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Elliptic Bernoulli Functions And Their Identities

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ABSTRACT. We introduce an elliptic analogue of the Bernoulli functions, which we call elliptic Bernoulli functions. They are defined by using the modified generating function of the elliptic polylogarithms. By degeneration of the elliptic Bernoulli functions, we obtain standard properties and new identities for the Bernoulli functions.

1. Introduction

As a generalization of the usual polylogarithms, A. Levin [1] studied elliptic polylogarithms purely analytically, and introduced their generating function. In this paper, we define elliptic Bernoulli functions by using the modified generating function of the elliptic polylogarithms, and show that they have symmetry, periodicity, an additive formula, and transformation properties. In addition, We get several identities for the generating function of the elliptic Bernoulli functions, which gives identities for the elliptic Bernoulli functions. Standard properties and new identities for the (classical) Bernoulli functions and polynomials are obtained by degeneration of the elliptic Bernoulli functions. We note that Fourier expansions of the elliptic Bernoulli functions are a sort of Eisenstein series.

The paper is organized as follows: In Section 2, we define the elliptic Bernoulli functions, and give their symmetry, periodicity, an additive formula, and transformation properties. By degeneration of the elliptic Bernoulli functions, standard properties of the Bernoulli functions and polynomials are induced. In Section 3, we show various identities for the generating function of the elliptic Bernoulli functions, which give birth to identities for the elliptic Bernoulli functions. By degeneration, new identities for the Bernoulli functions are deduced.

We will use the following notations: $e(x) := e^{2\pi\sqrt{-1}x}$, $q := e(\tau)$, and Jacobi's theta function

$$\theta(x; \tau) := \sum_{m \in \mathbb{Z}} e^{\pi\sqrt{-1}(m+\frac{1}{2})\tau + 2\pi\sqrt{-1}(m+\frac{1}{2})(x+\frac{1}{2})}.$$

Here τ is a complex number with positive imaginary part, i.e., $\text{Im } \tau > 0$.

2. Elliptic Bernoulli Functions

In this section, we give the definition and properties of elliptic Bernoulli functions, and degeneration into the Bernoulli functions.

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Consider the following function

$$(1) \quad F_x(\xi; \tau) = 2\pi\sqrt{-1} \left[\sum_{j=1}^{\infty} \frac{q^j}{e(x) - q^j} e(-j\xi) - \sum_{j=1}^{\infty} \frac{q^j}{e(-x) - q^j} e(j\xi) \right. \\ \left. + \frac{1}{e(x) - 1} + \frac{1}{e(\xi) - 1} + 1 \right], \quad (|\operatorname{Im} x|, |\operatorname{Im} \xi| < \operatorname{Im} \tau).$$

This function has another expression ([2] p.446 , [3] Sect.3):

$$F_x(\xi; \tau) = F_\xi(x; \tau) = \frac{\theta'(0; \tau)\theta(x + \xi; \tau)}{\theta(x; \tau)\theta(\xi; \tau)},$$

where $\theta'(x; \tau) = \frac{\partial}{\partial x}\theta(x; \tau)$. Hence it has simple poles on $\mathbb{Z} + \tau\mathbb{Z}$ and residue 1 at the origin with respect to x . Periodic properties and transformation properties of $F_x(\xi; \tau)$ are the following ([3] Sect.3):

$$(2) \quad F_x(\xi + 1; \tau) = F_x(\xi; \tau), \quad F_x(\xi + \tau; \tau) = e(-x)F_x(\xi; \tau), \\ F_x(\xi; \tau) = \frac{1}{c\tau + d} e\left(-\frac{cx\xi}{c\tau + d}\right) F_{\frac{x}{c\tau + d}}\left(\frac{\xi}{c\tau + d}; \frac{a\tau + b}{c\tau + d}\right),$$

where $a, b, c, d \in \mathbb{Z}$ such that $ad - bc = 1$.

