Variational approach for identifying the coefficient of wave equation

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Abstract

An inverse boundary value problem for identifying the coefficient of some second order hyperbolic equation by one boundary measurement is considered. The problem is transformed to a minimization problem of a functional. By computing the Gateaux derivative of the functional, an algorithm for identifying the coefficient is given based on the projected gradient method. A numerical result is given testing the algorithm.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3) be a bounded domain with smooth boundary $\partial\Omega$. Let $K(x) \in L^\infty(\Omega)$ satisfy

$$\begin{cases} 0 < C_1 \leq K(x) \leq C_2 & \text{in} \quad \bar{\Omega} \\ K(x) \text{ is } C^\infty & \text{in} \quad \Omega \setminus F \\ |\nabla K(x)| \leq C_3 & \text{in} \quad \Omega \setminus F, \end{cases} \tag{1}$$

where C_1, C_2, C_3 are fixed positive constants and $F \subset \Omega$ is a compact set with $\partial\Omega \cap F = \emptyset$. For a given $\bar{u} \in C^6([0,T]; H_{(\frac{5}{2})}(\partial\Omega))$ with $\partial_t^i \bar{u}(x,0) = 0$ ($x \in \Omega, 0 \leq i \leq 5$), we consider an initial boundary value problem :

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \nabla \cdot (K(x)\nabla u) = 0 & \text{in} \quad \Omega \times (0,T) \\ u(0) = 0, \quad \dfrac{\partial u}{\partial t}(0) = 0 & \text{in} \quad \Omega \\ u = \bar{u} & \text{on} \quad \partial\Omega \times (0,T). \end{cases} \tag{2}$$

Here $H_{(\frac{5}{2})}(\partial\Omega)$ is the Sobolev space of order $\dfrac{5}{2}$ defined over $\partial\Omega$ and $C^m([0,T]; X)$ with $m \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ denotes the set of all C^m class functions defined in $[0,T]$ taking their values in some Banach space X.

(2) admits a unique solution $u \in \bar{H}_{(5)}((0,T); \bar{H}_{(1)}(\Omega))$. We denote this u by $u = u[K](x,t)$ to clarify the dependency on K.

Here and hereafter, we define $\bar{H}_{(s)}(\Omega)$ for $s \in \mathbb{R}$ by $g \in \bar{H}_{(s)}(\Omega)$ if and only if there exists an extension $f \in H_{(s)}(\mathbb{R}^n)$ of g to the ambient space $\mathbb{R}^n$ of Ω and the norm $||g||_{\bar{H}_{(s)}(\Omega)}$ is defined by $||g||_{\bar{H}_{(s)}(\Omega)} := \inf \{||f||_{H_{(s)}(\mathbb{R}^n)}; f|_\Omega = g\}$. We also define $\dot{H}_{(s)}(\Omega) := \{g \in \bar{H}_{(s)}(\Omega); \text{supp } g \subset \bar{\Omega}\}$ with norm $||g||_{\dot{H}_{(s)}(\Omega)} := ||g||_{\bar{H}_{(s)}(\Omega)}$. These kind of Sobolev spaces are discussed systematically in [3].

Now, suppose we do not know $K(x)$, but we are given $\bar{q} := K(x)\dfrac{\partial u}{\partial n}$ on $\partial\Omega \times (0,T)$ beside $\bar{u}$ for T large enough. Then, we are interested in the following inverse problem (IP):

Inverse problem (IP): Reconstruct $K(x)$ from $\{\bar{u}, \bar{q}\}$.

Let $\mathcal{K}$ be the set of all $K(x) \in L^\infty(\Omega)$ satisfying (1). For any $L \in \mathcal{K}$, we define $J(L)$ by

$$J(L) = \int_0^T \int_{\partial\Omega} |L(x)\frac{\partial u[L]}{\partial n} - \bar{q}|^2 \, d\sigma dt,$$

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where $u[L]$ is the solution of (2) for $K = L$.

Since the absolute minimum of $J(L)$ is attained when $L(x) = K(x)$, we can expect to recover $K(x)$ by minimizing $J(L)$.

One of our coauthor Shirota started the numerical study of the inverse problem (IP) in his paper [9]. He used the projected gradient method to minimize the functional $J(L)$. There are many methods for minimizing $J(L)$. The projected gradient method is one of them. It needs to compute the Gateaux derivative $J'(L)$ of $J(L)$. Shirota computed $J'(L)$ approximately by a formal argument. Some of the numerical results in [9] were quite good.

The aim of this paper is to justify his formal argument, give the complete form of $J'(L)$ and provide some numerical results using $J'(L)$.

The complete form of $J'(L)$ is given by

Theorem 1.1
$$\begin{cases} J(L+\varepsilon M) - J(L) = \varepsilon J'(L)M + o(\varepsilon) \\ \\ J'(L)M = \displaystyle\int_0^T \int_\Omega M\nabla u[L]\cdot\nabla v dx dt + \int_\Omega \frac{\partial U}{\partial t}(T) w dx, \end{cases}$$
where $w \in \bar{H}_{(1)}(\Omega)$ is the weak solution of the elliptic equation
$$\begin{cases} \nabla\cdot(L\nabla w) = 0 & in \quad \Omega \\ w = 2\left(L\dfrac{\partial u}{\partial n}(T) - \bar{q}(T)\right) & on \quad \partial\Omega \end{cases} \tag{3}$$
and $v \in L^2((0,T);\bar{H}_{(1)}(\Omega))$ is the weak solution of the equation
$$\begin{cases} \dfrac{\partial^2}{\partial t^2}v - \nabla\cdot(L\nabla v) = 0 & in \quad \Omega\times(0,T) \\ v(T) = 0, \quad \dfrac{\partial v}{\partial t}(T) = 0 & in \quad \Omega \\ v = 2\left(L\dfrac{\partial u}{\partial n} - \bar{q}\right) & on \quad \partial\Omega\times(0,T) \end{cases} \tag{4}$$
and $U \in \bar{H}_{(5)}((0,T);L^2(\Omega))$ is the weak solution of the equation
$$\begin{cases} \dfrac{\partial^2}{\partial t^2}U - \nabla\cdot(L\nabla U) = \nabla\cdot(M\nabla u[L]) & in \quad \Omega\times(0,T) \\ U(0) = 0, \quad \dfrac{\partial U}{\partial t}(0) = 0 & in \quad \Omega \\ U = 0 & on \quad \partial\Omega\times(0,T). \end{cases} \tag{5}$$

Remark 1.2 *In the proof of Theorem 1.1 given later, we showed that the mapping $\Phi : L^\infty(\Omega) \ni M \mapsto \int_\Omega \frac{\partial U}{\partial t}(T) w\, dx \in \mathbb{R}$ is bounded linear. Hence, by Example 5 in page 118 of [12], there exists a unique $h \in L^1(\Omega)$ such that*
$$\int_\Omega \frac{\partial U}{\partial t}(T) w\, dx = \int_\Omega Mh\, dx. \tag{6}$$

The proof of this theorem is given in section 5, the existence of u, v and w are given in Appendix and that of U given in section 3.

To the best of our knowledge, there is not any paper other than [9] which tried projected gradient method to obtain some good numerical results for the inverse problem (IP). The inverse problem (IP) is only a prototype. The same method can be applied to similar inverse problem with different equation. When we consider an elastic equation, the problem becomes more practical because it really models the nondestructive testing of a material using ultrasound.

The rest of our paper is organized as follows. In section 2, we will show preliminary computation of the variation of $J(L)$ with respect to L as an intermediate step to get the complete form of the Gateaux derivative $J'(L)$ of $J(L)$. In section 3, we will present some theorems, which play a major role to prove Theorem 1.1. In sections 4 and 5, using the theorems given in section 3, we will prove Theorem 1.1. Finally in the last section, we show a numerical algorithm based on the complete form of $J'(L)$ and its example.

