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# Generalized Riesz Projections and Toeplitz Operators

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## Abstract

Let $1 < p < \infty$. In this paper, for a measurable function $v$ and a weight function $w$, the generalized Riesz projection $P^v$ is defined by $P^v f = vP(v^{-1}f)$, $(f \in L^p(w))$. If $P_0$ is the self-adjoint projection from $L^2(w)$ onto $H^2(w)$, then $P_0 = P^\alpha$ for some outer function $\alpha$ satisfying $w = |\alpha|^{-2}$. In this paper, $P^v$ on $L^p(w)$ is studied. As an application, the invertibility criterion for the generalized Toeplitz operator $T^v_\phi$ and the generalized singular integral operator $\phi P^v + Q^v$, $Q^v = I - P^v$ are investigated using the weighted norm inequality. The operator norm inequality for the generalized Hankel operator $H^v_\phi$ is also presented.

# 1 Introduction

Let $\mathcal{P} = \text{span}\{e^{in\theta}; n \geq 0\}$, and let $\mathcal{Q} = \text{span}\{e^{in\theta}; n < 0\}$. Then $\mathcal{P} + \mathcal{Q}$ is the set of all trigonometric polynomials. Let $dm(e^{i\theta}) = d\theta/2\pi$ be the normalized Lebesgue measure on the unit circle $\mathbf{T}$. Let $w$ be a positive function in $L^1 = L^1(dm)$. Let $1 \leq p < \infty$. Then $\mathcal{P} + \mathcal{Q}$ is dense in $L^p(w) = L^p(wdm)$ in norm. Let $H^p(w)$ denote the norm closure in $L^p(w)$ of $\mathcal{P}$, and let $\overline{H_0^p(w)}$ denote the norm closure in $L^p(w)$ of $\mathcal{Q}$. We will write $H^p(w) = H^p$ when $w = 1$, and then this is a usual Hardy space. The Riesz projection $P$ from $\mathcal{P} + \mathcal{Q}$ to $\mathcal{P}$ is an operator defined by

$$(Pf)(e^{i\theta}) = \sum_{k \geq 0} \hat{f}(k) e^{ik\theta}, \quad (f \in \mathcal{P} + \mathcal{Q}),$$

where $\hat{f}(k)$ denotes the $k$-th Fourier coefficient of $f$. Hence, the Riesz projection $P$ is a densely defined operator from $L^p(w)$ to $H^p(w)$. $P$ may not be extended to a bounded operator. $P$ can be extended to a bounded operator from $L^p(w)$ onto $H^p(w)$ if and only if $w$ satisfies the condition:

$$(A_p) \qquad \sup_I \left( \frac{1}{m(I)} \int_I w dm \right) \left( \frac{1}{m(I)} \int_I w^{-1/(p-1)} dm \right)^{p-1} < \infty$$

where the supremum is over all intervals $I$ of $\mathbf{T}$. This is the theorem of Hunt, Muckenhoupt and Wheeden (cf. [1, p.39], [4, p.255], [11, p.209, p.450], [12, p.119]) which is a generalization of the theorem of Helson and Szegő (cf. [4, p.147], [11, p.450], [12, p.99]). Let $v$ be a measurable function on the unit circle $\mathbf{T}$ satisfying $|v| > 0$. In this paper, the generalized Riesz projection $P^v$ is defined by

$$(P^v f)(e^{i\theta}) = v(e^{i\theta}) P(v^{-1} f)(e^{i\theta}) = v(e^{i\theta}) \sum_{k \geq 0} (v^{-1} f)\hat{}\,(k) e^{ik\theta},$$

$(f \in v\mathcal{P} + v\mathcal{Q})$. Then $v\mathcal{P} \cap v\mathcal{Q} = \{0\}$, and $P^v$ maps $v\mathcal{P} + v\mathcal{Q}$ onto $v\mathcal{P}$. Hence, $(P^v)^2 = P^v$. Let $w$ be an integrable function on $\mathbf{T}$ satisfying $w > 0$. Let $1 \leq p < \infty$. If $v \in L^p(w)$, then $v\mathcal{P} + v\mathcal{Q}$ is dense in $L^p(w)$. Let $1 < p < \infty$, and let $1/p + 1/q = 1$. In Section 2, we will consider the boundedness of the generalized Riesz projection $P^v$. It is well known that if $p = 2$ and $v$ is an outer function such that $|v|^2 = w$, then $P^v$ becomes a self-adjoint projection which maps $L^2(w)$ onto $H^2(w)$ (cf. [2], [7]). In particular, $P = P^1$ is a self-adjoint projection which maps $L^2$ onto $H^2$. We will prove that if $1 < p < \infty$ and $w, v$ satisfy some conditions, then $P^v$ is a bounded operator on $L^p(w)$ if and only if $|v|^p w \in (A_p)$.

In Section 3, we will consider the adjoint operator for $P^v$. We will give the form of $(P^v)^*$, and prove that if $1 < p < \infty$ and $w, v$ satisfy some conditions, then $(P^v)^* = P^v$ on $L^p(w) \cap L^q(w)$ if and only if $|v|^2 w$ is a constant function.

In Section 4, we will consider the invertibility of the Toeplitz operator $T_\phi^v$ and singular integral operator $\phi P^v + Q^v$, where $Q^v = I - P^v$. Let $1 < p < \infty$, and let $\phi \in L^\infty$. If $P^v \in B(L^p(w))$, then the operator $T_\phi^v$ from $\text{ran}P^v$ to $\text{ran}P^v$ is defined by

$$T_\phi^v f = P^v(\phi f), \quad (f \in \text{ran}P^v).$$

If $w \in (A_p)$, then Rochberg [13] established an invertibility criterion for the Toeplitz operator $T_\phi$ on $H^p(w)$ (cf. [1, p.216]). If $p = 2$ and $w = 1$, then this reduces to a theorem of Widom and Devinatz (cf. [1, p.59], [11, p.316], [12, p.250]).
In Section 5, we do not assume the boundedness of $P^v$ on $L^p(w)$. Hence, the results in Section 5 do not follow from the theorem of Rochberg and Simonenko or the theorem of Widom and Devinatz (cf. [13], [1, p.216], [12]). We will consider the invertibility of the quotient type Toeplitz operator $R^v_\phi$ for an outer function $v$. Let $1 < p < \infty$, and let $\phi \in L^\infty$. If $\log|v| \in L^1$, then an operator $R^v_\phi$ is defined as a bounded operator from $H^p(w)$ to $L^p(w)/\frac{v}{\bar{v}}\overline{H^p_0(w)}$ by

$$R^v_\phi f = \phi f + \frac{v}{\bar{v}}\overline{H^p_0(w)}, \quad (f \in H^p(w)).$$

If $P^v \in B(L^p(w))$, then $\ker P^v = \frac{v}{\bar{v}}\overline{H^p_0(w)}$. $R^v_\phi$ is always bounded. When $v = 1$, Nakazi ([8], [9]) considered the quotient type Toeplitz operator $R_\phi = R^1_\phi$ from $H^p(w)$ to $L^p(w)/\overline{H^p_0(w)}$ and proved Lemma 5.1. We use Lemma 5.1 to prove Theorem 5.2.
In Section 6, the operator norm inequality for the generalized Hankel operator $H^v_\phi$ is presented. Let $1 < p < \infty$, and let $\phi \in L^\infty$. If $P^v \in B(L^p(w))$, then the Hankel operator $H^v_\phi$ from $\mathrm{ran}P^v$ to $\mathrm{ran}Q^v$ is defined by

$$H^v_\phi f = Q^v(\phi f), \quad (f \in \mathrm{ran}P^v).$$

If $v = w = 1$, then this reduces to a theorem of Nehari (cf. [1, p.54], [11, p.181], [12, p.181]).

# 2 Boundedness of $P^v$

In this section, we discuss the condition such that the generalized Riesz projection $P^v$ is extended to $L^p(w)$ by continuity to a bounded operator. We will not distinguish between an operator's being bounded and being densely defined and extendable by continuity to a bounded operator. We use Lemmas 1.1 and 1.2 to prove Theorems 2.3 and 2.4.

**Lemma 2.1** *Let $1 \le p < \infty$. Let $w$ be a positive function in $L^1$.*
(1) *If $|v| > 0$ and $v \in L^p(w)$, then $v\mathcal{P} + v\mathcal{Q}$ is a dense subspace of $L^p(w)$.*
(2) *If $\log w \in L^1$ and $|v| = |k|$ for some outer function $k$ in $H^p(w)$, then $k\mathcal{P}$ is dense in $H^p(w)$.*

**Proof.** (1): Let $f \in L^p(w)$. Then, $v^{-1}L^p(w) = L^p(|v|^p w)$. Hence, $v^{-1}f \in L^p(|v|^p w)$. Since $\mathcal{P} + \mathcal{Q}$ is dense in $L^p(|v|^p w)$, it follows that there exists a sequence $f_n \in \mathcal{P} + \mathcal{Q}$ such that

$$\lim_{n\to\infty} \int |vf_n - f|^p w dm = \lim_{n\to\infty} \int |f_n - v^{-1}f|^p |v|^p w dm = 0.$$

(2): Let $g \in H^p(w)$. Since $k$ is an outer function such that $|k| = |v|$, it follows that $k^{-1}g \in H^p(|v|^p w)$. Since $\mathcal{P}$ is dense in $H^p(|v|^p w)$, it follows that there exists a sequence $g_n \in \mathcal{P}$ such that

$$\lim_{n\to\infty} \int |kg_n - g|^p w dm = \lim_{n\to\infty} \int |g_n - k^{-1}g|^p |v|^p w dm = 0.$$

