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Ground State Energy of the Polaron in the Relativistic Quantum Electrodynamics

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Abstract

We consider the polaron model in the relativistic quantum electrodynamics(QED). We prove that the ground state energy of the model is finite for all values of the fine-structure constant and the ultraviolet cutoff Λ . Moreover we give an upper bound and a lower bound of the ground state energy.

Key words: relativistic QED; ground state energy, polaron model.

1 Introduction and Main Results

We consider the relativistic quantum electrodynamics(QED) for a fixed total momentum — the polaron model of the relativistic QED. The Hamiltonian, which describes a Dirac particle minimally coupled to the quantized radiation field, commutes with the total momentum operator and has a direct integral decomposition with respect to the total momentum operator([1][2]). Each fibre in this direct integral decomposition is just the Hamiltonian of the polaron we consider. The Hilbert space of the polaron model is defined by

$$\mathcal{F} := \mathbb{C}^4 \otimes \mathcal{F}_b(L^2(\mathbb{R}^3 \times \{1, 2\})), \quad (1)$$

where

$$\mathcal{F}_b(L^2(\mathbb{R}^3 \times \{1, 2\})) := \bigoplus_{n=0}^{\infty} \left[\bigotimes_s^n (L^2(\mathbb{R}^3 \times \{1, 2\})) \right] \quad (2)$$

is the photon Fock space (\bigotimes_s^n denotes n -fold symmetric tensor product). For a closable operator T on $L^2(\mathbb{R}^3 \times \{1, 2\})$ we denote by $d\Gamma_b(T)$, the second quantization operator of T (see [4]). Let $a(f)$, $f \in L^2(\mathbb{R}^3 \times \{1, 2\})$ be the annihilation operator on the photon Fock space. For a function $g_j \in L^2(\mathbb{R}^3 \times \{1, 2\})$, $j = 1, 2, 3$, we set

$$A_j := a(g_j) + a(g_j)^*, \quad j = 1, 2, 3. \quad (3)$$

Let $\{\alpha_1, \alpha_2, \alpha_3, \beta\}$ be the 4×4 -Dirac matrices, i.e., $\{\alpha_i, \alpha_j\} = 2\delta_{i,j}$, $\{\alpha_i, \beta\} = 0$, $\beta^2 = 1$, $i, j = 1, 2, 3$. Here $\{A, B\} := AB + BA$. For three objects a_1, a_2, a_3 we set $\mathbf{a} = (a_1, a_2, a_3)$, and write $\mathbf{a} \cdot \mathbf{b} := \sum_{j=1}^3 a_j b_j$, provided that $a_j b_j$ and $\sum_{j=1}^3 a_j b_j$ are defined.

The Hamiltonian of the polaron model we consider is

$$H(\mathbf{p}) := \boldsymbol{\alpha} \cdot \mathbf{p} + M\beta + d\Gamma_b(\omega) - \boldsymbol{\alpha} \cdot d\Gamma_b(\mathbf{k}) - q\boldsymbol{\alpha} \cdot \mathbf{A}, \quad (4)$$

where $\mathbf{p} \in \mathbb{R}^3$ is the fixed total momentum, $M \geq 0$ is the mass of the Dirac particle, $q \in \mathbb{R}$ is a constant proportional to the fine-structure constant, and $\omega = |\mathbf{k}|$ is the 1-photon Hamiltonian ($\mathbf{k} \in \mathbb{R}^3$). Note that we omit the symbol \otimes between the Hilbert space for the Dirac matrices \mathbb{C}^4 and the photon Fock space $\mathcal{F}_b(L^2(\mathbb{R}^3 \times \{1, 2\}))$. The most important example of $\{g_j\}_{j=1}^3$ is of the form

$$f_j(\mathbf{k}, r) := \frac{\chi_\Lambda(\mathbf{k})}{|\mathbf{k}|^{1/2}} e_j^{(r)}(\mathbf{k}), \quad (5)$$

