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# Blow-up directions at space infinity for solutions of semilinear heat equations

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## Abstract

A blowing up solution of the semilinear heat equation  $u_t = \Delta u + f(u)$  with  $f$  satisfying  $\liminf f(u)/u^p > 0$  for some  $p > 1$  is considered when initial data  $u_0$  satisfies  $u_0 \leq M$ ,  $u_0 \not\equiv M$  and  $\lim_{m \rightarrow \infty} \inf_{x \in B_m} u_0(x) = M$  with sequence of ball  $\{B_m\}$  whose radius diverging to infinity. It is shown that the solution blows up only at space infinity. A notion of blow-up direction is introduced. A characterization for blow-up direction is also established.

## 1 Introduction and main theorems

We are interested in solutions of semilinear heat equations which blow up at space infinity.

In [8] we considered a nonnegative blowing up solution of

$$u_t = \Delta u + u^p \quad x \in \mathbf{R}^n, t > 0$$

with initial data  $u_0$  satisfying

$$0 \leq u_0(x) \leq M, \quad u_0 \not\equiv M \text{ and } \lim_{|x| \rightarrow \infty} u_0(x) = M,$$

where  $p > 1$  and  $M > 0$  is a constant. We proved in [8] that the solution  $u$  blows up exactly at the blow-up time for the spatially constant solution with initial data  $M$ . We moreover proved that  $u$  blows up only at the space infinity. In this paper we would like to generalize this result in following two directions.

(i) (Initial data) We consider more general initial data  $u_0$  which may not converge to  $M$  for some direction of  $x$ , for example  $u_0 \rightarrow M$  as  $|x| \rightarrow \infty$  only for  $x$  in some sector. It is convenient to introduce a notion of blow up direction at the space infinity, We are able to give necessary and sufficient conditions so that particular direction is a blow-up direction.

(ii) (Nonlinear term) We extend a class of nonlinear term. It includes  $e^u$  and  $u^p + u^q$  for  $p, q > 1$ .

We consider solutions of the initial value problem for the equation

$$\begin{cases} u_t = \Delta u + f(u), & x \in \mathbf{R}^n, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbf{R}^n. \end{cases} \quad (1)$$

The nonlinear term  $f$  is assumed to be locally Lipschitz in  $\mathbf{R}$  with the property that

$$\liminf_{s \rightarrow \infty} \frac{f(s)}{s^p} > 0 \quad \text{for some } p > 1, \quad f' \geq 0. \quad (2)$$

We take two constants  $M$  and  $N$  satisfying  $M + N > 0$  and

$$f(M) > 0. \quad (3)$$

The initial data  $u_0$  is assumed to be a measurable function in  $\mathbf{R}^n$  satisfying

$$-N \leq u_0 \leq M \text{ a.e.} \quad \text{and} \quad u_0 \neq M \text{ a.e.} \quad (4)$$

We are interested in initial data such that  $u_0 \rightarrow M$  as  $|x| \rightarrow \infty$  for  $x$  in some sector of  $\mathbf{R}^n$ . We assume that

$$\operatorname{ess\,inf}_{x \in B_m} (u_0(x) - M_m) \geq 0 \quad \text{for } m = 1, 2, \dots, \quad (5)$$

where

$$B_m = B_{r_m}(x_m) \quad (6)$$

with a sequence  $\{r_m\}$  and a sequence of constants  $M_m$  satisfying

$$\lim_{m \rightarrow \infty} r_m = \infty, \quad \lim_{m \rightarrow \infty} |M - M_m| = 0,$$

and  $\{x_m\}_{m=1}^{\infty}$  is some sequence of vectors. Here  $B_r(x)$  denotes the closed ball of radius  $r$  centered at  $x$ . (In fact, it follows from (4) that  $|x_m| \rightarrow \infty$  as  $m \rightarrow \infty$ .)

Problem (1) has a unique bounded solution at least locally in time. However, the solution may blow up in finite time. For a given initial value  $u_0$

and nonlinear term  $f$  let  $T^* = T^*(u_0, f)$  be the maximal existence time of the solution. If  $T^* = \infty$ , the solution exists globally in time. If  $T^* < \infty$ , we say that the solution blows up in finite time. It is well known that

$$\limsup_{t \rightarrow T^*} \|u(\cdot, t)\|_\infty = \infty, \quad (7)$$

where  $\|\cdot\|_\infty$  denotes the  $L^\infty$ -norm in space variables.

