Uniform local solvability for the Navier-Stokes equations with the Coriolis force

Dedicated to Professor Joel A. Smoller on the occasion of his 65th birthday.

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Abstract

The unique local existence is established for the Cauchy problem of the incompressible Navier-Stokes equations with the Coriolis force. The Coriolis operator restricted to divergence free vector fields is a zero order pseudo-differential operator with the skew-symmetric matrix symbol related to the Riesz operator. It leads to the additional term in the Navier-Stokes equations which has real parameter being proportional to the speed of rotation. For initial data as Fourier preimage of the space of all finite Radon measures with no point mass at the origin we prove uniform estimate for the existence time in the speed of rotation.

1 Introduction

In this paper we consider the Cauchy problem for the three dimensional Navier-Stokes equations with the Coriolis force:

$$u_t + (u \cdot \nabla)u + \Omega e_3 \times u - \Delta u = -\nabla p, \quad \nabla \cdot u = 0, \quad u|_{t=0} = u_0, \qquad (1.1)$$

where $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the unknown velocity vector field and $p = p(x,t)$ is the unknown scalar pressure at the point $x =$

$(x_1, x_2, x_3) \in \mathbb{R}^3$ in space and time $t > 0$, while $u_0 = u_0(x)$ is the given initial velocity vector field. Here, the real constant Ω represents the speed of rotation around the vertical unit vector $e_3 = (0, 0, 1)$ and it is called the Coriolis parameter. By $\times$ we denote the exterior product, hence, the Coriolis term is represented by $e_3 \times u \equiv \mathbf{J}u$ with the corresponding skew-symmetric 3×3 matrix $\mathbf{J}$.

For the equations (1.1) global existence and regularity results for large fixed Ω were obtained in L^2 setting for the periodic domains such as cylinder with infinite length rotating axis or spatial lattices ([2], [6]). In this regard, for the Euler equations, Nicolaenko, Bardos, Golse and the third author [7] proved uniform local existence and long-time regularity for initial data in H^4.

In the case of unbounded domains without periodicity nor decay at space infinity Sawada [9] and Hieber and Sawada [5] proved unique local existence for initial data in a homogeneous Besov space $\dot{B}^0_{\infty,1}$ which is strictly smaller than BUC, the space of all bounded uniformly continuous functions (see also [3] for recent improvement). However, they do not get uniform estimate for the existence time in the speed of rotation $\Omega \in \mathbb{R}$, the precondition to consider fast singular oscillating limits $\Omega \to +\infty$ and proving global existence for large fixed Ω.

In this paper we take initial data in a new space FM_0 (Fourier preimage of the space of all finite Radon measures with no point mass at the origin) that ensures uniform local solvability of (1.1). The space for initial data FM_0 is strictly smaller than the homogeneous Besov space $\dot{B}^0_{\infty,1}$ and its precise definition will be given in section 2.

Our key observation is a bound of $\exp(tR_j)$ in the space FM_0 uniform in time t, where R_j is a Riesz operator. This enables us to construct a local solution uniform in Ω.

Now we state the main result of this paper.

Theorem 1.1. Assume that $u_0 \in FM_0$ with div $u_0 = 0$. Then there exist $T_0 (\geq c/||u_0||^2_{FM}) > 0$ independent of the Coriolis parameter Ω and a unique mild solution $u = u(t) \in C([0, T_0]; FM_0)$ of (1.1), where $c > 0$ is a numerical constant (given in Remark 3.1). Moreover, the solution $u(t)$ belongs to $C([0, T_0]; FM)$.

From the above theorem on a mild solution we also have the following theorem on existence and uniqueness of the classical solution of (1.1). The proof is carried out in the same manner as [3].

Theorem 1.2. Assume that $u_0 \in FM_0$ with div $u_0 = 0$.
(1) *Let $u = u(t)$ be the mild solution obtained in Theorem 1.1. If we set the pressure $p = p(x, t)$ as*

$$\partial_i p(t) = \partial_i \sum_{j,k=1}^{3} R_j R_k u^j u^k(t) + \Omega R_i \ (R_2 u^1 - R_1 u^2)(t) \tag{1.2}$$

for $t > 0$ and $i = 1, 2, 3$, then the pair $(u, \nabla p)$ is a classical solution of (1.1).
(2) Let $u \in L^\infty((0,T) \times \mathbb{R}^3)$ and $p \in L^1_{loc}([0,T); BMO)$ be a solution of (1.1) in a distributional sense for some $T > 0$. Then the pair $(u, \nabla p)$ is unique. Moreover, the relation (1.2) holds.

Before concluding the introduction we remark that $u(x,t)$ is almost periodic in x if u_0 is almost periodic. This can be proved along the line of [4, Section 3] if one observes that FM-norm is invariant under translation in spatial variables $x \in \mathbb{R}^3$.

