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Instructions for use

On a uniform approximation of motion by anisotropic curvature by the Allen–Cahn equations

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Abstract

The convergence of solutions of anisotropic Allen–Cahn equations is studied when the interface thickness parameter (denoted by ε) tends to zero. It is shown that the convergence to a level set solution of the corresponding anisotropic interface equations is uniform with respect to the derivatives of a surface energy density function. As an application a crystalline motion of interfaces is shown to be approximated by anisotropic Allen–Cahn equations.

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1. Introduction

In this paper we consider an anisotropic Allen–Cahn equation with kinetic term. The convergence of solutions of anisotropic Allen–Cahn equations is already proved by [ElS1] (in the case the kinetic term is isotropic), [ElPS], and [ElS2]. However, their estimate of the convergence depend on the derivative of a surface energy density function which denotes an anisotropy of an equilibrium form of interfaces. In this paper we obtain the uniform estimate of the convergence with respect to the derivative of a surface energy density function. One of applications of our result is approximation of a crystalline motion of interfaces by anisotropic Allen–Cahn equations. We also propose the way to approximate a crystalline motion of interfaces by anisotropic Allen–Cahn equations.

An anisotropic Allen–Cahn equation is proposed by [MWBCS]. We consider the functional of the form

$$F_\varepsilon(v) = \int_{\mathbf{R}^n} \left[\frac{1}{2}\gamma(\nabla v)^2 + \frac{1}{\varepsilon^2}(W(v) - \varepsilon\lambda f v) \right] dx.$$

Here $\gamma \in C^2(\mathbf{R}^n \setminus \{0\})$ is positive on S^{n-1}, convex, positively homogeneous of degree one. Moreover, we assume that γ^2 is strictly convex. The function W is a double-well potential of the form $W(v) = (v^2 - 1)^2/2$. The quantity λ is a normalization constant determined by W. The quantity f is a given constant. We consider a weighted L^2-gradient flow of this functional, and obtain an anisotropic Allen–Cahn equation. Its explicit form is

$$\beta(\nabla v)\partial_t v - \mathrm{div}\,\gamma(\nabla v)\xi(\nabla v) + \frac{1}{\varepsilon^2}(W'(v) - \varepsilon\lambda f) = 0. \tag{1.1}$$

Here $\beta \in C(\mathbf{R}^n \setminus \{0\})$ is a positive on S^{n-1} and positively homogeneous of degree zero, and $\xi = D\gamma = (\partial_{p_1}\gamma(p), \ldots, \partial_{p_n}\gamma(p))$ for $p = (p_1, \ldots, p_n)$. A formal asymptotic analysis provided by [MWBCS], [WM] and [BP1] (the case $\beta \equiv 1$) says that the internal transition layer of (1.1) approximates the evolving interface $\{\Gamma_t\}_{t \geq 0}$ under the evolution law of the form

$$\beta(\mathbf{n})V = -\gamma(\mathbf{n})\{\mathrm{div}_{\Gamma_t}\xi(\mathbf{n}) + f\} \quad \text{on } \Gamma_t, \tag{1.2}$$

where $\mathbf{n}$ denotes the outer unit normal vector field of Γ_t, V denotes the normal velocity in the direction of $\mathbf{n}$, and the divergence operator in this equation denotes the surface divergence on Γ_t. The constant λ is taken so that the multiple constant in front of f in (1.2) equals one. Physically, the function γ is called a surface energy density, which induces an anisotropy of the equilibrium form of interfaces. The function ξ is called the Cahn–Hoffman vector. The function β expresses an anisotropy of kinetics. The quantity f is a driving force of the evolution. The quantity γ/β is called mobility.

If the initial data $v(x, 0)$ of (1.1) is positive in a region O_0 enclosed by Γ_0 and negative in $\mathbf{R}^n \setminus (O_0 \cup \Gamma_0)$, then one expects that

$$v \longrightarrow \begin{cases} +1 & \text{in a region } O_t \text{ enclosed by } \Gamma_t, \\ -1 & \text{in } \mathbf{R}^n \setminus (O_t \cap \Gamma_t) \end{cases} \tag{1.3}$$

locally uniformly as $\varepsilon \to 0$. This fact is rigorously proved by [ElS1] locally in time at least if the initial interface is smooth. Using a level set method due to [CGG1] and [ES] the authors of [ElPS] and [ElS2] proved (1.3) globally-in-time by interpreting Γ_t as a generalized solution of (1.2). They introduced a signed anisotropic distance function from Γ_t as outlined by [BP2] (see section 3). By using this distance, they constructed a sub- and supersolution of (1.1) to prove the convergence (1.3).

We here note that their convergence results depend on the smoothness of γ. One can find in [ElS2] that the way to determine ε for the estimate to obtain (1.3) depends at least on the 2nd derivatives of γ. Physically, however, there is a situation such that γ is not smooth so that an equilibrium form of interface of (1.2) may have a flat portion called a facet. If one tries to consider such a situation by (1.1) with γ_a approximating nonsmooth γ, their results are not enough.

In this paper we will show the convergence of internal transition layer is in some sense '*uniform*' with respect to derivatives of γ provided that $\gamma, 1/\gamma, \beta, 1/\beta$ on the unit

sphere is bounded. No control of derivatives of γ is necessary. This gives a way to approximate crystalline motion [T], [AG] in the plane by an anisotropic Allen-Cahn type equation in conjunction with a general level set method for nondifferentiable γ in [GG4], [GG5]. This will be explained in §2.5 as an application of our main result. In [BGN] anisotropic Allen–Cahn equations with crystalline γ and $\beta \equiv 1$ is considered. They derived even convergence rate of internal layer of the Allen–Cahn equation when the limit evolution is a crystalline motion. By the assumption $\beta \equiv 1$, (2.6) is considered as a variational inequality. (Several examples of solution are proposed in [TC].) Although we mollify γ and β, one advantage of our theory is that anisotropic β can be handled. Moreover, our uniform convergence result itself holds for arbitrary dimensional spaces. We approximate nonsmooth γ by smoother γ^τ while [BGN] studied the Allen–Cahn equation with nonsmooth γ.

The difficulty treating (1.1) directly is that (1.1) does not enjoy a comparison principle. This is caused by singularities at $\nabla v = 0$ which are due to nonconstant kinetic factor β. This difficulty is overcome in [ElS2] by adjusting a definition of solution to have a comparison principle. In this paper we overcome the difficulty caused by singularities of β by a way different from [ElS2]. We introduce a modified equation of (1.1) to remove singularities. The advantage of our idea is that the usual theory of viscosity solutions is available for a modified equation. We prove that the solution of a modified equation satisfies (1.3) and the convergence is '*uniform*' with respect to derivatives of γ.

The basic strategy of the proof of (1.3) is a combination of the method of [ESS] and [ElS2]. However, we need to estimate the time derivative of an anisotropic distance function in a different way. We construct a viscosity supersolution of (1.1) for estimate to obtain the convergence result by combining three ingredients: a distance function induced by Finsler geometry as in [BP2], its truncation as in [ESS] and the traveling wave as in [BSS]. The key estimate why we are able to prove the uniform convergence result with respect to the modulus of derivative of γ is in an estimate of the time derivative of a distance function from Γ_t. Although the time derivative is estimated by [ElS2], their bound depends on the second derivatives of γ on S^{n-1}. In this paper we will prove such an estimate by using a duality between γ and a support function of $\{p \in \mathbf{R}^n ; \gamma(p) \leq 1\}$ so that no derivatives of γ are involved.

Recently, [BS] and [BDL] provide the geometrical approach to approximate the motion of interfaces. However, their method do not provide our uniform convergence.

Finally, we note that, for the isotropic case ($\beta(p) \equiv 1$, $\gamma(p) = |p|$), the convergence problem has been well studied in various contexts, e.g., [BK], [C], [ESS], [BSS], [I], [S], etc.

2. Main Result

2.1. Equations

We now recall an anisotropic mean curvature flow. Let $\{\Gamma_t\}_{t \geq 0}$ be a family of closed hypersurfaces in $\mathbf{R}^n$. We consider an evolution law for Γ_t of the form

$$\beta(\mathbf{n})V = -\gamma(\mathbf{n})\{\mathrm{div}_{\Gamma_t}\xi(\mathbf{n}) + f\} \quad \text{on } \Gamma_t, \tag{2.1}$$

where V denotes the normal velocity of the surface Γ_t and $\mathbf{n}$ denotes the outer unit normal vector field of Γ_t. In this paper we assume that

(β1) $\beta \in C(\mathbf{R}^n \setminus \{0\})$,

(β2) β is positively homogeneous of degree 0,

(β3) there exists a positive constant Λ_β satisfying $\Lambda_\beta^{-1} \leq \beta \leq \Lambda_\beta$ on S^{n-1},

(γ1) $\gamma \in C^2(\mathbf{R}^n \setminus \{0\})$,

(γ2) γ is positively homogeneous of degree 1,

(γ3) there exists a positive constant Λ_γ satisfying $\Lambda_\gamma^{-1} \leq \gamma \leq \Lambda_\gamma$ on S^{n-1},

(γ4) γ is convex,

(γ5) $\alpha := \gamma^2/2$ is strictly convex,

(f1) f is a given constant satisfying $|f| \leq \Lambda_f$ with some $\Lambda_f > 0$,

(ε1) $\varepsilon \in (0, \bar{\varepsilon})$, where $\bar{\varepsilon}$ is such that the function $\sigma \mapsto W'(\sigma) - \varepsilon\lambda\Lambda_f$ has exactly three zeros,

where S^{n-1} is a unit sphere. The vector field ξ is the gradient field of γ i.e., $\xi = D\gamma = (\partial_{p_1}\gamma, \ldots, \partial_{p_n}\gamma)$, $\partial_{p_i}\gamma = \partial\gamma/\partial p_i$, $1 \leq i \leq n$. The divergence operator in (2.1) denotes the surface divergence on Γ_t. In this paper, we only consider the driving force term f is constant.

A level set formulation for (2.1) gives one of generalized notations of the motion of Γ_t (see [CGG1]). We introduce an auxiliary function $u: \mathbf{R}^n \times [0, T) \to \mathbf{R}$ and define

$$\Gamma_t = \{x \in \mathbf{R}^n;\ u(x, t) = 0\}. \tag{2.2}$$

The level set equation obtained from (2.1) is of the form

$$\beta(\nabla u)\partial_t u - \gamma(\nabla u)\{\mathrm{div}\,\xi(\nabla u) + f\} = 0 \quad \text{in } \mathbf{R}^n \times (0, T). \tag{2.3}$$

Here div denotes the divergence in $\mathbf{R}^n$, and ∇ denotes the spatial derivatives, i.e., $\nabla v = (\partial_{x_1}v, \ldots, \partial_{x_n}v)$, so we distinguish between the differential operator D and spatial derivative ∇. We define that $\{\Gamma_t\}_{t \in [0, T)}$ is a generalized solution of (2.1) if Γ_t is given by (2.2) for an auxiliary function $u \in C(\mathbf{R}^n \times [0, T))$ which is a viscosity supersolution of (2.3).

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We are interested in the motion of Γ_t, which started from some compact Γ_0, in finite time interval $(0, T)$. Then, since the viscosity solution of (2.4) is continuous, we may assume that there exists a big cube $\prod_{j=1}^{n}[a_j, b_j]$ satisfying $\Gamma_t \subset \prod_{j=1}^{n}[a_j, b_j]$ for $t \in [0, T)$. Therefore we consider the all equation with the periodic boundary condition, i.e., the equality $u(x + (b_j - a_j)e_j, t) = u(x, t)$ holds for $(x, t) \in \mathbf{R}^n \times [0, T)$ and $j = 1, 2, \ldots, n$. We now set $\mathbf{T}^n = \prod_{j=1}^{n} \mathbf{R}/(b_j - a_j)\mathbf{Z}$. We consider (2.3) on $\mathbf{T}^n \times (0, T)$, i.e.,

$$\beta(\nabla u)\partial_t u - \gamma(\nabla u)\{\mathrm{div}\xi(\nabla u) + f\} = 0 \quad \text{in } \mathbf{T}^n \times (0, T) \tag{2.4}$$

with initial data

$$u(\cdot, 0) = u_0(\cdot) \quad \text{on } \mathbf{T}^n. \tag{2.5}$$

Since (2.4) is degenerate parabolic and geometric, it is well-known that, for the periodic initial data, there exists a unique global periodic viscosity solution of (2.4) (See [CGG1] or [G2]).

There is another way to analyze the motion of Γ_t. In fact, there is the approximation of Γ_t by the internal transition layer of an anisotropic Allen–Cahn type equation introduced by [MWBCS]. The explicit form of the equation is

$$\beta(\nabla v)\partial_t v - \mathrm{div}\{\gamma(\nabla v)\xi(\nabla v)\} + \frac{1}{\varepsilon^2}(W'(v) - \varepsilon\lambda f) = 0 \quad \text{in } \mathbf{T}^n \times (0, T), \tag{2.6}$$

with initial data

$$v(\cdot, 0) = v_0(\cdot) \quad \text{on } \mathbf{T}^n. \tag{2.7}$$

Here W is a double-well potential of the form $W(\sigma) = (\sigma^2 - 1)^2/2$, and λ is a constant determined by W, in our case $\lambda = 2/3$. We choose a suitable v_0 to approximate an interfaces moving by (2.1). See section 2.4 and Theorem 2.2 to know how to choose v_0. The internal transition layers of (2.6) approximates the motion of Γ_t. This fact is already established rigorously by [ElS1], [ElPS] and [ElS2].

Our aim in this paper is to prove that an estimate of the convergence of internal transition layers is uniform with respect to modulus of derivatives of γ. For this purpose, we have to clarify quantities which determine the speed of the convergence of internal transition layers.

Traditionally as in [ElS2] or [ESS], we construct a supersolution and a subsolution of (2.6) for the estimate of the convergence. The key tool of this method is the comparison principle for viscosity solutions. Unfortunately, however, (2.6) has singularities so that we cannot apply the usual comparison principle for viscosity solutions. To overcome this difficulty, we modify the equation. We introduce a cut-off function $\zeta \in C^\infty([0, \infty))$ satisfying

$$\zeta(\sigma) = \begin{cases} 1 & \text{if } \sigma \le 1/2, \\ 0 & \text{if } \sigma \ge 3/4, \end{cases}$$

and $\zeta' \le 0$. Let $\tilde{\beta}$ be a function defined by

$$\tilde{\beta}(p) = (1 - \zeta(|p|))\beta(p) + \Lambda_\beta \zeta(|p|). \tag{2.8}$$

We take the coefficient $\tilde{\beta}(\nabla v)$ in front of $\partial_t v$ in (2.6) instead of $\beta(\nabla v)$, i.e.,

$$\tilde{\beta}(\nabla v)\partial_t v - \mathrm{div}\{\gamma(\nabla v)\xi(\nabla v)\} + \frac{1}{\varepsilon^2}(W'(v) - \varepsilon\lambda f) = 0 \quad \text{in } \mathbf{T}^n \times (0, T). \tag{2.9}$$

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The same type of modification appears in [ElPS]. The main advantage of (2.9) over (2.6) is that the singularity at $\nabla v = 0$ in the term involving β is removed. Since $\tilde{\beta}$ is positive and continuous on $\mathbf{R}^n$, we can apply the usual theory of viscosity solutions, in particular the comparison principle (see [CGG1] or [G2]). We treat (2.9) as the approximation model of an anisotropic mean curvature flow instead of (2.6). The solvability of (2.9) with initial data $v_0 \in C(\mathbf{T}^n)$ is already mentioned by [ElS2]. (See section 2.3 and Theorem 2.8 in [ElS2].)

2.2. Anisotropic distance function

We now recall an anisotropic *distance* function induced by Finsler (Minkowski) metric as in [BP2]. The distance is useful to construct an initial datum for (2.4) or (2.9).

