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Green polynomials at roots of unity and its application

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Abstract

We consider Green polynomials at roots of unity. We obtain a recursive formula for Green polynomials at appropriate roots of unity, which is described in a combinatorial manner. The coefficients of the recursive formula are realized by the number of permutations satisfying a certain condition, which leads to interpretation of a combinatorial property of certain graded modules of the symmetric group in terms of representation theory.

1 Introduction

The Green polynomial $Q^{\mu}_{\rho}(q)$ was introduced by J. A. Green [Gr] as a tool describing irreducible characters of the finite general linear group $GL_n(\mathbf{F}_q)$. They are polynomials in q with integer coefficients which are parametrized by two partitions μ, ρ of the same size. We also consider polynomials $X^{\mu}_{\rho}(q)$ obtained by reverting the sequences of coefficients of $Q^{\mu}_{\rho}(q)$, which are also called Green polynomials. These polynomials are characterized as the components of the transition matrix between the power-sum functions $p_\rho(x)$ and the Hall-Littlewood functions $P_\mu(x;q)$. A result of the Green polynomials at roots of unity was first obtained by A. Lascoux, B. Leclerc and J. -Y. Thibon [LLT], originally conjectured by A. Morris and N. Sultana [MS], which describes behavior of Green polynomials $X^{\mu}_{\rho}(q)$ corresponding to rectangle partitions μ at a certain special root of unity corresponding to μ. This result is founded on the properties of 'modified'Hall-Littlewood function $Q'_\mu(x;q)$ at roots of unity. The Hall-Littlewood functions at roots of unity were first considered by I. Schur. He considers Hall-Littlewood functions at $q = -1$, in connection with the theory of projective representations of the symmetric group. The study of Hall-Littlewood functions at other roots of unity was initiated by A. Morris [Mr2] in connection with modular representation theory of the symmetric group.

In this paper, we study the Green polynomials $Q^{\mu}_{\rho}(q)$ at roots of unity. We handle Green polynomials $Q^{\mu}_{\rho}(q)$ for any partition μ, and consider behavior of them at l-the roots of unity ζ_l, where l is not larger than the maximum multiplicity M_μ of μ. We describe a certain recursive formula of Green polynomials $Q^{\mu}_{\rho}(q)$ at $q = \zeta_l$ for the partition ρ satisfying $Q^{\mu}_{\rho}(\zeta_l) \neq 0$. The results of Lascoux-Leclerc-Thibon on Hall-Littlewood functions at roots of unity play an important role in the argument. Our result includes the result of Lascoux-Leclerc-Thibon on Green polynomials as a special case.

We also considers the recursive formula in terms of representation theory of the symmetric group S_n. It is known that the Green polynomials give the graded characters of a family of graded representations of the symmetric group, called the DeConcini-Procesi-Tanisaki algebras, which includes the coinvariant algebra as a special case. The DeConcini-Procesi-Tanisaki algebra R_μ was first introduced by C. DeConcini and C. Procesi [DP] as an algebraic model of the cohomology ring of a certain subvariety of the flag variety parametrized by a partition μ, and T. Tanisaki [T] gives simple generators of the defining ideal of the algebra, described by combinatorial information on the partition μ. The DeConcini-Procesi-Tanisaki algebra R_μ has a structure of graded S_n-modules, and the Green polynomial $Q^\mu_\rho(q)$ gives its graded character values at the conjugacy class of which cycle type is ρ. The recursive formula is equivalent to some representation theoretical interpretation of a certain combinatorial property on the Hilbert polynomial $\mathrm{Hilb}_{R_\mu}(q)$ of R_μ, that is, $\mathrm{Hilb}_{R_\mu}(q)$ has l-th roots of unity ζ_l^j $(j = 1, 2, \ldots, l-1)$ as its zeros for each positive integer l not larger than the maximum multiplicity M_μ of μ. This property of the Hilbert polynomial is equivalent to the fact that the direct sums $R_\mu(k;l)$ $(k = 0, 1, \ldots, l-1)$ of the homogeneous components of R_μ of which degrees are congruent to k modulo l, have the same dimension. The recursive formula shows that there exists a subgroup $H_\mu(l)$ of S_n and $H_\mu(l)$-modules $Z_\mu(k;l)$ of equal dimension such that each $R_\mu(k;l)$ is induced from the corresponding $H_\mu(l)$-modules $Z_\mu(k;l)$ for each $k = 0, 1, \ldots, l-1$, which could be regarded as a representation theoretical interpretation of the property 'coincidence of dimensions'.

A problem of this type on graded representations of reflection groups was first studied by W. Kraśkiewicz and J. Weyman [KW], essentially by T. A. Springer [Sp1], for the coinvariant algebra R_W of the Weyl groups W of type A, B, D. They consider the problem for $l = c$, the Coxeter element of W, and verify that the direct sums $R_W(k;c)$ is induced from the corresponding irreducible representation of the cyclic subgroup of W generated by a Coxeter element. (Recall that the Coxeter elements have the same order, and the order is called the Coxeter number.) In [MN1], we consider the problem for the coinvariant algebra of the symmetric group, and show similar result for every fundamental degrees l. Recently, C. Bonnafé, G. Lehrer and J. Michel [BLM] showed that the same situation holds for the coinvariant algebra of arbitrary finite complex reflection group. In [RSW], a similar problem is considered for finite reflection groups over an arbitrary field.

The study of the problem, representation theoretical interpretation for the coincidence of dimensions for DeConcini-Procesi-Tanisaki algebra, was started by [Mt]. We consider in [Mt] the algebras R_μ corresponding to hook partitions, and show that the same situation also holds here for each positive integer l not larger than M_μ. In this paper, we consider this problem for arbitrary μ and arbitrary possible l. In fact, we establish an isomorphism between the algebra R_μ and a module induced from a certain 'smaller'DeConcini-Procesi-Tanisaki algebra $R_{\bar{\mu}(l)}$ corresponding to μ and l. It is explicitly noticed in [MN2] the relation between the problem on the algebras R_μ and the result of Lascoux-Leclerc-Thibon [LLT] on Green polynomials at roots of unity. Their result describes the value $X^\mu_\rho(\zeta_l)$ by the inner product value of a power-sum function and a complete symmetric functions for the partition, of which multiplicities are all multiples of l. In [MN2], we obtain a explicitly formula for the inner product values, and show that their result essentially gives the answer to the problem on the DeConcini-Procesi-Tanisaki algebra R_μ. In the present article, founded on the LLT's result on modified Hall-Littlewood functions, we establish an isomorphism between the algebra R_μ and a induced module from a smaller algebra $R_{\bar{\mu}(l)}$ as $S_n \times C_l$-modules for an arbitrary μ and an arbitrary possible l.

2 Preliminaries

In this section, we recall the definition of Green polynomials, and fundamental facts on (modified) Hall-Littlewood symmetric functions which we shall use.

A partition μ of a positive integer n is a non-increasing sequence $\mu = (\mu_1, \mu_2, \ldots, \mu_d)$ of nonnegative integers of which total sum is n. If $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_d > 0$, then d is called the *length* of μ, denoted by $l(\mu)$. The positive integer n is called the *size* of μ, denoted by $|\mu|$. Let n be a positive integer and μ a partition of n. We employ the symbol $\mu \vdash n$ to denote that μ is a partition of n. Let $\mathcal{P}_n$ denote the set of partitions of n, and $\mathcal{P} = \cup_{n\geq 1}\mathcal{P}_n$ the set of all partitions. If we denote by $m_i(\mu)$ the multiplicity of i in a partition $\mu \vdash n$, then μ can be written in the form $\mu = (1^{m_1(\mu)}2^{m_2(\mu)} \cdots n^{m_n(\mu)})$. With this notation, remark that $n = 1m_1(\mu) + 2m_2(\mu) + \cdots + nm_n(\mu)$. Define M_μ the maximum multiplicity of the partition μ:

$$M_\mu := \max\{m_1(\mu), m_2(\mu), \cdots, m_n(\mu)\}.$$

Let q be an indeterminate, and $P_\mu(x;q)$ the *Hall-Littlewood symmetric function* corresponding to μ [M]. It is known that the Hall-Littlewood functions form a $\mathbf{Z}[q]$-basis of the ring $\Lambda[q] = \Lambda \otimes_{\mathbf{Z}} \mathbf{Z}[q]$ of symmetric function (with $\mathbf{Z}[q]$-coefficients) [M, III, (2.7)], which is orthogonal with respect to the Hall-Littlewood inner product $\langle \cdot, \cdot \rangle_q$ [M]. Let $p_\rho(x)$ be the *power-sum symmetric function* [M, p.24] corresponding to the partition $\rho \vdash n$, and we expand $p_\rho(x)$ as a linear combination of Hall-Littlewood functions as follows:

$$p_\rho(x) = \sum_{\mu \in \mathcal{P}} X_\rho^\mu(q) P_\mu(x;q).$$

Then the coefficients $X_\rho^\mu(q)$ are elements of $\mathbf{Z}[q]$, and it can be seen that $X_\rho^\mu(t) = 0$ unless $|\mu| = |\rho|$ [M, p.246]. The *Green polynomials* $Q_\rho^\mu(q)$ [Gr] (see also [M, III, (7.8)]) are defined by

$$Q_\rho^\mu(q) = q^{n(\mu)} X_\rho^\mu(q^{-1}),$$

where $n(\mu) = \sum_{i\geq 1}(i-1)\mu_i$ if $\mu = (\mu_1, \mu_2, \ldots)$. (Remark that the polynomial $X_\rho^\mu(q)$ is also called the Green polynomial, but for our sake it is more suitable to use $Q_\rho^\mu(q)$. Thus a word Green polynomials always means the polynomials $Q_\rho^\mu(q)$ otherwise stated.) The Green polynomial $Q_\rho^\mu(q)$ is a polynomial with integer coefficients whose degree is $n(\mu)$, which was introduced by J. A. Green [Gr] to describe irreducible character values of the general linear group $GL_n(\mathbf{F}_q)$ over a finite field $\mathbf{F}_q$. No explicit formula of Green polynomials are known in general, except for some special case [Mr1, Mt]. We have, however, a certain factorization formula as follows. For a partition $\mu \vdash n$, let $M_\mu \vdash n$ be the maximum value of the multiplicities $m_1(\mu), m_2(\mu), \ldots, m_n(\mu)$, and $e_\mu(q)$ the polynomial $(1-q)^{m_1(\mu)}(1-q^2)^{m_2(\mu)} \cdots (1-q^n)^{m_n(\mu)}$.

Proposition 1 ([Mt, Theorem 5]) *Let $\mu, \rho \vdash n$ be a partition. Then there exists a polynomial $G^\mu_\rho(q) \in \mathbf{Z}[q]$ such that*

$$Q^\mu_\rho(q) = \frac{\varphi_{M_\mu}(q)}{e_\rho(q)} G^\mu_\rho(q).$$

This proposition should not be best possible. We conjecture that the rational factor may be taken as $e_\mu(q)/e_\rho(q)$. The identity, however, is enough for our sake in the present article.

Let $\varphi_r(q)$ be the polynomial $(1-q)(1-q^2)\cdots(1-q^r)$, and $b_\mu(q)$ the polynomial

$$b_\mu(q) = \prod_{i \geq 1} \varphi_{m_i(\mu)}(q),$$

where $m_i(\mu)$ is the multiplicity of i in the partition μ. Define

$$Q_\mu(x;q) = b_\mu(q) P_\mu(x;q)$$

which is referred to, as well as the P_μ, as Hall-Littlewood functions. If we replace the variables $x = (x_1, x_2, \ldots)$ of $Q_\mu(x;q)$ by

$$x/(1-q) = (x_1, x_2, \ldots; qx_1, qx_2, \ldots; q^2x_1, q^2, x_2, \ldots),$$

then we obtain the *modified* Hall-Littlewood function, which is denoted by

$$Q'_\mu(x;q) \left(= Q_\mu\left(\frac{x}{1-q}; q\right)\right).$$

Equivalently, it is also defined by replacing $p_k(x)$ by $p_k(x)/(1-t^k)$ after expressing $Q_\mu(x;t)$ as a polynomial in $\{p_k(x) | k \geq 1\}$. It is known (see, e.g., [DLT]) that the Green polynomial $X^\mu_\rho(q)$ is obtained as the inner product value

$$X^\mu_\rho(x) = \langle Q'_\mu(x;q), p_\rho(x) \rangle$$

of the modified Hall-Littlewood function $Q'_\mu(x;q)$ and the power-sum function $p_\rho(x)$. The inner product $\langle \cdot, \cdot \rangle$ of the ring $\Lambda[q]$ is defined by $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$, where s_λ denotes the *Schur function* [M] corresponding to the partition λ, and $\delta_{\lambda\mu}$ the Kronecker delta. It should be remarked here [M] that the adjoint operator of the multiplication map

$$\times p_k : \Lambda \longrightarrow \Lambda : f \longmapsto f p_k$$

is obtained by $k\partial/\partial p_k$, i.e.,

$$\langle p_k f, g \rangle = \langle f, k \frac{\partial}{\partial p_k} g \rangle$$

for each $f, g \in \Lambda$.

In the rest of this section, we recall results on (modified) Hall-Littlewood functions at roots of unity due to Lascoux-Leclerc-Thibon. In [LLT], they consider modified Hall-Littlewood functions at roots of unity and find a factorization formula, which plays a crucial role in the present paper. Let $\mu \vdash n$ be a partition, and an integer l such that $2 \leq l \leq M_\mu$ fixed, and $m_i(\mu) = lq_i + r_i$, $0 \leq r_i \leq l-1$, for each i. Set $q = q_1 + 2q_2 + \cdots + nq_n$ and $r = r_1 + 2r_2 + \cdots + nr_n$. Let $\tilde{\mu}(l)$ and $\bar{\mu}(l)$ be the partitions

$$\tilde{\mu}(l) := (1^{lq_1} 2^{lq_2} \cdots n^{lq_n})$$

and

$$\bar{\mu}(l) := (1^{r_1} 2^{r_2} \cdots n^{r_n}).$$

It is clear that $\tilde{\mu}(l)$ and $\bar{\mu}(l)$ are partitions of $n - r = lq$ and r respectively, and the partition μ decomposes into the disjoint union $\mu = \tilde{\mu}(l) \cup \bar{\mu}(l)$. Also define

$$\tilde{\mu}(l)^{1/l} := (1^{q_1} 2^{q_2} \cdots n^{q_n}),$$

which is a partition of q.

Example 2 If $\mu = (3,3,3,2,2,1)$, then $M_\mu = 3$. Let $l = 2$ be fixed. Then $\tilde{\mu}(l) = (3,3,2,2)$, $\bar{\mu}(l) = (3,1)$, and $\mu = (3,3,2,2) \cup (3,1)$. Also the partition $\tilde{\mu}(l)^{1/l}$ is $(3,2)$.

Let μ be a partition, and l a positive integer such that $l \leq M_\mu$. The modified Hall-Littlewood function $Q'_\mu(x;q)$ at $q = \zeta_l$, a primitive l-th root of unity, is factorized in such a way consistent with the decomposition of the partition $\mu = \tilde{\mu}(l) \cup \bar{\mu}(l)$.

Proposition 3 ([LLT, Theorem 2.1.]) *Let $\mu = (1^{m_1} 2^{m_2} \cdots n^{m_n}) \vdash n$ be a partition, l an positive integer such that $l \leq M_\mu$, and $m_i = lq_i + r_i$, $0 \leq r_i \leq l-1$, for each $i = 1, 2, \ldots, n$. Let $\bar{\mu}(l)$ denote the partition $(1^{r_1} 2^{r_2} \cdots n^{r_n})$. Then, ζ_l being a primitive l-th root of unity, we have*

$$Q'_\mu(x;\zeta_l) = Q'_{\bar{\mu}(l)}(x;\zeta_l) \prod_{i \geq 1} \left(Q'_{(i^l)}(x;\zeta_l) \right)^{q_i}.$$

Example 4 Let $\mu = (3,3,3,2,1,1,1,1,1)$ and $l = 2$. Then $\bar{\mu}(l) = (3,2,1)$, and we have $Q'_{(3,3,3,2,1,1,1,1,1)}(x;\zeta_2) = Q'_{(3,2,1)}(x;\zeta_2) Q'_{(3^2)}(x;\zeta_2) \left(Q'_{(1^2)}(x;\zeta_2) \right)^2$.

Proposition 5 ([LLT, Theorem 2.2.])

$$Q'_{(i^l)}(x;\zeta_l) = (-1)^{(l-1)i} (p_l \circ h_i)(x),$$

where $(p_l \circ h_i)(x)$ denotes the plethysm.

For the definition of the plethysm, consult [M]. Remark that

$$(p_l \circ h_i)(x) = \sum_{\lambda \vdash i} z_\lambda^{-1} p_{l\lambda}(x), \tag{2.1}$$

which follows from the facts that $(p_l \circ f)(x) = f(x_1^l, x_2^l, \ldots)$ for any $f = f(x_1, x_2, \ldots) \in \Lambda$, and $h_i(x) = \sum_{\lambda \vdash i} z_\lambda^{-1} p_\lambda(x)$.

Example 6 $Q'_{(3^2)}(x;\zeta_2) = (-1)^{(2-1)3}(p_2 \circ h_3)(x) = -z_{(3)}^{-1}p_{(6)}(x) - z_{(2,1)}^{-1}p_{(4,2)} - z_{(1,1,1)}^{-1}p_{(2,2,2)}(x)$.

It follows from Proposition 3, Proposition 5 and (2.1) that the Green polynomial corresponding to a rectangular partition $\mu = (r^k)$ at a primitive k-th root of unity is described by a certain 'smaller'Green polynomial.

