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The no-response approach and its relation to non-iterative methods for the inverse scattering.

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Abstract

The goal of this work is to investigate the relation of the no-response approach to some other non-iterative reconstruction schemes. We will derive several equivalence statements and dependency results. For simplicity we consider the obstacle reconstruction problem from far field data.

In particular, we investigate two versions of the no-response test (NRT) for the inverse scattering problem. The first version is the *pure* NRT, the second combines the NRT with a rangetest element. We show convergence for these two versions without any eigenvalue assumption about the scatterer.

Second, we state the natural formulation of the probe method for far field data and reformulate the singular sources method. We show that these statements of the two methods coincide and they form one face of the first version of the no-response test. *Third*, we prove that the convergence of the linear sampling method implies the convergence of the second version of the no-response test. Precisely, we show that we can use the blowup sequence of the linear sampling method to create the blowup sequence of the second version of the no response test. *Fourth*, we show that the two versions of the no response method are equivalent with respect to their convergence. Thus, the convergence of the linear sampling method also implies the convergence of the pure no-response test.

1 Introduction

Let D be a bounded domain in $\mathbb{R}^3$ with C^2 boundary ∂D. We consider the acoustic inverse scattering problem. The propagation of time-harmonic acoustic fields in a homogeneous media is governed by the *Helmholtz equation*

$$\Delta u + \kappa^2 u = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D} \tag{1.1}$$

where κ is the real positive *wave number*. At the boundary of sound-soft scatterers the total field u satisfies the *Dirichlet boundary condition*

$$u = 0 \quad \text{on } \partial D. \tag{1.2}$$

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Given an incident field u^i which satisfies $\Delta u^i + \kappa^2 u^i = 0$ we look for solutions $u := u^i + u^s$ of (1.1) and (1.2) where the *scattered field* u^s is assumed to satisfy the Sommerfeld radiation condition

$$\lim_{r\to\infty} r(\frac{\partial u^s}{\partial r} - i\kappa u^s) = 0, \tag{1.3}$$

$r = |x|$ and the limit is uniform with respect to all the directions $\theta := \frac{x}{|x|}$. It is well known (see [P2]) that this reflected field satisfies the following asymptotic property,

$$u^s(x) \quad = \quad \frac{e^{i\kappa r}}{r} u^\infty(\theta) + O(r^{-2}), \quad r \to \infty, \tag{1.4}$$

where the function $u^\infty(\cdot)$ defined on the unit sphere $\mathbb{S}$ is called the far-field associated to the incident field u^i. Taking particular incident fields given by the plane waves, $u^i(x,d) := e^{i\kappa d\cdot x}$, $d \in \mathbb{S}$, we define the far-field pattern $u^\infty(\theta, d)$ for $(\theta, d) \in \mathbb{S}\times\mathbb{S}$. Analogously, for an incident point source $\Phi(\cdot, z)$, where

$$\Phi(x,y) := \frac{1}{4\pi}\frac{e^{i\kappa|x-y|}}{|x-y|}, x \neq y, x, y \in \mathbb{R}^3.$$

is the fundamental solution of $\Delta + \kappa^2$ on $\mathbb{R}^3$, we denote the scattered field by $\Phi^s(\cdot, z)$ and its far field pattern by $\Phi^\infty(\cdot, z)$. The problem we are concerned with is the following

Definition 1.1 **(Shape reconstruction problem.)** *Given $u^\infty(\cdot,\cdot)$ on $\mathbb{S}\times\mathbb{S}$ for the scattering problem (1.1) - (1.3) find the obstacle D.*

This problem has been well studied, see [C-K] or [P2] for more details. Several methods have been created to solve this problem. Often, the methods are classified into the categories of iterative methods and non-iterative methods. We are concerned with the non-iterative methods as *linear sampling method* [C-Ki], *probe method* [I1], *singular sources method* [P2], *no-response test* [L-P] and *range test* [P-S-K]. All these methods make different use of the given data. The goal of this paper is to clarify the relation between them.

In section 2, we give the first version of the no-response test which is the multiwave version of the one-wave method given in [L-P]. We will justify its convergence by reducing it to the convergence analysis of the singular sources method.

To deal with this inverse scattering problem via the probe method, in [I2] the author proceeds in two steps. The first step is to compute the near field from the far field and the second step is to detect the obstacle from this near field. The near field is given by the Dirichlet to Neumann map of the boundary problem stated on some artificially introduced domain Ω containing the unknown obstacle. In this paper, we state the natural far field version of the probe method. This version uses directly the far field (in one step) to detect the obstacle. We show also that the indicator of this far field version and the (original) one of near field version are equivalent regarding the blow up property.

The singular sources method computes the scattered field $\Phi^s(\cdot, z)$ of the incident point sources $\Phi(\cdot, z)$, where z is outside the obstacle D. We reformulate the singular sources method in a way which enables us to compute its indicator function for any point z inside or outside D. For z outside $\overline{D}$, it coincides with the original one , i.e. $\Phi^s(z,z)$.

We find out that the indicator function of the far field version of probe method and the one of the reformulated singular sources method coincide. The obstacle D is characterized by the set of points z for which this indicator function blows-up. This behavior of the indicator function with respect to the parameter z is similar to the one of the linear sampling method or the factorization

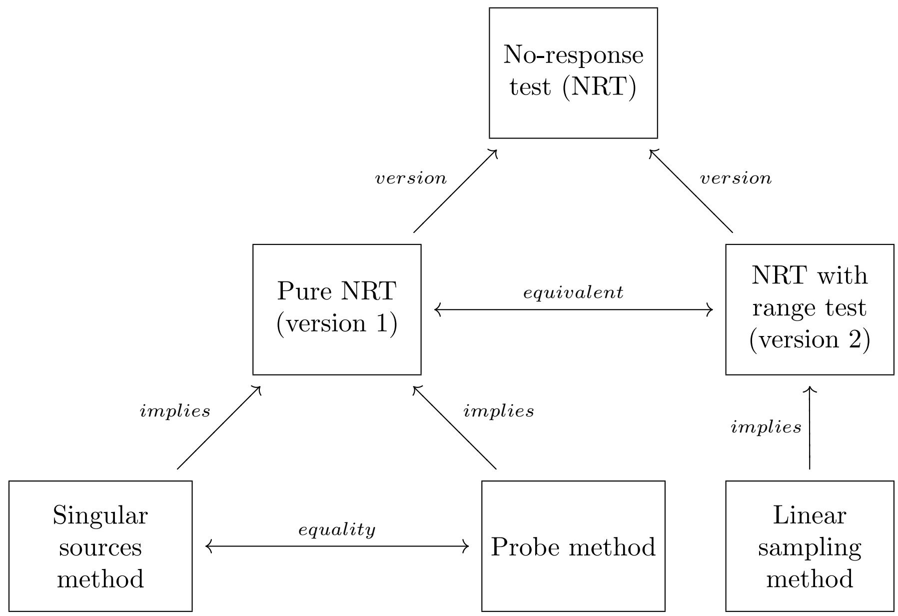

Figure 1: *The Diagramm shows the relation between the two versions of the no-response test and several non-iterative methods for the inverse scattering theory.*

method [Ki]. But it has an opposite behavior in the sense that it is bounded outside the obstacle D and becomes large when approaching D and stay unbounded inside D. This is the object of section 3.

This first version of the no-response test provides a general framework for the probe and the singular sources methods in the sense that these two last methods constitute one face of the first version of the no response test. Similar results have been achieved in [N-P-S], where it is shown that for the *conductivity problem* the no-response test unifies the singular sources method and the probe method. It is easy to see that the equivalence of these methods is based only on the application of Green's theorem and, thus, is also valid for any boundary value problem. Further, it is shown that the rate of blow-up of the indicator functions of these two methods has the order of the singularity of the Green function.

In section 4, we introduce a second version of the no-response test for reconstructing the obstacle D from the knowledge of the far-field pattern. We base this second version on a combination of the superposition principle with the range test idea given in [P-S-K]. This second version of the no-response test can also be seen as a *multiwave version* of the range test which is different from the *multiwave rangetest* as described in [P-S]. We give the justification of its convergence.

In section 5, we recall the linear sampling method, see [C-Ki] and [C-C], and show how it is related to the two proposed methods by explaining how the convergence of the linear sampling method implies the convergence of the second version of the no-response test. We will show that the singular sequence creating the blow-up for the linear sampling method can be used to create the blow-up for the no-response test. For the linear sampling we need an eigenvalue assumption on the unknown obstacle while for the no-response test we don't need.

Finally, in section 5, we show that the two versions of the no-response test are equivalent with respect to their convergence. Altogether, the relations between the different methods are graphically displayed in Figure 1.

2 The first version of the no-response test for the scattering problem

The goal of this section is to develop a multi-wave formulation for the *no response test* (NRT) for the acoustic scattering problem. We will introduce the basic idea of the NRT and then prove its convergence. We start with some preparations. We set $\mathbb{S}$ to be the unit sphere in $\mathbb{R}^3$.

It is well known (see [C-K] and [P2]) that the scattered field associated with the Herglotz incident field $v_g^i := v_g$ defined by

$$v_g(x) := \int_{\mathbb{S}} e^{i\kappa x\cdot d} g(d)\, ds(d), \quad x \in \mathbb{R}^3, \tag{2.5}$$

is given by

$$v_g^s(x) := \int_{\mathbb{S}} u^s(x,d) g(d)\, ds(d), \quad x \in \mathbb{R}^3 \setminus D, \tag{2.6}$$

and its far field is given by

$$v_g^\infty(\theta) := \int_{\mathbb{S}} u^\infty(\theta,d) g(d)\, ds(d), \quad \theta \in \mathbb{S}. \tag{2.7}$$

We base the method on the representation

$$u^\infty(\theta,d) = \frac{1}{4\pi} \int_{\partial D} \left\{ \frac{\partial u^s(y,d)}{\partial \nu} e^{-i\kappa\theta\cdot y} - \frac{\partial e^{-i\kappa\theta\cdot y}}{\partial \nu} u^s(y,d) \right\} ds(y) \tag{2.8}$$

given by using the Green's formula in $\mathbb{R}^3 \setminus \overline{D}$ for $u^s(\cdot,d)$ and $\Phi(\cdot,y)$ and their asymptotic behavior at infinity (see [C-K], Theorem 2.5) where the normal is directed into inside D. The representation of the scattered field $\Phi^s(x,z)$ for $x,z \in \mathbb{R}^3 \setminus \overline{D}$ is given by Green's formula

$$\Phi^s(x,z) = \int_{\partial D} \left\{ \frac{\partial \Phi^s(y,z)}{\partial \nu(y)} \Phi(x,y) - \Phi^s(y,z) \frac{\partial \Phi(x,y)}{\partial \nu(y)} \right\} ds(y), \quad x,z \in \mathbb{R}^3 \setminus \overline{D}. \tag{2.9}$$

Using

$$0 = \int_{\partial D} \left\{ \frac{\partial \Phi(y,z)}{\partial \nu(y)} \Phi(x,y) - \Phi(y,z) \frac{\partial \Phi(x,y)}{\partial \nu(y)} \right\} ds(y), \quad x,z \in \mathbb{R}^3 \setminus \overline{D} \tag{2.10}$$

this can be transformed into

$$\Phi^s(x,z) = \int_{\partial D} \frac{\partial (\Phi^s + \Phi)(y,z)}{\partial \nu(y)} \cdot \Phi(x,y)\, ds(y), \quad x,z \in \mathbb{R}^3 \setminus \overline{D}. \tag{2.11}$$

Now, consider a couple of densities $(f,g) \in L^2(\mathbb{S}) \times L^2(\mathbb{S})$. We replace θ by $-\theta$ in equation

(2.8), multiply the result by $f(\theta)g(d)$ and integrate on $\mathbb{S}\times\mathbb{S}$ to calculate

$$
\begin{aligned}
&\int_{\mathbb{S}}\int_{\mathbb{S}} u^{\infty}(-\theta,d)f(\theta)g(d)\,ds(\theta)ds(d)\\
=\;& \frac{1}{4\pi}\int_{\partial D}\Bigg\{\int_{\mathbb{S}}\frac{\partial u^s(y,d)}{\partial\nu}g(d)\,ds(d)\cdot\int_{\mathbb{S}} e^{i\kappa\theta\cdot y}f(\theta)\,ds(\theta)\\
&-\int_{\mathbb{S}}\frac{\partial e^{i\kappa\theta\cdot y}}{\partial\nu}f(\theta)\,ds(\theta)\cdot\int_{\mathbb{S}}\underbrace{u^s(y,d)}_{=-u^i(y,d)}g(d)\,ds(d)\Bigg\}\,ds(y)\\
=\;& \frac{1}{4\pi}\int_{\partial D}\left\{\frac{\partial v_g^s}{\partial\nu}(y)v_f^i(y)+\frac{\partial v_f^i}{\partial\nu}(y)v_g^i(y)\right\}\,ds(y).
\end{aligned}
\tag{2.12}
$$

We call a domain B with C^2-regular boundary such that κ^2 is not a Dirichlet eigenvalue of the operator $-\Delta$ on B and $\mathbb{R}^3\setminus B$ connected a *non-vibrating domain*.