2 Key function spaces

In this section we introduce a function space on which the semigroup generated by the Riesz operator is uniformly bounded; moreover, it contains various almost periodic functions, not necessarily periodic. Furthermore, operation of spatial averaging is defined for the elements of this space. The latter property and uniform boundedness with respect to spatial translations and rotations are essential in studies of statistical properties of turbulence (e.g. [8], [11]).
We now recall an important connection between the Coriolis rotation operator restricted to divergence free vector fields and classical Riesz operators. For the sake of clarity of notations, we present formulas for symbols of the corresponding pseudo-differential operators in $\mathbb{R}^3$ which is the main case. Let $\mathbf{P}$ be the projection operator on divergence free vector fields, represented by $\mathbf{P} = (P_{ij})_{ij} = (\delta_{ij} + R_i R_j)_{1 \le i,j \le 3}$. Here, δ_{ij} denotes the Kronecker delta. Applying $\mathbf{P}$ to (1.1), we note that the Coriolis term is represented by the Coriolis operator $\mathbf{S} = \mathbf{PJP}$ ($\mathbf{PJP}u = \mathbf{P}\mathbf{J}u$ since u is solenoidal).

Let $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. The operator $\mathbf{S}$ is a zero order pseudo-differential operator with the 3×3 matrix symbol $\mathbf{S}(\xi)$ ([1], [3]):

$$\sigma(\mathbf{S}) \equiv \mathbf{S}(\xi) = \frac{\xi_3}{|\xi|}\mathbf{R}(\xi), \tag{2.1}$$

where $\mathbf{R}(\xi)$ is the skew-symmetric matrix such that $\mathbf{R}(\xi)v = \frac{1}{|\xi|}\xi \times v$ for every vector $v \in \mathbb{R}^3$. The symbol of the operator $\exp(\Omega \mathbf{S} t)$ is given by

$$\exp(\Omega \mathbf{S}(\xi)t) = \cos(\frac{\xi_3}{|\xi|}\Omega t)\mathbf{I} + \sin(\frac{\xi_3}{|\xi|}\Omega t)\mathbf{R}(\xi), \tag{2.2}$$

where $\mathbf{I}$ is the identity matrix. We remark that dependence on the parameter Ω appears only in scalar terms $\cos(\frac{\xi_3}{|\xi|}\Omega t)$ and $\sin(\frac{\xi_3}{|\xi|}\Omega t)$, which are functions of the classical scalar Riesz operator R_3, $\sigma(R_3) = \frac{i\xi_3}{|\xi|}$. Then our goal in this section is to construct function spaces for initial data in which Fourier operators with the symbols $e^{\pm i\frac{\xi_3}{|\xi|}\Omega t}$ are uniformly bounded in Ω.

2.1 Spaces of measures

Let $M = M(\mathbb{R}^n)$ be the space of all complex-valued finite Radon measures on $\mathbb{R}^n$. This space is the (complex) Banach space equipped with the total variation norm $\| \cdot \|_M$. By the Riesz representation theorem M is identified with the dual space of $C_\infty(\mathbb{R}^n)$, the space of all (complex-valued) continuous functions that converge to zero at the space infinity (equipped with the supremum norm $\| \cdot \|_\infty$.) Moreover, for $\mu \in M$

$$\| \mu \|_M = \sup\{|\langle \mu, \psi \rangle|; \psi \in C_\infty(\mathbb{R}^n), \| \psi \|_\infty \leq 1\},$$

where $\langle , \rangle$ denotes a canonical pairing. Since the (Schwartz) space $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ of all rapidly decreasing functions in $\mathbb{R}^n$ is dense in $C_\infty(\mathbb{R}^n)$, $\mu \in M$ is regarded as a tempered distribution, i.e. $\mu \in \mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$. In other words, $M \subset \mathcal{S}'$. Let $|\mu|$ denote the total variation measure of μ. It is a nonnegative Radon measure defined by

$$|\mu|(O) = \sup\{|\langle \mu, \psi \rangle|; \psi \in C_0(O), \ \| \psi \|_\infty \leq 1\}$$

for an open set O, where $C_0(O)$ denotes the space of all (complex-valued) continuous functions with compact support in O. Since $C_0(\mathbb{R}^n)$ is dense in $C_\infty(\mathbb{R}^n)$, we see that $|\mu|(\mathbb{R}^n) = \| \mu \|_M$. For $\mathbb{C}^d$-valued (complex vector-valued) finite Radon measure $\mu = (\mu_1, \cdots, \mu_d)$ we still denote its total variation measure $|\mu|$ by the above duality by understanding that $||\psi||_\infty = || \ |\psi| \ ||_\infty$ for $\psi = (\psi_1, \cdots, \psi_d)$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$. The totality of $\mathbb{C}^d$-valued finite Radon measures on $\mathbb{R}^n$ is denoted by $M^d = (M(\mathbb{R}^n))^d$. (We often write M^d simply by M unless confusion is caused.) There are several ways to define a norm in $(M(\mathbb{R}^n))^d$. The most convenient one for our purpose is

$$||\mu||_{M^d} = |\mu|(\mathbb{R}^n).$$

Another norm we use here is

$$||\mu||'_{M^d} = (\sum_{i=1}^d ||\mu_i||_M^2)^{1/2}.$$

In the case that μ_i is absolutely continuous with respect to the Lebesgue measure, i.e. $\mu_i = \rho_i d\xi$, then

$$||\mu||_{M^d} = \int_{\mathbb{R}^n} (\sum_{i=1}^d |\rho_i|^2) d\xi \ , \ ||\mu||'_{M^d} = (\sum_{i=1}^d (\int_{\mathbb{R}^n} |\rho_i| d\xi)^2)^{1/2}.$$

As we shall see later these norms are equivalent. For the definition of total variations as well as several elementary properties of measures, the reader is referred to, for example, a book [10] of L. Simon.