2 Preliminary computation for $J'(L)$

In this section, we prove

Lemma 2.1

$$\left\{\begin{array}{l} J(L+\varepsilon M)-J(L) \\ \qquad =\varepsilon\displaystyle\int_0^T\int_\Omega M\nabla u[L]\cdot\nabla v dxdt+\int_\Omega\frac{\partial\delta u}{\partial t}(T)wdx \\ \qquad +\displaystyle\int_0^T\int_\Omega \varepsilon M\nabla\delta u\cdot\nabla v dxdt+\int_0^T\int_{\partial\Omega}|\delta q|^2 d\sigma dt, \end{array}\right. \tag{7}$$

where $\delta u=u[L+\varepsilon M]-u[L]$, $\delta q=(L+\varepsilon M)\dfrac{\partial u[L+\varepsilon M]}{\partial n}-L\dfrac{\partial u[L]}{\partial n}$,
$w\in\bar{H}_{(1)}(\Omega)$ *is the weak solution of the elliptic equation*

$$\left\{\begin{array}{lll} \nabla\cdot(L\nabla w)=0 & in & \Omega \\ w=2\left(L\dfrac{\partial u}{\partial n}(T)-\bar{q}(T)\right) & on & \partial\Omega, \end{array}\right. \tag{8}$$

and $v\in L^2((0,T);\bar{H}_{(1)}(\Omega))$ *is the weak solution of the equation*

$$\left\{\begin{array}{lll} \dfrac{\partial^2}{\partial t^2}v-\nabla\cdot(L\nabla v)=0 & in & \Omega\times(0,T) \\ v(T)=0,\quad \dfrac{\partial v}{\partial t}(T)=0 & in & \Omega \\ v=2\left(L\dfrac{\partial u}{\partial n}-\bar{q}\right) & on & \partial\Omega\times(0,T). \end{array}\right. \tag{9}$$

Proof Let $q[L]:=L\dfrac{\partial u[L]}{\partial n}\Big|_{\partial\Omega\times(0,T)}$. For $L,M\in\mathcal{K}$,

$$\left\{\begin{array}{l} J(L+\varepsilon M)-J(L) \\ \qquad =\displaystyle\int_0^T\int_{\partial\Omega}[(q[L+\varepsilon M]-\bar{q})^2-(q[L]-\bar{q})^2]d\sigma dt \\ \qquad =\displaystyle\int_0^T\int_{\partial\Omega}(q[L+\varepsilon M]+q[L]-2\bar{q})\delta q d\sigma dt \\ \qquad =\displaystyle\int_0^T\int_{\partial\Omega}2(q[L]-\bar{q})\delta q d\sigma+\int_0^T\int_{\partial\Omega}|\delta q|^2 d\sigma dt. \end{array}\right.$$

Integrating by parts and reminding $\dfrac{\partial u[L]}{\partial t}(0)=0$ and $v(T)=w$,

$$\left\{\begin{array}{l} \displaystyle\int_0^T\int_\Omega\frac{\partial u[L]}{\partial t}\frac{\partial v}{\partial t}dxdt=\int_\Omega\left(\left[\frac{\partial u[L]}{\partial t}v\right]_0^T-\int_0^T\frac{\partial^2 u[L]}{\partial t^2}vdt\right)dx \\ \qquad =\displaystyle\int_\Omega\frac{\partial u[L]}{\partial t}(T)wdx-\int_0^T\int_\Omega\frac{\partial^2 u[L]}{\partial t^2}vdxdt. \end{array}\right.$$

By another integration by parts,

$$\left\{\begin{array}{l} \displaystyle\int_0^T\int_\Omega L\nabla u[L]\cdot\nabla v dxdt \\ \qquad =\displaystyle\int_0^T\left(\int_{\partial\Omega}vq[L]d\sigma-\int_\Omega\nabla\cdot(L\nabla u[L])vdx\right)dt. \end{array}\right.$$

Hence,

$$\left\{\begin{array}{l} \displaystyle\int_0^T\int_\Omega\left(\frac{\partial u[L]}{\partial t}\frac{\partial v}{\partial t}-L\nabla u\cdot\nabla v\right)dxdt \\ \qquad =\displaystyle\int_\Omega\frac{\partial u[L]}{\partial t}(T)wdx-\int_0^T\int_{\partial\Omega}vq[L]d\sigma dt. \end{array}\right. \tag{10}$$

By the similar way, we have

$$\begin{cases} \displaystyle\int_0^T \int_\Omega \left(\frac{\partial u[L+\varepsilon M]}{\partial t} \frac{\partial v}{\partial t} - (L+\varepsilon M)\nabla u[L+\varepsilon M] \cdot \nabla v \right) dxdt \\ \displaystyle\qquad = \int_\Omega \frac{\partial u[L+\varepsilon M]}{\partial t}(T) w dx - \int_0^T \int_{\partial\Omega} vq[L+\varepsilon M] d\sigma dt. \end{cases} \tag{11}$$

Using (10) and (11), we get

$$\begin{cases} \displaystyle\int_0^T \int_\Omega \left[\frac{\partial \delta u}{\partial t} \frac{\partial v}{\partial t} - ((L+\varepsilon M)\nabla u[L+\varepsilon M] - L\nabla u[L] \cdot \nabla v) \right] dxdt \\ \displaystyle\qquad = \int_\Omega \frac{\partial \delta u}{\partial t}(T) w dx - \int_0^T \int_{\partial\Omega} v\delta q d\sigma dt. \end{cases} \tag{12}$$

Moreover, applying the integration by parts, we have

$$\begin{cases} \displaystyle\int_0^T \int_\Omega \frac{\partial \delta u}{\partial t} \frac{\partial v}{\partial t} dxdt = \int_\Omega \left(\left[\delta u \frac{\partial v}{\partial t} \right]_0^T - \int_0^T \delta u \frac{\partial^2 v}{\partial t^2} dt \right) dx \\ \displaystyle\qquad = -\int_0^T \int_\Omega \delta u \frac{\partial^2 v}{\partial t^2} dxdt, \end{cases}$$

$$\begin{cases} \displaystyle\int_0^T \int_\Omega L\nabla \delta u \cdot \nabla v dxdt \\ \displaystyle\qquad = \int_0^T \left[\int_{\partial\Omega} \delta u \left(L\frac{\partial v}{\partial n} \right) d\sigma - \int_\Omega \delta u \nabla \cdot (L\nabla v) dx \right] dt \\ \displaystyle\qquad = -\int_0^T \int_\Omega \delta u \nabla \cdot (L\nabla v) dxdt. \end{cases} \tag{13}$$

Hence,

$$\int_0^T \int_\Omega \left(\frac{\partial \delta u}{\partial t} \frac{\partial v}{\partial t} - L\nabla \delta u \cdot \nabla v \right) dxdt = 0. \tag{14}$$

Therefore, from (12) and (14),

$$\begin{cases} \displaystyle -\int_0^T \int_\Omega \varepsilon M \nabla u[L+\varepsilon M] \cdot \nabla v dxdt \\ \displaystyle\qquad = \int_\Omega \frac{\partial \delta u}{\partial t}(T) w dx - \int_0^T \int_{\partial\Omega} v\delta q d\sigma dt. \end{cases}$$

Reminding $v|_{\partial\Omega\times(0,T)} = 2(q[L] - \bar{q})$,

$$\begin{cases} \displaystyle\int_0^T \int_{\partial\Omega} 2(q[L]-\bar{q})\delta q d\sigma dt = \int_0^T \int_\Omega \varepsilon M \nabla u \cdot \nabla v dxdt \\ \displaystyle\qquad + \int_0^T \int_\Omega \varepsilon M \nabla \delta u \cdot \nabla v dxdt + \int_\Omega \frac{\partial \delta u}{\partial t}(T) w dx. \end{cases} \tag{15}$$

Then, substituting (15) into (13), we get (7). □

3 Continuous dependence of the solution on the coefficient

Fix $M \in \mathcal{K}$ and take a small $\varepsilon > 0$. We write $L_\varepsilon(x) := L(x) + \varepsilon M(x)$,

$$\mathcal{A}_L := \frac{\partial^2}{\partial t^2} - A_L,$$

$$\mathcal{A}_{L_\varepsilon} := \frac{\partial^2}{\partial t^2} - A_{L_\varepsilon},$$

where the notation A_L, A_{L_ε} are defined as in (37) of Appendix.

