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# A generalization of a curvature flow of graphs on $\mathbf{R}$

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**1. Introduction.** Gauss curvature flow is known as a mathematical model of the wearing process of a convex stone rolling on a beach and has been studied by many authors (see [3, 5, 6] and the reference therein). In [5] we proposed and studied the discrete stochastic approximations of nonconvex functions which evolve by a convexified Gauss curvature and the PDE which appears as the continuum limit of discrete stochastic processes (see [6] for the similar results on the convexified Gauss curvature flow of closed hypersurfaces). In this paper we study a class of PDE which gives a generalization of a curvature flow of graphs on  $\mathbf{R}$ .

We briefly describe [5] to discuss the results in this paper more precisely. Alexandrov-Bakelman's generalized curvature played a crucial role in [5].

**Definition 1** (see e.g. [1, section 9.6]). Let  $R \in L^1(\mathbf{R}^n : [0, \infty), dx)$  and  $u \in C(\mathbf{R}^n)$ . For  $A \in B(\mathbf{R}^n)(:=\text{Borel } \sigma\text{-field of } \mathbf{R}^n)$ , put

$$w(R, u, A) := \int_{\cup_{x \in A} \partial u(x)} R(y) dy, \quad (1)$$

where  $\partial u(x) := \{p \in \mathbf{R}^n | u(y) - u(x) \geq \langle p, y - x \rangle \text{ for all } y \in \mathbf{R}^n\}$  and  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $\mathbf{R}^n$ .

(It is known that  $w(R, u, \cdot) : B(\mathbf{R}^n) \mapsto [0, \infty)$  is completely additive.)

For  $R \in L^1(\mathbf{R}^n : [0, \infty), dx)$ , we showed the existence and the uniqueness of a solution  $u \in C([0, \infty) \times \mathbf{R}^n)$  to the following equation (see [5, Theorem 1]): for any  $\varphi \in C_o(\mathbf{R}^n)$  and any  $t \geq 0$ ,

$$\int_{\mathbf{R}^n} \varphi(x)(u(t, x) - u(0, x))dx = \int_0^t ds \int_{\mathbf{R}^n} \varphi(x)w(R, u(s, \cdot), dx). \quad (2)$$

In [5, Theorem 2], we proved that a continuous solution  $u$  to (2) sweeps in time  $t > 0$  a region with volume given by  $t \cdot w(R, u(0, \cdot), \mathbf{R}^n)$ , and that, for a continuous solution  $u$  to (2) with a convex  $u(0, \cdot)$ ,  $x \mapsto u(t, \cdot)$  is convex for all  $t > 0$ .

We also showed that a continuous solution  $u$  to (2) is a viscosity solution of the following PDE (see [5, Theorem 3]):

$$\begin{aligned} \partial_t u(t, x) &= \chi(u, Du(t, x), t, x)R(Du(t, x)) \\ &\quad \times \max(\text{Det}(D^2u(t, x)), 0) \quad ((0, \infty) \times \mathbf{R}^n), \end{aligned} \quad (3)$$

where  $Du(t, x) := (\partial u(t, x)/\partial x_i)_{i=1}^n$ ,  $D^2u(t, x) := (\partial^2 u(t, x)/\partial x_i \partial x_j)_{i,j=1}^n$  and

$$\chi(u, p, t, x) := \begin{cases} 1 & \text{if } p \in \partial u(t, x), \\ 0 & \text{otherwise.} \end{cases}$$

Here  $\partial u(t, x)$  denotes the subdifferential of the function  $x \mapsto u(t, x)$ . Conversely, we discussed under what conditions a viscosity solution to (3) is a solution to (2).

We briefly discuss what we study in this paper. In (1) we only considered the measure  $R(y)dy$  which is absolutely continuous with respect to the Lebesgue measure  $dy$ . Otherwise  $\omega(R, u, dx)$  is not generally completely additive.