Now, we state the definition of the first version of the no-response test.

DEFINITION 2.1 **(The first version of the no response method.)** *Let B be any* non-vibrating domain. *Hereafter, we call such B a test domain . We define the* indicator function *for the multi-wave no-response test by*

$$I_1(B) := \lim_{\epsilon\to 0}\sup\{I_{1,\epsilon}(f,g):\ ||v_f||_{L^2(\partial B)}<\epsilon, ||v_g||_{L^2(\partial B)}<\epsilon\} \tag{2.13}$$

with

$$I_{1,\epsilon}(f,g) := \Big|\int_{\mathbb{S}}\int_{\mathbb{S}} u^{\infty}(-\theta,d)f(\theta)g(d)\,ds(\theta)ds(d)\Big|. \tag{2.14}$$

For the set $\mathcal{G}$ of all the test domains B no response test calculates the indicator function $I_1(B)$ and builds the intersection

$$D_{rec,1} := \bigcap_{B\in\boldsymbol{B}_1} B, \tag{2.15}$$

where

$$\boldsymbol{B}_1 := \{B\in\mathcal{G}:\ I_1(B)=0\}. \tag{2.16}$$

After the above preparations we can prove the following characterization of D from the far field pattern, which provides a convergence result for the no response test for reconstructing the inclusion D.

THEOREM 2.2 **(Convergence of the first NRT version.)** *Let $\mathcal{G}$ as in Definition 2.1. We have*

1. *if $\overline{D}\subset B$ then $I_1(B)=0$ and*
2. *if $\overline{D}\not\subset B$ then $I_1(B)=\infty$.*

Thus the unknown scatterer is given by the intersection of all test domains B for which $I_1(B)$ is zero, i.e

$$D = D_{rec,1}.$$

Proof. *First*, consider the case where $\overline{D} \subset B$. For $||v_g||_{L^2(\partial B)} < \epsilon$ then from the regularity theory of the very weak solutions of the elliptic problems, see [N], we have $||v_g||_{L^2(B)} < c\epsilon$. Hence by interior estimates we have $||v_g||_{C^1(\partial D)} < c'\epsilon$ and then

$$||v_g^s||_{C(\partial D)} < c_1\epsilon, \qquad \Big|\Big|\frac{\partial v_g^s}{\partial \nu}\Big|\Big|_{C(\partial D)} < c_2\epsilon \tag{2.17}$$

with some constants c, c', c_1 and c_2. Using (2.12) and the fact that $||v_f||_{C^1(\overline{D})} < \tilde{c}\epsilon$, we obtain

$$|I_{1,\epsilon}(f,g)| \leq C\epsilon^2 \tag{2.18}$$

with some constant C and thus

$$I_1(B) = \lim_{\epsilon\to 0} \sup\{I_{1,\epsilon}(f,g) : \ ||v_g||_{L^2(\partial B)} < \epsilon, ||v_f||_{L^2(\partial B)} < \epsilon\} = 0. \tag{2.19}$$

Second, $\overline{D} \not\subset B$. Let $z \in \partial D$ such that z is on the boundary of the unbounded component of $\mathbb{R}^3 \setminus \overline{D \cup B}$. Then, there exists a sequence of points

$$(z_p)_{p\in\mathbb{N}} \subset \mathbb{R}^3 \setminus (\overline{B \cup D}) \tag{2.20}$$

such that z_p tends to z. We consider the sequence of point sources $\Phi(\cdot, z_p)$. We set B_p as a sequence of non-vibrating domains such that $\overline{B \cup D} \subset B_p$ and $z_p \in \mathbb{R}^n \setminus \overline{B_p}$. In this case, due to the denseness property of the Herglotz wave operator (see [P2], Lemma 3.1.3) we take g_n^p as a sequence such that for every p fixed

$$\Big|\Big|v_{g_n^p} - \frac{\epsilon}{2}\alpha_p\Phi(\cdot, z_p)\Big|\Big|_{L^2(\partial B_p)} \to 0, \ \ n \to \infty, \tag{2.21}$$

where

$$\alpha_p := ||\Phi(\cdot, z_p)||^{-1}_{L^2(\partial B)}. \tag{2.22}$$

Hence, by a combination of (2.21) and (2.22) and the well-posedness of the interior Dirichlet problem in B_p we derive that for every p fixed we have

$$||v_{g_n^p}||_{L^2(\partial B)} < \epsilon, \tag{2.23}$$

for n large enough. In this case, from (2.12), replacing (f, g) by (g_n^p, g_n^p), we deduce that

$$\begin{aligned} & \lim_{n\to\infty} \int_{\mathbb{S}}\int_{\mathbb{S}} u^\infty(-\theta, d)\ g_n^p(\theta)\ g_n^p(d)\ ds(\theta)ds(d) \\ = \ & \frac{\epsilon^2\alpha_p^2}{16\pi}\int_{\partial D} \frac{\partial(\Phi^s + \Phi)(y, z_p)}{\partial\nu(y)} \cdot \Phi(y, z_p)\ ds(y) && (2.24) \\ = \ & \frac{\epsilon^2\alpha_p^2}{16\pi}\Phi^s(z_p, z_p). && (2.25) \end{aligned}$$

Hence using the property

$$|\Phi^s(z_p, z_p)| \geq c_1[d(z_p, \partial D)]^{-1} \tag{2.26}$$

as shown in Theorem 2.1.15 of [P2] and the fact that

$$\alpha_p^2 := ||\Phi(\cdot, z_p)||^{-2}_{L^2(\partial B)} \geq c_2[\ln(d(z_p, \partial B))]^{-1} \tag{2.27}$$

for some positive constants c_1, c_2, we deduce that

$$\lim_{p\to\infty}\lim_{n\to\infty}\int_{\mathbb{S}}\int_{\mathbb{S}} u^\infty(-\theta, d)g_n^p(\theta)g_n^p(d)d\theta ds(d) = \infty. \tag{2.28}$$

Then $I_1(B) = \infty$. $\square$

3 The probe and singular sources methods for far field data coincide.

3.1 Singular sources method

The idea of the singular sources method is firstly to compute $\Phi^s(z,z)$ from the far field and secondly to evaluate its behavior with respect to the parameter z. The observation is that when z approaches ∂D then $\Phi^s(z,z)$ blows-up. To compute $\Phi^s(z,z)$ from the far field pattern, in [P2], the author uses the so-called back-projection operator, see ([P2], Definition 3.1.5 and Theorem 3.1.6). Its derivation is based on a mixed reciprocity relation. However, the calculation of $\Phi^s(z,z)$ can be justified directly by the use of the identity (2.8) and arguing as in (2.21)-(2.25). We obtain an approximation

$$4\pi\Phi^s(z,z) \approx \int_{\mathbb{S}}\int_{\mathbb{S}} u^\infty(-\theta,d)\ g(\theta)\ g(d)\ ds(\theta)ds(d) \tag{3.29}$$

with an appropriate function $g \in L^2(\mathbb{S})$ such that v_g approximates the point source on ∂D. Note that here we obtain an approximation with the quadratic form, whereas in [P2], Theorem 3.1.6, one needs the full bilinear form

$$\int_{\mathbb{S}}\int_{\mathbb{S}} u^\infty(-\theta,d)\ g(\theta)\ f(d)\ ds(\theta)ds(d) \tag{3.30}$$

with $f,g \in L^2(\mathbb{S})$ chosen appropriately to obtain the approximation of $\Phi^s(z,z)$. Thus, the proof via equation (2.8) yields better results than the application of the mixed reciprocity relation. But the computation of $\Phi^s(z,z)$ is meaningful just if $z \in \mathbb{R}^3 \setminus \overline{D}$. To state the singular sources method for any point $z \in \overline{D}$, we reformulate it in the following way:

Definition 3.1 *Let Ω be a large but bounded domain containing the unknown obstacle D. Let $z \in \Omega$. We take a curve from $\partial\Omega$ to reach z. We denote it c_z. We define Ω_z to be a C^2-regular domain contained strictly in $\Omega \setminus c_z$.*

On $\partial\Omega_z$, we approximate $\Phi(\cdot,z)$ by a sequence of Herglotz waves $v_{g_z^n}$. Then we compute the functional:

$$I_{ssm}(z) := \lim_{n\to\infty}\int_{\mathbb{S}}\int_{\mathbb{S}} u^\infty(-\theta,d)g_z^n(\theta)g_z^n(d)ds(\theta)ds(d).$$

We state the following claim.

Claim 3.2 *1) Let $z \in \Omega \setminus \overline{D}$. Then we can find a curve c_z and a domain Ω_z such that $\overline{D} \subset \Omega_z \subset \overline{\Omega_z} \subset \Omega \setminus c_z$. In this case we have:*

$$I_{ssm}(z) = 4\pi\Phi^s(z,z).$$

As a conclusion we have

$$\lim_{z\to\partial D} I_{ssm}(z) = \infty.$$

2) Let $z \in \overline{D}$. In this case for any curve c_z we cannot find Ω_z such that $\overline{D} \subset \Omega_z \subset \overline{\Omega_z} \subset \Omega \setminus c_z$. In this case:

$$I_{ssm}(z) = \infty.$$

3.2 Probe method

Consider some bounded domain Ω containing the obstacle D. As we said in the introduction, in [I2] the author proceeds in two steps to detect the obstacle from the far field. The first one is to go from the far field to the near field on $\partial\Omega$ i.e the boundary of Ω. In the second one from this computed near field he detects the obstacle.

The near field is given by the Dirichlet-Neumann map:

$$\Lambda_D : H^{\frac{1}{2}}(\partial\Omega) \Rightarrow H^{-\frac{1}{2}}(\partial\Omega),$$

where $\Lambda_D f := \frac{\partial u^f}{\partial \nu}\mid_{\partial\Omega}$ and $u^f \in H^1(\Omega \setminus \overline{D})$ satisfies

$$\left\{ \begin{array}{l} \Delta u^f + k^2 u^f = 0 \quad \text{in} \quad \Omega \setminus \overline{D}, \\ u^f = f \quad \text{on} \quad \partial\Omega, \\ u^f = 0 \quad \text{on} \quad \partial D. \end{array} \right. \tag{3.31}$$

From this data, the indicator function of the probe method is related to the following quadratic form:

$$\int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) f(x) \cdot f(x) ds(x) \tag{3.32}$$

where $\Lambda_\emptyset$ is the Dirichlet-Neumann map for (3.31) when $D = \emptyset$.

Let us now $z \in \Omega \setminus \overline{D}$, then we take Ω_z as in the subsection concerning the singular sources method. We approximate $\Phi(\cdot, z)$ on $\partial\Omega_z$ by a sequence of Herglotz waves $v_{g_z^n}$ (which is a reconstructive version of the Runge approximation used in the original probe method [I1]).