We introduce the support function γ° of the convex set $\{p \in \mathbf{R}^n; \ \gamma(p) \leq 1\}$ defined by

$$\gamma^\circ(p) = \sup\{\langle p, q \rangle; \ \gamma(q) \leq 1\}.$$

Here we remark that $\gamma^\circ \in C^2(\mathbf{R}^n \setminus \{0\})$, γ° is convex, positively homogeneous of degree 1. Moreover we observe that, for $p \in \mathbf{R}^n \setminus \{0\}$, there exists uniquely $q \in \{p \in \mathbf{R}^n; \ \gamma(p) \leq 1\}$ satisfying $\gamma^\circ(p) = \langle p, q \rangle$ since γ^2 is strictly convex. The important property of γ° for studying (2.6) or (2.9) is obtained by [BP2]. Here we shall list a part of them in section 3.

We define an anisotropic *distance* Ξ by

$$\Xi(x, y) = \gamma^\circ(x - y).$$

We remark that only the symmetry in the definition of distance does not hold for Ξ since γ° is not assumed to be symmetric. For the subset $\Gamma \subset \mathbf{R}^n$ we define

$$\Xi(x, \Gamma) = \inf\{\Xi(x, y); \ y \in \Gamma\}.$$

For later convenience we take an order of x and a subset $\Gamma \subset \mathbf{R^n}$ in the definition of Ξ. The following argument also apply to the reversed version of the anisotropic distance function of the form $\Xi(\Gamma, x)$.

2.3. Travelling wave

To derive an estimate for the convergence of internal transition layer of (2.9), it is convenient to introduce a traveling wave solution of (2.9) with initial data which has a layer around of Γ_0. In general, we consider a solution of (2.9) of the form $v(x, t) = Q(x \cdot e - ct)$ for the function Q, constant c and fixed $e \in S^{n-1}$. Then we observe that Q satisfies some ordinary differential equation. However, here it suffices to consider a equation of Q for the isometric case as in [BSS].

Here we introduce a generalized notion of a travelling wave. We shall consider the double-well potential of the form $W(\sigma) - z\sigma$ for $z \in \mathbf{R}$. For z satisfying $|z| < 4\sqrt{3}/9$, the function $\sigma \mapsto W'(\sigma) - z$ has exactly three zeros if and only if $|z| < 4\sqrt{3}/9$. We shall denote them by $h_- = h_-(z)$, $h_0 = h_0(z)$ and $h_+ = h_+(z)$, which satisfy $h_- < h_0 < h_+$.

Here we assume that z satisfies $|z| < 4\sqrt{3}/9$, and set

$$
\begin{aligned}
m(z) &= h_+(z) - h_-(z), \\
c(z) &= 2h_0(z) - (h_+(z) + h_-(z)), \\
Q(\sigma, z) &= h_-(z) + \frac{m(z)}{1 + \exp\{-m(z)(\sigma - \sigma_0(z))\}},
\end{aligned}
$$

where $\sigma_0(z)$ is taken so that Q satisfies $Q(0, z) = h_0(z)$. Since $h_\pm$ and h_0 are smooth, we observe that $Q \in C^\infty(\mathbf{R} \times (-4\sqrt{3}/9, 4\sqrt{3}/9))$ and it solves

$$
Q_{\sigma\sigma}(\sigma, z) + c(z)Q_\sigma(\sigma, z) = W'(Q(\sigma, z)) - z \quad \text{for } \sigma \in \mathbf{R}, \tag{2.10}
$$
$$
\lim_{\sigma \to \pm\infty} Q(\sigma, z) = h_\pm(z), \ Q(0, z) = h_0(z).
$$

Moreover, we observe that

$$
h_\pm(z) = \pm 1 + O(z), \ h_0(z) = 0 + O(z),
$$
$$
m(z) = 2 + O(z^2), \ \text{in particular } \sqrt{3} < m(z) \le 2,
$$
$$
\lim_{\varepsilon \to 0} \frac{c(z)}{z} = \frac{2}{W''(0)} - \left(\frac{1}{W''(1)} + \frac{1}{W''(-1)}\right) + O(z) = -\frac{1}{\lambda} + O(z),
$$

as $z \to 0$.

In our case we fix $z = \varepsilon\lambda f$ and set $Q(\sigma) = Q(\sigma, \varepsilon\lambda f)$. Here and hereafter we shall omit the dependence of z when $z = \varepsilon\lambda f$, and we express $Q'(\sigma) = Q_\sigma(\sigma, \varepsilon\lambda f)$ and $Q''(\sigma) = Q_{\sigma\sigma}(\sigma, \varepsilon\lambda f)$. We shall list properties of these.

Proposition 2.1. *Assume that f satisfies (f1) and ε satisfies (ε1). Then,*

  (i) $\lim_{\varepsilon \to 0} \sup_{|f| \le \Lambda_f} |c/\varepsilon + f| = 0$,

  (ii) $\lim_{\varepsilon \to 0} \sup\{|Q(\sigma) - \tanh\sigma|; \ \sigma \in \mathbf{R}, \ f \in [-\Lambda_f, \Lambda_f]\} = 0$,

  (iii) $\inf\{Q'(\sigma); \ \sigma \in [-b, b], \ \varepsilon \in (0, \bar{\varepsilon}), \ f \in [-\Lambda_f, \Lambda_f]\} > 0$ *for $b > 0$,*

  (iv) *There exist constants C_1, C_2 and C_3, which depend only on Λ_f, satisfying*

$$
|Q(\sigma)^2 - 1| \le C_1 \exp(-C_2|\sigma|) + C_3\varepsilon, \tag{2.11}
$$
$$
|Q'(\sigma)|, |Q''(\sigma)| \le C_1 \exp(-C_2|\sigma|). \tag{2.12}
$$

2.4. Main result

We now determine the moving interfaces by (2.1). Let O_0 be an open subset in $\mathbf{T}^n$ and $\Gamma_0 = \partial O_0$. Let d_0 be a signed anisotropic distance function from an initial interface Γ_0 defined by

$$
d_0(x) = \begin{cases} \Xi(x, \Gamma_0) & \text{if } x \in O_0 \cup \Gamma_0, \\ -\Xi(x, \Gamma_0) & \text{otherwise.} \end{cases} \tag{2.13}
$$

We note that d_0 is continuous on $\mathbf{T}^n$ and spatially periodic. Let u be a periodic viscosity solution of (2.4) with initial data $u_0 = d_0$. Then we obtain a generalized solution Γ_t of (2.1) started from Γ_0 by (2.2).

We assume that $\Gamma_t \neq \emptyset$ for $t \in [0, T)$. We define a signed anisotropic distance function $d : \mathbf{T}^n \times [0, T) \to \mathbf{R}$ from Γ_t by

$$d(x, t) = \begin{cases} \Xi(x, \Gamma_t) & \text{if } x \in \{y \in \mathbf{T}^n; \ u(y, t) \geq 0\}, \\ -\Xi(x, \Gamma_t) & \text{if } x \in \{y \in \mathbf{T}^n; \ u(y, t) < 0\}. \end{cases} \tag{2.14}$$

We are now in position to state our main result.

Theorem 2.2. *Assume that β, γ, f, and ε satisfy (β1)–(β3), (γ1)–(γ5), (f1), and (ε1) respectively. Let O_0 be an open set in $\mathbf{T}^n$ and $\Gamma_0 = \partial O_0$. Let d_0, $d(x, t)$ be the anisotropic signed distance function from Γ_0, Γ_t defined by (2.13), (2.14), respectively. Let v be a viscosity solution of (2.9) satisfying (2.8) with initial data $v_0(x) = Q(d_0(x)/\varepsilon)$ for $\varepsilon < \bar{\varepsilon}$. For $\theta > 0$, there exist positive constants $\delta = \delta(\theta)$, $\varepsilon_1 = \varepsilon_1(\theta, \Lambda_\beta, \Lambda_\gamma, \Lambda_f)$ and $C = C(\theta, \Lambda_\beta, \Lambda_\gamma, \Lambda_f)$ satisfying*

$$v(x, t) \leq -1 + C_1 \exp\left(-\frac{C_2 \delta}{\varepsilon}\right) + C\varepsilon \tag{2.15}$$

if $(x, t) \in \{(y, s) \in \mathbf{T}^n \times (0, T); \ d(y, s) \leq -\theta\}$ provided that $\varepsilon \in (0, \varepsilon_1)$, where C_1 and C_2 are numerical constants.

We remark that this result is a refined version of [ElS2] since the constants C_1, C_2, C and ε_0 are independent of first and 2nd derivatives of γ. It is useful to treat the approximating problem of (2.4) and (2.9) for nonsmooth γ.

The main strategy of the proof stems from [ESS] and [ElS2]. We construct a function $\psi = \psi_{\varepsilon, \delta}$ satisfying:

(i) for $\theta > 0$, there exist positive constants $\delta = \delta(\theta)$ and $C = C(\theta, \Lambda_\beta, \Lambda_\gamma)$ such that $\psi(x, t)$ satisfies (2.15) for $(x, t) \in \{(y, s) \in \mathbf{T}^n \times (0, T); \ d(y, s) < -\theta\}$,

(ii) for δ, there exists a positive constant ε_0 such that ψ is a supersolution of (2.9) provided that $\varepsilon \in (0, \varepsilon_0)$,

(iii) $\psi(x, t) \geq Q(d_0(x)/\varepsilon)$,

Then, by the comparison principle, we obtain Theorem 2.2. Unfortunately the construction by [ESS] and [ElS2] is suitable only to construct a supersolution of the unmodified equation (2.6). It is not enough to construct a supersolution of (2.9). To clarify the difficulty to obtain (i) we shall give a formal calculation. Set

$$R_\varepsilon = \beta(\nabla\psi) - \text{div}\{\gamma(\nabla\psi)\xi(\nabla\psi)\} + \frac{1}{\varepsilon^2}(W'(\psi) - \varepsilon\lambda f),$$

$$\tilde{R}_\varepsilon = \tilde{\beta}(\nabla\psi) - \text{div}\{\gamma(\nabla\psi)\xi(\nabla\psi)\} + \frac{1}{\varepsilon^2}(W'(\psi) - \varepsilon\lambda f).$$

Clearly the first quantity R_ε is easy to calculate. However we have to calculate $\tilde{R}_\varepsilon$. We observe that

$$\tilde{R}_\varepsilon = R_\varepsilon + (\Lambda_\beta - \beta(\nabla\psi))\zeta(|\nabla\psi|)\partial_t\psi.$$

Thus it suffices to derive the suitable estimate for $\partial_t\psi$ to calculate $\tilde{R}_\varepsilon$.

We summarize the way for constructing ψ:

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(i) (in §3.1 and §3.2) We verify that the anisotropic signed distance function d is a viscosity supersolution of (2.4) in $\{(x,t) \in \mathbf{R}^n \times (0,T); \ d(x,t) > 0\}$. We also give an estimate of $\partial_t d$.

(ii) (in §3.3) For fixed δ, we introduce the truncating function η as in [ESS] and consider $\omega = \eta(d)$. We give an estimate of $\beta(\nabla\omega)\partial_t\omega - \mathrm{div}\{\gamma(\nabla\omega)\xi(\nabla\omega)\}$. We also give an estimate of $\partial_t\omega$.

(iii) (in §4) We construct a function ψ by using ω. We verify that, for δ, there exists a positive constant $\varepsilon_0 = \varepsilon_0(\delta, \Lambda_\beta, \Lambda_\gamma)$ such that ψ is a viscosity supersolution of (2.6) provided that $\varepsilon \in (0, \varepsilon_0)$.

(iv) (in §4) We verify that, for δ, there exists a positive constant $\varepsilon_1 = \varepsilon_1(\delta, \Lambda_\beta, \Lambda_\gamma)$ such that ψ is a viscosity supersolution of (2.9) provided that $\varepsilon \in (0, \varepsilon_1)$.

We give the proof of Theorem 2.2 in §5.

Hereafter, we often use another representation of the second terms of (2.4), (2.6) and (2.9), i.e.,

$$\mathrm{div}\{\xi(\nabla u)\} = \mathrm{tr}\{D^2\gamma(\nabla u)\nabla^2 u\},$$

$$\mathrm{div}\{\gamma(\nabla v)\xi(\nabla v)\} = \mathrm{tr}\{D^2\alpha(\nabla v)\nabla^2 v\},$$

where $\alpha(p) = \gamma(p)^2/2$. We remark that α is positively homogeneous of degree 2.

Finally we remark that we only mention the estimate for solutions from above. This is because that the estimate from below is essentially same as this by considering (2.6) and (2.9) with $\tilde{\beta}(p) = \beta(-p)$, $\tilde{\alpha}(p) = \alpha(-p)$, $\tilde{W}(\sigma) = W(-\sigma)$, and $\tilde{f} = -f$ instead of $\beta(p)$, $\alpha(p)$, $W(\sigma)$, and f, respectively. By a standard argument, Theorem 2.2 and this remark yields (1.3).

2.5. Application

We now give an application of Theorem 2.2. Our result is useful to approximate solutions of (2.1) by (2.9) even when γ is not differentiable provided that (2.4) fulfills the following convergence ansatz.

Convergence ansatz. *Assume that $\beta^\tau \in C(\mathbf{R}^n \setminus \{0\})$, $\gamma^\tau \in C^2(\mathbf{R}^n \setminus \{0\})$ are positive outside of the origin, and $f^\tau \in \mathbf{R}$. Assume that β^τ and γ^τ is positively homogeneous of degree 0 and 1, respectively. Assume that γ^τ is convex. (We do not assume the differentiability of γ.) Assume that $\beta^\tau \to \hat{\beta}$, $\gamma^\tau \to \hat{\gamma}$ locally uniformly in $\mathbf{R}^n \setminus \{0\}$ and $f^\tau \to \hat{f}$ as $\tau \to 0$. Let u^τ be the periodic viscosity solution of*

$$\beta^\tau(\nabla u^\tau)\partial_t u^\tau - \gamma^\tau(\nabla u^\tau)\{\mathrm{div}\xi^\tau(\nabla u^\tau) + f^\tau\} = 0 \quad in \ \mathbf{R}^n \times (0,T),$$

with continuous periodic initial data $u^\tau(x,0) = u_0^\tau(x)$, where $\xi^\tau = D\gamma^\tau$. Assume that period is independent of τ. Assume that $u_0^\tau \to u_0$ uniformly in $\mathbf{R}^n$. Then u^τ converges to $\hat{u} \in C(\mathbf{R}^n \times [0,\infty))$ which "solves" (2.4) with $\beta = \hat{\beta}$, $\gamma = \hat{\gamma}$ and $\hat{u}(x,0) = u_0(x)$. The convergence is uniform in $\mathbf{R}^n \times [0,T]$ for every $T > 0$.

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If we further assume that there exists a function $H \in C(S^{n-1}; \mathbf{S}_n)$, where $\mathbf{S}_n$ denotes the space of real symmetric $n \times n$ matrices, such that $D\xi^\tau \to H$ on S^{n-1}, then γ is $C^2(S^{n-1})$ and $D\xi = H$. In this case, the convergence of a solution u^τ to (2.4) with $\beta = \beta^\tau$, $\gamma = \gamma^\tau$ and $f = f^\tau$ is well-known (cf. [CGG1], [Ca], and [GG5]). However, if do not assume the convergence of derivatives of γ^τ, it is quite recent that the convergence ansatz has been proved for $n = 2$ in [GG4] and [GG5]. Note that meaning of a solution to (2.4) is not clear at all for all nondifferentiable γ since the term $\mathrm{div}\xi(\nabla u)$ is not well-defined even for smooth u. The papers [GG4] and [GG5] provide a proper notion of the solution.