Suppose that  $n = 1$ . In (1), replace  $R(y)dy$  by a continuous Borel probability measure  $P(dy)$ . Then, using a similar notation,  $\omega(P, u, dx)$  turn out to be a measure and (2) has a unique continuous solution  $u$  (see Theorem 1 in section 2).

Since  $P(dy)$  is not generally absolutely continuous with respect to  $dy$ , we can not consider the PDE for  $u$ . For  $(t, x) \in [0, \infty) \times \mathbf{R}$ , put

$$U(t, x) := \int_{-\infty}^x (u(t, y) - u(0, y))dy + \int_0^x u(0, y)dy + tF(a), \quad (4)$$

where

$$F(x) := P((-\infty, x]), \quad a := \inf\{\cup_{x \in \mathbf{R}} \partial u(0, x)\}.$$

Then from (2),

$$U(t, x) - U(0, x) = \int_0^t F(D_+(D\hat{U}(s, x))) ds. \quad (5)$$

Here  $\hat{u}$  denotes a convex envelope of  $u$  and for  $\varphi$ ,  $D_+\varphi(t, x)$  denotes the right derivative of  $x \mapsto \varphi(t, x)$ .

When  $DU(0, x)$  is convex, we show that  $U(t, x)$  is a unique continuous viscosity solution in  $(0, \infty) \times \mathbf{R}$  of the following (see Theorems 2 and 3 in section 2):

$$\partial_t U(t, x) = F(D^2 U(t, x)). \quad (6)$$

**Definition 2 (Viscosity solution)** (1) Let  $\Omega = (0, \infty) \times \mathbf{R}$ .

(i). A function  $U \in USC(\Omega)$  is called a viscosity subsolution of (6) in  $\Omega$  if whenever  $\varphi \in C^{1,2}(\Omega)$ ,  $(s, y) \in \Omega$ , and  $U - \varphi$  attains a local maximum at  $(s, y)$ , then  $\partial_t \varphi(s, y) \leq F(D^2 \varphi(s, y))$ .

(ii). A function  $U \in LSC(\Omega)$  is called a viscosity supersolution of (6) in  $\Omega$  if whenever  $\varphi \in C^{1,2}(\Omega)$ ,  $(s, y) \in \Omega$ , and  $U - \varphi$  attains a local minimum at  $(s, y)$ , then  $\partial_t \varphi(s, y) \geq F(D^2 \varphi(s, y))$ .

(iii). A function  $U \in C(\Omega)$  is called a viscosity solution of (6) in  $\Omega$  if it is both a viscosity subsolution and a viscosity supersolution of (6) in  $\Omega$ .

(2) Let  $t_2 > t_1 > 0$ ,  $O$  be an open subset of  $\mathbf{R}$  and  $Q := (t_1, t_2] \times O$ . A function  $U \in C(\overline{Q})$  is called a viscosity solution of (6) in  $Q$  if (1,i)-(1,ii) with  $\Omega$  replaced by  $Q$  hold (see [4, p. 66]). Here  $\overline{Q}$  denotes the closure of  $Q$ .

**2. Main results.** We state assumptions under which we can generalize [5] when  $n = 1$ .

(A.1).  $P$  is a continuous Borel probability measure on  $\mathbf{R}$ .

(A.2).  $h \in C(\mathbf{R})$  and the set  $\partial h(\mathbf{R})$  has a positive Lebesgue measure.

(A.3). For any  $p \notin \partial h(\mathbf{R})$  and  $C \in \mathbf{R}$ ,

$$\int_{\mathbf{R}} \max(px + C - h(x), 0) dx = \infty.$$

**Theorem 1** Suppose that (A.1)-(A.3) hold. Then there exists a unique continuous solution  $u$  to (2) with  $u(0, \cdot) = h$ .

The following assumption implies (A.2)-(A.3).  
(A.2)'.  $h$  is convex and is not a constant.