Hence, $k\mathcal{P}$ is dense in $H^p(w)$. Lemma 2.1 is proved. □

**Lemma 2.2** *Let $1 \le p < \infty$. Let $w$ be a positive function in $L^1$. Suppose $|v| > 0$ and $v \in L^p(w)$. Then the following properties are equivalent.*
(1) *$P^v$ is a bounded operator on $L^p(w)$.*
(2) *$P^{|v|}$ is a bounded operator on $L^p(w)$.*
(3) *$P$ is a bounded operator $L^p(|v|^p w)$.*
*If one of these conditions holds, then*

$$\|P^v\|_{B(L^p(w))} = \|P^{|v|}\|_{B(L^p(w))} = \|P\|_{B(L^p(|v|^p w))}.$$

**Proof.** It is sufficient to prove the equivalence of (1) and (3). By (1), for all $f \in \mathcal{P}$ and $g \in \mathcal{Q}$,

$$\begin{aligned} \int |f|^p |v|^p w dm &= \int |vf|^p w dm \\ &\le \|P^v\|^p_{B(L^p(w))} \int |vf + vg|^p w dm \\ &= \|P^v\|^p_{B(L^p(w))} \int |f + g|^p |v|^p w dm. \end{aligned}$$

Hence, $\|P\|_{B(L^p(|v|^p w))} \le \|P^v\|_{B(L^p(w))} < \infty$. This implies (3). Conversely, by (3), for all $f \in \mathcal{P}$ and $g \in \mathcal{Q}$,

$$\begin{aligned} \int |vf|^p w dm &= \int |f|^p |v|^p w dm \\ &\le \|P\|^p_{B(L^p(|v|^p w))} \int |f + g|^p |v|^p w dm \\ &= \|P\|^p_{B(L^p(|v|^p w))} \int |vf + vg|^p w dm \end{aligned}$$

By Lemma 2.1(1), $v\mathcal{P}+v\mathcal{Q}$ is dense in $L^p(w)$. Hence, $\|P^v\|_{B(L^p(w))} \le \|P\|_{B(L^p(|v|^p w))} < \infty$, and hence (1) follows. Lemma 2.2 is proved. □

Suppose $w = |\alpha|^{-2}$ for some outer function $\alpha$. Then $P_0 = P^\alpha$ is a self-adjoint projection from $L^2(w)$ onto $H^2(w)$. Let $Q_0 = I - P_0$. If $a, b$ are constant functions, then $\|aP_0 + bQ_0\|_{B(L^2(w))} = \max(|a|, |b|)$ (cf. [5, Vol.I, p.79]). By the similar proof of Lemma 2.2, if $a, b \in L^\infty$, then $\|aP_0 + bQ_0\|_{B(L^2(w))} = \|aP + bQ\|_{B(L^2(|\alpha|^2 w))} = \|aP + bQ\|_{B(L^2)}$. Hence,

$$\begin{aligned}\|aP_0 + bQ_0\|_{B(L^2(w))} &= \|aP + bQ\|_{B(L^2)} \\ &= \inf_{k \in H^\infty} \left\| \frac{|a|^2 + |b|^2}{2} + \sqrt{|a\bar{b} - k|^2 + \left(\frac{|a|^2 - |b|^2}{2}\right)^2} \right\|_\infty .\end{aligned}$$

The infimum is attained (cf. [10]).

Let $1 < p < \infty$. There are many measurable functions $v$ and $w$ such that $v, \log v, \log w \notin L^1$, $w \in L^1$ and $P^v \in B(L^p(w))$. For example, let

$$v(e^{i\theta}) = \exp\left(\frac{1}{2\pi - \theta}\right), \quad (0 \le \theta < 2\pi),$$

and let $w = |v|^{-p}$. Since $\frac{p}{\theta - 2\pi} \le \frac{-p}{2\pi}$, it follows that $0 < w(e^{i\theta}) = \exp\left(\frac{p}{\theta - 2\pi}\right) \le \exp\left(\frac{-p}{2\pi}\right) < \infty$. Hence, $w \in L^\infty$. By Lemma 2.2 and the theorem of Gohberg, Krupnik, Hollenbeck and Verbitsky (cf. [5, Vol.II, p.102], [6]),

$$\|P^v\|_{B(L^p(w))} = \|P\|_{B(L^p)} = \frac{1}{\sin(\pi/p)} < \infty.$$

Then $\mathrm{ran}P^v \oplus \ker P^v = vH^p \oplus v\overline{H_0^p} = vL^p = L^p(w)$. If $p = 2$, then $P^v$ is a self-adjoint projection on $L^2(w)$.

**Theorem 2.3** *Let $w$ be a positive function in $L^1$.*
(1) *If $|v| > 0$ and $v \in L^1(w)$, then $P^v$ is an unbounded operator on $L^1(w)$.*
(2) *Let $1 < p < \infty$. If $|v| > 0$ and $v \in L^p(w)$, then $P^v \in B(L^p(w))$ if and only if $|v|^p w \in (A_p)$.*

**Proof.** (1): Suppose $P^v \in B(L^1(w))$. By Lemma 2.2, $|v| > 0$, and $P \in B(L^1(|v|w))$. By the theorem of Forelli (cf. [3]), $P \in B(L^1)$. This is a contradiction (cf. [5, Vol.I, p.78]).
(2): By Lemma 2.2, if $P^v \in B(L^p(w))$, then $P \in B(L^p(|v|^p w))$. By the theorem of Hunt, Muckenhoupt and Wheeden (cf. [1], [4], [11], [12]), this implies $|v|^p w \in (A_p)$. The converse is also true. Theorem 2.3 is proved. □

**Theorem 2.4** *Let $1 < p < \infty$. Let $w$ be a positive function in $L^1$. Suppose $|v| > 0$ and $v \in L^p(w)$.*
(1) *If $P^v \in B(L^p(w))$, then*

$$\mathrm{ran}P^v = \ker Q^v = vH^p(|v|^p w) = [v\mathcal{P}]_{L^p(w)},$$

$$\ker P^v = \mathrm{ran} Q^v = v\overline{H_0^p(|v|^p w)} = [v\mathcal{Q}]_{L^p(w)},$$

*where* $[\ \cdot\ ]_{L^p(w)}$ *denotes the norm closure in* $L^p(w)$.

(2) *Suppose* $\log w$ *and* $\log|v|$ *are in* $L^1$. *Let* $k$ *be an outer function such that* $|k| = |v|$. *Let* $Q^v = I - P^v$. *If* $P^v \in B(L^p(w))$, *then*

$$\mathrm{ran} P^v = \ker Q^v = \frac{v}{k} H^p(w) \subset L^p(w),$$

$$\ker P^v = \mathrm{ran} Q^v = \frac{v}{\bar{k}} \overline{H_0^p(w)} \subset L^p(w),$$

*and*

$$L^p(w) = H^p(w) \oplus \frac{k}{\bar{k}} \overline{H_0^p(w)}.$$

(3) *If there is an outer function* $k$ *such that* $|k| = |v|$ *and* $L^p(w) = H^p(w) \oplus \frac{k}{\bar{k}}\overline{H_0^p(w)}$, *then* $P^v \in B(L^p(w))$.

**Proof.** (1): Suppose $f \in \mathrm{ran} P^v$. Then there is a $g \in L^p(w)$ such that $f = P^v g$. By Lemma 2.1(1), there is a sequence $\{t_n\}$ in $\mathcal{P} + \mathcal{Q}$ such that

$$\int |vt_n - g|^p w dm \to 0,$$

as $n \to \infty$. Since $P^v \in B(L^p(w))$, it follows that

$$\begin{aligned}
\int |Pt_n - v^{-1} f|^p |v|^p w dm &= \int |vPt_n - f|^p w dm \\
&= \int |P^v(vt_n - g)|^p w dm \\
&\le \|P^v\|^p_{B(L^p(w))} \int |vt_n - g|^p w dm.
\end{aligned}$$

Hence,

$$\int |Pt_n - v^{-1} f|^p |v|^p w dm \to 0,$$

as $n \to \infty$. This implies that $v^{-1} f \in H^p(|v|^p w)$. Hence, $\mathrm{ran} P^v \subset vH^p(|v|^p w)$. Suppose $f \in vH^p(|v|^p w)$. Since $v^{-1} f \in H^p(|v|^p w)$, there is a sequence $\{g_n\}$ in $\mathcal{P}$ such that

$$\int |vg_n - f|^p w dm = \int |g_n - v^{-1} f|^p |v|^p w dm \to 0,$$

as $n \to \infty$. This implies that $f \in [v\mathcal{P}]_{L^p(w)}$. Hence, $vH^p(|v|^p w) \subset [v\mathcal{P}]_{L^p(w)}$. Therefore,

$$v\mathcal{P} \subset \mathrm{ran} P^v \subset vH^p(|v|^p w) \subset [v\mathcal{P}]_{L^p(w)}.$$

Since $(P^v)^2 = P^v$, $\mathrm{ran} P^v$ is a closed subspace of $L^p(w)$. Similarly

$$v\mathcal{Q} \subset \mathrm{ran} P^v \subset v\overline{H_0^p(|v|^p w)} \subset [v\mathcal{Q}]_{L^p(w)},$$

and $\mathrm{ran}Q^v$ is a closed subspace of $L^p(w)$. Hence (1) follows.
(2): By Theorem 2.3, if $P^v \in B(L^p(w))$, then $|v|^p w \in (A_p)$. Since $k$ is an outer function such that $|k| = |v|$, it follows that

$$\begin{aligned}\mathrm{ran}P^v &= P^v L^p(w) = vP(v^{-1}h^{-1}L^p) = vPL^p(|v|^p w)\\ &= vH^p(|v|^p w) = \frac{v}{k}kH^p(|k|^p w) = \frac{v}{k}H^p(w),\end{aligned}$$

and

$$\begin{aligned}\mathrm{ran}Q^v &= Q^v L^p(w) = vQ(v^{-1}h^{-1}L^p) = vQL^p(|v|^p w)\\ &= v\overline{H_0^p(|v|^p w)} = \frac{v}{\bar{k}}\overline{kH_0^p(|k|^p w)} = \frac{v}{\bar{k}}\overline{H_0^p(w)}.\end{aligned}$$

Hence,

$$L^p(w) = \mathrm{ran}P^v + \mathrm{ran}Q^v = \frac{k}{v}H^p(w) \oplus \frac{\bar{k}}{v}\overline{H_0^p(w)}.$$