Let $v \in \bar{H}_{(5)}((0,T);\dot{H}_{(1)}(\bar{\Omega}))$ and $v_\varepsilon \in \bar{H}_{(5)}((0,T);\dot{H}_{(1)}(\bar{\Omega}))$ be the solutions to

$$\begin{cases} \mathcal{A}_L v = f(L) \in C^4([0,T];\bar{H}_{(1)}(\Omega)) & \text{in} \quad \Omega\times(0,T) \\ v = 0 & \text{on} \quad \partial\Omega\times(0,T) \\ v(0) = \widetilde{u_0}, \quad \dfrac{\partial}{\partial t}v(0) = \widetilde{u_1} \end{cases} \tag{16}$$

and

$$\begin{cases} \mathcal{A}_{L_\varepsilon} v_\varepsilon = f(L_\varepsilon) \in C^4([0,T];\bar{H}_{(1)}(\Omega)) & \text{in} \quad \Omega\times(0,T) \\ v_\varepsilon = 0 & \text{on} \quad \partial\Omega\times(0,T) \\ v_\epsilon(0) = \widetilde{u_0}, \quad \dfrac{\partial}{\partial t}v_\varepsilon(0) = \widetilde{u_1}, \end{cases} \tag{17}$$

respectively. Here, $f(L)$, $f(L_\varepsilon)$, $\widetilde{u_0}$ and $\widetilde{u_1}$ are defined by the formula given in (40) of Appendix.

Then, we the following continuous dependency of the solution on the coefficient in a Gelfand triple (V,H,V'), which is defined as in (40) of Appendix.

Theorem 3.1

$v_\varepsilon \to v$ $(\varepsilon \to 0)$ *in* $C([0,T];V)\cap C^1([0,T];H)$

Proof If the inhomogeneous terms of the equations (16) and (17) are same, the continuous dependency of the solution on the coefficient is given as Theorem 2.8.1 and Theorem 2.8.2 in [10]. The proof for them can be also applied to the present situation without any essential change. So we omit giving further details of the proof. □

Now, for given $u_0 \in H$, $u_1 \in V'$ and $f \in L^2((0,T);V')$, we consider the Cauchy problem :

$$\begin{cases} \mathcal{A}_K u = f & \text{in} \quad \Omega\times(0,T) \\ u(0) = u_0, \quad \dfrac{\partial}{\partial t}u(0) = u_1 & \text{in} \quad \Omega. \end{cases} \tag{18}$$

In order to define a weak solution u to (18), we first define the test function space X.

Definition 3.2 *(test function space) The test function space X is the set of all $\varphi \in L^2((0,T);V)$ satisfying*

$$\begin{cases} \mathcal{A}_K\varphi \in L^2((0,T);H) \\ \varphi(T) = \dfrac{\partial\varphi}{\partial t}(T) = 0. \end{cases}$$

The definition of the weak solution is as follows.

Definition 3.3 $u \in L^2((0,T);H)$ *with* $u' \in L^2((0,T);V')$ *is called a weak solution of (18) if it satisfies*

$$\begin{cases} \displaystyle\int_0^T\int_\Omega u\mathcal{A}_K\varphi\,dxdt \\ \displaystyle\quad = \int_0^T\int_\Omega f\varphi\,dxdt + \int_\Omega\left(u_1\varphi(0) - u_0\frac{\partial\varphi}{\partial t}(0)\right)dx \quad \textit{for} \quad \forall\varphi\in X. \end{cases}$$

Then, we have the following existence and uniqueness result from [4].

Theorem 3.4 *Given* $(f,u_0,u_1) \in L^2((0,T);V')\times H\times V'$. *Then there is a unique weak solution u with* $\left(u,\dfrac{\partial u}{\partial t}\right) \in C([0,T];H)\times C([0,T];V')$.

For the continuous dependence of the solution to (18) on the coefficient, we have

Theorem 3.5 *Let v_ε and v be the weak solution of (18) with $K = L+\varepsilon M$ and $K = L$, respectively. Then,* $v_\varepsilon \to v$ $(\varepsilon\to 0)$ *in* $L^2((0,T);H)$ *and* $v'_\varepsilon \to v'$ $(\varepsilon \to 0)$ *in* $L^2((0,T);V')$.

Proof Let

$$E(t) := \frac{1}{2}(\langle v, v\rangle_H + \langle A_L^{-1}v', v'\rangle_H)$$

and

$$E_\varepsilon(t) := \frac{1}{2}(\langle v_\varepsilon, v_\varepsilon\rangle_H + \langle A_L^{-1}v_\varepsilon', v_\varepsilon'\rangle_H)\,.$$

Then, from the proof of Theorem 9.3 of chapter 3 in [4] and the same argument with Lemma 2.4.1 in [10], we have

$$E(t),\, E_\varepsilon(t) \le C\left(||u_0||_H^2 + ||u_1||_{V'}^2 + \int_0^T ||f||_{V'}^2\right) \tag{19}$$

and

$$\begin{cases} E(t) - E(0) = \displaystyle\int_0^t \langle A_L^{-1} f, v'\rangle_H \\ E_\varepsilon(t) - E_\varepsilon(0) = \displaystyle\int_0^t \langle A_{L_\varepsilon}^{-1} f, v_\varepsilon'\rangle_H, \end{cases} \tag{20}$$

respectively.
(1) Weak convergence of v_ε to v.
From the coercivity in (36), we have

$$E_\varepsilon(t) \ge C(||v_\varepsilon||_H^2 + ||v_\varepsilon'||_{V'}^2)$$

So, by (19), v_ε and v_ε' is uniformly bounded independent of ε in $L^2((0,T);H)$, $L^2((0,T);V')$, respectively. Therefore, by the weak compactness of these spaces, there are $\{\varepsilon(l)\}$, which is the subsequence of $\{\varepsilon\}$, $\alpha \in L^2((0,T);H)$ and $\beta \in L^2((0,T);V')$ satisfying

$$\begin{cases} v_{\varepsilon(l)} \rightharpoonup \alpha & \text{in} \quad L^2((0,T);H) \\ v_{\varepsilon(l)}' \rightharpoonup \beta & \text{in} \quad L^2((0,T);V'). \end{cases} \tag{21}$$

Furthermore, $\beta = \alpha'$, where $'$ means the derivative in distribution sense.

Now, we will prove this α is same with v by showing

$$\begin{cases} \displaystyle\int_0^T \int_\Omega \alpha \mathcal{A}_L \varphi \, dxdt \\ \quad = \displaystyle\int_0^T \int_\Omega f\varphi \, dxdt + \int_\Omega \left(u_1\varphi(0) - u_0 \frac{\partial \varphi}{\partial t}(0)\right) dx \quad \text{for} \quad \forall \varphi \in X. \end{cases} \tag{22}$$

Let $g := \mathcal{A}_L\varphi \in L^2((0,T);H)$ and extend g to the whole time interval by putting $g \equiv 0$ in $(-\infty, 0] \cup [T, \infty)$. Also, let $g^l := \chi_{[0,T-\frac{1}{l}]} g \in L^2((0,T);H)$, with the characteristic function $\chi_{[0,T-\frac{1}{l}]}$ of $[0, T-\frac{1}{l}]$ and $g^{l,m} := \rho_m * g^l \in C^\infty([0,T], H)$, where ρ_m is a mollifier $\rho_m(t) := m^{-1}\rho(m^{-1}t)$ with $\rho \in C_0^\infty(\mathbb{R})$ satisfying $0 \le \rho \le 1$, $\displaystyle\int_{\mathbb{R}} \rho(t)\, dt = 1$. Then, $g^{l,m}$ is flat at T and we have

$$\begin{cases} g^{l,m} \to g^l \quad (m \to 0) & \text{in} \quad L^2((0,T);H) \\ g^l \to g \quad (l \to 0) & \text{in} \quad L^2((0,T);H). \end{cases} \tag{23}$$

