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# Differential Geometry from the viewpoint of Lagrangian or Legendrian singularity theory

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## Abstract

This is a half survey on the classical results of extrinsic differential geometry of hypersurfaces in Euclidean space from the view point of Lagrangian or Legendrian singularity theory. Many results in this paper have been already obtained in some articles. However, we can discover some new information of geometric properties of hypersurfaces from this point of view.

## 1 Introduction

In this paper we revise the classical differential geometry from the view point of the theory of Lagrangian or Legendrian singularities. Recently we apply the theory of Lagrangian or Legendrian singularities to the extrinsic differential geometry on submanifolds of pseudo-spheres in Minkowski space[9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. As consequences, we have obtained several interesting geometric properties of such submanifolds mainly from the view point of contact with model hypersurfaces (i.e., totally umbilic hypersurfaces). The theory of contact between submanifolds has been systematically developed by Montaldi[23, 24] for the study of curves and surfaces in Euclidean space as an application of the theory of singularities of smooth mappings due to Mather[21, 20]. However, we have discovered that if we apply the theory of Lagrangian or Legendrian singularities, we might be able to have much more detailed geometric properties through the previous researches[9, 15, 18]. Although such researches were focused on submanifolds of pseudo-spheres in Minkowski space, this method also supplies new information on submanifolds of Euclidean space.

In §2 we give a quick review on the classical Gaussian differential geometry of hypersurfaces in Euclidean space. The fundamental concept is the Gauss map of a hypersurface whose Jacobian determinant is the Gauss-Kronecker curvature. Therefore the singularities of the Gauss map is the set of the points where the Gauss-Kronecker curvature vanishes (i.e., the parabolic

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points). We also have the notion of evolutes and pedal hypersurfaces whose singularities correspond to some important geometric properties (umbilical points, ridge points and parabolic points etc). The height functions family and the distance squared functions family are the fundamental tools for the study of classical differential geometry as applications of singularity theory. The importance of such families were originally pointed out by Thom and the idea of Thom has been first realized by Porteous[25]. See also [3, 4, 5, 19, 26, 27]. We review the basic properties of the height functions family and the distance squared functions family in §3. We can show that these families are Morse families in the theory of Lagrangian or Legendrian singularities which control the singularities of evolutes, Gauss maps and pedals of hypersurfaces (cf., §4). We also review the theory of contact between submanifolds due to Montaldi[23, 24] in §5. In [15] we have considered the contact of submanifolds with families of hypersurfaces for the study of contact of hypersurfaces with families of hyperspheres in hyperbolic space as an application of Goryunov's result([6], Appendix). This technique is also useful for the study of the contact of hypersurfaces with families of hyperspheres in Euclidean space. We apply Lagrangian or Legendrian singularity theory to these theories of contact and show some new results in §6. §7 is devoted to a more detailed study of the case  $n = 3$ .

We shall assume throughout the whole paper that all the maps and manifolds are  $C^\infty$  unless the contrary is explicitly stated.

## 2 Hypersurfaces in Euclidean space

In this section we review the classical theory of differential geometry on hypersurfaces in Euclidean space and introduce some singular mappings associated to geometric properties of hypersurfaces.

Let  $\mathbf{X} : U \rightarrow \mathbb{R}^n$  be an embedding, where  $U \subset \mathbb{R}^{n-1}$  is an open subset. We denote that  $M = \mathbf{X}(U)$  and identify  $M$  and  $U$  through the embedding  $\mathbf{X}$ . The tangent space of  $M$  at  $p = \mathbf{X}(u)$  is

$$T_p M = \langle \mathbf{X}_{u_1}(u), \mathbf{X}_{u_2}(u), \dots, \mathbf{X}_{u_{n-1}}(u) \rangle_{\mathbb{R}}.$$

For any  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{n-1} \in \mathbb{R}^n$ , we define

$$\mathbf{a}_1 \times \mathbf{a}_2 \times \dots \times \mathbf{a}_{n-1} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_n \\ a_1^1 & a_2^1 & \dots & a_n^1 \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ \vdots & \vdots & \dots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \dots & a_n^{n-1} \end{vmatrix},$$

where  $\{\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_n\}$  is the canonical basis of  $\mathbb{R}^n$  and  $\mathbf{a}_i = (a_1^i, a_2^i, \dots, a_n^i)$ . It follows that we can define the unit normal vector field

$$\mathbf{n}(u) = \frac{\mathbf{X}_{u_1}(u) \times \dots \times \mathbf{X}_{u_{n-1}}(u)}{\|\mathbf{X}_{u_1}(u) \times \dots \times \mathbf{X}_{u_{n-1}}(u)\|}$$

along  $\mathbf{X} : U \rightarrow \mathbb{R}^n$ . A map  $G : U \rightarrow S_1^n$  defined by  $G(u) = \mathbf{n}(u)$  is called the *Gauss map* of  $M = \mathbf{X}(U)$ . We can easily show that  $D_v \mathbf{n} \in T_p M$  for any  $p = \mathbf{X}(u) \in M$  and  $\mathbf{v} \in T_p M$ . Here  $D_v$  denotes the covariant derivative with respect to the tangent vector  $\mathbf{v}$ . Therefore the derivative of the Gauss map  $dG(u)$  can be interpreted as a linear transformation on the tangent

space  $T_p M$  at  $p = \mathbf{X}(u)$ . We call the linear transformation  $S_p = -dG(u) : T_p M \longrightarrow T_p M$  the *shape operator* (or *Weingarten map*) of  $M = \mathbf{X}(U)$  at  $p = \mathbf{X}(u)$ . We denote the eigenvalue of  $S_p$  by  $\kappa_p$  which we call a *principal curvature*. We call the eigenvector of  $S_p$  the *principal direction*. By definition,  $\kappa_p$  is a principal curvature if and only if  $\det(S_p - \kappa_p I) = 0$ . The *Gauss-Kronecker curvature* of  $M = \mathbf{X}(U)$  at  $p = \mathbf{X}(u)$  is defined to be  $K(u) = \det S_p$ .

We say that a point  $p = \mathbf{X}(u) \in M$  is an *umbilical point* if  $S_p = \kappa_p \text{id}_{T_p M}$ . We also say that  $M$  is *totally umbilic* if all points of  $M$  are umbilic. Then the following proposition is a well-known result:

**Proposition 2.1** *Suppose that  $M = \mathbf{X}(U)$  is totally umbilic, then  $\kappa_p$  is constant  $\kappa$ . Under this condition, we have the following classification:*

- 1) *If  $\kappa \neq 0$ , then  $M$  is a part of a hypersphere.*
- 2) *If  $\kappa = 0$ , then  $M$  is a part of a hyperplane.*

In the extrinsic differential geometry, totally umbilic hypersurfaces are considered to be the model hypersurfaces in Euclidean space. Since the set  $\{\mathbf{X}_{u_i} \mid (i = 1, \dots, n-1)\}$  is linearly independent, we induce the Riemannian metric (*first fundamental form*)  $ds^2 = \sum_{i=1}^{n-1} g_{ij} du_i du_j$  on  $M = \mathbf{X}(U)$ , where  $g_{ij}(u) = \langle \mathbf{X}_{u_i}(u), \mathbf{X}_{u_j}(u) \rangle$  for any  $u \in U$ . We define the *second fundamental invariant* by  $h_{ij}(u) = \langle -\mathbf{n}_{u_i}(u), \mathbf{X}_{u_j}(u) \rangle$  for any  $u \in U$ . We have the following *Weingarten formula*:

$$\mathbf{n}_{u_i}(u) = - \sum_{j=1}^{n-1} h_i^j(u) \mathbf{X}_{u_j}(u),$$

where  $(h_i^j(u)) = (h_{ik}(u))(g^{kj}(u))$  and  $(g^{kj}(u)) = (g_{kj}(u))^{-1}$ . By the Weingarten formula, the Gauss-Kronecker curvature is given by

$$K(u) = \frac{\det(h_{ij}(u))}{\det(g_{\alpha\beta}(u))}.$$

For a hypersurface  $\mathbf{X} : U \longrightarrow \mathbb{R}^n$ , we say that a point  $u \in U$  or  $p = \mathbf{X}(u)$  is a *flat point* if  $h_{ij}(u) = 0$  for all  $i, j$ . Therefore,  $p = \mathbf{X}(u)$  is a flat point if and only if  $p$  is an umbilic point with the vanishing principal curvature. We say that a point  $p = \mathbf{X}(u) \in M$  is a *parabolic point* if  $K(u) = 0$ . For a hypersurface  $\mathbf{X} : U \longrightarrow \mathbb{R}^n$ , we define the *evolute* of  $\mathbf{X}(U) = M$  by

$$\text{Ev}_M = \left\{ \mathbf{X}(u) + \frac{1}{\kappa(u)} \mathbf{n}(u) \mid \kappa(u) \text{ is a principal curvature at } p = \mathbf{X}(u), u \in U \right\}.$$

We define a smooth mapping  $\text{Ev}_\kappa : U \longrightarrow \mathbb{R}^n$  by

$$\text{Ev}_\kappa(u) = \mathbf{X}(u) + \frac{1}{\kappa(u)} \mathbf{e}(u),$$

where we fix a principal curvature  $\kappa(u)$  on  $U$  at  $u$  with  $\kappa(u) \neq 0$ . This map gives a parametrization of a component of  $\text{Ev}_M$ . We also define the *pedal hypersurface* of  $M = \mathbf{X}(U)$  by

$$\text{Pe}_M : U \longrightarrow \mathbb{R}^n ; \text{Pe}_M(u) = \langle \mathbf{X}(u), \mathbf{n}(u) \rangle \mathbf{n}(u).$$

Concerning on the pedal hypersurface in  $\mathbb{R}^n$ , we define the *cylindrical pedal* of  $M = \mathbf{X}(U)$  by

$$\text{CPe}_M : U \longrightarrow S^{n-1} \times \mathbb{R} ; \text{CPe}_M(u) = (\mathbf{n}(u), \langle \mathbf{X}(u), \mathbf{n}(u) \rangle).$$

We have the following well-known result:

**Proposition 2.2** *Let  $M = \mathbf{X}(U)$  be a hypersurface in  $\mathbb{R}^n$ .*

(a) *Suppose that there are no parabolic points or flat points, then the following are equivalent:*

- (1)  *$M$  is totally umbilic with  $\kappa \neq 0$ .*
- (2)  *$\text{Ev}_M$  is a point in  $\mathbb{R}^n$ .*
- (3)  *$M$  is a part of a hypersphere.*

(b) *The following are equivalent:*

- (1)  *$M$  is totally umbilic with  $\kappa = 0$ .*
- (2) *The Gauss map is a constant map.*
- (3)  *$M$  is a part of a hyperplane.*

We define a mapping  $\Psi : S^{n-1} \times (\mathbb{R} \setminus \{0\}) \longrightarrow \mathbb{R}^n \setminus \{\mathbf{0}\}$  by  $\Psi(\mathbf{v}, r) = r\mathbf{v}$ . We can easily show that  $\Psi$  is a double covering and  $\Psi(\text{CPe}_M(u)) = \text{Pe}_M(u)$  under the assumption that  $\langle \mathbf{X}(u), \mathbf{n}(u) \rangle \neq 0$ . If necessary, by applying a Euclidean motion in  $\mathbb{R}^n$ , we have the condition  $\langle \mathbf{X}(u), \mathbf{n}(u) \rangle \neq 0$ . Since we consider the geometric properties which are invariant under Euclidean motion, we might assume the above condition. Therefore the singularities of the pedal and the cylindrical pedal of a hypersurface are diffeomorphic. Although the notion of pedals are classically given, we consider the cylindrical pedal instead of the pedal of  $M = \mathbf{X}(U)$  by the above reason.

### 3 Height functions and distance squared functions

We now define two kinds of functions families in order to describe the Gauss map, the evolute and the pedal hypersurface of a hypersurface in  $\mathbb{R}^n$ .

For the purpose, we need some concepts and results in the theory of unfoldings of function germs. We shall give a brief review of the theory in the appendices.

We now define two families of functions

$$H : U \times S^{n-1} \longrightarrow \mathbb{R}$$

by  $H(u, \mathbf{v}) = \langle \mathbf{X}(u), \mathbf{v} \rangle$  and

$$D : U \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

by  $D(u, \mathbf{x}) = \|\mathbf{X}(u) - \mathbf{x}\|^2$ . We call  $H$  a *height function* and  $D$  *distance squared function* on  $M = \mathbf{X}(U)$ . We denote that  $h_{\mathbf{v}}(u) = H(u, \mathbf{v})$  and  $d_{\mathbf{x}}(u) = D(u, \mathbf{x})$ . These two families of functions are introduced by Thom for the study of parabolic points and umbilical points. Actually, Porteous and Montaldi realized Thom's program[22, 25, 26]. The following proposition follows from direct calculations:

**Proposition 3.1** *Let  $\mathbf{X} : U \longrightarrow \mathbb{R}^n$  be a hypersurface. Then*

- (1)  *$(\partial h_{\mathbf{v}} / \partial u_i)(u) = 0$  ( $i = 1, \dots, n-1$ ) if and only if  $\mathbf{v} = \pm \mathbf{n}(u)$ .*
- (2)  *$(\partial d_{\mathbf{v}} / \partial u_i)(u) = 0$  ( $i = 1, \dots, n-1$ ) if and only if there exist real numbers  $\lambda$  such that  $\mathbf{v} = \mathbf{x}(u) + \lambda \mathbf{n}(u)$ .*

By Proposition 3.1, we can detect both the catastrophe sets (cf., Appendix A) of  $H$  and  $D$  as follows:

$$C(H) = \left\{ (u, \mathbf{v}) \in U \times S^{n-1} \mid \mathbf{v} = \pm \mathbf{n}(u) \right\},$$

$$C(D) = \left\{ (u, \mathbf{x}) \in U \times \mathbb{R}^n \mid \mathbf{x} = \mathbf{X}(u) + \mu \mathbf{n}(u) \right\}.$$

For  $\mathbf{v} = \mathbf{n}(u)$ , We also calculate that

$$\frac{\partial^2 H}{\partial u_i \partial u_j}(u, \mathbf{v}) = \langle \mathbf{X}_{u_i u_j}(u), \mathbf{v} \rangle = \mp h_{ij}(u)$$

on  $C(H)$  and

$$\frac{\partial^2 D}{\partial u_i \partial u_j}(u, \mathbf{x}) = 2(\langle \mathbf{X}_{u_i u_j}(u), \mathbf{X}(u) - \mathbf{x} \rangle + \langle \mathbf{X}_{u_j}(u), \mathbf{X}_{u_j}(u) \rangle) = 2(-\lambda h_{ij}(u) + g_{ij}(u))$$

on  $C(D)$ .