Since $|k| = |v|$, it follows that

$$L^p(w) = \frac{k}{v}L^p(w) = H^p(w) \oplus \frac{k}{\bar{k}}\overline{H_0^p(w)}.$$

(3): Since $|v|^p w \in L^1$ and $L^p(w) = H^p(w) \oplus \frac{k}{\bar{k}}\overline{H_0^p(w)}$, it follows that

$$\begin{aligned}L^p(|v|^p w) = k^{-1}L^p(w) &= k^{-1}H^p(w) \oplus \overline{k^{-1}H_0^p(w)}\\ &= H^p(|v|^p w) \oplus \overline{H_0^p(|v|^p w)}.\end{aligned}$$

By the closed graph theorem, this implies that $P \in B(L^p(|v|^p w))$. Theorem 2.4 is proved. □

Let $1 \leq p < \infty$. If $f \in L^p(w)$ and $w \in L^1$, then $fw \in L^1$. Let

$$K^p(w) = \{f \in L^p(w)\ ;\ (fw)\hat{}\,(n) = 0,\ (n < 0)\},$$

and let

$$K_0^p(w) = \{f \in L^p(w)\ ;\ (fw)\hat{}\,(n) = 0,\ (n \leq 0)\}.$$

Hence, $K^p(w)$ and $K_0^p(w)$ are closed subspaces of $L^p(w)$ satisfying $K^p(w) = L^p(w) \cap w^{-1}H^1$. The shift operator maps $K^p(w)$ onto $K_0^p(w)$. If $p = 2$, then we have the orthogonal decomposition:

$$L^2(w) = H^2(w) \oplus \overline{K_0^2(w)}.$$

If $w = 1$, then $K^p(w) = H^p$. According to the Riesz representation theorem, for every bounded linear functional $\phi \in H^p(w)^*$, $1 < p < \infty$, there exists a unique function $g \in K^q(w)$, $1/p + 1/q = 1$, such that

$$\phi(f) = \int f\bar{g}w\,dm,\ (f \in H^p(w)).$$

We use Lemmas 2.5 and 2.6 to prove Theorem 2.7.

**Lemma 2.5** *Let $1 \le p < \infty$, and let $1/p + 1/q = 1$. Let $h$ be an outer function satisfying $w = |h|^p$.*

(1)
$$K^p(w) = \frac{h^{p-1}}{w} H^p = \frac{h^p}{w} H^p(w).$$

(2)
$$K^q(w) = \frac{h}{w} H^q = \frac{h^p}{w} H^q(w).$$

(3) *$K^p(w) = H^p(w)$ if and only if $w$ is a constant function.*

**Proof.** (1): Suppose $f \in K^p(w)$. Then $f \in L^p(w) \cap w^{-1}H^1$. Then $fh \in L^p$ and $(fw)/(h^{p-1}) \in H^p$. Then $f \in \frac{h^{p-1}}{w} H^p$. The converse is also true. Hence, $K^p(w) = \frac{h^{p-1}}{w} H^p$. Since $H^p(w) = H^p(|h|^p) = h^{-1}H^p$, it follows that $K^p(w) = \frac{h^p}{w} H^p(w)$.
(2): Suppose $f \in K^q(w)$. Then $f \in L^q(w) \cap w^{-1}H^1$. Then $fh^{p-1} \in L^q$ and $fw/h \in H^q$. Then $f \in \frac{h}{w} H^q$. The converse is also true. Hence, $K^q(w) = \frac{h}{w} H^q$. Since $H^q(w) = H^q(|h^p|) = h^{1-p}H^q$, it follows that $K^q(w) = \frac{h^p}{w} H^q(w)$.
(3): Suppose $K^p(w) = H^p(w)$. Since $1 \in H^p(w)$, $1 \in K^p(w)$. By (1), $1 \in \frac{h^{p-1}}{w} H^p$. Hence, there is an $f \in H^p$ such that $\frac{fh^{p-1}}{w} = 1$. Since $h^{p-1} \in H^q$, $fh^{p-1}$ is a positive function in $H^1$. Hence, $f$ and $h$ are constant functions. Hence, $w$ is a constant function. The converse is clear. Lemma 2.5 is proved. $\square$

**Lemma 2.6** *Let $w, \log w, w^{(2-p)/2} \in L^1$.*
(1) *$H^p(w) \oplus \overline{K_0^p(w)} = L^p(w)$ if and only if $H^p(w^{(2-p)/2}) \oplus \overline{H_0^p(w^{(2-p)/2})} = L^p(w^{(2-p)/2})$.*
(2) *There is a constant $C$ such that*

$$\int |f|^p w dm \le C \int |f+g|^p w dm, \ \ (f \in H^p(w), \ g \in \overline{K_0^p(w)})$$

*if and only if there is a constant $C$ such that*

$$\int |f|^p w^{(2-p)/2} dm \le C \int |f+g|^p w^{(2-p)/2} dm, \ \ (f \in \mathcal{P}, \ g \in \mathcal{Q}).$$

**Proof.** By the closed graph theorem, it is sufficient to prove (1). Since $\log w \in L^1$, there is an outer function $h$ satisfying $w = |h|^p$. Let $p = 2a$. Then $w = h^a \overline{h^a}$ and $w^{(2-p)/2} = w^{1-a}$. By Lemma 2.5,

$$\begin{aligned} h^a \left( H^p(w) \oplus \overline{K_0^p(w)} \right) &= h^{a-1} H^p \oplus \overline{h^{a-1} H_0^p} \\ &= H^p(|h^{1-a}|^p) \oplus \overline{H_0^p(|h^{1-a}|^p)} \\ &= H^p(w^{1-a}) \oplus \overline{H_0^p(w^{1-a})} \\ &= H^p(w^{(2-p)/2}) \oplus \overline{H_0^p(w^{(2-p)/2})}. \end{aligned}$$

Since $L^p(w^{(2-p)/2}) = L^p(w^{1-a}) = L^p(|h^{1-a}|^p) = h^{a-1}L^p = h^a L^p(w)$, this implies (1). Lemma 2.6 is proved. $\square$

**Theorem 2.7** *Let $w \in L^1$. Suppose $w = |\alpha|^{-2}$, for some outer function $\alpha$.*
(1) $P^\alpha \in B(L^p(w))$ *if and only if* $w^{(2-p)/2} \in (A_p)$. *Then* $\|P^\alpha\|_{B(L^p(w))} = \|P\|_{B(L^p(w^{(2-p)/2}))}$.
(2) $\mathrm{ran}P^\alpha = H^p(w)$, $\ker P^\alpha = \overline{K_0^p(w)}$, $\mathrm{ran}P^{\bar\alpha} = K^p(w)$, $\ker P^{\bar\alpha} = \overline{H_0^p(w)}$.
(3) *If* $w^{(2-p)/2} \in (A_p)$, *then* $L^p(w) = H^p(w) \oplus \overline{K_0^p(w)}$, *and* $P^\alpha$ *is a bounded projection from* $L^p(w)$ *onto* $H^p(w)$ *such that*

$$P^\alpha(f+g) = f, \quad (f \in H^p(w),\ g \in \overline{K_0^p(w)}).$$

(4) $P^\alpha$ *(resp. $I - P^\alpha$) is a self-adjoint projection from $L^2(w)$ onto $H^2(w)$ (resp. $\overline{K_0^2(w)}$).*
(5) $P^{\bar\alpha}$ *(resp. $I - P^{\bar\alpha}$) is a self-adjoint projection from $L^2(w)$ onto $K^2(w)$ (resp. $\overline{H_0^2(w)}$).*

**Proof.** (1): By Theorem 2.3(2), if $P^\alpha \in B(L^p(w))$, then $|\alpha|^p w \in (A_p)$. Hence, $w^{(2-p)/2} = w^{-p/2}w = |\alpha|^p w \in (A_p)$. The converse is also true.
(2): By Lemma 2.5(1), $H_0^p(w) = \frac{\alpha^2}{|\alpha|^2} K_0^p(w)$. By Theorem 2.4(2), $\mathrm{ran}P^\alpha = H^p(w)$ and

$$\ker P^\alpha = \frac{\alpha}{\bar\alpha}\overline{H_0^p(w)} = \frac{\alpha}{\bar\alpha}\frac{|\alpha|^2}{\alpha^2}\overline{K_0^p(w)} = \overline{K_0^p(w)}.$$

Similarly, $\ker P^{\bar\alpha} = \overline{H_0^p(w)}$ and

$$\mathrm{ran}P^{\bar\alpha} = \frac{\bar\alpha}{\alpha}\overline{H^p(w)} = \frac{\bar\alpha}{\alpha}\frac{\alpha^2}{|\alpha|^2}K^p(w) = K^p(w).$$

(3): If $w^{(2-p)/2} \in (A_p)$, then $L^p(w^{(2-p)/2}) = H^p(w^{(2-p)/2}) \oplus \overline{H_0^p(w^{(2-p)/2})}$. By Lemma 2.6(1), $L^p(w) = H^p(w) \oplus \overline{K_0^p(w)}$. Since $(P^\alpha)^2 = P^\alpha$, (3) follows.
(4): Since

$$\int f\bar g w dm = 0, \quad (f \in H^2(w),\ g \in \overline{K_0^2(w)}),$$

it follows that $L^2(w) = H^2(w) \oplus \overline{K_0^2(w)}$ is the orthogonal decomposition. Since $\mathrm{ran}P^\alpha = H^2(w)$, and $\ker P^\alpha = \overline{K_0^2(w)}$, it follows that $P^\alpha$ is a self-adjoint projection.
(5): Since

$$\int f\bar g w dm = 0, \quad (f \in K^2(w),\ g \in \overline{H_0^2(w)}),$$

it follows that $L^2(w) = K^2(w) \oplus \overline{H_0^2(w)}$ is the orthogonal decomposition. Since $\mathrm{ran}P^{\bar\alpha} = K^2(w)$, and $\ker P^{\bar\alpha} = \overline{H_0^2(w)}$, it follows that $P^{\bar\alpha}$ is a self-adjoint projection from $L^2(w)$ onto $K^2(w)$. Theorem 2.7 is proved. □

# 3 Adjoint operators for $P^v$

Let $1 < p < \infty$, and let $1/p + 1/q = 1$. In this section $P^v$ is supposed to be a bounded operator on $L^p(w)$. For functions $f \in L^p(w)$ and $g \in L^q(w)$, let

$$\langle f, g\rangle_w = \int f\bar{g}wdm.$$

To each $P^v \in B(L^p(w))$ corresponds a unique $(P^v)^* \in B(L^q(w))$ that satisfies

$$\langle P^v f, g\rangle_w = \langle f, (P^v)^* g\rangle_w, \quad (f \in L^p(w),\ g \in L^q(w)).$$

We use Lemmas 3.1 and 3.2 to prove Theorem 3.3.