Taking $t = T$ as an initial surface and $g^{l,m}$ is flat at $t = T$ into account, we have from Theorem A.1 that there exist a unique $\varphi^{l,m} \in L^2((0,T);V)$ with enough time regularity to

$$\begin{cases} \mathcal{A}_L \varphi^{l,m} = g^{l,m} \\ \varphi^{l,m}(T) = (\varphi^{l,m})'(T) = 0. \end{cases}$$

By defining $\varphi_\varepsilon^{l,m} := A_{L_\varepsilon}^{-1} A_L \varphi^{l,m}$, it has enough time regularity and satisfies

$$\begin{cases} \mathcal{A}_{L_\varepsilon} \varphi_\varepsilon^{l,m} = A_{L_\varepsilon}^{-1} A_L (\varphi^{l,m})'' + A_L \varphi^{l,m} \in L^2((0,T);H) \\ \varphi_\varepsilon^{l,m}(T) = (\varphi_\varepsilon^{l,m})'(T) = 0, \end{cases}$$

Hence, $\varphi_\varepsilon^{l,m}$ is a test function. By (17) and taking $u = v_\varepsilon$ in (3.3),

$$\left\{ \begin{aligned} &\int_0^T \int_\Omega v_\varepsilon((\varphi_\varepsilon^{l,m})'' + A_{L_\varepsilon}\varphi_\varepsilon^{l,m})\, dxdt \\ &\quad = \int_0^T \int_\Omega f\varphi_\varepsilon^{l,m}\, dxdt + \int_\Omega \left(u_1\varphi_\varepsilon^{l,m}(0) - u_0(\varphi_\varepsilon^{l,m})'(0)\right)\, dx. \end{aligned} \right.$$

From $A_L - A_{L_\varepsilon} = A_{L_\varepsilon}(A_{L_\varepsilon}^{-1} - A_L^{-1})A_L$, we have

$$\left\{ \begin{aligned} &||\varphi_\varepsilon^{l,m} - \varphi^{l,m}||_V \\ &\quad = ||A_{L_\varepsilon}^{-1}A_L\varphi^{l,m} - \varphi^{l,m}||_V \;=\; ||(A_{L_\varepsilon}^{-1} - A_L^{-1})A_L\varphi^{l,m}||_V \\ &\quad = ||A_{L_\varepsilon}^{-1}(A_L - A_{L_\varepsilon})\varphi^{l,m}||_V \;\leq\; C||(A_L - A_{L_\varepsilon})\varphi^{l,m}||_{V'} \\ &\quad \to 0 \quad (\varepsilon \to 0) \quad \text{uniformly with respect to} \quad t \in [0,T]. \end{aligned} \right. \tag{24}$$

Similarly, we can show

$$||(\varphi_\varepsilon^{l,m})' - (\varphi^{l,m})'||_V,\; ||(\varphi_\varepsilon^{l,m})'' - (\varphi^{l,m})''||_V \to 0. \tag{25}$$

Using (21), (24), (25) and uniform boundedness of $\{v_\varepsilon\}$, we reach

$$\left\{ \begin{aligned} &\int_0^T \int_\Omega \alpha((\varphi^{l,m})'' + A_{L_\varepsilon}\varphi^{l,m})\, dxdt \\ &\quad = \int_0^T \int_\Omega f\varphi^{l,m}\, dxdt + \int_\Omega \left(u_1\varphi^{l,m}(0) - u_0(\varphi^{l,m})'(0)\right)\, dx. \end{aligned} \right.$$

Furthermore, by (23) and the continuous dependency on the source term in Theorem A.1, we get (22).

Also, we can show v_ε itself converges to v weakly. In fact, if not, then, we can find $\varepsilon > 0$, one subsequence $\{v_{\varepsilon(m)}\}$ and $\varphi \in X$ satisfying

$$\int_0^T \langle v - v_{\varepsilon(m)}, \varphi\rangle_H > \epsilon. \tag{26}$$

However, $\{v_{\varepsilon(m)}\}$ is also uniformly bounded in $L^2((0,T);H)$. As a result, it has convergent subsequence to v, which is contradiction to (26).

(2) Strong convergence of v_ε to v.
Reminding $\langle A_{L_\varepsilon}^{-1}f, v_\varepsilon'\rangle_H - \langle A_L^{-1}f, v'\rangle_H = \langle A_{L_\varepsilon}^{-1}f, v_\varepsilon' - v'\rangle_H + \langle (A_{L_\varepsilon}^{-1} - A_L^{-1})f, v_\varepsilon'\rangle_H$ and $A_{L_\varepsilon}^{-1}(A_L - A_{L_\varepsilon})A_L^{-1} = (A_{L_\varepsilon}^{-1} - A_L^{-1})$, we can show in (20)

$$E_\varepsilon(0) \to E(0), \quad \int_0^t \langle A_{L_\varepsilon}^{-1}f, v_\varepsilon'\rangle_H\, dt \to \int_0^t \langle A_L^{-1}f, v'\rangle_H\, dt \quad (\varepsilon \to 0).$$

As a result, $E_\varepsilon(t) \to E(t)$ $(\varepsilon \to 0)$.

Now, let $\xi(t) = \langle v_\varepsilon - v, v_\varepsilon - v\rangle_H + \langle A_L^{-1}(v_\varepsilon' - v'), v_\varepsilon' - v'\rangle_H$. Then, by expanding the right side of $\xi(t)$, we have

$$\xi(t) = 2E(t) + 2E_\varepsilon(t) - \langle (A_{L_\varepsilon}^{-1} - A_L^{-1})v_\varepsilon', v_\varepsilon'\rangle_H - 2(\langle v, v_\varepsilon\rangle_H + \langle A_L^{-1}v', v_\varepsilon'\rangle_H). \tag{27}$$

On the other hand, by coercivity (36),

$$\xi(t) \geq C(||v_\varepsilon - v||_H^2 + ||v_\varepsilon' - v'||_{V'}^2). \tag{28}$$

Therefore, using uniform boundedness of $\{v_\varepsilon\}$ and $\{v\}$, (21), (27) and (28), we get the strong convergence of v_ε and v_ε' to v and v' in $L^2((0,T);H)$, $L^2((0,T);V')$, respectively. □

4 Asymptotic of $\varepsilon^{-1}\dfrac{\partial \delta u}{\partial t}(T)\ (\varepsilon \to 0)$

This section is devoted to the following theorem which gives a representation formula of the second term on the right side in (7).

Theorem 4.1

$$\int_\Omega \frac{\partial \delta u}{\partial t}(T) w\, dx \; = \; \varepsilon \int_\Omega \frac{\partial U}{\partial t}(T) w\, dx + o(\varepsilon),$$

where U is the weak solution of

$$\begin{cases} \mathcal{A}_L U \; = \nabla \cdot (M\nabla) u[L] & in \quad \Omega \times (0,T) \\ U(0) = \dfrac{\partial U}{\partial t}(0) = 0 & in \quad \Omega. \end{cases}$$

Proof Let $u_\varepsilon := u[L+\varepsilon M]$, $u := u[L]$. Then, likewise (39) and (40) in Appendix, we transform from u, u_ε into $\tilde{u}$, $\widetilde{u_\varepsilon}$ through $\phi := \Lambda \bar{u}$ with the inverse trace operator Λ given by (38), respectively, i.e. $\widetilde{u_\varepsilon} = u_\varepsilon - \phi$ and $\tilde{u} = u - \phi$. Then we have $\nabla \cdot (M\nabla) u = \nabla \cdot (M\nabla)\tilde{u} + \nabla \cdot (M\nabla)\phi \in \bar{H}^{(5)}((0,T);V')$, and $u_\varepsilon - u = \widetilde{u_\varepsilon} - \tilde{u}$.