DEFINITION 2.1. We define the generating function $\underline{F}(x, y; \xi; \tau)$ of elliptic Bernoulli functions and the m -th elliptic Bernoulli function $B_m(x, y; \tau)$ as

$$\underline{F}(x, y; \xi; \tau) := e(y\xi)F_{-x+y\tau}(\xi; \tau) = \sum_{m=0}^{\infty} \frac{B_m(x, y; \tau)}{m!} (2\pi\sqrt{-1})^m \xi^{m-1}.$$

Because of (1), we have $B_0(x, y; \tau) = 1$ and

$$(3) \quad B_m(x, y; \tau) = m \left(\sum_{j=1}^{\infty} (y - j)^{m-1} \frac{q^j}{e(-x + y\tau) - q^j} \right. \\ \left. - \sum_{j=1}^{\infty} (y + j)^{m-1} \frac{q^j}{e(x - y\tau) - q^j} + y^{m-1} \frac{e(-x + y\tau)}{e(-x + y\tau) - 1} \right) \\ + B_m(y)$$

for every positive integer m . Here $B_m(y)$ is the m -th Bernoulli polynomial:

$$2\pi\sqrt{-1} \frac{e(y\xi)}{e(\xi) - 1} = \sum_{m=0}^{\infty} \frac{B_m(y)}{m!} (2\pi\sqrt{-1})^m \xi^{m-1}.$$

Remark 1 The origin of the definition of elliptic Bernoulli functions is the following. We can formally regard the function $2\pi\sqrt{-1} \int e(y\xi)/(e(\xi) - 1)d\xi$ as the generating function of the Debye polylogarithms $\Lambda_m(\xi)$ in the following sense (the definition of the Debye polylogarithms is in [1] Sect.1):

$$\begin{aligned}
 & 2\pi\sqrt{-1} \int e(y\xi) \frac{1}{e(\xi) - 1} d\xi \\
 &= 2\pi\sqrt{-1} \int e(y\xi) Li_0(e(-\xi)) d\xi \\
 &= -e(y\xi) Li_1(e(-\xi)) + 2\pi\sqrt{-1}y \int e(y\xi) Li_1(e(-\xi)) d\xi \\
 &= \dots\dots \\
 &= -e(y\xi) \left(\sum_{m=0}^{\infty} Li_{m+1}(e(-\xi)) y^m \right) \\
 &= - \sum_{m=0}^{\infty} \left(\sum_{k=0}^m Li_{k+1}(e(-\xi)) \frac{(2\pi\sqrt{-1})^{-k} \xi^{m-k}}{(m-k)!} \right) (2\pi\sqrt{-1}y)^m \\
 &= \sum_{m=0}^{\infty} \Lambda_{m+1}(-\xi) (2\pi\sqrt{-1})^{m+1} (-y)^m
 \end{aligned}$$

Here $Li_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m}$ is the Euler polylogarithm, and assume that the series is formal. A. Levin introduced the modified generating function $\underline{\Lambda}(\xi, \tau; x, y)$ of the elliptic polylogarithms ([1] Sect.2):

$$\underline{\Lambda}(\xi, \tau; x, y) = \sum_{m \geq 0, n \geq 1} \Lambda_{m,n}(\xi; \tau) (-Y)^{n-1} X^m + \frac{e^{-Y\xi}}{Y(Y\tau - X)} + \frac{1}{(e^Y - 1)X},$$

where $\Lambda_{m,n}(\xi; \tau)$ is the (Debye) elliptic polylogarithm with index (m, n) . He also give the equation ([1] Prop.3.1)

$$\underline{F}(x, y; \xi; \tau) = \frac{\partial}{\partial \xi} \underline{\Lambda}(\xi, \tau; -2\pi\sqrt{-1}x, -2\pi\sqrt{-1}y).$$

Integrating both sides of the equation above with respect to ξ , we get the formula

$$\int \underline{F}(x, y; \xi; \tau) d\xi = \underline{\Lambda}(\xi, \tau; -2\pi\sqrt{-1}x, -2\pi\sqrt{-1}y)$$

as a matter of form. Two facts that $\int e(y\xi)/(e(\xi) - 1)d\xi$ is formally regarded as the generating function of the Debye polylogarithms and that $e(y\xi)/(e(\xi) - 1)$ is the generating function of the Bernoulli functions (polynomials) suggest that $\underline{F}(x, y; \xi; \tau)$ is the generating function of elliptic Bernoulli functions (polynomials).