**Theorem 2** *Suppose that (A.1) and (A.2)' hold. Then the function  $U$  defined, by (4), from  $u$  in Theorem 1 is a continuous viscosity solution of (6) in  $(0, \infty) \times \mathbf{R}$ .*

As a regularity result, we have

**Proposition 1** *Suppose that (A.1) holds. Then for a continuous viscosity solution  $v$  of (6) in  $(0, \infty) \times \mathbf{R}$ ,*

$$0 \leq v(t, x) - v(s, x) \leq t - s \quad (0 \leq s < t, x \in \mathbf{R}). \quad (7)$$

*In particular,  $t \mapsto v(t, x)$  is absolutely continuous for all  $x \in \mathbf{R}$ .*

We state an additional assumption and an asymptotic behavior of a viscosity solution of (6).

(A.4). (i)  $v_0 : \mathbf{R} \mapsto \mathbf{R}$  is twice continuously differentiable.  
(ii)  $\lim_{x \rightarrow -\infty} D^2 v_0(x)$  and  $\lim_{x \rightarrow \infty} D^2 v_0(x)$  exist and

$$a := \inf_{x \in \mathbf{R}} D^2 v_0(x) = \lim_{x \rightarrow -\infty} D^2 v_0(x),$$

$$b := \sup_{x \in \mathbf{R}} D^2 v_0(x) = \lim_{x \rightarrow \infty} D^2 v_0(x).$$

**Proposition 2** *Suppose that (A.1) and (A.4,i) hold. Then for a continuous viscosity solution  $v$  of (6) with  $v(0, \cdot) = v_0(\cdot)$  in  $(0, \infty) \times \mathbf{R}$ , the following holds: for any  $t \geq 0$  and  $x \in \mathbf{R}$ ,*

$$F(a)t \leq v(t, x) - v(0, x) \leq F(b)t. \quad (8)$$

*Suppose in addition that (A.4,ii) holds. Then for any  $T \geq 0$ ,*

$$\lim_{x \rightarrow -\infty} \left( \sup_{0 \leq t \leq T} |v(t, x) - v(0, x) - F(a)t| \right) = 0, \quad (9)$$

$$\lim_{x \rightarrow \infty} \left( \sup_{0 \leq t \leq T} |v(t, x) - v(0, x) - F(b)t| \right) = 0. \quad (10)$$

Since  $F$  is nondecreasing, (6) is a degenerate elliptic PDE and we can use the maximum principle for this equation in a bounded domain (see [4, p. 244, Theorem 8.1] and also [2]). From Prop. 2, we immediately obtain

**Theorem 3** *Suppose that (A.1) and (A.4) hold. Then the viscosity solution  $v$  of (6) with  $v(0, \cdot) = v_0(\cdot)$  is unique in  $C([0, \infty) \times \mathbf{R})$ .*

From Theorems 2 and 3, we also obtain

**Corollary 1** *Suppose that (A.1) and (A.2)' hold and that  $h$  is continuously differentiable. Then  $U$  in Theorem 2 is the unique continuous viscosity solution of (6) with  $U(0, x) = \int_0^x h(y)dy$  in  $(0, \infty) \times \mathbf{R}$ .*

In particular, we have

**Corollary 2** *Suppose that (A.1) and (A.4, i) hold and that  $Dv_0$  is convex. Then (6) with  $v(0, \cdot) = v_0(\cdot)$  has the unique viscosity solution in  $C([0, \infty) \times \mathbf{R})$ .*

(Proof of Theorem 1). The proof can be done almost in the same way as in [5, Theorem 1] (In [5, (A.3)], “ $x \in \mathbf{R}^d$ ” should be “ $(x, \hat{h}(x)) \in \mathbf{R}^{d+1}$ ”). The only thing we have to prove is the following (i)-(ii):

- (i) For a convex  $u \in C(\mathbf{R})$ ,  $w(P, u, dx)$  is completely additive,
- (ii) For  $u \in C(\mathbf{R})$  for which  $\partial u(\mathbf{R}) \neq \emptyset$ ,  $w(P, u, dx) = w(P, \hat{u}, dx)$ .