Proposition 7 ([LLT, Theorem 3.2.]) *Let $\mu = (r^k)$ be a rectangular partition, ζ_k a primitive k-th root of unity. If $m_i(\mu) \geq 1$ for some $i \geq 1$ divisible by k, then it holds that*

$$X_\rho^\mu(\zeta_k) = (-1)^{(k-1)j} k X_{\rho\setminus\{i\}}^{((r-j)^k)}(\zeta_k), \tag{2.2}$$

where $i = jk$.

If we rewrite the identity (2.2) in terms of the polynomial $Q_\rho^\mu(x)$, then the signature $(-1)^{(k-1)j}$ is vanished and we have [Mt, Lemma 7 or Proposition 5]

$$Q_\rho^\mu(\zeta_k) = k Q_{\rho\setminus\{i\}}^{((r-j)^k)}(\zeta_k). \tag{2.3}$$

Applying this identity repeatedly, we also have

$$Q_\rho^\mu(\zeta_k) = k^{l(\rho)},$$

if the partition ρ consists of multiples of k.

3 Root of unity

In this section, we shall describe behavior of the Green polynomial $Q_\rho^\mu(q)$, $\rho \vdash n$, at l-th roots of unity for each $l = 2, 3, \ldots, M_\mu$. The result in this section generalizes the formula of Lascoux-Leclerc-Thibon, which treats the case where μ is a rectangle and $l = M_\mu$, to the case where μ is any partition and $l = 2, 3, \ldots, M_\mu$.

Let μ be a partition of n and a positive integer l such that $2 \leq l \leq M_\mu$ fixed, and $m_i(\mu) = lq_i + r_i$, $0 \leq r_i \leq l-1$, for each i. Set $q = q_1 + 2q_2 + \cdots + nq_n$ and $r = r_1 + 2r_2 + \cdots + nr_n$. Let $\tilde{\mu}(l)$, $\bar{\mu}(l)$, and $\tilde{\mu}(l)^{1/l}$ be as in the previous section. We define 'partitions of a partition'as follows. Let $\nu = (\nu_1, \nu_2, \ldots, \nu_d)$ be a partition of n. A *partition* of the partition ν is by definition a sequence of partitions

$$\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(d)})$$

such that $\lambda^{(i)} \vdash \nu_i$ for each $i = 1, 2, \ldots, d$, which is denoted by $\lambda \vdash \nu$. We distinguish any nontrivial permutation of $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(d)})$ from the original one. For example, we consider that the following two partitions $((2), (1, 1)), ((1, 1), (2))$ are different as partitions of $(2, 2)$. The *length* $l(\lambda)$ of $\lambda \vdash \nu$ is defined by

$$l(\lambda) = \sum_{i=1}^{d} l(\lambda^{(i)}),$$

and the *size* $|\lambda|$ is defined by the sum of sizes of the components $\lambda^{(i)}$ of λ, which is equal to $n = |\nu|$. Also define

$$z_\lambda := \prod_{i \geq 1} z_{\lambda^{(i)}},$$

where z_π is defined by

$$z_\pi = 1^{m_1} m_1! 2^{m_2} m_2! \cdots n^{m_n} m_n!$$

for a partition $\pi \vdash n$ of positive integer as usual. Let $\nu = (\nu_i)$ be a partition of n and $\lambda = (\lambda^{(i)})$ a partition of ν. Let

$$m_k(\lambda) := \sum_{i=1}^{d} m_k(\lambda^{(i)})$$

for each possible $k \geq 1$. Then define

$$m_\lambda := \prod_{k \geq 1} \binom{m_k(\lambda)}{m_k(\lambda^{(1)}), m_k(\lambda^{(2)}), \ldots, m_k(\lambda^{(d)})}.$$

Also, for each positive integer l, let $l\lambda$ denotes the partition whose components are those of λ multiplied by l.

Example 8 Let $\nu = (4, 2)$. Then the partitions λ of ν are $((4), (2))$, $((3, 1), (2))$, $((2, 2), (1, 1))$, $((2, 1, 1), (1, 1))$ and so on. Suppose that $\lambda = ((2, 1, 1), (2)) \vdash \nu$. Then m_λ is computed as follows: $m_{((2,1,1),(2))} = \binom{m_1(\lambda)}{m_1(\lambda^{(1)}), m_1(\lambda^{(2)})} \binom{m_2(\lambda)}{m_2(\lambda^{(1)}), m_2(\lambda^{(2)})} \cdots = \binom{2}{2,0}\binom{2}{1,1} = 2$. For the same λ, if $l = 2$ for example, the partition $l\lambda = 2\lambda$ is $(4, 4, 2, 2)$.

Let ρ be a partition and ν a *subpartition* of ρ, i.e., $m_i(\nu) \leq m_i(\rho)$ for each possible $i \geq 1$. Then we define the *binomial coefficient* $\binom{\rho}{\nu}$ by

$$\binom{\rho}{\nu} := \prod_{i \geq 1} \binom{m_i(\rho)}{m_i(\nu)}.$$

Let μ be a partition, and an integer l such that $2 \leq l \leq M_\mu$ fixed. For a partition ν of $|\tilde{\mu}(l)|$, define

$$C(\nu, \mu; l) := \sum_{\substack{\pi \vdash \tilde{\mu}(l)^{1/l} \\ l\pi = \nu}} m_\pi.$$

If there exists no $\pi \vdash \tilde{\mu}(l)^{1/l}$ such that $l\pi = \nu$, then it is convinced that $C(\nu, \mu; l) = 0$.

Example 9 Let $\mu = (5,4,4,2,2,1)$, and l such that $2 \le l \le M_\mu$ fixed, say $l = 2$. Then $\tilde{\mu}(l) = (4,4,2,2)$ and $\tilde{\mu}(l)^{1/2} = (4,2)$. Suppose that $\nu = (4,4,4) \vdash |\tilde{\mu}(l)|$. Then there exists only one $\pi \vdash \tilde{\mu}(l)^{1/2}$ such that $2\pi = \nu$, i.e., $\pi = ((2,2),(2))$. Hence $C(\nu,\mu;2) = m_{((2,2),(2))} = \binom{3}{2,1} = 3$. On the other hand, if $\nu = (4,4,2,2)$, then there exist two $\pi \vdash (4,2)$ such that $2\pi = \nu$, i.e., $\pi = ((2,2),(1,1)),((2,1,1),(2))$. Hence we have $C(\nu,\mu;2) = m_{((2,2),(1,1))} + m_{((2,1,1),(2))} = \binom{2}{0,1}\binom{2}{2,0} + \binom{2}{2,0}\binom{2}{1,1} = 1 + 2 = 3$. On the other hand, in the case where $\tilde{\mu}(l)$ is given by $(4,4)$ for $l = 2$ and $\nu = (4,2,2)$, the partitions $\pi \vdash \tilde{\mu}(l)^{1/l}$ satisfying $l\pi = \nu$ are $\pi = ((2),(1,1)),((1,1)(2))$. Since we distinguish these two partition, $C(\nu,\mu;l)$ is obtained by $m_{((2),(1,1))} + m_{((1,1),(2))} = 1 + 1 = 2$.

Now we can state our main result, which retrieves LLT's result, Proposition 7, if we consider the case where μ is a rectangle and $l = M_\mu$.

Theorem 10 *Let $\mu = (1^{m_1}2^{m_2}\cdots n^{m_n})$ be a partition of n, a positive integer $l = 1,2,\ldots,M_\mu$ fixed, and ζ_l an l-th primitive root of unity. Let $m_i = lq_i + r_i$, $0 \le r_i \le l-1$, for each $i = 1,2,\ldots,n$. Let $r = r_1 + 2r_2 + \cdots + nr_n$, and $\bar{\mu}(l) = (i^{r_i}) \vdash r$.*

Then we have:

1. *$Q^\mu_\rho(\zeta_l) \neq 0 \Longrightarrow \rho = l\tilde{\rho} \cup \bar{\rho}$ for some $\tilde{\rho} \vdash \tilde{\mu}(l)^{1/l}$ and $\bar{\rho} \vdash r$.*

2. *For such a partition $\rho = l\tilde{\rho} \cup \bar{\rho}$, it holds that:*

$$Q^\mu_\rho(\zeta_l) = \sum_{\substack{\nu \vdash |\tilde{\mu}(l)| \\ \nu \subset \rho}} \binom{\rho}{\nu} C(\nu,\mu;l)\, l^{l(\nu)} Q^{\bar{\mu}(l)}_{\rho\setminus\nu}(\zeta_l).$$