Therefore, for any  $\mathbf{v} = \mathbf{n}(u)$ ,  $\det(\mathcal{H}(h_v)(u)) = \det(\partial^2 H / \partial u_i \partial u_j)(u, \mathbf{v}) = 0$  if and only if  $K(p) = 0$  (i.e.,  $p = \mathbf{X}(u)$  is a parabolic point). Moreover, for any  $\mathbf{x} = \mathbf{X}(u) + \lambda \mathbf{n}(u)$ ,  $\det(\mathcal{H}(d_x)(u)) = \det(\partial^2 D / \partial u_i \partial u_j)(u, \mathbf{x}) = 0$  if and only if  $\kappa(u) = \frac{1}{\lambda}$  is a principal curvature. By the above calculation, we have the following well-known results:

**Proposition 3.2** *For any  $p = \mathbf{X}(u)$ , we have the following assertions:*

*Suppose that  $\mathbf{v} = \mathbf{n}(u)$ , then*

- (a)  *$p$  is a parabolic point if and only if  $\det(\mathcal{H}(h_v)(u)) = 0$ .*
- (b)  *$p$  is a flat point if and only if  $\text{rank } \mathcal{H}(h_v)(u) = 0$ .*

*Suppose that  $p$  is not a flat point and  $\mathbf{x} = \mathbf{X}(u) + (1/\kappa(u))\mathbf{n}(u)$  for a non-zero principal curvature  $\kappa(u)$ . Then*

- (c)  *$p$  is an umbilical point if and only if  $\text{rank } \mathcal{H}(d_x)(u) = 0$ .*

We say that  $u$  is a *ridge point* if  $h_v$  has the  $A_{k \geq 3}$ -type singular point at  $u$ , where  $\mathbf{v} \in \text{Ev}_M(U)$ . For a function germ  $f : (\mathbb{R}^{n-1}, \mathbf{x}_0) \rightarrow \mathbb{R}$ ,  $f$  has  $A_k$ -type singular point at  $\mathbf{x}_0$  if  $f$  is  $\mathcal{R}^+$ -equivalent to the germ  $x_1^{k+1} \pm x_2^2 \pm \cdots \pm x_{n-1}^2$ . We say that two function germs  $f_i : (\mathbb{R}^{n-1}, \mathbf{x}_i) \rightarrow \mathbb{R}$  ( $i = 1, 2$ ) are  $\mathcal{R}^+$ -equivalent if there exists a diffeomorphism germ  $\Phi : (\mathbb{R}^{n-1}, \mathbf{x}_1) \rightarrow (\mathbb{R}^{n-1}, \mathbf{x}_2)$  and a real number  $c$  such that  $f_2 \circ \Phi(\mathbf{x}) = f_1(\mathbf{x}) + c$ . The notion of ridge points was introduced by Porteous[25] as an application of the singularity theory of unfoldings to the evolute and the geometric meaning of ridge points is given as follows: Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  be a function and  $\mathbf{X} : U \rightarrow \mathbb{R}^n$  a hypersurface. We say that  $\mathbf{X}$  and  $F^{-1}(0)$  have a *corank  $r$  contact* at  $p = \mathbf{X}(u)$  if the Hessian of the function  $g(u) = F \circ \mathbf{X}(u)$  has corank  $r$  at  $u$ . We also say that  $\mathbf{X}$  and  $F^{-1}(0)$  have an  *$A_k$ -type contact* at  $p = \mathbf{X}(u)$  if the function  $g(u) = F \circ \mathbf{X}(u)$  has the  $A_k$ -type singularity at  $u$ . By definition, if  $\mathbf{X}$  and  $F^{-1}(0)$  have an  *$A_k$ -type contact* at  $p = \mathbf{X}(u)$ , then these have a corank 1 contact. For any  $r \in \mathbb{R}$  and  $\mathbf{a}_0 \in \mathbb{R}^n$ , we consider a function  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by  $F(\mathbf{x}) = \|\mathbf{x} - \mathbf{a}_0\|^2 - r^2$ . We denote that

$$S^{n-1}(\mathbf{a}, r) = F^{-1}(0) = \{\mathbf{u} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{a}\|^2 = r^2\}.$$

It follows that  $S^{n-1}(\mathbf{a}, r)$  is a hypersphere with the center  $\mathbf{a}$  and the radius  $|r|$ . We put  $\mathbf{a} = \text{Ev}_\kappa(u)$  and  $r = 1/\kappa(u)$ , where we fix a principal curvature  $\kappa(u)$  on  $U$  at  $u$ , then we have the following simple proposition:

**Proposition 3.3** *Under the above notations, there exists an integer  $\ell$  with  $1 \leq \ell \leq n - 1$  such that  $M = \mathbf{X}(U)$  and  $S^{n-1}(\mathbf{a}, r)$  have corank  $\ell$  contact at  $u$ .*

In the above proposition,  $S^{n-1}(\mathbf{a}, r)$  is called an *osculating hypersphere* of  $M = \mathbf{X}(U)$ . We also call  $\mathbf{a}$  the *center of the principal curvature*  $\kappa(u)$ . By Proposition 3.2,  $M = \mathbf{X}(U)$  and the osculating hypersphere has corank  $n - 1$  contact at an umbilic point. Therefore the ridge point is not an umbilic point.

By the general theory of unfoldings of function germs, the bifurcation set  $B_F$  is non-singular at the origin if and only if the function  $f = F|\mathbb{R}^n \times \{0\}$  has the  $A_2$ -type singularity (i.e., the fold type singularity). Therefore we have the following proposition:

**Proposition 3.4** *Under the same notations as in the previous proposition, the evolute  $\text{Ev}_M$  is non-singular at  $\mathbf{a} = \text{Ev}_\kappa(u)$  if and only if  $M = \mathbf{X}(U)$  and  $S^{n-1}(\mathbf{a}, r)$  have  $A_2$ -type contact at  $u$ .*

All results mentioned in the above paragraphs on the evolute have been shown by Porteous and Montaldi[22, 25].

We also define a family of functions  $\tilde{H} : U \times (S^{n-1} \times \mathbb{R}) \longrightarrow \mathbb{R}$  by

$$\tilde{H}(u, \mathbf{v}, r) = \langle \mathbf{X}(u), \mathbf{v} \rangle - r.$$

We call it the *extended height function* of  $M = \mathbf{X}(U)$ . By the previous calculations, we have

$$D_{\tilde{H}} = \{ \pm \text{CPe}_M(u) \mid u \in U \} \quad \text{and} \quad B_D = \text{Ev}_M.$$

Moreover, the catastrophe map of  $H$  is  $\pi_{C(H)}(u, \pm \mathbf{n}(u)) = \pm \mathbf{n}(u) = \pm G(u)$ . Therefore, we can identify the Gauss map of  $M = \mathbf{X}(U)$  with plus components of the catastrophe map  $\pi_{C(H)}$ .

## 4 Evolutes and Cylindrical pedals as Caustics and Wavefronts

In this section we naturally interpret the evolute (respectively, the cylindrical pedal) of a hypersurface as a caustics (respectively, a wave front) in the framework of symplectic (respectively, contact) geometry and consider the geometric meaning of those singularities. In Appendix A (respectively, Appendix B) we give a brief survey of the theory of Lagrangian (respectively, Legendrian) singularities. For notions and basic results on the theory of Lagrangian or Legendrian singularities, please refer to these appendices.

For a hypersurface  $\mathbf{X} : U \longrightarrow \mathbb{R}^n$ , we consider the distance squared function  $D$  and the height function  $H$ . We have the following propositions:

**Proposition 4.1** *Both of the distance squared function  $D : U \times \mathbb{R}^n \longrightarrow \mathbb{R}$  and the height function  $H : U \times S^{n-1} \longrightarrow \mathbb{R}$  of  $M = \mathbf{X}(U)$  are Morse families of functions.*

*Proof.* First we consider the distance squared function.

For any  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ , we have  $D(u, \mathbf{x}) = \sum_{i=1}^n (x_i(u) - x_i)^2$ , where  $\mathbf{X}(u) = (x_1(u), \dots, x_n(u))$ . We will prove that the mapping

$$\Delta D = \left( \frac{\partial D}{\partial u_1}, \dots, \frac{\partial D}{\partial u_{n-1}} \right)$$

is non-singular at any point. The Jacobian matrix of  $\Delta D$  is given as follows:

$$\begin{pmatrix} A_{11} & \cdots & A_{1(n-1)} & -2x_{1u_1}(u) & \cdots & -2x_{nu_1}(u) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{(n-1)1} & \cdots & A_{(n-1)(n-1)} & -2x_{1u_{n-1}}(u) & \cdots & -2x_{nu_{n-1}}(u) \end{pmatrix},$$

where  $A_{ij} = 2(\langle \mathbf{X}_{u_i u_j}(u), \mathbf{X}(u) - \mathbf{x} \rangle + \langle \mathbf{X}_{u_i}(u), \mathbf{X}_{u_j}(u) \rangle)$ . Since  $\mathbf{X} : U \rightarrow \mathbb{R}^n$  is an embedding, the rank of the matrix

$$X = \begin{pmatrix} 2x_{1u_1}(u) & \cdots & -2x_{nu_1}(u) \\ \vdots & \vdots & \vdots \\ 2x_{1u_{n-1}}(u) & \cdots & -2x_{nu_{n-1}}(u) \end{pmatrix}$$

is  $n - 1$  at any  $u \in U$ .

Therefore the rank of the Jacobian matrix of  $\Delta D$  is  $n - 1$ .

Next we consider the height function. The proof is also given by direct calculations but a bit more carefully than in the previous case. For any  $\mathbf{v} \in S^{n-1}$ , we have  $v_1^2 + \cdots + v_n^2 = 1$ . Without loss of the generality, we might assume that  $v_n > 0$ . We have  $v_n = \sqrt{1 - v_1^2 - \cdots - v_{n-1}^2}$ , so that

$$H(u, \mathbf{v}) = x_1(u)v_1 + \cdots + x_{n-1}(u)v_{n-1} + x_n(u)\sqrt{1 - v_1^2 - \cdots - v_{n-1}^2}.$$

We also prove that the mapping

$$\Delta H = \left( \frac{\partial H}{\partial u_1}, \dots, \frac{\partial H}{\partial u_{n-1}} \right)$$

is non-singular at any point. The Jacobian matrix of  $\Delta H$  is given as follows:

$$\begin{pmatrix} \langle \mathbf{X}_{u_1 u_1}, \mathbf{v} \rangle & \cdots & \langle \mathbf{X}_{u_1 u_{n-1}}, \mathbf{v} \rangle & x_{1u_1}(u) - x_{nu_1} \frac{v_1}{v_n} & \cdots & x_{n-1u_1} - x_{nu_1} \frac{v_{n-1}}{v_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle \mathbf{X}_{u_{n-1} u_1}, \mathbf{v} \rangle & \cdots & \langle \mathbf{X}_{u_{n-1} u_{n-1}}, \mathbf{v} \rangle & x_{1u_{n-1}} - x_{nu_{n-1}} \frac{v_1}{v_n} & \cdots & x_{n-1u_{n-1}} - x_{nu_{n-1}} \frac{v_{n-1}}{v_n} \end{pmatrix}.$$

We will show that the rank of the matrix

$$\tilde{X} = \begin{pmatrix} x_{1u_1} - x_{nu_1} \frac{v_1}{v_n} & \cdots & x_{n-1u_1} - x_{nu_1} \frac{v_{n-1}}{v_n} \\ \vdots & \vdots & \vdots \\ x_{1u_{n-1}} - x_{nu_{n-1}} \frac{v_1}{v_n} & \cdots & x_{n-1u_{n-1}} - x_{nu_{n-1}} \frac{v_{n-1}}{v_n} \end{pmatrix}$$

is  $n - 1$  at  $(u, \mathbf{v}) \in C(H)$ . We denote that  $\mathbf{a}_i = \begin{pmatrix} x_{iu_1} \\ \vdots \\ x_{iu_{n-1}} \end{pmatrix}$  for  $i = 0, \dots, n$ .

It should be proven that the rank of the matrix

$$\tilde{A} = \left( \mathbf{a}_1 - \mathbf{a}_n \frac{v_1}{v_n}, \dots, \mathbf{a}_{n-1} - \mathbf{a}_n \frac{v_{n-1}}{v_n} \right)$$



is  $n - 1$  at  $(u, \mathbf{v}) \in C(H)$ .