**Lemma 3.1** *Let $1 < p < \infty$, and let $1/p + 1/q = 1$. Let $w \in L^1$, $w > 0$, and let $v$ be a measurable function.*
(1) $|v|^p w \in (A_p)$ *if and only if* $|v|^{-q} w^{1-q} \in (A_q)$.
(2) $w^{(2-p)/2} \in (A_p)$ *if and only if* $w^{(2-q)/2} \in (A_q)$.

**Proof.** (1): If $|v|^p w \in (A_p)$, then $(|v|^p w)^{-1/(p-1)} \in (A_q)$. Since $(p-1)(q-1) = 1$, it follows that $|v|^{-q} w^{1-q} \in (A_q)$. The converse is also true.
(2): If $w^{(2-p)/2} \in (A_p)$, then $(w^{(2-p)/2})^{-1/(p-1)} \in (A_q)$. Since $(p-1)(q-1) = 1$, it follows that

$$\left(\frac{2-p}{2}\right)\left(\frac{-1}{p-1}\right) = \frac{p}{2(p-1)} - \frac{1}{p-1} = \frac{q}{2} - (q-1) = \frac{2-q}{2}.$$

Hence, $w^{(2-q)/2} \in (A_q)$. The converse is also true. Lemma 3.1 is proved. □

**Lemma 3.2** *Let $1 < p < \infty$, and let $1/p + 1/q = 1$. Let $w \in L^1$, $w > 0$, and let $|v|^p w \in (A_p)$. Then $(P^v)^* \in B(L^q(w))$, $((P^v)^*)^2 = (P^v)^*$, and*

$$(1) \qquad (P^v)^*(g) = \frac{1}{\bar{v}w} P(\bar{v}wg), \quad (g \in L^q(w)).$$

(2) *If* $\log w,\ \log|v| \in L^1$, *then*

$$(P^v)^*(g_1 + g_2) = g_1, \quad \left(g_1 \in \frac{1}{k\bar{v}w} H^q(w),\ g_2 \in \frac{1}{\overline{kvw}} \overline{H_0^q(w)}\right),$$

*where $k$ is an outer function satisfying $|k| = |vw|^{-1}$.*

**Proof.** (1): By Theorem 2.3(2), $P^v \in B(L^p(w))$. Hence, $(P^v)^* \in B(L^q(w))$. If $|v|^p w \in (A_p)$, then there is a constant $\delta > 0$ satisfying $(|v|^p w)^{1+\delta} \in L^1$ (cf. [4, p.262]). Since $1/p + 1/q = 1$, there is a constant $r > 1$ satisfying

$$\frac{1}{p(1+\delta)} + \frac{1}{q} = \frac{1}{r}.$$

Then

$$\int |vgw|^r dm \leq \left(\int (|v|^p w)^{1+\delta} dm\right)^{\frac{r}{p(1+\delta)}} \left(\int |g|^q w dm\right)^{\frac{r}{q}}.$$

For all $f \in v\mathcal{P} + v\mathcal{Q}$ and all $g \in L^q(w)$,

$$\begin{aligned}
\langle f,\ (P^v)^* g\rangle_w &= \langle P^v f,\ g\rangle_w = \int (P^v f)\bar{g} w dm \\
&= \int vP(v^{-1}f)\bar{g} w dm = \int P(v^{-1}f)\overline{P(\bar{v} g w)} dm \\
&= \int v^{-1} f \overline{P(\bar{v} g w)} dm = \int \frac{1}{vw} f \overline{P(\bar{v} g w)} w dm \\
&= \left\langle f,\ \frac{1}{\bar{v} w} P(\bar{v} w g)\right\rangle_w .
\end{aligned}$$

By Lemma 2.1(1), $v\mathcal{P} + v\mathcal{Q}$ is dense in $L^p(w)$.
(2): By Lemma 3.1(1), if $|v|^p w \in (A_p)$, then $|v|^{-q} w^{1-q} \in (A_q)$. Hence, $|(vw)^{-1}|^q w = |v|^{-q} w^{1-q} \in L^1$. By (1) and Theorem 2.4(2),

$$\operatorname{ran}(P^v)^* = \frac{1}{k\bar{v}w} H^q(w), \quad \ker(P^v)^* = \frac{1}{\overline{kvw}} \overline{H_0^q(w)}.$$

Since $((P^v)^*)^2 = (P^v)^*$,

$$L^q(w) = \operatorname{ran}(P^v)^* \oplus \ker(P^v)^* = \frac{1}{k\bar{v}w} H^q(w) \oplus \frac{1}{\overline{kvw}} \overline{H_0^q(w)}.$$

Lemma 3.2 is proved. $\square$

By Lemma 3.2, if $v = 1$ and $w$ satisfies the Muckenhoupt condition $(A_p)$, then $P^* f = P^{1/w} f$, $(f \in L^q(w))$.

**Theorem 3.3** *Let $1 < p < \infty$, and let $1/p + 1/q = 1$. Suppose $v,\ w \in L^1$, $w > 0$, $|v|^p w \in (A_p)$ and $|v|^q w \in L^1$. Then the following two properties are equivalent.*
(1) $(P^v)^* g = P^v g$, $(g \in L^p(w) \cap L^q(w))$.
(2) $|v|^2 w$ *is a constant function.*

**Proof.** Suppose (1) holds. Since $v \in L^p(w) \cap L^q(w)$, $(P^v)^* v = P^v v = vP1 = v$. By Lemma 3.2, $(\bar{v}w)^{-1} P(\bar{v}wv) = v$. Hence, $P(|v|^2 w) = |v|^2 w$. By Lemma 3.1(1), if $|v|^p w \in (A_p)$, then there is a constant $\delta > 0$ satisfying $(|v|^p w)^{1+\delta} \in L^1$ (cf. [4, p.262]). Since $1/p + 1/q = 1$, there is a constant $r > 1$ satisfying

$$\frac{1}{p(1+\delta)} + \frac{1}{q} = \frac{1}{r}.$$

Then

$$\begin{aligned}\int (|v|^2 w)^r dm &= \int |v|^r w^{r/p} |v|^r w^{r/q} dm \\ &\leq \left(\int (|v|^p w)^{1+\delta} dm\right)^{\frac{r}{p(1+\delta)}} \left(\int |v|^q w dm\right)^{\frac{r}{q}} < \infty.\end{aligned}$$

Hence, $|v|^2 w$ is a positive function satisfying $|v|^2 w \in H^r$, $r > 1$. This implies (2). Conversely, suppose (2) holds. By Lemma 3.2, for all $g \in L^p(w) \cap L^q(w)$,

$$(P^v)^* g = \frac{1}{\bar{v} w} P(\bar{v} w g) = \frac{v}{|v|^2 w} P\left(\frac{|v|^2 w}{v} g\right) = \frac{v}{c} P\left(\frac{c}{v} g\right) = v P(v^{-1} g) = P^v g.$$

Theorem 3.3 is proved. □

By Theorem 3.3, if $v = 1$ and $w$ satisfies the Muckenhoupt condition $(A_p)$, then $P^* = P$ on $L^p(w) \cap L^q(w)$ if and only if $w$ is a constant.

**Corollary 3.4** *Let $1 < p < \infty$, and let $1/p + 1/q = 1$. Let $\alpha$ be an outer function such that $w = |\alpha|^{-2}$. If $w^{(2-p)/2} \in (A_p)$, then $P^\alpha$ is a bounded operator on $L^p(w)$, and $(P^\alpha)^*$ is a bounded operator on $L^q(w)$ such that*

$$(P^\alpha)^*(g_1 + g_2) = g_1, \quad (g_1 + g_2 \in H^q(w) \oplus \overline{K_0^q(w)}).$$

*and*

$$(P^\alpha)^* = P^\alpha, \quad \text{on} \quad L^p(w) \cap L^q(w).$$

**Proof.** By Theorem 2.7, if $w^{(2-p)/2} \in (A_p)$, then $P^\alpha \in B(L^p(w))$ and $H^p(w) \oplus \overline{K_0^p(w)} = L^p(w)$. By Lemma 3.1(2), if $w^{(2-p)/2} \in (A_p)$, then $w^{(2-q)/2} \in (A_q)$, and hence $H^q(w) \oplus \overline{K_0^q(w)} = L^q(w)$. For all $f_1 + f_2 \in H^p(w) \oplus \overline{K_0^p(w)}$, and all $g_1 + g_2 \in H^q(w) \oplus \overline{K_0^q(w)}$,

$$\begin{aligned}\langle f_1 + f_2,\ (P^\alpha)^*(g_1 + g_2)\rangle_w &= \langle P^\alpha(f_1 + f_2),\ g_1 + g_2\rangle_w \\ &= \langle f_1,\ g_1 + g_2\rangle_w \\ &= \langle f_1,\ g_1\rangle_w \\ &= \langle f_1 + f_2,\ g_1\rangle_w.\end{aligned}$$

On the other hand, by Theorem 3.3, $(P^\alpha)^* = P^\alpha$. Corollary 3.4 is proved. □

## 4 Invertibility of $T_\phi^v$ and $\phi P^v + Q^v$

In this section, the invertibility criterion for the generalized Toeplitz operator $T_\phi^v$ and the generalized singular integral operator $\phi P^v + Q^v$, $Q^v = I - P^v$ are investigated using the weighted norm inequality. By the theorem of Hunt, Muckenhoupt and Wheeden (cf. [1], [4], [11], [12]), $w \in (A_p)$ if and only if $P$ is a bounded projection from $L^p(w)$ onto $H^p(w)$. For $\phi \in L^\infty$, the Toeplitz operator $T_\phi$ is defined as a bounded operator from $H^p(w)$ to $H^p(w)$ by

$$T_\phi f = P(\phi f), \quad (f \in H^p(w)).$$

By Theorem 2.3, if $|v|^p w \in (A_p)$, then $P^v \in B(L^p(w))$. Since $(P^v)^2 = P^v$, $\mathrm{ran}P^v$ is a closed subspace of $L^p(w)$. For $\phi \in L^\infty$, the generalized Toeplitz operator $T_\phi^v$ is defined as a bounded operator from $\mathrm{ran}P^v$ to $\mathrm{ran}P^v$ by

$$T_\phi^v f = P^v(\phi f), \quad (f \in \mathrm{ran}P^v).$$

We use Lemma 4.1 to prove Lemma 4.2.