Theorem 2.3. ([GG4], [GG5]) *Assume that $n = 2$. Assume that $\gamma|_{S^1}$ is C^2 except finitely many points $P = \{P_i\}_{i=1}^m$. Assume that the angular second derivatives of $\gamma|_{S^1}$ is bounded on $S^1 \setminus P$. Then the convergence ansatz is actually verified.*

This is a very special version of results in [GG4] or [GG5], where level set equations to more general equation of the form $V = g(\mathbf{n}, -\mathrm{div}_{\Gamma_t}(\mathbf{n}))$ is studied. The main idea of the proof is to reduce the problem to graph-like solutions of (2.1) which is studied in [GG2] and [GG3]. If the convergence ansatz is fulfilled, by a standard argument, Theorem 2.2 yields:

Theorem 2.4. *Assume that the convergence ansatz is true. Assume that β^τ, γ^τ and f^τ satisfy $(\beta 1)$–$(\beta 3)$, $(\gamma 1)$–$(\gamma 5)$, $(f1)$ and $(\varepsilon 1)$ with $\beta = \beta^\tau$, $\gamma = \gamma^\tau$ and $f = f^\tau$ uniformly in τ. Let v^τ be a solution of (2.9) with $\beta = \beta^\tau$, $\gamma = \gamma^\tau$ and $f = f^\tau$ with initial data $v^\tau(x, 0) = Q(d(x, 0)/\varepsilon)$. Then*

$$v^\tau(x,t) \to \begin{cases} 1 & \text{if} \quad x \in \{y \in \mathbf{R}^n; \ u(y,t) > 0\}, \\ -1 & \text{if} \quad x \in \{y \in \mathbf{R}^n; \ u(y,t) < 0\} \end{cases}$$

as τ, $\varepsilon \to 0$. Here u is a solution of (2.4) with β and γ.

Of course, there is always a way to approximate γ by γ_τ having required properties. Let $\gamma \in C(\mathbf{R}^n)$ be convex, positive outside of the origin, and positively homogeneous of degree one. This situation includes, for examples, $\gamma(p) = \max\{|p_j|; \ j = 1, \ \ldots, \ n\}$ or $\gamma(p) = \sum_{j=1}^n |p_j|$. We shall approximate γ by smooth γ_τ with $(\gamma 1)$–$(\gamma 5)$.

We take the heat kernel $G(p, \tau) = (4\pi\tau)^{-n/2} \exp(-|p|^2/4\tau)$ and define

$$\tilde{\gamma}(p, \tau) := (\gamma * G(\cdot, \tau))(p) = \int_{\mathbf{R}^n} \gamma(q)G(p - q, \tau)dq.$$

We get $\tilde{\gamma} \in C^\infty(\mathbf{R}^n \times (0, \infty))$ and $\tilde{\gamma}(\cdot, \tau) \to \gamma$ as $\tau \to 0$ locally uniformly by standard arguments. Moreover we see that $\tilde{\gamma}$ is strictly convex, more strongly, $\langle \nabla^2 \tilde{\gamma}(p, \tau)\xi, \xi \rangle > 0$ for $(p, \tau) \in \mathbf{R}^n \times (0, \infty)$ and $\xi \in \mathbf{R}^n \setminus \{0\}$. In fact, we obtain the strict convexity of $\tilde{\gamma}$ by the convexity of γ since $G > 0$. To see $\langle \nabla^2 \tilde{\gamma}(p, \tau)\xi, \xi \rangle > 0$ for $(p, \tau) \in \mathbf{R}^n \times (0, \infty)$ and $\xi \in \mathbf{R}^n \setminus \{0\}$, we define the function $\phi(p, \tau) := \langle \nabla^2 \tilde{\gamma}(p, \tau)\xi, \xi \rangle = (\gamma * \frac{\partial^2 G}{\partial \xi^2}(\cdot, \tau))(p)$. We shall assume that there exists $(p_0, \tau_0) \in \mathbf{R}^n \times (0, \infty)$ with $\phi(p_0, \tau_0) = 0$ and derive

a contradiction. We observe that ϕ is a solution of the heat equation and $\phi \geq 0$ in $\mathbf{R}^n \times (0, \infty)$. Using the strong maximum principle for the heat equation, we get $\phi \equiv 0$ for $\mathbf{R}^n \times (0, \tau_0]$. This implies that the function $\sigma \mapsto \tilde{\gamma}(p + \sigma\xi, \tau)$ is linear for $\tau \in (0, \tau_0]$ which contradicts the strict convexity of $\tilde{\gamma}$.

The sequence $\{\tilde{\gamma}(\cdot, \tau)\}$ gives an approximation of γ. However, unfortunately $\tilde{\gamma}(\cdot, \tau)$ is not positively homogeneous of degree 1. By using $\tilde{\gamma}$, we shall give a function which satisfies $(\gamma1)$–$(\gamma5)$ and approximates γ. We take $\bar{\tau} > 0$ satisfying $0 \in \mathcal{F}_\tau \setminus \partial\mathcal{F}_\tau$ for $\tau \leq \bar{\tau}$, where $\mathcal{F}_\tau = \{p;\ \tilde{\gamma}(p, \tau) \leq 1\}$. We define

$$\gamma_\tau(p) = \inf\{r;\ r > 0,\ p/r \in \mathcal{F}_\tau\}.$$

As we see later, γ_τ is a desired function, i.e., γ_τ satisfies the properties $(\gamma1)$–$(\gamma5)$, there exists a uniform bound Λ_γ in $(\gamma3)$ for γ_τ, and $\gamma_\tau \to \gamma$ as $\tau \to 0$ locally uniformly.

The homogeneity $(\gamma2)$ and the convexity $(\gamma4)$ easily follow from definition of γ_τ.

The smoothness $(\gamma1)$ follows from an estimate of $D\tilde{\gamma}$ on $\partial\mathcal{F}_\tau$. Since $\tilde{\gamma}$ is strictly convex and $\tilde{\gamma}(0, \tau) < 1$ for $\tau < \bar{\tau}$, we get $|D\tilde{\gamma}(p, \tau)| \neq 0$, in particular, $\langle D\tilde{\gamma}(p, \tau), p \rangle > 0$ for $(p, \tau) \in \partial\mathcal{F}_\tau \times (0, \bar{\tau})$. We define $g(r, q) = f(rq, \tau) - 1$ for $r > 0$ and $q \in S^{n-1}$, and get

$$\frac{\partial g}{\partial r}(r, q) = \frac{1}{r}\langle D\tilde{\gamma}(p, \tau), p \rangle > 0$$

for $p = rq \in \partial\mathcal{F}_\tau$. This implies that there exists a smooth function $\varphi = \varphi(q)$ for $q \in S^{n-1}$ with $g(\varphi(q), q) = 0$ so that $\varphi(q)q \in \partial\mathcal{F}_\tau$ since $\partial\mathcal{F}_\tau = \{p;\ \tilde{\gamma}(p, \tau) = 1\} = \{p;\ \gamma_\tau(p) = 1\}$. This yields that $\gamma_\tau(p) = |p|(\varphi(p/|p|))^{-1}$ so that γ_τ is smooth outside the origin.

The property $(\gamma5)$ follows from the strict convexity of $\mathcal{F}_\tau$. In fact, these two conditions are equivalent. (See revised version of [G2, Remark 1.7.5]) We indicate here the proof that the strict convexity of $\mathcal{F}_\tau$ implies the strict convexity of γ_τ^2. By $(\gamma4)$ we get $\{p;\ \gamma_\tau(p) \leq c\} = \{cp;\ \gamma_\tau(p) \leq 1\} = \{cp;\ \tilde{\gamma}(p, \tau) \leq 1\}$ for $c > 0$. This and $D^2\tilde{\gamma} > 0$ yield that

$$\langle R_\xi D^2\gamma_\tau(p)R_\xi\eta, \eta \rangle > 0 \quad \text{for } p,\ \eta \in \mathbf{R}^n \setminus \{0\} \text{ with } \langle \xi, \eta \rangle = 0,$$

where $\xi = D\gamma_\tau(p)$ and $R_\xi = I - (\xi \otimes \xi)/|\xi|^2$. Since $R_\xi\eta = \eta$ we obtain

$$\begin{aligned}
\langle R_\xi D^2\gamma_\tau(p)R_\xi\eta, \eta \rangle &= \langle D^2\gamma_\tau(p)\eta - |\xi|^{-2}\xi \otimes \xi D^2\gamma_\tau(p)\eta, \eta \rangle \\
&= \langle D^2\gamma_\tau(p)\eta, \eta \rangle - |\xi|^{-2}\langle \xi, D^2\gamma_\tau(p)\eta \rangle \langle \xi, \eta \rangle \\
&= \langle D^2\gamma_\tau(p)\eta, \eta \rangle,
\end{aligned}$$

i.e., $\langle D^2\gamma_\tau(p)\eta, \eta \rangle > 0$ for $p, \eta \in \mathbf{R}^n \setminus \{0\}$ with $\langle \xi, \eta \rangle = 0$. By using this inequality we get $\mathrm{Ker} D^2\gamma_\tau(p) = \mathbf{R}p = \{cp;\ c \in \mathbf{R}\}$. (See revised version of [G2, Remark 1.7.5].) In fact, we obtain $D^2\gamma_\tau(p)p = 0$ since γ_τ is positively homogeneous of degree 1. To see $\mathrm{Ker} D^2\gamma_\tau(p) = \mathbf{R}p$ we shall assume that there exists $q \in \mathrm{Ker} D^2\gamma_\tau(p)$ with $\langle p, q \rangle = 0$ and derive a contradiction. We set $x := -\langle \xi, q \rangle p + \gamma_\tau(p)q. = -\langle \xi, q \rangle p + \langle \xi, p \rangle q$. Then we obtain $x \neq 0$, $\langle \xi, x \rangle = 0$ and $D^2\gamma_\tau(p)x = 0$. However, we get $\langle D^2\gamma_\tau(p)x, x \rangle > 0$ since $\langle \xi, x \rangle = 0$. This is a contradiction.

For $x \neq 0$, we set $x = c_1 p + c_2 q$ for $p, q \in \mathbf{R}^n$ and $c_1, c_2 \in \mathbf{R}$ with $\langle p, q \rangle = 0$ and shall prove that $\langle D^2\gamma_\tau(p)^2 x, x \rangle > 0$. If $c_2 = 0$, then we obtain $\langle D^2\gamma_\tau(p)^2 x, x \rangle = 2c_1^2\gamma_\tau(p)^2 > 0$. If $c_2 \neq 0$, then we observe that $\langle D^2\gamma(p)^2 x, x \rangle \geq c_2^2\langle D^2\gamma_\tau(p)q, q \rangle > 0$ since $\mathrm{Ker} D^2\gamma_\tau(p) = \mathbf{R}p$. We thus establish $(\gamma5)$.

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The uniform bound ($\gamma 3$) and the convergence $\gamma_\tau \to \gamma$ as $\tau \to 0$ locally uniformly easily follow from the convergence $\tilde{\gamma}(\cdot, \tau) \to \gamma$ as $\tau \to 0$ locally uniformly and the homogeneity of each functions.

3.   Properties of anisotropic distance function

In this section, we prepare some properties of the anisotropic distance function, which follow from those of the support function. Most of them are already been proved by [BP2] for the support function and by [ElS2] for the anisotropic distance function. However, we need to refine some of them for our purpose. Especially, a refined version of an estimate of $\partial_t d$ is crucial for the proof of our uniform convergence result.

We list some properties of γ°.

$$\gamma(D\gamma^\circ(p)) = \gamma^\circ(D\gamma(p)) = 1 \quad \text{for } p \neq 0, \tag{3.1}$$

$$\gamma(p)D\gamma^\circ(D\gamma(p)) = \gamma^\circ(p)D\gamma(D\gamma^\circ(p)) = p \quad \text{for } p \neq 0, \tag{3.2}$$

$$D^2\gamma^\circ(p)D\gamma(D\gamma^\circ(p)) = D^2\gamma(p)D\gamma^\circ(D\gamma(p)) = 0 \quad \text{for } p \neq 0. \tag{3.3}$$

We only give a few remarks for the proof; see [BP2] for the detailed proof.

Since $\{p \in \mathbf{R}^n; \ \gamma(p) \leq 1\}$ is convex, we see that $(\gamma^\circ)^\circ = \gamma$ by the convex analysis. The first equalities of (3.1)–(3.3) easily follow from this duality formulas. Moreover the identity of (3.3) follows from (3.1) by its differentiation. In [BP2], to prove (3.1), one needs to assume that, for $p \neq 0$, there exists unique $q \in \{p \in \mathbf{R}^n; \ \gamma(p) \leq 1\}$ satisfying $\gamma^\circ(p) = \langle p, q \rangle$. In our situation, it is fulfilled since γ^2 is strictly convex.

3.1.   Properties of d

We state the general properties of the anisotropic distance from a subset in $\mathbf{R}^n$.

Lemma 3.1. *Assume that γ satisfies ($\gamma 1$)–($\gamma 5$). Let $\Gamma \subset \mathbf{R}^n$ be a closed subset. We define $d(x) = \Xi(x, \Gamma)$. Then d is a viscosity supersolution of*

$$\left. \begin{array}{l} \gamma(\nabla d) = 1, \quad -\gamma(\nabla d) = -1, \\ -\langle \nabla^2 d D\gamma(\nabla d), D\gamma(\nabla d) \rangle = 0, \end{array} \right\} \quad in \ \{x \in \mathbf{R}^n; \ d(x) > 0\}.$$

Lemma 3.1 stems from (3.1) and the derivative of (3.1) in the direction $D\gamma(\nabla d)$. Fortunately, however, we can prove the first equation without the differentiability of d by using a viscosity sense. We shall give the proof for completeness. In the theory of viscosity solutions, we often consider the upper and lower semicontinuous envelope of functions to show it is a viscosity subsolution and a supersolution of an equation, respectively. However, since

$$-\gamma^\circ(y - x) \leq d(x) - d(y) \leq \gamma^\circ(x - y) \quad \text{for all } x, \ y \in \mathbf{R}^n,$$

d is Lipschitz continuous. Therefore we do not have to consider a lower semicontinuous envelope.