Therefore we have

$$\begin{aligned}
\det \tilde{A} &= (-1)^{n+1} \frac{v_1}{v_n} \det(\mathbf{a}_2, \dots, \mathbf{a}_n) \\
&\quad + \dots + (-1)^{2n} \frac{v_n}{v_n} \det(\mathbf{a}_1, \dots, \mathbf{a}_{n-1}) \\
&= (-1)^{n-1} \left\langle \left( \frac{v_1}{v_n}, \dots, \frac{v_n}{v_n} \right), \mathbf{X}_{u_1} \times \dots \times \mathbf{X}_{u_{n-1}} \right\rangle \\
&= \frac{(-1)^{n-1}}{v_n} \langle \pm \mathbf{n}, \mathbf{X}_{u_1} \times \dots \times \mathbf{X}_{u_{n-1}} \rangle \\
&= \pm \frac{(-1)^{n-1}}{v_n} \|\mathbf{X}_{u_1} \times \dots \times \mathbf{X}_{u_{n-1}}\| \neq 0
\end{aligned}$$

for  $(u, \mathbf{v}) = (u, \pm \mathbf{n}(u)) \in C(H)$ . This completes the proof of the proposition.  $\square$

By the method for constructing the Lagrangian immersion germ from Morse family of functions (cf., Appendix A), we can define a Lagrangian immersion germ whose generating family is the distance squared function or the height function of  $M = \mathbf{X}(U)$  as follows: For a hypersurface  $\mathbf{X} : U \rightarrow \mathbb{R}^n$  with  $\mathbf{X}(u) = (x_1(u), \dots, x_n(u))$ , We define a smooth mapping

$$L(D) : C(D) \rightarrow T^*\mathbb{R}^n$$

by

$$L(D)(u, \mathbf{x}) = (\mathbf{x}, -2(x_1(u) - x_1), \dots, -2(x_n(u) - x_n)),$$

where  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ . Here we have used the triviality of the cotangent bundle  $T^*\mathbb{R}^n$ . For the  $(n - 1)$ -sphere  $S^{n-1}$ , we consider the local coordinate  $U_i = \{\mathbf{v} = (v_1, \dots, v_n) \in S^{n-1} \mid v_i \neq 0\}$ . Since  $T^*S^{n-1}|_{U_i}$  is a trivial bundle, we define a map

$$L_i(H) : C(H) \rightarrow T^*S^{n-1}|_{U_i} \quad (i = 0, 1, \dots, n)$$

by

$$L_i(H)(u, \mathbf{v}) = (\mathbf{v}, x_1(u) - x_i(u) \frac{v_1}{v_i}, \dots, \widehat{x_i(u) - x_i(u) \frac{v_i}{v_i}}, \dots, x_n(u) - x_i(u) \frac{v_n}{v_i}),$$

where  $\mathbf{v} = (v_1, \dots, v_n) \in S^{n-1}$  and we denote  $(x_1, \dots, \hat{x}_i, \dots, x_n)$  as a point in the  $(n - 1)$ -dimensional space such that the  $i$ -th component  $x_i$  is removed. We can show that if  $U_i \cap U_j \neq \emptyset$  for  $i \neq j$ , then  $L_i(H)$  and  $L_j(H)$  are Lagrangian equivalent which are given by the local coordinate transformation of  $S^{n-1}$  and Lagrangian lift of it. Indeed, we denote that the local coordinate change of  $S^{n-1}$  for  $i < j$ ;  $\varphi_{ij} : U_i \rightarrow U_j$ , defined by

$$\varphi_{ij}(v_1, \dots, \hat{v}_i, \dots, v_n) = (v_1, \dots, v_i = \sqrt{1 - v_1^2 - \dots - \hat{v}_i^2 - \dots - v_n^2}, \dots, \hat{v}_j, \dots, v_n),$$

and  $\tilde{\varphi}_{ij} : T^*S^{n-1} \rightarrow T^*S^{n-1}$  are Lagrangian lift of  $\varphi_{ij}$  which defined by  $\tilde{\varphi}_{ij}(\xi) = (\varphi_{ij}^{-1})^* \xi$ . Then  $\tilde{\varphi}_{ij}$  are symplectic diffeomorphism germs (c.f [1]). Also we define diffeomorphism germs  $\hat{\sigma}_{ij} : U \times U_i \rightarrow U \times U_j$  by  $\hat{\sigma}_{ij}(u, \mathbf{v}) = (u, \varphi_{ij}(\mathbf{v}))$  and  $\sigma_{ij} = \hat{\sigma}_{ij}|_{C(H)}$ , then  $\tilde{\varphi}_{ij} \circ L_i(H) = L_j(H) \circ \sigma_{ij}$  and  $\varphi_{ij} \circ \pi = \pi \circ \tilde{\varphi}_{ij}$ . Therefore we can define a global Lagrangian immersion,  $L(H) : C(H) \rightarrow T^*S^{n-1}$ .

By definition, we have the following corollary of the above proposition:

**Corollary 4.2** *Under the above notations,  $L(D)$  (respectively,  $L(H)$ ) is a Lagrangian immersion such that the distance squared function  $D : U \times \mathbb{R}^n \rightarrow \mathbb{R}$  (respectively, height function  $H : U \times S^{n-1} \rightarrow \mathbb{R}$ ) of  $M = \mathbf{X}(U)$  is a generating family of  $L(D)$  (respectively,  $L(H)$ ).*

Therefore, we have the Lagrangian immersion  $L(D)$  whose caustics is the evolute of  $M = \mathbf{X}(U)$ . We call  $L(D)$  the *Lagrangian lift* of the evolute  $\text{Ev}_M$  of  $M = \mathbf{X}(U)$ . Moreover, the plus component of the Lagrangian map  $\pi \circ L(H)$  can be identified with the Gauss map of  $M = \mathbf{X}(U)$ . We also call  $L(H)$  the *Lagrangian lift* of the Gauss map  $G : U \rightarrow S^{n-1}$  of  $M = \mathbf{X}(U)$ .

On the other hand, we consider the extended height function  $\tilde{H} : U \times (S^{n-1} \times \mathbb{R}) \rightarrow \mathbb{R}$  of  $M = \mathbf{X}(U)$ . We have the following proposition.

**Proposition 4.3** *The extended height function  $\tilde{H} : U \times (S^{n-1} \times \mathbb{R}) \rightarrow \mathbb{R}$  on  $M = \mathbf{X}(U)$  is a Morse family of hypersurfaces.*

*Proof.* The proof is given by almost the similar calculation as the case for the height function. For any  $\mathbf{v} \in S^{n-1}$ , we have  $v_1^2 + \dots + v_n^2 = 1$ . Without loss of the generality, we also assume that  $v_n > 0$ . We have  $v_n = \sqrt{1 - v_1^2 - \dots - v_{n-1}^2}$ , so that

$$\tilde{H}(u, \mathbf{v}, r) = x_1(u)v_1 + \dots + x_{n-1}(u)v_{n-1} + x_n(u)\sqrt{1 - v_1^2 - \dots - v_{n-1}^2} - r.$$

We also prove that the mapping

$$\Delta^* \tilde{H} = \left( \tilde{H}, \frac{\partial \tilde{H}}{\partial u_1}, \dots, \frac{\partial \tilde{H}}{\partial u_{n-1}} \right)$$

is non-singular at any point in  $\Sigma_*(\tilde{H}) = \Delta^* \tilde{H}^{-1}(0)$ . The Jacobian matrix of  $\Delta^* \tilde{H}$  is given as follows:

$$\begin{pmatrix} \langle \mathbf{X}_{u_1}, \mathbf{v} \rangle & \cdots & \langle \mathbf{X}_{u_n}, \mathbf{v} \rangle & x_1 - x_n \frac{v_1}{v_n} & \cdots & x_{n-1} - x_n \frac{v_{n-1}}{v_n} & 1 \\ \langle \mathbf{X}_{u_1 u_1}, \mathbf{v} \rangle & \cdots & \langle \mathbf{X}_{u_1 u_{n-1}}, \mathbf{v} \rangle & x_{1u_1} - x_{nu_1} \frac{v_1}{v_n} & \cdots & x_{n-1u_1} - x_{nu_1} \frac{v_{n-1}}{v_n} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle \mathbf{X}_{u_{n-1} u_1}, \mathbf{v} \rangle & \cdots & \langle \mathbf{X}_{u_{n-1} u_{n-1}}, \mathbf{v} \rangle & x_{1u_{n-1}} - x_{nu_{n-1}} \frac{v_1}{v_n} & \cdots & x_{n-1u_{n-1}} - x_{nu_{n-1}} \frac{v_{n-1}}{v_n} & 0 \end{pmatrix}.$$

it is enough to show that the rank of the matrix

$$\tilde{X} = \begin{pmatrix} x_{1u_1} - x_{nu_1} \frac{v_1}{v_n} & \cdots & x_{n-1u_1} - x_{nu_1} \frac{v_{n-1}}{v_n} \\ \vdots & \vdots & \vdots \\ x_{1u_{n-1}} - x_{nu_{n-1}} \frac{v_1}{v_n} & \cdots & x_{n-1u_{n-1}} - x_{nu_{n-1}} \frac{v_{n-1}}{v_n} \end{pmatrix}$$

is  $n - 1$  at  $(u, \mathbf{v}, r) \in \Sigma_*(\tilde{H})$ . It has been done in the proof of Proposition 4.1. This completes the proof of the proposition.  $\square$

We can also define a Legendrian immersion germ whose generating family is the extended height function of  $M = \mathbf{X}(U)$  as follows (cf., Appendix B): For the  $(n - 1)$ -sphere  $S^{n-1}$ , we

consider the local coordinate  $U_i = \{\mathbf{v} = (v_1, \dots, v_n) \in S^{n-1} \mid v_i \neq 0\}$ . Since  $PT^*(S^{n-1} \times \mathbb{R})|(U_i \times \mathbb{R})$  is a trivial bundle, we define a map

$$\mathcal{L}_i(\tilde{H}) : \Sigma_*(\tilde{H})|U \times (U_i \times \mathbb{R}) \longrightarrow PT^*(S^{n-1} \times \mathbb{R})|(U_i \times \mathbb{R}) \quad (i = 0, 1, \dots, n)$$

by

$$\mathcal{L}_i(\tilde{H})(u, \mathbf{v}, r) = \left( \mathbf{v}, r, [x_1(u) - x_i(u) \frac{v_1}{v_i} : \dots : x_i(u) - x_i(u) \frac{v_i}{v_i} : \dots : x_n(u) - x_i(u) \frac{v_n}{v_i} : -1] \right),$$

where  $v = (v_1, \dots, v_n) \in S^{n-1}$  and we denote  $(x_1, \dots, \hat{x}_i, \dots, x_n)$  as a point in the  $(n-1)$ -dimensional space such that the  $i$ -th component  $x_i$  is removed. We can also show that if  $U_i \cap U_j \neq \emptyset$  for  $i \neq j$ , then  $\mathcal{L}_i(\tilde{H})$  and  $\mathcal{L}_j(\tilde{H})$  are Legendrian equivalent which are given by the local coordinate transformation of  $S^{n-1} \times \mathbb{R}$  and Legendrian lift of it by exactly the same method as the case for Lagrangian equivalence. Therefore we can define a global Legendrian immersion,  $\mathcal{L}(\tilde{H}) : \Sigma_*(\tilde{H}) \longrightarrow PT^*(S^{n-1} \times \mathbb{R})$ .

By definition, we have the following corollary of the above proposition:

**Corollary 4.4** *Under the above notations,  $\mathcal{L}(\tilde{H})$  is a Legendrian immersion such that the extended height function  $\tilde{H} : U \times (S^{n-1} \times \mathbb{R}) \longrightarrow \mathbb{R}$  of  $M = \mathbf{X}(U)$  is a generating family of  $\mathcal{L}(\tilde{H})$ .*

Therefore, we have the Legendrian immersion  $\mathcal{L}(\tilde{H})$  whose wave front is the cylindrical pedal of  $M = \mathbf{X}(U)$ . We call  $\mathcal{L}(\tilde{H})$  the *Legendrian lift* of the cylindrical pedal  $\text{CPe}_M$  of  $M = \mathbf{X}(U)$ .

## 5 Contact with model hypersurfaces and families of model hypersurfaces

In [23, 24] Montaldi studied the contact of surfaces with hyperplanes or hyperspheres in  $\mathbb{R}^n$  ( $n = 3, 4$ ). For the purpose, he has developed a general theory of contact between submanifolds. Let  $X_i, Y_i$  ( $i = 1, 2$ ) be submanifolds of  $\mathbb{R}^n$  with  $\dim X_1 = \dim X_2$  and  $\dim Y_1 = \dim Y_2$ . We say that the *contact* of  $X_1$  and  $Y_1$  at  $y_1$  is of the *same type* as the *contact* of  $X_2$  and  $Y_2$  at  $y_2$  if there is a diffeomorphism germ  $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$  such that  $\Phi(X_1) = X_2$  and  $\Phi(Y_1) = Y_2$ . In this case we write  $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$ . It is clear that in the definition  $\mathbb{R}^n$  could be replaced by any manifold. In his paper[23], Montaldi gives a characterization of the notion of contact by using the terminology of Singularity theory.

**Theorem 5.1** *Let  $X_i, Y_i$  ( $i = 1, 2$ ) be submanifolds of  $\mathbb{R}^n$  with  $\dim X_1 = \dim X_2$  and  $\dim Y_1 = \dim Y_2$ . Let  $g_i : (X_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$  be immersion germs and  $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}^p, 0)$  be submersion germs with  $(Y_i, y_i) = (f_i^{-1}(0), y_i)$ . Then  $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$  if and only if  $f_1 \circ g_1$  and  $f_2 \circ g_2$  are  $\mathcal{K}$ -equivalent. For the definition of the  $\mathcal{K}$ -equivalence and the basic properties, see Appendix B or [20].*

On the other hand, we now briefly describe the theory of contact with foliations. Here we consider the relationship between the contact of submanifolds with foliations and the  $\mathcal{R}^+$ -class of functions. Let  $X_i$  ( $i = 1, 2$ ) be submanifolds of  $\mathbb{R}^n$  with  $\dim X_1 = \dim X_2$ ,  $g_i : (X_i, \bar{x}_i) \longrightarrow (\mathbb{R}^n, \bar{y}_i)$  be immersion germs and  $f_i : (\mathbb{R}^n, \bar{y}_i) \longrightarrow (\mathbb{R}, 0)$  be submersion germs. For

a submersion germ  $f : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ , we denote that  $\mathcal{F}_f$  be the regular foliation defined by  $f$ ; i.e.,  $\mathcal{F}_f = \{f^{-1}(c) | c \in (\mathbb{R}, 0)\}$ . We say that *the contact of  $X_1$  with the regular foliation  $\mathcal{F}_{f_1}$  at  $\bar{y}_1$  is of the same type as the contact of  $X_2$  with the regular foliation  $\mathcal{F}_{f_2}$  at  $\bar{y}_2$*  if there is a diffeomorphism germ  $\Phi : (\mathbb{R}^n, \bar{y}_1) \longrightarrow (\mathbb{R}^n, \bar{y}_2)$  such that  $\Phi(X_1) = X_2$  and  $\Phi(Y_1(c)) = Y_2(c)$ , where  $Y_i(c) = f_i^{-1}(c)$  for each  $c \in (\mathbb{R}, 0)$ . In this case we write  $K(X_1, \mathcal{F}_{f_1}; \bar{y}_1) = K(X_2, \mathcal{F}_{f_2}; \bar{y}_2)$ . It is also clear that in the definition  $\mathbb{R}^n$  could be replaced by any manifold. We apply the method of Goryunov[6] to the case for  $\mathcal{R}^+$ -equivalences among function germs, so that we have the following:

**Proposition 5.2** ([6, Appendix]) *Let  $X_i$  ( $i = 1, 2$ ) be submanifolds of  $\mathbb{R}^n$  with  $\dim X_1 = \dim X_2 = n - 1$  (i.e. hypersurface),  $g_i : (X_i, \bar{x}_i) \longrightarrow (\mathbb{R}^n, \bar{y}_i)$  be immersion germs and  $f_i : (\mathbb{R}^n, \bar{y}_i) \longrightarrow (\mathbb{R}, 0)$  be submersion germs. Then  $K(X_1, \mathcal{F}_{f_1}; \bar{y}_1) = K(X_2, \mathcal{F}_{f_2}; \bar{y}_2)$  if and only if  $f_1 \circ g_1$  and  $f_2 \circ g_2$  are  $\mathcal{R}^+$ -equivalent.*

Golubitsky and Guillemin[7] have given an algebraic characterization for the  $\mathcal{R}^+$ -equivalence among function germs. We denote  $C_0^\infty(X)$  is the set of function germs  $(X, 0) \longrightarrow \mathbb{R}$ . Let  $J_f$  be the Jacobian ideal in  $C_0^\infty(X)$  (i.e.,  $J_f = \langle \partial f / \partial x_1, \dots, \partial f / \partial x_n \rangle_{C_0^\infty(X)}$ ). Let  $\mathcal{R}_k(f) = C_0^\infty(X) / J_f^k$  and  $\bar{f}$  be the image of  $f$  in this local ring. We say that  $f$  satisfies the *Milnor Condition* if  $\dim_{\mathbb{R}} \mathcal{R}_1(f) < \infty$ .