where the measurable functions $\mathbf{e}^{(1)}(\mathbf{k})$, $\mathbf{e}^{(2)}(\mathbf{k})$ are the polarization vectors:

$$\mathbf{k} \cdot \mathbf{e}^{(r)}(\mathbf{k}) = 0, \quad \mathbf{e}^{(r)}(\mathbf{k}) \cdot \mathbf{e}^{(s)}(\mathbf{k}) = \delta_{r,s}, \quad \text{a.e. } \mathbf{k} \in \mathbb{R}^3, \quad r, s = 1, 2, \quad (6)$$

and $\chi_\Lambda(\mathbf{k})$ is the characteristic function of the ball $\{\mathbf{k} \in \mathbb{R}^3 \mid |\mathbf{k}| < \Lambda\}$, $\Lambda > 0$.

We define

$$E_0(\mathbf{p}) := \inf_{\substack{\Psi \in \text{Dom}(H(\mathbf{p})) \\ \|\Psi\|=1}} \langle \Psi, H(\mathbf{p})\Psi \rangle \quad (7)$$

the ground state energy of $H(\mathbf{p})$, where ‘‘Dom’’ means operator domain. We assume the following:

Hypothesis I. $g_j \in \text{Dom}(\omega^{-1/2}) \cap \text{Dom}(\omega)$, $\langle g_j, g_\ell \rangle \in \mathbb{R}$, $j, \ell = 1, 2, 3$.

It should be noted that it is highly non-trivial whether or not $E_0(\mathbf{p})$ is finite, because $H(\mathbf{p})$ contains the term $-\boldsymbol{\alpha} \cdot d\Gamma_b(\mathbf{k})$. This is the main problem discussed in the present paper.

We prove that the ground state energy $E_0(\mathbf{p})$ is finite under suitable conditions:

Theorem 1.1. *Assume Hypothesis I, and*

$$G(\mathbf{g}) := \inf_{\mathbf{k} \in \mathbb{R}^3 \setminus \{0\}} \frac{1}{|\mathbf{k}|} \sum_{r=1,2} \int_{\mathbb{R}^3} \frac{|\mathbf{k} \cdot \mathbf{g}(\mathbf{k}', r)|^2}{|\mathbf{k}| |\mathbf{k}'| - \mathbf{k} \cdot \mathbf{k}'} d\mathbf{k}' < \infty. \quad (8)$$

Then, the ground state energy $E_0(\mathbf{p})$ is finite:

$$E_0(\mathbf{p}) > -\infty. \quad (9)$$

In particular, if $g_j = f_j$, $j = 1, 2, 3$, the ground state energy $E_0(\mathbf{p})$ is finite.

For a vector $u \in \mathbb{C}^4$, we set $a_j := \langle u, \alpha_j u \rangle_{\mathbb{C}^4}$, and

$$\mathcal{E}(\Lambda, u) := \mathbf{p} \cdot \mathbf{a} + M \langle u, \beta u \rangle + 4\pi\Lambda q^2 \frac{1 - |\mathbf{a}|^2}{|\mathbf{a}|} \log \left(\frac{1 + |\mathbf{a}|}{1 - |\mathbf{a}|} \right) - 4\pi\Lambda q^2.$$

In the physical case (i.e. the function g_j 's are given by (5)), the lower bound of $E_0(\mathbf{p}) + \sqrt{|\mathbf{p}|^2 + M^2}$ are proportional to Λ :