**Lemma 4.1** *Let $1 < p < \infty$. Suppose $\phi \in L^\infty$, $w, \log w \in L^1$, and $|v|^p w \in (A_p)$. Then the following properties are equivalent.*
*(1) $T_\phi^v$ is a left invertible operator on $\mathrm{ran}P^v$.*
*(2) $T_\phi$ is a left invertible operator on $H^p(|v|^p w)$.*
*(3) $\phi P + Q$ is a left invertible operator on $L^p(|v|^p w)$.*
*(4) $\phi P^v + Q^v$ is a left invertible operator on $L^p(w)$.*

**Proof.** Let $w' = |v|^p w$. By Theorem 2.3, $T_\phi^v$, $T_\phi$, $\phi P + Q$, $\phi P^v + Q^v$ are bounded operators on each spaces. Suppose (1) holds. Then there is an $\varepsilon_1 > 0$ such that

$$\int |T_\phi^v f|^p w dm \geq \varepsilon_1 \int |f|^p w dm, \quad (f \in \mathrm{ran}P^v)\,.$$

Suppose $f \in H^p(w')$. Since $\log w \in L^1$, there is an outer function $h$ satisfying $w = |h|^p$. Since $\log |v| \in L^1$, there is an outer function $k$ satisfying $|k| = |v|$. Since $w' = |v|^p w$, $H^p(w') = H^p(|kh|^p) = \frac{1}{kh} H^p = k^{-1} H^p(w)$. By Theorem 2.4, $\mathrm{ran}P^v = \frac{v}{k} H^p(w) = v H^p(w')$. Hence, there is a $g \in \mathrm{ran}P^v$ such that $g = vf$. By (1), there is an $\varepsilon_1 > 0$ such that

$$\begin{aligned}
\int |T_\phi f|^p w' dm &= \int |P(\phi f)|^p |v|^p w dm \\
&= \int |v P(\phi v^{-1} g)|^p w dm \\
&= \int |P^v(\phi g)|^p w dm
\end{aligned}$$

$$
\begin{aligned}
&= \int |T_\phi^v g|^p w dm \\
&\geq \varepsilon_1 \int |g|^p w dm \\
&= \varepsilon_1 \int |f|^p w' dm.
\end{aligned}
$$

This implies (2). Suppose (2) holds. Then there is an $\varepsilon_2 > 0$ such that

$$\int |T_\phi f|^p w' dm \geq \varepsilon_2 \int |f|^p w' dm, \quad (f \in H^p(w')).$$

Suppose $f \in L^p(w')$. Let $g = (I + Q\phi P)f$. Since $\phi \in L^\infty$ and $w' \in (A_p)$, it follows that $g \in L^p(w')$, and there is a $C_1 > 0$ such that

$$
\begin{aligned}
\int |Qg|^p w' dm &= \int |Q(P\phi P + Q)g|^p w' dm \\
&\leq C_1 \int |(P\phi P + Q)g|^p w' dm.
\end{aligned}
$$

By the theorem of Hunt, Muckenhoupt and Wheeden (cf. [1], [4], [11], [12]), if $w' \in (A_p)$, then $P, Q \in B(L^p(w'))$, and there is a $C_2 > 0$ such that

$$
\begin{aligned}
\varepsilon_2 \int |Pg|^p w' dm &\leq \int |T_\phi Pg|^p w' dm \\
&= \int |P(P\phi P + Q)g|^p w' dm \\
&\leq C_2 \int |(P\phi P + Q)g|^p w' dm.
\end{aligned}
$$

Since $Q = I - P$, it follows that $Q \in B(L^p(w'))$, and there is a $C_3 > 0$ such that

$$\int |g|^p w' dm \leq C_3 \int |(P\phi P + Q)g|^p w' dm.$$

Since $P, Q \in B(L^p(w'))$, it follows that $(I + Q\phi P)f \in L^p(w')$, and there is a $C_4 > 0$ such that

$$
\begin{aligned}
\int |f|^p w' dm &= \int |(I - Q\phi P)(I + Q\phi P)f|^p w' dm \\
&\leq C_4 \int |(I + Q\phi P)f|^p w' dm \\
&\leq C_3 C_4 \int |(P\phi P + Q)(I + Q\phi P)f|^p w' dm \\
&= C_3 C_4 \int |(\phi P + Q)f|^p w' dm.
\end{aligned}
$$

This implies (3). Suppose (3) holds. Then there is an $\varepsilon_3 > 0$ such that

$$\int |(\phi P + Q)f|^p w' dm \geq \varepsilon_3 \int |f|^p w' dm, \quad (f \in L^p(w')).$$

Suppose $f \in L^p(w)$. Then $vf \in L^p(|v|^p w) = L^p(w')$. Since $P^v f = vP(v^{-1}f)$ and $Q^v f = vQ(v^{-1}f)$, it follows that

$$\begin{aligned}\int |(\phi P^v + Q^v)f|^p w dm &= \int |v(\phi P + Q)(v^{-1}f)|^p w dm \\ &= \int |(\phi P + Q)(v^{-1}f)|^p |v|^p w dm \\ &\geq \varepsilon_3 \int |v^{-1}f|^p |v|^p w dm = \varepsilon_3 \int |f|^p w dm.\end{aligned}$$

This implies (4). Suppose (4) holds. Then there is an $\varepsilon_4 > 0$ such that

$$\int |(\phi P^v + Q^v)f|^p w dm \geq \varepsilon_4 \int |f|^p w dm, \quad (f \in L^p(w)).$$

By Theorem 2.3, $P^v \in B(L^p(w))$. Suppose $f \in \mathrm{ran} P^v$. Since $Q^v = I - P^v$, it follows that $P^v f = f$, $Q^v f = 0$, and there is an $\varepsilon_5 > 0$ such that

$$\begin{aligned}\int |T_\phi^v f|^p w dm &= \int |P^v(\phi f)|^p w dm \\ &= \int |(P^v \phi P^v + Q^v)f|^p w dm \\ &= \int |(\phi P^v + Q^v)(I - Q^v \phi P^v)f|^p w dm \\ &\geq \varepsilon_4 \int |(I - Q^v \phi P^v)f|^p w dm \\ &\geq \varepsilon_5 \int |(I + Q^v \phi P^v)(I - Q^v \phi P^v)f|^p w dm \\ &= \varepsilon_5 \int |f|^p w dm.\end{aligned}$$

This implies (1). Lemma 4.1 is proved. □

We use Lemma 4.2 to prove Theorem 4.3.

**Lemma 4.2** *Let* $1 < p < \infty$. *Suppose* $w, \log w \in L^1$ *and* $|v|^p w \in (A_p)$. *Suppose* $w = |h|^p$ *and* $|v| = |k|$ *for some outer functions* $h$ *and* $k$. *Let* $\phi$ *be a nonzero function in* $L^\infty$ *and let*

$$\psi = \phi \frac{\overline{kh}}{kh}.$$

*Then the following properties are equivalent.*
(1) $T_\phi^v$ *is a left invertible operator on* $\mathrm{ran} P^v$.
(2) $T_\psi$ *is a left invertible operator on* $H^p$.

**Proof.** Suppose (1) holds. By Lemma 4.1,

$$\begin{aligned}\int |(\phi P+Q)f|^p|v|^p w dm &\geq \varepsilon_1 \int |f|^p|v|^p w dm \\ &\geq \varepsilon_2 \int |Pf|^p |v|^p w dm, \quad (f \in L^p(|v|^p w)).\end{aligned}$$

Hence,

$$\int |\phi f_0 + \overline{g_0}|^p |v|^p w dm \geq \varepsilon_2 \int |f_0|^p |v|^p w dm, \quad (f_0 \in H^p(|v|^p w),\ g_0 \in H_0^p(|v|^p w)).$$

Hence,

$$\int \left| \phi \frac{\overline{kh}}{kh} kh f_0 + \overline{khg_0} \right|^p dm \geq \varepsilon_2 \int |khf_0|^p\, dm, \quad (f_0 \in H^p(|kh|^p),\ g_0 \in H_0^p(|kh|^p)).$$

Since $khH^p(|kh|^p) = H^p$, it follows that

$$\int |\psi f_1 + \overline{g_1}|^p dm \geq \varepsilon \int |f_1|^p dm, \quad (f_1 \in H^p,\ g_1 \in H_0^p).$$

Hence,

$$\int |(\psi P + Q) f|^p dm \geq \varepsilon_3 \int |f|^p dm, \quad (f \in L^p).$$

By Lemma 4.1 with $v = w = 1$,

$$\int |T_\psi f|^p dm \geq \varepsilon_4 \int |f|^p dm, \quad (f \in H^p).$$

This implies (2). The converse is also true. Lemma 4.2 is proved. □

If $P^v \in B(L^p(w))$, then $T_\phi^v$ is an invertible operator on $\mathrm{ran}P^v$ if and only if $P^v\phi P^v + Q^v$ is an invertible operator on $L^p(w)$ if and only if $\phi P^v + Q^v$ is an invertible operator on $L^p(w)$, since $P^v \phi P^v + Q^v = T_\phi^v P^v + Q^v$ , $(\phi P^v + Q^v)(I - Q^v \phi P^v) = P^v \phi P^v + Q^v$, and $(I - Q^v \phi P^v)^{-1} = I + Q^v \phi P^v$ (cf. [11, p.393], [12, Vol.1, p.274]). Hence, we consider only the invertibility of $T_\phi^v$. Corollary 4.4 is the theorem of Rochberg and Simonenko (cf. [13], [1, p.216], [12]). Their proof did not use the theorem of Widom and Devinatz. We use the theorem of Widom and Devinatz to prove Theorem 4.3.