By defining $U_\varepsilon := \dfrac{\widetilde{u_\varepsilon} - \tilde{u}}{\varepsilon}$, we have

$$\begin{cases} \mathcal{A}_{L_\varepsilon} U_\varepsilon = \dfrac{1}{\varepsilon}(\mathcal{A}_{L_\varepsilon}\widetilde{u_\varepsilon} - \mathcal{A}_{L_\varepsilon}\tilde{u}) \\ \quad = \dfrac{1}{\varepsilon}(\mathcal{A}_L \tilde{u} + f(L_\varepsilon) - f(L) - \mathcal{A}_{L_\varepsilon}\tilde{u}) \\ \quad = \nabla \cdot (M\nabla \tilde{u}) + \nabla \cdot (M\nabla\phi) \\ U_\varepsilon(0) = U'_\varepsilon(0) = 0. \end{cases}$$

Moreover, we have

$$\begin{cases} \mathcal{A}_{L_\varepsilon} \dfrac{\partial^i}{\partial t^i} U_\varepsilon \\ \quad = \dfrac{\partial^i}{\partial t^i}\left(\nabla \cdot (M\nabla\tilde{u}) + \nabla \cdot (M\nabla\phi)\right) \\ \dfrac{\partial^i}{\partial t^i} U_\varepsilon(0) = \dfrac{\partial^{i+1}}{\partial t^{i+1}} U_\varepsilon(0) = 0 \qquad (0 \le i \le 5). \end{cases}$$

Therefore, by Theorem 3.5, we have

$$U_\varepsilon \; \to \; U \ (\varepsilon \to 0) \text{ in } \bar{H}_{(5)}((0,T);H).$$

As a result, we have

$$\frac{\partial \delta u}{\partial t}(T) = \frac{\partial \widetilde{u_\varepsilon}}{\partial t}(T) - \frac{\partial \tilde{u}}{\partial t}(T) = \varepsilon \frac{\partial U}{\partial t}(T) + o(\varepsilon) \text{ in } H,$$

which completes the proof. □

5 Completion of the proof of Theorem 1.1

First, we show that the third and fourth terms on right side in (7) are $o(\varepsilon)$. That is

Theorem 5.1

$$\int_0^T \int_\Omega \epsilon M \nabla \delta u \cdot \nabla v dx dt + \int_0^T \int_{\partial\Omega} |\delta q|^2 d\sigma dt \; = \; o(\varepsilon) \ (\varepsilon \to 0).$$

Proof Let u, u_ε, $\tilde{u}$ and $\widetilde{u_\varepsilon}$ be those given in the proof of Theorem 4.1. Using Theorem 3.1, we can show easily the first term is $o(\varepsilon)$.

Let $z_\varepsilon := u_\varepsilon - u = \widetilde{u_\varepsilon} - \tilde{u} \in \bar{H}_{(5)}((0,T);\dot{H}_{(1)}(\bar{\Omega}))$. If we prove $||z_\varepsilon||_{L^2((0,T);\bar{H}_{(2)}(\Omega\setminus F))} = O(\varepsilon)$ $(\varepsilon \to 0)$, then the proof is done. To begin proving this, we first observe

$$\begin{cases} \mathcal{A}_L z_\varepsilon = \mathcal{A}_L \widetilde{u_\varepsilon} - \mathcal{A}_L \tilde{u} \\ \quad = \mathcal{A}_{L_\varepsilon} \widetilde{u_\varepsilon} + (\mathcal{A}_L - \mathcal{A}_{L_\varepsilon})\widetilde{u_\varepsilon} - \mathcal{A}_L \tilde{u} \\ \quad = (f(L_\varepsilon) - f(L)) + (\mathcal{A}_L - \mathcal{A}_{L_\varepsilon})\widetilde{u_\varepsilon} \\ \quad = \varepsilon(\nabla \cdot (M\nabla\phi) + \nabla \cdot (M\nabla\widetilde{u_\varepsilon})) \\ \quad = \varepsilon h \in \bar{H}_{(5)}((0,T);V') \cap \bar{H}_{(5)}((0,T);\bar{H}_{(1)}(\Omega\setminus F)), \end{cases} \tag{29}$$

where $h := \nabla \cdot (M\nabla u_\varepsilon)$. From (29), we have, for $0 \le i \le 5$,

$$\begin{cases} \mathcal{A}_L \dfrac{\partial^i}{\partial t^i} z_\varepsilon = \varepsilon \dfrac{\partial^i}{\partial t^i} h & \text{in} \quad \Omega \times (0,T) \\ \dfrac{\partial^i}{\partial t^i} z_\varepsilon(0) = \dfrac{\partial^{i+1}}{\partial t^{i+1}} z_\varepsilon(0) = 0 & \text{in} \quad \Omega. \end{cases}$$

By (19) together with (36) given in Appendix and the uniform boundedness of $\{\widetilde{u_\varepsilon}\}$, we can show

$$\left|\left|\frac{\partial^i}{\partial t^i} z_\varepsilon\right|\right|_{L^2((0,T);H)} \le \varepsilon C \left|\left|\frac{\partial^i}{\partial t^i} h\right|\right|_{L^2((0,T);V')} \le \varepsilon C \quad (0 \le i \le 5). \tag{30}$$

Now, let $Z_\varepsilon := \alpha z_\varepsilon$ where $\alpha \in C_0^\infty(\mathbb{R}^n)$ satisfying supp $\alpha \subset \mathbb{R}^n \setminus F$ and $\alpha \equiv 1$ near $x_0 \in \partial\Omega$. Then, we have

$$\begin{cases} \mathcal{A}_L Z_\varepsilon = Z_\varepsilon'' + A_L Z_\varepsilon \\ \quad = \varepsilon\alpha h - z_\varepsilon \nabla L \cdot \nabla\alpha - 2L\nabla\alpha \cdot \nabla z_\varepsilon - L z_\varepsilon \triangle\alpha. \end{cases}$$

By defining $g := \varepsilon\alpha h - z_\varepsilon \nabla L \cdot \nabla\alpha - 2L\nabla\alpha \cdot \nabla z_\varepsilon - L z_\varepsilon \triangle\alpha$,

$$A_L Z_\varepsilon = \alpha z_\varepsilon'' + g \in L^2((0,T);H). \tag{31}$$

Moreover, reminding $||z_\varepsilon||_{L^2((0,T);V)} \le C||z_\varepsilon'' - \varepsilon h||_{L^2((0,T);V')}$ from $A_L z_\varepsilon = z_\varepsilon'' - \varepsilon h$, by (30) we can show

$$||\alpha z_\varepsilon'' + g||_{L^2((0,T);H)} \le \varepsilon C. \tag{32}$$

To get exact inequality of Z_ε in $L^2((0,T);\bar{H}_{(2)}(\Omega))$, we change the coordinate into a boundary normal coordinate near x_0. For example, in the case of dimension 3, by a transform $F : \Omega \to \mathbb{R}^n$ with $F(x) := y(x) = (y_1(x), y_2(x), y_3(x))$, we have near x_0

$$\begin{cases} \nabla_y \cdot (\tilde{L}\nabla_y Z_\varepsilon) = \alpha z_\varepsilon'' + g \\ \{y_1 > 0\} = F(\Omega), \quad \{y_1 = 0\} = F(\partial\Omega), \end{cases} \tag{33}$$

where $\tilde{L} = (\widetilde{L_{rs}}) = (Lg)(F^{-1}(y))$, $g = (g_{rs}) = \displaystyle\sum_{j=1}^{3} \frac{\partial y_r}{\partial x_j}\frac{\partial y_s}{\partial x_j}$ and we used the same notation Z_ε, z_ε, g to denote their pull back F^{-1}. Then, the principal part of $\nabla_y \cdot (\tilde{L}\nabla_y Z_\varepsilon)$ is $L(\partial_{y_1}^2 + \displaystyle\sum_{i,j=2}^{3} g_{ij}\partial_{y_i}\partial_{y_j})$ due to $g_{11} = 1$, $g_{1s} = 0\,(s \ne 1)$ where we used the same notation L to denote its pull back by F^{-1}. By defining $[Z_\varepsilon]_k := \dfrac{Z_\varepsilon(y + ke_j) - Z_\varepsilon(y)}{k}$ and letting $\tilde{H} := L^2(\mathbb{R}^n_+)$, $\tilde{V} := \dot{H}_{(1)}(\overline{\mathbb{R}^n_+})$ $((\tilde{V}, \tilde{H}, \widetilde{V'})$ becomes Gelfand triple.), we have

$$\begin{cases} \nabla_y \cdot \tilde{L}\nabla_y [Z_\varepsilon]_k \\ \quad = [\alpha z''_\varepsilon + g]_k - \nabla_y \cdot [\tilde{L}]_k \nabla_y Z_\varepsilon(y + ke_j) \\ \quad = [\alpha z''_\varepsilon]_k + [g]_k - \nabla_y \cdot [\tilde{L}]_k \nabla_y Z_\varepsilon(y + ke_j) \ \in \ L^2((0,T); \widetilde{V'}). \end{cases}$$