In this paper, we are interested in behavior of a blowing up solution near space infinity as well as location of blow-up directions defined below. A point  $x_{BU} \in \mathbf{R}^n$  is called a *blow-up point* (with value  $\pm\infty$ ) if there exists a sequence  $\{(x_m, t_m)\}_{m=1}^\infty$  such that

$$t_m \uparrow T^*, \quad x_m \rightarrow x_{BU} \quad \text{and} \quad u(x_m, t_m) \rightarrow \pm\infty \quad \text{as} \quad m \rightarrow \infty.$$

If there exists a sequence  $\{(x_m, t_m)\}_{m=1}^\infty$  such that

$$t_m \uparrow T^*, \quad |x_m| \rightarrow \infty \quad \text{and} \quad u(x_m, t_m) \rightarrow \pm\infty \quad \text{as} \quad m \rightarrow \infty,$$

then we say that the solution blows up to  $\pm\infty$  at space infinity.

A direction  $\psi \in S^{n-1}$  is called a *blow-up direction for the value  $\pm\infty$*  if there exists a sequence  $\{(x_m, t_m)\}_{m=1}^\infty$  with  $x_m \in \mathbf{R}^n$  and  $t_m \in (0, T^*)$  such that  $u(x_m, t_m) \rightarrow \pm\infty$  (as  $m \rightarrow \infty$ ) and

$$\frac{x_m}{|x_m|} \rightarrow \psi \quad \text{as} \quad m \rightarrow \infty. \quad (8)$$

We consider the solution  $v(t)$  of an ordinary differential equation

$$\begin{cases} v_t = f(v), & t > 0, \\ v(0) = M. \end{cases} \quad (9)$$

Let  $T_v = T^*(M, f)$  be the maximal existence time of solutions of (9), i. e.,

$$T_v = \int_M^\infty \frac{ds}{f(s)}.$$

We are now in position to state our main results.

**Theorem 1.** *Assume that  $f$  is locally Lipschitz in  $\mathbf{R}$  and satisfies (2) and (3). Let  $u_0$  be a continuous function satisfying (4) and (5), and  $T_v \leq T^*(-N, f)$ . Then there exists a subsequence of  $\{x_m\}_{m=1}^\infty$  (still denote by  $\{x_m\}$ , independent of  $t$ ) such that*

$$\lim_{m \rightarrow \infty} u(x_m, t) = v(t).$$

*The convergence is uniform in every compact subset of  $\{t : 0 \leq t < T_v\}$ . Moreover, the solution blows up at  $T_v$ .*

**Remark.** Our assumption  $T_v \leq T^*(-N, f)$  says that the solution does not blow up to minus infinity before it blows up to plus infinity. From the condition (4), it follows that  $\lim_{m \rightarrow \infty} |x_m| = \infty$ .

This result in particular implies that

$$\sup_{0 < t < T^*} v^{-1}(t) \|u(\cdot, t)\|_\infty < \infty. \quad (10)$$

When we set  $f(u) = |u|^{p-1}u$ , such a blow-up rate estimate is known for subcritical  $p$ ; see e.g. [4], [6], [7] for general bounded initial data without assuming (4) and (5). Such a blow-up estimate is very fundamental to analyze the behavior of solution near blow-up point as noted in [3]. However, for supercritical  $p$  such a blow-up rate estimate (10) may not hold in general; see e.g. [1], [9]. If one considers only radial solutions of (1) for supercritical  $p$  less than  $1 + 4/(n - 4 - 2(n - 1)^{1/2})$  or  $n \leq 10$ , then the estimate (10) holds [11]. We would like to emphasize that Theorem 1 requires no restriction on  $p$ .

Our second main result is on the location of blow-up points.

**Theorem 2.** *Assume the same hypotheses of Theorem 1. Then the solution of (1) has no blow-up points with  $+\infty$  in  $\mathbf{R}^n$ . (It blows up only at space infinity.)*

There is a huge literature on location of blow-up points since the work of Weissler [13] and Friedman-McLeod [2]. (We do not intend to list references exhaustively in this paper.) However, most results consider either bounded domains or solutions decaying at space infinity; such a solution does not blow up at space infinity [5].