Remark 2 The Fourier expansion of the elliptic Bernoulli function of degree $m \geq 3$ with respect to x, y is (cf. [4], [5])

$$B_m(x, y; \tau) = -\frac{m!}{(2\pi\sqrt{-1})^m} \sum_{\substack{i,j=-\infty \\ (i,j) \neq (0,0)}}^{\infty} \frac{e(ix + jy)}{(i\tau + j)^m}, \quad (0 \leq x, y < 1).$$

We introduce properties of the elliptic Bernoulli functions.

PROPOSITION 2.1. *Let a, b, c, d be integers such that $ad - bc = 1$, and m be a non negative integer. Then we have the following equations:*

(i) (*transformation property*)

$$B_m(x, y; \tau) = \frac{1}{(c\tau + d)^m} B_m(ax + by, cx + dy; \frac{a\tau + b}{c\tau + d}).$$

(ii) (*symmetry*) $B_m(x, y; \tau) = (-1)^m B_m(-x, -y; \tau)$.

(iii) (*periodicity*) $B_m(x + 1, y; \tau) = B_m(x, y + 1; \tau) = B_m(x, y; \tau)$.

(iv) (*additive formula*) $B_m(x, y + z; \tau) = \sum_{i=0}^m \binom{m}{i} B_i(x - z\tau, y; \tau) z^{m-i}$.

PROOF. By transformation properties of $F_x(\xi; \tau)$, we can get

$$\underline{F}(x, y; \xi; \tau) = \frac{1}{c\tau + d} \underline{F}(ax + by, cx + dy; \frac{\xi}{c\tau + d}; \frac{a\tau + b}{c\tau + d}).$$

It follows (i) from comparing the coefficient of ξ^m of the above equation. One can prove the other cases in a similar way. \square

We will show that the elliptic Bernoulli functions degenerate into the Bernoulli functions. Let $\{y\}$ denotes the fractional part of a real number y , i.e., $0 \leq \{y\} < 1$.

THEOREM 2.1. *Let y be a real number and m, k positive integers. Then the following equations hold:*

$$(i) \quad \lim_{\tau \rightarrow \infty \sqrt{-1}} B_m(x, y; \tau) = \begin{cases} B_m(\{y\}) = (-1)^m B_m(\{-y\}), & (y \notin \mathbb{Z}), \\ B_1 + \frac{e(-x)}{e(-x) - 1}, & (y \in \mathbb{Z}, m = 1), \\ B_m, & (y \in \mathbb{Z}, m > 1), \end{cases}$$

where $B_m = B_m(0)$.

$$(ii) \quad \lim_{k \rightarrow \infty} B_m(x, \frac{y}{\tau}; \frac{k}{|y|} \tau) = B_m(\frac{y}{\tau}) + m \left(\frac{y}{\tau}\right)^{m-1} \frac{e(-x)}{e(-x) - 1}, \quad (y \neq 0).$$

PROOF. If τ approaches $\infty \sqrt{-1}$, then $e(y\tau)q^j$ and $e(-y\tau)q^j$ ($j \geq 1$) approach 0. Therefore one can prove the equation (i) for $0 \leq y < 1$ because of the formula (3), which give the complete proof of (i) by virtue of periodicity and symmetry of the elliptic Bernoulli functions. The formula (ii) is derived from the equation

$$B_m(x, \frac{y}{\tau}; \frac{k}{|y|} \tau) = m \left(\sum_{j=1}^{\infty} \left(\frac{y}{\tau} - j\right)^{m-1} \frac{q'^{jk}}{e(-x) - q'^{jk}} - \sum_{j=1}^{\infty} \left(\frac{y}{\tau} + j\right)^{m-1} \frac{q'^{jk}}{e(x) - q'^{jk}} + \left(\frac{y}{\tau}\right)^{m-1} \frac{e(-x)}{e(-x) - 1} \right) + B_m\left(\frac{y}{\tau}\right),$$

where $q' = e\left(\frac{\tau}{|y|}\right)$. \square

We can deduce some properties of the Bernoulli functions and polynomials from the above proposition and theorem (i).