We first prove (i). The set

$$S(u) := \{p \in \mathbf{R} \mid \{x \in \mathbf{R} \mid p \in \partial u(x)\} \text{ contains more than one point}\}$$

contains at most countably many points. Indeed, if  $p \in S(u)$ , then for  $x_p$  for which  $p \in \partial u(x_p)$ , the graph of  $y = u(x)$  and the straight line  $y = p(x - x_p) + u(x_p)$  contains a line segment with a positive length and the interiors of such line segments are disjoint. Hence for each  $n, m \geq 1$ , on the set  $S(u) \cap [D_-u(-n), D_+u(n)]$ , such line segments with the length  $\geq 1/m$  are finitely many. Here  $D_-u(x)$  denotes the left derivative of  $x \mapsto u(x)$ . Since a continuous measure does not have a point mass, we obtain (i) (see [1, p. 118]).

Next we prove (ii). If  $u(x) = \hat{u}(x)$ , then  $\partial u(x) = \partial \hat{u}(x)$ . If  $u(x) \neq \hat{u}(x)$ , then  $\partial u(x) = \emptyset$  and  $\partial \hat{u}(x) = D\hat{u}(x) \in S(\hat{u})$ . Since  $P(S(\hat{u})) = 0$  as we explained above, we obtain (ii).  $\square$

(Proof of Theorem 2). (A.2)' implies that  $x \mapsto DU(t, x)$  is convex for all  $t \geq 0$  (see [5, Theorem 2]). We first prove that  $U$  is a viscosity subsolution to (6) in  $(0, \infty) \times \mathbf{R}$ . Suppose that  $\varphi \in C^{1,2}((0, \infty) \times \mathbf{R})$ ,  $(s, y) \in (0, \infty) \times \mathbf{R}$ , and  $U - \varphi$  attains a local maximum at  $(s, y)$ . Then for  $x$  and  $y \in \mathbf{R}$  for which  $x - y$  is positive,

$$\begin{aligned} \partial_t \varphi(s, y) &\leq F\left(\frac{U(s, x) - U(s, y) - DU(s, y)(x - y)}{(x - y)^2/2}\right) & (11) \\ &\leq F\left(\frac{\varphi(s, x) - \varphi(s, y) - D\varphi(s, y)(x - y)}{(x - y)^2/2}\right) \\ &\rightarrow F(D^2\varphi(s, y)) \quad (x \downarrow y). \end{aligned}$$

Indeed, from (5), for  $t$  and  $s \geq 0$  for which  $s - t$  is positive and is sufficiently small,

$$\varphi(s, y) - \varphi(t, y) \leq U(s, y) - U(t, y) = \int_t^s F(D_+(DU(\alpha, y)))d\alpha,$$

$$\begin{aligned} &U(\alpha, x) - U(\alpha, y) - DU(\alpha, y)(x - y) \\ &= \int_y^x (DU(\alpha, z) - DU(\alpha, y))dz \\ &\geq \int_y^x D_+(DU(\alpha, y))(z - y)dz = D_+(DU(\alpha, y))(x - y)^2/2. \end{aligned}$$

Since  $U$  and  $DU \in C([0, \infty) \times \mathbf{R})$  from Theorem 1, we obtain the first inequality in (11). Since  $DU(s, y) = D\varphi(s, y)$  and  $F$  is nondecreasing, the second inequality of (11) holds.

Next we prove that  $U$  is a viscosity supersolution to (6) in  $(0, \infty) \times \mathbf{R}$ . Suppose that  $\varphi \in C^{1,2}((0, \infty) \times \mathbf{R})$ ,  $(s, y) \in (0, \infty) \times \mathbf{R}$ , and  $U - \varphi$  attains a local minimum at  $(s, y)$ . Then in the same way as in (11), for  $x$  and  $y \in \mathbf{R}$  for which  $y - x$  is positive,

$$\begin{aligned}
\partial_t \varphi(s, y) &\geq F\left(\frac{U(s, x) - U(s, y) - DU(s, y)(x - y)}{(x - y)^2/2}\right) \\
&\geq F\left(\frac{\varphi(s, x) - \varphi(s, y) - D\varphi(s, y)(x - y)}{(x - y)^2/2}\right) \\
&\rightarrow F(D^2\varphi(s, y)) \quad (x \uparrow y). \square
\end{aligned}$$