**Proposition 5.3** ([7, Proposition 4.1]) *Let  $f$  and  $g$  be germs of functions at 0 in  $X$  satisfying the Milnor condition with  $df(0) = dg(0) = 0$ . Then  $f$  and  $g$  are  $\mathcal{R}^+$ -equivalent if*

- (1) *The rank and signature of the Hessians  $\mathcal{H}(f)(0)$  and  $\mathcal{H}(g)(0)$  are equal, and*
- (2) *There is an isomorphism  $\gamma : \mathcal{R}_2(f) \longrightarrow \mathcal{R}_2(g)$  such that  $\gamma(\bar{f}) = \bar{g}$ .*

On the other hand, we define the following functions:

$$\begin{aligned} \mathcal{H} & : \mathbb{R}^n \times S^{n-1} \longrightarrow \mathbb{R} ; \mathcal{H}(\mathbf{x}, \mathbf{v}) = \langle \mathbf{x}, \mathbf{v} \rangle, \\ \tilde{\mathcal{H}} & : \mathbb{R}^n \times (S^{n-1} \times \mathbb{R}) \longrightarrow \mathbb{R} ; \mathcal{H}(\mathbf{x}, \mathbf{v}, r) = \langle \mathbf{x}, \mathbf{v} \rangle - r, \\ \mathcal{D} & : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R} ; \mathcal{D}(\mathbf{y}, \mathbf{x}) = \|\mathbf{y} - \mathbf{x}\|^2. \end{aligned}$$

We now consider the contact of hypersurfaces with hyperplane. For any  $\mathbf{v} \in S^{n-1}$  we denote that  $\mathfrak{h}_v(\mathbf{x}) = \mathcal{H}(\mathbf{x}, \mathbf{v})$  and we have a hyperplane  $\mathfrak{h}_v^{-1}(r)$ . We denote it as  $H(\mathbf{v}, r)$ . For any  $u \in U$ , we consider the unit normal vector  $\mathbf{v} = \mathbf{n}(u)$  and  $r = \langle \mathbf{X}(u), \mathbf{n}(u) \rangle$ , then we have

$$\mathfrak{h}_v \circ \mathbf{X}(u) = \mathcal{H} \circ (\mathbf{X} \times id_{S^{n-1}})(u, \mathbf{v}) = H(u, \mathbf{n}(u)) = r.$$

We also have relations that

$$\frac{\partial \mathfrak{h}_v \circ \mathbf{X}}{\partial u_i}(u) = \frac{\partial H}{\partial u_i}(u, \mathbf{n}(u)) = 0$$

for  $i = 1, \dots, n-1$ . This means that the hyperplane  $\mathfrak{h}_v^{-1}(r) = H(\mathbf{v}, r)$  is tangent to  $M = \mathbf{X}(U)$  at  $p = \mathbf{X}(u)$ . Therefore,  $H(\mathbf{v}, r)$  is the *tangent hyperplane* of  $M = \mathbf{X}(U)$  at  $p = \mathbf{X}(u)$  (or,  $u$ ), which we write  $H(\mathbf{X}(U), u)$ . Let  $\mathbf{v}_1, \mathbf{v}_2$  be unit vectors. If  $\mathbf{v}_1, \mathbf{v}_2$  are linearly dependent, then corresponding hyperplanes  $H(\mathbf{v}_1, r_1), H(\mathbf{v}_2, r_2)$  are parallel. Then we have the following simple lemma.

**Lemma 5.4** *Let  $\mathbf{X} : U \longrightarrow \mathbb{R}^n$  be a hypersurface. Consider two points  $u_1, u_2 \in U$ . Then*

- (1)  $\text{CPe}_M(u_1) = \text{CPe}_M(u_2)$  *if and only if*  $H(\mathbf{X}(U), u_1) = H(\mathbf{X}(U), u_2)$ .
- (2)  $G(u_1) = G(u_2)$  *if and only if*  $H(\mathbf{X}, u_1), H(\mathbf{X}, u_2)$  *are parallel.*

We also consider the family of hyperplanes which contains the tangent hyperplane of  $M = \mathbf{X}(U)$ . Since  $\mathfrak{h}_v$  is a submersion, we have a regular foliation  $\mathcal{F}_{\mathfrak{h}_v} = \{H(\mathbf{v}, c) \mid c \in (\mathbb{R}, r)\}$  whose leaves are hyperplanes such that the case  $c = r$  corresponds to the tangent hyperplane  $H(\mathbf{X}(U), u)$ . It follows that we have a singular foliation germ  $(\mathbf{X}^{-1}(\mathcal{F}_{\mathfrak{h}_v}), u)$  which we call the *Dupin foliation germ* of  $M = \mathbf{X}(U)$  at  $u$ . We denote it by  $\mathcal{DF}(\mathbf{X}(U), u)$ . We remark that the Dupin foliation germ is diffeomorphic to the germ of the Dupin indicatrices family in the classical sense at a non-parabolic point ([28], page 136).

We consider the function  $\mathcal{D} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ . For any  $\mathbf{x} \in \mathbb{R}^n \setminus M$ , we denote that  $\mathfrak{d}_x(\mathbf{y}) = \mathcal{D}(\mathbf{y}, \mathbf{x})$  and we have a hypersphere  $\mathfrak{d}_x^{-1}(r^2) = S^{n-1}(\mathbf{x}, r)$ . It is easy to show that  $\mathfrak{d}_x$  is a submersion. For any  $u \in U$ , we consider a point  $\mathbf{x} = \mathbf{X}(u) + r\mathbf{n}(u) \in \mathbb{R}^n \setminus M$ , then we have

$$\mathfrak{d}_x \circ \mathbf{X}(u) = \mathcal{D} \circ (\mathbf{X} \times id_{\mathbb{R}^n})(u, \mathbf{x}) = r,$$

and

$$\frac{\partial \mathfrak{d}_x \circ \mathbf{X}}{\partial u_i}(u) = \frac{\partial \mathcal{D}}{\partial u_i}(u, \mathbf{x}) = 0.$$

for  $i = 1, \dots, n-1$ . This means that the hypersphere  $\mathfrak{d}_x^{-1}(r) = S^{n-1}(\mathbf{x}, r)$  is tangent to  $M = \mathbf{X}(U)$  at  $p = \mathbf{X}(u)$ . In this case, we call  $S^{n-1}(\mathbf{x}, r)$  a *tangent hypersphere* at  $p = \mathbf{X}(u)$  with the center  $\mathbf{x}$ . However, there are infinitely many tangent hyperspheres at a general point  $p = \mathbf{X}(u)$  depending on the real number  $r$ . If  $\mathbf{x}$  is a point of the hyperbolic evolute, the tangent hypersphere with the center  $\mathbf{x}$  is called the *osculating hypersphere* at  $p = \mathbf{X}(u)$  which is uniquely determined. For  $\mathbf{x} = \mathbf{X}(u) + r\mathbf{n}(u)$ , we also have a regular foliation

$$\mathcal{F}_{\mathfrak{d}_x} = \left\{ S^{n-1}(\mathbf{x}, c) \mid c \in (\mathbb{R}, r) \right\}$$

whose leaves are hyperspheres with the center  $\mathbf{x}$  such that the case  $c = r$  corresponding to the tangent hypersphere with radius  $|r|$ . Moreover, if  $r = 1/\kappa(u)$ , then  $S^{n-1}(\mathbf{x}, 1/\kappa(u))$  is the osculating hypersphere. In this case  $(\mathbf{X}^{-1}(\mathcal{F}_{\mathfrak{d}_x}), u)$  is a singular foliation germ at  $u$  which is called a *osculating hyperspherical foliation* of  $M = \mathbf{X}(U)$  at  $p = \mathbf{X}(u)$  (or,  $u$ ). We denote it by  $\mathcal{OF}(\mathbf{X}(U), u)$ .

## 6 The theory of contact from the view point of Lagrangian or Legendrian singularity theory

In this section we apply Lagrangian or Legendrian singularity theory to the study of contact of hypersurfaces with hyperplanes or hyperspheres.

First we consider the contact of hypersurfaces with hyperplanes. Let  $\text{CPe}_{M_i} : (U, u_i) \longrightarrow (S^{n-1} \times \mathbb{R}, (\mathbf{v}_i, r_i))$  ( $i = 1, 2$ ) be two cylindrical pedal germs of hypersurface germs  $\mathbf{X}_i : (U, u_i) \longrightarrow (\mathbb{R}^n, \mathbf{X}_i(u_i))$  and  $M_i = \mathbf{X}_i(U)$ . We say that two map germs  $f_i : (\mathbb{R}^n, \mathbf{x}_i) \longrightarrow (\mathbb{R}^p, \mathbf{y}_i)$  ( $i = 1, 2$ ) are  *$\mathcal{A}$ -equivalent* if there exist diffeomorphism germs  $\phi : (\mathbb{R}^n, \mathbf{x}_1) \longrightarrow (\mathbb{R}^n, \mathbf{x}_2)$  and  $\psi : (\mathbb{R}^p, \mathbf{y}_1) \longrightarrow (\mathbb{R}^p, \mathbf{y}_2)$  such that  $\psi \circ f_1 = f_2 \circ \phi$ . If for both  $i = 1, 2$  the regular set of  $\text{CPe}_{M_i}$  is dense in  $(U, u_i)$ , it follows from Proposition B.2 that  $\text{CPe}_{M_1}$  and  $\text{CPe}_{M_2}$  are  *$\mathcal{A}$ -equivalent* if

and only if the corresponding Legendrian immersion germs  $\mathcal{L}(\tilde{H}_1) : (U, u_1) \longrightarrow PT^*(S^{n-1} \times \mathbb{R})$  and  $\mathcal{L}(\tilde{H}_2) : (U, u_2) \longrightarrow PT^*(S^{n-1} \times \mathbb{R})$  are Legendrian equivalent, where  $\tilde{H}_i$  is the extended height function germ of  $M_i = \mathbf{X}_i(U)$ . This condition is also equivalent to the condition that two generating families  $\tilde{H}_1$  and  $\tilde{H}_2$  are  $P\mathcal{K}$ -equivalent by Theorem B.3.

On the other hand, we consider the case that  $\mathbf{v}_i = \mathbf{n}_i(u)$ ,  $r_i = \langle \mathbf{X}_i(u), \mathbf{n}_i(u) \rangle$ . We denote that  $\tilde{h}_{i,(v_i,r_i)}(u) = \tilde{H}_i(u, \mathbf{v}_i, r_i)$ , then we have  $\tilde{h}_{i,(v_i,r_i)}(u) = \mathfrak{h}_{v_i} \circ \mathbf{X}_i(u) - r_i$ . By Theorem 5.1,  $K(\mathbf{X}_1(U), H(\mathbf{X}_1(U), u_1), p_1) = K(\mathbf{X}_2(U), H(\mathbf{X}_2(U), u_2), p_2)$  if and only if  $\tilde{h}_{1,(v_1,r_1)}$  and  $\tilde{h}_{1,(v_2,r_2)}$  are  $\mathcal{K}$ -equivalent, where  $p_i = \mathbf{X}(u_i)$ . Therefore, we can apply the arguments in Appendix B to our situation. We denote  $Q(\mathbf{X}, u)$  the local ring of the function germ  $\tilde{h}_{v_0,r_0} : (U, u_0) \longrightarrow \mathbb{R}$ , where  $(\mathbf{v}_0, r_0) = \text{CPe}_M(u_0)$ . We remark that we can explicitly write the local ring as follows:

$$Q(\mathbf{X}(U), u_0) = \frac{C_{u_0}^\infty(U)}{\langle \langle \mathbf{X}(u), \mathbf{n}(u_0) \rangle - r_0 \rangle_{C_{u_0}^\infty(U)}},$$

where where  $r_0 = \langle \mathbf{X}(u_0), \mathbf{n}(u_0) \rangle$  and  $C_{u_0}^\infty(U)$  is the local ring of function germs at  $u_0$  with the unique maximal ideal  $\mathfrak{M}_{u_0}(U)$ .

**Theorem 6.1** *Let  $\mathbf{X}_i : (U, u_i) \longrightarrow (\mathbb{R}^n, p_i)$  ( $i = 1, 2$ ) be hypersurfaces germs such that the corresponding Legendrian immersion germs  $\mathcal{L}(\tilde{H}_i) : (U, u_i) \longrightarrow PT^*(S^{n-1} \times \mathbb{R})$  are Legendrian stable. Then the following conditions are equivalent:*

- (1) *Cylindrical pedal germs  $\text{CPe}_{M_1}$  and  $\text{CPe}_{M_2}$  are  $\mathcal{A}$ -equivalent.*
- (2)  *$\tilde{H}_1$  and  $\tilde{H}_2$  are  $P\mathcal{K}$ -equivalent.*
- (3)  *$\tilde{h}_{1,(v_1,r_1)}$  and  $\tilde{h}_{1,(v_2,r_2)}$  are  $\mathcal{K}$ -equivalent, where  $(\mathbf{v}_i, r_i) = \text{CPe}_{M_i}(u_i)$ .*
- (4)  *$K(\mathbf{X}_1(U), H(\mathbf{X}_1(U), u_1), p_1) = K(\mathbf{X}_2(U), H(\mathbf{X}_2(U), u_2), p_2)$ .*
- (5)  *$Q(\mathbf{X}_1, u_1)$  and  $Q(\mathbf{X}_2, u_2)$  are isomorphic as  $\mathbb{R}$ -algebras.*

*Proof.* By the previous arguments (mainly from Theorem 5.1), it has been already shown that conditions (3) and (4) are equivalent. Other assertions follow from Proposition B.4.  $\square$

As an application of a kind of the transversality theorems, we can show that the assumption of the theorem is generic in the case when  $n \leq 6$ . In general we have the following proposition.