As far as the authors know, before the result of [8] the only paper discussing blow-up at space infinity is the work of Lacey [10]. He considered the Dirichlet problem in a half line. He studied various nonlinear terms and proved that a solution blows up only at space infinity.

In particular, his result implies that the solution of

$$\begin{cases} u_t = u_{xx} + f(u), & x > 0, t > 0, \\ u(0, t) = 1, & t > 0, \\ u(x, 0) = u_0(x) \geq 1, & 4x > 0 \end{cases}$$

blows up only at space infinity, where  $u_0$  satisfies  $0 \leq u_0 \leq M$  with  $M > 1$ , and  $f(s) = s^p$  and  $e^s$ .

His method is based on construction of suitable subsolutions and supersolutions. However, the construction heavily depends on the Dirichlet condition at  $x = 0$  and does not apply to the Cauchy problem even for the case  $n = 1$ .

As previously described, the authors [8] proved the statement of Theorems 1 and 2 assuming that  $\lim_{|x| \rightarrow \infty} u_0(x) = M$  for positive solutions of  $u_t = \Delta u + u^p$ . Later, Simozyo[12] had the same results as in [8] by relaxing the assumptions of initial data  $u_0 \geq 0$  which is similar to that in the present paper. His approach is a construction of a suitable supersolution which implies that  $a \in \mathbf{R}^n$  is not a blow-up point. Although he restricted himself for  $f(s) = s^p$ , his idea works our  $f$  under slightly strong assumption on  $u_0$ . Here we give a different approach.

By Simozyo's results[12] it is natural to consider a problem of "blow-up direction" defined in (8). We next study this "blow-up direction" for the value  $+\infty$ . Our third result is on this blow-up direction. It is convenient to introduce the function  $A_m$  defined by

$$A_m(s) = \frac{1}{|B_s(y_m)|} \int_{B_s(y_m)} u_0(z) dz \quad (11)$$

for a given sequence  $\{y_m\}_{m=1}^{\infty}$ . This  $A_m(s)$  represents the mean value of  $u_0$  over the ball  $B_s(y_m)$ .

**Theorem 3.** *Assume the same hypotheses of Theorem 1 and let  $\{s_m\}_{m=1}^{\infty}$  be a sequence diverging to  $\infty$  in  $\mathbf{R}$ . For a given direction  $\psi \in S^{n-1}$ , the following alternatives hold.*

(i) *If there exists a sequence  $\{y_m\}_{m=1}^{\infty}$  satisfying  $\lim_{m \rightarrow \infty} y_m/|y_m| = \psi$  such that*

$$\limsup_{m \rightarrow \infty} \inf_{s \in (1, s_m)} A_m(s) = M,$$

*then  $\psi$  is a blow-up direction.*

(ii) *If there exists a constant  $s_c \in (1/(M + N), \infty)$  such that for any sequence  $\{y_m\}_{m=1}^{\infty}$  satisfying  $\lim_{m \rightarrow \infty} y_m/|y_m| = \psi$  such that*

$$\limsup_{m \rightarrow \infty} \inf_{s \in (1, s_c)} A_m(s) \leq M - \frac{1}{s_c},$$

*then  $\psi$  is not a blow-up direction.*

This characterizes blow up directions by profiles of initial data. This is a new result even if  $f(u) = |u|^{p-1}u$  or  $n = 1$ .

Here are main ideas of the proofs . To prove Theorem 1 we construct a suitable subsolution. To prove Theorem 2 we derive a non blow-up criterion. We do not appeal any energy arguments for rescaled function as is done in

our previous paper [8]. Our argument consists of two parts. First we observe that

$$u(x, t) \leq \delta v(t)$$

near a point  $a \in \mathbf{R}^n$  with some  $\delta \in (0, 1)$  when  $t$  is close to blow-up time. By a bootstrap argument we derive that  $u$  is actually bounded near  $a$  when  $t$  is close to the blow up time. To prove Theorem 3 we use comparison argument as in Theorems 1,2 and non blow-up criterion which is established in the proof of Theorem 2. We also note that there is no situation which is not covered by assumption of (i) and (ii) of Theorem 3.

This paper is organized as follows. In section 2 we prove Theorem 1 by using the Green kernel of the heat equation. The proof of Theorem 2 is given in section 3 by a priori estimate. In section 4 we show Theorem 3 using Theorems 1 and 2.