Proof. Fix $x_0 \in \{x;\ d(x) > 0\}$. Let $\varphi \in C^2(\mathbf{R}^n)$ satisfy

$$d(x) - \varphi(x) \geq d(x_0) - \varphi(x_0) \quad \text{for } x \in \mathbf{R}^n.$$

Since Γ is closed, then there exists $y_0 \in \Gamma$ satisfying

$$d(x_0) = \gamma^\circ(x_0 - y_0).$$

Then we observe that

$$\gamma^\circ(x - y_0) - \varphi(x) \geq d(x) - \varphi(x) \geq d(x_0) - \varphi(x_0) = \gamma^\circ(x_0 - y_0) - \varphi(x_0).$$

We first shall show the first equation. Since γ is strictly convex, for x, there exists an unique vector $q_x \in \{p;\ \gamma(p) \leq 1\}$ satisfying

$$\gamma(x - y_0) = \langle x - y_0, q_x \rangle, \ \text{and } \gamma(q_x) = 1.$$

We set $q_0 = q_{x_0}$. Then we observe that $q_x \to q_0$ as $x \to x_0$ by taking a sequence of $\{q_x\}$ if it is necessary. By a calculation we observe that

$$\begin{aligned}
\langle x - y_0, q_x \rangle - \varphi(x) &= \gamma^\circ(x - y_0) - \varphi(x) \\
&\geq \gamma^\circ(x_0 - y_0) - \varphi(x_0) \geq \langle x_0 - y_0, q_x \rangle - \varphi(x).
\end{aligned}$$

Thus we obtain

$$\langle x - x_0, q_x \rangle \geq \varphi(x) - \varphi(x_0) = \langle x - x_0, \nabla\varphi(x_0) \rangle + o(|x - x_0|)$$

as $|x - x_0| \to 0$. We now divide by $|x - x_0|$ and send $x \to x_0$ to get for $e \in S^{n-1}$,

$$\langle e, q_0 - \nabla\varphi(x_0) \rangle \geq 0.$$

Thus we obtain $q_0 = \nabla\varphi(x_0)$. We obtain from $\gamma(q_0) = 1$,

$$\gamma(\nabla\varphi(x_0)) \geq 1, \ \text{and } -\gamma(\nabla\varphi(x_0)) \geq -1.$$

Next we shall show the second equation. We differentiate (3.1) in the direction $D\gamma(D\gamma^\circ(p))$, to get

$$\langle D^2\gamma^\circ(p)D\gamma(D\gamma^\circ(p)), D\gamma(D\gamma^\circ(p)) \rangle = 0 \quad \text{for } p \neq 0.$$

Here we remark that $x_0 \in \{x;\ d(x) > 0\}$ implies $x_0 \neq y_0$. We now can calculate the derivatives of $\gamma^\circ(\cdot - y_0)$ at x_0 and obtain

$$D\gamma^\circ(x_0 - y_0) = \nabla\varphi(x_0), \ D^2\gamma^\circ(x_0 - y_0) \geq \nabla\varphi(x_0).$$

We thus obtain

$$\begin{aligned}
-\langle &\nabla^2\varphi(x_0)D\gamma(\nabla\varphi(x_0)), D\gamma(\nabla\varphi(x_0)) \rangle \\
&\geq \langle D^2\gamma^\circ(x_0 - y_0)D\gamma(D\gamma^\circ(x_0 - y_0)), D\gamma(D\gamma^\circ(x_0 - y_0)) \rangle = 0,
\end{aligned}$$

which yields the second equation. $\square$

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We remark that, for reversed version of the orientation of x and Γ such as $d(x) = \Xi(\Gamma, x)$, we obtain a similar result. However, the sign of ∇d is reversed, i.e., the reversed version of a distance is a viscosity supersolution of $\gamma(-\nabla d) = 1$, $-\gamma(-\nabla d) = -1$, and $-\langle \nabla^2 d D\gamma(-\nabla d), D\gamma(-\nabla d)\rangle = 0$. We remark that we do not treat the reversed version of a distance in this paper.

We next obtain properties of anisotropic distance functions from the moving interface Γ_t.

Lemma 3.2. *Assume that β, γ and f satisfy (β1)–(β3), (γ1)–(γ5), and (f1), respectively. Let u be a viscosity solution of (2.4) with initial data $u(x,0) = d_0(x)$. Let $d(x,t)$ be an anisotropic distance function defined by*

$$d(x,t) = \begin{cases} \Xi(x, \Gamma_t) & \text{for} \quad x \in \{u(x,t) \geq 0\}, \\ -\Xi(x, \Gamma_t) & \text{for} \quad x \in \{u(x,t) < 0\}, \end{cases}$$

where $\Gamma_t = \{x \in \mathbf{R}^n; \ u(x,t) = 0\}$.

Then d is a viscosity supersolution of (2.4) in $\{(x,t) \in \mathbf{R}^n \times (0,T); \ d(x,t) > 0\}$.

This lemma is already proved by [ElS2]. (See Lemma 3.3 in [ElS2].) Their lemma has an error term $C(\Lambda_\gamma)|\nabla d|d$. However this term is disappeared if f is independent of space variable x.

3.2. Estimate of $\partial_t d$

In this section we prepare an estimate of $\partial_t d$ which is useful to construct our supersolution.

Lemma 3.3. *Assume that β, γ and f satisfy (β1)–(β3), (γ1)–(γ5) and (f1), respectively. Let d be the anisotropic distance function defined by Lemma 3.2.*

(i) *Let μ be a function defined by $\mu(\sigma) = \int_0^\sigma s/(1+s)ds$. Then the following holds;*

$$\mu(d(\hat{x}, t)) \geq \mu(d(\hat{x}, \hat{t})) - L_{\beta,f}(t - \hat{t})$$

for $(\hat{x}, t)$, $(\hat{x}, \hat{t}) \in \{(x,t); \ d(x,t) > 0\}$ provided that $0 \leq \hat{t} \leq t < T$, where $L_{\beta,f}$ is a positive constant depends only on n, Λ_β, and Λ_f.

(ii) *The anisotropic distance function d is a viscosity supersolution of*

$$\partial_t d = -L_{\beta,f}\left(1 + \frac{1}{d}\right) \quad \text{in} \ \{(x,t); \ d(x,t) > 0\}.$$

This lemma is a refined version of that in [ElS2]. Especially, the constant $L_{\beta,f}$ is independent of any derivatives of γ. This is the main advantage over [ElS2] so that we obtain our uniform convergence result.

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Proof. Fix $(\hat{x}, \hat{t}) \in \{(x,t); \ d(x,t) > 0\}$ and let $\hat{r} = d(\hat{x}, \hat{t})$. We define the function $z \colon \mathbf{R}^n \times [0, T) \to \mathbf{R}$ by

$$z(x,t) = \mu(\hat{r}) - L(t - \hat{t}) - \mu(\gamma^\circ(\hat{x} - x)),$$

where L is a positive constant determined later. Since $\gamma^\circ \in C^2(\mathbf{R}^n \setminus \{0\})$ we observe that $z \in C^{2,1}((\mathbf{R}^n \setminus \{\hat{x}\}) \times [0, T)) \cap C^{1,1}(\mathbf{R}^n \times [0, T))$. By the straightforward calculation we obtain

$$
\begin{aligned}
\partial_t z(x,t) &= -L, \\
\nabla z(x,t) &= \mu'(\gamma^\circ(\hat{q})) D\gamma^\circ(\hat{q}), \\
\nabla^2 z(x,t) &= -\mu''(\gamma^\circ(\hat{q})) D\gamma^\circ(\hat{q}) \otimes D\gamma^\circ(\hat{q}) - \mu'(\gamma^\circ(\hat{q})) D^2 \gamma^\circ(\hat{q})
\end{aligned}
$$

for $x \neq \hat{x}$, where $\hat{q} = \hat{x} - x$. We observe that $z \in C^{1,1}(\mathbf{R}^n \times [0, T))$ and $\nabla z(\hat{x}, t) = 0$.

In the following argument, we shall verify that z is a viscosity subsolution of (2.4). For this purpose, we give an estimate of the second term of (2.4) for z provided that $\hat{x} - x \neq 0$. First we remark that $\mu' > 0$ on $\mathbf{R}$. Then we obtain

$$
\begin{aligned}
\gamma(\nabla z) &= \mu'(\gamma^\circ(\hat{q}))\gamma(D\gamma^\circ(\hat{q})) = \mu'(\gamma^\circ(\hat{q})), \\
D\gamma(\nabla z) &= D\gamma(D\gamma^\circ(\hat{q})), \\
D^2\gamma(\nabla z) &= \frac{1}{\mu'(\gamma^\circ(\hat{q}))} D^2\gamma(D\gamma^\circ(\hat{q})).
\end{aligned}
$$

Therefore, by straightforward calculation, we obtain

$$
\begin{aligned}
\operatorname{tr}\{\gamma(\nabla z) D^2\gamma(\nabla z)\nabla^2 z\} = &-\mu''(\gamma^\circ(\hat{q}))\langle D^2\gamma(D\gamma^\circ(\hat{q}))D\gamma^\circ(\hat{q}), D\gamma^\circ(\hat{q})\rangle \\
&+ \mu'(\gamma^\circ(\hat{q}))\operatorname{div}\{D\gamma(D\gamma^\circ(\hat{q}))\}.
\end{aligned} \tag{3.4}
$$

By calculating the derivative of the second equality of (3.1), we obtain

$$D^2\gamma(p)D\gamma^\circ(D\gamma(p)) = 0 \quad \text{for } p \neq 0.$$

By taking $p = D\gamma^\circ(\hat{q})$ we obtain

$$D^2\gamma(D\gamma^\circ(\hat{q}))D\gamma^\circ(D\gamma(D\gamma^\circ(\hat{q}))) = 0.$$

By (3.2) and since $D\gamma^\circ$ is positively homogeneous of degree 0, we obtain

$$D^2\gamma(D\gamma^\circ(\hat{q}))D\gamma^\circ(\hat{q}) = 0,$$

i.e., the first term of (3.4) is disappeared. Moreover, from (3.2) and since γ° is positively homogeneous of degree 1, we obtain

$$\operatorname{div} D\gamma(D\gamma^\circ(p)) = \operatorname{div}\left(\frac{p}{\gamma^\circ(p)}\right) = \frac{n\gamma^\circ(p) - \langle p, D\gamma^\circ(p)\rangle}{\gamma^\circ(p)^2} = \frac{n-1}{\gamma^\circ(p)}.$$

Combining these and $\mu'(\sigma) = \sigma/(1+\sigma)$ we obtain

$$\operatorname{tr}\{\gamma(\nabla z) D^2\gamma(\nabla z)\nabla^2 z\} = -\mu'(\gamma^\circ(\hat{q}))\frac{n-1}{\gamma^\circ(\hat{q})} = -\frac{n-1}{1+\gamma^\circ(\hat{q})} \geq -(n-1). \tag{3.5}$$

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We are now in position to verify that z is a viscosity subsolution of (2.4). First we verify it in $(\mathbf{R}^n \setminus \{\hat{x}\}) \times (0, T)$. By (3.5) and $\mu' < 1$ we obtain

$$\beta(\nabla z)\partial_t z - \operatorname{tr}\{\gamma(\nabla z)D^2\gamma(\nabla z)\nabla^2 z\} - \gamma(\nabla z)f \leq -\frac{L}{\Lambda_\beta} + n - 1 + |f|.$$

We take $L > 0$ satisfying $-L/\Lambda_\beta + n - 1 + |f| \leq 0$ so that we obtain z is a viscosity subsolution of (2.4) in $(\mathbf{R}^n \setminus \{\hat{x}\}) \times (0, T)$. Here we take $L = \Lambda_\beta(n + \Lambda_f) =: L_{\beta,f}$.

Next we verify that z is a viscosity subsolution including $\{\hat{x}\} \times (0, T)$. Let $\hat{s} \in (0, T)$ and let $\varphi \in C^2(\mathbf{R}^n \times (0, T))$ satisfy

$$z(x, t) - \varphi(x, t) < z(\hat{x}, \hat{s}) - \varphi(\hat{x}, \hat{s}) \quad \text{for } (x, t) \in \mathbf{R}^n \times (0, T) \setminus \{(\hat{x}, \hat{s})\}.$$

Then we observe that $\nabla\varphi(\hat{x}, \hat{s}) = \nabla z(\hat{x}, \hat{s}) = 0$. Fix $e \in S^{n-1}$ and define

$$\varphi_\tau(x, t) = \varphi(x, t) + \tau\langle e, x\rangle.$$

Then, for sufficiently small $\tau > 0$, there exists $x_\tau \in \{x; \ d(x, \hat{s}) > 0\}$ satisfy

$$z(\cdot, \hat{s}) - \varphi_\tau(\cdot, \hat{s}) \leq z(x_\tau, \hat{s}) - \varphi_\tau(x_\tau, \hat{s}) \quad \text{in some neighborhood of } \hat{x},$$
$$x_\tau \neq \hat{x}, \text{ and } x_\tau \to \hat{x} \text{ as } \tau \to 0.$$

We now verify only that $x_\tau \neq \hat{x}$. If $x_\tau = \hat{x}$, then we obtain $\nabla\varphi_\tau(x_\tau, \hat{s}) = \nabla\varphi(\hat{x}, \hat{s}) + \tau e = \tau e \neq 0$. However we also obtain $\nabla\varphi_\tau(x_\tau, \hat{s}) = \nabla z(x_\tau, \hat{s}) = \nabla z(\hat{x}, \hat{s}) = 0$. This is a contradiction.

We now observe that

$$\left.\begin{array}{rcl}\partial_t z(x_\tau, \hat{s}) & \to & \partial_t z(\hat{x}, \hat{s}) = \partial_t \varphi(\hat{x}, \hat{s}), \\ \nabla\varphi_\tau(x_\tau, \hat{s}) & \to & \nabla\varphi(\hat{x}, \hat{s}), \\ \nabla^2\varphi_\tau(x_\tau, \hat{s}) & \to & \nabla^2\varphi(\hat{x}, \hat{s})\end{array}\right\} \quad \text{as } \tau \to 0.$$

Moreover we observe that $\nabla\varphi_\tau(x_\tau, \hat{s}) = \nabla z(x_\tau, \hat{s})$ and $\nabla^2\varphi_\tau(x_\tau, \hat{s}) \geq \nabla^2 z(x_\tau, \hat{s})$ since $z - \varphi_\tau(\cdot, \hat{s})$ attains its maximum at $\hat{x}$. Therefore we now obtain that

$$[\beta(\nabla\varphi)\partial_t\varphi - \operatorname{tr}\{\gamma(\nabla\varphi)D^2\gamma(\nabla\varphi)\nabla^2\varphi\} - \gamma(\nabla\varphi)f]_*(\hat{x}, \hat{s})$$
$$\leq \lim_{\tau \to 0}[\beta(\nabla\varphi_\tau)\partial_t z - \operatorname{tr}\{\gamma(\nabla\varphi_\tau)D^2\gamma(\nabla\varphi_\tau)\nabla^2\varphi_\tau\} - \gamma(\nabla\varphi_\tau)f](x_\tau, \hat{s})$$
$$\leq \lim_{\tau \to 0}[\beta(\nabla z)\partial_t z - \operatorname{tr}\{\gamma(\nabla z)D^2\gamma(\nabla z)\nabla^2 z\} - \gamma(\nabla z)f](x_\tau, \hat{s})$$
$$\leq 0.$$

We now conclude that z is a viscosity subsolution of (2.4) in $\mathbf{R}^n \times (0, T)$.

We now verify (i). From [CGG1], we know that $u_k^+(x, t) := \min(k\max(u(x, t), 0), 1)$ is a viscosity solution of (2.4) for $k > 0$ since $u(x, t)$ is a viscosity solution of (2.4). Moreover, $u^\infty(x, t) := \chi_{\{u>0\}}(x, t) = \lim_{k\to\infty}\inf\{u_j^+(y, s); \ |y-x|+|s-t| < 1/k, \ j > k\}$, is a viscosity supersolution of (2.4) (see [CIL, Lemma 6.1]). We now consider the set $U = \{x; \ \gamma^\circ(\hat{x} - x) < \hat{r}\}$. Then we obtain

$$z(x, \hat{t}) = \mu(\hat{r}) - \mu(\gamma^\circ(\hat{x} - x)) \leq \mu(\hat{r}) = \mu(\hat{r})u^\infty(x, \hat{t}) \quad \text{for } x \in U$$

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since $\mu > 0$ and $d(x, \hat{t}) > 0$ for $x \in U$. Moreover, we obtain for $(x, t) \in \partial U \times [\hat{t}, T)$,

$$z(x, t) = -L_{\beta, f}(t - \hat{t}) \leq 0 \leq \mu(\hat{r}) u^\infty(x, t).$$

Therefore, by the comparison principle, we obtain

$$z(x, t) \leq \mu(\hat{r}) u^\infty(x, t) \quad \text{for } (x, t) \in \overline{U} \times [\hat{t}, T).$$

For $t \in [\hat{t}, T)$, fix $\hat{y} \in \mathbf{R}^n$ satisfying $u(\hat{y}, t) = 0$ and $d(\hat{x}, t) = \gamma^\circ(\hat{x} - \hat{y})$. If $\hat{y} \in \overline{U}$, then we obtain

$$\begin{aligned}
0 = \mu(\hat{r}) u^\infty(\hat{y}, t) &\geq z(\hat{y}, t) \\
&= \mu(\hat{r}) - L_{\beta, f}(t - \hat{t}) - \mu(\gamma^\circ(\hat{x} - \hat{y})) \\
&= \mu(d(\hat{x}, \hat{t})) - L_{\beta, f}(t - \hat{t}) - \mu(d(\hat{x}, t))
\end{aligned}$$

or

$$\mu(d(\hat{x}, t)) \geq \mu(d(\hat{x}, \hat{t})) - L_{\beta, f}(t - \hat{t}).$$

If $\hat{y} \notin \overline{U}$, then we observe

$$\mu(d(\hat{x}, t)) = \mu(\gamma^\circ(\hat{x} - \hat{y})) \geq \mu(\hat{r}) = \mu(d(\hat{x}, \hat{t})) \geq \mu(d(\hat{x}, \hat{t})) - L_{\beta, f}(t - \hat{t}),$$

which yields (i).