For a bounded Borel measurable function ψ and $\mu \in M$ we associate a new Radon measure $\mu \lfloor \psi \in M$ defined by

$$(\mu \lfloor \psi)(O) = \int_O \psi(\xi) \mu(d\xi).$$

If ψ is a characteristic function χ_B of a Borel set B, we simply write $\mu\lfloor\chi_B$ by $\mu\lfloor B$. Note that ψ is allowed to be any $|\mu|$-measurable functions. We here prove that above two norms of M^d are equivalent. More precisely,

$$||\mu||'_{M^d} \leq ||\mu||_{M^d} \leq \sqrt{d}||\mu||'_{M^d} \ , \ \mu \in M^d.$$

Indeed, let h_i be the Radon-Nikodym derivative of μ_i with respect to $|\mu|$; this is $|\mu|$-a.e. well-defined since μ_i is absolutely continuous with respect to $|\mu|$. Then

$$||\mu||'_{M^d} = \{\sum_{i=1}^{d}(\int_{\mathbb{R}^n} |h_i||\mu|(d\xi))^2\}^{1/2}.$$

Applying Minkowski's inequality, we have

$$||\mu||'_{M^d} \leq \int_{\mathbb{R}^n} (\sum_{i=1}^{d} |h_i|^2)^{1/2}|\mu|(d\xi) = |\mu|(\mathbb{R}^n) = ||\mu||_{M^d}$$

since $\sum_{i=1}^{d} |h_i|^2(\xi) = 1$ for $|\mu|$-a.e. ξ. If is easy to see that $||\mu||_{M^d} \leq \sum_{i=1}^{d} ||\mu_i||_M$, so we have $||\mu||_{M^d} \leq \sqrt{d}||\mu||'_{M^d}$ by Schwarz' inequality.

We next consider an important subspace M_0 of $M(\mathbb{R}^n)$ for our purpose defined by

$$M_0 = \{\mu \in M \ ; \ \mu\lfloor\{0\} = 0\}$$

(By M_0^d we mean $(M_0(\mathbb{R}^n))^d$). In other words $\mu \in M$ belongs to M_0 if and only if there is no point mass at the origin. The condition $\mu\lfloor\{0\} = 0$ is equivalent to say that

$$\lim_{r\downarrow 0}|\mu|\big(\overset{\circ}{B}_r(0)\big) = 0, \tag{2.3}$$

where $\overset{\circ}{B}_r(0)$ is the open ball of radius r centered at the origin. Thus the space M_0 is a closed subspace of M_0. Indeed, if $\mu_m \to \mu$ in M and μ_m satisfies (2.3), then

$$\begin{aligned} |\mu|\big(\overset{\circ}{B}_r(0)\big) &\leq |\mu_m - \mu|\big(\overset{\circ}{B}_r(0)\big) + |\mu_m|\big(\overset{\circ}{B}_r(0)\big) \\ &\leq \| \mu_m - \mu \|_M + |\mu_m|\big(\overset{\circ}{B}_r(0)\big). \end{aligned}$$

Sending r to 0, we have

$$\limsup_{r\downarrow 0}|\mu|\big(\overset{\circ}{B}_r(0)\big) \leq \| \mu_m - \mu \|_M \ .$$

Since $\mu_m \to \mu$ in M, this implies that μ satisfies (2.3), i.e., $\mu \in M_0$.

Lemma 2.1. Let $\sigma \in C(\mathbb{R}^n\backslash\{0\})$ be bounded on $\mathbb{R}^n\backslash\{0\}$. Assume that $\mu \in M_0$ and that $\{\sigma_m\}_{m=1}^{\infty} \subset \mathcal{S}(\mathbb{R}^n)$ is an approximate sequence of σ in the sense that $\sigma_m \to \sigma$ pointwise in $\mathbb{R}^n\backslash\{0\}$ and $\| \sigma_m \|_\infty$ is uniformly bounded. Then

$\{\mu\lfloor\sigma_m\}_{m=1}^{\infty} \subset M$ *is a convergent sequence in* M_0. *Moreover, its limit equals* $\big(\mu\lfloor(\mathbb{R}^n\setminus\{0\})\big)\lfloor\sigma \in M_0$ *and is independent of the choice of the approximate sequence* $\{\sigma_m\}_{m=1}^{\infty}$.

Remark 2.1. Let $\overline{\sigma}$ be any extension of $\sigma \in C(\mathbb{R}^n \setminus \{0\})$ to the origin. Since $\mu \in M_0$ so that $\mu = \mu\lfloor(\mathbb{R}^n\setminus\{0\})$, we see that $\big(\mu\lfloor(\mathbb{R}^n\setminus\{0\})\big)\lfloor\sigma = \mu\lfloor\overline{\sigma}$. In particular, the measure $\mu\lfloor\overline{\sigma}$ is independent of the way of the extension of σ.