**Theorem 4.3** *Let $1 < p < \infty$. Suppose $w, \log w \in L^1$ and $|v|^p w \in (A_p)$. Let $\phi$ be a nonzero function in $L^\infty$. Then the following properties are equivalent.*
(1) $T_\phi^v$ *is an invertible operator on* $\mathrm{ran}P^v$.
(2) $\phi = \gamma \exp(U - i\tilde{V})$, *where $\gamma$ is a constant with $|\gamma| = 1$, $U$ is a bounded real function, $V$ is a real function in $L^1$ and $|v|^p w \exp(pV/2) \in (A_p)$. ($\tilde{V}$ denote the harmonic conjugate function of $V$.)*

**Proof.** Suppose (1) holds. Since $\log w \in L^1$, there is an outer function $h$ satisfying $w = |h|^p$. Since $\log|v| \in L^1$, there is an outer function $k$ satisfying $|k| = |v|$. By Theorem 2.4, $\mathrm{ran}P^v = \frac{v}{k}H^p(w) = vH^p(|v|^p w)$ and $L^p(w) = H^p(w) \oplus \frac{k}{\bar{k}}\overline{H_0^p(w)} = \mathrm{ran}P^v \oplus \frac{v}{\bar{k}}\overline{H_0^p(w)}$. Since $1 \in H^p(|v|^p w)$, $v \in vH^p(|v|^p w) = \mathrm{ran}P^v$. Since $T_\phi^v$ is invertible, there is an $f \in \mathrm{ran}P^v$ such that $T_\phi^v f = v$. Hence, $P^v(\phi f) = v$. Hence, $\phi f - v = Q^v(\phi f)$. Hence, there is a $g \in \mathrm{ran}Q^v$ such that $\phi f = v + g$. Let

$$\psi = \phi\frac{\overline{kh}}{kh}.$$

Then $\psi f kh = \phi f \overline{kh} = (v+g)\overline{kh}$. Since $f \in \mathrm{ran}P^v = \frac{v}{k}H^p(w)$, it follows that $\frac{fk}{v} \in H^p(w) = h^{-1}H^p$. Hence, $\frac{fkh}{v} \in H^p$. Since $g \in \mathrm{ran}Q^v = \frac{v}{\bar{k}}\overline{H_0^p(w)} = \frac{v}{\overline{kh}}\overline{H_0^p}$, it follows that $\frac{g\overline{kh}}{v} \in \overline{H_0^p}$. Let $F_0 = \frac{fkh}{v}$. Then $F_0 \in H^p$, and

$$\psi F_0 - \overline{kh} = \frac{g\overline{kh}}{v} \in \overline{H_0^p}.$$

Let $c$ be the 0th Fourier coefficient of $kh$. Since $kh$ is an outer function, $c \neq 0$. Then $\psi F_0 - \bar{c} \in \overline{H_0^p}$. Hence, $T_\psi F_0 = \bar{c}$. Hence, $1 \in \mathrm{ran}T_\psi$. Hence, there is an $F \in H^p$ such that $\psi F - 1 \in \overline{H_0^p}$. Hence, $\psi z F - z \in \overline{H^p}$. Hence, $T_\psi(zF) - z$ is a constant. Since $1 \in \mathrm{ran}T_\psi$, this implies that $z \in \mathrm{ran}T_\psi$. Suppose $1, z, ..., z^n \in \mathrm{ran}T_\psi$ and there are constants $c_1, c_2, ..., c_n$ such that $\psi z^n F - z^n - (c_1 z^{n-1} + c_2 z^{n-2} + ... + c_n) \in \overline{H_0^p}$. Then

$$\psi z^{n+1} F - z^{n+1} - (c_1 z^n + c_2 z^{n-1} + ... + c_n z) \in \overline{H^p}.$$

Let $c_{n+1}$ be the 0th Fourier coefficient of this function. Then

$$\psi z^{n+1} F - z^{n+1} - (c_1 z^n + c_2 z^{n-1} + ... + c_n z + c_{n+1}) \in \overline{H_0^p}.$$

Hence,

$$T_\psi(z^{n+1}F) - z^{n+1} - (c_1 z^n + c_2 z^{n-1} + ... + c_n z + c_{n+1}) = 0.$$

Since $1, z, ..., z^n \in \mathrm{ran}T_\psi$, it follows that $z^{n+1} \in \mathrm{ran}T_\psi$. Hence, $1, z, z^2, ... \in \mathrm{ran}T_\psi$. Hence, $\mathrm{ran}T_\psi$ is dense in $H^p$ (cf. [9]). By Lemma 4.2, $T_\psi$ is left invertible. Hence, $T_\psi$ is an invertible operator on $H^p$. By the theorem of Widom and Devinatz (cf. [1], [11], [12]), $\psi = \gamma_1 \exp(U - i\tilde{V_0})$, where $\gamma_1$ is a constant with $|\gamma_1| = 1$, $U$ is a bounded real function, $V_0$ is a real function in $L^1$ and $\exp(pV_0/2) \in (A_p)$. Hence,

$$\phi\frac{\overline{kh}}{kh} = \psi = \gamma_1 \exp(U - i\tilde{V_0}).$$

There are constants $\gamma_2$ and $\gamma_3$ with $|\gamma_2| = |\gamma_3| = 1$ such that

$$h^p = \gamma_2 \exp\left(\log w + i(\log w)\tilde{}\,\right),$$

$$k = \gamma_3 \exp\left(\log|v| + i(\log|v|)\tilde{}\,\right).$$

Hence, there is a constant $\gamma_4$ with $|\gamma_4| = 1$ such that

$$\phi = \gamma_1 \frac{(kh)^2}{|kh|^2} \exp(U - i\tilde{V_0}) = \gamma_4 \exp\left(U - i(V_0 - \log|v|^2 - \log w^{2/p})\tilde{}\right).$$

Let $V = V_0 - \log|v|^2 - \log w^{2/p}$. Then $\phi = \gamma_4 \exp(U - i\tilde{V})$ and $|v|^p w = \exp(p(V_0 - V)/2)$. Hence, $|v|^p w \exp(pV/2) = \exp(pV_0/2) \in (A_p)$. This implies (2).
Conversely, suppose (2) holds. By the similar calculation, (2) implies that $\psi = \gamma_1 \exp(U - i\tilde{V_0})$, where $\gamma_1$ is a constant with $|\gamma_1| = 1$, $U$ is a bounded real function, $V_0$ is a real function in $L^1$ and $\exp(pV_0/2) \in (A_p)$. By the theorem of Widom and Devinatz (cf. [1], [11], [12]), $T_\psi$ is an invertible operator on $H^p$. By Lemma 4.2, $T_\phi^v$ is a left invertible operator on $\mathrm{ran}P^v$. It is sufficient to prove that $\mathrm{ran}T_\phi^v$ is dense in $\mathrm{ran}P^v$. Let $n$ be a nonnegative integer. Then there is an $F \in H^p$ such that $T_\psi F = P(z^n \overline{kh})$. Since $P(\psi F - z^n \overline{kh}) = 0$, it follows that $\psi F - z^n \overline{kh} = \phi \frac{\overline{kh}}{kh} F - z^n \overline{kh} \in \overline{H_0^p}$. Hence, $\frac{\phi \bar{k} F}{kh} - z^n \bar{k} \in \overline{H_0^p(w)}$. By Theorem 2.4, $\frac{\phi v F}{kh} - z^n v \in \frac{v}{k} \overline{H_0^p(w)} = \ker P^v$. Let $G = \frac{vF}{kh}$. Then $G \in \frac{v}{k} H^p(w) = \mathrm{ran}P^v$. Since $\phi G - z^n v \in \ker P^v$, it follows that $T_\phi^v G = P^v(\phi g) = P^v(z^n v) = z^n v$. Hence, $z^n v \in \mathrm{ran}T_\phi^v$, $(n = 0, 1, 2...)$. Let $g \in \mathrm{ran}P^v$. Then $v^{-1} g \in k^{-1} H^p(w) = H^p(|v|^p w)$. Hence, there is a sequence of analytic polynomials $f_n$ such that $\|f_n - v^{-1} g\|_{L^p(|v|^p w)} \to 0, (n \to \infty)$. Hence, $\|v f_n - g\|_{L^p(w)} \to 0$. Therefore $\mathrm{ran}T_\phi^v$ is dense in $\mathrm{ran}P^v$. This implies (1). Theorem 4.3 is proved. □

By Theorem 4.3, $T_\phi^v$ is invertible on $\mathrm{ran}P^v$ if and only if $T_\phi$ is invertible on $H^p(|v|^p w)$. Hence, it is proved that the condition "$T_\psi$ is an invertible operator on $H^p$" is also equivalent in the theorem.