**Proposition 6.2** *Let  $\mathbf{X}_i : (U, u_i) \longrightarrow (\mathbb{R}^n, p_i)$  ( $i = 1, 2$ ) be hypersurface germs such that their sets of parabolic points have no interior points as subspaces of  $U$ . If cylindrical pedal germs  $\text{CPe}_{M_1}$ ,  $\text{CPe}_{M_2}$  are  $\mathcal{A}$ -equivalent, then*

$$K(\tilde{\mathbf{X}}_1(U), H(\mathbf{X}_1(U), u_1), p_1) = K(\mathbf{X}_2(U), H(\mathbf{X}_2(U), u_2), p_2).$$

*In this case,  $(\mathbf{X}_1^{-1}(H(\mathbf{X}_1(U), u_1)), u_1)$  and  $(\mathbf{X}_2^{-1}(H(\mathbf{X}_2(U), u_2), u_2)$  are diffeomorphic as set germs.*

*Proof.* The set of parabolic points is the set of singular points of the cylindrical pedal. So the corresponding Legendrian lifts  $\mathcal{L}(\tilde{H}_i)$  satisfy the hypothesis of Proposition B.2. If cylindrical pedal germs  $\text{CPe}_{M_1}$ ,  $\text{CPe}_{M_2}$  are  $\mathcal{A}$ -equivalent, then  $\mathcal{L}(\tilde{H}_1)$ ,  $\mathcal{L}(\tilde{H}_2)$  are Legendrian equivalent, so that  $\tilde{H}_1$ ,  $\tilde{H}_2$  are  $P\mathcal{K}$ -equivalent. Therefore,  $\tilde{h}_{1,(v_1,r_1)}$ ,  $\tilde{h}_{1,(v_2,r_2)}$  are  $\mathcal{K}$ -equivalent, where  $r_i = \langle \mathbf{X}_i(u), \mathbf{n}_i(u) \rangle$ . By Theorem 5.1, this condition is equivalent to the condition that  $K(\mathbf{X}_1(U), H(\mathbf{X}_1(U), u_1), p_1) = K(\mathbf{X}_2(U), H(\mathbf{X}_2(U), u_2), p_2)$ .

On the other hand, we have  $(\mathbf{X}_i^{-1}(H(\mathbf{X}_i(U), u_i)), u_i) = (\tilde{h}_{i,(v_i,r_i)}^{-1}(0), u_i)$ . It follows from this fact that  $(\mathbf{X}_1^{-1}(H(\mathbf{X}_1(U), u_1)), u_1)$  and  $(\mathbf{X}_2^{-1}(H(\mathbf{X}_2(U), u_2)), u_2)$  are diffeomorphic as set germs because the  $\mathcal{K}$ -equivalence preserve the zero level sets.  $\square$

For a hypersurface germ  $\mathbf{X} : (U, u) \longrightarrow (\mathbb{R}^n, p)$ , we call  $(\mathbf{X}^{-1}(H(\mathbf{X}(U), u)), u)$  the *tangent indicatrix germ* of  $M = \mathbf{X}(U)$  at  $u$  (or  $p$ ). By Proposition 6.2, the diffeomorphism type of the tangent indicatrix germ is an invariant of the  $\mathcal{A}$ -classification of the cylindrical pedal germ of  $\mathbf{X}$ . Moreover, by the above results, we can borrow some basic invariants from the singularity theory on function germs. We need  $\mathcal{K}$ -invariants for function germ. The local ring of a function germ is a complete  $\mathcal{K}$ -invariant for generic function germs. It is, however, not a numerical invariant. The  $\mathcal{K}$ -codimension (or, Tyurina number) of a function germ is a numerical  $\mathcal{K}$ -invariant of function germs[20]. We denote that

$$\text{T-ord}(\mathbf{X}(U), u_0) = \dim \frac{C_{u_0}^\infty(U)}{\langle \langle \mathbf{X}(u), \mathbf{n}(u_0) \rangle - r_0, \langle \mathbf{X}_{u_i}(u), \mathbf{n}(u_0) \rangle \rangle_{C_{u_0}^\infty}},$$

where  $r_0 = \langle \mathbf{X}(u_0), \mathbf{n}(u_0) \rangle$ . Usually  $\text{T-ord}(\mathbf{x}(U), u_0)$  is called the  $\mathcal{K}$ -codimension of  $\tilde{h}_{(v_0,r_0)}$ . However, we call it the *order of contact with the tangent hyperplane* at  $\mathbf{X}(u_0)$ . We also have the notion of corank of function germs.

$$\text{T-corank}(\mathbf{X}(U), u_0) = (n - 1) - \text{rank Hess}(h_{v_0}(u_0)),$$

where  $\mathbf{v}_0 = \mathbf{n}(u_0)$ .

By Proposition 3.2,  $\mathbf{X}(u_0)$  is a parabolic point if and only if  $\text{T-corank}(\mathbf{X}(U), u_0) \geq 1$ . Moreover  $\mathbf{X}(u_0)$  is a flat point if and only if  $\text{T-corank}(\mathbf{X}(U), u_0) = n - 1$ .

On the other hand, a function germ  $f : (\mathbb{R}^{n-1}, \mathbf{a}) \longrightarrow \mathbb{R}$  has the  $A_k$ -type singularity if and only if  $f$  is  $\mathcal{K}$ -equivalent to the germ  $x_1^{k+1} \pm x_1^2 \cdots \pm x_{n-1}^2$ . If  $\text{T-corank}(\mathbf{X}(U), u_0) = n - 2$ , the height function  $h_{v_0}$  has the  $A_k$ -type singularity at  $u_0$  in generic. In this case we have  $\text{T-ord}(\mathbf{X}(U), u_0) = k$ . This number is equal to the order of contact in the classical sense (cf., [5]). This is the reason why we call  $\text{T-ord}(\mathbf{X}(U), u_0)$  the order of contact with the tangent hyperplane at  $\mathbf{X}(u_0)$ .

We now consider the contact of hypersurfaces with families of hyperplane. Let  $\mathbf{X}_i : (U, \bar{u}_i) \longrightarrow (\mathbb{R}^n, p_i)$  ( $i = 1, 2$ ) be hypersurface germs. We consider height functions  $H_i : (U \times S^{n-1}, (\bar{u}_i, \mathbf{v}_i)) \longrightarrow \mathbb{R}$  of  $\mathbf{X}_i(U)$ , where  $\mathbf{v}_i = \mathbf{n}(\bar{u}_i)$  respectively. We denote that  $h_{i,v_i}(u) = H_i(u, \mathbf{v}_i)$ , then we have  $h_{i,v_i}(u) = \mathfrak{h}_{v_i} \circ \mathbf{X}_i(u)$ . Then we have the following theorem:

**Theorem 6.3** *Let  $\mathbf{X}_i : (U, \bar{u}_i) \longrightarrow (\mathbb{R}^n, p_i)$  be hypersurface germs such that the corresponding Lagrangian immersion germs  $L(H_i) : (C(H_i), (\bar{u}_i, \mathbf{v}_i)) \longrightarrow T^*S^{n-1}$  are Lagrangian stable, where  $\mathbf{v}_i = \mathbf{n}(\bar{u}_i)$  respectively. Then the following conditions are equivalent:*

- (1)  $K(\mathbf{X}_1(U), \mathcal{F}_{\mathfrak{h}_{v_1}}; p_1) = K(\mathbf{X}_2(U), \mathcal{F}_{\mathfrak{h}_{v_2}}; p_2)$ .
- (2)  $h_{1,v_1}$  and  $h_{2,v_2}$  are  $\mathcal{R}^+$ -equivalent.
- (3)  $H_1$  and  $H_2$  are  $P\text{-}\mathcal{R}^+$ -equivalent.
- (4)  $L(H_1)$  and  $L(H_2)$  are Lagrangian equivalent.
- (5) (a) The rank and signature of the  $\mathcal{H}(h_{1,v_1})(\bar{u}_1)$  and  $\mathcal{H}(h_{2,v_2})(\bar{u}_2)$  are equal,  
(b) There is an isomorphism  $\gamma : \mathcal{R}_2(h_{1,v_1}) \longrightarrow \mathcal{R}_2(h_{2,v_2})$  such that  $\gamma(\overline{h_{1,v_1}}) = \overline{h_{2,v_2}}$ .

*Proof.* By Proposition 5.2, the condition (1) is equivalent to the condition (2). Since both of  $L(H_i)$  are Lagrangian stable, both of  $H_i$  are  $\mathcal{R}^+$ -versal unfoldings of  $h_{i,v_i}$  respectively. By the uniqueness theorem on the  $\mathcal{R}^+$ -versal unfolding of a function germ, the condition (2) is equivalent to the condition (3). By Theorem A.2, the condition (3) is equivalent to the condition (4). It also follows from Theorem A.2 that both of  $h_i$  satisfy the Milnor condition. Therefore we can apply Proposition 5.3 to our situation, so that the condition (2) is equivalent to the condition (5). This completes the proof.  $\square$

We remark that if  $L(H_1)$  and  $L(H_2)$  are Lagrangian equivalent, then the corresponding Lagrangian map germs  $\pi \circ L(H_1)$  and  $\pi \circ L(H_2)$  are  $\mathcal{A}$ -equivalent. The Gauss map of a hypersurface  $\mathbf{x}(U) = M$  is considered to be the Lagrangian map germ of  $L(H)$  (or, the catastrophe map germ of  $H_1$ ). Moreover, if  $h_{1,v_1}$  and  $h_{2,v_2}$  are  $\mathcal{R}^+$ -equivalent then the level set germs of function germs  $h_{1,v_1}$  and  $h_{2,v_2}$  are diffeomorphic. Therefore, we have the following corollary.

**Corollary 6.4** *Under the same assumptions as those of the above theorem for hypersurface germs  $\mathbf{X}_i : (U, \bar{u}_i) \longrightarrow (\mathbb{R}^n, p_i)$  ( $i = 1, 2$ ), we have the following: If one of the conditions of the above theorem is satisfied, then*

- (1) *The Gauss map germs  $G_1, G_2$  are  $\mathcal{A}$ -equivalent.*
- (2) *The Dupin foliation germs  $\mathcal{DF}(\mathbf{X}_1(U), \bar{u}_1), \mathcal{DF}(\mathbf{X}_2(U), \bar{u}_2)$  are diffeomorphic.*

We also consider the contact of hypersurfaces with families of hyperspheres. Let  $\mathbf{X}_i : (U, \bar{u}_i) \longrightarrow \mathbb{R}^n, p_i$  ( $i = 1, 2$ ) be hypersurface germs. We consider distance squared functions  $D_i : (U \times \mathbb{R}^n, (\bar{u}_i, \mathbf{x}_i)) \longrightarrow \mathbb{R}$  of  $\mathbf{X}_i(U)$ , where  $\mathbf{x}_i = \text{Ev}_{\kappa_i}(\bar{u}_i)$ . We denote that  $d_{i,v_i}(u) = D_i(u, \mathbf{x}_i)$ , then we have  $d_{i,x_i}(u) = \mathfrak{d}_{x_i} \circ \mathbf{X}_i(u)$ . Then we have the following theorem:

**Theorem 6.5** *Let  $\mathbf{X}_i : (U, \bar{u}_i) \longrightarrow \mathbb{R}^n, p_i$  ( $i = 1, 2$ ) be hypersurface germs such that the corresponding Lagrangian immersion germs  $L(D_i) : (C(D_i), (\bar{u}_i, \mathbf{x}_i)) \longrightarrow T^*\mathbb{R}^n$  are Lagrangian stable, where  $\mathbf{x}_i = \text{Ev}_{\kappa_i}(\bar{u}_i)$  are centers of the osculating hyperspheres of  $\mathbf{X}_i(U)$  respectively. Then the following conditions are equivalent:*

- (1)  $K(\mathbf{X}_1(U), \mathcal{F}_{\mathfrak{d}_{x_1}}; p_1) = K(\mathbf{X}_2(U), \mathcal{F}_{\mathfrak{d}_{x_2}}; p_2)$ .
- (2)  $d_{1,x_1}$  and  $d_{2,x_2}$  are  $\mathcal{R}^+$ -equivalent.
- (3)  $D_1$  and  $D_2$  are  $P\text{-}\mathcal{R}^+$ -equivalent.
- (4)  $L(D_1)$  and  $L(D_2)$  are Lagrangian equivalent.
- (5) (a) *The rank and signature of the  $\mathcal{H}(d_{1,x_1})(\bar{u}_1)$  and  $\mathcal{H}(d_{2,x_2})(\bar{u}_2)$  are equal,*  
(b) *There is an isomorphism  $\gamma : \mathcal{R}_2(d_{1,x_1}) \longrightarrow \mathcal{R}_2(d_{2,x_2})$  such that  $\gamma(\overline{d_{1,x_1}}) = \overline{d_{2,x_2}}$ .*

The proof of the theorem is parallel to those of Theorem 6.3, so that we omit it.

We remark that if  $L(D_1)$  and  $L(D_2)$  are Lagrangian equivalent, then the corresponding evolutes are diffeomorphic. Since the evolute of a hypersurface  $M = \mathbf{X}(U)$  is considered to be the caustic of  $L(D)$ , the above theorem gives a symplectic interpretation for the contact of hypersurfaces with family of hyperspheres (cf., Appendix A). We have the following corollary.

**Corollary 6.6** *Under the same assumptions as those of the above theorem for hypersurface germs  $\mathbf{X}_i : (U, \bar{u}_i) \longrightarrow (\mathbb{R}^n, p_i)$  ( $i = 1, 2$ ), we have the following: If one of the conditions of the above theorem is satisfied, then*



(1) The evolutes  $\text{Ev}_{M_1}$  and  $\text{Ev}_{M_2}$  are diffeomorphic as set germs.

(2) The osculating hyperspherical foliation germs  $\mathcal{OF}(\mathbf{X}_1(U), \bar{u}_1)$ ,  $\mathcal{OF}(\mathbf{X}_2(U), \bar{u}_2)$  are diffeomorphic.