We shall denote $\big(\mu\lfloor(\mathbb{R}^n\setminus\{0\})\big)\lfloor\sigma$ simply by $\mu\lfloor\sigma \in M_0$ for $\mu \in M_0$. (The measure $\mu\lfloor\overline{\sigma}$ in general depends on the value $\overline{\sigma}\lfloor 0$ when $\mu \in M$.)
Proof. By the Lebesgue dominated convergence theorem we see that

$$\lim_{m\to\infty}\int_{\mathbb{R}^n\setminus\{0\}} |\sigma_m - \sigma|\mu(d\xi) = 0.$$

Since $\mu \in M_0$ so that $\mu\lfloor\sigma_m = (\mu\lfloor\sigma_m)\lfloor(\mathbb{R}^n\setminus\{0\})$, this implies

$$\mu\lfloor\sigma_m \to \big(\mu\lfloor(\mathbb{R}^n\setminus\{0\})\big)\lfloor\sigma \in M_0.$$

2.2. Fourier images and Riesz operator

We shall consider Fourier (pre)image of spaces M and M_0 introduced above. For $f \in \mathcal{S}$ we define its Fourier transform and its inverse Fourier transform by

$$F\,f(\xi) = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)dx,$$
$$F^{-1}f(x) = \check{f}(x) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)d\xi,$$

where $\xi \cdot x$ denotes the standard inner product in $\mathbb{R}^n$ and $i = \sqrt{-1}$. The operator F and F^{-1} can be extended to an isomorphism of $\mathcal{S}'$ as topological vector spaces. Let FM be the image of M by F. It is the preimage of M by F^{-1}. Since $F^{-1}f(x) = (Ff)(-x)$, the space FM is also the preimage of M by F, i.e.,

$$FM = \{f \in \mathcal{S}'; \hat{f} \in M\}.$$

This space is a Banach space equipped with the norm

$$\| f \|_{FM} = (2\pi)^{-n/2} \| \hat{f} \|_M .$$

Let FM_0 be the closed subspace of FM of the form

$$FM_0 = \{f \in \mathcal{S}'; \hat{f} \in M_0\}.$$

The space FM is included in the space BUC, the space of all bounded uniformly continuous functions. The space FM_0 does not contain nonzero constant function. In fact, FM has a topological direct sum decomposition of the form

$$FM = FM_0 \oplus \mathbb{C},$$

where $\mathbb{C}$ denotes the space of all (complex) constant functions. Moreover, as we shall see in Appendix, FM_0 is strictly included in the Besov space $\dot{B}^0_{\infty,1}$. However, FM_0 still includes various almost periodic functions not necessarily periodic. For example,

$$f(x) = \sum_{j=1}^{\infty} \alpha_j e^{i\lambda_j \cdot x} \quad \lambda_j \in \mathbb{R}^n \backslash \{0\}, \quad \alpha_j \in \mathbb{C}, \quad \lambda_j \neq \lambda_k \text{ if } j \neq k$$

belongs to FM_0 if $\sum_{j=1}^{\infty} |\alpha_j| < \infty$. Indeed,

$$\hat{f}(\xi) = (2\pi)^{n/2} \sum_{j=1}^{\infty} \alpha_j \delta(\xi - \lambda_j) \in M_0.$$

Its norm $\| f \|_{FM} = \sum_{j=1}^{\infty} |\alpha_j|$ if $\lambda_j \neq \lambda_k$ for $j \neq k$.

We next study an operator whose symbol may not be continuous at the origin. We shall prove that

$$(\Sigma f)(x) = F^{-1}\big(\sigma(\xi)((Ff)(\xi))\big)(x)$$

is a well-defined operator for $f \in FM_0$ if the symbol $\sigma \in C(\mathbb{R}^n \backslash \{0\})$ is bounded. We approximate σ by $\{\sigma_m\} \subset \mathcal{S}$ such that $\| \sigma_m \|_\infty$ is bounded and $\sigma_m \to \sigma$ pointwise in $\mathbb{R}^n \backslash \{0\}$. (This is of course possible.) The quantity $F^{-1}(\sigma_m F f)$ is well-defined quantity in $\mathcal{S}'$ if $f \in \mathcal{S}'$. By Lemma 2.1 $F^{-1}\sigma_m F f$ converges to $F^{-1}\big((Ff)\lfloor\sigma\big)$ in FM_0 if $f \in FM_0$. So we shall define Σf by

$$\Sigma f = F^{-1}\big((Ff)\lfloor\sigma\big). \tag{2.3}$$

Lemma 2.2. Assume that $\sigma \in C(\mathbb{R}^n \backslash \{0\})$ is bounded in $\mathbb{R}^n \backslash \{0\}$. Then Σ in (2.3) is a bounded linear operator in FM_0 and

$$\| \Sigma f \|_{FM} \leq \| \sigma \|_\infty \| f \|_{FM} \tag{2.4}$$

for $f \in FM_0$. If, moreover σ is continuous at the origin, then Σ is a bounded linear operator in FM and (2.4) holds for all $f \in FM$.

This is clear by definition of the norm. As a simple application we obtain the boundedness on FM_0 of the Riesz operator

$$(R_j f) = F^{-1}\big(Ff\lfloor\sigma(R_j)\big)\ ,\ \sigma(R_j) = i\xi_j/|\xi|, j = 1, \cdots, n$$

although the symbol $\sigma(R_j)$ is not continuous at $\xi = 0$.