COROLLARY 2.1.

(i) (*symmetry*) $B_m(\{y\}) = (-1)^m B_m(\{-y\}), \quad (y \in \mathbb{R})$.

(ii) (*quasi periodicity*) $B_m(y + 1) = B_m(y) + my^{m-1}$.

$$(iii) \text{ (additive formula)} \quad B_m(x+y) = \sum_{i=0}^m \binom{m}{i} B_i(y) x^{m-i}.$$

PROOF. The formula (i) is already proved in the above theorem. When $0 < y < 1$, the equation (ii) is deduced from

$$\lim_{\tau \rightarrow \infty \sqrt{-1}} B_m(x, y+1; \tau) = -my^{m-1} + B_m(y+1).$$

One get (ii) by analyticity. As the proof of (ii), (iii) is derived from the additive formula of the elliptic Bernoulli functions. \square

3. Identities

We will give identities for the generating function of the elliptic Bernoulli functions, which imply identities for the elliptic Bernoulli functions and the (classical) Bernoulli functions. Throughout this section, we suppose that l, k, e_i, \dots , are positive integers, that m, m_i are non negative integers, and that empty product is regarded as one. we prepare the lemma bellow:

LEMMA 3.1. *If $e_0 = e'_0 = 1$, then we have*

$$\begin{aligned} & \sum_{a=0}^k \frac{1}{e_a} \left[\sum_{j=0}^{e'_a-1} \left(\sum_{i=0}^{e_a-1} F_{e'_0 \xi_0 + \dots + e'_k \xi_k} \left(l \left(x_a - \frac{i}{e_a} + \frac{j\tau}{e'_a} \right); \frac{l}{e'_a} \tau \right) \right. \right. \\ & \quad \left. \left. \times \prod_{\substack{b=0 \\ (b \neq a)}}^k F_{l' \xi_b} \left(e_b \left(x_b - x_a + \frac{i}{e_a} - \frac{j\tau}{e'_a} \right); \frac{e_b}{e'_b} \tau \right) \right) e(jl' \xi_a) \right] \\ & = \frac{1}{l} \sum_{j=0}^{l'-1} \left(\sum_{i=0}^{l-1} \prod_{b=0}^k F_{l' \xi_b} \left(e_b \left(x_b - \frac{i}{l} + \frac{j\tau}{l'} \right); \frac{e_b}{e'_b} \tau \right) \right) e(j(e'_0 \xi_0 + \dots + e'_k \xi_k)). \end{aligned}$$

PROOF. Let $f(x_0)$ be (the left hand side) - (the right hand side). It has quasi periodicity 1 and $e(-l' \xi_0)$ as x_0 goes to $x_0 + 1$ and $x_0 + \tau$ respectively. The left hand side has possible poles on the following lattices:

$$\begin{aligned} & \frac{1}{l} \mathbb{Z} + \frac{\tau}{l'} \mathbb{Z}; \quad x_a + \frac{1}{e_a} \mathbb{Z} + \frac{\tau}{e'_a} \mathbb{Z}, (a = 1, \dots, k); \\ & x_a - \frac{i}{e_a} + \frac{j\tau}{e'_a} + \mathbb{Z} + \tau \mathbb{Z}, (a = 1, \dots, k, i = 0, \dots, e_a - 1, j = 0, \dots, e'_a - 1). \end{aligned}$$

The right hand side has those on the following lattices:

$$\frac{i}{l} - \frac{j\tau}{l'} + \mathbb{Z} + \tau \mathbb{Z}, (i = 0, \dots, l-1, j = 0, \dots, l'-1).$$