Proof. Recall that

$$X^\mu_\rho(\zeta_l) = \langle Q'_\mu(x;\zeta_l), p_\rho(x)\rangle, \tag{3.1}$$

where $Q'_\mu(x;\zeta_l)$ is the modified Hall-Littlewood function at the primitive l-th root of unity. By Proposition 3 and Proposition 5, we have

$$\begin{aligned} Q'_\mu(x;\zeta_l) &= \left(\prod_{i=1}^n Q'_{(i^l)}(x;\zeta_l)^{q_i}\right) Q'_{\bar{\mu}(l)}(x;\zeta_l) \\ &= \prod_{i=1}^n \left((-1)^{(l-1)i} p_l \circ h_i\right)^{q_i} Q'_{\bar{\mu}(l)}(x;\zeta_l) \\ &= (-1)^s \prod_{i=1}^n \left(\sum_{\lambda^{(i)} \vdash i} z^{-1}_{\lambda^{(i)}} p_{l\lambda^{(i)}}(x)\right)^{q_i} Q'_{\bar{\mu}(l)}(x;\zeta_l), \end{aligned}$$

where $s = \sum_{i=1}^n (l-1)iq_i$ and $l\lambda^{(i)}$ is the partition obtained by multiplying the components of $\lambda^{(i)}$ by l. The third identity follows from (2.1). Thus we have

$$X^\mu_\rho(\zeta_l) = (-1)^s \sum_{\lambda \vdash \tilde{\mu}(l)^{1/l}} z^{-1}_\lambda \left\langle p_{l\lambda}(x) Q'_{\bar{\mu}(l)}(x;\zeta_l), p_\rho(x)\right\rangle.$$

Recall that the adjoint operator of the multiplication map $\times p_k$ is given by the differential operator $k\partial/\partial p_k$. Since

$$\prod_{i=1}^{n} p_{l\lambda^{(i)}}(x) = p_{l\lambda}(x),$$

we have

$$\left\langle p_{l\lambda}(x) Q'_{\bar{\mu}(l)}(x;\zeta_l), p_\rho(x) \right\rangle = 0$$

if $l\lambda$ does not contained in ρ. Hence we have

$$X^\mu_\rho(\zeta_l) = (-1)^s \sum_{\substack{\lambda \vdash \bar{\mu}(l)^{1/l} \\ l\lambda \subset \rho}} z_\lambda^{-1} \left\langle p_{l\lambda}(x) Q'_{\bar{\mu}(l)}(x;\zeta_l), p_\rho(x) \right\rangle.$$

Noting that $Q'_{\bar{\mu}(l)}(x;\zeta_l)$ is a linear combination of power-sums $p_\tau(x)$, where $\tau \vdash |\bar{\mu}(l)| = r$, the identity (3.1) proves 1., since power-sums $\{p_\tau | \tau \in \mathcal{P}\}$ are orthogonal each other with respect to the inner product $\langle \cdot, \cdot \rangle$.

Let ρ be a partition of n satisfying the condition 1. Then we have

$$\begin{aligned} X^\mu_\rho(\zeta_l) &= (-1)^s \sum_{\substack{\lambda \vdash \bar{\mu}(l)^{1/l} \\ l\lambda \subset \rho}} z_\lambda^{-1} \left\langle p_{l\lambda}(x) Q'_{\bar{\mu}(l)}(x;\zeta_l), p_\rho(x) \right\rangle \\ &= (-1)^s \sum_{\substack{\rho \vdash |\bar{\mu}(l)| \\ \nu \subset \rho}} \sum_{\substack{\lambda \vdash \bar{\mu}(l)^{1/l} \\ l\lambda = \nu}} z_\lambda^{-1} \left\langle p_\nu(x) Q'_{\bar{\mu}(l)}(x;\zeta_l), p_\rho(x) \right\rangle \end{aligned}$$

In the following, we shall consider the value $z_\lambda^{-1} \left\langle p_\nu(x) Q'_{\bar{\mu}(l)}(x;\zeta_l), p_\rho(x) \right\rangle$. Recall that $z_\lambda = \prod_{i\geq 1} z_{\lambda^{(i)}}$ if $\lambda = (\lambda^{(i)}) \vdash \tilde{\mu}(l)^{1/l}$. Define

$$\nu \frac{\partial}{\partial p_\nu} := \prod_{i \geq 1} \nu_i \frac{\partial}{\partial p_{\nu_i}}$$

for a partition $\nu = (\nu_1, \nu_2, \ldots)$ of a positive integer. Since the adjoint operator of the multiplication $\times p_i$ with respect to the inner product is $i\,(\partial/\partial p_i)$, we have

$$z_\lambda^{-1} \left\langle p_\nu(x) Q'_{\bar{\mu}(l)}(x;\zeta_l), p_\rho(x) \right\rangle = z_\lambda^{-1} \left\langle Q'_{\bar{\mu}(l)}(x;\zeta_l), \nu \frac{\partial}{\partial p_\nu} p_\rho(x) \right\rangle.$$

It is easy to see that the right hand side coincides with

$$z_\lambda^{-1} \left. \left(\nu \frac{\partial}{\partial p_\nu} p_\rho(x) \right) \right|_{p_1(x)=p_2(x)=\cdots=1} \langle Q'_{\bar{\mu}(l)}(x;\zeta_l), p_{\rho\backslash\nu}(x) \rangle,$$

and slight consideration shows that the coefficient $z_\lambda^{-1} \left. \left(\nu \frac{\partial}{\partial p_\nu} p_\rho(x) \right) \right|_{p_1(x)=p_2(x)=\cdots=1}$ equals

$$\binom{\rho}{\nu} m_\lambda l^{l(\nu)},$$

if the partition $\lambda \vdash \tilde{\mu}(l)^{1/l}$ satisfies $\nu = l\lambda \subset \rho$. Therefore we have

$$X^{\mu}_{\rho}(\zeta_l) = (-1)^s \sum_{\substack{\nu \vdash |\bar{\mu}(l)| \\ \nu \subset \rho}} \binom{\rho}{\nu} C(\nu, \mu; l) l^{l(\nu)} X^{\bar{\mu}(l)}_{\rho \backslash \nu}(\zeta_l).$$

By the definition, we have

$$Q^{\mu}_{\rho}(\zeta_l) = q^{n(\mu)-n(\bar{\mu}(l))}\big|_{q=\zeta_l} (-1)^s \sum_{\substack{\nu \vdash |\bar{\mu}(l)| \\ \nu \subset \rho}} \binom{\rho}{\nu} C(\nu, \mu; l) l^{l(\nu)} Q^{\bar{\mu}(l)}_{\rho \backslash \nu}(\zeta_l).$$

Finally, it is not difficult to show that

$$q^{n(\mu)-n(\bar{\mu}(l))}|_{q=\zeta_l} = (-1)^s,$$

which completes the proof.

Example 11 Let $\mu = (5, 4, 4, 2, 2, 1) \vdash 18$ and $l = 2$. In this case, we have $\mu(2) = (4, 4, 2, 2)$ and $\mu(2)^{1/2} = (4, 2)$. If $Q^{\mu}_{\rho}(\zeta_2) \neq 0$, then ρ should be of the form $\rho = 2\tilde{\rho} \cup \bar{\rho}$, where $\bar{\rho}$ is a partition of 6 and $\tilde{\rho} \vdash \mu(2)^{1/2}$ is one of the following partitions: ((4),(2)), ((3,1),(2)), ((2,2),(2)), ((2,1,1),(2)), ((1,1,1,1),(2)), ((4),(1,1)), ((3,1),(1,1)), ((2,2),(1,1)), ((2,1,1),(1,1)), ((1,1,1,1),(1,1)). Suppose that $\rho = (4, 4, 2, 2) \cup (4, 2) = (4, 4, 4, 2, 2, 2)$. Then subpartitions ν of ρ which satisfy $\nu \vdash |\mu(2)| = 12$ are $\nu = (4, 4, 4), (4, 4, 2, 2)$. Consider the case where $\nu = (4, 4, 4)$. Then $\binom{\rho}{\nu} = \binom{3+0}{0}\binom{0+3}{3} = 1$. There exists only one $\lambda \vdash \mu(2)^{1/2} = (4, 2)$ such that $2\lambda = (4, 4, 4)$, i.e., $\lambda = ((2, 2), (2))$, and we have $m_{\lambda} = \binom{2+1}{2,1} = 3$. Thus $C(\nu, \mu; 2) = 3$. If $\nu = (4, 4, 2, 2)$, then $\binom{\rho}{\nu} = \binom{2+1}{2}\binom{2+1}{2} = 9$. The corresponding λ's satisfying $2\lambda = \nu$ are $\lambda = ((2, 2), (1, 1)), ((2, 1, 1), (2))$, and $m_{((2,2),(1,1))} = \binom{2}{0,2}\binom{2}{2,0} = 1$, $m_{((2,1,1),(2))} = \binom{2}{2,0}\binom{2}{1,1} = 2$. Hence we have $C(\nu, \mu; 2) = 3$ in this case. Thus we have $Q^{(4,4,4,2,2,2)}_{(4,4,4,2,2,2)}(\zeta_2) = \binom{\rho}{(4,4,4)} C((4, 4, 4), \mu; 2) 2^{l(4,4,4)} Q^{\bar{\mu}(l)}_{\rho \backslash (4,4,4)}(\zeta_2) + \binom{\rho}{(4,4,2,2)} C((4, 4, 2, 2), \mu; 2) 2^{l(4,4,2,2)} Q^{\bar{\mu}(l)}_{\rho \backslash (4,4,2,2)}(\zeta_2) = 3Q^{(4,2)}_{(2,2,2)}(\zeta_2) + 27Q^{(4,2)}_{(4,2)}(\zeta_2)$.