Theorem 1.2. *Let $g_j = f_j$, $j = 1, 2, 3$. Then*

$$C_1 \Lambda - \sqrt{|\mathbf{p}|^2 + M^2} \leq E_0(\mathbf{p}), \quad (10)$$

$$E_0(\mathbf{p}) \leq C_2(\Lambda) \quad (11)$$

where

$$C_1 := \inf_{\epsilon, \epsilon' > 0} \left\{ \epsilon |q| + 16\pi q^2 + \left(\epsilon' + \frac{1}{\epsilon'} \right) 4\pi q^2, \sqrt{\frac{4\pi |q|}{3\epsilon} + \left(1 + \frac{1}{\epsilon'} \right) 4\pi q^2} \right\}, \quad (12)$$

$$C_2(\Lambda) := \inf_{\substack{u \in \mathbb{C}^4 \\ \|u\|_{\mathbb{C}^4} = 1}} \mathcal{E}(\Lambda, u). \quad (13)$$

2 Proof of Theorem 1.1 and 1.2

Lemma 2.1. *Let A be a positive self-adjoint operator on a Hilbert space \mathcal{H} . Let B be a symmetric operator with $\text{Dom}(A) \subset \text{Dom}(B)$ and*

$$\|B\Psi\| \leq \|A\Psi\|, \quad \Psi \in \text{Dom}(A). \quad (14)$$

Then, for all $\Psi \in D(A)$, $\langle \Psi, (A + B)\Psi \rangle \geq 0$.

Proof. By the Kato-Rellich theorem([4]), for all $\epsilon \in (-1, 1)$, $A + \epsilon B$ is self-adjoint and $A + \epsilon B \geq 0$. Therefore $\langle \Psi, (A + B)\Psi \rangle \geq 0$ for all $\Psi \in \text{Dom}(A)$. \blacksquare

By this lemma, it suffices to show that there exists a constant $E \geq 0$ such that

$$\|(\text{d}\Gamma_b(\omega) + E)\Psi\|^2 \geq \|\boldsymbol{\alpha} \cdot (\text{d}\Gamma_b(\mathbf{k}) + q\mathbf{A})\Psi\|^2, \quad \Psi \in \text{Dom}(\text{d}\Gamma_b(\omega)). \quad (15)$$

We use the following representation for $\boldsymbol{\alpha}$ -matrices:

$$\alpha_j = \begin{bmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{bmatrix}, \quad j = 1, 2, 3,$$

with $(\sigma_1, \sigma_2, \sigma_3)$ being the Pauli matrices. Using anticommutation relation of α_1, α_2 and α_3 , we have

$$\|\boldsymbol{\alpha} \cdot (\text{d}\Gamma_b(\mathbf{k}) + q\mathbf{A})\Psi\|^2 = \sum_{j=1}^3 \|(\text{d}\Gamma_b(k_j) + qA_j)\Psi\|^2 - q\langle \Psi, \mathbf{S} \cdot [a(i\mathbf{k} \times \mathbf{g}) + a(i\mathbf{k} \times \mathbf{g})^*]\Psi \rangle, \quad (16)$$

where $S_j = \sigma_j \oplus \sigma_j$, $j = 1, 2, 3$. The Hilbert space $\mathbb{C}^4 \otimes [\otimes_s^n L^2(\mathbb{R}^3 \times \{1, 2\})]$ is naturally embedded in $L^2(\mathbb{R}^3 \times \{1, 2\}; \mathbb{C}^4 \otimes [\otimes_s^{(n-1)} L^2(\mathbb{R}^3 \times \{1, 2\})])$. For a vector $\Psi \in \mathbb{C}^4 \otimes [\otimes_s^n L^2(\mathbb{R}^3 \times \{1, 2\})]$, we denote its value at point $(\mathbf{k}, r) \in \mathbb{R}^3 \times \{1, 2\}$ by $\Psi(\mathbf{k}, r, \cdot)$.

For $\Psi = (\Psi^{(n)})_{n=0}^\infty \in \mathcal{F}$, we define

$$a^{(r)}(\mathbf{k})\Psi := \left(\Psi^{(1)}(\mathbf{k}, r), \sqrt{2}\Psi^{(2)}(\mathbf{k}, r, \cdot), \dots, \sqrt{n}\Psi^{(n)}(\mathbf{k}, r, \cdot), \dots \right) \in \mathcal{F}, \quad \mathbf{k} \in \mathbb{R}^3, r = 1, 2, \quad (17)$$

a Fock space valued function. This operator $a^{(r)}(\mathbf{k})$ is the distributional kernel of the annihilation operator.