**Corollary 4.4** *Let $1 < p < \infty$. Suppose $w \in (A_p)$. Let $\phi$ be a nonzero function in $L^\infty$. Then the following properties are equivalent.*
*(1) $T_\phi$ is an invertible operator on $H^p(w)$.*
*(2) $\phi = \gamma \exp(U - i\tilde{V})$, where $\gamma$ is a constant with $|\gamma| = 1$, $U$ is a bounded real function, $V$ is a real function in $L^1$ and $w \exp(pV/2) \in (A_p)$.*

**Proof.** Let $v = k = 1$. By Theorem 2.4, $\mathrm{ran}P = \mathrm{ran}P^v = \frac{v}{k} H^p(w) = H^p(w)$. Theorem 4.3 proves Corollary 4.4. □

**Corollary 4.5** *Let $1 < p < \infty$, and let $1/p + 1/q = 1$. Let $\alpha$ be an outer function such that $w = |\alpha|^{-2}$. If $w^{(2-p)/2} \in (A_p)$, then $P^\alpha$ is a bounded projection from $L^p(w)$ onto $H^p(w)$ such that $(P^\alpha)^* = P^\alpha$, on $L^p(w) \cap L^q(w)$. Then the following properties are equivalent.*

(1) $T_\phi^\alpha$ *is an invertible operator on* $H^p(w)$.
(2) $\phi = \gamma \exp(U - i\tilde{V})$, *where* $\gamma$ *is a constant with* $|\gamma| = 1$, $U$ *is a bounded real function,* $V$ *is a real function in* $L^1$ *and* $w^{(2-p)/2}\exp(pV/2) \in (A_p)$.

**Proof.** By Corollary 3.4, $P^\alpha$ is a bounded projection from $L^p(w)$ onto $H^p(w)$ such that $(P^\alpha)^* = P^\alpha$. In the proof of Theorem 4.3, let $v = k = \alpha$. Then

$$\psi = \phi\frac{\overline{kh}}{kh}.$$

By Theorem 4.3, $T_\phi^\alpha$ is invertible on $H^p(w)$ if and only if $\phi = \gamma\exp(U - i\tilde{V})$, where $\gamma$ is a constant with $|\gamma| = 1$, $U$ is a bounded real function, $V$ is a real function in $L^1$ and $w^{(2-p)/2}\exp(pV/2) = |v|^p w \exp(pV/2) \in (A_p)$. Corollary 4.5 is proved. □

**Corollary 4.6** *Let* $\phi$ *be a nonzero function in* $L^\infty$. *Let* $w \in L^1$. *Suppose* $w = |\alpha|^{-2}$ *for some outer function* $\alpha$. *Then* $P^\alpha$ *is a self-adjoint projection from* $L^2(w)$ *onto* $H^2(w)$. *Then the following properties are equivalent.*
(1) $T_\phi^\alpha$ *is an invertible operator on* $H^2(w)$.
(2) $T_\phi$ *is an invertible operator on* $H^2$.
(3) $\phi = \gamma\exp(U - i\tilde{V})$, *where* $\gamma$ *is a constant with* $|\gamma| = 1$, $U$ *is a bounded real function,* $V$ *is a real function in* $L^1$ *and* $e^V \in (A_2)$.

**Proof.** By Theorem 4.3, $T_\phi^v$ is invertible on ran$P^v$ if and only if $T_\phi$ is invertible on $H^p(|v|^p w)$. Hence (1) is equivalent to (2). By Theorem 2.7, $P^\alpha$ is a self-adjoint projection from $L^2(w)$ onto $H^2(w)$. Since $p = 2$, it follows that $w^{(2-p)/2}\exp(pV/2) = e^V \in (A_p)$. By Corollary 4.5, (1) is equivalent to (3). Corollary 4.6 is proved. □

By the theorem of Widom (cf. [1, p.68], [12, p.260]), the spectrum of $T_\phi^\alpha \in B(H^2(w))$ is connected.

# 5 Invertibility of $R_\phi^v$

In this section, we assume that $v$ is an outer function. We do not assume that $P^v \in B(L^p(w))$. Hence, the results in this section do not follow from the theorem of Rochberg and Simonenko or the theorem of Widom and Devinatz (cf. [13], [1, p.216], [12]). Let $1 < p < \infty$. Let $w$, $\log w \in L^1$. Let $\phi \in L^\infty$. The operator $R_\phi^v$ is defined as a bounded operator from $H^p(w)$ to $L^p(w)/\frac{v}{\bar{v}}\overline{H_0^p(w)}$ by

$$R_\phi^v f = \phi f + \frac{v}{\bar{v}}\overline{H_0^p(w)}, \quad (f \in H^p(w)).$$

If $P^v \in B(L^p(w))$, then $\ker P^v = \frac{v}{\overline{v}}\overline{H_0^p(w)}$. If $w = |\alpha|^{-2}$ for some outer function $\alpha$, then $R_\phi^\alpha$ is a bounded operator from $H^p(w)$ to $L^p(w)/\overline{K_0^p(w)}$ such that

$$R_\phi^\alpha f = \phi f + \overline{K_0^p(w)}, \quad (f \in H^p(w)).$$

If $P^v \in B(L^p(w))$, then $T_\phi^v$ is an invertible operator on ran$P^v$ if and only if $R_\phi^v$ is an invertible operator from $H^p(w)$ onto $L^p(w)/\frac{v}{\overline{v}}\overline{H_0^p(w)}$. Theorem 4.3 for an outer function $v$ follows from Theorem 5.2. Theorem 5.2 with $P^v \in B(L^p(w))$ follows from Theorem 4.3. We use Lemma 5.1 to prove Theorem 5.2. Nakazi [9] considered the case when $v = 1$, and proved Lemma 5.1. We use Lemma 5.1 to prove Theorem 5.2.

**Lemma 5.1** *Let $1 < p < \infty$. Suppose $w = |h|^p$ for some outer function $h \in H^p$, $\phi \in L^\infty$ and $v$ is an outer function. Then the following conditions are equivalent.*
(1) $R_\phi^1$ *is an invertible operator from $H^p(w)$ onto $L^p(w)/\overline{H_0^p(w)}$.*
(2) $\phi = k_0(\overline{h_0}/h_0)(h/\overline{h})$, *where $k_0$ is an invertible function in $H^\infty$ and $h_0$ is an outer function in $H^p$ with $|h_0|^p \in (A_p)$.*
(3) $\phi = \gamma \exp(U - i\tilde{V})$, *where $\gamma$ is a constant with $|\gamma| = 1$, $U$ is a bounded real function, $V$ is a real function in $L^1$ and $w \exp(pV/2) \in (A_p)$.*

**Theorem 5.2** *Let $1 < p < \infty$. Suppose $w = |h|^p$ for some outer function $h \in H^p$, $\phi \in L^\infty$ and $v$ is an outer function. Let*

$$\psi = \phi \frac{\overline{v}}{v}.$$

*Then the following conditions are equivalent.*
(1) $R_\phi^v$ *is an invertible operator from $H^p(w)$ onto $L^p(w)/\frac{v}{\overline{v}}\overline{H_0^p(w)}$.*
(2) $\phi = \gamma \exp(U - i\tilde{V})$, *where $\gamma$ is a constant with $|\gamma| = 1$, $U$ is a bounded real function, $V$ is a real function in $L^1$ and $|v|^p w \exp(pV/2) \in (A_p)$.*

**Proof.** If $R_\phi^v$ is left invertible, then for any $f \in H^p(w)$ and $g \in H_0^p(w)$,

$$\int \left| \phi \frac{\bar{k}}{k} f + \overline{g} \right|^p w dm = \int \left| \phi f + \frac{v}{\overline{v}} \overline{g} \right|^p w dm \geq \varepsilon \int |f|^p w dm,$$

where $\varepsilon$ is a positive constant. This implies that $R_{\phi\bar{k}/k}^1$ is left invertible. The converse is also true. Hence, $R_\phi^v$ is left invertible if and only if $R_{\phi\bar{k}/k}^1$ is left invertible. Since

$$\left( L^p(w)/\frac{v}{\overline{v}}\overline{H_0^p(w)} \right)^* = \frac{v}{\overline{v}} K^q(w), \quad (H^p(w))^* = L^q(w)/\overline{K_0^q(w)},$$

it follows that $(R^v_\phi)^*$ is a bounded operator from $\frac{v}{\bar v}K^q(w)$ to $L^q(w)/\overline{K^q_0(w)}$. For all $F \in K^q(w)$ and all $g \in H^p(w)$,

$$\begin{aligned}\left\langle (R^v_\phi)^*\left(\frac{v}{\bar v}F\right),\, g\right\rangle &= \left\langle \frac{v}{\bar v}F,\ R^v_\phi g\right\rangle \\ &= \int \frac{v}{\bar v}F\overline{\phi g}wdm \\ &= \left\langle \bar\phi\frac{v}{\bar v}F + \overline{K^q_0(w)},\ g\right\rangle .\end{aligned}$$

Hence,

$$(R^v_\phi)^*\left(\frac{\bar v}{v}F\right) = \bar\phi\frac{v}{\bar v}F + \overline{K^q_0(w)}, \quad (F \in K^q(w)).$$

If $R^v_\phi$ is a right invertible operator from $H^p(w)$ to $L^p(w)/\frac{v}{\bar v}\overline{H^p_0(w)}$, then $(R^v_\phi)^*$ is a left invertible operator from $\frac{v}{\bar v}K^q(w)$ to $L^q(w)/\overline{K^q_0(w)}$. Hence,

$$\int \left|\bar\phi\frac{v}{\bar v}F + G\right|^q wdm \geq \varepsilon\int |F|^q wdm, \quad (F \in K^q(w),\ G \in \overline{K^q_0(w)}).$$

Hence $(R^1_{\phi\bar v/v})^*$ is a left invertible operator from $K^q(w)$ to $L^q(w)/\overline{K^q_0(w)}$. Hence, $R^1_{\phi\bar v/v}$ is a right invertible operator from $H^p(w)$ to $L^p(w)/\overline{H^p_0(w)}$. The converse is also true. Hence, $R^v_\phi$ is right invertible if and only if $R^1_{\phi\bar v/v}$ is right invertible. Hence, $R^v_\phi$ is invertible if and only if $R^1_{\phi\bar v/v}$ is invertible. By Lemma 5.1, $R^1_{\phi\bar v/v}$ is invertible if and only if

$$\phi\frac{|v|^2}{v^2} = \phi\frac{\bar v}{v} = \gamma_0\exp(U - i\tilde V_0),$$

where $\gamma_0$ is a constant with $|\gamma| = 1$, $U$ is a bounded real function, $V_0$ is a real function in $L^1$ and $w\exp(pV/2) \in (A_p)$. Since $v$ is an outer function,

$$v^2 = \gamma_1\exp(\log|v|^2 + i(\log|v|^2)\tilde{\ }).$$

Hence,

$$\phi = \gamma_2\exp\left(U - i(V_0 - \log|v|^2)\tilde{\ }\right).$$

Let $V = V_0 - \log|v|^2$. Then $\phi = \gamma_2\exp(U - i\tilde V)$, and

$$|v|^p w\exp(pV/2) = w\left(|v|^2e^V\right)^{p/2} = w\exp(pV_0/2) \in (A_p).$$

Theorem 5.2 is proved. □

By Theorem 4.3 and Theorem 5.2, if $P^v \in B(L^p(w))$ and $v$ is an outer function, then $T^v_\phi$ is invertible if and only if $R^v_\phi$ is invertible.