Evaluating the probe functional (3.32) for $f = v_{g_z^n}\mid_{\partial\Omega}$, using the Alessandrini identity on $\Omega \setminus \overline{D}$ (see, [Al] or [Is], Chap 3), we find:

$$I_{pb}(z) := \lim_{n\to\infty} \int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) v_{g_z^n}(x) \cdot v_{g_z^n}(x) ds(x) = \lim_{n\to\infty} \int_{\partial D} (\frac{\partial v^s_{g_z^n}}{\partial \nu}(x) - \frac{\partial v_{g_z^n}}{\partial \nu}(x)) \cdot v_{g_z^n}(x) ds(x),$$

and taking the limit with respect to n, we find:

$$I_{pb}(z) = \int_{\partial D} \left\{ \frac{\partial \tilde{\Phi}_\Omega}{\partial \nu}(x, z) - \frac{\partial \Phi}{\partial \nu}(x, z) \right\} \cdot \Phi(x, z) ds(x)$$

where we denoted by $v^s_{g_z^n}$ and $\tilde{\Phi}$ the solutions of (3.31) replacing f by $v_{g_z^n}|_{\partial\Omega}$ and $\Phi(\cdot, z)|_{\partial\Omega}$ respectively. The function

$$\Phi^s_\Omega(x, z) := \tilde{\Phi}(x, z) - \Phi(x, z)$$

is called the reflected solution for the problem (3.31). It satisfies the problem:

$$\left\{ \begin{array}{l} \Delta \Phi^s_\Omega + k^2 \Phi^s_\Omega = 0 \quad \text{in} \quad \Omega \setminus \overline{D}, \\ \Phi^s_\Omega(\cdot, z) = 0 \quad \text{on} \quad \partial\Omega, \\ \Phi^s_\Omega(\cdot, z) = -\Phi(\cdot, z) \quad \text{on} \quad \partial D. \end{array} \right. \tag{3.33}$$

Using the Green's formula on $\Omega \setminus \overline{D}$ for $\Phi^s_\Omega(\cdot, z)$ and $\Phi(\cdot, z)$ we deduce that

$$I_{pb}(z) = \Phi^s_\Omega(z, z) - \int_{\partial\Omega} \frac{\partial}{\partial \nu} \Phi^s_\Omega(x, z) \Phi(x, z) ds(x).$$

We are interested with z near $\overline{D}$. Since $\Phi^s_\Omega(\cdot,z)+\Phi(\cdot,z)$ satisfies

$$
\left\{\begin{array}{l} (\Delta+k^2)(\Phi^s_\Omega(\cdot,z)+\Phi(\cdot,z)) = -\delta(z) \quad \text{in} \quad \Omega\setminus\overline{D}, \\ (\Phi^s_\Omega(\cdot,z)+\Phi(\cdot,z)) = \Phi(\cdot,z), \quad \text{on} \quad \partial\Omega, \\ (\Phi^s_\Omega(\cdot,z)+\Phi(\cdot,z)) = 0, \quad \text{on} \quad \partial D \end{array}\right. \tag{3.34}
$$

then the function $\Phi^s_\Omega(\cdot,z)+\Phi(\cdot,z)$ can be seen as a sum of the Dirichlet Green's function of $\Delta+k^2$ on $\Omega\setminus\overline{D}$ and the solution of (3.34) replacing δ by zero. Since both of these two functions are bounded with respect to z, near $\overline{D}$, with values in $C^1(\partial\Omega)$, we deduce that

$$
\int_{\partial\Omega}\frac{\partial}{\partial\nu}\Phi^s_\Omega(x,z)\Phi(x,z)ds(x)
$$

is bounded with respect to z. This means that

$$
|I_{pb}(z)-\Phi^s_\Omega(z,z)| = O(1) \text{ for } z \text{ near } \overline{D}. \tag{3.35}
$$

Let us now compare $\Phi^s_\Omega(z,z)$ with $\Phi^s(z,z)$. We set $\Psi(x,z) := \Phi^s_\Omega(x,z)+\Phi^s(x,z)$. Then $\Psi(\cdot,z)$ satisfies:

$$
\left\{\begin{array}{l} \Delta\Psi+k^2\Psi = 0 \quad \text{in} \quad \Omega\setminus\overline{D}, \\ \Psi(\cdot,z) = \Phi(\cdot,z) \quad \text{on} \quad \partial\Omega, \\ \Psi(\cdot,z) = 0 \quad \text{on} \quad \partial D. \end{array}\right. \tag{3.36}
$$

For z near $\overline{D}$, $\Phi^s(\cdot,z)$ is bounded in $C^2(\partial\Omega)$. We can see it by remarking that $\Phi^s(x,z)+\Phi(x,z)$ is the Dirichlet Green's function of our equation on $\mathbb{R}^3\setminus\overline{D}$. Then the wellposedness of the problem (3.31) implies that $\Psi(\cdot,z)$ is bounded in $C^2(\Omega)$ hence $\Psi(z,z)$ is bounded. This means that

$$
|\Phi^s_\Omega(z,z)+\Phi^s(z,z)| = O(1) \text{ for } z \text{ near } \overline{D} \tag{3.37}
$$

Conclusion 3.3 *The indicator function of the original probe method is constructed from the Dirichlet-Neumann map by using a sequence of functions approximating the fundamental solution and the Alessandrini identity.*
Now, in place of using the Dirichlet to Neumann map as a starting point, we use the far field data $u^\infty(\theta,d), (\theta,d)\in\mathbb{S}^2$. Using the same sequence of functions approximating the fundamental solution and the identity (2.12), which is the far field counter part of the Alessandrini identity, we end up with:

$$
|I_{pb}(z)+4\pi\lim_{n\to\infty}\int_{\mathbb{S}}\int_{\mathbb{S}}u^\infty(-\theta,d)g^n_z(\theta)g^n(d)ds(\theta)ds(d)| = |I_{pb}(z)+\Phi^s(z,z)| = O(1),
$$

for z near $\overline{D}$. The second equality is given by a combination of (3.35) and (3.37).

This conclusion suggests that the natural farfield version of the probe method is the one given in Definition 3.1.

We state these two remarks on the singular sources and the probe methods as a theorem:

theorem 3.4 *The natural farfield versions of the probe method and the singular sources method are identical. This common version is given by Definition 3.1.*

3.3 Some comments

1. In [N-P-S] we defined the near field version of the singular sources method and we showed that it has the same convergence behavior as the probe method. Here, in this particular far field setting the two methods in fact are the same.

2. The full convergence proof of this natural far field version is not achieved yet. The first point of the claim is shown to be true, i.e $\Phi^s(z,z) \to \infty$ when $z \to \partial D$ (see [P2]). The justification of the second point of Definition 3.1 is not yet proven. A first result in this direction is the recent work [I3], where the obstacle boundary value problem is considered.

3. For the this far field version, we need to take some domain Ω containing the unknown obstacle. This is also the case for the linear sampling and the factorization methods. But such domain Ω can always be find using the first version of the no response test. It is given by testing if for any Ω we have $I_1(\Omega) = 0$. This shows how the combination of these methods can be useful...

4. The way of defining the set Ω_z in Definition 3.1 is nothing but the needle approach introduced in [I1].

4 The second version of the no-response test.

In this section, we develop the second version of the no-response test which combines the *superposition technique* and the *range test* idea [P-S-K].

Again, consider a bounded domain $B \subset \mathbb{R}^3$ with boundary of class C^2. The basic idea of the range test is to test the solvability of the equation

$$\frac{1}{4\pi}\int_{\partial B} e^{-i\kappa\theta\cdot x}\psi(x)ds(x) = u^\infty(\theta,d), \quad (\theta,d) \in \mathbb{S}\times\mathbb{S}. \tag{4.38}$$

Here, we will use this technique applied to the far field pattern v_g^∞ of the Herglotz wave functions used in the definition of the no response test above.

For regularization of the ill-posed integral equation (4.38) we use the *Tikhonov regularization scheme*

$$\psi_\alpha := (\alpha + S^{\infty,*}S^\infty)^{-1}S^{\infty,*}u^\infty \tag{4.39}$$

with *regularization parameter* $\alpha > 0$ and the far field operator $S^\infty : L^2(\partial B) \to L^2(\mathbb{S})$ is given by

$$(S^\infty\varphi)(\theta) := \frac{1}{4\pi}\int_{\partial B} e^{-i\kappa\theta\cdot y}\varphi(y)\,ds(y), \ \ \theta \in \mathbb{S}. \tag{4.40}$$

Definition 4.1 **(The second version of the no-response method.)** *For a non-vibrating domain B we define the indicator function*

$$\begin{aligned} I_2(B) \ &:= \ \lim_{\epsilon\to 0}\sup\Big\{\lim_{\alpha\to 0}||\psi_g^\alpha||_{L^2(\partial B)} : \ \ \psi_g^\alpha \textit{ is the regularized solution (4.39)} \\ &\qquad \textit{of } (4.38) \textit{ with } u^\infty = v_g^\infty,\ g \in L^2(\mathbb{S}) \textit{ and } ||v_g||_{L^2(\partial B)} < \epsilon\Big\} \end{aligned} \tag{4.41}$$

For the set $\mathcal{G}$ of non-vibrating domains B this second version of no response test calculates the indicator function $I_2(B)$ and builds the intersection

$$D_{rec,2} := \bigcap_{B\in\boldsymbol{B}_2} B, \tag{4.42}$$

where

$$\boldsymbol{B}_2 := \{B \in \mathcal{G} :\ I_2(B) = 0\}. \tag{4.43}$$

The convergence of the no response test is given by the following result.

THEOREM 4.2 **(Convergence of the second NRT version.)** *Let $\mathcal{G}$ be as in Definition 4.1. We have*

1. *if $\overline{D} \subset B$ then $I_2(B) = 0$ and*
2. *if $\overline{D} \not\subset B$ then $I_2(B) = \infty$.*

Thus, the obstacle D can be characterized by

$$D = D_{rec,2}.$$

The following lemmas are the key tools to prove Theorem 4.2. The proof of the first one can be found in [C-K] and [P-S-K].

LEMMA 4.3 *Let B be a domain with boundary of class C^2. We consider an injective integral operator with continuous kernel and dense range*

$$(A\psi)(\theta) := \int_{\partial B} k(\theta, y)\psi(y)ds(y), \theta \in \mathbb{S} \tag{4.44}$$

from $L^2(\partial B)$ to $L^2(\mathbb{S})$. Then the Tikhonov regularized solution of the equation $A\psi = f$ given by

$$\psi_\alpha := (\alpha I + A^* A)^{-1} A^* f \tag{4.45}$$

where α is the regularized parameter and A^* is the adjoint of A satisfies

$$\lim_{\alpha \to 0} \|\psi_\alpha\|_{L^2(\partial B)} = \begin{cases} \infty, & \text{if } f \text{ is not in } A(L^2(\partial B)), \\ \|\psi^*\|_{L^2(\partial B)}, & \text{if } A\psi^* = f. \end{cases} \tag{4.46}$$

From Rellich's Lemma we immediately obtain the following result.

LEMMA 4.4 *If the equation (4.38) is solvable, i.e. there exists $\psi \in L^2(\partial B)$ such that*

$$S^\infty \psi = u^\infty$$

then the scattered field u^s of u^∞ is given by

$$u^s = \int_{\partial B} \Phi(\cdot, y)\psi(y)ds(y) \ \text{in } \mathbb{R}^3 \setminus (B \cup D).$$

Also, we collect basic mapping properties of the single-layer operator.

LEMMA 4.5 *The operator $S : L^2(\partial B) \to H^1(\partial B)$ defined by:*

$$(S\psi)(x) := \int_{\partial B} \Phi(x, y)\psi(y)ds(y), \ \ x \in \partial B \tag{4.47}$$

is an isomorphism if B is a non-vibrating domain and ∂B has C^3-regularity.

Proof of Lemma 4.5. In [ML], Theorem 7.17, it is proved that for such regular domains B the operator $S : L^2(\partial B) \to H^1(\partial B)$ is Fredholm with index zero. The injectivity and hence the surjectivity of S is given by the assumption that B is a non-vibrating domain.