Then, from (32),

$$||[\alpha z''_\varepsilon]_k + [g]_k - \nabla_y \cdot [\tilde{L}]_k \nabla_y Z_\varepsilon(y + ke_j)||_{L^2((0,T);\widetilde{V'})} \ \le \ \varepsilon C$$

Hence, $||[Z_\varepsilon]_k||_{L^2((0,T);\tilde{V})} \le \varepsilon C$. By uniform boundedness of $\{[Z_\varepsilon]_k\}$, it has a subset which converges to a function $W \in L^2((0,T); \tilde{V})$ weakly. Moreover, $W = \partial_{y_j} Z_\varepsilon$ $(2 \le j \le 3)$ and $||W||_{L^2((0,T);\tilde{V})} \le \varepsilon C$. So $\partial^\beta_{y_j} Z_\varepsilon \in L^2((0,T), \tilde{H})$ for $|\beta| \le 2$ and $2 \le j \le 3$. Also, we can show $\partial^2_{y_1} Z_\varepsilon \in L^2((0,T), H)$ from (31) and observing the principal part of $\nabla_y \cdot (\tilde{L}\nabla_y Z_\varepsilon)$. Then, using the interpolation theorem of Proposition 3.8 in [5] we have

$$||Z_\varepsilon(y)||_{L^2((0,T),\bar{H}_{(2)}(\mathbb{R}^n_+))} \le \varepsilon C. \tag{34}$$

Since we can easily show $||\alpha z_\varepsilon||_{L^2((0,T),\bar{H}_{(2)}(\Omega))} \le C||Z_\varepsilon(y)||_{L^2((0,T),\bar{H}_{(2)}(\mathbb{R}^n_+))}$ with some constant $C > 0$ independent of ε, $||\alpha z_\varepsilon||_{L^2((0,T),\bar{H}_{(2)}(\Omega))} \le \varepsilon C$ with another constant $C > 0$. Reminding $\delta q = L_\varepsilon \dfrac{\partial u[L_\varepsilon]}{\partial n} - L\dfrac{\partial u[L]}{\partial n} = L\left(\dfrac{\partial u[L_\varepsilon]}{\partial n} - \dfrac{\partial u[L]}{\partial n}\right) + \varepsilon M \dfrac{\partial u[L_\varepsilon]}{\partial n}$ and (34), we can prove the second term is $o(\varepsilon)$. □

Next we finish the proof of Theorem 1.1. From Lemma 2.1, Theorem 4.1 and Theorem 5.1, we can have a representation in Theorem 1.1. We clearly have the linearity of the mapping : $L^\infty(\Omega) \ni M \mapsto J'(L)M \in \mathbb{R}$. By using (19), we easily have the boundedness of this mapping. Furthermore, $J'(L)$ gives the Gateaux derivative of $J(L)$ at L. □

6 Numerical algorithm and example

To find the minimum of the functional J, we make use of the projected gradient method[6]:

$$L_{k+1} = P_C\left(L_k - \alpha_k \nabla J(L_k)\right) \quad (k = 0,\, 1,\, 2,\, \ldots\ldots), \tag{35}$$

where α_k $(0 < \alpha_k \le 1)$ is a suitable step size and $\nabla J(L)$ is a search direction defined by

$$\langle \nabla J(L),\, M \rangle = J'(L)\, M \quad \text{for } \forall M \in L^\infty(\Omega)\,.$$

Here the map P_C is a clip-off operator such that

$$P_C L(\boldsymbol{x}) = \begin{cases} C_1 & (L(\boldsymbol{x}) < C_1) \\ L(\boldsymbol{x}) & (C_1 \le L(\boldsymbol{x}) \le C_2) \\ C_2 & (L(\boldsymbol{x}) > C_2) \end{cases}.$$

From Theorem 1.1 and Remark 1.2, we notice that

$$\nabla J(L) = \int_0^T \nabla u[L] \cdot \nabla v\, dx + h\,.$$

We have to discuss about obtaining numerically the function h in order to use (35).

Let $\{B_i\}_{i=1}^N$ be a division of the domain Ω such that

$$\Omega = \bigcup_{i=1}^N B_i, \quad B_i \cap B_j = \emptyset \ \ (i \ne j)\,.$$

We denote by χ_i a characteristic function, namely,

$$\chi_i(\boldsymbol{x}) := \begin{cases} 1 & (\boldsymbol{x} \in B_i) \\ 0 & (\boldsymbol{x} \notin B_i) \end{cases} .$$

Then, we consider to find the approximation of the density function h in $X_B = \text{span}\,\{\chi_1, \chi_2, \cdots, \chi_N\}$. By using the relation (6) and the Galerkin method, we can get

$$\int_\Omega h_B\, \chi_i\, dx = \int_\Omega \frac{\partial U}{\partial t}[\chi_i](T)\, w\, dx$$

for $i = 1, 2, \ldots, N$. Here $h_B \in X_B$ is the approximation of the density function and the function $U[\chi_i]$ is the solution to (5) with the source term $\nabla \cdot (\chi_i \nabla u[L])$. We represent h_B by the linear combination of χ_i, namely, $h_B = \sum_{j=1}^N h_j \chi_j$. Then, the linear system can be obtained as follows:

$$\sum_{j=1}^N h_j \int_\Omega \chi_i\, \chi_j\, dx = \int_\Omega \frac{\partial U}{\partial t}[\chi_i](T)\, w\, dx$$

for $i = 1, 2, \ldots, N$. Since χ_i has the orthogonal relation with respect to L^2 inner product, we have

$$h_i = \frac{1}{|B_i|} \int_\Omega \frac{\partial U}{\partial t}[\chi_i](T)\, w\, dx \quad (i = 1, 2, \ldots, N)\,,$$

where $|B_i|$ means the area of B_i. Therefore we can get the approximation h_B by solving N initial-boundary value problem (5). By using this approximation, we define the approximated search direction as follows:

$$\widetilde{\nabla} J(L) := \int_\Omega \nabla u[L] \cdot \nabla v\, dx + h_B\,.$$

Here we notice that our method with $\widetilde{\nabla} J(L)$ is not the projected gradient method exactly but its calculation is very easy.

Hence we summurize an algorithm for our inverse problem as follows:

Algorithm for coefficient identification

Given the division $\{B_i\}$.

1. *Pick an initial coefficient function L_0 which belongs to the admissible set $\mathcal{K}$.*
2. *For $k = 0, 1, 2, \ldots$; do*
   - (a) *Solve (1) to find $\nabla u[L_k]$ and $L_k \dfrac{\partial u}{\partial n}\Big|_{\partial\Omega \times (0,T)}$.*
   - (b) *Solve the boundary value problem (3) to find w.*
   - (c) *Solve the initial-boundary value problem (4) to find ∇v.*
   - (d) *For $i = 1, 2, \ldots, N$; do*
     - i. *Solve the initial-boundary value problem (5) with the source term $\nabla \cdot (\chi_i \nabla u[L_k])$ to find $\dfrac{\partial U}{\partial t}[\chi_i](T)$.*
     - ii. *Calculate h_i by*

       $$h_i = \frac{1}{|B_i|} \int_\Omega \frac{\partial U}{\partial t}[\chi_i](T)\, w\, dx\,.$$
   - (e) *Calculate the approximated search direction $\widetilde{\nabla} J(L_k)$ by*

     $$\widetilde{\nabla} J(L_k) = \int_0^T \nabla u[L_k] \cdot \nabla v\, dt + \sum_{i=1}^N h_i \chi_i\,.$$

(f) *Choose the step size α_k by using some method.*

(g) *Update the coefficient function:* $L_{k+1} = P_C\left(L_k - \alpha_l \widetilde{\nabla} J(L_k)\right)$.