Lemma 2.3. The Riesz operator R_j is bounded in FM_0 for $j = 1, 2, \cdots, n$. Moreover,

$$(i) \ \| R_j f \|_{FM} \leq \| f \|_{FM} \quad t > 0, \qquad (ii) \ \| e^{tR_j} f \|_{FM} \leq \| f \|_{FM} \quad t > 0$$

for all $f \in FM_0$.

2.3 Multiplication and heat operators

It is rather clear that FM is an algebra whose unit is a constant function 1. The space FM_0 is not an algebra since the multiplication of $e^{i\lambda\cdot x}$ and $e^{-i\lambda\cdot x}$ equals $1 \notin FM_0$ for $\lambda \in \mathbb{R}^n \backslash \{0\}$.

Lemma 2.4. If f *and* g *are in* FM, *so is* fg. *The norm of the unit 1 equals 1. Moreover,*

$$\| fg \|_{FM} \leq \| f \|_{FM} \| g \|_{FM} .$$

Proof. Since $\widehat{fg} = (2\pi)^{-n/2} \hat{f} * \hat{g}$, Fubini's theorem implies that

$$\| \widehat{fg} \|_M \leq (2\pi)^{-n/2} \| \hat{f} \|_M \| \hat{g} \|_M,$$

which yields the desired estimate. Since $\hat{1} = (2\pi)^{n/2}\delta(x)$, by definition $\| 1 \|_{FM} = 1$. □

We next study the heat semigroup on FM and FM_0. For $f \in FM$ we define

$$e^{t\Delta} f = F^{-1}\big(e^{-t|\xi|^2}(Ff)\big) = F^{-1}(\hat{f} \lfloor e^{-t|\xi|^2}), \quad t \geq 0.$$

Similarly, we also define

$$\partial_{x_j} e^{t\Delta} f = F^{-1}(i\xi_j e^{-t|\xi|^2}(Ff)) = F^{-1}(\hat{f} \lfloor i\xi_j e^{-t|\xi|^2}), \quad t \geq 0,$$

which is of course consistent with $\partial_{x_j}(e^{t\Delta} f)$.

Lemma 2.5.

(i) The family $\{e^{t\Delta}\}_{t\geq 0}$ *is a bounded* C_0*-semigroup in* FM *and* FM_0. *Moreover,* $\| e^{t\Delta} f \|_{FM} \leq \| f \|_{FM}$ *for* $f \in FM$, $t \geq 0$.

(ii) Let A *be a closed linear operator defined by*

$$Af = \Delta f$$

for $f \in D(A) = \{f \in FM; \frac{\partial^2}{\partial x_i \partial x_j} f \in FM, (1 \leq i, j \leq n)\}$, *Then* A *is the infinitesimal generator of the semigroup* $\{e^{t\Delta}\}_{t\geq 0}$ *in* FM. *The restriction* A *on* FM_0 *(denoted by* A_0*) is the infinitesimal generator of* $\{e^{t\Delta}\}_{t\geq 0}$ *in* FM_0.

(iii) The semigroup $\{e^{t\Delta}\}_{t\geq 0}$ *is an analytic semigroup in* FM *and* FM_0. *Moreover,*

$$\| \partial_{x_j} e^{t\Delta} f \|_{FM} \leq (2te)^{-1/2} \| f \|_{FM}, \ f \in FM, \ t \geq 0, \ i = 1, \cdots, n$$

and $\partial_{x_j} e^{t\Delta} f \in FM_0$, *where* $\partial_{x_j} = \partial/\partial x_j$.

Proof. All properties are easy to prove if one works on the Fourier images of f and $e^{t\Delta}f$. Necessary estimates are obtained by Lemma 2.2.

2.4. Averaging

An element $f \in FM$ always has a vertical averaging in the sense that

$$\lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} f(x_1, \cdots, x_{n-1}, x_n)dx_n =: f_a(x'), \quad x' = (x_1, \cdots, x_{n-1})$$

exists at least for almost every $x' \in \mathbb{R}^{n-1}$. We call the function f_a the vertical average of f. The function $f - f_a$ is denoted by $f^\perp$. Actually, we have a stronger result. Let χ^L be a function defined by

$$\chi^L(x_n) = \frac{1}{2L}\chi_{(-L,L)}(x_n)$$

and $\chi_{(-L,L)}(x_n)$ is the characteristic function of $(-L, L)$, i.e., $\chi_{(-L,L)}(x_n) = 1$ for $x_n \in (-L, L)$ and otherwise $\chi^L(x) = 0$.

Lemma 2.6.

(i) Assume that $f \in FM$ fulfills $\hat{f}\lfloor\{\xi_n = 0\} = 0$. (In particular $f \in FM_0$). Then $\chi^L * f \to 0$ in FM as $L \to \infty$.

(ii) Assume that $f \in FM$ fulfills $\hat{f}\lfloor\{\xi_n \neq 0\} = 0$. (This means f is independent of x_n.) Then $\chi^L * f = f$ for all $L > 0$.

