \tag{B.7}$$

$$\lim_{T-t\to\infty,\ t\to\infty} R(t;T) = \bar{R}, \tag{B.8}$$

$$\lim_{T\to\infty} \sup_{\delta T \le t \le (1-\epsilon)T} |R(t;T) - \bar{R}| = 0 \quad \text{for } \delta, \epsilon \in (0,\infty) \text{ such that } \delta + \epsilon < 1. \tag{B.9}$$

When we do not have the condition (B.5), we have the following result.

Theorem B.2. *We assume* (B.1)–(B.4)*. Suppose that, for every* $T \in (0, \infty)$*, the equation* (B.6) *has a solution* $R(t) = R(t;T)$ *and that* $R(t;T)$ *is bounded in* Δ*. Then it satisfies* (B.8) and (B.9)*.*

The proof of Theorem B.2 is almost the same as that of Theorem B.1 in [15]; so that we omit it.

We turn to the following one-dimensional backward linear differential equation:

$$\dot{v}(t) - b_1(t;T)v(t) + b_2(t;T) = 0, \quad 0 \le t \le T, \quad v(T) = 0, \tag{B.10}$$

where we assume that, for some constants $\bar{b}_1$ and $\bar{b}_2$, the following hold:

$$b_i(\cdot;T) \in C([0,T] \to \mathbf{R}) \text{ for } T \in (0,\infty) \text{ and } i = 1,2; \tag{B.11}$$

$$\lim_{T-t\to\infty,\ t\to\infty} b_i(t;T) = \bar{b}_i, \quad i = 1,2; \tag{B.12}$$

$$\bar{b}_1 > 0. \tag{B.13}$$

Theorem B.3 ([15], Section 5)**.** *For* $T \in (0,\infty)$*, the equation* (B.5) *has a unique solution* $v(t) = v(t;T)$*, and it satisfies the following:*

$$v(t;T) \text{ is bounded in } \Delta,$$

$$\lim_{T-t\to\infty,\ t\to\infty} v(t;T) = \bar{v},$$

$$\lim_{T\to\infty} \sup_{\delta T \le t \le (1-\epsilon)T} |v(t;T) - \bar{v}| = 0 \quad \text{for } \delta, \epsilon \in (0,\infty) \text{ such that } \delta + \epsilon < 1,$$

where $\bar{v}$ *is the solution of the linear equation* $\bar{b}_1\bar{v} - \bar{b}_2 = 0$*.*

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