A generalization of a curvature flow of graphs on $\mathbb{R}$

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1. Introduction. Gauss curvature flow is known as a mathematical model of the wearing process of a convex stone rolling on a beach and has been studied by many authors (see [3, 5, 6] and the reference therein). In [5] we proposed and studied the discrete stochastic approximations of nonconvex functions which evolve by a convexified Gauss curvature and the PDE which appears as the continuum limit of discrete stochastic processes (see [6] for the similar results on the convexified Gauss curvature flow of closed hypersurfaces). In this paper we study a class of PDE which gives a generalization of a curvature flow of graphs on $\mathbb{R}$.

We briefly describe [5] to discuss the results in this paper more precisely. Alexandrov-Bakelman’s generalized curvature played a crucial role in [5].

Definition 1 (see e.g. [1, section 9.6]). Let $R \in L^1(\mathbb{R}^n : [0, \infty), dx)$ and $u \in C(\mathbb{R}^n)$. For $A \in B(\mathbb{R}^n)(:=\text{Borel } \sigma\text{-field of } \mathbb{R}^n)$, put

$$w(R, u, A) := \int_{\cup_{x \in A} \partial u(x)} R(y) dy,$$

(1)

where $\partial u(x) := \{ p \in \mathbb{R}^n | u(y) - u(x) \geq < p, y - x > \text{ for all } y \in \mathbb{R}^n \}$ and $< ., . >$ denotes the inner product in $\mathbb{R}^n$.

(It is known that $w(R, u, \cdot) : B(\mathbb{R}^n) \mapsto [0, \infty)$ is completely additive.)

For $R \in L^1(\mathbb{R}^n : [0, \infty), dx)$, we showed the existence and the uniqueness of a solution $u \in C([0, \infty) \times \mathbb{R}^n)$ to the following equation (see [5, Theorem 1]): for any $\varphi \in C_0^0(\mathbb{R}^n)$ and any $t \geq 0$, .
\[
\int_{\mathbb{R}^n} \varphi(x)(u(t, x) - u(0, x))dx = \int_0^t ds \int_{\mathbb{R}^n} \varphi(x)w(R, u(s, \cdot), dx). \tag{2}
\]

In [5, Theorem 2], we proved that a continuous solution \( u \) to (2) sweeps in time \( t > 0 \) a region with volume given by \( t \cdot w(R, u(0, \cdot), \mathbb{R}^n) \), and that, for a continuous solution \( u \) to (2) with a convex \( u(0, \cdot) \), \( x \mapsto u(t, \cdot) \) is convex for all \( t > 0 \).

We also showed that a continuous solution \( u \) to (2) is a viscosity solution of the following PDE (see [5, Theorem 3]):

\[
\partial_t u(t, x) = \chi(u, Du(t, x), t, x) R(Du(t, x))
\times \max(\text{Det}(D^2 u(t, x)), 0) \quad ((0, \infty) \times \mathbb{R}^n),
\]

where \( Du(t, x) := (\partial u(t, x)/\partial x_i)_{i=1}^n \), \( D^2 u(t, x) := (\partial^2 u(t, x)/\partial x_i \partial x_j)_{i,j=1}^n \) and

\[
\chi(u, p, t, x) := \begin{cases} 
1 & \text{if } p \in \partial u(t, x), \\
0 & \text{otherwise}.
\end{cases}
\]

Here \( \partial u(t, x) \) denotes the subdifferential of the function \( x \mapsto u(t, x) \). Conversely, we discussed under what conditions a viscosity solution to (3) is a solution to (2).

We briefly discuss what we study in this paper. In (1) we only considered the measure \( R(y)dy \) which is absolutely continuous with respect to the Lebesgue measure \( dy \). Otherwise \( \omega(R, u, dx) \) is not generally completely additive.

Suppose that \( n = 1 \). In (1), replace \( R(y)dy \) by a continuous Borel probability measure \( P(dy) \). Then, using a similar notation, \( \omega(P, u, dx) \) turn out to be a measure and (2) has a unique continuous solution \( u \) (see Theorem 1 in section 2).