(iii) The space FM has a direct sum decomposition

$$\begin{aligned} & FM = FM^\perp \oplus FM_a \\ \text{with} \quad & FM^\perp = \{f \in FM; \hat{f}\lfloor\{\xi_n \neq 0\} = 0\} \\ & FM_a = \{f \in FM; \hat{f}\lfloor\{\xi_n = 0\} = 0\}. \end{aligned}$$

Proof.

(i) Since $\hat{\chi}^L(\xi_n) = \sin(L\xi_n)/L\xi_n$, we see that $\hat{\chi}^L(\xi_n) \to 0$ as $L \to \infty$ for $\xi_n \neq 0$ and $|\hat{\chi}^L(\xi_n)| \leq 1$. Since

$$\| f * \chi^L \|_{FM} = \| \hat{\chi}^L \hat{f} \|_M = \| (\hat{f}\lfloor\chi^L)\lfloor\{\xi_n \neq 0\} \|_M,$$

the Lebesgue dominated convergence theorem yields that $\chi^L * f \to 0$ for $f \in FM$ if $\hat{f}\lfloor\{\xi_n = 0\} = 0$.

(ii) This is trivial.

(iii) For $f \in FM$ we define mappings

$$f \mapsto F^{-1}(\hat{f}\lfloor\{\xi_n = 0\}), \quad f \mapsto F^{-1}(\hat{f}\lfloor\{\xi_n \neq 0\})$$

which give projections to $FM^\perp$ and FM_a.

2.5. Divergence free spaces

We shall study the space of vector fields in $\mathbb{R}^n$ whose components belong to FM. To simplify notation we still denote such a space by FM. We shall apply such convention for other spaces like FM_0, $FM^{\perp}$, FM_a.
Let $\mathbf{P}$ be defined by

$$(\mathbf{P}f)_j = \sum_{k=1}^{n} F^{-1}\big(\hat{f}_k \lfloor \sigma_{jk}(\mathbf{P})\big) \quad \text{for} \quad f = (f_1, \cdots, f_n) \in FM_0, j = 1, \cdots, n$$

with the symbol $\sigma_{jk}(\mathbf{P})(\xi) = \delta_{jk} - \xi_j\xi_k/|\xi|^2$, $\xi \in \mathbb{R}^n$. By Lemma 2.2 this operator is bounded in FM_0. Let PFM_0 denote the $\mathbf{P}$-image of FM_0. We set $\mathbf{Q} = I - \mathbf{P}$ and observe that

$$(\mathbf{Q}f)_j = \sum_{k=1}^{n} F^{-1}(\hat{f}_k \lfloor -\xi_j\xi_k/|\xi|^2) \text{ for } f = (f_1, \cdots, f_n) \in FM_0.$$

Let QFM_0 denote the $\mathbf{Q}$-image of FM_0. As usual, we have a Helmholtz decomposition of FM_0.

Lemma 2.7. The space FM_0 has a topological direct sum decomposition

$$FM_0 = PFM_0 \oplus QFM_0.$$

The space PFM_0 agrees with the space of all divergence free vector fields in FM_0.

Proof. Since $\mathbf{P}^2 = \mathbf{P}$ so that $\mathbf{PQ} = \mathbf{QP} = 0$ and since $\mathbf{P}$ is bounded in FM_0 the first statement is clear. If f satisfies $\mathrm{div} f = 0$ so that $\sum_{k=1}^{n} \hat{f}_k \lfloor \xi_k = 0$, then $\mathbf{Q}f = 0$ so $f \in PFM_0$. If $f \in PFM_0$ so that $f = \mathbf{P}f$, then $\mathrm{div} f = \mathrm{div}\mathbf{P}f = 0$ by a calculation of symbols. We shall estimate norms of several operators acting in vector valued FM spaces. For $f \in FM = (FM)^d$ we define

$$||f||_{FM} = (2\pi)^{-n/2}||\hat{f}||_{M^d}\ ,\ \ ||f||'_{FM} = (2\pi)^{-n/2}||\hat{f}||'_{M^d},$$

where $\hat{f} = (\hat{f}_1, \cdots, \hat{f}_d)$, $f = (f_1, \cdots, f_d)$. We shall state a vectorial version of Lemma 2.2. For $d \times d$ complex matrix valued function $\sigma = (\sigma_{jk})_{1 \le j,k \le d}$ we define

$$(\Sigma f)_j = \sum_{k=1}^{d} F^{-1}(\hat{f}_k \lfloor \sigma_{jk}) \qquad (j = 1, \cdots, d) \tag{2.5}$$

instead of (2.3). For $d \times d$ matrix A let $|A|$ be the operator norm form $\mathbb{C}^d$ to $\mathbb{C}^d$ equipped with standard inner product.

Lemma 2.8. (i) Assume that $\sigma = (\sigma_{jk}) \in C(\mathbb{R}^n \backslash \{0\})$ is bounded in $\mathbb{R}^n \backslash \{0\}$. Then Σ in (2.3) is a bounded linear operator in FM_0 with an estimate

$$||\Sigma f||_{FM^d} \le ||\ |\sigma|\ ||_{\infty} ||f||_{FM^d} \tag{2.6}$$

for $f \in FM_0$. If, moreover σ is continuous at the origin, then σ is a bounded linear operator in FM satisfying (2.6) for all $f \in FM$.