4 Permutation enumeration

In the previous section, we show that Green polynomials $Q^{\mu}_{\rho}(q)$ enjoy a recursive relation on a primitive l-th root of unity ζ_l, where l is not larger than M_{μ}. In this section, we consider a combinatorial characterization of each coefficients of the formula

$$\binom{\rho}{\nu} C(\nu, \mu; l) l^{l(\nu)},$$

and we see that these coefficients are exactly the numbers of certain permutations.

Let μ be a partition of a positive integer n, and an integer $l \in \{2, 3, \dots, M_{\mu}\}$ fixed. We define the *cyclic permutation product* $a = a_{\mu}(l)$ corresponding to μ and l. To avoid abuse of notation, we shall see the definition by the following example. It is clear from the definition that the element $a_{\mu}(l)$ has the order l.

Example 12 (Definition of $a_\mu(l)$) Let $\mu = (3,3,2,2,2,1)$ and $l = 2(\leq M_\mu = 3)$. We fix the numbering of the Young diagram of μ

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & \bullet & \\ \bullet & \bullet & \\ \bullet & & \end{array}$$

as follows:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & \\ 9 & 10 & \\ 11 & 12 & \\ 13 & & . \end{array}$$

Corresponding to the number $l = 2$, we extract subtableaux

$$\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}, \quad \begin{array}{cc} 7 & 8 \\ 9 & 10 \end{array}$$

Then the cyclic permutation product $a_\mu(2)$ is defined by using the letters corresponding to $\tilde{\mu}(l)$ as follows:

$$a_\mu(2) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 7 & 8 & 9 & 10 \\ 9 & 10 & 7 & 8 \end{pmatrix}$$

□

Let $n = ql + r$, $0 \leq r \leq l-1$. Recall that $\tilde{\mu}(l)$ is a partition of $n-r$. Let $S_{\tilde{\mu}(l)}$ be the Young subgroup which permutes the letters corresponding to $\tilde{\mu}(l)$ in the preceding tableau, and let S_r be the subgroups which permutes the remaining letters. It is obvious that elements of these groups commute each other. In the preceding definition (Example 12), these groups are the following:

$$\begin{aligned} S_{\tilde{\mu}(l)} &= S_{\{1,2,3\}} \times S_{\{4,5,6\}} \times S_{\{7,8\}} \times S_{\{9,10\}}, \\ S_r &= S_{\{11,12,13\}}, \end{aligned}$$

where $\tilde{\mu}(l) = (3,3,2,2)$, $r = 3$ and $S_{\{i,j,\ldots,k\}}$ denotes the symmetric group of the letters $\{i,j,\ldots,k\}$. Consider the subgroup of S_n

$$H_\mu(l) := \big(S_{\tilde{\mu}(l)} \times S_r\big) \rtimes \langle a_\mu(l) \rangle = \big(S_{\tilde{\mu}(l)} \rtimes \langle a_\mu(l) \rangle\big) \times S_r.$$

We should note here that:

Lemma 13 *The cycle types ρ of elements of the subgroup $H_\mu(l)$ are of the form*

$$\rho = l\tilde{\rho} \cup \bar{\rho},$$

where $\tilde{\rho} \vdash \tilde{\mu}(l)^{1/l}$ and $\bar{\rho} \vdash r$. Conversely, if ρ is a partition of such a form, then there exists an element of $H_\mu(l)$ whose cycle type is ρ.

Proof. Note that the cycle type of an element $\tau a_\mu(l) \in S_{\tilde{\mu}(l)} \rtimes \langle a_\mu(l) \rangle$ is of the form $l\tilde{\rho}$ for some $\tilde{\rho} \vdash \tilde{\mu}(l)^{1/l}$. $\square$

Example 14 Consider the case $\mu = (3,3,2,2,2,1)$ and $l = 2$. Then the corresponding cyclic permutation product is $a_\mu(2) = (1,4)(2,5)(3,6)(7,9)(8,10)$. If we consider $w = (1,2)(7,8)a(11,13) \in H_\mu(2)$, then $w = (1,4,2,5)(3,6)(7,9,8,10)(11,13)$ and its cycle type is $(4,4,2,2)$, which is the union of $(4,4,2)$ and (2). The partition $(4,4,2)$ is written in the form $(4,4,2) = 2((2,1),(2))$ for $((2,1),(2)) \vdash (3,2) = \tilde{\mu}(l)^{1/2}$. Conversely, if we consider $\rho = 2((2,1,1),(1,1)) \cup (3) = (4,3,2,2,2,2)$, then choose $\tau_1 = (1,2) \in S_{\tilde{\mu}(l)}$ and $\tau_2 = (11,12,13) \in S_r$ for example. It is easy to see that the cycle type of $w = \tau_1\tau_2 a_\mu(2)$ coincide with ρ. $\square$

Proposition 15 *Let $\mu \vdash n$ be a partition, $l = 2,3,\ldots,M_\mu$ fixed, and $a = a_\mu(l)$ the cyclic permutation product corresponding to μ and l. Let $\rho \vdash n$ be a partition of the form $\rho = l\tilde{\rho} \cup \bar{\rho}$ where $\tilde{\rho} \vdash \tilde{\mu}(l)^{1/l}$ and $\bar{\rho} \vdash r$. Suppose that $w \in S_n$ be a permutation whose cycle type is ρ. Then the number of permutations $\sigma \in S_n/S_{\tilde{\mu}(l)} \times S_r$ satisfying the condition*

$$w\sigma a^{-1} \equiv \sigma \quad \textit{modulo } S_{\tilde{\mu}(l)} \times S_r$$

coincides with the coefficient

$$\binom{\rho}{\tilde{\rho}} C(\tilde{\rho}, \mu; l) l^{l(\tilde{\rho})}$$

in the recursive formula:

$$\binom{\rho}{\tilde{\rho}} C(\tilde{\rho}, \mu; l) l^{l(\tilde{\rho})} = \sharp\{\sigma \in S_n/S_{\tilde{\mu}(l)} \times S_r | w\sigma a^{-1} \equiv \sigma \ mod\ S_{\tilde{\mu}(l)} \times S_r\}.$$

Proof. Let ρ be a partition of the form $\rho = l\tilde{\rho} \cup \bar{\rho}$, where $\tilde{\rho} \vdash \tilde{\mu}(l)^{1/l}$ and $\bar{\rho} \vdash r$. Let $w \in S_n$ be a permutation of cycle type ρ. Since such a permutation is conjugate to an element of the subgroup $H_\mu(l) = \big(S_{\tilde{\mu}(l)} \times S_r\big) \rtimes \langle a \rangle = \big(S_{\tilde{\mu}(l)} \rtimes \langle a \rangle\big) \times S_r$, we may assume that

$$w = \tau_1\tau_2 a = \tau_1 a \tau_2$$

where $\tau_1 \in S_{\tilde{\mu}(l)}$ and $\tau_2 \in S_r$.

Our problem is to enumerate permutations $\sigma \in S_n/S_{\tilde{\mu}(l)} \times S_r$ satisfying $w\sigma a^{-1} \equiv \sigma$ modulo $S_{\tilde{\mu}(l)} \times S_r$, i.e.,

$$\tau_1\tau_2 a\sigma = \sigma\pi_1\pi_2 a \tag{4.1}$$

for some $\pi_1 \in S_{\tilde{\mu}(l)}$ and $\pi_2 \in S_r$. By (4.1), the cycle type of $\pi_1\pi_2 a$ coincide with ρ.

Let $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_n]$. This means that the permutation σ is the bijection $i \mapsto \sigma_i$ for each $i = 1, 2, \ldots, n$. The left multiplication by $\tau_1\tau_2 a$ acts on the letters $\sigma_1, \sigma_2, \ldots, \sigma_n$ of σ, and the right multiplication by $\pi_1\pi_2 a$ acts on positions $1, 2, \ldots, n$ of the components. Thus the condition (4.1) means that the action of $\tau_1\tau_2 a$ of cycle type ρ on the letters of σ coincide with the action of $\pi_1\pi_2 a$ of cycle type ρ on the positions of components of σ. Let $\tilde{\rho} = (1^{\tilde{m}_1}2^{\tilde{m}_2}\cdots)$ $\bar{\rho} = (1^{\bar{m}_1}2^{\bar{m}_2}\cdots)$, and hence $\rho = (l^{\tilde{m}_1}(2l)^{\tilde{m}_2}\cdots) \cup (1^{\bar{m}_1}2^{\bar{m}_2}\cdots)$.