Remark 4.6 *Since we are free to choose B, then C^3-regularity of ∂B is enough. In [A-K], Theorem 1.8 the lemma is proved for Lipschitz regularity of ∂B where the proof is given for the case $k = 0$.*

Lemma 4.7 *The operator $S^\infty : L^2(\partial B) \to L^2(\mathbb{S})$ has a dense range if B is a non-vibrating domain.*

Proof of Lemma 4.7. We define $\mathcal{F}_B : H^{\frac{1}{2}}(\partial B) \to L^2(\mathbb{S})$ to be the far field map for the artificial obstacle B, i.e for $u \in H^{\frac{1}{2}}(\partial B)$, $\mathcal{F}_B u$ is the far field of the solution u^s of the $(\Delta + \kappa^2)u^s = 0$ satisfying the Sommerfeld radiation condition and $u^s = u$ on ∂B.
We write $S^\infty = \mathcal{F}_B S$. We consider the operator $S : H^{-\frac{1}{2}}(\partial B) \to H^{\frac{1}{2}}(\partial B)$, which is an isomorphism [ML], and we set also the Herglotz wave operator $H : L^2(\mathbb{S}) \to H^{\frac{1}{2}}(\partial B)$, $Hg := v_g|_{\partial B}$ where v_g is the Herglotz wave function (2.5). We denote by H^* its dual operator from $H^{-\frac{1}{2}}(\partial B) \to L^2(\mathbb{S})$. For $\phi \in L^2(\partial B)$, we have

$$H^*\phi = \int_{\partial B} \phi(x) e^{-ikd\cdot x} ds(x) = 4\pi S^\infty \phi.$$

which means that

$$S^\infty \phi := \mathcal{F}_B S \phi = \frac{1}{4\pi} H^* \phi \tag{4.48}$$

for every $\phi \in L^2(\partial B)$ and hence for every $\phi \in H^{-\frac{1}{2}}(\partial B)$ by the continuity of $\mathcal{F}_B$, S, H^* and the denseness of $L^2(\partial B)$ in $H^{-\frac{1}{2}}(\partial B)$.

Now since B is a non-vibrating domain then H is injective, hence H^* has a dense range. From (4.48), we see that the operator S^∞ stated on $H^{-\frac{1}{2}}(\partial B)$ has a dense range. Finally, S^∞ stated on $L^2(\partial B)$ has a dense range since $L^2(\partial B)$ is dense in $H^{-\frac{1}{2}}(\partial B)$ and S^∞ is continuous on $H^{-\frac{1}{2}}(\partial B)$ (since $\mathcal{F}_B : H^{\frac{1}{2}}(\partial B) \to L^2(\mathbb{S})$ and $S : H^{-\frac{1}{2}}(\partial B) \to H^{\frac{1}{2}}(\partial B)$ are continuous). $\square$

Proof of Theorem 4.2. We will investigate the two cases $\overline{D} \subset B$ and $\overline{D} \not\subset B$ in two steps.

I. Case One. Consider the case where $\overline{D} \subset B$. We take any $g \in L^2(\mathbb{S})$ satisfying $||v_g||_{L^2(\partial B)} < \epsilon$. As for the case one of the first version, using the regularity of very week solution for elliptic problems and interior estimates, this implies that

$$||v_g||_{C^1(\partial D)} < c\epsilon \tag{4.49}$$

with some positive constant c. Since $\overline{D} \subset B$ the scattered field v_g^s has a trace on ∂B which is in $C^1(\partial B)$ and $||v_g^s||_{C^1(\partial B)} < C\epsilon$ with some appropriate constant C depending on the scatterer and on ∂B. In this case, by lemma 4.5 the single-layer equation

$$S\psi = v_g^s \text{ on } \partial B \tag{4.50}$$

has a solution $\psi \in L^2(\partial B)$ which satisfies

$$||\psi||_{L^2(\partial B)} \leq \tilde{c}\epsilon \tag{4.51}$$

with a further constant $\tilde{c}$. Hence, also the equation (4.38) with $u^\infty = v_g^\infty$ is solvable with ψ as the solution and we obtain $I_2(B) = 0$.

II. Case Two. Assume that $\overline{D} \not\subset B$. II.A) We first assume that $D \not\subset \overline{B}$. Then, there is a point $z \in \partial D \setminus \overline{B}$. We choose some arbitrary $\epsilon > 0$. As in section 2, we take a sequence of points $z_p \in \mathbb{R}^3 \setminus (\overline{D \cup B})$ such that $z_p \to z$ and construct Herglotz wave functions which approximate

$\epsilon/2 \cdot \alpha_p$ times the point source $\Phi(\cdot, z_p)$ in $L^2(\partial B_p)$ where $\overline{D \cup B} \subset B_p$ and $z_p \notin B_p$. We recall that $\alpha_p := (||\Phi(\cdot, z_p)||_{L^2(\partial B)})^{-1}$. Also, using the well-posedness of the scattering problem we obtain that

$$v^{\infty}_{g^p_n} \to \frac{\epsilon}{2}\alpha_p \Phi^{\infty}(\cdot, z_p), \quad n \to \infty, \tag{4.52}$$

in $L^2(\mathbb{S})$.

Next, we need to consider the solvability of the equation (4.38) with right-hand side $u^{\infty} = v^{\infty}_{g^p_n}$. Here, we will distinguish two possibilities:

$II.A.\alpha$) There exists a couple (p_0, n_0) such that $\|v_{g^{p_0}_{n_0}}\|_{L^2(\partial B)} < \epsilon$ and (4.38) is not solvable. In this case by the Tikhonov regularization, we find a regularized sequence of solutions $\psi^{\alpha}_{g^{p_0}_{n_0}}$ of (4.38) such that $\|\psi^{\alpha}_{g^{p_0}_{n_0}}\|_{L^2(\partial B)}$ tends to ∞ as α tends to zero, where α is the regularization parameter. Hence we obtain the desired statement.

$II.A.\beta$) For every couple (p, n) such that $\|v_{g^p_n}\|_{L^2(\partial B)} < \epsilon$ the equation (4.38) is solvable. By Lemma 4.4 for every such couple (p, n), $v^s_{g^p_n}$ is extendable up to ∂B such that its trace is in $H^1(\partial B)$. Now solving the equation (4.50) with right-hand side $v^s_{g^p_n}$ we get a sequence of solutions $\psi_{g^p_n}$ of (4.38), with right-hand side given by $v^{\infty}_{g^p_n}$ i.e.

$$S^{\infty}\psi^p_n = v^{\infty}_{g^p_n}. \tag{4.53}$$

We distinguish two cases. The first one is that there exists p_0 such that the sequence $\|\psi_{g^{p_0}_n}\|_{L^2(\partial B)}$ is unbounded. In this case we obtain the desired statement. The second case is that for every p the sequence $||\psi_{g^p_n}||_{L^2(\partial B)}$ is bounded. In this case for every p fixed we can find a function $\psi_p \in L^2(\partial B)$ such that $\psi_{g^p_n}$ tends weakly to ψ_p in $L^2(\partial B)$. For every p fixed, we take the limit in (4.53) with respect to n. We obtain,

$$S^{\infty}\psi_p = \frac{\epsilon}{2}\alpha_p \Phi^{\infty}(\cdot, z_p) \tag{4.54}$$

on $\mathbb{S}$. This means that (4.38) is solvable for u^{∞} being the far field pattern of $\Phi^s(\cdot, z_p)$. Hence using again Lemma 4.4, we deduce that $\Phi^s(\cdot, z_p)$ is extendable up to ∂B with boundary values in $H^1(\partial B)$. The solution of (4.54) is then given by the solution of

$$S\psi_p = \frac{\epsilon}{2}\alpha_p \Phi^s(\cdot, z_p) \quad \text{on } \partial B. \tag{4.55}$$

In this case the sequence α_p is bounded from below by a positive constant since $(z_p)_{p\in\mathbb{N}}$ is in $\partial D \setminus \overline{B}$. We show that $\Phi^s(\cdot, z_p)$ is unbounded in $H^1(\partial B)$. Indeed suppose that $||\Phi^s(\cdot, z_p)||_{H^1(\partial B)}$ is bounded. Since $\Phi^s(x, z_p)$ satisfies $-\Delta\Phi^s(x, z_p) + \kappa^2\Phi^s(x, z_p) = 0$ in $\mathbb{R}^3 \setminus \overline{B}$ then $\Phi^s(\cdot, z_p)$ is bounded in $H^{\frac{3}{2}}(\mathbb{R}^3 \setminus \overline{B})$ hence also in $H^1(\partial D \cap (\mathbb{R}^3 \setminus \overline{B}))$. But $\Phi^s(\cdot, z_p) = \Phi(\cdot, z_p)$ on ∂D and $\Phi(\cdot, z_p)$ in unbounded in $H^1(\partial D \cap (\mathbb{R}^3 \setminus \overline{B}))$. This gives a contradiction.

II.B) We now assume that $D \subset B$. From $\overline{D} \not\subset B$ we obtain that there is a point $z \in \partial B \cap \partial D$. Let z_p be a sequence of points in $\mathbb{R}^3 \setminus \overline{(B \cup D)}$ tending to z. Applying the Green's Theorem on $\mathbb{R}^3 \setminus \overline{B}$ for $\Phi^s(\cdot, z_p)$ and $\Phi(\cdot, z_p)$, we get

$$\alpha_p \int_{\partial B} \Phi^s(\cdot, z_p)\frac{\partial \Phi}{\partial \nu}(\cdot, z_p)ds(x) - \alpha_p \int_{\partial B} \Phi(\cdot, z_p)\frac{\partial \Phi^s}{\partial \nu}(\cdot, z_p)ds(x) = \alpha_p \Phi^s(z_p, z_p). \tag{4.56}$$

Hence if $\alpha_p\Phi^s(\cdot, z_p)$ is bounded in $H^1(\partial B)$ then $\alpha_p \frac{\partial \Phi^s}{\partial \nu}(\cdot, z_p)$ is also bounded in $L^2(\partial B)$. Using the estimate

$$||\frac{\partial \Phi}{\partial \nu}(\cdot, z_p)||_{H^{-1}(\partial B)} \leq ||\Phi(\cdot, z_p)||_{L^2(\partial B)} \tag{4.57}$$

the second member of (4.56) behaves like $||\Phi(\cdot, z_p)||_{L^2(\partial B)} \approx (\ln(d(z_p, \partial B)))^{\frac{1}{2}}$. This fact, the estimate

$$\Phi^s(z_p, z_p) \approx \frac{1}{d(z_p, \partial B)},$$

(see Theorem 2.1.15 of [P2]) and 4.56 imply

$$O((\ln(d(z_p, \partial B)))^{\frac{1}{2}}) = O(\frac{(\ln(d(z_p, \partial B)))^{\frac{1}{2}}}{d(z_p, \partial B)})$$

which is impossible. Thus, we have

$$||\alpha_p \Phi^s(\cdot, z_p)||_{H^1(\partial B)} \to \infty, \;\; p \to \infty. \tag{4.58}$$

Then, also $\psi_p = \frac{\epsilon}{2}\alpha_p S^{-1}\Phi^s(\cdot, z_p)$ cannot be bounded in $L^2(\partial B)$.

Finally in all these cases, we constructed sequences g_n^p such that $\|v_{g_n^p}\|_{L^2(\partial B)} < \epsilon$ and $\lim_{\alpha \to 0} \|\psi^{\alpha}_{g_n^p}\|_{L^2(\partial D)}$ is unbounded. This means that $I_2(B) = \infty$. $\square$

Remark 4.8 *In the definition of the two versions of the no-response test, we can replace in* (2.13) *and* (4.41) *the* $L^2(\partial B)$ *norm by the* $C^1(\partial B)$ *norm. For these changes, the convergence of these two methods is as follows:*
$1')$ *If* $D \subset B$ *then* $I_1(B) = I_2(B) = 0$.
$2'$ *If* $D \not\subset B$ *then* $I_1(B) = I_2(B) = \infty$.
The difference with the original versions is that $\overline{D} \subset B$ *implies that* $I_1(B) = I_2(B) = 0$. *This means that if* $\partial D \not\subset B$ *then* $I_1(B) = I_2(B) = 0$ *which is not the case for original versions we gave. This is due to* (2.12) *and* (4.50) *respectively and the fact that the* $C^1(\partial B)$*-norm estimate of* v_g *implies the* $C^1(\partial B)$*-norm estimate of* v_g^s.