Finally we verify (ii). Let $(x_0, t_0) \in \{(x, t); \ d(x, t) > 0\}$ and let $\varphi \in C^2(\mathbf{R}^n \times (0, T))$ satisfy

$$d(x, t) - \varphi(x, t) \geq d(x_0, t_0) - \varphi(x_0, t_0) = 0 \quad \text{for } (x, t) \in \mathbf{R}^n \times (0, T).$$

From (i) we observe that $d(x, t)$ is left continuous in time in the sense

$$\lim_{x \to x_0} \lim_{t \uparrow t_0} d(x, t) = d(x_0, t_0).$$

(see [ElS2], Proposition 3.5.) Then there exists a constant $r > 0$ satisfying

$$d(x_0, t) > 0 \quad \text{for } t \in (t_0 - r, t_0].$$

By using (i) we see

$$\mu(\varphi(x_0, t_0)) = \mu(d(x_0, t_0)) \geq \mu(d(x_0, t)) - L_{\beta, f}(t_0 - t) \geq \mu(\varphi(x_0, t)) - L_{\beta, f}(t_0 - t)$$

or

$$\frac{\mu(\varphi(x_0, t_0)) - \mu(\varphi(x_0, t))}{t_0 - t} \geq -L_{\beta, f}.$$

Sending $t \to t_0$ yields

$$\mu'(\varphi(x_0, t_0)) \partial_t \varphi(x_0, t_0) \geq -L_{\beta, f}.$$

Since $\varphi(x_0, t_0) = d(x_0, t_0)$ and $\mu'(\sigma) = \sigma/(1 + \sigma)$ we obtain

$$\partial_t \varphi(x_0, t_0) \geq -L_{\beta, f}\left(1 + \frac{1}{d(x_0, t_0)}\right). \quad \square$$

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3.3. Truncated anisotropic distance function

In this section we shall prepare several estimates of truncated anisotropic distance functions to construct our supersolution having an uniform estimate.

We first recall a function η introduced by [ESS]. We fix $\delta \in (0,1)$ and we consider the function $\eta \in C^{\infty}(\mathbf{R})$ satisfying

$$\eta(\sigma) = \begin{cases} \sigma - \delta & \text{if} \quad \sigma \geq \delta/2, \\ -\delta & \text{if} \quad \sigma \leq \delta/4, \end{cases} \tag{3.6}$$

$$0 \leq \eta' \leq C_\eta, \quad |\eta''| \leq \frac{C_\eta}{\delta}, \tag{3.7}$$

where C_η is a constant independent of σ and δ.

We now introduce a truncated anisotropic distance function. Let u be a viscosity solution of (2.4) with initial data $u_0(x) = d_0(x)$, where d_0 is the anisotropic signed distance function defined by (2.13). We determine subsets O_t, D_t and Γ_t by

$$
\begin{aligned}
O_t &= \{x \in \mathbf{R}^n;\ u(x,t) > 0\}, \\
D_t &= \{x \in \mathbf{R}^n;\ u(x,t) < 0\}, \\
\Gamma_t &= \{x \in \mathbf{R}^n;\ u(x,t) = 0\}.
\end{aligned}
$$

We now define the anisotropic signed distance function $d\colon \mathbf{R}^n \times [0,T) \to \mathbf{R}$ from moving interface Γ_t by

$$d(x,t) = \begin{cases} \Xi(x,\Gamma_t) & \text{if} \quad x \in O_t \cup \Gamma_t, \\ -\Xi(x,\Gamma_t) & \text{if} \quad x \in D_t. \end{cases}$$

The *truncated anisotropic distance function* $\omega\colon \mathbf{R}^n \times [0,T) \to \mathbf{R}$ is defined by

$$\omega(x,t) = \eta(d(x,t)).$$

Here we state some properties of ω.

Lemma 3.4. *Assume that β, γ and f satisfy (β1)–(β3), (γ1)–(γ5) and (f1), respectively. Then the truncated anisotropic distance function $\omega(x,t) = \eta(d(x,t))$ is a viscosity supersolution of*

$$\left. \begin{aligned} \beta(\nabla\omega)\partial_t\omega - \mathrm{tr}\{D^2\alpha(\nabla\omega)\nabla^2\omega\} - \gamma(\nabla\omega)f &= -\frac{C_\eta}{\delta}, \\ -|\nabla\omega| &= -C_\gamma \end{aligned} \right\} \quad \text{in } \mathbf{R}^n \times (0,T), \tag{3.8}$$

where C_γ is a positive constant depends only on Λ_γ. Moreover ω is a viscosity supersolution of

$$\left. \begin{aligned} \beta(\nabla\omega)\partial_t\omega - \mathrm{tr}\{D^2\alpha(\nabla\omega)\nabla^2\omega\} - \gamma(\nabla\omega)f &= 0, \\ \pm\gamma(\nabla\omega) &= \pm 1 \end{aligned} \right\} \quad \text{in } \left\{(x,t);\ d(x,t) > \frac{\delta}{2}\right\}. \tag{3.9}$$

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We remark that the equation in Lemma 3.4 is a different from (2.4). We replace the second term of (2.4) by that of the Allen–Cahn equation (2.6) to get (3.8) or (3.9). This estimate is useful to construct a supersolution for (2.9).

Proof. Let $(x_0, t_0) \in \mathbf{R}^n \times (0, T)$ and let $\varphi \in C^2(\mathbf{R}^n \times (0, T))$ satisfy

$$\omega(x, t) - \varphi(x, t) > \omega(x_0, t_0) - \varphi(x_0, t_0) \quad \text{whenever } (x, t) \neq (x_0, t_0).$$

We divide the situation into two cases: $d(x_0, t_0) > 0$ and $d(x_0, t_0) \leq 0$.

Case 1. Assume that $d(x_0, t_0) > 0$. Let $\tau \in (0, 1)$ be a small parameter. We introduce a function η_τ approximating η, defined by

$$\eta_\tau(\sigma) = \eta(\sigma) + \tau \sigma.$$

Then there exist a neighborhood $U = U(x_0, t_0)$ of (x_0, t_0) and $(x_\tau, t_\tau) \in U$ satisfying

$$\eta_\tau(d(x, t)) - \varphi(x, t) \geq \eta_\tau(d(x_\tau, t_\tau)) - \varphi(x_\tau, t_\tau) \quad \text{for } (x, t) \in U,$$
$$(x_\tau, t_\tau) \to (x_0, t_0) \quad \text{as } \tau \to 0.$$

Since $\eta_\tau' = \eta' + \tau \geq \tau > 0$, there exists $\rho_\tau = (\eta_\tau)^{-1}$ and $\rho_\tau' > 0$. Here we define the function $\bar{\varphi}_\tau$ by
$$\bar{\varphi}_\tau(x, t) = \rho_\tau(\varphi(x, t) - \varphi(x_\tau, t_\tau) + \eta_\tau(d(x_\tau, t_\tau))).$$

Then we obtain

$$d(x, t) - \bar{\varphi}_\tau(x, t) \geq d(x_\tau, t_\tau) - \bar{\varphi}_\tau(x_\tau, t_\tau) = 0 \quad \text{for } (x, t) \in U.$$

By straightforward calculation we obtain

$$
\begin{aligned}
\partial_t \bar{\varphi}_\tau &= \rho_\tau'(\kappa) \partial_t \varphi, & (3.10) \\
\nabla \bar{\varphi}_\tau &= \rho_\tau'(\kappa) \nabla \varphi, & (3.11) \\
\nabla^2 \bar{\varphi}_\tau &= \rho_\tau''(\kappa) \nabla \varphi \otimes \nabla \varphi + \rho'(\kappa) \nabla^2 \varphi, & (3.12)
\end{aligned}
$$

where $\kappa = \kappa(x, t) = \varphi(x, t) - \varphi(x_\tau, t_\tau) + \eta_\tau(d(x_\tau, t_\tau))$.
By Lemmas 3.1 and 3.2 we observe that

$$
\left.
\begin{aligned}
\gamma(\nabla \bar{\varphi}_\tau) = 1, \text{ in particular, } \nabla \bar{\varphi}_\tau \neq 0, \\
\beta(\nabla \bar{\varphi}_\tau) \partial_t \bar{\varphi}_\tau - \text{tr}\{\gamma(\nabla \bar{\varphi}_\tau) D^2 \gamma(\nabla \bar{\varphi}_\tau) \nabla^2 \bar{\varphi}_\tau\} - \gamma(\nabla \bar{\varphi}_\tau) f \geq 0
\end{aligned}
\right\} \text{ at } (x_\tau, t_\tau), \quad (3.13)
$$

where we remark that we do not need to consider the place where the gradient of unknown of (3.13) equals zero since $\nabla \bar{\varphi}_\tau \neq 0$ at (x_τ, t_τ). By calculating the term including trace and using Lemma 3.1, we observe that

$$
\begin{aligned}
\text{tr}\{\gamma(\nabla \bar{\varphi}_\tau) D^2 \gamma(\nabla \bar{\varphi}_\tau) \nabla^2 \bar{\varphi}_\tau\} &= \text{tr}\{D^2 \alpha(\nabla \bar{\varphi}_\tau) \nabla^2 \bar{\varphi}_\tau\} \\
&\quad - \langle \nabla^2 \bar{\varphi}_\tau D\gamma(\nabla \bar{\varphi}_\tau), D\gamma(\nabla \bar{\varphi}_\tau) \rangle \\
&\geq \text{tr}\{D^2 \alpha(\nabla \bar{\varphi}_\tau) \nabla^2 \bar{\varphi}_\tau\}.
\end{aligned}
$$

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From (3.13) it now follows

$$\beta(\nabla\bar\varphi_\tau)\partial_t\bar\varphi_\tau - \mathrm{tr}\{D^2\alpha(\nabla\bar\varphi_\tau)\nabla^2\bar\varphi_\tau\} - \gamma(\nabla\bar\varphi_\tau)f \geq 0 \quad \text{at } (x_\tau, t_\tau). \tag{3.14}$$

By the homogeneity of β, γ and α we obtain

$$\beta(\nabla\bar\varphi_\tau) = \beta(\nabla\varphi), \ \ \gamma(\nabla\bar\varphi_\tau) = \rho_\tau'(\kappa)\gamma(\nabla\varphi), \ \ D^2\alpha(\nabla\bar\varphi_\tau) = D^2\alpha(\nabla\varphi).$$

Combining (3.10)–(3.12) and above, we obtain from (3.14)

$$\beta(\nabla\varphi)\partial_t\varphi - \mathrm{tr}\{D^2\alpha(\nabla\varphi)\nabla^2\varphi\} - \gamma(\nabla\varphi)f$$
$$\geq \frac{\rho_\tau''(\kappa)}{\rho_\tau'(\kappa)}\mathrm{tr}\{D^2\alpha(\nabla\varphi)\nabla\varphi \otimes \nabla\varphi\} \quad \text{at } (x_\tau, t_\tau).$$

Since α is positively homogeneous of degree 2, we obtain

$$\mathrm{tr}\{D^2\alpha(\nabla\varphi)\nabla\varphi \otimes \nabla\varphi\} = \langle D^2\alpha(\nabla\varphi)\nabla\varphi, \nabla\varphi\rangle = 2\alpha(\nabla\varphi) = \gamma(\nabla\varphi)^2.$$

Moreover, we remark that $\rho_\tau''(\kappa)/\rho_\tau'(\kappa) = -\eta_\tau''(\rho_\tau(\kappa))\rho_\tau'(\kappa)^2 = -\eta_\tau''(\bar\varphi_\tau)\rho_\tau'(\kappa)^2$. Then we obtain

$$\frac{\rho_\tau''(\kappa)}{\rho_\tau'(\kappa)}\mathrm{tr}\{D^2\alpha(\nabla\varphi)\nabla\varphi \otimes \nabla\varphi\} = -\eta_\tau''(\bar\varphi_\tau)(\rho_\tau'(\kappa)\gamma(\nabla\varphi))^2$$
$$= -\eta_\tau''(\bar\varphi_\tau)\gamma(\nabla\bar\varphi_\tau)^2 = -\eta_\tau''(\bar\varphi_\tau).$$

Combining these, we obtain

$$\beta(\nabla\varphi)\partial_t\varphi - \mathrm{tr}\{D^2\alpha(\nabla\varphi)\nabla^2\varphi\} - \gamma(\nabla\varphi)f \geq -\eta_\tau''(\bar\varphi_\tau) \quad \text{at } (x_\tau, t_\tau). \tag{3.15}$$

Case 1.1. We verify (3.8) for the case $(x_0, t_0) \in \{(x, t); \ d(x, t) > 0\}$.
   By (3.7) we obtain

$$-\eta_\tau''(\bar\varphi_\tau) \geq -\frac{C_\eta}{\delta} \quad \text{at } (x_\tau, t_\tau).$$

We apply this estimate to (3.15) and send $\tau \to 0$ to get

$$\beta(\nabla\varphi)\partial_t\varphi - \mathrm{tr}\{D^2\alpha(\nabla\varphi)\nabla^2\varphi\} - \gamma(\nabla\varphi)f \geq -\frac{C_\eta}{\delta} \quad \text{at } (x_0, t_0).$$

Moreover, by (3.13) we observe that

$$1 = \gamma(\nabla\bar\varphi_\tau) = |\nabla\bar\varphi_\tau|\gamma\left(\frac{\nabla\bar\varphi_\tau}{|\nabla\bar\varphi_\tau|}\right) \geq \frac{|\nabla\bar\varphi_\tau|}{\Lambda_\gamma} \quad \text{at } (x_\tau, t_\tau).$$

By definition we have $|\nabla\varphi| = |\nabla\bar\varphi_\tau|/\rho_\tau'(\kappa)$ and $1/\rho_\tau'(\cdot) = \eta_\tau'(\rho_\tau(\cdot)) \leq C_\eta + \tau$. We thus obtain

$$|\nabla\varphi| \leq (C_\eta + \tau)|\nabla\bar\varphi_\tau| \leq (C_\eta + \tau)\Lambda_\gamma \quad \text{at } (x_\tau, t_\tau).$$

We send $\tau \to 0$ to get

$$|\nabla\varphi| \leq C_\eta\Lambda_\gamma =: C_\gamma \quad \text{at } (x_0, t_0).$$

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Case 1.2. We verify (3.9).