## 2 Behavior at space infinity

In this section, we prove Theorem 1. We may assume  $r_m \leq r_{m+1}$  and  $M_m \leq M_{m+1}$  for  $m \in \mathbf{N}$  without loss of generality.

*Proof of Theorem 1.* Let  $G_{B_R(z)}(x, y, t)$  be the Green kernel of the Dirichlet problem of the heat equation in the domain  $B_R(z)$  and  $G(x, y, t)$  be that of  $\mathbf{R}^n$ . We set

$$G_R(x, y, t) = \begin{cases} G_{B_R(z)}(x, y, t), & x \in B_R(z), t > 0, \\ 0, & x \in \mathbf{R}^n \setminus B_R(z). \end{cases}$$

It is easily seen that

$$\lim_{R \rightarrow \infty} \int_{\mathbf{R}^n} G_R(x, y, t) \psi(y) dy = \int_{\mathbf{R}^n} G(x, y, t) \psi(y) dy$$

uniformly in  $B_r(z) \times [0, a]$  for any measurable function  $\psi(y)$  with any  $r$  and  $a$  satisfying  $0 < r < \infty$  and  $0 < a < \infty$ .

For the  $m$ -th ball  $B_m$  defined in (6), let  $\underline{u}_m$  be the subsolution of (1) satisfying

$$\begin{cases} (\underline{u}_m)_t = \Delta \underline{u}_m + f(\underline{u}_m), & x \in B_m, t > 0, \\ \underline{u}_m(x, 0) \equiv M_m, & x \in B_m, \\ \underline{u}_m = w & x \in \mathbf{R}^n \setminus B_m, t > 0, \end{cases}$$

where  $w$  is the solution of

$$\begin{cases} w_t = f(w), & t > 0, \\ w(0) = -N. \end{cases}$$

Our goal is to prove  $\lim_{m \rightarrow \infty} \underline{u}_m(x_m, t) = v(t)$ . We set  $X_m = \underline{u}_m - w$  and observe that  $X_m$  satisfies

$$\begin{cases} (X_m)_t = \Delta X_m + f(X_m + w) - f(w), & x \in B_m, t > 0, \\ X_m(x, 0) \equiv M_m - N, & x \in B_m, \\ X_m = 0 & x \in \mathbf{R}^n \setminus B_m, t > 0. \end{cases}$$

It is easily seen that  $X_m \leq X_{m+1}$  for any  $m \in \mathbf{N}$ . It is well known that  $X_m$  satisfies the integral equation

$$\begin{aligned} X_m(x, t) &= \int_{\mathbf{R}^n} G_m(x, y, t)(M_m + N)dy \\ &\quad + \int_0^t \int_{\mathbf{R}^n} G_m(x, y, t-s) \{f(X_m(y, s) + w(s)) - f(w(s))\} ds. \end{aligned}$$

when  $G_m(x, y, t)$  be the Green kernel of the Dirichlet problem of the heat equation in the domain  $B_{R_m}$ .

We shall prove that  $\lim_{m \rightarrow \infty} X_m(x_m, t) = v(t) - w(t)$ . Since  $v$  satisfies (9), we have

$$\begin{aligned} v(t) - w(t) - X_m(x_m, t) &= M + N - \int_{\mathbf{R}^n} G_m(x_m, y, t)(M_m + N)dx \\ &\quad - \int_0^t \left[ f(v(s)) - f(w(s)) - \int_{\mathbf{R}^n} \{f(X_m(y, s) + w(s)) - f(w(s))\} dy \right] ds. \end{aligned}$$

Since  $\int_{\mathbf{R}^n} G(x, y, t)dy = 1$ , we have

$$\begin{aligned} &M + N - \int_{\mathbf{R}^n} G_m(x_m, y, t)(M_m + N)dy \\ &= \int_{\mathbf{R}^n} \{G(x_m, y, t) - G_m(x_m, y, t)\}(M + N) + G_m(x_m, y, t)(M - M_m)dy \end{aligned}$$

and

$$\begin{aligned} &f(v(s)) - f(w(s)) - \int_{\mathbf{R}^n} \{f(X_m(y, s) + w(s)) - f(w(s))\} dy \\ &= \int_{\mathbf{R}^n} \left[ \{(G(x_m, y, t-s) - G_m(x_m, y, t-s))(f(v(s)) - f(w(s)))\} \right. \\ &\quad \left. + G_m(x_m, y, t-s) \{f(x(s)) - f(X_m(y, s) + w(s))\} \right] dy. \end{aligned}$$