Since \( P(dy) \) is not generally absolutely continuous with respect to \( dy \), we can not consider the PDE for \( u \). For \( (t, x) \in [0, \infty) \times \mathbb{R} \), put

\[
U(t, x) := \int_{-\infty}^x (u(t, y) - u(0, y))dy + \int_0^x u(0, y)dy + tF(a), \tag{4}
\]

where
\[ F(x) := P((-\infty, x]), \quad a := \inf \{ \cup_{x \in \mathbb{R}} \partial u(0, x) \}. \]

Then from (2),
\[ U(t, x) - U(0, x) = \int_0^t F(D_+(\hat{D}U(s, x))) ds. \]

(5)

Here \( \hat{u} \) denotes a convex envelope of \( u \) and for \( \varphi \), \( D_+\varphi(t, x) \) denotes the right derivative of \( x \mapsto \varphi(t, x) \).

When \( DU(0, x) \) is convex, we show that \( U(t, x) \) is a unique continuous viscosity solution in \((0, \infty) \times \mathbb{R}\) of the following (see Theorems 2 and 3 in section 2):
\[ \partial_t U(t, x) = F(D^2U(t, x)). \]

(6)

**Definition 2 (Viscosity solution)**

(i) A function \( U \in USC(\Omega) \) is called a viscosity subsolution of (6) in \( \Omega \) if whenever \( \varphi \in C^{1,2}(\Omega) \), \((s, y) \in \Omega\), and \( U - \varphi \) attains a local maximum at \((s, y)\), then \( \partial_t \varphi(s, y) \leq F(D^2\varphi(s, y)) \).

(ii) A function \( U \in LSC(\Omega) \) is called a viscosity supersolution of (6) in \( \Omega \) if whenever \( \varphi \in C^{1,2}(\Omega) \), \((s, y) \in \Omega\), and \( U - \varphi \) attains a local minimum at \((s, y)\), then \( \partial_t \varphi(s, y) \geq F(D^2\varphi(s, y)) \).

(iii) A function \( U \in C(\Omega) \) is called a viscosity solution of (6) in \( \Omega \) if it is both a viscosity subsolution and a viscosity supersolution of (6) in \( \Omega \).

(2) Let \( t_2 > t_1 > 0 \), \( O \) be an open subset of \( \mathbb{R} \) and \( Q := (t_1, t_2] \times O \). A function \( U \in C(Q) \) is called a viscosity solution of (6) in \( Q \) if (1,i)- (1,ii) with \( \Omega \) replaced by \( Q \) hold (see [4, p. 66]). Here \( \overline{Q} \) denotes the closure of \( Q \).

**2. Main results.** We state assumptions under which we can generalize [5] when \( n = 1 \).

(A.1) \( P \) is a continuous Borel probability measure on \( \mathbb{R} \).

(A.2) \( h \in C(\mathbb{R}) \) and the set \( \partial h(\mathbb{R}) \) has a positive Lesbegue measure.

(A.3) For any \( p \notin \partial h(\mathbb{R}) \) and \( C \in \mathbb{R} \),
\[ \int_{\mathbb{R}} \max(px + C - h(x), 0) dx = \infty. \]

**Theorem 1** Suppose that (A.1)-(A.3) hold. Then there exists a unique continuous solution \( u \) to (2) with \( u(0, \cdot) = h \).
The following assumption implies (A.2)-(A.3).

(A.2)’  h is convex and is not a constant.

**Theorem 2** Suppose that (A.1) and (A.2)’ hold. Then the function \( U \) defined by (4), from \( u \) in Theorem 1 is a continuous viscosity solution of (6) in \( (0, \infty) \times \mathbb{R} \).

As a regularity result, we have

**Proposition 1** Suppose that (A.1) holds. Then for a continuous viscosity solution \( v \) of (6) in \( (0, \infty) \times \mathbb{R} \),

\[
0 \leq v(t, x) - v(s, x) \leq t - s \quad (0 \leq s < t, x \in \mathbb{R}).
\]  

(7)

In particular, \( t \mapsto v(t, x) \) is absolutely continuous for all \( x \in \mathbb{R} \).

We state an additional assumption and an asymptotic behavior of a viscosity solution of (6).

(A.4). (i) \( v_0 : \mathbb{R} \mapsto \mathbb{R} \) is twice continuously differentiable.