## 7 Surfaces in 3-space

In this section we consider the case  $n = 3$ . Before we start to consider the case  $n = 3$ , we study generic properties of hypersurfaces in  $\mathbb{R}^n$  for general  $n$ . The main tool is a kind of transversality theorems. We consider the space of embeddings  $\text{Emb}(U, \mathbb{R}^n)$  with Whitney  $C^\infty$ -topology. We also consider the functions:

$$\begin{aligned}\mathcal{H} &: \mathbb{R}^n \times S^{n-1} \longrightarrow \mathbb{R}, \\ \tilde{\mathcal{H}} &: \mathbb{R}^n \times (S^{n-1} \times \mathbb{R}) \longrightarrow \mathbb{R}, \\ \mathcal{D} &: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}.\end{aligned}$$

which are given in §5. We claim that  $\mathfrak{h}_v$ ,  $\tilde{\mathfrak{h}}_{(v,r)}$  and  $\mathfrak{d}_x$  are respectively submersions for any  $\mathbf{v} \in S^{n-1}$ ,  $(\mathbf{v}, r) \in S^{n-1} \times \mathbb{R}$  and  $\mathbf{x} \in \mathbb{R}^n \setminus M$  respectively. where  $\mathfrak{h}_v(\mathbf{x}) = \mathcal{H}(\mathbf{x}, \mathbf{v})$ ,  $\tilde{\mathfrak{h}}_{(v,r)}(\mathbf{x}) = \tilde{\mathcal{H}}(\mathbf{x}, \mathbf{v}, r)$  and  $\mathfrak{d}_x(\mathbf{y}) = \mathcal{D}(\mathbf{y}, \mathbf{x})$ . For any  $\mathbf{X} \in \text{Emb}(U, \mathbb{R}^n)$ , we have

$$H = \mathcal{H} \circ (\mathbf{X} \times id_{S^{n-1}}), \quad \tilde{H} = \tilde{\mathcal{H}} \circ (\mathbf{X} \times id_{S^{n-1} \times \mathbb{R}}) \quad \text{and} \quad D = \mathcal{D} \circ (\mathbf{X} \times id_{\mathbb{R}^n}).$$

We also have the  $\ell$ -jet extensions:

$$\begin{aligned}j_1^\ell H &: U \times S^{n-1} \longrightarrow J^\ell(U, \mathbb{R}) ; j_1^\ell H(u, \mathbf{v}) = j^\ell h_v(u), \\ j_1^\ell \tilde{H} &: U \times (S^{n-1} \times \mathbb{R}) \longrightarrow J^\ell(U, \mathbb{R}) ; j_1^\ell \tilde{H}(u, (\mathbf{v}, r)) = j^\ell \tilde{h}_{(v,r)}(u), \\ j_1^\ell D &: U \times \mathbb{R}^n \longrightarrow J^\ell(U, \mathbb{R}) ; j_1^\ell D(u, \mathbf{x}) = j^\ell d_x(u).\end{aligned}$$

We consider the trivialization  $J^\ell(U, \mathbb{R}) \equiv U \times \mathbb{R} \times J^\ell(n-1, 1)$ . For any submanifold  $Q \subset J^\ell(n-1, 1)$ , we denote that  $\tilde{Q} = U \times \{0\} \times Q$ . Then we have the following proposition as a corollary of Lemma 6 in Wassermann [29]. (See also Montaldi [24]).

**Proposition 7.1** *Let  $Q$  be a submanifold of  $J^\ell(n-1, 1)$ . Then the set*

$$T_Q(F) = \{ \mathbf{X} \in \text{Emb}(U, \mathbb{R}^n) \mid j_1^\ell F \text{ is transversal to } \tilde{Q} \}$$

*is a residual subset of  $\text{Emb}(U, \mathbb{R}^n)$ . If  $Q$  is a closed subset, then  $T_Q$  is open. Here,  $F$  is  $H$ ,  $\tilde{H}$  or  $D$ .*

As a corollary of the above proposition and classification results of function germs[1], we have the following theorem.

**Theorem 7.2** *Suppose that  $n \leq 6$ . There exists an open dense subset  $\mathcal{O} \subset \text{Emb}(U, \mathbb{R}^n)$  such that for any  $\mathbf{X} \in \mathcal{O}$ , the germ of the corresponding the germs of the Lagrangian lifts  $L(D)$  and  $L(H)$  of the evolute  $\text{Ev}_M$  and the Gauss map  $G$  at each point are Lagrangian stable. Moreover the germ of the Legendrian lift  $\mathcal{L}(\tilde{H})$  of the cylindrical pedal  $\text{CPe}_M$  at each point is Legendrian stable.*

We now stick to the case when  $n = 3$ . In this case we call  $\mathbf{X} : U \rightarrow \mathbb{R}^3$  a *surface*,  $S^2$  a *sphere* and  $H(\mathbf{X}(U), u)$  the *tangent plane* and etc. By Theorem 7.2 and the classification of function germs [1], we have the following theorem.

**Theorem 7.3** *There exists an open dense subset  $\mathcal{O} \subset \text{Emb}(U, \mathbb{R}^3)$  such that for any  $\mathbf{X} \in \mathcal{O}$ , the following conditions hold:*

- (1) *The parabolic set  $K^{-1}(0)$  is a regular curve. We call such a curve the parabolic curve.*
- (2) *The Gauss map  $G$  along the parabolic curve are the folds except at isolated points. At this point  $G$  is the cusp.*

Here, a map germ  $f : (\mathbb{R}^2, \mathbf{a}) \rightarrow (\mathbb{R}^2, \mathbf{b})$  is called a *fold* if it is  $\mathcal{A}$ -equivalent to the germ  $(x_1, x_2^2)$  (cf., Fig. 1) and a *cuspidal* if it is  $\mathcal{A}$ -equivalent to the germ  $(x_1, x_2^3 + x_1x_2)$  (cf., Fig. 1).

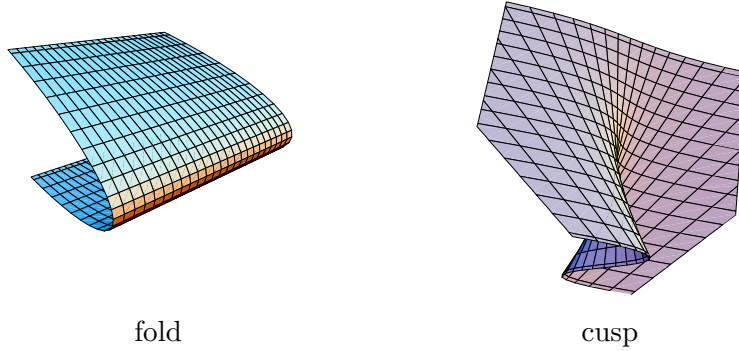


Fig. 1.

(3) *A parabolic point  $u \in U$  is a fold of the Gauss map  $G$  if and only if it is the cuspidal edge of the cylindrical pedal  $\text{CPe}_M$ .*

(4) *A parabolic point  $u \in U$  is a cusp of the Gauss map  $G$  if and only if it is the swallowtail of the cylindrical pedal  $\text{CPe}_M$ .*

Here, a map germ  $f : (\mathbb{R}^2, \mathbf{a}) \rightarrow (\mathbb{R}^3, \mathbf{b})$  is called a *cuspidal edge* if it is  $\mathcal{A}$ -equivalent to the germ  $(x_1, x_2^2, x_2^3)$  (cf., Fig. 2) and a *swallowtail* if it is  $\mathcal{A}$ -equivalent to the germ  $(3x_1^4 + x_1^2x_2, 4x_1^3 + 2x_1x_2, x_2)$  (cf., Fig.2).

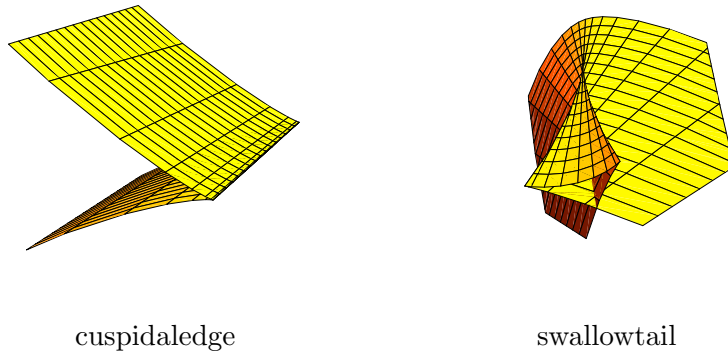


Fig. 2.

The assertion (1) and (2) can be interpreted that the Lagrangian lift  $L(H)$  of the Gauss map  $G$  of  $\mathbf{X} \in \mathcal{O}$  is Lagrangian stable at each point. Since  $\mathcal{L}(\tilde{H})$  is the Legendrian covering of the

Lagrangian map  $L(H)$  whose Lagrangian map is the Gauss map  $G$ , it has been known that the corresponding singularities of the wavefront of  $\mathcal{L}(\tilde{H})$  are the cuspidaledge or the swallowtail[1]. Therefore we have the assertion (3) and (4).

Following the terminology of Whitney [30], we say that a surface  $\mathbf{X} : U \rightarrow \mathbb{R}^3$  has the *excellent Gauss map*  $G$  if  $L(H)$  is a stable Lagrangian immersion germ at each point. In this case, the Gauss map  $G$  has only folds and cusps as singularities. Theorem 7.3 asserts that a surface with the excellent Gauss map is generic in the space of all surfaces in  $\mathbb{R}^3$ . We now consider the geometric meanings of folds and cusps of the Gauss map. We have the following results the main part of which is given by Banchoff et al [2]. However, we add few new information from the view point of Legendrian singularity theory.

**Theorem 7.4** *Let  $G : (U, u_0) \rightarrow (\mathbb{R}^3, \mathbf{v}_0)$  be the excellent Gauss map of a surface  $\mathbf{X}$  and  $h_{\mathbf{v}_0} : (U, u_0) \rightarrow \mathbb{R}$  be the height function germ at  $\mathbf{v}_0 = G(u_0) = \mathbf{n}(u_0)$ . Then we have the following:*

(1)  *$u$  is a parabolic point of  $\mathbf{X}$  if and only if  $\text{T-corank}(\mathbf{X}(U), u_0) = 1$  (i.e.,  $u_0$  is not a flat point of  $\mathbf{X}$ ).*

(2) *If  $u_0$  is a parabolic point of  $\mathbf{X}$ , then  $\tilde{h}_{(u_0, r_0)}$  has the  $A_k$ -type singularity for  $k = 2, 3$ , where  $\tilde{h}_{(u_0, r_0)}(u) = h_{\mathbf{v}_0}(u) - r_0$ .*

(3) *Suppose that  $u_0$  is a parabolic point of  $\mathbf{X}$ . Then the following conditions are equivalent:*

(a) *The cylindrical pedal  $\text{CPe}_M$  is the cuspidaledge at  $u_0$*

(b)  *$\tilde{h}_{(u_0, r_0)}$  has the  $A_2$ -type singularity.*

(c)  *$\text{T-ord}(\mathbf{X}(U), u_0) = 2$ .*

(d) *Tangent indicatrix  $(\mathbf{X}^{-1}(H(\mathbf{X}(U), u_0), u_0))$  is a ordinary cusp, where a curve  $C \subset \mathbb{R}^2$  is called an ordinary cusp if it is diffeomorphic to the curve given by  $\{(x_1, x_2) \mid x_1^2 - x_2^3 = 0\}$ .*

(e) *For each  $\varepsilon > 0$ , there exist two distinct points  $u_1, u_2 \in U$  such that  $|u_0 - u_i| < \varepsilon$  for  $i = 1, 2$ , both of  $u_1, u_2$  are not parabolic points and the tangent planes to  $M = \mathbf{x}(U)$  at  $u_1, u_2$  are parallel.*

(f) *The Gauss map  $G$  is the fold at  $u_0$ .*

(4) *Suppose that  $u_0$  is a parabolic point of  $\mathbf{X}$ . Then the following conditions are equivalent:*

(a) *The cylindrical pedal  $\text{CPe}_M$  is the swallowtail at  $u_0$*

(b)  *$\tilde{h}_{(u_0, r_0)}$  has the  $A_3$ -type singularity.*

(c)  *$\text{T-ord}(\mathbf{X}(U), u_0) = 3$ .*

(d) *Tangent indicatrix  $(\mathbf{X}^{-1}(H(\mathbf{X}(U), u_0), u_0))$  is a point or a tachnodal, where a curve  $C \subset \mathbb{R}^2$  is called a tachnodal if it is diffeomorphic to the curve given by  $\{(x_1, x_2) \mid x_1^2 - x_2^4 = 0\}$ .*

(e) *For each  $\varepsilon > 0$ , there exist three distinct points  $u_1, u_2, u_3 \in U$  such that  $|u_0 - u_i| < \varepsilon$  for  $i = 1, 2, 3$ , both of  $u_1, u_2, u_3$  are not parabolic points and the tangent planes to  $M = \mathbf{x}(U)$  at  $u_1, u_2, u_3$  are parallel.*

(f) *For each  $\varepsilon > 0$ , there exist two distinct points  $u_1, u_2 \in U$  such that  $|u_0 - u_i| < \varepsilon$  for  $i = 1, 2$ , both of  $u_1, u_2$  are not parabolic points and the tangent planes to  $M = \mathbf{x}(U)$  at  $u_1, u_2$  are equal.*

(g) *The Gauss map  $G$  is the cusp at  $u_0$ .*

*Proof.* We have shown in §6 that  $u_0$  is a parabolic point if and only if  $\text{T-corank}(\mathbf{X}(U), u_0) \geq 1$ . Since  $n = 3$ , we have  $\text{T-corank}(\mathbf{X}(U), u_0) \leq 2$ . Since the extended height function germ  $\tilde{H} : (U \times (S^{n-1} \times \mathbb{R}), (u_0, (\mathbf{v}_0, r_0))) \rightarrow \mathbb{R}$  can be considered as a generating family of the

Legendrian immersion germ  $\mathcal{L}(\tilde{H}), \tilde{h}_{(v_0, r_0)}$  has only the  $A_k$ -type singularities ( $k = 1, 2, 3$ ). This means that the corank of the Hessian matrix of  $\tilde{h}_{(v_0, r_0)}$  at a parabolic point is 1. The assertion (2) also follows. By the same reason, the conditions (3);(a),(b),(c) (respectively, (4); (a),(b),(c)) are equivalent. If the height function germ  $\tilde{h}_{(v_0, r_0)}$  has the  $A_2$ -type singularity, it is  $\mathcal{K}$ -equivalent to the germ  $\pm x_1^2 + x_2^3$ . Since the  $\mathcal{K}$ -equivalence preserves the zero level sets, the tangent indicatrix is diffeomorphic to the curve given by  $\pm x_1^2 + x_2^3 = 0$ . This is the ordinary cusp. The normal form for the  $A_3$ -type singularity is given by  $\pm x_1^2 + x_2^4$ , so the tangent indicatrix is diffeomorphic to the curve  $\pm x_1^2 + x_2^4 = 0$ . This means that the condition (3),(d) (respectively, (4),(d)) is also equivalent to the other conditions.