Lemma 2.2. *For all $\Psi \in \text{Dom}(\text{d}\Gamma_b(\omega))$ and $\epsilon > 0$, the following inequality holds:*

$$|q\langle \Psi, \mathbf{S} \cdot [a(i\mathbf{k} \times \mathbf{g}) + a(i\mathbf{k} \times \mathbf{g})^*]\Psi \rangle| \leq |q|\epsilon\langle \Psi, \text{d}\Gamma_b(\omega)\Psi \rangle + \frac{|q|}{\epsilon}\langle \mathbf{g}, \omega\mathbf{g} \rangle \|\Psi\|^2, \quad (18)$$

where $\langle \mathbf{g}, \omega\mathbf{g} \rangle := \sum_{j=1}^3 \langle g_j, \omega g_j \rangle$.

Proof.

$$\begin{aligned}
\text{l.h.s of (18)} &= 2|q| \left| \operatorname{Re} \int_{\mathbb{R}^3} \langle \Psi, -i\mathbf{S} \cdot (\mathbf{k} \times \mathbf{g}(\mathbf{k}, r)) a^{(r)}(\mathbf{k}) \Psi \rangle d\mathbf{k} \right| \\
&\leq 2|q| \int_{\mathbb{R}^3} \|\mathbf{S} \cdot (\mathbf{k} \times \mathbf{g}(\mathbf{k}, r)) \Psi\| \|a^{(r)}(\mathbf{k}) \Psi\| d\mathbf{k} \\
&= 2|q| \int_{\mathbb{R}^3} |\mathbf{k}|^{1/2} |\mathbf{g}(\mathbf{k}, r)| \cdot \|\mathbf{k}|^{1/2} a^{(r)}(\mathbf{k}) \Psi\| \cdot \|\Psi\| d\mathbf{k} \\
&\leq 2|q| \langle \mathbf{g}, \omega \mathbf{g} \rangle^{1/2} \left[\int_{\mathbb{R}^3} \|\mathbf{k}|^{1/2} a^{(r)}(\mathbf{k}) \Psi\|^2 \right]^{1/2} \|\Psi\| \\
&\leq |q| \epsilon \langle \Psi, d\Gamma_b(\omega) \Psi \rangle + \frac{|q|}{\epsilon} \langle \mathbf{g}, \omega \mathbf{g} \rangle \|\Psi\|^2,
\end{aligned}$$

where $\int := \sum_{r=1,2} \int$. ■

Lemma 2.3. *For all $\Psi \in \operatorname{Dom}(d\Gamma_b(\omega))$ and $\epsilon > 0$, the following inequality holds:*

$$\langle \Psi, \mathbf{A}^2 \Psi \rangle \leq \left(2 + \epsilon + \frac{1}{\epsilon} \right) \langle \omega^{-1/2} \mathbf{g}, \omega^{-1/2} \mathbf{g} \rangle \langle \Psi, d\Gamma_b(\omega) \Psi \rangle + \left(1 + \frac{1}{\epsilon} \right) \langle \mathbf{g}, \mathbf{g} \rangle \|\Psi\|^2. \quad (19)$$

Proof.

$$\begin{aligned}
\langle \Psi, \mathbf{A}^2 \Psi \rangle &\leq \sum_{j=1}^3 \left[(1 + \epsilon) \|a(g_j) \Psi\|^2 + \left(1 + \frac{1}{\epsilon} \right) \|a(g_j)^* \Psi\|^2 \right] \\
&\leq \sum_{j=1}^3 \left[(1 + \epsilon) \|\mathbf{k}|^{-1/2} g_j\|^2 \cdot \|d\Gamma_b(\omega)^{1/2} \Psi\|^2 \right. \\
&\quad \left. + \left(1 + \frac{1}{\epsilon} \right) \|\mathbf{k}|^{-1/2} g_j\|^2 \cdot \|d\Gamma_b(\omega)^{1/2} \Psi\|^2 + \left(1 + \frac{1}{\epsilon} \right) \|g_j\|^2 \cdot \|\Psi\|^2 \right].
\end{aligned}$$
■