Fix a permutation $\sigma \in S_n/S_{\tilde{\mu}(l)} \times S_r$ satisfying the condition (4.1), for example, $\sigma = [1, 2, \ldots, n]$. If $\tau_1 a$ has a cycle of length dl, say $(1, 2, \ldots, dl)$ for example, then multiplying σ from the left by the permutation

$$\begin{pmatrix} 1 & 2 & \ldots & l-1 & l & l+1 & l+2 & \ldots & 2l-1 & 2l & \ldots \\ 2 & 3 & \ldots & l & 1 & l+2 & l+3 & \ldots & 2l & l+1 & \ldots \end{pmatrix}$$

of order l produces another element of $S_n/S_{\tilde{\mu}(l)} \times S_r$ satisfying the condition. Moreover, for such a fixed representative σ, the number of elements of $S_n/S_{\tilde{\mu}(l)} \times S_r$ obtained by exchanging the letters corresponding to products of cycles in $\tau_1 a$ with the same total length in σ is

$$\sum_{\substack{\lambda \vdash \tilde{\mu}(l)^{1/l} \\ l\lambda = l\tilde{\rho}}} m_\lambda.$$

It is clear that permutations produced in these two ways from a fixed representative σ do not overlap. Therefore there are

$$l^{l(\tilde{\rho})} C(l\tilde{\rho}, \mu; l)$$

permutations for each fixed representative.

Finally, if σ satisfies the condition (4.1), then exchanging a cycle of $\tau_1 a$ and τ_2 of the same length produces $\binom{\rho}{l\tilde{\rho}}$ appropriate permutations which do not appeared in the preceding process. $\square$

Example 16 Let $\mu = (2, 2, 2, 2, 2, 1)$ and $l = 2, \ldots, M_\mu (= 5)$ be fixed, say $l = 2$. Then the corresponding product of cyclic permutations is $a = (13)(24)(57)(68)$. The subgroups $S_{\tilde{\mu}(l)}$ and $S_r = S_3$ are $S_{\{1,2\}} \times S_{\{3,4\}} \times S_{\{5,6\}} \times S_{\{7,8\}}$ and $S_{\{9,10,11\}}$ respectively. Let us consider the case $w = (12)a(9, 10) = (1324)(57)(68)(9, 10)$ ($\tau_1 = (12)$, $\tau_2 = (9, 10)$). The cycle type ρ of w is $\rho = (4, 2, 2, 2, 1)$. If we let $\tilde{\rho} = ((2), (1, 1)) \vdash \tilde{\mu}(l)^{1/2} = (2, 2)$ and $\bar{\rho} = (2, 1) \vdash r = 3$, we have $\rho = 2\tilde{\rho} \cup \bar{\rho}$. Consider $\sigma = [1, 2, \ldots, 11] \in S_n/S_{\tilde{\mu}(l)} \times S_r$. It is clear that σ satisfies the condition $w\sigma = \sigma\pi_1\pi_2 a, \pi_1 \in S_{\tilde{\mu}(l)}, \pi_2 \in S_r$. If we replace the components $1, 2, 3, 4$, which is the cycle range of the cycle $(1, 3, 2, 4)$, by $3, 4, 1, 2$ respectively, the resulting permutation $[3, 4, 1, 2, 5, 6, \ldots, 11]$ also satisfies the condition. Similarly, exchanging 5 and 7, or 6 and 8 works. Exchanging two different cycles in $\tau_1 a = (1324)(57)(68)$ of the same length also works. In this example, we can exchange 5 (resp. 7) and 6 (resp. 8). The resulting permutation, for example $[1, 2, 3, 4, 6, 5, 7, 8, 9, 10, 11]$ can be easily checked that it satisfies the condition. This change of components in σ, however, is killed by the right action of the Young subgroup $S_{\tilde{\mu}(l)}$. This is the consequence of the fact $m_{((2),(1,1))} = \binom{2}{0,2}\binom{1}{1,0} = 1$. On the other hand, exchanging the letters of σ corresponding to products of cycles in $\tau_1 a$ with the same total length are valid, which produces in this example new appropriate permutation $[5, 6, 7, 8, 1, 2, 3, 4, 9, 10, 11]$. This reflect the fact

$$\sum_{\substack{\lambda \vdash \tilde{\mu}(l)^{1/2}=(2,2) \\ 2\lambda=(4,2,2)}} m_\lambda = m_{((2),(1,1))} + m_{((1,1),(2))} = 2.$$

Finally, we can exchange the letters of σ corresponding to (57) or (68) by $(9, 10)$, which produces new $\binom{\rho}{\tilde{\rho}} = \binom{2+1}{2} = 3$ appropriate permutations. These new permutations satisfying the condition obtained by these three methods do not overlap. Thus we have

$$\sharp\{\sigma \in S_{11}/S_{(2^4)} \times S_3 | w\sigma a^{-1} \equiv \sigma \bmod S_{(2^4)} \times S_3\} = \binom{3}{2} \left(m_{((2),(1,1))} + m_{((1,1),(2))}\right) 2^3 = 48.$$

5 Representation theory of symmetric group

In this final section, we understand the main result in terms of representation theory of the symmetric group.

It is known that the Green polynomial $Q^\mu_\rho(q)$ gives the graded character value of a certain graded S_n-module, called the *DeConcini-Procesi-Tanisaki algebra* [DP]. It is known that the algebra R_μ is isomorphic as an S_n-module to the cohomology ring

$$H^*(X_\mu, \mathbf{C})$$

of a certain algebraic variety X_μ, called the *fixed point subvariety*. The symmetric group S_n has a natural action on $H^*(X_\mu, \mathbf{C})$, the representation of S_n afforded by which is called the Springer representation [Sp2, L]. The fixed point subvariety X_μ is a subvariety of the flag variety $X_n = GL_n/B$, where GL_n is the general linear group and B a Borel subgroup, defined as the set of fixed point of the left multiplication by a unipotent matrix of which sizes of Jordan blocks form the partition μ. For special μ's, it is known that

$$R_{(n)} \cong \mathbf{C},$$

the trivial representation, and

$$R_{(1^n)} \cong R_n,$$

the coinvariant algebra of S_n, a graded version of the left regular representation of S_n. For general μ, it is known that R_μ is isomorphic to the representation of S_n induced from the trivial representation of the Young subgroup S_μ corresponding to the partition μ:

$$R_\mu \cong_{S_n} \mathrm{Ind}^{S_n}_{S_\mu} 1.$$

The DeConcini-Procesi-Tanisaki algebra R_μ is defined as the quotient algebra of the polynomial ring $\mathbf{C}[x_1, x_2, \ldots, x_n]$ by a certain S_n-invariant homogeneous ideal I_μ, on which the symmetric group S_n naturally acts as permutation of the variables [DP, T, GP]. Let

$$R_\mu = \bigoplus_{d=0}^{n(\mu)} R_\mu^d,$$

be the homogeneous decomposition of R_μ, where $n(\mu) = \sum_{i\geq 1}(i-1)\mu_i$ if $\mu = (\mu_i)$, and $R_\mu^0 = \mathbf{C}$. It is clear that each homogeneous space R_μ^d is also an S_n-module, i.e., R_μ is an graded S_n-module. The *graded character* $\mathrm{char}_q R_\mu$ of the graded module R_μ, evaluated on the conjugacy class corresponding to $\rho \vdash n$, is by definition a polynomial in q

$$\mathrm{char}_q R_\mu(\rho) = \sum_{d\geq 0} q^d \mathrm{char} R_\mu^d(\rho)$$

with integer coefficients. It is known that the graded character value $\mathrm{char}_q R_\mu(\rho)$ coincide with the Green polynomial

$$Q_\rho^\mu(q) = \mathrm{char}_q R_\mu(\rho)$$

for each $\rho \vdash n$.

The aim of this section is to rephrase the recursive formula of the Green polynomials $Q_\rho^\mu(q)$ in the main theorem, in terms of the graded algebra R_μ. The formula gives a representation theoretical interpretation of a certain combinatorial property of the algebra R_μ. This property concerns with the Hilbert polynomial of the algebra R_μ (or the Betti numbers of the variety X_μ). Let q be an indeterminate. Then the *Hilbert polynomial* $\mathrm{Hilb}_\mu(q)$ of the algebra R_μ is defined by

$$\mathrm{Hilb}_\mu(q) = \sum_{d\geq 0} q^d \dim R_\mu^d.$$

Since the character value of each S_n-module R_μ^d evaluated at the identity element $e \in S_n$ coincides with its dimension $\dim R_\mu^d$, the Hilbert polynomial $\mathrm{Hilb}_\mu(q)$ coincides with the Green polynomial $Q_\rho^\mu(q)$ with $\rho = (1^n)$ the cycle type of e:

$$\mathrm{Hilb}_\mu(q) = Q_{(1^n)}^\mu(q).$$

By Proposition 1, we have

Proposition 17 *Let μ be a partition. Then it holds that*

$$\mathrm{Hilb}_\mu(q) = \frac{(1-q)(1-q^2)\cdots(1-q^{M_\mu})}{(1-q)^n} G_{(1^n)}^\mu(q),$$

where $G_{(1^n)}^\mu(q)$ is the polynomial in q with integer coefficients.