5 Convergence of the linear sampling method implies the convergence of the no response method

In this section we recall the linear sampling method and show how its convergence implies the convergence of the second version of the no response test.

The linear sampling method. The fundamental object of the linear sampling method is the following linear integral operator $F : L^2(\mathbb{S}) \to L^2(\mathbb{S})$, given by

$$Fg(\theta) := \int_{\mathbb{S}} u^{\infty}(\theta, d) g(d) ds(d), \;\; \theta \in \mathbb{S}.$$

This operator is called the far-field operator. Let $g \in L^2(\mathbb{S})$ and $v_g := \int_{\mathbb{S}} e^{i\kappa d \cdot x} g(\theta) ds(\theta), x \in \mathbb{R}^3$. From the asymptotic behavior of this fundamental solution we know that the far-field of $\Phi(\cdot, z)$, $z \in \mathbb{R}^3$, is given by

$$\Phi_{\infty}(\theta, z) = \frac{1}{4\pi} e^{-i\kappa\theta\cdot z}, \;\; \theta \in \mathbb{S}.$$

When $z \in D$, then $\Phi_{\infty}(\hat{x}, z) = \Phi^{\infty}(\hat{x}, z)$, where we used $\Phi^{\infty}(\hat{x}, z)$ to be the far field pattern of the scattered field $\Phi^s(x, z)$ created by the obstacle D using $\Phi(x, z)$ as the incident wave.

The idea of the linear sampling method is to approximately solve the following integral equation, called the far field equation:

$$Fg_z = \Phi_{\infty}(\cdot, z) \tag{5.59}$$

for a grid of points z and to look at the behavior of the norms of g_z. It is observed that these norms blow up near and outside ∂D. The behavior of the norms of g_z given for a grid of points z is used to localize D.

A detailed version is given in the following theorem, see [C-C].

THEOREM 5.1 *Assume that κ^2 is not a Dirichlet eigenvalue of $-\Delta$ in D. We have*

1) If $z \in D$, then for every $\epsilon > 0$ there exists a solution $g^\epsilon(\cdot, z)$ in $L^2(\mathbb{S})$ of the inequality

$$\|Fg^\epsilon(\cdot, z) - \Phi^\infty(\cdot, z)\|_{L^2(\mathbb{S})} < \epsilon$$

such that

$$\lim_{z \to \partial D} \|g^\epsilon(\cdot, z)\|_{L^2(\mathbb{S})} = \lim_{z \to \partial D} \|v_{g^\epsilon}(\cdot, z)\|_{H^1(D)} = \infty.$$

2) If $z \in \mathbb{R}^3 \setminus \overline{D}$, then for every $\epsilon > 0$ and $\delta > 0$ there exists a solution $g^{\epsilon,\delta}(\cdot, z)$ in $L^2(\mathbb{S})$ of the inequality

$$\|Fg^{\epsilon,\delta}(\cdot, z) - \Phi^\infty(\cdot, z)\|_{L^2(\mathbb{S})} < \epsilon + \delta$$

such that

$$\lim_{\delta \to 0} \|g^{\epsilon,\delta}(\cdot, z)\|_{L^2(\mathbb{S})} = \lim_{\delta \to 0} \|v_{g^{\epsilon,\delta}}(\cdot, z)\|_{H^1(D)} = \infty.$$

Using the density g of the linear sampling method (LSM) for the second NRT version. To prove the convergence of the linear sampling method one has to assume that k^2 is not a Dirichlet eigenvalue for $-\Delta$ on D, i.e D is a non-vibrating domain. With this assumption, from Theorem 5.1 we have a sequence g^ϵ which creates the blow-up. Using this sequence g^ϵ we will now justify the blow-up in the case two of the second version of the no-response test. This means that in any case where the linear sampling method converges then the no-response converges too. In addition, the no-response converges even if the obstacle D is vibrating. This has been shown in section 3.

Let us now explain how we can use the singular sequence g^ϵ of the linear sampling method to create the blow-up for the no-response test. To this end we will go into the explicit construction of the sequence as carried out in [C-C] and use its properties. Note that for some general solution g^ϵ of (5.59) it is not yet proven that it will coincide with this particular solution whose existence is stated by the above theorem. However, our assumption will be that the linear sampling method is convergent in the sense that it picks this particular solution plus some additive function $\tilde{g}$, for which the Herglotz wave function is bounded on $\overline{D}$.

THEOREM 5.2 *Consider a test domain B such that $\overline{D} \not\subset B$. Given the densities $g^\epsilon(\cdot, z)$ provided by Theorem 5.1 as the basic ingredient for the indicator function of the linear sampling method, there is a density $\tilde{g}(\cdot, z)$ such that the Herglotz wave function $v_{\tilde{g}(\cdot,z)}$ is bounded in a neighbourhood of z and the density $g_{RT}(\cdot, z) := g^\epsilon(\cdot, z) + \tilde{g}(\cdot, z)$ leads to a blow-up of the functional I_2 of the second NRT version.*

Remark. The modification by $\tilde{g}$ is necessary only to tailor the Herglotz wave function of the linear sampling method to the normalization assumptions on B demanded by the no-response test. Alternatively, we could just neglect the normalization assumption of the no response test and feed the density $g^\epsilon(\cdot, z)$ into the functional $I_{1,\epsilon}$ defined in (2.14). We consider this to be an interesting question for the further analysis of the linear sampling method in its connection to the no response test.

Proof. By assumption we have a non-vibrating domain B such that $\overline{D} \not\subset B$. The situation $\overline{D} \not\subset B$ means that either $\partial D \setminus \overline{B} \neq \emptyset$ or $\partial D \subset \overline{B}$.

I. We consider first the case where $\partial D\setminus\overline{B}\neq\emptyset$. For this case, we may choose a point $a\in\partial D\setminus\overline{B}$ and a sequence $z_p\to a$ for $p\to\infty$ with $z_p\in D$. From the part 1) of Theorem 5.1, there is a sequence $g^\epsilon(\cdot,z_p)$ such that we have

$$\|Fg^\epsilon(\cdot,z_p)-\Phi^\infty(\cdot,z_p)\|_{L^2(\mathbb{S})}<\epsilon$$

and

$$\lim_{z_p\to a}\|v_{g^\epsilon}(\cdot,z_p)\|_{H^1(D)}=\infty. \tag{5.60}$$

In [C-C], page 416, the sequence $v_{g^\epsilon(\cdot,z_p)}$ is constructed such that it tends to some single layer potential

$$S\phi_{z_p}(x):=\int_{\partial D}\Phi(x,y)\phi_{z_p}(y)ds(y),\quad x\in\partial D, \tag{5.61}$$

in $H^{\frac{1}{2}}(\partial D)$ where ϕ_{z_p} is the solution of the integral equation $S\phi_{z_p}(x)=-\Phi(x,z_p)$.

Let us now consider the sequence $(v^\infty_{g^\epsilon}(\cdot,z_p))_{p\in\mathbb{N}}$. With this sequence taken as the right-hand side of (4.38) either the equation (4.38) is not solvable for some p_0 and hence its regularized solution $\psi^\epsilon_{p_0,\alpha}$ satisfies

$$\lim_{\alpha\to 0}\|\psi^\epsilon_{p_0,\alpha}\|_{L^2(\partial B)}=\infty$$

or (4.38) with right-side $v^\infty_{g^\epsilon}(\cdot,z_p)$ is solvable for every p and then via Lemma 4.4 the scattered fields $v^s_{g^\epsilon}(\cdot,z_p)$ will be extendable up to ∂B such that its trace is in $H^1(\partial B)$. Now the corresponding sequence of solutions ψ^ϵ_p of (4.38) satisfies the equation

$$\int_{\partial B}\Phi(x,y)\psi^\epsilon_p(y)ds(y)=v^s_{g^\epsilon}(x,z_p)\text{ on }\partial B. \tag{5.62}$$

Next, we will prove that the sequence $v^s_{g^\epsilon}(\cdot,z_p)$ is not bounded in $H^{\frac{1}{2}}(\partial B)$ (and hence in $H^1(\partial B)$). Indeed, suppose that the sequence $(v^s_{g^\epsilon}(\cdot,z_p))_{p\in\mathbb{N}}$ is bounded in $H^{\frac{1}{2}}(\partial B)$. The wellposedness of the forward scattering problem on $\mathbb{R}^3\setminus B$ implies that $(v^s_{g^\epsilon}(\cdot,z_p))_{p\in\mathbb{N}}$ is bounded in $H^1_{loc}(\mathbb{R}^3\setminus B)$ hence $v_{g^\epsilon}(\cdot,z_p)\mid_{\partial D\cap\mathcal{V}(a)}$ $(=-v^s_{g^\epsilon}(\cdot,z_p)\mid_{\partial D\cap\mathcal{V}(a)})$, where $\mathcal{V}(a)$ is a neighborhood of a such that $\mathcal{V}(a)\cap\overline{B}=\emptyset$, is bounded in $L^2(\partial D\cap\mathcal{V}(a))$. We recall that $v_{g^\epsilon}(\cdot,z_p)$ approximates $S\phi_{z_p}$ in $H^{\frac{1}{2}}(\partial D)$, hence also in $L^2(\partial D\cap\mathcal{V}(a))$. Since $S\phi_{z_p}=-\Phi(\cdot,z_p)$ on ∂D we deduce that the sequence $(\Phi(\cdot,z_p))_{p\in\mathbb{N}}$ is bounded in $L^2(\partial D\cap\mathcal{V}(a))$, which is not true. Then the sequence $(v^s_{g^\epsilon}(\cdot,z_p))_{p\in\mathbb{N}}$ is unbounded in $H^{\frac{1}{2}}(\partial B)$.

From (5.62), the sequence of solutions, ψ^ϵ_p, satisfies

$$\lim_{p\to\infty}\|\psi^\epsilon_p\|_{L^2(\partial B)}=\infty. \tag{5.63}$$

Normalization of the sequence $v_{g^\epsilon}(\cdot,z_p)$. To finish the proof for this case, we need to normalize the

sequence $v_{g^\epsilon}(\cdot,z_p)$, i.e. to have $||v_{g^\epsilon}(\cdot,z_p)||_{L^2(\partial B)}\leq\epsilon$. It is enough to prove that this sequence is bounded. Hence, since every step in the argument is linear, multiplying it by ϵ we get the desired property.

We start by proving that $v_{g^\epsilon}(\cdot,z_p)$ is bounded in $H^1(D\cap K)$, for any C^2-regular domain K not containing a neighborhood of the point a.

Indeed, The function $W_{z_p}:=S\phi_{z_p}+\Phi(\cdot,z_p)$ satisfies

$$\begin{cases}\Delta W_z+\kappa^2W_z=-\delta(z)\quad\text{in}\quad D,\\ W_z=0\quad\text{on }\partial D,\end{cases} \tag{5.64}$$

i.e W_z is the Green's function on D. From the estimates of this Green's function we deduce that the sequence $\|W_{z_p}\|_{H^1(D\cap K)}$, $p\in\mathbb{N}$, is bounded, hence $S\phi_{z_p}$ also has the same property since the sequence $\Phi(\cdot,z_p)$ does. This implies that for ϵ fixed the sequence $\|v_{g^\epsilon(\cdot,z_p)}\|_{H^1(D\cap K)}$ is bounded.

Let us now consider its $L^2(\partial B)$-norm. The only information we know is that $||v_{g^\epsilon}(\cdot,z_p)-S\phi_{z_p}||_{H^1(D)}\leq\epsilon$ and we have no information on the behavior of the sequence $(g^\epsilon(\cdot,z_p))_p$ in $B\setminus\overline{D}$. Thus we cannot affirm its boundedness in $L^2(\partial B)$. For this reason, we modify it by another sequence which has this property and which behaves as $v_{g^\epsilon}(\cdot,z_p)$ near the point a.