By the monotone convergence theorem we have

$$\lim_{m \rightarrow \infty} \left\{ M + N - \int_{\mathbf{R}^n} G_m(x_m, y, t)(M_m + N)dy \right\} = 0$$



and

$$\begin{aligned} & \lim_{m \rightarrow \infty} \left\{ \int_0^t \int_{\mathbf{R}^n} G_m(x, y, t-s) \{f(X_m(y, s) + w(s)) - f(w(s))\} ds \right\} \\ &= \lim_{m \rightarrow \infty} \int_0^t \int_{\mathbf{R}^n} G_m(x_m, y, t-s) \{f(v(s)) - f(X_m(y, s) + w(s))\} dy ds. \end{aligned}$$

Thus we have

$$\begin{aligned} & \lim_{m \rightarrow \infty} \{v(t) - w(t) - X_m(x_m, t)\} \\ &= \lim_{m \rightarrow \infty} \left[ \int_0^t \int_{\mathbf{R}^n} G_m(x_m, y, t-s) \{f(v(s)) - f(X_m(y, s) + w(s))\} dy ds \right] \\ &\leq \lim_{m \rightarrow \infty} \left[ \int_0^t C \{v(s) - w(s) - X_m(x_m, s)\} ds \right] \end{aligned}$$

for  $t \in [0, T_0]$  with  $T_0 \in (0, T^*)$  and some constant  $C = C(T_0)$ . We set  $X(t) = \lim_{m \rightarrow \infty} X_m(x_m, t)$ . The monotone convergence theorem yields

$$v(t) - w(t) - X(t) \leq \int_0^t C(v(s) - w(s) - X(s)) ds.$$

Since  $v(t) \geq w(t) + X(t)$ , we have

$$v(t) - w(t) - X(t) \equiv 0 \quad \text{for } t \in [0, T_0].$$

We thus obtain that

$$\lim_{m \rightarrow \infty} \underline{u}_m(x_m, t) \equiv v(t) \quad \text{for } t \in [0, T_0].$$

Since  $\underline{u}_m(x, t) \leq u(x, t) \leq v(t)$ , we conclude

$$\lim_{m \rightarrow \infty} u_m(x_m, t) \equiv v(t) \quad \text{for } t \in [0, T_0].$$

Thus, since  $T_v \leq T^*(-N, f)$  and  $T_v \leq T^*(u_0, f)$ , we take

$$\lim_{m \rightarrow \infty} u(x_m, t) = v(t) \quad \text{for } t \in [0, T_v]. \quad (12)$$

It remains to prove that  $u$  blows up at  $t = T_v$ . For this purpose it suffices to prove that  $\lim_{m \rightarrow \infty} u(x_m, t_m) = \infty$  for some sequence  $t_m \rightarrow T_v$ . We argue by contradiction. Suppose that  $\lim_{m \rightarrow \infty} u(x_m, t_m) \leq C$  for some  $C \in [M, \infty)$ . Then we could take  $t_0 \in (0, T_v)$  satisfying  $v(t_0) \geq C$  and  $v_t(t) > 0$  for  $t \geq t_0$ . By (12) we have

$$\lim_{m \rightarrow \infty} u \left( x_m, \frac{t_0 + T_v}{2} \right) = v \left( \frac{t_0 + T_v}{2} \right) > C,$$

which yields a contradiction. We thus proved that  $\lim_{m \rightarrow \infty} u(x_m, t_m) = \infty$ , so that  $u(x, t)$  blows up at  $T_v$ .  $\square$

### 3 Non blow-up point in $\mathbf{R}^n$

In this section we prove Theorem 2. We may assume that  $f(u_0(x)) \geq 0$  for any  $x \in \mathbf{R}^n$  without loss of generality.

**Lemma 3.1.** *Let  $u$  and  $v$  be solutions of (1) and (9) with  $u_0$ ,  $M$  and  $f$  satisfying (2), (3) and (4). Then there exist  $\delta = \delta(a, r, t_0, u_0, f) \in (0, 1)$  such that for  $(x, t) \in B_r(a) \times [t_0, T^*)$ ,*

$$u(x, t) \leq \delta v(t)$$

with  $t_0 \in [0, T^*)$ .