For generic x_1, \dots, x_k , the order of the possible poles of $f(x_0)$ is 1 or less. By direct calculation, the function $f(x_0)$ is an entire function with quasi periodicity 1 and $e(-l' \xi_0)$ as x_0 goes to $x_0 + 1$ and $x_0 + \tau$ respectively. Therefore it must vanish for generic ξ_0, x_1, \dots, x_k . It also vanish for all ξ_0, x_1, \dots, x_k by analyticity. \square

Recall the generating function $\underline{F}(x, y; \xi)$ of the elliptic Bernoulli functions. Calculating the residue of the formula in the above lemma at $\xi_0 = 0$, and replacing x_a

with $-x_a + y_a\tau$, we obtain the following identity for $\underline{F}(x, y; \xi)$:

$$\begin{aligned} & \sum_{a=1}^k \frac{1}{e_a} \left[\sum_{j=0}^{e'_a-1} \sum_{i=0}^{e_a-1} \underline{F}\left(l(x_a + \frac{i}{e_a}), l'(y_a + \frac{j}{e'_a}); e'_1\xi_1 + \cdots + e'_k\xi_k; \frac{l}{l'}\tau\right) \right. \\ & \quad \left. \times \prod_{\substack{b=1 \\ (b \neq a)}}^k \underline{F}\left(e_b(x_b - x_a - \frac{i}{e_a}), e'_b(y_b - y_a - \frac{j}{e'_a}); l'\xi_b; \frac{e_b}{e'_b}\tau\right) \right] \\ & = \frac{1}{l} \sum_{j=0}^{l'-1} \sum_{i=0}^{l-1} \prod_{b=1}^k \underline{F}\left(e_b(x_b + \frac{i}{l}), e'_b(y_b + \frac{j}{l'}); l'\xi_b; \frac{e_b}{e'_b}\tau\right). \end{aligned}$$

Multiplying $(e'_1\xi_1 + \cdots + e'_k\xi_k) \prod_{b=1}^k (l'\xi_b)$ both sides of the equation above, one can compare the coefficient of $\xi_1^{m_1} \cdots \xi_k^{m_k}$ ($m_1, \dots, m_k \geq 0$). Since

$$\begin{aligned} & (e'_1\xi_1 + \cdots + e'_k\xi_k) \underline{F}\left(l(x_a + \frac{i}{e_a}), l'(y_a + \frac{j}{e'_a}); e'_1\xi_0 + \cdots + e'_k\xi_k; \frac{l}{l'}\tau\right) \\ & = \sum_{m_1, \dots, m_k=0}^{\infty} B_{\mathbf{M}}\left(l(x_a + \frac{i}{e_a}), l'(y_a + \frac{j}{e'_a}); \frac{l}{l'}\tau\right) (2\pi\sqrt{-1})^{\mathbf{M}} \frac{(e'_1\xi_1)^{m_1} \cdots (e'_k\xi_k)^{m_k}}{m_1! \cdots m_k!} \end{aligned}$$

where $\mathbf{M} = m_1 + \cdots + m_k$, we reach identities for the elliptic Bernoulli functions.

THEOREM 3.1. *If $k \geq 2$, then the following identities hold:*

$$\begin{aligned} \text{(i)} \quad & \frac{l^{m-1}}{l} \sum_{j=0}^{l'-1} \sum_{i=0}^{l-1} B_m\left(e\left(x + \frac{i}{l}\right), e'\left(y + \frac{j}{l'}\right); \frac{e}{e'}\tau\right) \\ & = \frac{e^{l^{m-1}}}{e} \sum_{j=0}^{e'-1} \sum_{i=0}^{e-1} B_m\left(l\left(x + \frac{i}{e}\right), l'\left(y + \frac{j}{e'}\right); \frac{l}{l'}\tau\right). \end{aligned}$$