Let E be any C^2-regular domain containing $D\cup B$ such that $\mathcal{V}(a)\subset\partial E\cap(\partial D\setminus B)$. We also assume that E is a non-vibrating domain. We take any $C^\infty(\mathbb{R}^3)$ function χ equal to 1 in a domain containing $\mathcal{V}(a)$ and zero in a domain containing some neighborhood $\mathcal{V}(B)$ of B. We set $\tilde{v}_g(\cdot,z_p):=\chi v_{g^\epsilon}(\cdot,z_p)$. For p fixed, let $(v_n^p)_{n\in\mathbb{N}}$ be a sequence of Herglotz functions approximating $\tilde{v}_g(\cdot,z_p)$ in $L^2(\partial E)$.

The sequence $(\tilde{v}_{g^\epsilon}(\cdot,z_p)-v_n^p)_{(p,n)\in\mathbb{N}^2}$ satisfies

$$\Delta[\tilde{v}_{g^\epsilon}(\cdot,z_p)-v_n^p]+k^2[\tilde{v}_{g^\epsilon}(\cdot,z_p)-v_n^p]=\Delta\chi v_{g_\epsilon}(\cdot,z_p)+\nabla\chi\cdot\nabla v_{g_\epsilon}(\cdot,z_p)\text{ in }E. \tag{5.65}$$

Since $\nabla\chi=0$ in a neighborhood of $\mathcal{V}(a)$ and in a neighborhood of B and $v_{g^\epsilon}(\cdot,z_p)$ is bounded in $H^1(D\cap K)$, for every domain K such that $K\cap\mathcal{V}(a)=\emptyset$, we deduce that the right hand side of (5.65) is bounded in $L^2(E)$. Using the regularity of the week solution for elliptic problems with L^2- Dirichlet boundary condition the sequence $(\tilde{v}_{g^\epsilon}(\cdot,z_p)-v_n^p)_{p,n}$ is bounded in $L^2(E)$ (see [N]). Again from (5.65) and using the interior estimates we conclude that the sequence $(\tilde{v}_{g^\epsilon}(\cdot,z_p)-v_n^p)_{n,p}$ is bounded in $H^2(B)$ and in $H^2(D)$.
Since $\tilde{v}_{g^\epsilon}=0$ in $\mathcal{V}(B)$ then the sequence $(v_n^p)_{n,p}$ is bounded in $H^2(B)$, and in particular in $L^2(\partial B)$. Also since $v_{g^\epsilon}(\cdot,z_p)=\tilde{v}_{g^\epsilon}(\cdot,z_p)$ in a neighborhood of a, i.e. $\mathcal{V}(a)$, then $[v_n^p-v_{g^\epsilon}(\cdot,z_p)]_{n,p}$ is bounded in $H^2(\mathcal{V}(a))$. We set $g_{RT}:=g_n^p$.

Now repeating the arguments applied for $v^\infty_{g^\epsilon}(\cdot,z_p)$ before replacing it by $v^\infty_{g_{RT}}$, we deduce that $I_2(B)=\infty$.

This gives a justification of the second case of Theorem 4.2.

II. Consider now the case where $\partial D\subset\overline{B}$ and assume that $a\in\partial D\cap\partial B$. We have

$$||v_{g^\epsilon}(\cdot,z_p)+\Phi(\cdot,z_p)||_{H^{\frac{1}{2}}(\partial D)}<\epsilon.$$

The wellposedness of the forward scattering problem gives

$$||v^s_{g^\epsilon}(\cdot,z_p)+\Phi(\cdot,z_p)||_{H^{\frac{1}{2}}(\partial B)}<C\epsilon \tag{5.66}$$

with some positive constant C. From this estimate we deduce that

$$||v^s_{g^\epsilon}(\cdot,z_p)||_{H^1(\partial B)}\to\infty\ (z_p\to a).$$

This property gives the justification of the convergence of the second version of the no-response test. But as for the case I, we need to normalize it in the $L^2(\partial B)$-norm. In this case the argument given in I. doesn't work. We give another way to justify it which uses the information $\partial D\subset B$. For this, we argue as follows. We take a particular sequence $z_p:=a+\frac{1}{p}\nu(a)$, where $a\in\partial D\cap\partial B$ and $\nu(a)$ is the exterior unit normal at a of ∂B. We set $z_p^*:=a-\frac{1}{p}\nu(a)$. We need the following lemma.

Lemma 5.3 *There exists a constant $C>0$ such that*

$$||\Phi(\cdot,z_p)-\Phi(\cdot,z_p^*)||_{H^{\frac{1}{2}}(\partial B)}\leq C,\forall p\in\mathbb{N} \tag{5.67}$$

Proof. We set $\Psi_\kappa(\cdot, z_p) := \Phi_\kappa(\cdot, z_p) - \Phi_\kappa(\cdot, z_p^*)$. The index κ is used to distinguish between the fundamental solution of the Helmholtz equation and the one of the Laplace equation, i.e $\kappa = 0$.

The distribution $\Psi(\cdot, z_p) := \Psi_\kappa(\cdot, z_p) - \Psi_0(\cdot, z_p)$ satisfies

$$(\Delta + \kappa^2)\Psi(\cdot, z_p) = \kappa^2[\Phi_0(\cdot, z_p^*) - \Phi_0(\cdot, z_p)] \text{ in } \mathbb{R}^3 \tag{5.68}$$

where $\Phi_0(x, z_p) := \frac{1}{|x-z_p|}$. We take a large domain K containing the sequences $(z_p)_{p\in\mathbb{N}}$ and $(z_p^*)_{p\in\mathbb{N}}$. Hence from (5.68), we get that $\Psi(\cdot, z_p)$ is bounded in $H^1(K)$ since the second member of (5.68) is bounded in $L^2(K)$ and the boundary conditions on ∂K are bounded in $C^m(\partial K)$, for any m in $\mathbb{N}$, which is due to the fact that the sequence z_p is a way from ∂K.

This implies that $\Psi\mid_{\partial B}$ is bounded $H^{\frac{1}{2}}(\partial B)$. Then it is enough to prove (5.67) for $\Psi_0(\cdot, z_p)$. We consider $||\Psi_0(\cdot, z_p)||_{L^2(\partial B)}$.

$$|\frac{1}{|x-z_p|} - \frac{1}{|x-z_p^*|}| = |\frac{|x-z_p^*| - |x-z_p|}{|x-z_p||x-z_p|}| \leq \frac{|z_p - z_p^*|}{|x-z_p||x-z_p^*|}.$$

Hence

$$\int_{\partial B} |\frac{1}{|x-z_p|} - \frac{1}{|x-z_p^*|}|^2 ds(x) \leq |z_p - z_p^*|^2 \int_{\partial B} \frac{1}{|x-z_p|^2|x-z_p^*|^2}$$

$$\leq C|z_p - z_p^*|^2 \frac{1}{|z_p - z_p^*|} = C|z_p - z_p^*|.$$

Then

$$||\Psi_0(\cdot, z_p)||^2_{L^2(\partial B)} \leq C|z_p - z_p^*| \tag{5.69}$$

with some constant $C > 0$.

We consider now $||\Psi_0(\cdot, z_p)||^2_{H^1(\partial B)}$. It is enough to consider the estimate of the tangential derivative $||\nabla_T\Psi_0(\cdot, z_p)||_{L^2(\partial B)}$. We have

$$\nabla\Psi_0(\cdot, z_p) = \frac{(x-z_p)}{|x-z_p|^3} - \frac{(x-z_p^*)}{|x-z_p^*|^3} = \frac{(x-z_p)}{|x-z_p|^3} + \frac{(z_p-z_p^*)}{|x-z_p|^3} - \frac{(x-z_p^*)}{|x-z_p^*|^3} =$$

$$= \frac{(z_p-z_p^*)}{|x-z_p|^3} + (x-z_p^*)[\frac{|x-z_p^*|^3 - |x-z_p|^3}{|x-z_p^*|^3|x-z_p|^3}]$$

hence the tangential derivative satisfies

$$|\nabla_T\Psi_0(\cdot, z_p)| \leq \frac{|z_p-z_p^*|}{|x-z_p|^3} + |x-z_p^*| \frac{C_1|z_p-z_p^*|}{|x-z_p^*|^2|x-z_p|^3}$$

where $c_1 > 0$ is a constant. Then the square integrals of the first and the second terms of the left hand side of the last inequality give

$$\int_{\partial B} |\nabla\Psi_0(\cdot, z_p)|^2 ds(x) \leq \frac{C_2}{|z_p - z_p^*|}$$

which means that

$$||\Psi_0(\cdot, z_p)||^2_{H^1(\partial B)} \leq \frac{C_3}{|z_p - z_p^*|} \tag{5.70}$$

with some positive constants C_2, C_3. From (5.69) and (5.70) and the interpolation theorem between $L^2(\partial B) := H^0(\partial B)$ and $H^1(\partial B)$, we get

$$||\Psi_0(\cdot, z_p)||_{H^{\frac{1}{2}}(\partial B)} \leq C_4$$

with some positive constant C_4. This ends the proof of Lemma 5.67.

From (5.67) and (5.66), we deduce that

$$||v^s_{g^\epsilon}(\cdot, z_p) + \Phi(\cdot, z^*_p)||_{H^{\frac{1}{2}}(\partial B)} \leq C_5 \tag{5.71}$$

for some positive constant C_5.

For every p fixed, let $v_{g^p_n}$ be a Herglotz sequence of functions tending to $v^s_{g^\epsilon}(\cdot, z_p)$ in $H^{\frac{1}{2}}(\partial B)$, i.e.

$$\forall p \in \mathbb{N}, \exists N(p)/\forall n \geq N(p) \text{ we have } ||v_{g^p_n} - v^s_{g^\epsilon}(\cdot, z_p)||_{H^{\frac{1}{2}}(\partial B)} \leq \epsilon \tag{5.72}$$

From (5.67) and (5.72) we have

$$\forall p \in \mathbb{N}, \exists N(p)/\forall n \geq N(p) \text{ we have } ||v_{g^p_n} - \Phi(\cdot, z^*_p)||_{H^{\frac{1}{2}}(\partial B)} \leq C_6 \tag{5.73}$$

hence, since $\partial D \subset B$, by the wellposedness of the scattering problem we get

$$||v^s_{g^p_n} - \Phi^s(\cdot, z^*_p)||_{H^{\frac{1}{2}}(\partial B)} \leq C_7$$

where $C_6, C_7 > 0$ are constants. From (5.73), we have $||v_{g^p_n}||^2_{L^2(\partial B)} \leq C_8 \ln p$ for p large enough and $C_8 > 0$ is a constant. We set $\beta_p := ||v_{g^p_n}||^{-1}_{L^2(\partial B)}$ then

$$||\beta_p v^s_{g^p_n}(\cdot, z_p)||^2_{H^{\frac{1}{2}}(\partial B)} \geq \frac{||\Phi^s(\cdot, z^*_p)||^2_{H^{\frac{1}{2}}(\partial B)}}{C_8 \ln p} - \frac{C^2_7}{C_8}(\ln p)^{-1}.$$

Since $||\Phi^s(\cdot, z^*_p)||_{H^{\frac{1}{2}}(\partial B)} \geq C_9 d(z^*_p, \partial B) = C_9 \frac{1}{p}$, we deduce that

$$||\beta_p v^s_{g^p_{N(p)}}(\cdot, z_p)||_{H^{\frac{1}{2}}(\partial B)} \to \infty.$$

We set $v_{g_{RT}} := \beta_p v_{g^p_{N(p)}}$. Then

$$||v_{g_{RT}}||_{L^2(\partial B)} = 1 \text{ and } ||v^s_{g_{RT}}||_{H^1(\partial B)} \to \infty.$$

Arguing as in the part I, taking $v^\infty_{g_{RT}}$ instead of $v^\infty_{g^\epsilon(\cdot, z_p)}$, we deduce that $I_2(B) = \infty$.