*Proof.* By (2) there exist  $M_f = M_f(f) > M$  and  $\delta_f = \delta_f(f) \in (0, 1)$  satisfying for  $r \geq M_f$  and  $\delta \in (\delta_f, 1)$ ,

$$f(\delta r) \leq \delta f(r). \quad (13)$$

Let  $T_0 = T_0(u_0, f) \in (0, T^*)$  such that  $v(T_0) = M_f$ . Since  $u_0 \leq M$  and  $u_0 \not\equiv M$ , we have  $u(x, T_0) < v(T_0)$ . Note that  $u(x, t) < v(t)$  for  $t \in (0, T_0]$ . Let  $w$  be the solution of

$$\begin{cases} w_t = \Delta w, & x \in \mathbf{R}^n, t \in (T_0, T^*), \\ w(x, T_0) = \max\{u(x, T_0)/v(T_0), \delta_f\}, & x \in \mathbf{R}^n. \end{cases}$$

Put  $\bar{u} = vw$ . Then we have

$$\begin{cases} \bar{u}_t = \Delta \bar{u} + wf(v), & x \in \mathbf{R}^n, t \in (T_0, T^*), \\ \bar{u}(x, T_0) = \max\{u(x, T_0), \delta_f v(T_0)\}, & x \in \mathbf{R}^n. \end{cases}$$

Since  $w(x, t) \in [\delta_f, 1)$  and  $v(t) \geq M_f$ , we have

$$wf(v) \geq f(wv) = f(\bar{u})$$

by (13). This  $\bar{u}$  is supersolution of (1).

Since for any  $x \in \mathbf{R}^n$ ,  $\sup_{t \in [T_0, T^*)} w(x, t) < 1$ , we can take  $\delta = \delta(a, r, T_0, u_0, f) \in (0, 1)$  satisfying  $w(x, t) \leq \delta$  for  $(x, t) \in B_r(a) \times [T_0, T^*)$ . Thus, we obtain

$$u(x, t) \leq \bar{u}(x, t) = w(x, t)v(t) \leq \delta v(t)$$

and Lemma 3.1 is proved. □

We consider the equation

$$\begin{cases} \hat{u}_t = \Delta \hat{u}, & x \in B_1, t > 0, \\ \hat{u}(x, 0) = u_0(x), & x \in B_1, \\ \hat{u}(x, t) \leq v(t), & x \in \partial B_1, \end{cases} \quad (14)$$

where  $B_1 = B_1(a)$  with some  $a \in \mathbf{R}^n$ , and  $v$  is the solution of (9) and  $T$  is maximum existence time for  $v$ .

**Lemma 3.2.** *Let  $\hat{u}$  be the solutions of (14) with a bounded continuous function  $u_0$  satisfying  $u_0 \leq v(0)$  and  $B_1 = B_1(a)$  with some  $a \in \mathbf{R}^n$ . And let  $v = v(t) \in C^1([0, T])$  blow up at  $t = T$ . Then, for any  $\epsilon > 0$  and  $\zeta \in (0, 1)$ , there exist  $r \in (0, \sqrt{T^*})$  depending only on the space dimension,  $\epsilon$  and  $\zeta$  such that*

$$\sup_{(x,t) \in B_\zeta \times [T-r^2, T]} \hat{u}(x, t)/v(t) < \epsilon, \quad (15)$$

where  $B_\zeta = B_\zeta(a)$ .

*Proof.* We consider the function

$$w = v - \hat{u}.$$

Then, the function  $w$  satisfies

$$\begin{cases} w_t \geq \Delta w + v_t, & x \in B_1, t > 0, \\ w(x, 0) = v(0) - u_0(x), & x \in B_1, \\ w(x, t) = 0, & x \in \partial B_1, \end{cases} \quad (16)$$

or its integral form:

$$\begin{aligned} w(x, t) &\geq \int_{B_1} G_1(x, y, t)(v(0) - u_0(y))dy \\ &\quad + \int_0^t \int_{B_1} G_1(x, y, t-s)v_t(s)dyds, \end{aligned}$$

where  $G_1(x, y, t)$  is the Green kernel of the Dirichlet problem of the heat equation in the domain  $B_1(0)$ .