(ii) \( \lim_{x \to -\infty} D^2 v_0(x) \) and \( \lim_{x \to \infty} D^2 v_0(x) \) exist and

\[
a := \inf_{x \in \mathbb{R}} D^2 v_0(x) = \lim_{x \to -\infty} D^2 v_0(x),
\]

\[
\quad \quad b := \sup_{x \in \mathbb{R}} D^2 v_0(x) = \lim_{x \to \infty} D^2 v_0(x).
\]

**Proposition 2** Suppose that (A.1) and (A.4,i) hold. Then for a continuous viscosity solution \( v \) of (6) with \( v(0, \cdot) = v_0(\cdot) \) in \( (0, \infty) \times \mathbb{R} \), the following holds: for any \( t \geq 0 \) and \( x \in \mathbb{R} \),

\[
F(a)t \leq v(t, x) - v(0, x) \leq F(b)t. \tag{8}
\]

Suppose in addition that (A.4,ii) holds. Then for any \( T \geq 0 \),

\[
\lim_{x \to -\infty} \left( \sup_{0 \leq t \leq T} |v(t, x) - v(0, x) - F(a)t| \right) = 0, \tag{9}
\]

\[
\lim_{x \to \infty} \left( \sup_{0 \leq t \leq T} |v(t, x) - v(0, x) - F(b)t| \right) = 0. \tag{10}
\]
Since $F$ is nondecreasing, (6) is a degenerate elliptic PDE and we can use the maximum principle for this equation in a bounded domain (see [4, p. 244, Theorem 8.1] and also [2]). From Prop. 2, we immediately obtain

**Theorem 3** Suppose that (A.1) and (A.4) hold. Then the viscosity solution $v$ of (6) with $v(0, \cdot) = v_0(\cdot)$ is unique in $C([0, \infty) \times \mathbb{R})$.

From Theorems 2 and 3, we also obtain

**Corollary 1** Suppose that (A.1) and (A.2), hold and that $h$ is continuously differentiable. Then $U$ in Theorem 2 is the unique continuous viscosity solution of (6) with $U(0, x) = \int_0^x h(y)dy$ in $(0, \infty) \times \mathbb{R}$.

In particular, we have

**Corollary 2** Suppose that (A.1) and (A.4, i) hold and that $Dv_0$ is convex. Then (6) with $v(0, \cdot) = v_0(\cdot)$ has the unique viscosity solution in $C([0, \infty) \times \mathbb{R})$.

(Proof of Theorem 1). The proof can be done almost in the same way as in [5, Theorem 1] (In [5, (A.3)], “$x \in \mathbb{R}^{d+}$” should be “$(x, \hat{h}(x)) \in \mathbb{R}^{d+1}$”). The only thing we have to prove is the following (i)-(ii):

(i) For a convex $u \in C(\mathbb{R})$, $w(P, u, dx)$ is completely additive,

(ii) For $u \in C(\mathbb{R})$ for which $\partial u(\mathbb{R}) \neq \emptyset$, $w(P, u, dx) = w(P, \hat{u}, dx)$.

We first prove (i). The set

$$S(u) := \{p \in \mathbb{R} | \{x \in \mathbb{R} | p \in \partial u(x)\} \text{ contains more than one point}\}$$

contains at most countably many points. Indeed, if $p \in S(u)$, then for $x_p$ for which $p \in \partial u(x_p)$, the graph of $y = u(x)$ and the straight line $y = p(x - x_p) + u(x_p)$ contains a line segment with a positive length and the interiors of such line segments are disjoint. Hence for each $n$, $m \geq 1$, on the set $S(u) \cap [D_- u(-n), D_+ u(n)]$, such line segments with the length $\geq 1/m$ are finitely many. Here $D_- u(x)$ denotes the left derivative of $x \mapsto u(x)$. Since a continuous measure does not have a point mass, we obtain (i) (see [1, p. 118]).
Next we prove (ii). If \( u(x) = \hat{u}(x) \), then \( \partial u(x) = \partial \hat{u}(x) \). If \( u(x) \neq \hat{u}(x) \), then \( \partial u(x) = \emptyset \) and \( \partial \hat{u}(x) = D\hat{u}(x) \in S(\hat{u}) \). Since \( P(S(\hat{u})) = 0 \) as we explained above, we obtain (ii). □