(Proof of Proposition 1) Without loss of generality, we can put  $s = 0$ . We first prove the first inequality of (7). Suppose that there exists  $(t_0, x_0) \in (0, \infty) \times \mathbf{R}$  such that

$$v(t_0, x_0) - v(0, x_0) < 0. \quad (12)$$

Put

$$\begin{aligned}
C_0 &:= \min\{v(t, x) - v(0, x) \mid 0 \leq t \leq t_0, |x - x_0| \leq 1\} (< 0), \quad (13) \\
\underline{v}(t, x) &:= v(0, x) + C_0(x - x_0)^2.
\end{aligned}$$

Then it is easy to see that  $\underline{v}$  is a viscosity subsolution of (6) in  $(0, t_0] \times (x_0 - 1, x_0 + 1)$  since  $F \geq 0$ . By the maximum principle (see [4, p.244, Th. 8.1]),

$$\begin{aligned}
&\min\{v(t, x) - \underline{v}(t, x) \mid 0 \leq t \leq t_0, |x - x_0| \leq 1\} \quad (14) \\
&= \min\{v(t, x) - \underline{v}(t, x) \mid 0 \leq t \leq t_0, |x - x_0| = 1 \text{ or } t = 0, |x - x_0| \leq 1\} \\
&\geq 0,
\end{aligned}$$

from (13). This contradicts (12).

Next we prove the second inequality of (7). Suppose that there exists  $(\bar{t}_0, \bar{x}_0) \in (0, \infty) \times \mathbf{R}$  such that

$$v(\bar{t}_0, \bar{x}_0) - v(0, \bar{x}_0) - \bar{t}_0 > 0. \quad (15)$$

Put

$$\begin{aligned}
\bar{C}_0 &:= \max\{v(t, x) - v(0, x) - t \mid 0 \leq t \leq \bar{t}_0, |x - \bar{x}_0| \leq 1\} (> 0), \quad (16) \\
\bar{v}(t, x) &:= v(0, x) + t + \bar{C}_0(x - \bar{x}_0)^2.
\end{aligned}$$



Then it is easy to see that  $\bar{v}$  is a viscosity supersolution of (6) in  $(0, \bar{t}_0] \times (\bar{x}_0 - 1, \bar{x}_0 + 1)$  since  $F \leq 1$ . By the maximum principle,

$$\begin{aligned} & \max\{v(t, x) - \bar{v}(t, x) | 0 \leq t \leq \bar{t}_0, |x - \bar{x}_0| \leq 1\} \\ &= \max\{v(t, x) - \bar{v}(t, x) | 0 \leq t \leq \bar{t}_0, |x - \bar{x}_0| = 1 \text{ or } t = 0, |x - \bar{x}_0| \leq 1\} \\ &\leq 0, \end{aligned} \tag{17}$$

from (16). This contradicts (15).  $\square$

(Proof of Proposition 2) First of all, we prove the first inequality in (8). Suppose that there exists  $(t_0, x_0) \in (0, \infty) \times \mathbf{R}$  for which  $F(a)t_0 > v(t_0, x_0) - v(0, x_0)$ . Take  $\varepsilon_0 > 0$  so that

$$v(t_0, x_0) - (v(0, x_0) + F(a)t_0 - \varepsilon_0 t_0) < 0. \tag{18}$$

For  $n \geq 1$ , put

$$\begin{aligned} C_n &:= \min\{v(t, x) - (v(0, x) + F(a)t) | 0 \leq t \leq t_0, |x - x_0| \leq n\}, \\ \psi_n(t, x) &:= v(t, x) - \left(v(0, x) + F(a)t - \varepsilon_0 t + \frac{C_n(x - x_0)^2}{n^2}\right). \end{aligned} \tag{19}$$