We take  $\epsilon_1 > 0$  and  $\epsilon_2 > 0$  small enough such that

$$(1 - \epsilon_1)(1 - \epsilon_2) \geq (1 - \epsilon). \quad (17)$$

For these  $\epsilon_1$  and  $\epsilon_2$ , and for any  $\zeta \in (0, 1)$ , we are able to take  $\delta_1 = \delta(\zeta, \epsilon_2) > 0$  and  $\delta_2 = \delta_2(\delta_1, \epsilon_1) \in (0, \delta_1)$  such that

$$\inf_{\substack{x \in B_\zeta \\ s \in [0, \delta_1]}} \int_{B_1} G_1(x, y, s)dy \geq 1 - \epsilon_2 \quad (18)$$

and

$$v(T - \delta_3) - v(T - \delta_1) \geq (1 - \epsilon_1)v(T - \delta_3). \quad (19)$$

for any  $\delta_3 \in (0, \delta_2]$ . Then,

$$\begin{aligned} w(x, T - \delta_3) &\geq \int_0^{T-\delta_3} \int_{B_1} G_1(x, y, T - \delta_3 - s) v_t(s) dy ds \\ &= \int_0^{T-\delta_3} v_t(s) \int_{B_1} G_1(x, y, T - \delta_3 - s) dy ds \\ &\geq \int_{T-\delta_1}^{T-\delta_3} v_t(s) \int_{B_1} G_1(x, y, T - \delta_3 - s) dy ds. \end{aligned}$$

Since  $\inf_{x \in D} \int_a^b h(x, s) ds \geq \int_a^b \inf_{x \in D} h(x, s) ds$  for any constants  $a, b$  satisfying  $a \leq b$ , any integrable function  $h$  in  $[a, b]$  and any domain  $D$ , we have

$$\inf_{x \in B_\zeta} w(x, T - \delta_2) \geq \int_{T-\delta_1}^{T-\delta_2} v_t(s) \inf_{x \in B_\zeta} \int_{B_1} G_1(x, y, T - \delta_2 - s) dy ds.$$

From (18) we obtain

$$\begin{aligned} \inf_{x \in B_\zeta} w(x, T - \delta_2) &\geq (1 - \epsilon_2) \int_{T-\delta_1}^{T-\delta_2} v_t(s) ds \\ &\geq (1 - \epsilon_2) (v(T - \delta_2) - v(T - \delta_1)). \end{aligned}$$

From (19) and (17) we get

$$\begin{aligned} \inf_{x \in B_\zeta} w(x, T - \delta_2) &\geq (1 - \epsilon_2)(1 - \epsilon_1)v(T - \delta_2) \\ &\geq (1 - \epsilon)v(T - \delta_2). \end{aligned}$$

Then we have

$$\sup_{x \in B_\zeta} \hat{u}(x, T - \delta_3) \leq \epsilon v(T - \delta_3)$$

for any  $\delta_3 \in (0, \delta_2]$ .

We take  $r = \delta_3^{1/2}$ , and observe that

$$\sup_{(x,t) \in B_\zeta \times [T-r^2, T]} \hat{u}(x, t)/v(t) < \epsilon.$$

□

**Proposition 3.3.** For  $p > 1$ ,  $\delta \in (0, 1)$  and  $\epsilon_m > 0$  ( $m = 1, 2, \dots, l$ ) with  $l > 0$ , let  $\{a_m\}_{m=1}^\infty$  be a sequence defined by

$$a_1 = \delta^p + \epsilon_1, \quad a_m = a_{m-1}^p + \epsilon_{m-1}, \quad (m = 2, 3, \dots, l),$$

Then for any  $\epsilon > 0$ ,  $p > 1$  and  $\delta \in (0, 1)$ , there exist  $l$  and  $\epsilon_m > 0$  ( $m = 1, 2, \dots, l$ ) satisfying  $a_l < \epsilon$ .

*Proof.* First, we take  $\theta_1 = \delta^{(p-1)/2}$  and  $\epsilon_1 = \theta_1\delta - \delta^p > 0$  to get

$$a_1 = \theta_1\delta \text{ and } \theta_1 \in (\delta^{p-1}, 1).$$

Next, we take  $\theta_2 = (\theta_1\delta)^{(p-1)/2}$  and  $\epsilon_2 = \theta_1\theta_2\delta - (\theta_1\delta)^{(p-1)/2}$ . Then  $\theta_2$  and  $\epsilon_2$  satisfy

$$a_2 = a_1^p + \epsilon_2 = \theta_1\theta_2\delta \quad \text{and} \quad \theta_2 \in ((\theta_1\delta)^{p-1}, 1).$$

By repeating these arguments, we have

$$a_l = (\theta_1\theta_2 \dots \theta_{l-1}\delta)^p + \epsilon_l = \theta_1\theta_2 \dots \theta_l\delta$$

and  $\theta_n \in ((\theta_1\theta_2 \dots \theta_{n-1})^{p-1}, 1)$ .