Let $\mu \vdash n$ be a partition and $l \in \{2, 3, \ldots, M_\mu\}$ fixed. For each $k = 0, 1, \ldots, l-1$, define

$$R_\mu(k;l) := \bigoplus_{d\equiv k \bmod l} R_\mu^d.$$

It is clear that these $R_\mu(k;l)$'s are S_n-submodules of R_μ. Then, by virtue of a lemma due to T. Oshima (see [MN1, Lemma 3]), it is shown that:

Corollary 18 *The dimensions of the submodules $R_\mu(k;l)$ $(k = 0, 1, \ldots, l-1)$ coincides with each other.*

Our problem is to give an interpretation to this property "coincidence of dimensions" in terms of representation theory, i.e., find a subgroup $H(l)$ and its modules $Z(k;l)$ $(k = 0, 1, \ldots, l-1)$ of equal dimension such that

$$R_\mu(k;l) \cong_{S_n} \mathrm{Ind}_{H(l)}^{S_n} Z(k;l), \quad k = 0, 1, \ldots, l-1.$$

Since the dimension of the induced representation $\mathrm{Ind}_{H(l)}^{S_n} Z(k;l)$ is $\dim Z(k;l)|S_n|/|H(l)|$, we can convince ourselves that these isomorphisms are representation theoretical interpretation of the coincidence of dimensions. Let $\mu \vdash n$ be a partition, $l \in \{2, 3, \ldots, M_\mu\}$ fixed, $a = a_\mu(l)$ the cyclic permutation product corresponding to μ and l, and $C_l = \langle a \rangle$ the cyclic subgroup of S_n generated by a. Recall that the subgroup $H_\mu(l)$ is defined by $H_\mu(l) = \big(S_{\tilde{\mu}(l)} \rtimes C_l\big) \times S_r$. Consider, for each $k = 0, 1, \ldots, l-1$, $H_\mu(l)$-modules $Z_\mu(k;l)$ defined as follows:

$$Z_\mu(k;l) = \bigoplus_{d=1}^{n(\bar{\mu}(l))} \varphi_l^{(k-d)} \otimes R_{\bar{\mu}(l)}^d,$$

where $\varphi_l^{(r)}$ is the irreducible representation of the cyclic group $C_l = \langle a \rangle$ such that $a \longmapsto \zeta_l^r$. The Young subgroup $S_{\tilde{\mu}(l)}$ acts trivially on $Z_\mu(k;l)$. Since $\varphi_l^{(r)}$'s are one dimensional, it is obvious that the dimension of $Z_\mu(k;l)$ does not depend on k, i.e., it is equal to $\dim R_{\bar{\mu}(l)}$ for each $k = 0, 1, \ldots, l-1$. We shall show that

$$R_\mu(k;l) \cong_{S_n} \mathrm{Ind}_{H_\mu(l)}^{S_n} Z_\mu(k;l), \quad k = 0, 1, \ldots, l-1.$$

Actually, we shall show a certain $S_n \times C_l$-module isomorphism between R_μ and $\mathrm{Ind}_{S_{\tilde{\mu}(l)} \times S_r}^{S_n} R_{\bar{\mu}(l)}$, originally suggested by T. Shoji, which is equivalent to those isomorphisms.

We define $S_n \times C_l$-modules structures on R_μ and $\mathrm{Ind}_{S_{\tilde{\mu}(l)} \times S_r}^{S_n} R_{\bar{\mu}(l)}$ as follows. In both cases, the S_n-actions are natural ones. The action of C_l on R_μ is defined by

$$a.x = \zeta_l^d x, \quad x \in R_\mu^d.$$

Recall that the induced modules $\mathrm{Ind}_{S_{\tilde{\mu}(l)} \times S_r}^{S_n} R_{\bar{\mu}(l)}$ has the following realization:

$$\mathrm{Ind}_{S_{\tilde{\mu}(l)} \times S_r}^{S_n} R_{\bar{\mu}(l)} = \bigoplus_{\sigma \in S_n / S_{\tilde{\mu}(l)} \times S_r} \sigma \otimes R_{\bar{\mu}(l)}. \tag{5.1}$$

Then the C_l-action is defined by

$$a.\sigma \otimes x = \sigma a^{-1} \otimes a.x, \quad \sigma \in S_n / S_{\tilde{\mu}(l)} \times S_r,\ x \in R_{\bar{\mu}(l)}. \tag{5.2}$$

It is easy to see that these S_n-action and C_l-action commute on each module.

Theorem 19 *Let μ be a partition of a positive integer n, and l an integer such that $2 \le l \le M_\mu$ fixed. Suppose that $n = ql + r$, $0 \le r \le l-1$, and let C_l be the cyclic group generated by the element $a = a_\mu(l)$. Then there exists an isomorphism of $S_n \times C_l$-modules*

$$R_\mu \cong \mathrm{Ind}_{S_{\bar\mu(l)}\times S_r}^{S_n} R_{\bar\mu(l)}. \tag{5.3}$$

Proof. Since we work on a field of characteristic zero, it is enough to show that the character values on both sides coincide on $S_n \times C_l$. We shall show the following identity

$$\mathrm{char} R_\mu(w, a^j) = \mathrm{char}\ \mathrm{Ind}_{S_{\bar\mu(l)}\times S_r}^{S_n} R_{\bar\mu(l)}(w, a^j)$$

for each $(w, a^j) \in S_n \times C_l$ $(j = 0, 1, \dots, l-1)$. Let $\rho(w)$ be the cycle type of w. Recall that the Green polynomial $Q^\mu_\rho(q)$ gives the graded character $\mathrm{char}_q R_\mu(w) = \sum_{d\ge 0} q^d \mathrm{char} R^d_\mu(w)$, a slight consideration shows that

$$Q^\mu_{\rho(w)}(q)|_{q=\zeta_l^j} = \mathrm{char} R_\mu(w, a^j).$$

Suppose that ζ_l^j is a primitive p-th root of unity. In this case, the order of a^j is also p. Thus our problem is reduced to show that

$$Q^\mu_\rho(\zeta_p) = \mathrm{char}\ \mathrm{Ind}_{S_{\bar\mu(l)}\times S_r}^{S_n} R_{\bar\mu(l)}(w, a^j),$$

where $w \in S_n$ lies in the conjugacy class corresponding to ρ, and the order of a^j is p. Since the order p satisfies $p|l$, we have $p \le M_\mu$. By Theorem 10, it suffices to show that

1. $\mathrm{char}\ \mathrm{Ind}_{S_{\bar\mu(l)}\times S_r}^{S_n} R_{\bar\mu(l)}(w, a^j) \ne 0 \Longrightarrow \rho = p\tilde\rho \cup \bar\rho, \quad \tilde\rho \vdash \tilde\mu(l)^{1/l},\ \bar\rho \vdash r,$

2. For an element $w \in S_n$ with the cycle type ρ, $\rho = p\tilde\rho \cup \bar\rho, \tilde\rho \vdash \tilde\mu(l)^{1/l}, \bar\rho \vdash r$, it holds that
$$\mathrm{char}\ \mathrm{Ind}_{S_{\bar\mu(l)}\times S_r}^{S_n} R_{\bar\mu(l)}(w, a^j) = \sum_{\substack{\nu\vdash|\bar\mu(l)| \\ \nu\subset\rho}} \binom{\rho}{\nu} C(\nu, \mu; p) p^{l(\nu)} Q^{\bar\mu(l)}_{\rho\setminus\nu}(\zeta_p),$$

where the symbols $\tilde\rho$, $\bar\rho$ and $\binom{\rho}{\nu}$ are considered for p.