Then from (18)-(19),

$$\begin{aligned} & \min\{\psi_n(t, x) | 0 \leq t \leq t_0, |x - x_0| \leq n\} \\ &= \min\{\psi_n(t, x) | 0 < t \leq t_0, |x - x_0| < n\}. \end{aligned} \tag{20}$$

Take  $(t_n, x_n) \in (0, t_0] \times (x_0 - n, x_0 + n)$  which attains the minimum in (20). Since  $v$  is a viscosity supersolution of (6) and since  $|C_n| \leq t_0$  from Prop.1,

$$F(a) - \varepsilon_0 \geq F\left(D^2v(0, x_n) + \frac{2C_n}{n^2}\right) \geq F\left(a + \frac{2C_n}{n^2}\right) \rightarrow F(a), \tag{21}$$

as  $n \rightarrow \infty$ , which is a contradiction.

Next we prove the second inequality in (8). Suppose that there exists  $(\bar{t}_0, \bar{x}_0) \in (0, \infty) \times \mathbf{R}$  for which  $F(b)\bar{t}_0 < v(\bar{t}_0, \bar{x}_0) - v(0, \bar{x}_0)$ . Take  $\bar{\varepsilon}_0 > 0$  so that

$$v(\bar{t}_0, \bar{x}_0) - (v(0, \bar{x}_0) + F(b)\bar{t}_0 + \bar{\varepsilon}_0\bar{t}_0) > 0. \quad (22)$$

For  $n \geq 1$ , put

$$\begin{aligned} \bar{C}_n &:= \max\{v(t, x) - (v(0, x) + F(b)t) \mid 0 \leq t \leq \bar{t}_0, |x - \bar{x}_0| \leq n\}, \\ \bar{\psi}_n(t, x) &:= v(t, x) - \left(v(0, x) + F(b)t + \bar{\varepsilon}_0t + \frac{\bar{C}_n(x - \bar{x}_0)^2}{n^2}\right). \end{aligned} \quad (23)$$

Then from (22)-(23),

$$\begin{aligned} &\max\{\bar{\psi}_n(t, x) \mid 0 \leq t \leq \bar{t}_0, |x - \bar{x}_0| \leq n\} \\ &= \max\{\bar{\psi}_n(t, x) \mid 0 < t \leq \bar{t}_0, |x - \bar{x}_0| < n\}. \end{aligned} \quad (24)$$

Take  $(\bar{t}_n, \bar{x}_n) \in (0, \bar{t}_0] \times (\bar{x}_0 - n, \bar{x}_0 + n)$  which attains the maximum in (24). Since  $v$  is a viscosity subsolution of (6) and since  $|\bar{C}_n| \leq \bar{t}_0$  from Prop.1,

$$F(b) + \bar{\varepsilon}_0 \leq F\left(D^2v(0, \bar{x}_n) + \frac{2\bar{C}_n}{n^2}\right) \leq F\left(b + \frac{2\bar{C}_n}{n^2}\right) \rightarrow F(b), \quad (25)$$

as  $n \rightarrow \infty$ , which is a contradiction.

Next we prove (9)-(10). From (8), we only have to prove the following:

$$\limsup_{x \rightarrow -\infty} \left( \sup_{0 \leq t \leq T} \{v(t, x) - v(0, x) - F(a)t\} \right) \leq 0, \quad (26)$$

$$\liminf_{x \rightarrow \infty} \left( \inf_{0 \leq t \leq T} \{v(t, x) - v(0, x) - F(b)t\} \right) \geq 0. \quad (27)$$

We first prove (26). Suppose that (26) does not hold. Then there exists  $\varepsilon_1 > 0$  so that

$$\limsup_{x \rightarrow -\infty} \left( \sup_{0 \leq t \leq T} \{v(t, x) - v(0, x) - F(a)t - \varepsilon_1 t\} \right) > 0. \quad (28)$$