Suppose that  $u_0$  is a parabolic point, then the Gauss map has only folds or cusps. If the point  $u_0$  is the fold point, there is a neighborhood of  $u_0$  on which the Gauss map is 2 to 1 except the parabolic curve (i.e, fold curve). By Lemma 5.4, the condition (3), (e) is satisfied. If the point  $u_0$  is the cusp, the critical value set is the ordinary cusp. By the normal form, we can understand that the Gauss map is 3 to 1 inside region of the critical values. Moreover, the point  $u_0$  is in the closure of the region. This means that the condition (4),(e) holds. We can also observe that near by the cusp point, there are 2 to 1 points which near to the cusp  $u_0$ . However, one of those points is always a parabolic point. Since no other singularities appear for in this case, we have the condition (3),(e) (respectively, (4),(e)) characterizes the fold (respectively, the cusp).

If we consider the cylindrical pedal instead of the Gauss map, the only singularities are cuspidaledges or swallowtails. For a swallowtail point  $u_0$ , there is a self intersection curve (cf., Fig. 1) approaching to  $u_0$ . On this curve, there are two distinct point  $u_1, u_2$  such that  $\text{CPe}_M(u_1) = \text{CPe}_M(u_2)$ . By Lemma 5.4, this means that the tangent planes to  $M = \mathbf{x}(U)$  at points  $u_1, u_2$  are equal. Since there are no other singularities in this case, the condition (4),(f) characterizes a swallowtail point of  $\text{CPe}_M$ . This completes the proof.  $\square$

We now apply Theorem 6.3 to the above theorem and obtain new information from the view point of Lagrangian singularity theory.

**Proposition 7.5** *Let  $G : (U, u_0) \longrightarrow (\mathbb{R}^3, \mathbf{v}_0)$  be the excellent Gauss map of a surface  $\mathbf{X}$  and  $h_{v_0} : (U, u_0) \longrightarrow \mathbb{R}$  be the height function germ at  $\mathbf{v}_0 = G(u_0) = \mathbf{n}(u_0)$ . Then the Dupin foliation germ  $\mathcal{DF}(\mathbf{X}(U), u_0)$  is diffeomorphic to a foliation germ  $(\mathcal{F}_f, \mathbf{0})$  where  $f$  is one of the germs in the following list:*

- (1)  $x_1^3 + x_2^2$  (fold)
- (2)  $\pm x_1^4 + x_2^2$  ( $\pm$ cusp)

By Theorems 7.2, A.2 and the classification of function germs under  $\mathcal{R}^+$ -codimension  $\leq 3$ , we have the following classification theorem:

**Theorem 7.6** *There exists an open dense subset  $\mathcal{O} \subset \text{Emb}(U, \mathbb{R}^3)$  such that for any  $\mathbf{X} \in \mathcal{O}$ , the corresponding Lagrangian immersion germ  $L(D)$  at any point  $(u_0, \mathbf{x}_0) \in U \times (\mathbb{R}^3 \setminus M)$  is Lagrangian equivalent to a Lagrangian immersion germ  $L(F) : (C(F), 0) \longrightarrow T^*\mathbb{R}^3$  whose generating family  $F(x_1, x_2, v, q)$  ( $\mathbf{q} = (q_1, q_2, q_3) \in \mathbb{R}^3$ ) is one of the germs in the following list:*

- (1)  $x_1^3 + x_2^2 + q_1 x_1$  (fold)
- (2)  $\pm x_1^4 + x_2^2 + q_1 x_1 + q_2 x_1^2$  ( $\pm$ cusp)
- (3)  $x_1^5 + x_2^2 + q_1 x_1 + q_2 x_1^2 + q_3 x_1^3$  (swallowtail)
- (4)  $x_1^3 - x_1 x_2^2 + q_1 x_1 + q_2 x_2 + q_3 (x_1^2 + x_2^2)$  (pyramid)
- (5)  $x_1^3 + x_2^3 + q_1 x_1 + q_2 x_1 + q_3 x_1 x_2$  (purse).

We now apply Corollary 6.6 to the above classification theorem. Let  $F(x_1, x_2, \mathbf{q})$  be one of the germs in the above list. We write  $f(x_1, x_2) = F(x_1, x_2, 0)$ . As a corollary of the above classification theorem and Corollary 6.6, we have the following:

**Corollary 7.7** *There exists an open dense subset  $\mathcal{O} \subset \text{Emb}(U, \mathbb{R}^3)$  such that for any  $\mathbf{X} \in \mathcal{O}$  and any point  $(u_0, \mathbf{x}_0) \in U \times (\mathbb{R}^3 \setminus M)$ , we have the following assertions :*

(1) *The evolute germ  $(\text{Ev}_M, \mathbf{x}_0)$  is diffeomorphic to the cuspidal edge, the swallowtail, the pyramid or the purse.*

(2) *The osculating spherical foliation germ  $\mathcal{OF}(\mathbf{X}(U), u_0)$  is diffeomorphic to a foliation germ  $(\mathcal{F}_f, \mathbf{0})$  where  $F(x_1, x_2, \mathbf{q})$  is one of the germs in the list of Theorem 7.3.*

Here, the purse and the pyramid are depicted in Figure 3.

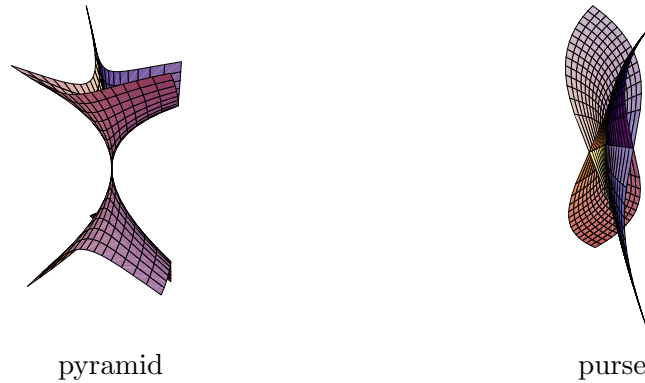


Fig. 3.

We can draw the pictures of the foliation germs  $\mathcal{F}_f$  for the germs  $f$  in Theorem 7.3:

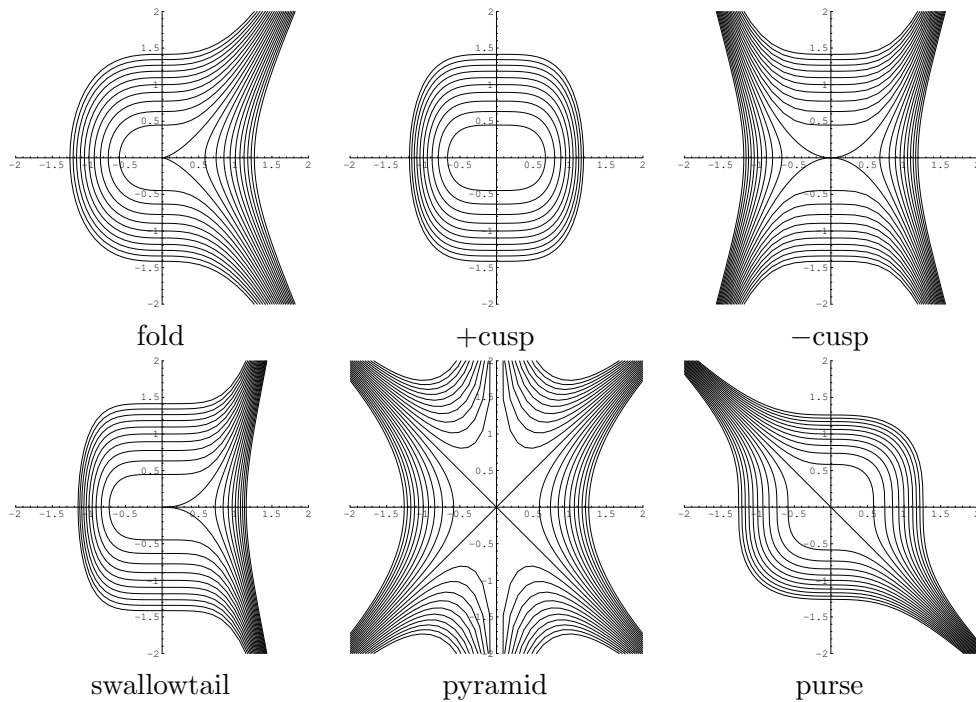


Fig. 4.

## Appendix A. The theory of Lagrangian singularities

In this section we give a brief review on the theory of Lagrangian singularities due to [1, 31]. We consider the cotangent bundle  $\pi : T^*\mathbb{R}^r \longrightarrow \mathbb{R}^r$  over  $\mathbb{R}^r$ . Let  $(u, p) = (u_1, \dots, u_r, p_1, \dots, p_r)$  be the canonical coordinate on  $T^*\mathbb{R}^r$ . Then the canonical symplectic structure on  $T^*\mathbb{R}^r$  is given by the *canonical two form*  $\omega = \sum_{i=1}^r dp_i \wedge du_i$ . Let  $i : L \longrightarrow T^*\mathbb{R}^r$  be an immersion. We say that  $i$  is a *Lagrangian immersion* if  $\dim L = r$  and  $i^*\omega = 0$ . In this case the critical value of  $\pi \circ i$  is called the *caustic* of  $i : L \longrightarrow T^*\mathbb{R}^r$  and it is denoted by  $C_L$ . The main result in the theory of Lagrangian singularities is to describe Lagrangian immersion germs by using families of function germs. Let  $F : (\mathbb{R}^n \times \mathbb{R}^r, (\mathbf{0}, \mathbf{0})) \longrightarrow (\mathbb{R}, 0)$  be an  $r$ -parameter unfolding of function germs. We call

$$C(F) = \left\{ (x, u) \in (\mathbb{R}^n \times \mathbb{R}^r, (\mathbf{0}, \mathbf{0})) \mid \frac{\partial F}{\partial x_1}(x, u) = \dots = \frac{\partial F}{\partial x_n}(x, u) = 0 \right\},$$

the *catastrophe set* of  $F$  and

$$B_F = \left\{ u \in (\mathbb{R}^r, 0) \mid \text{there exist } (x, u) \in C(F) \text{ such that } \text{rank} \left( \frac{\partial^2 F}{\partial x_i \partial x_j}(x, u) \right) < n \right\}$$

the *bifurcation set* of  $F$ . Let  $\pi_r : (\mathbb{R}^n \times \mathbb{R}^r, 0) \longrightarrow (\mathbb{R}^r, 0)$  be the canonical projection, then we can easily show that the bifurcation set of  $F$  is the critical value set of  $\pi_r|C(F)$ . We call  $\pi_{C(F)} = \pi|C(F) : (C(F), 0) \longrightarrow \mathbb{R}$  a *catastrophe map* of  $F$ . We say that  $F$  is a *Morse family of functions* if the map germ

$$\Delta F = \left( \frac{\partial F}{\partial u_1}, \dots, \frac{\partial F}{\partial u_r} \right) : (\mathbb{R}^n \times \mathbb{R}^r, 0) \longrightarrow (\mathbb{R}^r, 0)$$

is non-singular, where  $(x, u) = (x_1, \dots, x_n, u_1, \dots, u_r) \in (\mathbb{R}^n \times \mathbb{R}^r, 0)$ . In this case we have a smooth submanifold germ  $C(F) \subset (\mathbb{R}^n \times \mathbb{R}^r, 0)$  and a map germ  $L(F) : (C(F), 0) \longrightarrow T^*\mathbb{R}^r$  defined by

$$L(F)(x, u) = \left( u, \frac{\partial F}{\partial u_1}, \dots, \frac{\partial F}{\partial u_r} \right).$$

We can show that  $L(F)$  is a Lagrangian immersion. Then we have the following fundamental theorem ([1], page 300).

**Proposition A.1** *All Lagrangian submanifold germs in  $T^*\mathbb{R}^r$  are constructed by the above method.*

Under the above notation, we call  $F$  a *generating family* of  $L(F)$ .

We define an equivalence relation among Lagrangian immersion germs. Let  $i : (L, x) \longrightarrow (T^*\mathbb{R}^r, p)$  and  $i' : (L', x') \longrightarrow (T^*\mathbb{R}^r, p')$  be Lagrangian immersion germs. Then we say that  $i$  and  $i'$  are *Lagrangian equivalent* if there exist a diffeomorphism germ  $\sigma : (L, x) \longrightarrow (L', x')$ , a symplectic diffeomorphism germ  $\tau : (T^*\mathbb{R}^r, p) \longrightarrow (T^*\mathbb{R}^r, p')$  and a diffeomorphism germ  $\bar{\tau} : (\mathbb{R}^r, \pi(p)) \longrightarrow (\mathbb{R}^r, \pi(p'))$  such that  $\tau \circ i = i' \circ \sigma$  and  $\pi \circ \tau = \bar{\tau} \circ \pi$ , where  $\pi : (T^*\mathbb{R}^r, p) \longrightarrow (\mathbb{R}^r, \pi(p))$  is the canonical projection and a symplectic diffeomorphism germ is a diffeomorphism germ which preserves symplectic structure on  $T^*\mathbb{R}^r$ . In this case the caustic  $C_L$  is diffeomorphic to the caustic  $C_{L'}$  by the diffeomorphism germ  $\bar{\tau}$ .

A Lagrangian immersion germ into  $T^*\mathbb{R}^r$  at a point is said to be *Lagrangian stable* if for every map with the given germ there is a neighborhood in the space of Lagrangian immersions (in

the Whitney  $C^\infty$ -topology) and a neighborhood of the original point such that each Lagrangian immersion belonging to the first neighborhood has in the second neighborhood a point at which its germ is Lagrangian equivalent to the original germ.