The following Lemma is the most important fact in the proof of Theorem 1.1:

Lemma 2.4. *For all $\Psi \in \operatorname{Dom}(d\Gamma_b(\omega))$, the following inequality holds:*

$$\begin{aligned}
&\|d\Gamma_b(\omega) \Psi\|^2 - \sum_{j=1}^3 \|d\Gamma_b(k_j) \Psi\|^2 - q \langle d\Gamma_b(\mathbf{k}) \Psi, \mathbf{A} \Psi \rangle - q \langle \mathbf{A} \Psi, d\Gamma_b(\mathbf{k}) \Psi \rangle \\
&\geq -4q^2 G(\mathbf{g}) \langle \Psi, d\Gamma_b(\omega) \Psi \rangle - q \langle \Psi, (a(\mathbf{k} \cdot \mathbf{g})^* + a(\mathbf{k} \cdot \mathbf{g})) \Psi \rangle.
\end{aligned} \quad (20)$$

Proof. We define

$$F := \frac{\mathbf{k} \cdot \mathbf{g}(\mathbf{k}', \mu)}{|\mathbf{k}| \cdot |\mathbf{k}| - \mathbf{k} \cdot \mathbf{k}'} \quad (21)$$

For all $\Psi \in \text{Dom}(d\Gamma_b(\omega))$, we have

$$\begin{aligned} \text{l.h.s of (20)} &= \int_{\mathbb{R}^3} d\mathbf{k} \int_{\mathbb{R}^3}' d\mathbf{k}' (|\mathbf{k}| \cdot |\mathbf{k}'| - \mathbf{k} \cdot \mathbf{k}') \|(b - 2qF)a\Psi\|^2 \\ &\quad - q\langle \Psi, [a(\mathbf{k} \cdot \mathbf{g}) + a(\mathbf{k} \cdot \mathbf{g})^*] \Psi \rangle \\ &\quad - 4q^2 \int_{\mathbb{R}^3} d\mathbf{k} \left[\int_{\mathbb{R}^3}' d\mathbf{k}' (|\mathbf{k}| \cdot |\mathbf{k}'| - \mathbf{k} \cdot \mathbf{k}')^{-1} |\mathbf{k} \cdot \mathbf{g}(\mathbf{k}', \mu)|^2 \right] \|a\Psi\|^2, \end{aligned}$$

where $a := a^{(r)}(\mathbf{k})$, $b := a^{(\mu)}(\mathbf{k}')$, and $\int' := \sum_{\mu=1,2} \int$. Since $|\mathbf{k}| \cdot |\mathbf{k}'| - \mathbf{k} \cdot \mathbf{k}' \geq 0$, the inequality (20) holds. \blacksquare

Proof of Theorem 1.1. Using Lemmas 2.2 - 2.4, we get

$$\begin{aligned} (d\Gamma_b(\omega) + E)^2 &- \sum_{j=1}^3 (d\Gamma_b(k_j) + qA_j)^2 - q\mathbf{S} \cdot [a(i\mathbf{k} \times \mathbf{g}) + a(i\mathbf{k} \times \mathbf{g})^*] \\ &\geq 2Ed\Gamma_b(\omega) + E^2 - 4q^2G(\mathbf{g})d\Gamma_b(\omega) - q[a(\mathbf{k} \cdot \mathbf{g})^* + a(\mathbf{k} \cdot \mathbf{g})] - |q|d\Gamma(\omega) \\ &\quad - |q|\langle \mathbf{g}, \omega \mathbf{g} \rangle - 4\langle \omega^{-1/2} \mathbf{g}, \omega^{-1/2} \mathbf{g} \rangle d\Gamma_b(\omega) - 2\langle \mathbf{g}, \mathbf{g} \rangle, \end{aligned}$$