Proof. It suffices to estimate Fourier transforms. We notice that

$$\sum_{k=1}^{d} \hat{f}_k \lfloor \sigma_{jk} = |\hat{f}| \lfloor \sum_{k=1}^{d} h_k \sigma_{jk},$$

where h_k is the Radon-Nikodym derivative of $\hat{f}_k$ with respect to $|\hat{f}|$. Since $|h|(\xi) = 1$ for $|\hat{f}|$-a.e.ξ, by definition of σ

$$|\sum_{k=1}^{d} h_b(\xi)\sigma_{jk}(\xi)| \le |\sigma(\xi)|\ |h(\xi)| = |\sigma(\xi)|$$

for $h = (h_1, \cdots, h_d)$. We now obtain (2.6) by using the supremum norm of $|\sigma|$. □

Lemma 2.8 provides a stronger consequence for $\mathbf{P}$ or $\exp(\mathbf{S}t)$ than that follows from Lemma 2.3.

Lemma 2.9. (i) Let $\mathbf{P}$ be defined in the beginning of §2.5. Then $\mathbf{P}$ is bounded from FM_0^n to FM_0^n. Moreover,

$$||\mathbf{P}f||_{FM^n} \le ||f||_{FM^n} \quad \text{for all} \quad f \in FM_0^n.$$

(ii) Let $\mathbf{S} = \mathbf{PJP}$ be as in (2.1). Then $\mathbf{S}$ is bounded from FM_0^3 into itself and so is $exp(\mathbf{S}t)$. Moreover,

$$||\exp\ (\mathbf{S}t)f||_{FM^3} \le ||f||_{FM^3} \quad \text{for all} \quad f \in FM_0^3,\ t \in \mathbb{R},$$

where $\exp\ (\mathbf{S}t) = \sum_{k=0}^{\infty} \frac{1}{k!}(t\mathbf{S})^k$.

Proof. (i) Since $\sigma(P) = I - \xi \otimes \xi/|\xi|^2$, it is an orthogonal projection from $\mathbb{C}^n$ into itself. Thus its operator norm is one so that $||\ |\sigma|\ ||_\infty = 1$. From (2.6) it now follows the estimate of (i). The property $\mathbf{P}f \in FM_0^n$ for $f \in FM_0^n$ is clear.

(ii) Since $\sigma(\mathbf{S})$ is skew symmetric ([1], [3]), $\exp(\sigma(\mathbf{S})t)$ is an unitary matrix so $|\exp\ \sigma(\mathbf{S})t| = 1$ as a matrix. Thus (2.6) yields the desired estimate for $\exp(\mathbf{S}t)$. The boundedness of $\mathbf{S}$ follows from boundedness of its symbol with respect to ξ. It is clear that its image is in FM_0^3. □

3 Proof of Theorem 1.1

We work on the integral equation which is formally equivalent to (1.1) of the form

$$u(t) = \exp(t\Delta)\ \exp(-\Omega t\mathbf{S})u_0 + N(u,u)(t) \text{ for } t \ge 0. \tag{3.1}$$

Here $\exp(t\Delta) = e^{t\Delta}$ is the heat semigroup and

$$N(u,u)(t) = -\int_0^t \exp((t-s)\Delta)\ \exp(-\Omega(t-s)\mathbf{S})\mathbf{P}\ \mathrm{div}(u \otimes u)(s)ds.$$

There are two key estimates.
(i) estimate for linear term : By Lemma 2.5 and 2.9 (ii) we have

$$||\exp(t\Delta)\ \exp(-\Omega t\mathbf{S})u_0||_{FM^3} \le ||u_0||_{FM^3}.$$

(ii) estimate for nonlinear term : Since we have by Lemma 2.5 (iii)

$$\nabla\cdot\ \exp(-\Omega t\mathbf{S})\mathbf{P}\ \exp(t\Delta)F \in FM_0 \text{ for any } F=(F_{ij})_{1\le i,j\le 3}\in FM^{3\times 3},$$

applying Lemma 2.9 together with Lemma 2.5 (iii) yields

$$\begin{aligned}
& ||\exp(t\Delta)\ \exp(-\Omega t\mathbf{S})\mathbf{P}\ \text{div}\ F||_{FM^3} \quad (\text{div}\ F=\sum_{i=1}^{3}\partial_{x_i}F_{ij}) \\
\le\ & ||\exp(\Omega t\mathbf{S})\mathbf{P}\nabla\cdot\ \exp(t\Delta)F||_{FM^3} \\
\le\ & 1\cdot 1\cdot||\nabla\cdot\ \exp(t\Delta)F||_{FM^3} \\
\le\ & (2te)^{-1/2}\sum_{i=1}^{3}||F_i||_{FM^3} \quad , \quad F_i=(F_{i1},F_{i2},F_{i3})
\end{aligned}$$

for all $F\in FM^{3\times 3}$.