Since d is lower semicontinuous, there exists a positive constant $\tau_0 > 0$ such that $\tau < \tau_0$ implies $d(x_\tau, t_\tau) > \delta/2$. Since $\eta_\tau''(\sigma) = 0$ for $\sigma > \delta/2$, we obtain

$$\eta_\tau''(\varphi_\tau(x_\tau, t_\tau)) = \eta_\tau''(d(x_\tau, t_\tau)) = 0 \quad \text{for } \tau < \tau_0.$$

We apply this equality to (3.15) and send $\tau \to 0$ to get

$$\beta(\nabla\varphi)\partial_t\varphi - \text{tr}\{D^2\alpha(\nabla\varphi)\nabla^2\varphi\} - \gamma(\nabla\varphi)f \geq 0 \quad \text{at } (x_0, t_0).$$

Moreover, since $\nabla\varphi = \eta_\tau'(\bar\varphi_\tau)\nabla\bar\varphi_\tau = (1 + \tau)\nabla\bar\varphi_\tau$ we obtain by (3.13)

$$1 = \gamma\left(\frac{\nabla\varphi}{1 + \tau}\right) \quad \text{at } (x_\tau, t_\tau).$$

Sending $\tau \to 0$, we obtain

$$\gamma(\nabla\varphi) = 1 \quad \text{at } (x_0, t_0).$$

Case 2. Assume that $d(x_0, t_0) \leq 0$. Since d is left continuous in time in the sense

$$\lim_{x \to x_0} \lim_{t \uparrow t_0} d(x, t) = d(x_0, t_0),$$

there exist a positive constant r_0 and a some neighborhood $U_0(x_0)$ of x_0 satisfying

$$d(x, t) \leq \frac{\delta}{4} \quad \text{for } (x, t) \in U_0(x_0) \times (t_0 - r_0, t_0).$$

For $(x, t) \in U_0(x_0) \times (t_0 - r_0, t_0)$ we have $\omega(x, t) = -\delta$, i.e., ω is a constant there. This implies

$$\partial_t\varphi(x_0, t_0) \geq 0, \quad \nabla\varphi(x_0, t_0) = 0, \quad \nabla^2\varphi(x_0, t_0) \leq 0.$$

Since $\nabla\varphi(x_0, t_0) = 0$, we need to take an upper semicontinuous envelope of the equation (3.8).

We observe that

$$\beta(p)\partial_t\varphi \geq 0, \quad -\text{tr}\{D^2\alpha(p)\nabla^2\varphi(x_0, t_0)\} \geq 0 \quad \text{for } p \neq 0,$$

since $\beta > 0$ and γ^2 is strictly convex. We have $\lim_{p \to 0} \gamma(p)f = 0$. Therefore we obtain

$$[\beta(\nabla\varphi)\partial_t\varphi - \text{tr}\{D^2\alpha(\nabla\varphi)\nabla^2\varphi\} - \gamma(\nabla\varphi)f]^*(x_0, t_0)$$

$$\geq \lim_{p \to 0}[\beta(p)\partial_t\varphi(x_0, t_0) - \text{tr}\{D^2\alpha(p)\nabla^2\varphi(x_0, t_0)\} - \gamma(p)f] \geq 0 \geq -\frac{C_\eta}{\delta}.$$

Moreover we obtain $|\nabla\varphi(x_0, t_0)| = 0 \leq C_\gamma$. $\square$

We next give an estimate of $\partial_t\omega$ by using Lemma 3.3.

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Lemma 3.5. *Assume that β, γ and f satisfy (β1)–(β3), (γ1)–(γ5), and (f1), respectively. There exists a positive constant $C_{\beta,f}$ which depends only on n, Λ_β, and Λ_f such that the truncated anisotropic distance function $\omega(x,t) = \eta(d(x,t))$ is a viscosity supersolution of*

$$\partial_t \omega = -\frac{C_{\beta,f}}{\delta} \quad in \ \mathbf{R}^n \times (0,T). \tag{3.16}$$

Proof. We continue to use notations in the proof of Lemma 3.4.

Case 1. Assume that $d(x_0,t_0) > \delta/8$. Since d is a lower semicontinuous, there exists a positive constant $\tau_1 > 0$ satisfying

$$d(x_\tau, t_\tau) \geq \frac{\delta}{8} \quad \text{for } \tau < \tau_1.$$

Then we obtain from Lemma 3.3 that

$$\partial_t \bar\varphi_\tau \geq -L_{\beta,f}\left(1 + \frac{1}{d}\right) \quad \text{at } (x_\tau, t_\tau).$$

Since $\partial_t \bar\varphi_\tau = \rho'_\tau(\kappa)\partial_t \varphi$ and $d(x_\tau, t_\tau) \geq \delta/8$, we obtain

$$\partial_t \varphi \geq -\frac{L_{\beta,f}}{\rho'_\tau(\kappa)}\left(1 + \frac{8}{\delta}\right) \geq -\frac{9L_{\beta,f}(\|\eta'\|_\infty + 1)}{\delta} \quad \text{at } (x_\tau, t_\tau),$$

where we have invoked that $\tau \in (0,1)$ and $\delta \in (0,1)$. We take $C_{\beta,f} = 9L_{\beta,f}(C_\eta + 1)$ and send $\tau \to 0$ to get a desired conclusion in $\{(x,t); \ d(x,t) \geq \delta/8\}$.

Case 2. Assume that $d(x_0,t_0) \leq \delta/8$. By the similar argument as in Case 2 in the proof of Lemma 3.4, we obtain, in particular,

$$\partial_t \varphi(x_0,t_0) \geq 0 \geq -\frac{C_{\beta,f}}{\delta}. \ \square$$

4. Supersolution estimating internal layer

In this section we construct a supersolution for estimating a solution of (2.9). The basic strategy for the construction stems from [ESS]. We shall follow them.

Let u be a viscosity solution of (2.4) with initial data $u(x,0) = d_0(x)$. We remark that the set $\Gamma_t^\delta = \{x; \ u(x,t) = -2\delta\}$ is also a generalized solution of (2.1). So we introduce an anisotropic signed distance function $d_\delta(x,t)$ defined by

$$d_\delta(x,t) = \begin{cases} \Xi(x,\Gamma_t^\delta) & \text{if} \quad x \in \{y; \ u(y,t) \geq -2\delta\}, \\ -\Xi(x,\Gamma_t^\delta) & \text{if} \quad x \in \{y; \ u(y,t) < -2\delta\}. \end{cases}$$

By the definition of d_δ the properties in §3 still hold for d_δ and $\omega_\delta = \eta(d_\delta)$.

Combining this and the traveling wave in §2.3 we introduce a candidate of our viscosity supersolution for (2.9). We define a function $\psi_\varepsilon : \mathbf{R}^n \times (0,T) \to \mathbf{R}$ by

$$\psi_\varepsilon(x,t) = Q\left(\frac{\omega_\delta(x,t) + K_1 t}{\varepsilon}\right) + \varepsilon K_2,$$

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where K_1 and K_2 are positive constants determined later. We shall verify the following propositions.

Proposition 4.1. *Assume that β, γ, f and ε satisfy (β1)–(β3), (γ1)–(γ5), (f1) and (ε1), respectively. Then, for $\delta > 0$, there exist positive constants $K_1 = K_1(\delta)$, $K_2 = K_2(\delta, \Lambda_\beta, \Lambda_f)$ and $\varepsilon_0 = \varepsilon_0(\delta, \Lambda_\beta, \Lambda_\gamma, \Lambda_f)$ such that ψ_ε is a viscosity supersolution of*

$$\beta(\nabla\psi_\varepsilon)\partial_t\psi_\varepsilon - \mathrm{div}\{\gamma(\nabla\psi_\varepsilon)\xi(\nabla\psi_\varepsilon)\} + \frac{1}{\varepsilon^2}(W'(\psi_\varepsilon) - \varepsilon\lambda f) = \frac{K_{\beta,\delta,f}}{\varepsilon} \quad \text{in } \mathbf{R}^n \times (0, T)$$

provided that $\varepsilon \in (0, \varepsilon_0)$, where $K_{\beta,\delta,f}$ is a numerical positive constant depending only on Λ_β, Λ_f and δ.

This proposition says not only ψ_ε is a viscosity supersolution of (2.6) but also the left hand side of (2.6) with $v = \psi_\varepsilon$ goes up to $+\infty$ by the order $1/\varepsilon$.

Proof. We shall take ε_0 small 7 times in our proof; i.e., (4.5), (4.10), (4.11), (4.12), (4.14), (4.18), and (4.19). It suffices to take a minimum of these choices to obtain a conclusion.

Let $(x_\varepsilon, t_\varepsilon) \in \mathbf{R}^n \times (0, T)$ and let $\varphi \in C^2(\mathbf{R}^n \times (0, T))$ satisfy

$$\psi_\varepsilon(x, t) - \varphi(x, t) > \psi_\varepsilon(x_\varepsilon, t_\varepsilon) - \varphi(x_\varepsilon, t_\varepsilon) = 0 \quad \text{whenever } (x, t) \neq (x_\varepsilon, t_\varepsilon).$$

Since $Q' > 0$ in $\mathbf{R}$ we have $Q^{-1} \in C^\infty(\mathbf{R})$ and $(Q^{-1})' > 0$. Then we observe that a function $\tilde{\varphi}$ defined by

$$\tilde{\varphi}(x, t) = \varepsilon Q^{-1}(\varphi(x, t) - \varepsilon K_2) - K_1 t$$

satisfy $\tilde{\varphi} \in C^{2,1}(\mathbf{R}^n \times (0, T))$ and

$$\omega_\delta(x, t) - \tilde{\varphi}(x, t) \geq \omega_\delta(x_\varepsilon, t_\varepsilon) - \tilde{\varphi}(x_\varepsilon, t_\varepsilon) \quad \text{for } (x, t) \in \mathbf{R}^n \times (0, T),$$

$$\varphi(x, t) = Q\left(\frac{\tilde{\varphi}(x, t) + K_1 t}{\varepsilon}\right) + \varepsilon K_2.$$

By the straightforward calculation we obtain

$$\partial_t\varphi = \frac{1}{\varepsilon}Q'(h)(\partial_t\tilde{\varphi} + K_1), \tag{4.1}$$

$$\nabla\varphi = \frac{1}{\varepsilon}Q'(h)\nabla\tilde{\varphi}, \tag{4.2}$$

$$\nabla^2\varphi = \frac{1}{\varepsilon^2}Q''(h)\nabla\tilde{\varphi} \otimes \nabla\tilde{\varphi} + \frac{1}{\varepsilon}Q'(h)\nabla^2\tilde{\varphi}, \tag{4.3}$$

where $h = h(x, t) = (\tilde{\varphi}(x, t) + K_1 t)/\varepsilon$. Moreover, we observe that

$$W'(\psi_\varepsilon) = W'(\varphi) = W'(Q(h)) + \varepsilon K_2 W''(Q(h)) + O(\varepsilon^2 K_2^2) \quad \text{at } (x_\varepsilon, t_\varepsilon) \tag{4.4}$$

as $\varepsilon \to 0$. We now take $\varepsilon_0 = \varepsilon_0(K_2)$ small so that

$$|\varepsilon K_2| \leq 1 \quad \text{provided } \varepsilon \in (0, \varepsilon_0). \tag{4.5}$$

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Case 1. We assume that $(x_\varepsilon, t_\varepsilon)$ satisfies $d_\delta(x_\varepsilon, t_\varepsilon) > \delta/2$. By Lemma 3.4 we have

$$\left.\begin{array}{l} \gamma(\nabla\tilde\varphi) = 1, \text{ in particular } \nabla\tilde\varphi \neq 0, \\[4pt] \beta(\nabla\tilde\varphi)\partial_t\tilde\varphi - \mathrm{tr}\{D^2\alpha(\nabla\tilde\varphi)\nabla^2\tilde\varphi\} - \gamma(\nabla\tilde\varphi)f \geq 0 \end{array}\right\} \quad \text{at } (x_\varepsilon, t_\varepsilon). \qquad (4.6)$$

We observe that $\nabla\varphi \neq 0$ since $\nabla\tilde\varphi \neq 0$. We set

$$R_\varepsilon = R_\varepsilon(x,t) = \beta(\nabla\varphi)\partial_t\varphi - \mathrm{tr}\{D^2\alpha(\nabla\varphi)\nabla^2\varphi\} + \frac{1}{\varepsilon^2}(W'(\psi_\varepsilon) - \varepsilon\lambda f). \qquad (4.7)$$

Our aim is to show that there exists a positive constant $K_{\beta,\delta,f}$, which depends only on Λ_β, Λ_f and δ, satisfying $R_\varepsilon \geq K_{\beta,\delta,f}/\varepsilon$ at $(x_\varepsilon, t_\varepsilon)$.

By the homogeneity of β, γ and α we observe that

$$\beta(\nabla\varphi) = \beta(\nabla\tilde\varphi), \ \gamma(\nabla\varphi) = \frac{1}{\varepsilon}Q'(h)\gamma(\nabla\tilde\varphi), \ D^2\alpha(\nabla\varphi) = D^2\alpha(\nabla\tilde\varphi).$$

By (4.1)–(4.3) we now obtain

$$\beta(\nabla\varphi)\partial_t\varphi = \beta(\nabla\tilde\varphi)Q'(h)\frac{\partial_t\tilde\varphi + K_1}{\varepsilon},$$

and

$$\begin{aligned} \mathrm{tr}\{D^2\alpha(\nabla\varphi)\nabla^2\varphi\} &= \frac{Q''(h)}{\varepsilon^2}\langle D^2\alpha(\nabla\tilde\varphi)\nabla\tilde\varphi, \nabla\tilde\varphi\rangle + \frac{Q'(h)}{\varepsilon}\mathrm{tr}\{D^2\alpha(\nabla\tilde\varphi)\nabla^2\tilde\varphi\} \\ &= \frac{Q''(h)}{\varepsilon^2}\gamma(\nabla\tilde\varphi)^2 + \frac{Q'(h)}{\varepsilon}\mathrm{tr}\{D^2\alpha(\nabla\tilde\varphi)\nabla^2\tilde\varphi\}. \end{aligned}$$

Here we have invoked the property that $\langle D^2\alpha(p)p, p\rangle = 2\alpha(p) = \gamma(p)^2$ for $p \neq 0$ since α is positively homogeneous of degree 2. Combining (4.4) and above, we conclude that

$$\begin{aligned} R_\varepsilon &= \frac{1}{\varepsilon^2}I_{-2} + \frac{1}{\varepsilon}I_{-1} + O(K_2^2), \\ I_{-2} &= -Q''(h)\gamma(\nabla\tilde\varphi)^2 + W'(Q(h)) - \varepsilon\lambda f, \qquad\qquad\qquad (4.8) \\ I_{-1} &= K_2 W''(Q(h)) + Q'(h)[\beta(\nabla\tilde\varphi)K_1 \\ &\qquad\qquad + \beta(\nabla\tilde\varphi)\partial_t\tilde\varphi - \mathrm{tr}\{D^2\alpha(\nabla\tilde\varphi)\nabla^2\tilde\varphi\}]. \qquad (4.9) \end{aligned}$$

By (2.10) and since $\gamma(\nabla\tilde\varphi) = 1$, we obtain

$$I_{-2} = -Q''(h) + W'(Q(h)) - \varepsilon\lambda f = cQ'(h).$$

Then, by using (4.6), we obtain

$$\begin{aligned} R_\varepsilon &= \frac{1}{\varepsilon}\Big(K_2 W''(Q(h)) + Q'(h)\Big[f + \frac{c}{\varepsilon} + \beta(\nabla\tilde\varphi)K_1 \\ &\qquad\qquad + \beta(\nabla\tilde\varphi)\partial_t\tilde\varphi - \mathrm{tr}\{D^2\alpha(\nabla\tilde\varphi)\nabla^2\tilde\varphi\} - \gamma(\nabla\tilde\varphi)f\Big]\Big) + O(K_2^2) \\ &\geq \frac{1}{\varepsilon}\Big[K_2 W''(Q(h)) + Q'(h)\Big(f + \frac{c}{\varepsilon} + \frac{K_1}{\Lambda_\beta}\Big)\Big] + O(K_2^2) \quad \text{at } (x_\varepsilon, t_\varepsilon). \end{aligned}$$

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We now determine K_1. We take

$$K_1 = \frac{\delta}{4T}.$$

The reason why we take such K_1 is clarified in the Case 2. By Proposition 2.1(i) we take $\varepsilon_0 = \varepsilon_0(\delta, \Lambda_f)$ smaller so that

$$f + \frac{c}{\varepsilon} \geq -\frac{K_1}{2\Lambda_\beta} = -\frac{\delta}{8\Lambda_\beta T} \quad \text{provided } \varepsilon \in (0, \varepsilon_0). \tag{4.10}$$

Then we obtain

$$R_\varepsilon \geq \frac{1}{\varepsilon}[K_2 W''(Q(h)) + Q'(h)C_{\beta,\delta}] + O(K_2^2) \quad \text{at } (x_\varepsilon, t_\varepsilon),$$

where $C_{\beta,\delta} = \delta/(8\Lambda_\beta T)$.