Since the case one is always justified as the forward problem is well posed, we deduce the convergence of the second version of the no-response test.

6 Equivalence of the two versions

We will show that with respect to the convergence properties the two versions are equivalent.

THEOREM 6.1 *Consider the versions (2.13) and (4.41) of the no-response test. Then the convergence of the first version implies the convergence of the second version.*

Proof. According to our assumption the first NRT version is convergent, i.e. we have $\overline{D} \subset B$ implies $I_1(B) = 0$ and $\overline{D} \not\subset B$ implies $I_1(B) = \infty$. We will show that the same implications hold for the indicator function I_2.

For the case $\overline{D} \subset B$ we have $I_1(B) = I_2(B) = 0$ which is justified by the wellposedness of the direct problem. We will show that for $\overline{D} \not\subset B$ we obtain the logical implication

$$\Big(I_1(B) = \infty\Big) \implies \Big(I_2(B) = \infty\Big),$$

which implies the above theorem.

We suppose that $I_1(B) = \infty$. Let (f_n, g_n) be a sequence in $L^2(\mathbb{S}) \times L^2(\mathbb{S})$ such that $||v_{f_n}||_{L^2(\partial B)} < \frac{1}{n}$, $||v_{g_n}||_{L^2(\partial B)} < \frac{1}{n}$ and

$$\lim_{n\to\infty} \int_{\mathbb{S}^2} u^\infty(-\theta, d) f_n(\theta) g_n(d)\, ds(\theta) ds(d) = \infty. \tag{6.74}$$

We will investigate the functions $v^i_{g_n}$ and its far field $v^\infty_{g_n}$. To this end consider the solvability of the integral equation (4.38) with v^∞_g on the right-hand side as described in (4.41). As in section 3, we distinguish two cases.

1) We have the case where for some $v^\infty_{g_{n_0}}$ the equation (4.38) is not solvable. In this case we deduce that $I_2(B) = \infty$.

2) The second case is when for every $v^\infty_{g_n}$, (4.38) is solvable. In this case Lemma 4.4 gives

$$v^s_{g_n}(x) = \int_{\partial B} \Phi(x, y)\psi_n(y) ds(y), \quad x \in \mathbb{R}^3 \setminus (B \cup D),$$

hence the far field patterns enjoy the property

$$v^\infty_{g_n}(\theta) = \frac{1}{4\pi} \int_{\partial B} \psi_n(y) e^{-i\kappa\theta\cdot y} ds(y).$$

Now multiplying the last equality by f_n and integrating over $\mathbb{S}$, we get

$$\int_{\mathbb{S}} v^\infty_{g_n}(-\theta) f_n(\theta) ds(\theta) = \frac{1}{4\pi} \int_{\partial B} \psi_n(y) v_{f_n}(y) ds(y).$$

If ψ_n is bounded in $L^2(\partial B)$ then $\int_{\partial B} \psi_n(y) v_{f_n}(y) ds(y)$ is also bounded. Hence

$$\int_{\mathbb{S}} \int_{\mathbb{S}} u^\infty(-\theta, d) f_n(\theta) g_n(d) ds(\theta) ds(d) = \int_{\mathbb{S}} v^\infty_{g_n}(-\theta) f_n(\theta) ds(\theta)$$

is also bounded. This contradicts (6.74) and implies that the sequence $||\psi_n||_{L^2(\partial B)}$ is unbounded. We have proven that $I_2(B) = \infty$. □

6.1 The second version implies the first version.

To prove this implication we need to use an assumption on the testing domains. Precisely, for any testing domain B we assume that we can find a homotopy of domains $\lambda \to B_\lambda$, $\lambda \geq 1$ such that $\overline{B_{\lambda_2}} \subset B_{\lambda_1}$ for $\lambda_2 < \lambda_1$, B_λ has boundary of class C^2, is simply connected and its open exterior is connected, $B_1 = B$. Also, for every $r > 0$ there is $\lambda \in \mathbb{R}$ such that $\big\{y \in \mathbb{R}^3 : |y| \leq r\big\} \subset B_\lambda$.

THEOREM 6.2 *Consider the versions (2.13) and (4.41) of the no-response test. Then the convergence of the second version implies the convergence of the first version.*

Proof. Suppose that the second NRT version is convergent, i.e. we have $\overline{D} \subset B$ implies $I_2(B) = 0$ and $\overline{D} \not\subset B$ implies $I_2(B) = \infty$. We will show that the same implications hold for the indicator function I_1.

The case $\overline{D} \subset B$ is treated as in the proof of Theorem 6.1. We suppose that $\overline{D} \not\subset B$, B is a non-vibrating domain and $I_2(B) = \infty$. Our goal is to show that under these assumptions $I_1(B) = \infty$.

Since $I_2(B) = \infty$, there exists a sequence $(\psi_n^\alpha) \subset L^2(\partial B)$ of regularized solution of (4.38) with right-side $v_{g_n}^\infty$ such that $||v_{g_n}^i||_{L^2(\partial B)} < \frac{1}{n}$ and

$$\limsup_{n\to\infty} \lim_{\alpha\to 0} ||\psi_n^\alpha||_{L^2(\partial B)} = \infty.$$

We split the proof into two parts which investigate the following cases.

1) For every n in $\mathbb{N}$, the integral equation (4.38) with right-side $v_{g_n}^\infty$ is solvable. In this case according to Lemma 4.3 we pass to the limit $\alpha \to 0$ to obtain

$$\limsup_{n\to\infty} ||\psi_n||_{L^2(\partial B)} = \infty \tag{6.75}$$

where ψ_n is the sequence of the solutions of (4.38) with right-hand side $v_{g_n}^\infty$.

2) There exists n_0 such that $\lim_{\alpha\to 0} ||\psi_{n_0}^\alpha||_{L^2(\partial B)} = \infty$, i.e the integral equation (4.38) with right-side $v_{g_{n_0}}^\infty$ is not solvable.

First case. We multiply the equation

$$S^\infty \psi_n = v_{g_n}^\infty$$

by $f \in L^2(\mathbb{S})$ and integrate over $\mathbb{S}$ to obtain

$$\int_{\mathbb{S}} S^\infty \psi_n(-\theta) f(\theta) ds(\theta) = \int_{\mathbb{S}} v_{g_n}^\infty(-\theta) f(\theta) ds(\theta). \tag{6.76}$$

The left-hand side is transformed by

$$\begin{aligned}
\int_{\mathbb{S}} S^\infty \psi_n(-\theta) f(\theta) ds(\theta) &= \int_{\mathbb{S}} \int_{\partial B} e^{ik\theta\cdot x} \psi_n(x) ds(x) f(\theta) ds(\theta) \\
&= \int_{\partial B} \int_{\mathbb{S}} e^{ik\theta\cdot x} f(\theta) ds(\theta) \psi_n(x) ds(x) \\
&= \int_{\partial B} v_f(x) \psi_n(x) ds(x). \qquad (6.77)
\end{aligned}$$

For every n fixed, by the denseness of the range of the Herglotz operator in $L^2(\partial B)$, we take $(v_{f_n^p})_{p\in\mathbb{N}}$ such that

$$v_{f_n^p} \longrightarrow \frac{\epsilon}{2} \frac{\overline{\psi_n}}{||\psi_n||_{L^2(\partial B)}}, \quad p \to \infty, \tag{6.78}$$

in $L^2(\partial B)$. For n fixed, there exists $N_0(n)$, such that $\forall p > N_0(n)$,

$$||v_{f_n^p}||_{L^2(\partial B)} < \epsilon. \tag{6.79}$$

Then for every n fixed, we have

$$\begin{aligned}
\lim_{p\to\infty} \int_{\mathbb{S}} S^\infty \psi_n(\theta) f_p^n(\theta)\, ds(\theta) &= \lim_{p\to\infty} \int_{\partial B} v_{f_p^n}(x) \psi_n(x) ds(x) \\
&= \int_{\partial B} \frac{\epsilon}{2} \frac{|\psi_n|^2(x)}{||\psi_n||_{L^2(\partial B)}} ds(x) \\
&= \frac{\epsilon}{2} ||\psi_n||_{L^2(\partial B)}. \qquad (6.80)
\end{aligned}$$

Hence

$$\limsup_{n\to\infty}\lim_{p\to\infty}\int_{\mathbb{S}} v^\infty_{g_n}(\theta)f^n_p(\theta)ds(\theta)=\infty. \tag{6.81}$$

The right-hand side of (6.76) with f chosen such that (6.79) is satisfied is a lower bound for $I_1(B)$. Now, from (6.81) we obtain $I_1(B)=\infty$.

Second case. Since $v^\infty_{g_{n_0}}$ is not in the range of S^∞, then $v^s_{g_{n_0}}$ cannot be analytically extended up to $\mathbb{R}^3\setminus B$. Indeed, if it is analytically extendable up to $\mathbb{R}^3\setminus B$, then it satisfies $\Delta v^s_{g_{n_0}}+k^2v^s_{g_{n_0}}=0$ in a neighborhood of $\mathbb{R}^3\setminus B$. In particular $v^s_{g_{n_0}}|_{\partial B}\in H^1(\partial B)$. Then the equation $S\psi=v^s_{g_{n_0}}|_{\partial B}$ is solvable in $L^2(\partial B)$ and then $S^\infty\psi=v^\infty_{g_{n_0}}$, which is a contradiction.

We now consider a homotopy of domains $\lambda\to B_\lambda$, $\lambda\geq 1$ such that $\overline{B_{\lambda_2}}\subset B_{\lambda_1}$ for $\lambda_2<\lambda_1$, B_λ has boundary of class C^2, is simply connected and the open exterior is connected, $B_1=B$ and for every $r>0$ there is $\lambda\in\mathbb{R}$ such that $\{y\in\mathbb{R}^3:|y|\leq r\}\subset B_\lambda$.

Since $v^s_{g_{n_0}}$ is not extendable into $\mathbb{R}^3\setminus B$, there must be some minimal $\lambda_0>1$ such that v^s_g is extensible into $\mathbb{R}^3\setminus B_\lambda$ for all $\lambda>\lambda_0$, but not into $\mathbb{R}^3\setminus B_{\lambda_0}$. Hence there exist $z_0\in\partial B_{\lambda_0}$ and an open neighborhood $N(z_0)$ of z_0 such that $\overline{N(z_0)}\cap\overline{B}=\emptyset$ and $v^s_{g_{n_0}}$ is not analytically extendable into any open subset of $N(z_0)$.

LEMMA 6.3 *Let z_p be a sequence of points in $\mathbb{R}^3\setminus B_{\lambda_0}$ such that z_p tends to z_0. If for some positive $\rho\in\mathbb{R}$ the set*

$$\left\{\sup_{|h|=1}\rho^\mu\frac{|(h\cdot\nabla)^\mu v^s_{g_{n_0}}(z_p)|}{\mu!},\mu\in\mathbb{Z}_+\right\} \tag{6.82}$$

is uniformly bounded by a constant $c>0$, then v can be analytically extended near the point z_o.

Proof of Lemma 6.3. From the boundedness of (6.82) we derive that the series

$$\sum_{\mu\in\mathbb{Z}_+}\frac{1}{\mu!}((x-z)\cdot\nabla)^\mu v^s_{g_{n_0}}(z)\leq\sum_{\mu\in\mathbb{Z}_+}c|\frac{x-z_p}{\rho}|^\mu \tag{6.83}$$

is absolutely converging for $x\in B(z_p,\rho)$ and for all the points z_p. Taking z_p near enough to z_0, we deduce that v is analytically extendable into some open neighborhood of z_0. However, in this case we would obtain an extension into $\mathbb{R}^3\setminus B_{\lambda_1}$ with some $\lambda_1<\lambda_0$. □

As a consequence of the preceding lemma, for any sequence z_p of points in $\mathbb{R}^3\setminus B_{\lambda_0}$ tending to z_0 there exist a sequence $h_p\in\mathbb{S}$, $\mu_p\in\mathbb{N}$ such that

$$\lim_{p\to\infty}\rho^{\mu_p}\frac{|\,(h_p\cdot\nabla)^{\mu_p}v^s_{g_{n_0}}(z_p)\,|}{\mu_p!}=\infty. \tag{6.84}$$

Let $\epsilon>0$ be fixed. We set

$$\psi_p(x):=\frac{\epsilon}{2\beta(z_p,\mu)}(h_p\cdot\nabla_z)^{\mu_p}\Phi(x,z_p)$$

where

$$\beta(p):=\sup_{y\in\overline{B}}\{|(h_p\cdot\nabla_z)^{\mu_p}\Phi(y,z_p)|\}$$

is introduced for normalization.