We show a numerical example for our algorithm. Let Ω be a unit disk. The coefficient K is given by

$$K(\boldsymbol{x}) = \begin{cases} 1.25 & (|\boldsymbol{x}| < 0.15) \\ 1.0 & (|\boldsymbol{x}| > 0.15) \end{cases}$$

as shown in Fig.1. Here $|\cdot|$ means the Euclidean norm on $\boldsymbol{R}^2$.

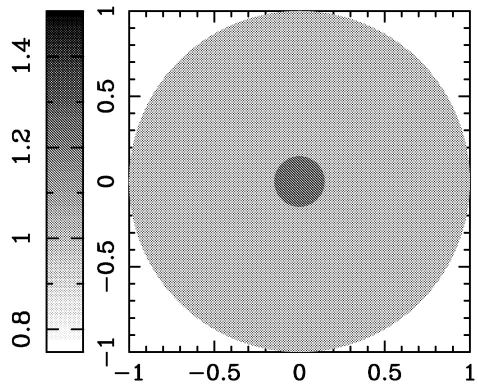

Figure 1: Exact coefficient

The constants in the constrained condition are given by $C_1 = 0.90$ and $C_2 = 1.35$. The Neumann boundary value for this example are supposed to be given by

$$\overline{q}(t) = \begin{cases} -p(t) & \text{on } \partial\Omega_m \times (0,\, T] \\ 0.0 & \text{on } (\partial\Omega \setminus \overline{\partial\Omega_m}) \times (0,\, T] \end{cases},$$

where

$$p(t) = \begin{cases} 0.25 \sin(12.5\pi\, t) & (0 \le t \le 0.16) \\ 0.0 & (t > 0.16) \end{cases}.$$

Here $\partial\Omega_m$ $(m = 1,\, 2,\, \ldots,\, 5)$ are set as

$$\partial\Omega_m = \left\{ (\cos\theta,\, \sin\theta) \,\Big|\, -\frac{\pi}{50} < \theta - (m-1)\frac{\pi}{4} < \frac{\pi}{50} \right\}.$$

The Dirichlet boundary value $\overline{u}$ is generated by solving numerically the wave equation with the exact coefficient K and the Neumann boundary value $\overline{q}$. In order to solve this problem numerically, we make use of the Newmark method[2] for time integration with linear triangular finite elements in space. To avoid the inverse inclusion, the measured value $\overline{u}$ is given by $\overline{u}(\boldsymbol{x},\, t) = u_{\mathrm{cal}}(\boldsymbol{x},\, t) + \delta(\boldsymbol{x},\, t)$, where u_{cal} means the calculated value on the circle and $\delta(\boldsymbol{x},\, t)$ is a random small valued function satisfied $|\delta(\boldsymbol{x},\, t)| < 10^{-10}|u_{\mathrm{cal}}|$ on the boundary $\partial\Omega$ for any $t > 0$. The length of time is set as $T = 4.0$. The division $\{B_i\}$ is supposed to be given by

$$B_i = \{\boldsymbol{x} \in \Omega \mid 0.1(i-1) \le |\boldsymbol{x}| < 0.1i\}$$

for $1 \le i \le 10$. We employ the Armijo criterion[1] in order to find the step size α_k in our algorithm.

We assume that $L_0(\boldsymbol{x}) = K|_{\partial\Omega} = 1.0$ in the whole domain. After 100 times of iterations, we have the calculated coefficient as shown in Figure 2. Figure 3 shows the distribution of the relative error for calculated coefficient. The maximum value of the relative error is about 8.94%. These figures show that calculated coefficient is in good agreement with the exact one.

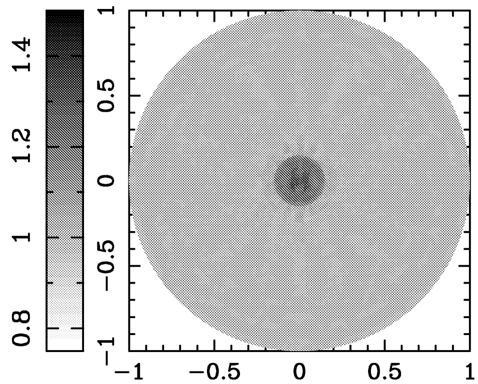

Figure 2: Calculated coefficient

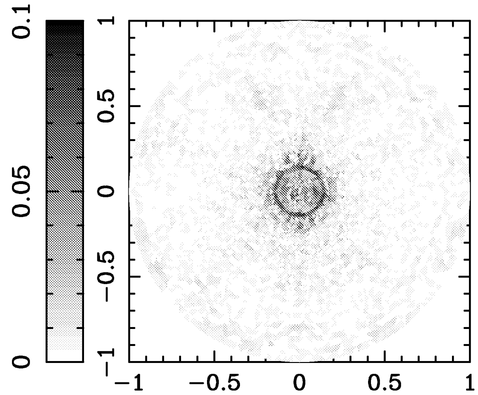

Figure 3: Relative error

A Proof for existence of the solutions to (2), (4) and (3)

In this Appendix, we show the existence of solutions $u \in \bar{H}_{(5)}((0,T);\bar{H}_{(1)}(\Omega))$, $v \in L^2((0,T);\bar{H}_{(1)}(\Omega))$ and $w \in \bar{H}_{(1)}(\Omega)$ to (2), (4) and (3), respectively.

To begin with we cite from [11] the existence and regularity theorem for the abstract hyperbolic evolution equation of second order in the time variable.

Let V, H be real Hilbert spaces and V be separable. Suppose the embedding $i : V \hookrightarrow H$ is continuous, injective and its image is dense in H. Then, the dual $i' : H \hookrightarrow V'$ of i is continuous, injective and has a dense range. Such a triple (V, H, V') is called a Gelfand triple. For $T > 0$, $k \in \mathbb{Z}_+$ and $X = H$ or V, the Sobolev space $W_2^k((0,T);X)$ is the collection of measurable functions $\varphi : (0,T) \to X$ with $\dfrac{d^l\varphi}{dt^l} \in L^2((0,T);X)$ $(0 \le l \le k)$, where the differentiation is in the distributional sense. The norm of $\varphi \in W_2^k((0,T);X)$ is given by $||\varphi||_k^2 = \displaystyle\sum_{l=0}^{k} \int_0^T \left|\left|\frac{d^l\varphi}{dt^l}(t)\right|\right|_X^2 dt.$

Let $a(\varphi, \psi)$ $((\varphi, \psi) \in V)$ be a continuous, symmetric sesquilinear form satisfying the coercivity :

$$\text{there exist } k_0,\ \alpha > 0 \text{ with } a(\varphi,\varphi) + k_0||\varphi||_H^2 \ge \alpha||\varphi||_V^2 \quad (\varphi \in V) \tag{36}$$

Then, it is well known that there exists a unique $A_K \in L(V, V')$ (i.e. the set of all bounded linear operators from V to V') such that

$$a_K(\varphi, \psi) = (A_K\varphi, \psi)_H. \tag{37}$$

For the Cauchy problem for the abstract hyperbolic equation $\dfrac{d^2y}{dt^2} + A_K y = f$, we have

Theorem A.1 *Let y_0, $y_1 \in V$ and $f \in W_2^{k-1}((0,T);H)$ with $k \in \mathbb{N}$ satisfy the compatibility condition of degree $k-1$. That is*