Here we shall determine K_2. We take suitable K_2 to estimate R_ε in the case that $W''(Q(h)) < 0$. The basic strategy stems from the fact that $Q(\sigma) \to \tanh\sigma$ uniformly with respect to $\sigma \in \mathbf{R}$ and f satisfying $(f1)$ as $\varepsilon \to 0$, $W'(\tanh\sigma) \geq 0$ for enough large $|\sigma|$, and an local uniform bound of Q' from below with respect to f and ε satisfying $(f1)$ and $(\varepsilon 1)$.

By Proposition 2.1(ii) we take $\varepsilon_0 = \varepsilon_0(\Lambda_f)$ smaller so that

$$|Q(\sigma) - \tanh\sigma| \leq \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right) =: \nu \quad \text{for } \sigma \in \mathbf{R} \text{ provided } \varepsilon \in (0, \varepsilon_0). \tag{4.11}$$

We take $b = -\sup\{\sigma; \ \tanh\sigma + \nu \leq -1/\sqrt{2}\} = \inf\{\sigma; \ \tanh\sigma - \nu \geq 1/\sqrt{2}\}$ and let

$$K_2 = \frac{a_2 C_{\beta,\delta}}{2a_1},$$
$$a_1 = |\inf_{|\sigma| \leq 1+\nu} W''(\sigma)|, \ a_2 = \inf\{Q'(\sigma); \ |\sigma| \leq b, \ \varepsilon \in (0, \bar{\varepsilon}), \ |f| \leq \Lambda_f\}.$$

We remark that there exists such a $a_2 > 0$ by Proposition 2.1 (iii). Moreover, we remark that $\varepsilon_0 = \varepsilon_0(K_2)$ implies that ε_0 depends on δ, Λ_β and Λ_f.

We divide the situation into two cases.

Case 1.1. Assume that $(x_\varepsilon, t_\varepsilon) \in \{(x,t); \ |h(x,t)| \leq b\}$. Then we observe that $Q(h(x_\varepsilon, t_\varepsilon)) \leq 1/\sqrt{2}$ or $W''(Q(h(x_\varepsilon, t_\varepsilon))) \leq 1$. Therefore we obtain

$$R_\varepsilon \geq \frac{1}{\varepsilon}(-K_2 a_1 + a_2 C_{\beta,\delta}) + O(K_2^2) \geq \frac{a_2 C_{\beta,\delta}}{2\varepsilon} + O(K_2^2) \quad \text{at } (x_\varepsilon, t_\varepsilon).$$

This is the reason why we take K_2 as above. We now take $\varepsilon_0 = \varepsilon_0(\delta, \Lambda_\beta, \Lambda_f)$ smaller so that

$$|\varepsilon O(K_2^2)| \leq \frac{a_2 C_{\beta,\delta}}{4} \quad \text{provided } \varepsilon \in (0, \varepsilon_0). \tag{4.12}$$

Then we obtain

$$R_\varepsilon \geq \frac{a_2 C_{\beta,\delta}}{4\varepsilon} > 0 \quad \text{at } (x_\varepsilon, t_\varepsilon). \tag{4.13}$$

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Case 1.2. Assume that $(x_\varepsilon, t_\varepsilon) \in \{(x,t); \ |h(x,t)| > b\}$. Then we observe that $W''(Q(h(x_\varepsilon, t_\varepsilon))) > 1$ and

$$R_\varepsilon \geq \frac{K_2}{\varepsilon} + O(K_2^2) \quad \text{at } (x_\varepsilon, t_\varepsilon).$$

We now take $\varepsilon_0 = \varepsilon_0(\delta, \Lambda_\beta, \Lambda_f)$ smaller so that

$$|\varepsilon O(K_2^2)| \leq \frac{K_2}{2} \quad \text{provided } \varepsilon \in (0, \varepsilon_0). \tag{4.14}$$

Then we obtain

$$R_\varepsilon \geq \frac{K_2}{2\varepsilon} > 0 \quad \text{at } (x_\varepsilon, t_\varepsilon). \tag{4.15}$$

Case 2. We assume that $(x_\varepsilon, t_\varepsilon) \in \{(x,t); \ d_\delta(x,t) \leq \delta/2\}$. By Lemma 3.4 we have

$$|\nabla\tilde{\varphi}(x_\varepsilon, t_\varepsilon)| \leq C_\gamma,$$

$$[\beta(\nabla\tilde{\varphi})\partial_t\tilde{\varphi} - \text{tr}\{D^2\alpha(\nabla\tilde{\varphi})\nabla^2\tilde{\varphi}\} - \gamma(\nabla\tilde{\varphi})f]^*(x_\varepsilon, t_\varepsilon) \geq -\frac{C_\eta}{\delta}. \tag{4.16}$$

We first observe that

$$h(x_\varepsilon, t_\varepsilon) = \frac{\eta(d_\delta(x_\varepsilon, t_\varepsilon)) + K_1 t_\varepsilon}{\varepsilon} \leq -\frac{\delta}{4\varepsilon} < 0, \tag{4.17}$$

i.e., $h \to -\infty$ as $\varepsilon \to 0$. This is the reason why we set K_1 near (4.10). Therefore we take $\varepsilon_0 = \varepsilon_0(\delta)$ smaller so that

$$(x_\varepsilon, t_\varepsilon) \in \{(x,t); \ W''(Q(h(x,t))) \geq 1\}$$

$$\text{for } (x_\varepsilon, t_\varepsilon) \in \left\{(x,t); \ d_\delta(x,t) < \frac{\delta}{2}\right\} \quad \text{provided } \varepsilon \in (0, \varepsilon_0). \tag{4.18}$$

Case 2.1. Assume that $\nabla\tilde{\varphi}(x_\varepsilon, t_\varepsilon) \neq 0$. By the same argument in Case 1.1, it suffices to see

$$R_\varepsilon = \frac{1}{\varepsilon^2}I_{-2} + \frac{1}{\varepsilon}I_{-1} + O(K_2^2) \geq \frac{K_{\delta,\beta,f}}{\delta} \quad \text{at } (x_\varepsilon, t_\varepsilon),$$

where R_ε, I_{-2} and I_{-1} are defined by (4.7), (4.8), and (4.9), respectively. We remark that $\gamma(\nabla\tilde{\varphi}(x_\varepsilon, t_\varepsilon)) \neq 1$ in this case. Therefore we obtain

$$I_{-2} = -Q''(h)(\gamma(\nabla\tilde{\varphi})^2 - 1) + cQ'(h).$$

Then we observe from (4.16)

$$R_\varepsilon \geq -\frac{1}{\varepsilon^2}Q''(h)(\gamma(\nabla\tilde{\varphi})^2 - 1) + \frac{1}{\varepsilon}\left[K_2 + Q'(h)\left\{C_{\beta,\delta} + (\gamma(\nabla\tilde{\varphi}) - 1)f - \frac{C_\eta}{\delta}\right\}\right]$$
$$+ O(K_2^2) \quad \text{at } (x_\varepsilon, t_\varepsilon).$$

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By the homogeneity of γ we obtain

$$\gamma(\nabla\tilde{\varphi}) = |\nabla\tilde{\varphi}|\gamma\left(\frac{\nabla\tilde{\varphi}}{|\nabla\tilde{\varphi}|}\right) \leq C_\gamma\Lambda_\gamma \quad \text{at } (x_\varepsilon, t_\varepsilon).$$

Therefore we obtain

$$R_\varepsilon \geq -\frac{1}{\varepsilon^2}(C_\gamma^2\Lambda_\gamma^2 + 1)|Q''(h)| + \frac{1}{\varepsilon}\{K_2 - C_{\beta,\gamma,\delta}|Q'(h)|\} + O(K_2^2) \quad \text{at } (x_\varepsilon, t_\varepsilon),$$

where $C_{\beta,\gamma,\delta} := C_{\beta,\delta} + (C_\gamma\Lambda_\gamma + 1)|f| + C_\eta/\delta$ is a constant. By (2.12) and (4.17) we obtain

$$R_\varepsilon \geq \frac{1}{\varepsilon}\left\{K_2 - \left(C_{\beta,\gamma,\delta} + \frac{C_\gamma^2\Lambda_\gamma^2 + 1}{\varepsilon}\right)C_1\exp(-C_2|h|)\right\} + O(K_2^2)$$

$$\geq \frac{1}{\varepsilon}\left\{K_2 - \left(C_{\beta,\gamma,\delta} + \frac{C_\gamma^2\Lambda_\gamma^2 + 1}{\varepsilon}\right)C_1\exp\left(-\frac{C_2\delta}{4\varepsilon}\right)\right\} + O(K_2^2) \quad \text{at } (x_\varepsilon, t_\varepsilon).$$

We take $\varepsilon_0 = \varepsilon_0(\delta, \Lambda_\beta, \Lambda_\gamma, \Lambda_f)$ smaller so that

$$\left.\begin{array}{l}\left|\left(C_{\beta,\gamma,\delta} + \dfrac{C_\gamma^2\Lambda_\gamma^2 + 1}{\varepsilon}\right)C_1\exp\left(-\dfrac{C_2\delta}{4\varepsilon}\right)\right| \leq \dfrac{K_2}{4}, \\[4mm] |\varepsilon O(K_2^2)| \leq \dfrac{K_2}{4}\end{array}\right\} \quad \text{provided } \varepsilon \in (0, \varepsilon_0). \qquad (4.19)$$

Then we obtain

$$R_\varepsilon \geq \frac{K_2}{2\varepsilon} > 0 \quad \text{at } (x_\varepsilon, t_\varepsilon). \qquad (4.20)$$

Case 2.2. Assume that $\nabla\tilde{\varphi}(x_\varepsilon, t_\varepsilon) = 0$. We need to consider the equations in week sense. We now set $\hat{\sigma}_\varepsilon = \psi_\varepsilon(x_\varepsilon, t_\varepsilon)$, $\hat{s}_\varepsilon = \partial_t\varphi(x_\varepsilon, t_\varepsilon)$, $\hat{p}_\varepsilon = \nabla\varphi(x_\varepsilon, t_\varepsilon)$, $\hat{X}_\varepsilon = \nabla^2\varphi(x_\varepsilon, t_\varepsilon)$, and

$$\bar{R}_\varepsilon = \varlimsup_{r\to 0}\{\beta(p)s - \text{tr}\{D^2\alpha(p)X\} + \frac{1}{\varepsilon^2}(W'(\sigma) - \varepsilon\lambda f);$$

$$|\sigma - \hat{\sigma}_\varepsilon| < r, \ |s - \hat{s}_\varepsilon| < r, \ |p - \hat{p}_\varepsilon| < r, \ |X - \hat{X}_\varepsilon| < r\},$$

We shall prove $\bar{R}_\varepsilon \geq K_{\beta,\delta,f}/\delta$. By (4.16) there exists a sequence $\{(\tau_j, q_j, Y_j)\}_{j=1}^\infty$ satisfying

$$\lim_{j\to\infty}(\tau_j, q_j, Y_j) = (\partial_t\tilde{\varphi}(x_\varepsilon, t_\varepsilon), 0, \nabla^2\tilde{\varphi}(x_\varepsilon, t_\varepsilon)),$$

$$q_j \neq 0, \ \varlimsup_{j\to\infty}|q_j| \leq C_\gamma,$$

$$\lim_{j\to\infty}[\beta(q_j)\tau_j - \text{tr}\{D^2\alpha(q_j)Y_j\} - \gamma(q_j)f] \geq 0.$$

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We now set

$$\begin{cases} \sigma_j = Q(h(x_\varepsilon, t_\varepsilon)) + \varepsilon K_2 = \hat{\sigma}_\varepsilon, \\[2mm] s_j = \dfrac{1}{\varepsilon} Q'(h(x_\varepsilon, t_\varepsilon))(\tau_j + K_1) \to \hat{s}_\varepsilon, \\[2mm] p_j = \dfrac{1}{\varepsilon} Q'(h(x_\varepsilon, t_\varepsilon)) q_j \to 0 = \hat{p}_\varepsilon, \\[2mm] X_j = \dfrac{1}{\varepsilon^2} Q''(h(x_\varepsilon, t_\varepsilon)) q_j \otimes q_j + \dfrac{1}{\varepsilon} Q'(h(x_\varepsilon, t_\varepsilon)) Y_j \to \hat{X}_\varepsilon, \end{cases} \tag{4.21}$$

where limits are taken as $j \to \infty$. Moreover, let R_ε^j be defined by

$$R_\varepsilon^j := \beta(p_j) s_j - \operatorname{tr}\{D^2 \alpha(p_j) X_j\} + \frac{1}{\varepsilon^2}(W'(\sigma_j) - \varepsilon \lambda f).$$

From the arguments in Case 2.1 it follows that

$$\bar{R}_\varepsilon \geq \lim_{j \to \infty} R_\varepsilon^j \geq \frac{K_2}{2\varepsilon} > 0. \tag{4.22}$$

Set $K_{\beta, \delta, f} = \min\{K_2/2, a_2 C_{\beta, \delta}/4\}$. Then we conclude from (4.13), (4.15), (4.20), and (4.22),

$$\left[\beta(\nabla\varphi)\partial_t\varphi - \operatorname{tr}\{D^2 \alpha(\nabla\varphi)\nabla^2\varphi\} + \frac{1}{\varepsilon^2}(W'(\psi_\varepsilon) - \varepsilon\lambda f)\right]^*(x_\varepsilon, t_\varepsilon) \geq \frac{K_{\beta, \delta, f}}{\varepsilon} > 0. \ \Box \ (4.23)$$

We are now in position to see ψ_ε is a viscosity supersolution of (2.9).

Proposition 4.2. *Assume that β, γ, f and ε satisfy $(\beta 1)$–$(\beta 3)$, $(\gamma 1)$–$(\gamma 5)$, $(f1)$ and $(\varepsilon 1)$, respectively. Then, for $\delta > 0$, there exists a positive constant $\varepsilon_1 = \varepsilon_1(\delta, \Lambda_\beta, \Lambda_\gamma, \Lambda_f)$ such that ψ_ε is a viscosity supersolution of (2.9) in $\mathbf{R}^n \times (0, T)$ provided that $\varepsilon \in (0, \varepsilon_1)$.*

Proof. We continue the proof from Proposition 4.1. We first fix $\varepsilon_1 < \varepsilon_0$. In this proof, we take ε_1 twice, (4.25) and (4.26). It suffices to take their minimum.