(ii)

$$\begin{aligned} & \sum_{a=1}^k \frac{e_a^{l^{m_a-1}}}{e_a} m_a \sum_{j=0}^{e'_a-1} \sum_{i=0}^{e_a-1} \left[\sum_{\substack{c=1 \\ (c \neq a)}}^k \sum_{n_c=0}^{m_c} B_{(\mathbf{N}, a) + m_a - 1}\left(l(x_a + \frac{i}{e_a}), l'(y_a + \frac{j}{e'_a}); \frac{l}{l'}\tau\right) \right. \\ & \quad \left. \times \prod_{\substack{b=1 \\ (b \neq a)}}^k \binom{m_b}{n_b} B_{m_b - n_b}\left(e_b(x_b - x_a - \frac{i}{e_a}), e'_b(y_b - y_a - \frac{j}{e'_a}); \frac{e_b}{e'_b}\tau\right) e_b^{n_b} l^{m_b - n_b} \right] \\ & = \sum_{a=1}^k \frac{e_a^{l^{m_1 + \cdots + m_k - 2}}}{l} m_a \sum_{j=0}^{l'-1} \sum_{i=0}^{l-1} \left[B_{m_a - 1}\left(e_a\left(x_a + \frac{i}{l}\right), e'_a\left(y_a + \frac{j}{l'}\right); \frac{e_a}{e'_a}\tau\right) \right. \\ & \quad \left. \times \prod_{\substack{b=1 \\ (b \neq a)}}^k B_{m_b}\left(e_b\left(x_b + \frac{i}{l}\right), e'_b\left(y_b + \frac{j}{l'}\right); \frac{e_b}{e'_b}\tau\right) \right], \end{aligned}$$

where $(\mathbf{N}, a) = \sum_{d=1(d \neq a)}^k n_d$ and $B_{-1}(x, y; \tau) = 0$.

we obtain new identities for the Bernoulli functions due to degeneration.

COROLLARY 3.1. *Let $k \geq 2$ and $y \in \mathbb{R}$. Then we have*

$$(i) \quad l'^{m-1} \sum_{j=0}^{l'-1} B_m(\{e'(y + \frac{j}{l'})\}) = e'^{m-1} \sum_{j=0}^{e'-1} B_m(\{l'(y + \frac{j}{e'})\}), \quad (m \geq 2).$$

(ii)

$$\begin{aligned} & \sum_{a=1}^k e_a'^{m_a-1} m_a \sum_{j=0}^{e_a'-1} \left[\sum_{\substack{c=1 \\ (c \neq a)}}^k \sum_{n_c=0}^{m_c} B_{(\mathbf{n}, a) + m_a - 1}(\{l'(y_a + \frac{j}{e_a'})\}) \right. \\ & \quad \left. \times \prod_{\substack{b=1 \\ (b \neq a)}}^k \binom{m_b}{n_b} B_{m_b - n_b}(\{e_b'(y_b - y_a - \frac{j}{e_a'})\}) e_b'^{m_b} l'^{m_b - n_b} \right] \\ & = \sum_{a=1}^k e_a' l'^{m_1 + \dots + m_k - 2} m_a \sum_{j=0}^{l'-1} \left[B_{m_a - 1}(\{e_a'(y_a + \frac{j}{l'})\}) \prod_{\substack{b=1 \\ (b \neq a)}}^k B_{m_b}(\{e_b'(y_b + \frac{j}{l'})\}) \right], \end{aligned}$$

where $(\mathbf{n}, a) = \sum_{d=1}^k (d \neq a) n_d$, $B_{-1}(y) = 0$, and

$$l'(y_a + \frac{j}{e_a'}), e_b'(y_b - y_a - \frac{j}{e_a'}), e_a'(y_a + \frac{j}{l'}), e_b'(y_b + \frac{j}{l'}) \notin \mathbb{Z}, (a, b = 1, \dots, k, a \neq b).$$

Note. After this work was done, the author came to know of the preprint, "Lamé Equation, Quantum Top and Elliptic Bernoulli Polynomials" by M-P. Grosset and A.P. Veselov (arXiv:math-ph/0508068), which gives another elliptic analogue of the Bernoulli polynomials.

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