in the sense of quadratic form on $\text{Dom}(d\Gamma_b(\omega))$. Since $a(\mathbf{k} \cdot \mathbf{g}) + a(\mathbf{k} \cdot \mathbf{g})^*$ is $d\Gamma_b(\omega)^{1/2}$ -bounded, for a large $E > 0$ we have

$$(d\Gamma_b(\omega) + E)^2 - \sum_{j=1}^3 (d\Gamma_b(k_j) + qA_j)^2 - q\mathbf{S} \cdot [a(i\mathbf{k} \times \mathbf{g}) + a(i\mathbf{k} \times \mathbf{g})] \geq 0.$$

By Lemma 2.1, for a large $E \geq 0$, we obtain

$$d\Gamma_b(\omega) - \boldsymbol{\alpha} \cdot d\Gamma_b(\mathbf{k}) - q\boldsymbol{\alpha} \cdot \mathbf{A} \geq -E, \quad (22)$$

in the sense of quadratic form on $\text{Dom}(d\Gamma_b(\omega))$. This inequality implies that $E_0(\mathbf{p})$ is finite.

Next we show that $G(\mathbf{f}) < \infty$ if $g_j = f_j (j = 1, 2, 3)$. By the definitions of $\mathbf{e}^{(r)}(\mathbf{k})$, the vectors $\mathbf{k}/|\mathbf{k}|, \mathbf{e}^{(1)}(\mathbf{k}), \mathbf{e}^{(2)}(\mathbf{k})$ are the orthonormal basis of \mathbb{C}^3 . Therefore

$$G(\mathbf{f}) = \sup_{\mathbf{k} \in \mathbb{R}^3 \setminus \{0\}} \frac{1}{|\mathbf{k}|} \int_{\mathbb{R}^3} \frac{\chi_\Lambda(\mathbf{k}') d\mathbf{k}'}{|\mathbf{k}| |\mathbf{k}'| - \mathbf{k} \cdot \mathbf{k}'} \cdot \frac{1}{|\mathbf{k}'|} \left[|\mathbf{k}|^2 - \frac{(\mathbf{k} \cdot \mathbf{k}')^2}{|\mathbf{k}'|^2} \right] = \int_{\mathbb{R}^3} \frac{\chi_\Lambda(\mathbf{k}')}{|\mathbf{k}'|^2} d\mathbf{k}' = 4\pi\Lambda. \quad \blacksquare$$

Proof of Theorem 1.2. First we show (10). We set $g_j = f_j (j = 1, 2, 3)$. It is easy to see that

$$H(\mathbf{p}) \geq -\sqrt{|\mathbf{p}|^2 + M^2} + d\Gamma_b(\omega) - \boldsymbol{\alpha} \cdot d\Gamma_b(\mathbf{k}) - q\boldsymbol{\alpha} \cdot \mathbf{A}. \quad (23)$$

By the definition of $\mathbf{e}^{(r)}(\mathbf{k})$, we have $\mathbf{k} \cdot \mathbf{f}(\mathbf{k}, r) = 0$ ($\mathbf{k} \in \mathbb{R}^3, r = 1, 2$). Therefore, using Lemmas 2.2 – 2.4, we have