(Proof of Theorem 2). (A.2)’ implies that \( x \mapsto DU(t, x) \) is convex for all \( t \geq 0 \) (see [5, Theorem 2]). We first prove that \( U \) is a viscosity subsolution to (6) in \( (0, \infty) \times \mathbb{R} \). Suppose that \( \varphi \in C^{1,2}((0, \infty) \times \mathbb{R}), (s, y) \in (0, \infty) \times \mathbb{R}, \) and \( U - \varphi \) attains a local maximum at \( (s, y) \). Then for \( x \) and \( y \in \mathbb{R} \) for which \( x - y \) is positive,

\[
\partial_t \varphi(s, y) \leq F\left( \frac{U(s, x) - U(s, y) - DU(s, y)(x - y)}{(x - y)^2/2} \right) \tag{11}
\leq F\left( \varphi(s, x) - \varphi(s, y) - D\varphi(s, y)(x - y) \right) \frac{(x - y)^2/2}{(x - y)^2/2}
\rightarrow F(D^2\varphi(s, y)) \left( \frac{x - y}{x + y} \right).
\]

Indeed, from (5), for \( t \) and \( s \geq 0 \) for which \( s - t \) is positive and is sufficiently small,

\[
\varphi(s, y) - \varphi(t, y) \leq U(s, y) - U(t, y) = \int_t^s F(D_+(DU(\alpha, y)))d\alpha,
\]

\[
U(\alpha, x) - U(\alpha, y) - DU(\alpha, y)(x - y)
\]

\[
= \int_y^\alpha (DU(\alpha, z) - DU(\alpha, y))dz
\]

\[
\geq \int_y^\alpha D_+(DU(\alpha, y))(z - y)dz = D_+(DU(\alpha, y))(x - y)^2/2.
\]

Since \( U \) and \( DU \in C([0, \infty) \times \mathbb{R}) \) from Theorem 1, we obtain the first inequality in (11). Since \( DU(s, y) = D\varphi(s, y) \) and \( F \) is nondecreasing, the second inequality of (11) holds.

Next we prove that \( U \) is a viscosity supersolution to (6) in \( (0, \infty) \times \mathbb{R} \). Suppose that \( \varphi \in C^{1,2}((0, \infty) \times \mathbb{R}), (s, y) \in (0, \infty) \times \mathbb{R}, \) and \( U - \varphi \) attains a local minimum at \( (s, y) \). Then in the same way as in (11), for \( x \) and \( y \in \mathbb{R} \) for which \( y - x \) is positive,
\[ \partial_t \varphi(s, y) \geq F \left( \frac{U(s, x) - U(s, y) - DU(s, y)(x - y)}{(x - y)^2/2} \right) \]
\[ \geq F \left( \frac{\varphi(s, x) - \varphi(s, y) - D\varphi(s, y)(x - y)}{(x - y)^2/2} \right) \]
\[ \rightarrow F(D^2\varphi(s, y)) \quad (x \uparrow y). \] 

(Proof of Proposition 1) Without loss of generality, we can put \( s = 0 \). We first prove the first inequality of (7). Suppose that there exists \((t_0, x_0) \in (0, \infty) \times \mathbb{R}\) such that

\[ v(t_0, x_0) - v(0, x_0) < 0. \] 

Put

\[ C_0 := \min\{v(t, x) - v(0, x)|0 \leq t \leq t_0, |x - x_0| \leq 1\} \] (13)
\[ \underline{\varphi}(t, x) := v(0, x) + C_0(x - x_0)^2. \]

Then it is easy to see that \( \varphi \) is a viscosity subsolution of (6) in \((0, t_0] \times (x_0 - 1, x_0 + 1)\) since \( F \geq 0 \). By the maximum principle (see [4, p.244, Th. 8.1]),

\[ \min\{v(t, x) - \underline{\varphi}(t, x)|0 \leq t \leq t_0, |x - x_0| \leq 1\} \]
\[ = \min\{v(t, x) - \underline{\varphi}(t, x)|0 \leq t \leq t_0, |x - x_0| = 1 \text{ or } t = 0, |x - x_0| \leq 1\} \]
\[ \geq 0, \] 

from (13). This contradicts (12).