This estimate enables us to construct the solution of the integral equation (3.1) by a successive approximation $\{u_j\}_{j=1,2,\cdots}$ given by

$$\begin{aligned}
u_1(t) &= \exp(t\Delta)\ \exp(-\Omega t\mathbf{S})u_0 \\
u_{j+1}(t) &= u_1(t)+N(u_j,u_j)(t) \quad j=1,2,\cdots .
\end{aligned}$$

We now define

$$K_j(T)=\sup_{0\le s\le T}||u_j(s)||_{FM}, L_j(T)=\sup_{0\le s\le T}||u_j(s)-u_{j-1}(s)||_{FM}$$

for $T>0$. Applying (i), (ii), we have

$$K_{j+1}\le ||u_0||_{FM^3}+(2e)^{-1/2}\int_0^t(t-s)^{-1/2}\sum_{i=1}^{3}||F_i||_{FM^3}ds \quad (j\ge 1)$$

with $F_i=u_iu_j$. By Lemma 2.4 and the equivalence of norms we observe

$$\begin{aligned}
&||vu||_{FM^3}\le\sqrt{3}||vu||'_{FM^3} \\
&\le\sqrt{3}||v||_{FM}||u||'_{FM^3}\le\sqrt{3}||v||_{FM}||u||_{FM^3}
\end{aligned}$$

for $u\in FM^3,\ v\in FM$ so

$$\sum_{i=1}^{3}||F_i||_{FM^3}\le\sqrt{3}\sum_{i=1}^{3}||u_i||_{FM}||u||_{FM^3}\le 3||u||'_{FM^3}||u||_{FM^3}\le 3||u||^2_{FM^3}.$$

We thus obtain

$$K_{j+1}\le||u_0||_{FM^3}+3(2/e)^{-1/2}T^{1/2}K_j^2 \qquad (j\ge 1).$$

The remaining argument is now standard e.g. [3]. From this estimate one easily sees the uniform boundedness $\sup_{j\geq 1} K_j(T) \leq 2||u_0||_{FM}$ if $T < 1/\{4 \cdot 3(2/e)^{-1/2}||u_0||_{FM^3}\}^2$. Similar calculation for $u_{j+1} - u_j$ and the uniform boundedness give us

$$L_{j+1} \leq 2 \cdot 3(2/e)^{-1/2}(2||u_0||_{FM^3})T^{1/2}L_j \quad \text{for } j \geq 1,$$

that implies $L_{j+1}/L_j < 1/2$ if $T \leq 1/[2\{2 \cdot 3(2/e)^{-1/2}(2||u_0||_{FM^3})\}^2]$. Therefore, we see there exists a unique limit u such that $u_j \to u$ in $C([0,T_0];FM_0)$ as $j \to \infty$. It is easy to see that the limit u solves the equation. Uniqueness comes from similar calculation for the subtraction $w = u - v$. Theorem 1.1 has been proved

Remark 3.1. From the above proof the existence time of the solution u, denoted as T_0 in Theorem 1.1, is estimated from below as follows:

$$T_0 \geq 1/[2\{2 \cdot 3(2/e)^{-1/2}(2||u_0||_{FM^3})\}^2] = e/(576||u_0||^2_{FM^3}).$$

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A Appendix

We shall prove that the space FM_0 is strictly included in the Besov space $\dot{B}^0_{\infty,1}$.

Theorem A. The space FM_0 is continuously embedded in $\dot{B}^0_{\infty,1}$. Moreover, the inclusion $FM_0 \hookrightarrow \dot{B}^0_{\infty,1}$ is strict.

Proof. For $f \in FM_0$ we get

$$\begin{aligned} ||f||_{\dot{B}^0_{\infty,1}} &= \sum_{j=-\infty}^{\infty} ||\phi_j * f||_\infty = \sum_{j=-\infty}^{\infty} ||F^{-1}(\hat{\phi}_j \hat{f})||_\infty \\ &\leq C \sum_{j=-\infty}^{\infty} ||\hat{\phi}_j \hat{f}||_M \leq C \sum_{j=-\infty}^{\infty} ||\hat{\phi}_j \hat{f}||_{M(E_j)} \\ &\leq 3C||\hat{f}||_M = 3C||f||_{FM}. \end{aligned}$$

Here, $E_j = \{2^{j-1} \leq |\xi| \leq 2^{j+1}\}$. Hence the embedding is continuous. The inclusion is strict. We shall prove it for $n = 1$. Consider $f = h * e^{ix} i \text{ sgn } x \in L^\infty(\mathbb{R})$, where h is a smooth bounded function whose Fourier transform $\hat{h}$ is supported in $\{\xi \in \mathbb{R} \ ; \ |\xi - 1| \leq 1/2\}$. Here $\text{sgn}(s)$ is the signature function with value 1 if $s > 0$ and -1 otherwise. The Fourier transform $\hat{f}$ equals a constant multiple of $\hat{h}$ p.v. $1/(\xi - 1)$, which is not in M ; here p.v. denotes the Cauchy principal value. So f does not belong to FM. However, since the support of $\hat{f}$ is in $\{|\xi - 1| \leq 1/2\}$, away from the origin and infinity, the sum $\sum \phi_j * f$ in the Besov norm is finite. Since $f \in L^\infty(\mathbb{R})$, this implies $f \in \dot{B}^0_{\infty,1}$. For general n it suffices to consider $F = f(x_1) \cdots f(x_n)$ to observe that $F \notin FM$ but $F \in \dot{B}^0_{\infty,1}$. $\square$

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