# 6 Norms of Hankel Operators $H_\phi^v$

In this section, the operator norm inequality for the generalized Hankel operator $H_\phi^v$ is presented. Let $Q^v = I - P^v$. By Theorem 2.3, if $Q^v \in B(L^p(w))$, then $|v|^p w \in (A_p)$. For $\phi \in L^\infty$, the generalized Hankel operator $H_\phi^v$ is defined as a bounded operator from $\mathrm{ran} P^v$ to $\ker P^v$ by

$$H_\phi^v f = Q^v(\phi f), \quad (f \in \mathrm{ran} P^v).$$

If $w \in (A_p)$, $Q = I - P$, and $\phi \in L^\infty$, then the original Hankel operator $H_\phi$ is defined as a bounded operator from $H^p(w)$ to $H^p(w)$ by

$$H_\phi f = Q(\phi f), \quad (f \in H^p(w)).$$

We use Lemma 6.1 to prove Theorem 6.2.

**Lemma 6.1** *Let $1 < p < \infty$, and let $1/p + 1/q = 1$. Suppose $w \in L^1, w > 0, \log w \in L^1$. For a function $k$, the following two properties are equivalent.*
*(1) $k \in H_0^1$, and $\|k\|_1 \leq 1$.*
*(2) There are $f \in H^p(w)$ and $g \in K_0^q(w)$ such that $\|f\|_{p,w} = \|g\|_{q,w} \leq 1$, and $k = fgw$.*

**Proof.** Suppose (1) holds. By the factorization theorem, there exists an inner function $j$ and an outer function $k_0 \in H^1$ such that $k = zjk_0$. Let $h \in H^p$ be an outer function such that $w = |h|^p$. If $f = h^{-1} j k_0^{1/p}$, then $f \in H^p(w)$. By Lemma 2.5, if $g = w^{-1} h z k_0^{1/q}$, then

$$g \in \frac{h^p}{w} h^{1-p} H_0^q = \frac{h^p}{w} H_0^q(w) = K_0^q(w),$$

$\|f\|_{p,w} = \|g\|_{q,w} = \|k\|_1 \leq 1$, and $k = fgw$. This implies (2). Conversely, suppose (2) holds. Since $K_0^q(w) = \frac{h^p}{w} H_0^q(w)$, it follows from (1) that $gw \in h^p H_0^q$. Hence, $fgw \in h^p H^p(w) H_0^q(w) = h^p H_0^1(w) = H_0^1$. By the Hölder inequality, $\|k\|_1 = \|fgw\|_1 \leq \|f\|_{p,w} \|g\|_{p,w}$. This implies (1). Lemma 6.1 is proved. □

**Theorem 6.2** *Let $1 < p < \infty$. Suppose $\phi \in L^\infty$, and $w$ is a positive function such that $w$, $\log w \in L^1$. Let $\log |v| \in L^1$. If $|v|^p w \in (A_p)$, then the following inequality holds.*

$$\|H_\phi\|_{B(L^2)} \leq \|H_\phi^v\|_{B(L^p(w))} \leq \|Q^v\|_{B(L^p(w))} \|H_\phi\|_{B(L^2)}.$$

**Proof.** By Theorem 2.3, if $|v|^p w \in (A_p)$, then $P^v \in B(L^p(w))$. We shall prove the first inequality. Let $k$ be an outer function such that $|v| = |k|$. Hence,

$$\begin{aligned}
\|H_\phi^v\|_{B(L^p(w))} &= \sup_{f\in \mathrm{ran}P^v, \|f\|_{p,w}\le 1} \|H_\phi^v f\|_{p,w} \\
&\ge \sup_{f\in \mathrm{ran}P^v, \|f\|_{p,w}\le 1, g\in K_0^q(w), \|g\|_{q,w}\le 1} \left|\int (H_\phi^v f)\frac{k}{v} gwdm\right| \\
&= \sup_{f,g} \left|\int Q^v(\phi f)\frac{k}{v} gwdm\right|.
\end{aligned}$$

By Theorem 2.4, $\mathrm{ran}P^v = \frac{v}{k}H^p(w)$. Hence, $P^v(\phi f)\frac{k}{v} \in H^p(w)$. By Lemma 2.5, if $g \in K_0^q(w)$, then $g \in \frac{h^p}{w}H_0^q(w)$. Hence, $gw \in h^p H_0^q(w)$. Hence, $P^v(\phi f)\frac{k}{v}gw \in h^p H_0^1(w) = H_0^1$. Hence,

$$\|H_\phi^v\|_{B(L^p(w))} \ge \sup_{f,g} \left|\int \phi\frac{k}{v} fgwdm\right|.$$

Let $F = \frac{k}{v}f$. Since $\mathrm{ran}P^v = \frac{v}{k}H^p(w)$, it follows that $f \in \mathrm{ran}P^v$ if and only if $F \in H^p(w)$, and $\|f\|_{p,w} = \|F\|_{p,w}$. Hence,

$$\|H_\phi^v\|_{B(L^p(w))} \ge \sup_{F\in H^p(w), \|F\|_{p,w}\le 1, g\in K_0^q(w), \|g\|_{q,w}\le 1} \left|\int \phi F gdm\right|.$$

By Lemma 6.1 and the theorem of Nehari (cf. [1], [11], [12]),

$$\|H_\phi^v\|_{B(L^p(w))} \ge \sup_{k\in H_0^1, \|k\|_1\le 1} \left|\int \phi kdm\right| = \mathrm{dist}(\phi, H^\infty).$$

Next we shall prove the second inequality. If $f \in \mathrm{ran}P^v$ and $G \in H^\infty$, then $Gf \in \mathrm{ran}P^v = \frac{v}{k}H^p(w)$. Hence, $v^{-1}Gf \in k^{-1}H^p(w) = H^p(|v|pw)$. Since $|v|^p w \in (A_p)$, it follows that $P^v(Gf) = vP(v^{-1}Gf) = vv^{-1}Gf = Gf$. Hence, $Q^v(Gf) = (I - P^v)(Gf) = 0$. Hence,

$$\begin{aligned}
\|H_\phi^v\|_{B(L^p(w))} &= \sup_{f\in \mathrm{ran}P^v, \|f\|_{p,w}\le 1} \|H_\phi^v f\|_{p,w} \\
&= \sup_{f\in \mathrm{ran}P^v, \|f\|_{p,w}\le 1} \|Q^v(\phi f)\|_{p,w} \\
&= \sup_{f\in \mathrm{ran}P^v, \|f\|_{p,w}\le 1} \|Q^v((\phi - G)f)\|_{p,w} \\
&\le \|Q^v\|_{B(L^p(w))}\|\phi - G\|_\infty.
\end{aligned}$$

Hence,

$$\|H_\phi^v\|_{B(L^p(w))} \le \|Q^v\|_{B(L^p(w))} \inf_{G\in H^\infty} \|\phi - G\|_\infty = \|Q^v\|_{B(L^p(w))}\mathrm{dist}(\phi, H^\infty).$$

Hence,
$$\mathrm{dist}(\phi, H^\infty) \le \|H_\phi^v\|_{B(L^p(w))} \le \|Q^v\|_{B(L^p(w))}\mathrm{dist}(\phi, H^\infty).$$
If $v = w = 1$, then the equalities hold, and hence we have the Nehari theorem: $\|H_\phi\|_{B(L^2)} = \|H_\phi^1\|_{B(L^2)} = \mathrm{dist}(\phi, H^\infty)$. Theorem 6.2 is proved. $\square$

**Corollary 6.3** *Let* $1 < p < \infty$. *Suppose* $\phi \in L^\infty$, *and* $w$ *is a positive function such that* $w$, $\log w \in L^1$.
(1) *If* $w = |v|^{-p}$ *for some function* $v$, *then* $H_\phi^v$ *is a bounded operator from* $\mathrm{ran}P^v$ *to* $\ker P^v$ *satisfying*
$$\|H_\phi\|_{B(L^2)} \le \|H_\phi^v\|_{B(L^p(w))} \le \frac{1}{\sin(\pi/p)}\|H_\phi\|_{B(L^2)}.$$
(2) *If* $w = |\alpha|^{-2}$ *for some outer function* $\alpha$, *then* $H_\phi^\alpha$ *is a bounded operator from* $H^2(w)$ *to* $\overline{K_0^2(w)}$ *satisfying*
$$\|H_\phi^\alpha\|_{B(L^2(w))} = \|H_\phi\|_{B(L^2)}.$$

**Proof.** It is sufficient to prove (1). By Lemma 2.2, if $|v|^p w$ is a constant, then $\|P^v\|_{B(L^p(w))} = \|P\|_{B(L^p)}$. By the similar proof, it follows that $\|Q^v\|_{B(L^p(w))} = \|Q\|_{B(L^p)}$. It is known that $\|P\|_{B(L^p)} = \|Q\|_{B(L^p)}$ (cf. [5, Vol.I, p.79]). By the theorem of Gohberg, Krupnik, Hollenbeck and Verbitsky (cf. [5], [6]), $\|P\|_{B(L^p)} = \frac{1}{\sin(\pi/p)}$. By Theorem 6.2, Corollary 6.3 is proved. $\square$

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