Next we prove the second inequality of (7). Suppose that there exists \((\bar{t}_0, \bar{x}_0) \in (0, \infty) \times \mathbb{R}\) such that

\[ v(\bar{t}_0, \bar{x}_0) - v(0, \bar{x}_0) - \bar{t}_0 > 0. \] 

Put

\[ \overline{C}_0 := \max\{v(t, x) - v(0, x) - t|0 \leq t \leq \bar{t}_0, |x - \bar{x}_0| \leq 1\} \] (16)
\[ \overline{\varphi}(t, x) := v(0, x) + t + \overline{C}_0(x - \bar{x}_0)^2. \]

7
Then it is easy to see that $\overline{v}$ is a viscosity supersolution of (6) in $(0, \tilde{t}_0] \times (\overline{x}_0 - 1, \overline{x}_0 + 1)$ since $F \leq 1$. By the maximum principle,

$$\max\{v(t, x) - \overline{v}(t, x)|0 \leq t \leq \tilde{t}_0, |x - \overline{x}_0| \leq 1\}$$

$\leq 0,$

from (16). This contradicts (15). $\square$

(Proof of Proposition 2) First of all, we prove the first inequality in (8). Suppose that there exists $(t_0, x_0) \in (0, \infty) \times \mathbb{R}$ for which $F(a)t_0 > v(t_0, x_0) - v(0, x_0)$. Take $\varepsilon_0 > 0$ so that

$$v(t_0, x_0) - (v(0, x_0) + F(a)t_0 - \varepsilon_0 t_0) < 0. \tag{18}$$

For $n \geq 1$, put

$$C_n := \min\{v(t, x) - (v(0, x) + F(a)t)|0 \leq t \leq t_0, |x - x_0| \leq n\}, \tag{19}$$

$$\psi_n(t, x) := v(t, x) - \left(v(0, x) + F(a)t - \varepsilon_0 t + \frac{C_n(x - x_0)^2}{n^2}\right).$$

Then from (18)-(19),

$$\min\{\psi_n(t, x)|0 \leq t \leq t_0, |x - x_0| \leq n\}$$

$$= \min\{\psi_n(t, x)|0 < t \leq t_0, |x - x_0| < n\}. \tag{20}$$

Take $(t_n, x_n) \in (0, t_0] \times (x_0 - n, x_0 + n)$ which attains the minimum in (20). Since $v$ is a viscosity supersolution of (6) and since $|C_n| \leq t_0$ from Prop.1,

$$F(a) - \varepsilon_0 \geq F\left(D^2v(0, x_n) + \frac{2C_n}{n^2}\right) \geq F\left(a + \frac{2C_n}{n^2}\right) \rightarrow F(a), \tag{21}$$

as $n \rightarrow \infty$, which is a contradiction.

Next we prove the second inequality in (8). Suppose that there exists $(\tilde{t}_0, \overline{x}_0) \in (0, \infty) \times \mathbb{R}$ for which $F(b)\tilde{t}_0 < v(\tilde{t}_0, \overline{x}_0) - v(0, \overline{x}_0)$. Take $\varepsilon_0 > 0$ so that
\[ v(\bar{t}_0, \bar{x}_0) - (v(0, \bar{x}_0) + F(b)\bar{t}_0 + \bar{x}_0) > 0. \] 

(22)

For \( n \geq 1 \), put

\[
C_n := \max \{ v(t, x) - (v(0, x) + F(b)t) | 0 \leq t \leq \bar{t}_0, |x - \bar{x}_0| \leq n \},
\]

(23)

\[
\psi_n(t, x) := v(t, x) - \left( v(0, x) + F(b)t + \bar{x}_0 + \frac{C_n(x - \bar{x}_0)^2}{n^2} \right).
\]

Then from (22)-(23),

\[
\max \{ \psi_n(t, x) | 0 \leq t \leq \bar{t}_0, |x - \bar{x}_0| \leq n \} = \max \{ \psi_n(t, x) | 0 < t \leq \bar{t}_0, |x - \bar{x}_0| < n \}. \tag{24}
\]

Take \((\bar{t}_n, \bar{x}_n) \in (0, \bar{t}_0] \times (\bar{x}_0 - n, \bar{x}_0 + n)\) which attains the maximum in (24).

Since \( v \) is a viscosity subsolution of (6) and since \(|C_n| \leq \bar{t}_0\) from Prop.1,

\[ F(b) + \bar{x}_0 \leq F \left( D^2v(0, \bar{x}_n) + \frac{2C_n}{n^2} \right) \leq F \left( b + \frac{2C_n}{n^2} \right) \rightarrow F(b), \tag{25} \]

as \( n \to \infty \), which is a contradiction.