Let p be fixed, we can find a sequence of densities $g_n^p \in L^2(\mathbb{S})$ such that $v_{g_n^p}$ tends to ψ_p in $L^2(E_{z_p})$ for some non-vibrating domain E_{z_p} such that $\overline{B_{\lambda_0}} \subset E_{z_p}$, $z_p \notin E_{z_p}$ and $\mathbb{R}^3 \setminus \overline{E_{z_p}}$. Hence also $v_{g_n^p}$ tends to ψ_p in $C^1(\overline{B_{\lambda_0}})$.

Since

$$||\psi_p||_{C(\overline{B})} \leq \frac{\epsilon}{2},$$

taking n large enough we deduce that

$$||v_{g_n^p}||_{C(\overline{B})} < \epsilon.$$

Now from the identity (2.8) applied on $\mathbb{R}^3 \setminus \overline{B_{\lambda_0}}$ for $v^s_{g_{n_0}}$ and $e^{-ik\theta\cdot x}$, we have:

$$\int_{\mathbb{S}^2} u^\infty(-\theta, d) g_{n_0}(d) g_n^p(\theta)\, ds(\theta) ds(d) = \frac{1}{4\pi}\int_{\partial B_{\lambda_0}} \left\{ \frac{\partial v^s_{g_{n_0}}}{\partial \nu} v_{g_n^p} - \frac{\partial v_{g_n^p}}{\partial \nu} v^s_{g_{n_0}} \right\} ds(x)$$

Taking the limit with respect to n, we have:

$$\begin{aligned} &\lim_{n\to\infty} \int_{\mathbb{S}^2} u^\infty(-\theta, d) g_{n_0}(d) g_n^p(\theta)\, ds(\theta) ds(d) \\ &= \frac{1}{4\pi}\int_{\partial B_{\lambda_0}} \left\{ \frac{\partial v^s_{g_{n_0}}}{\partial \nu} \psi_p - \frac{\partial \psi_p}{\partial \nu} v^s_{g_{n_0}} \right\} ds(x) \end{aligned} \tag{6.85}$$

We take an open ball Ω_R with radius centered at the origin such that $\overline{B_{\lambda_0}} \subset \Omega_R$. Applying the Green's formula in $\Omega_R \setminus \overline{D}$ for $v^s_{g_{n_0}}$ and ψ_p, we deduce that

$$\int_{\partial B_{\lambda_0}} \left\{ \frac{\partial v^s_{g_{n_0}}}{\partial \nu} \psi_p - \frac{\partial \psi_p}{\partial \nu} v^s_{g_{n_0}} \right\} ds(x) =$$

$$\frac{\epsilon}{2\beta(p)} (h_p \cdot \nabla_z)^{\mu_p} v^s_{g_{n_0}}(z_p) - \int_{\partial \Omega_R} \left\{ \frac{\partial v^s_{g_{n_0}}}{\partial \nu} \psi_p - \frac{\partial \psi_p}{\partial \nu} v^s_{g_{n_0}} \right\} ds(x) =$$

$$\frac{\epsilon}{2\beta(p)} (h_p \cdot \nabla_z)^{\mu_p} v^s_{g_{n_0}}(z_p) - \int_{\partial \Omega_R} \left\{ (\frac{\partial v^s_{g_{n_0}}}{\partial \nu} - i\kappa v^s_{g_{n_0}}) \psi_p - (\frac{\partial \psi_p}{\partial \nu} - i\kappa\psi_p) v^s_{g_{n_0}} \right\} ds(x) \tag{6.86}$$

It is easy to see that

$$\psi_p(x) = \frac{\epsilon}{8\pi\beta_p} (\frac{-i\kappa h_p \cdot (x - z_p)}{|x - z_p|})^{|\mu_p|} \frac{e^{i\kappa|x-z_p|}}{|x - z_p|} + O(\frac{1}{|x|^2}) \text{ for } |x| \to \infty.$$

By

$$\frac{d}{dr} h_p \cdot (x - z_p) = \frac{x \cdot h_p}{|x|}$$

and

$$\frac{d}{dr}|x - z_p| = 1 + O(\frac{1}{|x|}), |x - z_p| = |x| - \frac{x}{|x|}\cdot z_p + O(\frac{1}{|x|})$$

where $r := |x| \to \infty$, we have

$$\frac{d}{dr}\psi_p(x) - i\kappa\psi_p(x) = O(\frac{1}{|x|^2}),\ (|x| \to \infty). \tag{6.87}$$

From its definition, $v^s_{g_{n_0}}$ satisfies

$$v^s_{g_{n_0}} = O(\frac{1}{|x|}) \text{ and } \frac{d}{dr}v^s_{g_{n_0}}(x) - i\kappa v^s_{g_{n_0}} = O(\frac{1}{|x|^2}),\ (|x| \to \infty). \tag{6.88}$$

From (6.87) and (6.88), the second term of the right hand side of (6.86) tends to zero as R tends to ∞. Hence, we get

$$\int_{\partial B_{\lambda_0}} \left\{ \frac{\partial v^s_{g_{n_0}}}{\partial \nu}\psi_p - \frac{\partial \psi_p}{\partial \nu} v^s_{g_{n_0}} \right\} ds(x) = \frac{\epsilon}{2\beta(p)}(h_p \cdot \nabla_z)^{\mu_p} v^s_{g_{n_0}}(z_p). \tag{6.89}$$

Next, in order to analyze the behavior of the right hand side of (6.89), we have to estimate $\beta(p)$.

To begin with, we use $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3) \in \mathbb{C}^3$ for complexifying $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ such that the real part of $\hat{x}_j$ is $\mathrm{Re}(\hat{x}_j) = x_j$.

Since $\overline{B} \subset B_{\lambda_0}$, then the distance $d(z_0, B)$ is positive. Then for any $x \in \overline{B}$, there exist an open neighborhood of $U_x \subset \mathbb{R}^3$ of x and complex open polydiscs $V_{z_0} \subset \mathbb{C}^3$ centered z_0 such that

$$\begin{cases} U_x \cap V_{z_0} = \emptyset \text{ and} \\ (y - \hat{z}) \cdot (y - \hat{z}) := \sum_{j=i}^{3} (\hat{y}_j - \hat{z}_j)^2 \neq 0 \text{ for any } y \in U_x \text{ and } \hat{z} \in V_z \end{cases} \tag{6.90}$$

By the compactness of $\overline{B}$ in $\mathbb{R}^3$, there exist $x^{(k)} \in \overline{B}$ $(1 \le k \le K)$ such that $\overline{B} \subset \cup_{k=1}^{K} U_{x^{(k)}}$. Let $V^{(k)}_{z_0}$ be the polydisc centered at z_0 associated with $U_{x^{(k)}}$ satisfying (6.90).

Now, we define a complex open polydisc V centered at z_0 by $V := \cap_{k=1}^{K} V^{(k)}_{z_0}$. Then, we have

$$\begin{cases} U_{x^{(k)}} \cap V = \emptyset (1 \le k \le K) \text{ and} \\ (x - \hat{z}) \cdot (x - \hat{z}) \neq 0 \text{ for any } x \in U_{x^{(k)}} \text{ and } \hat{z} \in V; 1 \le k \le K. \end{cases} \tag{6.91}$$

Moreover, by considering large p, we can assume that $z_p \in V$.

For each k $(1 \le k \le K)$, $\Phi(x, z)$ has a natural analytic extension $\tilde{\Phi}(x, \hat{z})$:

$$\hat{\Phi}(x, \hat{z}) = \frac{e^{i\kappa\{(x-\hat{z})\cdot(x-\hat{z})\}^{\frac{1}{2}}}}{4\pi\{(x - \hat{z}) \cdot (x - \hat{z})\}^{\frac{1}{2}}} \text{ for } x \in U_{x^{(k)}} \text{ and } \hat{z} \in V, \tag{6.92}$$

where we take the branch of $\{(x - \hat{z}) \cdot (x - \hat{z})\}^{\frac{1}{2}}$ such that it is equal to $|x - z|$ when $\hat{z} = z$.

By the Cauchy integral formula, there exist $C_k > 0$ and $\rho_k > 0$ such that

$$|(h_p \cdot \nabla_z)^{\mu_p} \Phi(x, z_p)| \le C_k \frac{\mu_p!}{\rho_k^{\mu_p}},\ \text{for } x \in U_{x^{(k)}}, p \in \mathbb{N}. \tag{6.93}$$

Hence by taking

$$\rho := \min_{1 \le k \le K} \rho_k \text{ and } C := \max_{1 \le k \le K} C_k,$$

we have

$$|(h_p \cdot \nabla_z)^{\mu_p} \Phi(x, z_p)| \le C \frac{\mu_p!}{\rho^{\mu_p}} \text{ for } x \in \overline{B} \text{ and } p \in \mathbb{N}. \tag{6.94}$$

By (6.94), the right hand side of (6.89) is estimated from below by

$$\frac{\epsilon}{2C}\frac{\rho^{\mu_p}}{\mu_p!}|(h_p \cdot \nabla_z)^{(\mu_p)} v_{g_{n_0}}(z_p)|. \tag{6.95}$$

This last sequence is unbounded by (6.84).

Finally from (6.85), we deduce that there exists a subsequence $v_{g_n^p}$ such that:

$$\lim_{p\to\infty}\lim_{n\to\infty}\int_{\mathbb{S}^2} u^\infty(-\theta,d)v_{g_{n_0}}(d)v_{g_n^p}(\theta)\;ds(\theta)ds(d) \;=\; \infty,$$

i.e. $I_1(B)=\infty$.

To finish the proof, let us show how we get (6.93). Indeed, from the Cauchy formula, we represent $\hat{\Phi}(x,\hat{z})$ by

$$\hat{\Phi}(x,\hat{z}) = \frac{1}{(2\pi i)^3}\int_\Gamma \frac{\hat{\Phi}(x,\zeta)}{(\zeta_1-\hat{z}_1)\cdot(\zeta_2-\hat{z}_2)\cdot(\zeta_3-\hat{z}_3)}d\zeta$$

where $\Gamma := \{\zeta=(\zeta_1,\zeta_2,\zeta_3); |\zeta_j-(z_0)_j| = r \text{ for } 1\leq j\leq 3\}$ such that $[\Gamma] := \{\hat{z} : |\hat{z}_j-(z_0)_j| \leq r \text{ for } 1\leq j\leq 3\}\subset V$. We set $D(\rho) := \{\hat{z} : |\hat{z}_j-(z_0)_j| \leq \rho \text{ for } 1\leq j\leq 3\}$ with $\rho<r$. Since

$$|\zeta_j-\hat{z}_j| \geq |\zeta_j-(z_o)_j| - |\hat{z}_j-(z_o)_j| \geq r-\rho$$

and

$$|\hat{\Phi}(x,\zeta)|\leq M,\ \text{for } x\in U_{x^{(k)}}, \zeta\in V$$

with some positive constant M, then we have

$$|\partial_{\hat{z}}^\alpha \hat{\Phi}(x,\hat{z})| \leq \frac{M\alpha!\rho^3}{(r-\rho)^{|\alpha|}+3},\ \text{for } x\in U_{x^{(k)}}, \hat{z}\in D(\rho)$$

Hence

$$|\partial_{\hat{z}}^\alpha \hat{\Phi}(x,\hat{z})| \leq \frac{M\alpha!}{(\frac{r}{2})^{|\alpha|}},\ \text{for } x\in U_{x^{(k)}}, \hat{z}\in D(\frac{r}{2}).$$

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