$y_l \in V \ (0 \le l \le k-1),\ y_k \in H,$

where
$$\begin{cases} y_{2l-1} = f^{(2l-3)}(0) - A_K f^{(2l-5)}(0) + \cdots + (-1)^{l-2} A_K^{l-2} f'(0) + (-1)^{l-1} A_K^{l-1} y_1 \\ y_{2l} = f^{(2l-2)}(0) - A_K f^{(2l-4)}(0) + \cdots + (-1)^{l-1} A_K^{l-1} f(0) + (-1)^l A_K^l y_0. \end{cases}$$

Then, the Cauchy problem :

$$\begin{cases} \dfrac{d^2y}{dt^2}(t) + A_K y(t) = f(t) \qquad in \qquad (0,T) \\ y(0) = y_0, \quad y'(0) = y_1 \end{cases}$$

admits a unique solution $y(t)$ *such that*

$$y \in L^2((0,T);V),\ \frac{dy}{dt} \in L^2((0,T);H)$$

and they depend linearly and continuously on

$$(f, y_0, y_1) \in L^2((0,T);H) \times V \times H.$$

Moreover, y *has the regularity :*

$$y \in W_2^{k-1}((0,T);V),\ y^{(k)} \in L^2((0,T);H),\ y^{(k+1)} \in L^2((0,T);V'),$$

where $y^{(k)} := \dfrac{d^k y}{dt^k}$.

For $\delta > 0$ small enough let $B_\delta := \{x \in \mathbb{R}^n;\ \text{dist}\,(x, \partial\Omega) < \delta\}$.
Let (V_j, Φ_j) $(1 \le j \le J)$ be patches of manifold B_δ where collection is an atlas of B_δ. We can assume that $V_j \cap \partial\Omega \ne \emptyset$, $\Phi_j(V_j \cap \partial\Omega) \subset \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$ and $\Phi_j(V_j \cap \Omega) \subset \mathbb{R}^n_+$ for each j $(1 \le j \le J)$. Let $\{\xi_j\}_{1 \le j \le J}$, $\{\eta_j\}_{1 \le j \le J} \subset C_0^\infty(\mathbb{R}^n)$ be partition of unities subordinated to $\{V_j\}_{1 \le j \le J}$. Now, we can construct an inverse trace operator $\Lambda : C^6([0,T]; H_{(\frac{5}{2})}(\partial\Omega)) \to C^6([0,T]; \bar{H}_{(3)}(\Omega))$ i.e. $(\Lambda\ell)|_{\partial\Omega \times [0,T]} = \ell \in C^6([0,T]; H_{(\frac{5}{2})}(\partial\Omega))$ in the following way.

For $\ell \in C^6([0,T]; H_{(\frac{5}{2})}(\partial\Omega))$, let $\ell_j := \xi_j \ell \in C^6([0,T]; H_{(\frac{5}{2})}(\partial\Omega))$, $m_j := \ell_j \circ (I \times (\Phi_j|_{\partial\Omega})^{-1}) \in C^6([0,T]; H_{(\frac{5}{2})}(\mathbb{R}^{n-1}))$, where $I : [0,T] \to [0,T]$ is the identity operator. We define an inverse trace operator $\Lambda_0 : C^6([0,T]; H_{(\frac{5}{2})}(\mathbb{R}^{n-1})) \to C^6([0,T]; H_{(3)}(\mathbb{R}^n_+))$ by

$$\begin{cases} (\Lambda_0 m)(t,x) = \dfrac{1}{(2\pi)^{n-1} d} \displaystyle\int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{(1+|\xi'|^2)^{\frac{5}{2}}}{(1+|\xi|^2)^3} \hat{m}(t,\xi') d\xi \\ \qquad\qquad\qquad\qquad (m \in C^6([0,T]; H_{(\frac{5}{2})}(\mathbb{R}^{n-1}))), \end{cases}$$

where $\xi' = (\xi_1, \cdots, \xi_{n-1})$ for $\xi = (\xi_1, \cdots, \xi_n)$, $d := \displaystyle\int_{-\infty}^{\infty} (1+\tau^2)^{-3} d\tau$ and $\hat{m}(t,\xi') := \displaystyle\int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} m(t,x') dx'$.
Then, Λ can be given by

$$\Lambda\ell := \sum_{j=1}^{J} (\eta_j((\Lambda_0 m_j) \circ (I \times \Phi_j)))\Bigg|_{\bar{\Omega} \times [0,T]} \tag{38}$$

for any $\ell \in C^6([0,T]; H_{(\frac{5}{2})}(\partial\Omega))$. (See [8] for the details.)

We first prove the existence of the solution $u \in \bar{H}_{(5)}((0,T); \bar{H}_{(1)}(\bar{\Omega}))$ to (2) with $u^{(6)} \in L^2((0,T); L^2(\Omega))$, $u^{(7)} \in L^2((0,T); (\dot{H}_{(1)}(\bar{\Omega}))')$. Let $\tilde{u} := u - \phi$ with $\phi := \Lambda\overline{u}$. Then, $\tilde{u}$ has to satisfy

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}\tilde{u} - \nabla \cdot (K \nabla \tilde{u}) = f \\ \tilde{u}(0) = \widetilde{u_0},\ \frac{\partial}{\partial t}\tilde{u}(0) = \widetilde{u_1} \end{cases} \tag{39}$$

with

$$\begin{cases} f = f(K) := \nabla \cdot (K\nabla\phi) - \dfrac{\partial^2}{\partial t^2}\phi \in C^4([0,T]; \bar{H}_{(1)}(\Omega)) \\ \widetilde{u_0} := -\phi|_{t=0}, \quad \widetilde{u_1} := -\left.\dfrac{\partial \phi}{\partial t}\right|_{t=0} \in V := \dot{H}_{(1)}(\bar{\Omega}). \end{cases} \tag{40}$$

Now $(V, H := L^2(\Omega), V')$ is clearly a Gelfand triple and $a(\psi, \omega) := \int_\Omega K\nabla\psi \cdot \nabla\omega \, dx$ $(\psi, \omega \in V)$ is a continuous sesquiliear form satisfying the coercivity condition with $k_0 = 0$. Moreover, it is easy to see that f, $\widetilde{u_0}$, $\widetilde{u_1}$ satisfy the compatibility condition of degree 5. Then, the existence of u to (2) with the desired properties immediately follows by applying Theorem A.1 to (39).

By observing that $\tilde{u} \in \bar{H}_{(5)}((0,T); \dot{H}_{(1)}(\bar{\Omega}))$ satisfies

$$\nabla \cdot (K\nabla\tilde{u}) = \frac{\partial^2 \tilde{u}}{\partial t^2} - f \in \bar{H}_{(3)}((0,T); \dot{H}_{(1)}(\bar{\Omega})) \subset C^2((0,T); \dot{H}_{(1)}(\bar{\Omega})),$$

we have $\tilde{u} \in C^2([0,T]; \bar{H}_{(3)}(\Omega \setminus F))$ and hence $u \in C^2([0,T]; \bar{H}_{(3)}(\Omega \setminus F))$ by the regularity theorem near the boundary of solutions to the Dirichlet boundary value problem for strongly elliptic equations (See [5], Chapter 3, Proposition 3.7). This implies that $2(L\frac{\partial u}{\partial n}(T) - \bar{q}(T)) \in H_{(\frac{3}{2})}(\partial\Omega)$ and

$$2(L\frac{\partial u}{\partial n} - \bar{q}) \in C^2([0,T]; H_{(\frac{3}{2})}(\partial\Omega)). \tag{41}$$

By the well-posedness of (8), we immediately have $w \in \bar{H}_{(1)}(\Omega)$.

For the existence of $v \in L^2((0,T); \bar{H}_{(1)}(\Omega))$, we argue likewise we did for the solution u to (2) using the inverse trace operator transforming (9) to an initial boundary value problem with Dirichlet boundary condition. Then, by (41), the second term of equation of this initial boundary value problem belongs to $L^2((0,T); H)$ with $H = L^2(\Omega)$. Therefore, by Theorem A.1, we have the existence of $v \in L^2((0,T); \bar{H}_{(1)}(\Omega))$.

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