Case 1. We assume that $(x_\varepsilon, t_\varepsilon) \in \{(x, t); \ d_\delta(x, t) > \delta/2\}$. Since $\gamma(\nabla\tilde{\varphi}) = 1$ we observe that $\nabla\varphi \neq 0$. We now set

$$\tilde{R}_\varepsilon = \tilde{\beta}(\nabla\varphi)\partial_t\varphi - \operatorname{tr}\{D^2 \alpha(\nabla\varphi)\nabla^2\varphi\} + \frac{1}{\varepsilon^2}(W'(\varphi) - \varepsilon\lambda f). \tag{4.24}$$

Our aim is to show $\tilde{R}_\varepsilon \geq 0$ at $(x_\varepsilon, t_\varepsilon)$. By straightforward calculation we obtain

$$\begin{aligned} \tilde{R}_\varepsilon &= R_\varepsilon + \tilde{R}_\varepsilon - R_\varepsilon \\[1mm] &\geq \frac{K_{\beta, \delta, f}}{\varepsilon} + (\tilde{\beta}(\nabla\varphi) - \beta(\nabla\varphi))\partial_t\varphi \\[1mm] &= \frac{1}{\varepsilon}[K_{\beta, \delta, f} + Q'(h)\zeta(|\nabla\varphi|)(\Lambda_\beta - \beta(\nabla\varphi))(\partial_t\tilde{\varphi} + K_1)] \quad \text{at } (x_\varepsilon, t_\varepsilon). \end{aligned}$$

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We observe that
$$Q'(h)\zeta(|\nabla\varphi|)(\Lambda_\beta - \beta(\nabla\varphi))K_1 \geq 0.$$
Moreover, we obtain from Lemma 3.5,
$$Q'(h)\zeta(|\nabla\varphi|)(\Lambda_\beta - \beta(\nabla\varphi))\partial_t\tilde\varphi \geq -Q'(h)\zeta(|\nabla\varphi|)(\Lambda_\beta - \beta(\nabla\varphi))\frac{C_\beta}{\delta}$$
$$\geq -Q'(h)\zeta(|\nabla\varphi|)\left(\Lambda_\beta - \frac{1}{\Lambda_\beta}\right)\frac{C_\beta}{\delta} \quad \text{at } (x_\varepsilon, t_\varepsilon).$$

Thus we obtain
$$\tilde R_\varepsilon \geq \frac{1}{\varepsilon}\left[K_{\beta,\delta,f} - M_{\beta,\delta}Q'(h)\zeta(|\nabla\varphi|)\right] \quad \text{at } (x_\varepsilon, t_\varepsilon),$$

where $M_{\beta,\delta} := (\Lambda_\beta - \frac{1}{\Lambda_\beta})\frac{C_\beta}{\delta}$. We now study $\tilde R_\varepsilon$ in two pieces.

Case 1.1. Assume that $(x_\varepsilon, t_\varepsilon) \in \{(x,t); \ M_{\beta,\delta}Q'(h(x,t)) < K_{\beta,\delta,f}/2\}$. In this case it is easy to see
$$\tilde R_\varepsilon \geq \frac{K_{\beta,\delta,f}}{2\varepsilon} > 0 \quad \text{at } (x_\varepsilon, t_\varepsilon).$$

Case 1.2. Assume that $(x_\varepsilon, t_\varepsilon) \in \{(x,t); \ M_{\beta,\delta}Q'(h(x,t)) \geq K_{\beta,\delta,f}/2\}$. We remark that $|\nabla\varphi| = Q'(h)|\nabla\tilde\varphi|/\varepsilon$. Since $\gamma(\nabla\tilde\varphi) = 1$ we obtain
$$|\nabla\tilde\varphi| \geq \frac{1}{\Lambda_\gamma} \quad \text{at } (x_\varepsilon, t_\varepsilon).$$

Then we obtain
$$|\nabla\varphi| = \frac{Q'(h)|\nabla\tilde\varphi|}{\varepsilon} \geq \frac{K_{\beta,\delta,f}}{2\varepsilon\Lambda_\gamma M_{\beta,\delta}} \quad \text{at } (x_\varepsilon, t_\varepsilon).$$
We take $\varepsilon_1 = \varepsilon_1(\delta, \Lambda_\beta, \Lambda_f)$ smaller so that
$$\frac{K_{\beta,\delta,f}}{2\varepsilon\Lambda_\beta M_{\beta,\delta}} \geq 1 \quad \text{provided } \varepsilon \in (0, \varepsilon_1). \tag{4.25}$$

Then we obtain $|\nabla\varphi| \geq 1 \geq 3/4$, i.e., $\zeta(|\nabla\varphi|) = 0$ at $(x_\varepsilon, t_\varepsilon)$. Thus we obtain
$$\tilde R_\varepsilon \geq \frac{K_{\beta,\delta,f}}{\varepsilon} > 0 \quad \text{at } (x_\varepsilon, t_\varepsilon).$$

Case 2. We assume that $(x_\varepsilon, t_\varepsilon) \in \{(x,t); \ d_\delta(x,t) \leq \delta/2\}$. By (4.17) there exists $\varepsilon_1 = \varepsilon_1(\delta, \Lambda_\beta, \Lambda_f)$ satisfying
$$(x_\varepsilon, t_\varepsilon) \in \left\{(x,t); \ M_{\beta,\delta}Q'(h(x,t)) < \frac{K_{\beta,\delta,f}}{2}\right\} \quad \text{provided } \varepsilon \in (0, \varepsilon_1). \tag{4.26}$$

We take ε_1 satisfying (4.26).

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Case 2.1. Assume that $\nabla\varphi(x_\varepsilon, t_\varepsilon) \neq 0$. From the same argument in Case 1.1 of this proof we obtain

$$\tilde{R}_\varepsilon \geq \frac{K_{\beta,\delta,f}}{2\varepsilon} > 0 \quad \text{at } (x_\varepsilon, t_\varepsilon).$$

Case 2.2. Assume that $\nabla\varphi(x_\varepsilon, t_\varepsilon) = 0$. Since $\overline{\lim}_{j\to\infty} R_\varepsilon^j \geq K_{\beta,\delta,f}/\varepsilon$, there exists a positive number $N_1 \in \mathbf{N}$ satisfying

$$R_\varepsilon^j \geq \frac{7K_{\beta,\delta,f}}{8\varepsilon} \quad \text{for } j > N_1$$

by taking a subsequence of $\{R_\varepsilon^j\}$ if it is necessary. Set

$$\tilde{R}_\varepsilon^j = \tilde{\beta}(p_j)s_j - \text{tr}\{D^2\alpha(p_j)X_j\} + \frac{1}{\varepsilon^2}(W'(\sigma_j) - \varepsilon\lambda f),$$

where σ_j, s_j, p_j and X_j is as in (4.21). By (4.21) there exists a positive number $N_2 \in \mathbf{N}$ satisfying

$$Q'(h(x_\varepsilon, t_\varepsilon))\left(\Lambda_\beta - \frac{1}{\Lambda_\beta}\right)\tau_j \geq -\frac{5K_{\beta,\delta,f}}{8} \quad \text{for } j > N_2$$

since $\tau_j \to \partial_t\tilde{\varphi}(x_\varepsilon, t_\varepsilon)$ as $j \to \infty$ and by (4.26). Then we obtain

$$\tilde{R}_\varepsilon^j = R_\varepsilon^j + \tilde{R}_\varepsilon^j - R_\varepsilon^j \geq \frac{1}{\varepsilon}\left(\frac{7K_{\beta,\delta,f}}{8} + M_{\beta,\delta}Q'(h(x_\varepsilon, t_\varepsilon))\zeta(|p_j|)\tau_j\right) \geq \frac{K_{\beta,\delta,f}}{4\varepsilon} > 0$$

for $j > N = \max\{N_1, N_2\}$. We thus obtain

$$(\tilde{R}_\varepsilon)^*(x_\varepsilon, t_\varepsilon) \geq \overline{\lim_{j\to\infty}} R_\varepsilon^j \geq \frac{K_{\beta,\delta,f}}{4\varepsilon} > 0. \ \square$$

5. Uniform Estimate

In this section, we shall prove Theorem 2.2

Proof of Theorem 2.2. Let v be a solution of (2.9) with $v(x,0) = Q(d_0(x)/\varepsilon)$, and ψ_ε be that is defined in §4. We first verify that, for $\delta > 0$,

$$\psi_\varepsilon(x,0) \geq v(x,0) \quad \text{for } x \in \mathbf{R}^n. \tag{5.1}$$

We remark that $\omega_\delta(x,0) = \eta(d_\delta(x,0)) \geq d_0(x)$. Let $y \in \Gamma_0^\delta$ be such that $d_\delta(x,0) = \gamma^\circ(x-y)$. Then we observe that

$$d_0(x) - d_0(y) \leq \gamma^\circ(x-y) = d_\delta(x,0).$$

By the definition of Γ_0^δ, we observe that $d_0(y) = -2\delta$. Then we obtain

$$d_\delta(x,0) \geq d_0(x) + 2\delta.$$

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If $d_\delta(x, 0) \geq \delta/2$, then we obtain

$$\eta(d_\delta(x, 0)) = d_\delta(x, 0) - \delta \geq d_0(x) + \delta > d_0(x).$$

If $d_\delta(x, 0) < \delta/2$, then we observe that

$$d_0(x) \leq d_\delta(x, 0) - 2\delta < -\frac{3\delta}{2} < -\delta \leq \eta(d_\delta(x, 0)).$$

Thus we obtain

$$\psi_\varepsilon(x, 0) = Q\left(\frac{\omega_\delta(x, 0)}{\varepsilon}\right) + \varepsilon K_2 \geq Q\left(\frac{d_0(x)}{\varepsilon}\right) = v(x, 0).$$

By the comparison principle and (5.1) we obtain, for $\delta > 0$,

$$\psi_\varepsilon(x, t) \geq v(x, t) \quad \text{for } (x, t) \in \mathbf{R}^n \times (0, T).$$

Fix $\theta > 0$. We take δ satisfying $d_\delta(x, t) < 0$ if $d(x, t) < -\theta$. We recall (4.17) that is

$$\frac{\omega_\delta(x, t) + K_1 t}{\varepsilon} < -\frac{\delta}{4\varepsilon} \quad \text{for } (x, t) \in \{(x, t); \ d_\delta(x, t) < \delta/2\}.$$

Therefore we obtain from (2.11),

$$\psi_\varepsilon(x, t) = Q\left(\frac{\omega_\delta(x, t) + K_1 t}{\varepsilon}\right) + \varepsilon K_2$$
$$\leq Q\left(-\frac{\delta}{4\varepsilon}\right) + \varepsilon K_2 \leq -1 + C_1 \exp\left(-\frac{C_2\delta}{4\varepsilon}\right) + \varepsilon(K_2 + C_3)$$

for $(x, t) \in \{(x, t); \ d(x, t) < -\theta\}$. Combining all above inequalities, we consequently obtain

$$v(x, t) \leq -1 + C_1 \exp\left(-\frac{C_2\delta}{4\varepsilon}\right) + C\varepsilon \quad \text{for } (x, t) \in \{(x, t); \ d(x, t) < -\theta\},$$

provided $\varepsilon \in (0, \varepsilon_1)$, where C_1 and C_2 are numerical constants, $C = K_2 + C_3$ is a positive constant depending only on Λ_β and δ, and ε_0 is a positive constant depending only on Λ_β, Λ_γ, $|u|$ and δ. $\square$

6. Concluding remarks

We shall explain the difference between [ElS2] and ours and also discuss remaining problems. Here we keep our notation α, γ, β and f which are denoted by A, B, β and u in [ElS2].

(i) (Essential Difference.) If we further assume the driving force f is constant in [ElS2], a lot of propositions in ours have something in common with those of [ElS2], for examples our Lemma 3.2 and [ElS2, Lemma 3.3]. The crucial difference is found in

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the proof of our Lemma 3.3 and [ElS2, Lemma 3.4]. In [ElS2] the authors estimate that, for $F(p, X) = -\gamma(p)\{\text{tr}(D^2\gamma(p)X) + f\}$,

$$F(p, I) \leq C(\tilde{\Lambda}_\gamma)(1 + |p|) \quad \text{for } p \neq 0, \tag{6.1}$$

where $C(\tilde{\Lambda}_\gamma)$ is a constant depending on $\tilde{\Lambda}_\gamma := \|\gamma\|_{C^2(B_2(0)\setminus B_{1/2}(0))}$. By using this estimate they prove that $z = z(x, t)$ in the proof of our Lemma 3.3 is a viscosity subsolution of (2.4), and consequently determine L. In the case f is a constant, this dependence of L with respect to the second derivatives of γ is the crucial reason why the estimate of convergence depends on the derivatives of γ. In this paper, without using the estimate (6.1), we rather calculate the quantity

$$F(D\gamma^\circ(p), D^2\gamma^\circ(p)) = -\frac{n-1}{\gamma^\circ(p)} - f \quad \text{for } p \neq 0,$$

by using the convex analysis and determine the constant L_β by using this formula. Evidently, this calculation is independent of the second derivatives of γ.

(ii) (Technical difference.) There is a difference on the strategies between [ElS2] and ours. In [ElS2] the authors consider the approximation of each problems to clarify the relation of (2.6) and (2.4). In our paper we introduce a modified Allen–Cahn equation (2.9) instead of (2.6) to remove some technical difficulties. However, since we would like to have a detailed estimate rather than the convergence result, we need more detailed computation.

(iii) (Inhomogeneity.) If the driving force f depends on the spatial variables x even in C^2, the method in our paper is not enough to achieve our goal. In fact, in the case $f = f(x)$, the traveling wave Q in the §2.3 depends on the spatial variable x, i.e., $Q = Q(\sigma, x)$. Then we obtain formally

$$\begin{aligned}
\nabla\psi &= \frac{Q_\sigma}{\varepsilon}\nabla\omega + Q_x, \\
\nabla^2\psi &= \frac{Q_{\sigma\sigma}}{\varepsilon^2}\nabla\omega \otimes \nabla\omega + \frac{1}{\varepsilon}(Q_{\sigma x} \otimes \nabla\omega + Q_{x\sigma} \otimes \nabla\omega) + Q_{xx}.
\end{aligned}$$

We cannot use the homogeneity of α, β and γ to estimate R_ε or $\tilde{R}_\varepsilon$ because of the form of $\nabla\psi$. Moreover, it is not clear how we estimate the ε^{-1}-term of $\nabla^2\psi$. In [ElS2] the authors assume that the highest order derivative of α, β and γ is Lipschitz continuous, and calculate that, for examples,

$$D^2\alpha(\nabla\psi) = \frac{1}{\varepsilon}D^2\alpha(\nabla\omega) + O(1) \quad \text{as } \varepsilon \to 0.$$

The bound of the last term depends on the Lipschitz constant of $D^2\alpha$.

(iv) (Time-dependent driving force.) It is easy to apply our methods to the estimate of internal layer with time-dependent driving force $f(t)$ satisfying, for examples, $f \in C^1([0, T])$. Essentially, to apply our method for our problem with driving force $f(t)$, we need the following properties:

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(a) $Q = Q(\sigma, t)$, $Q^{-1}(\sigma, t) \in C^{2,1}(\mathbf{R} \times [0, T])$,

(b) $\|Q_t\|_{L^\infty(\mathbf{R} \times [0, T])} < \infty$,

(c) the convergences as in Proposition 2.1 (i) and (ii) are uniformly with respect to $t \in [0, T]$.

The traveling wave $Q = Q(\sigma, t)$ by the equation (2.10) with $f \in C^1([0, T])$ satisfies above conditions. In the proof of propositions, we should be careful for the limiting procedure for sequences of times, in particular, in the proof of Propositions 4.1 and 4.2.

Fortunately, when we verify that ψ_ε is a viscosity supersolution of (2.6) and (2.9), this generalization yields only one extra term of the time derivative of ψ_ε, i.e.,

$$\partial_t \psi_\varepsilon = \frac{Q_\sigma}{\varepsilon}(\partial_t \omega_\delta + K_1) + Q_t.$$

The last term is included only in the term of the order ε^0 of R_ε.

(v) (Application for the driving force $f = f(t)$.) We remark that an application in §2.5 is still valid for $f = f(t)$ depending on t. Suppose that f_0 is continuous. It is easy to approximate f_0 by a smooth function f^τ converging to f_0 locally uniformly. Then convergence ansatz extends to this situation is proved in [GG4] and [GG5]. However, by the remark(iv), we need the bound of $\|f\|_{C^1([0,T])}$ to verify that our function ψ_ε is a viscosity supersolution of (2.9) by the method developed in this paper. Another method seems to be necessary to prove our uniform convergence for $f = f(t)$ without a bound for f'.

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