Next we prove (9)-(10). From (8), we only have to prove the following:

\[
\limsup_{x \to -\infty} \left( \sup_{0 \leq t \leq T} \left\{ v(t, x) - v(0, x) - F(a)t \right\} \right) \leq 0, \tag{26}
\]

\[
\liminf_{x \to -\infty} \left( \inf_{0 \leq t \leq T} \left\{ v(t, x) - v(0, x) - F(b)t \right\} \right) \geq 0. \tag{27}
\]

We first prove (26). Suppose that (26) does not hold. Then there exists \( \varepsilon_1 > 0 \) so that

\[
\limsup_{x \to -\infty} \left( \sup_{0 \leq t \leq T} \left\{ v(t, x) - v(0, x) - F(a)t - \varepsilon_1 t \right\} \right) > 0. \tag{28}
\]

In particular, there exists \((s_n, y_n) \in (0, T] \times (-\infty, -n^2)\) for which

\[ v(s_n, y_n) - v(0, y_n) - F(a)s_n - \varepsilon_1 s_n > 0 \quad (n \geq 1). \tag{29} \]
For \( n \geq 1 \), put

\[
\gamma_n := \max\{v(t, x) - (v(0, x) + F(a)t)|0 \leq t \leq T, |x - y_n| \leq n\},
\]

\[
\phi_n(t, x) := v(t, x) - \left( v(0, x) + F(a)t + \varepsilon_1 t + \frac{\gamma_n(x - y_n)^2}{n^2} \right).
\]

Then from (29)-(30).

\[
\max\{\phi_n(t, x)|0 \leq t \leq T, |x - y_n| \leq n\} = \max\{\phi_n(t, x)|0 < t \leq T, |x - y_n| < n\}. \tag{31}
\]

Take \((r_n, z_n) \in (0, T] \times (y_n - n, y_n + n)\) which attains the maximum in (31). Since \( v \) is a viscosity subsolution of (6), \( z_n < -n^2 + n \to -\infty \) as \( n \to \infty \) and \( |\gamma_n| \leq T \) from Prop. 1,

\[
F(a) + \varepsilon_1 \leq F\left( D^2v(0, z_n) + \frac{2\gamma_n}{n^2} \right) \to F(a) \quad (n \to \infty), \tag{32}
\]

which is a contradiction.

Next we prove (27). Suppose that (27) does not hold. Then there exists \( \varpi_1 > 0 \) so that

\[
\liminf_{x \to \infty} \left( \inf_{0 \leq t \leq T} \{v(t, x) - v(0, x) - F(b)t + \varpi_1 t\} \right) < 0. \tag{33}
\]

In particular, there exists \((\varpi_n, \varpi_n) \in (0, T] \times (n^2, \infty)\) for which

\[
v(\varpi_n, \varpi_n) - v(0, \varpi_n) - F(b)\varpi_n + \varpi_1 \varpi_n < 0 \quad (n \geq 1). \tag{34}
\]

Put

\[
\varpi_n := \min\{v(t, x) - (v(0, x) + F(b)t)|0 \leq t \leq T, |x - \varpi_n| \leq n\}, \tag{35}
\]

\[
\overline{\phi}_n(t, x) := v(t, x) - \left( v(0, x) + F(b)t - \varpi_1 t + \frac{\varpi_n(x - \varpi_n)^2}{n^2} \right).
\]

Then from (34)-(35),
\[
\min\{\varphi_n(t, x) | 0 \leq t \leq T, |x - \bar{y}_n| \leq n\} = \min\{\varphi_n(t, x) | 0 < t \leq T, |x - \bar{y}_n| < n\}. \tag{36}
\]

Take \((\tau_n, \bar{z}_n) \in (0, T] \times (\bar{y}_n - n, \bar{y}_n + n)\) which attains the minimum in (36). Since \(v\) is a viscosity supersolution of (6), \(\bar{z}_n > n^2 - n \rightarrow \infty\) as \(n \rightarrow \infty\) and \(|\bar{z}_n| \leq T\) from Prop. 1,
\[
F(b) - \bar{z}_1 \geq F\left(D^2 v(0, \bar{z}_n) + \frac{2\bar{y}_n}{n^2}\right) \rightarrow F(b) \quad (n \rightarrow \infty), \tag{37}
\]
which is a contradiction. \(\square\)

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References