We can interpret the Lagrangian equivalence by using the notion of generating families. We denote  $\mathcal{E}_m$  the local ring of function germs  $(\mathbb{R}^m, 0) \rightarrow \mathbb{R}$  with the unique maximal ideal  $\mathfrak{M}_m = \{h \in \mathcal{E}_m | h(0) = 0\}$ . Let  $F, G : (\mathbb{R}^n \times \mathbb{R}^r, 0) \rightarrow (\mathbb{R}, 0)$  be function germs. We say that  $F$  and  $G$  are  $P\text{-}\mathcal{R}^+$ -equivalent if there exists a diffeomorphism germ  $\Phi : (\mathbb{R}^n \times \mathbb{R}^r, 0) \rightarrow (\mathbb{R}^n \times \mathbb{R}^r, 0)$  of the form  $\Phi(x, u) = (\Phi_1(x, u), \phi(u))$  and a function germ  $h : (\mathbb{R}^r, 0) \rightarrow \mathbb{R}$  such that  $G(x, u) = F(\Phi(x, u)) + h(u)$ . For any  $F_1 \in \mathfrak{M}_{n+r}$  and  $F_2 \in \mathfrak{M}_{n'+r}$ ,  $F_1, F_2$  are said to be *stably  $P\text{-}\mathcal{R}^+$ -equivalent* if they become  $P\text{-}\mathcal{R}^+$ -equivalent after the addition to the arguments to  $x_i$  of new arguments  $y_i$  and to the functions  $F_i$  of nondegenerate quadratic forms  $Q_i$  in the new arguments (i.e.,  $F_1 + Q_1$  and  $F_2 + Q_2$  are  $P\text{-}\mathcal{R}^+$ -equivalent).

Let  $F : (\mathbb{R}^n \times \mathbb{R}^r, 0) \rightarrow (\mathbb{R}, 0)$  be a function germ. We say that  $F$  is an  $\mathcal{R}^+$ -versal deformation of  $f = F|_{\mathbb{R}^n \times \{0\}}$  if

$$\mathcal{E}_n = J_f + \left\langle \frac{\partial F}{\partial u_1} |_{\mathbb{R}^n \times \{0\}}, \dots, \frac{\partial F}{\partial u_r} |_{\mathbb{R}^n \times \{0\}} \right\rangle_{\mathbb{R}} + \langle 1 \rangle_{\mathbb{R}},$$

where

$$J_f = \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle_{\mathcal{E}_n}.$$

**Theorem A.2** *Let  $F_1 \in \mathfrak{M}_{n+r}$  and  $F_2 \in \mathfrak{M}_{n'+r}$  be Morse families. Then we have the following:*

- (1)  $L(F_1)$  and  $L(F_2)$  are Lagrangian equivalent if and only if  $F_1, F_2$  are stably  $P\text{-}\mathcal{R}^+$ -equivalent.
- (2)  $L(F)$  is Lagrangian stable if and only if  $F$  is a  $\mathcal{R}^+$ -versal deformation of  $F|_{\mathbb{R}^n \times \{0\}}$ .

For the proof of the above theorem, see ([1], page 304 and 325). The following proposition describes the well-known relationship between bifurcation sets and equivalence among unfoldings of function germs:

**Proposition A.3** *Let  $F, G : (\mathbb{R}^n \times \mathbb{R}^r, 0) \rightarrow (\mathbb{R}, 0)$  be function germs. If  $F$  and  $G$  are  $P\text{-}\mathcal{R}^+$ -equivalent then there exist a diffeomorphism germ  $\phi : (\mathbb{R}^r, 0) \rightarrow (\mathbb{R}^r, 0)$  such that  $\phi(B_F) = B_G$*

## Appendix B. The theory of Legendrian singularities

In which we give a quick survey on the Legendrian singularity theory mainly due to Arnol'd-Zakalyukin [1, 31]. Almost all results have been known at least implicitly. Let  $\pi : PT^*(M) \rightarrow M$  be the projective cotangent bundle over an  $n$ -dimensional manifold  $M$ . This fibration can be considered as a Legendrian fibration with the canonical contact structure  $K$  on  $PT^*(M)$ . We now review geometric properties of this space. Consider the tangent bundle  $\tau : TPT^*(M) \rightarrow PT^*(M)$  and the differential map  $d\pi : TPT^*(M) \rightarrow N$  of  $\pi$ . For any  $X \in TPT^*(M)$ , there exists an element  $\alpha \in T^*(M)$  such that  $\tau(X) = [\alpha]$ . For an element  $V \in T_x(M)$ , the property  $\alpha(V) = \mathbf{0}$  does not depend on the choice of representative of the class  $[\alpha]$ . Thus we can define the canonical contact structure on  $PT^*(M)$  by

$$K = \{X \in TPT^*(M) | \tau(X)(d\pi(X)) = 0\}.$$

For a local coordinate neighborhood  $(U, (x_1, \dots, x_n))$  on  $M$ , we have a trivialization  $PT^*(U) \cong U \times P(\mathbb{R}^{n-1})^*$  and we call

$$((x_1, \dots, x_n), [\xi_1 : \dots : \xi_n])$$

*homogeneous coordinates*, where  $[\xi_1 : \dots : \xi_n]$  are homogeneous coordinates of the dual projective space  $P(\mathbb{R}^{n-1})^*$ .

It is easy to show that  $X \in K_{(x, [\xi])}$  if and only if  $\sum_{i=1}^n \mu_i \xi_i = 0$ , where  $d\tilde{\pi}(X) = \sum_{i=1}^n \mu_i \frac{\partial}{\partial x_i}$ . An immersion  $i : L \rightarrow PT^*(M)$  is said to be a *Legendrian immersion* if  $\dim L = n$  and  $di_q(T_q L) \subset K_{i(q)}$  for any  $q \in L$ . We also call the map  $\pi \circ i$  the *Legendrian map* and the set  $W(i) = \text{image } \pi \circ i$  the *wave front* of  $i$ . Moreover,  $i$  (or, the image of  $i$ ) is called the *Legendrian lift* of  $W(i)$ .

The main tool of the theory of Legendrian singularities is the notion of generating families. Here we only consider local properties, we may assume that  $M = \mathbb{R}^n$ . Let  $F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \rightarrow (\mathbb{R}, \mathbf{0})$  be a function germ. We say that  $F$  is a *Morse family* if the mapping

$$\Delta^* F = \left( F, \frac{\partial F}{\partial q_1}, \dots, \frac{\partial F}{\partial q_k} \right) : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \rightarrow (\mathbb{R} \times \mathbb{R}^k, \mathbf{0})$$

is non-singular, where  $(q, x) = (q_1, \dots, q_k, x_1, \dots, x_n) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ . In this case we have a smooth  $(n-1)$ -dimensional submanifold

$$\Sigma_*(F) = \left\{ (q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \mid F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \dots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\}$$

and the map germ  $\mathcal{L}(F) : (\Sigma_*(F), \mathbf{0}) \rightarrow PT^*\mathbb{R}^n$  defined by

$$\mathcal{L}(F)(q, x) = \left( x, \left[ \frac{\partial F}{\partial x_1}(q, x) : \dots : \frac{\partial F}{\partial x_n}(q, x) \right] \right)$$

is a Legendrian immersion germ. Then we have the following fundamental theorem of Arnol'd-Zakalyukin [1, 31].

**Proposition B.1** *All Legendrian submanifold germs in  $PT^*\mathbb{R}^n$  are constructed by the above method.*

We call  $F$  a *generating family* of  $\mathcal{L}(F)(\Sigma_*(F))$ . Therefore the wave front is

$$W(\mathcal{L}(F)) = \left\{ x \in \mathbb{R}^n \mid \text{there exists } q \in \mathbb{R}^k \text{ such that } F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \dots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\}.$$

We sometime denote  $\mathcal{D}_F = W(\mathcal{L}(F))$  and call it the *discriminant set* of  $F$ .

On the other hand, for any map  $f : N \rightarrow P$ , we denote by  $\Sigma(f)$  the set of singular points of  $f$  and  $D(f) = f(\Sigma(f))$ . In this case we call  $f|_{\Sigma(f)} : \Sigma(f) \rightarrow D(f)$  the *critical part* of the mapping  $f$ . For any Morse family  $F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \rightarrow (\mathbb{R}, \mathbf{0})$ ,  $(F^{-1}(0), \mathbf{0})$  is a smooth hypersurface, so we define a smooth map germ  $\pi_F : (F^{-1}(0), \mathbf{0}) \rightarrow (\mathbb{R}, 0)$  by  $\pi_F(q, x) = x$ . We can easily show that  $\Sigma_*(F) = \Sigma(\pi_F)$ . Therefore, the corresponding Legendrian map  $\pi \circ \mathcal{L}(F)$  is the critical part of  $\pi_F$ .

We now introduce an equivalence relation among Legendrian immersion germs. Let  $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$  and  $i' : (L', p') \subset (PT^*\mathbb{R}^n, p')$  be Legendrian immersion germs. Then we say that  $i$  and  $i'$  are *Legendrian equivalent* if there exists a contact diffeomorphism germ



$H : (PT^*\mathbb{R}^n, p) \longrightarrow (PT^*\mathbb{R}^n, p')$  such that  $H$  preserves fibers of  $\pi$  and that  $H(L) = L'$ . A Legendrian immersion germ  $i : (L, p) \subset PT^*\mathbb{R}^n$  (or, a *Legendrian map*  $\pi \circ i$ ) at a point is said to be *Legendrian stable* if for every map with the given germ there is a neighborhood in the space of Legendrian immersions (in the Whitney  $C^\infty$  topology) and a neighborhood of the original point such that each Legendrian immersion belonging to the first neighborhood has in the second neighborhood a point at which its germ is Legendrian equivalent to the original germ.

Since the Legendrian lift  $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$  is uniquely determined on the regular part of the wave front  $W(i)$ , we have the following simple but significant property of Legendrian immersion germs:

**Proposition B.2** *Let  $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$  and  $i' : (L', p') \subset (PT^*\mathbb{R}^n, p')$  be Legendrian immersion germs such that regular sets of  $\pi \circ i, \pi \circ i'$  are dense respectively. Then  $i, i'$  are Legendrian equivalent if and only if wave front sets  $W(i), W(i')$  are diffeomorphic as set germs.*

This result has been firstly pointed out by Zakalyukin [32]. The assumption in the above proposition is a generic condition for  $i, i'$ . Specially, if  $i, i'$  are Legendrian stable, then these satisfy the assumption.

We can interpret the Legendrian equivalence by using the notion of generating families. We denote  $\mathcal{E}_n$  the local ring of function germs  $(\mathbb{R}^n, \mathbf{0}) \longrightarrow \mathbb{R}$  with the unique maximal ideal  $\mathfrak{M}_n = \{h \in \mathcal{E}_n \mid h(0) = 0\}$ . Let  $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$  be function germs. We say that  $F$  and  $G$  are *P-K-equivalent* if there exists a diffeomorphism germ  $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$  of the form  $\Psi(x, u) = (\psi_1(q, x), \psi_2(x))$  for  $(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$  such that  $\Psi^*(\langle F \rangle_{\mathcal{E}_{k+n}}) = \langle G \rangle_{\mathcal{E}_{k+n}}$ . Here  $\Psi^* : \mathcal{E}_{k+n} \longrightarrow \mathcal{E}_{k+n}$  is the pull back  $\mathbb{R}$ -algebra isomorphism defined by  $\Psi^*(h) = h \circ \Psi$ . If  $n = 0$ , we simply say these germs are *K-equivalent*.

Let  $F : (\mathbb{R}^k \times \mathbb{R}^3, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$  be a function germ. We say that  $F$  is a *K-versal deformation* of  $f = F|_{\mathbb{R}^k \times \{\mathbf{0}\}}$  if

$$\mathcal{E}_k = T_e(\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial x_1} \Big|_{\mathbb{R}^k \times \{\mathbf{0}\}}, \dots, \frac{\partial F}{\partial x_n} \Big|_{\mathbb{R}^k \times \{\mathbf{0}\}} \right\rangle_{\mathbb{R}},$$

where

$$T_e(\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \dots, \frac{\partial f}{\partial q_k}, f \right\rangle_{\mathcal{E}_k}.$$

(See [20].)

The main result in Arnol'd-Zakalyukin's theory [1, 31] is the following:

**Theorem B.3** *Let  $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$  be Morse families. Then*

- (1)  $\mathcal{L}(F)$  and  $\mathcal{L}(G)$  are Legendrian equivalent if and only if  $F, G$  are *P-K-equivalent*.
- (2)  $\mathcal{L}(F)$  is Legendrian stable if and only if  $F$  is a *K-versal deformation* of  $F|_{\mathbb{R}^k \times \{\mathbf{0}\}}$ .

Since  $F, G$  are function germs on the common space germ  $(\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ , we do not need the notion of stably *P-K-equivalences* under this situation (cf., [1]). By the uniqueness result of the *K-versal deformation* of a function germ, Proposition B.2 and Theorem B.3, we have the following classification result of Legendrian stable germs. For any map germ  $f : (\mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^p, \mathbf{0})$ , we define the *local ring of f* by  $Q(f) = \mathcal{E}_n / f^*(\mathfrak{M}_p)\mathcal{E}_n$ .

**Proposition B.4** Let  $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$  be Morse families. Suppose that  $\mathcal{L}(F), \mathcal{L}(G)$  are Legendrian stable. The the following conditions are equivalent.

- (1)  $(W(\mathcal{L}(F)), \mathbf{0})$  and  $(W(\mathcal{L}(G)), \mathbf{0})$  are diffeomorphic as germs.
- (2)  $\mathcal{L}(F)$  and  $\mathcal{L}(G)$  are Legendrian equivalent.
- (3)  $Q(f)$  and  $Q(g)$  are isomorphic as  $\mathbb{R}$ -algebras, where  $f = F|_{\mathbb{R}^k \times \{\mathbf{0}\}}$ ,  $g = G|_{\mathbb{R}^k \times \{\mathbf{0}\}}$ .

*Proof.* Since  $\mathcal{L}(F), \mathcal{L}(G)$  are Legendrian stable, these satisfy the generic condition of Proposition B.2, so that the conditions (1) and (2) are equivalent. The condition (3) implies that  $f, g$  are  $\mathcal{K}$ -equivalent [20, 21]. By the uniqueness of the  $\mathcal{K}$ -versal deformation of a function germ,  $F, G$  are  $P\mathcal{K}$ -equivalent. This means that the condition (2) holds. By Theorem B.3, the condition (2) implies the condition (3).  $\square$

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