$$\begin{aligned} & (\mathrm{d}\Gamma_{\mathbf{b}}(\omega) + C_1\Lambda)^2 - (\mathrm{d}\Gamma_{\mathbf{b}}(\mathbf{k}) + q\mathbf{A})^2 - q\mathbf{S} \cdot [a(i\mathbf{k} \times \mathbf{f}) + a(i\mathbf{k} \times \mathbf{f})^*] \\ & \geq \left(2C_1\Lambda - 2|q|\epsilon\Lambda - 4q^2G(\mathbf{f}) - \left(2 + \epsilon' + \frac{1}{\epsilon'} \right) \langle \omega^{-1/2}\mathbf{f}, \omega^{-1/2}\mathbf{f} \rangle \right) \mathrm{d}\Gamma_{\mathbf{b}}(\omega) \\ & \quad + C_1^2\Lambda^2 - \frac{|q|}{2\epsilon\Lambda} \langle \mathbf{f}, \omega\mathbf{f} \rangle - \left(1 + \frac{1}{\epsilon'} \right) q^2 \langle \mathbf{f}, \mathbf{f} \rangle, \quad \epsilon, \epsilon' > 0. \end{aligned} \quad (24)$$

It is easy to see that $\langle \omega^{-1/2}\mathbf{f}, \omega^{-1/2}\mathbf{f} \rangle = 8\pi\Lambda$, $\langle \mathbf{f}, \omega\mathbf{f} \rangle = 8\pi\Lambda^3/3$, $\langle \mathbf{f}, \mathbf{f} \rangle = 4\pi\Lambda^2$. Hence, by the definition of C_1 , the left hand side of (24) is positive for suitable $\epsilon, \epsilon' > 0$. Thus, using Lemma 2.1 (and (16)), we have

$$H(\mathbf{p}) \geq -\sqrt{|\mathbf{p}|^2 + M^2} - C_1\Lambda. \quad (25)$$

For normalized vectors $u \in \mathbb{C}^4$, $\psi \in \mathrm{Dom}(\mathrm{d}\Gamma_{\mathbf{b}}(\omega))$ we define

$$\begin{aligned} a_j & := \langle u, \alpha_j u \rangle, \quad h(\mathbf{a}) := \mathrm{d}\Gamma_{\mathbf{b}}(\omega - \mathbf{a} \cdot \mathbf{k}) - q\mathbf{a} \cdot \mathbf{A}, \\ \Psi & := u \otimes \psi \in \mathcal{F}. \end{aligned}$$

Note that $\omega - \mathbf{a} \cdot \mathbf{k} \geq 0$ and $\omega - \mathbf{a} \cdot \mathbf{k}$ is injective as a multiplication operator. We have

$$\langle \Psi, H(\mathbf{p})\Psi \rangle = \mathbf{a} \cdot \mathbf{p} + M\langle u, \beta u \rangle + \langle \psi, h(\mathbf{a})\psi \rangle \quad (26)$$

Since $h(\mathbf{a})$ is a van Hove type Hamiltonian, we have

$$\begin{aligned} \inf \sigma(h(\mathbf{a})) & = -q^2 \| (|\mathbf{k}| - \mathbf{a} \cdot \mathbf{k})^{-1/2} \mathbf{a} \cdot \mathbf{f} \|^2 \\ & = -4\pi\Lambda q^2 + q^2(1 - |\mathbf{a}|^2) \int_{\mathbb{R}^3} \mathrm{d}\mathbf{k} \frac{\chi_{\Lambda}(\mathbf{k})}{|\mathbf{k}|^2 - (\mathbf{a} \cdot \mathbf{k})^2} \\ & = -4\pi\Lambda q^2 + 2\pi\Lambda q^2(1 - |\mathbf{a}|^2) \frac{1}{|\mathbf{a}|} \log \left(\frac{1 - |\mathbf{a}|}{1 + |\mathbf{a}|} \right), \end{aligned}$$

where σ means the spectrum (e.g. [3]). Thus we have

$$E_0(\mathbf{p}) \leq \inf_{u \in \mathbb{C}^4; \|u\|=1} \inf_{\psi \in \mathrm{Dom}(\mathrm{d}\Gamma_{\mathbf{b}}(\omega))} \langle \Psi, H(\mathbf{p})\Psi \rangle = \inf_{u \in \mathbb{C}^4; \|u\|=1} \mathcal{E}(u)$$

■

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