In particular, there exists  $(s_n, y_n) \in (0, T] \times (-\infty, -n^2)$  for which

$$v(s_n, y_n) - v(0, y_n) - F(a)s_n - \varepsilon_1 s_n > 0 \quad (n \geq 1). \quad (29)$$

For  $n \geq 1$ , put

$$\begin{aligned}\gamma_n &:= \max\{v(t, x) - (v(0, x) + F(a)t) \mid 0 \leq t \leq T, |x - y_n| \leq n\}, \quad (30) \\ \phi_n(t, x) &:= v(t, x) - \left(v(0, x) + F(a)t + \varepsilon_1 t + \frac{\gamma_n(x - y_n)^2}{n^2}\right).\end{aligned}$$

Then from (29)-(30).

$$\begin{aligned}& \max\{\phi_n(t, x) \mid 0 \leq t \leq T, |x - y_n| \leq n\} \\ &= \max\{\phi_n(t, x) \mid 0 < t \leq T, |x - y_n| < n\}.\end{aligned}\quad (31)$$

Take  $(r_n, z_n) \in (0, T] \times (y_n - n, y_n + n)$  which attains the maximum in (31). Since  $v$  is a viscosity subsolution of (6),  $z_n < -n^2 + n \rightarrow -\infty$  as  $n \rightarrow \infty$  and  $|\gamma_n| \leq T$  from Prop. 1,

$$F(a) + \varepsilon_1 \leq F\left(D^2v(0, z_n) + \frac{2\gamma_n}{n^2}\right) \rightarrow F(a) \quad (n \rightarrow \infty), \quad (32)$$

which is a contradiction.

Next we prove (27). Suppose that (27) does not hold. Then there exists  $\bar{\varepsilon}_1 > 0$  so that

$$\liminf_{x \rightarrow \infty} \left( \inf_{0 \leq t \leq T} \{v(t, x) - v(0, x) - F(b)t + \bar{\varepsilon}_1 t\} \right) < 0. \quad (33)$$

In particular, there exists  $(\bar{s}_n, \bar{y}_n) \in (0, T] \times (n^2, \infty)$  for which

$$v(\bar{s}_n, \bar{y}_n) - v(0, \bar{y}_n) - F(b)\bar{s}_n + \bar{\varepsilon}_1 \bar{s}_n < 0 \quad (n \geq 1). \quad (34)$$

Put

$$\begin{aligned}\bar{\gamma}_n &:= \min\{v(t, x) - (v(0, x) + F(b)t) \mid 0 \leq t \leq T, |x - \bar{y}_n| \leq n\}, \quad (35) \\ \bar{\phi}_n(t, x) &:= v(t, x) - \left(v(0, x) + F(b)t - \bar{\varepsilon}_1 t + \frac{\bar{\gamma}_n(x - \bar{y}_n)^2}{n^2}\right).\end{aligned}$$

Then from (34)-(35),

$$\begin{aligned} & \min\{\bar{\phi}_n(t, x) | 0 \leq t \leq T, |x - \bar{y}_n| \leq n\} \\ = & \min\{\bar{\phi}_n(t, x) | 0 < t \leq T, |x - \bar{y}_n| < n\}. \end{aligned} \quad (36)$$

Take  $(\bar{r}_n, \bar{z}_n) \in (0, T] \times (\bar{y}_n - n, \bar{y}_n + n)$  which attains the minimum in (36). Since  $v$  is a viscosity supersolution of (6),  $\bar{z}_n > n^2 - n \rightarrow \infty$  as  $n \rightarrow \infty$  and  $|\bar{\gamma}_n| \leq T$  from Prop. 1,

$$F(b) - \bar{\varepsilon}_1 \geq F\left(D^2v(0, \bar{z}_n) + \frac{2\bar{\gamma}_n}{n^2}\right) \rightarrow F(b) \quad (n \rightarrow \infty), \quad (37)$$

which is a contradiction.  $\square$

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