Suppose that $\mathrm{char}\ \mathrm{Ind}_{S_{\bar\mu(l)}\times S_r}^{S_n} R_{\bar\mu(l)}(w, a^j) \ne 0$. By (5.1), there should be $\sigma \in S_n/S_{\bar\mu(l)} \times S_r$, such that $\mathrm{char}\ \sigma \otimes R_{\bar\mu(l)}(w, a^j) \ne 0$. Let $\mathcal{B}_{\bar\mu(l)}$ be a homogenous basis of $R_{\bar\mu(l)}$. Then, (5.2) implies that there exists $x \in \mathcal{B}_{\bar\mu(l)}$ such that

$$(w, a^j)(\sigma \otimes x)|_{\sigma\otimes x} \ne 0.$$

The symbol $(w, a^j)(\sigma \otimes x)|_{\sigma\otimes x}$ indicates the coefficient of $\sigma \otimes x$ in the linear expansion of $(w, a^j)(\sigma \otimes x)$ with the basis $\{\sigma \otimes x | x \in \mathcal{B}_{\bar\mu(l)}\}$. By the definition of the action of $S_n \times C_l$, we have

$$(w, a^j)(\sigma \otimes x) = w\sigma a^{-j} \otimes a^j.x = \zeta_p w\sigma a^{-j} \otimes x.$$

It follows from the condition $(w, a^j)(\sigma \otimes x)|_{\sigma\otimes x} \neq 0$ that $w\sigma a^{-j} \equiv \sigma$ modulo $S_n/(S_{\tilde{\mu}(l)} \times S_r)$. Therefore w is conjugate to an element of the form $\tau_1\tau_2 a^j$, $\tau_1 \in S_{\tilde{\mu}(l)}$ and $\tau_2 \in S_r$. Since the order of a^j is p, it follows from Lemma 13 that the cycle type ρ of w is of the form $\rho = p\tilde{\rho} \cup \bar{\rho}$, for some $\tilde{\rho} \vdash \tilde{\mu}(l)^{1/p}$ and $\bar{\rho} \vdash r$.

Let ρ be a partition of the form $\rho = p\tilde{\rho} \cup \bar{\rho}$, for some $\tilde{\rho} \vdash \tilde{\mu}(l)^{1/p}$ and $\bar{\rho} \vdash r$, and suppose that $w \in S_n$ is an element of which cycle type is ρ. Then, again by Lemma 13, w is conjugate to some element $\tau_1\tau_2 a^j$, where $\tau_1 \in S_{\tilde{\mu}(l)}$ and $\tau_2 \in S_r$. We may assume that $w = \tau_1\tau_2 a^j$, $\tau_1 \in S_{\tilde{\mu}(l)}$ and $\tau_2 \in S_r$, without loss of generality. Fix a set of complete representatives $\{\sigma_1, \sigma_2, \dots, \sigma_t\}$ of $S_n/S_{\tilde{\mu}(l)} \times S_r$. Then we have

$$\operatorname{char} \operatorname{Ind}_{S_{\tilde{\mu}(l)}\times S_r}^{S_n} R_{\bar{\mu}(l)}(w, a^j) = \sum_{i=1}^{t} \operatorname{char}\left(\sigma_i \otimes R_{\bar{\mu}(l)}\right)(w, a^j).$$

Suppose that

$$\operatorname{char}\,(\sigma_i \otimes R_{\bar{\mu}(l)})(w, a^j) \neq 0.$$

Then it follows that $w\sigma_i a^{-j} = \sigma_i\pi_1\pi_2$ for some $\pi_1 \in S_{\tilde{\mu}(l)}$ and $\pi_2 \in S_r$. Since $w = \tau_1\tau_2 a^j$, $\tau_1 \in S_{\tilde{\mu}(l)}$ and $\tau_2 \in S_r$, it is trivial that $\tau_1\tau_2 a^j$ and $\pi_1\pi_2 a^j$ are conjugate in S_n. Since the cycle types of these elements are coincide, if the cycle type of $\pi_1 a^j$ is $\nu \vdash n - r$, then that of π_2 is $\rho \setminus \nu$. Moreover, the cycle type of $\pi_1 a^j$ is of the form $p\lambda$, $\lambda \vdash \tilde{\mu}(l)^{1/p}$. With these notation, it holds that

$$\begin{aligned}
\operatorname{char}\left(\sigma_i \otimes R_{\bar{\mu}(l)}\right)(w, a^j) &= \sum_{x\in\mathcal{B}_{\bar{\mu}(l)}} (w, a^j)(\sigma_i \otimes x)|_{\sigma_i\otimes x} \\
&= \sum_{x\in\mathcal{B}_{\bar{\mu}(l)}} w\sigma_i a^{-j} \otimes a^j.x|_{\sigma_i\otimes x} \\
&= \sum_{x\in\mathcal{B}_{\bar{\mu}(l)}} \zeta_p^{\deg x}\sigma_i \otimes \pi_2 x,
\end{aligned}$$

where $\deg x$ denotes the degree of x in $R_{\bar{\mu}(l)}$. By the definition of the graded character of $R_{\bar{\mu}(l)}$, it immediately follows that

$$\operatorname{char}_q R_{\bar{\mu}(l)}(\pi_2)|_{q=\zeta_p} = Q^{\bar{\mu}(l)}_{\rho\setminus\nu}(\zeta_p).$$

Let $\nu \vdash r$ be a partition such that $\nu \subset \rho$. By Proposition 15, the number of permutations $\sigma \in S_n/S_{\tilde{\mu}(l)} \times S_r$ satisfying $w\sigma a^{-j} = \sigma\pi_1\pi_2$, $\pi_1 \in S_{\tilde{\mu}(l)}$, $\pi_2 \in S_r$, such that $\rho(\pi_2) = \nu$ is given by

$$\binom{\rho}{\nu} m(\nu, \mu; p) p^{l(\rho\setminus\nu)}.$$

Thus we have

$$\sum_{\sigma\in S_n/S_{\tilde{\mu}(l)}\times S_r} \operatorname{char}(\sigma \otimes R_{\bar{\mu}(l)})(w, a^j) = \sum_{\substack{\sigma\in S_n/S_{\tilde{\mu}(l)}\times S_r \\ w\sigma a^{-j}\equiv\sigma \mod S_{\tilde{\mu}(l)}\times S_r}} \operatorname{char}(\sigma \otimes R_{\bar{\mu}(l)})(w, a^j)$$

$$
\begin{aligned}
&= \sum_{\substack{\nu\vdash r\\ \nu\subset\rho}} \sum_{\substack{\sigma\in S_n/S_{\tilde\mu}\times S_r\\ w\sigma a^{-j}\equiv\sigma \bmod S_{\bar\mu(l)}\times S_r\\ \rho(\pi_2^\sigma)=\nu}} \mathrm{char}_q R_{\bar\mu(l)}(\pi_2^\sigma)|_{q=\zeta_l^j}\\
&= \sum_{\substack{\nu\vdash r\\ \nu\subset\rho}} \binom{\rho}{\nu} m(\nu,\mu;p)p^{l(\rho\backslash\nu)}Q^\mu_{\rho\backslash\nu}(\zeta_p),
\end{aligned}
$$

which proves the theorem. □

Corollary 20 *Let $\mu\vdash n$ be partition and an integer $l\in\{2,3,\dots,M_\mu\}$ fixed. Then there exist $H_\mu(l)$-modules $Z_\mu(k;l)$ ($k=0,1,\dots,l-1$) of equal dimension such that*

$$R_\mu(k;l)\cong_{S_n}\mathrm{Ind}_{H_\mu(l)}^{S_n}Z_\mu(k;l)$$

for each $k=0,1,\dots,l-1$.

Proof. Consider the eigenspace decomposition of the action of a in the $S_n\times C_l$-isomorphism (5.3). □

Example 21 Let $\mu=(5,4,4,2,2,1)$ and $l=2$. Then $\mu(2)=(4,4,2,2)$, $\bar\mu(l)=(5,1)$, and

$$a=a_\mu(2)=\begin{pmatrix}6&7&8&9&10&11&12&13\\10&11&12&13&6&7&8&9\end{pmatrix}\begin{pmatrix}14&15&16&17\\16&17&14&15\end{pmatrix}.$$

The dimensions of $R_\mu(k;2)$, $k=0,1$, equals $\dim R_\mu/2=\binom{18}{5,4,4,2,2,1}/2=18!/5!4!4!2!2!1!2$. The subgroup $H_\mu(2)$ is defined by $H_\mu(2)=S_{\mu(2)}\rtimes\langle a\rangle\times S_6$, where $S_{\mu(2)}=S_{\{6,7,8,9\}}\times S_{\{10,11,12,13\}}\times S_{\{14,15\}}\times S_{\{16,17\}}$ and $S_3=S_{\{1,2,3,4,5,18\}}$. Define $H_\mu(2)$-modules $Z_\mu(k;l)$ ($k=0,1$) by $Z_\mu(k;2):=\bigoplus_{d\equiv k \bmod 2}\varphi_2^{(k-d)}\otimes R^d_{\bar\mu(l)}$. These spaces are considered as $H_\mu(2)$-module, where $S_{\mu(2)}$ acts on them trivially. The dimension of these modules are both equal to $\dim R_{\bar\mu(l)}=\binom{6}{5,1}=6!/5!1!$. Then, for each $k=0,1$, we have an isomorphism of S_{18}-modules $R_\mu(k;2)\cong\mathrm{Ind}^{S_{18}}_{(S_{(4,4,2,2)}\rtimes C_2)\times S_6}Z_\mu(k;2)$. The induced modules are of dimension $18!/4!4!2!2!6!2\times 6!/5!1!=18!/5!4!4!2!2!1!2=\dim R_\mu(k;2)$ for each $k=0,1$.

Remark 22 Recently, the author was informed by T. Shoji that the problem considered in this section is given an affirmative answer in a largely generalized setting [Sh].

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