Then it is shown that  $\epsilon_m$  ( $m = 1, 2, \dots, n$ ) satisfy

$$a_l = \theta_1\theta_2 \dots \theta_l\delta = \delta^{\frac{p-1}{2}l+1} \theta_1^{\frac{p-1}{2}(l-1)+1} \theta_2^{\frac{p-1}{2}(l-2)+1} \dots \theta_{l-1}^{\frac{p-1}{2}+1} < \delta^{\frac{p-1}{2}l+1}$$

Take  $l$  large enough satisfying  $\delta^{\frac{p-1}{2}l+1} < \epsilon$ . Then we have

$$a_l < \epsilon.$$

□

**Lemma 3.4.** *Assume that  $f$  satisfies (2) and (3). Let  $u$  and  $v$  be the solutions of*

$$\begin{cases} u_t = \Delta u + f(u), & x \in B_1, t > 0, \\ u(x, 0) = u_0(x), & x \in B_1 \end{cases}$$

and (9). Assume that there exists a  $\delta \in (0, 1)$  satisfying  $u \leq \delta v$  in  $B_1 \times [t_0, T)$  with some  $t_0 \in (0, T)$ . Then, for any  $\epsilon > 0$  and  $\zeta \in (0, 1)$ , there exists  $r$  depending only on the space dimension such that

$$\sup_{(x,t) \in B_\zeta \times [T-r^2, T)} u(x, t)/v(t) < \epsilon.$$

*Proof.* By (2), one can take some  $q \in (1, p)$ ,  $\delta_0 = \delta_0(f)$  and  $b_0 = b_0(f, \delta_0)$  such that for any  $\delta_1 \in (\delta_0, 1)$ ,  $v \geq b_0$  and  $u \leq \delta_1 v$ ,

$$f(u) \leq \delta^q f(v). \tag{20}$$

And from Lemma 3.1, for any  $a \in \mathbf{R}^n$ , we can take  $\delta = \delta(u_0, f, a, \tau) \geq \delta_0 = \delta_0(f)$  such that  $u(x, t) \leq \delta v(t)$  for  $t > \tau \in (0, T)$  and  $x \in B_1(a)$ . Put  $\hat{w} = u - \hat{u}$ , where  $\hat{u}$  is solution of (16). Then by (20), we have

$$\hat{w}_t = u_t - \hat{u}_t = \Delta u - \Delta \hat{u} + f(u) = \Delta \hat{w} + f(u).$$

We define  $\tau = \min\{t : v(s) \geq b_0 \text{ for any } s \geq t\}$  and observe that

$$\begin{aligned}
\hat{w}(x, t) &= \int_0^t \int_{B_\zeta} G(x, y, t-s) f(u(y, s)) dy ds \\
&\leq \int_0^\tau \int_{B_\zeta} G(x, y, t-s) f(v(s)) dy ds \\
&\quad + \delta^q \int_\tau^t \int_{B_\zeta} G(x, y, t-s) f(v(s)) dy ds \\
&\leq \int_0^\tau f(v(s)) \int_{B_\zeta} G(x, y, t-s) dy ds \\
&\quad + \delta^q \int_\tau^t f(v(s)) \int_{B_\zeta} G(x, y, t-s) dy ds.
\end{aligned}$$

Since

$$\lim_{t \rightarrow T} v(t) = \lim_{t \rightarrow T} \int_M^t f(v(s)) ds = \infty,$$

there exist  $r_1 = r_1(\epsilon_1) > 0$  small enough such that for  $t \in [T - r_1^2, T)$ ,

$$\begin{aligned}
&\int_0^\tau f(v(s)) \int_{B_\zeta} G(x, y, t-s) dy ds \\
&\quad + \delta^q \int_\tau^t f(v(s)) \int_{B_\zeta} G(x, y, t-s) dy ds \\
&\leq \left( \delta^q + \frac{\epsilon_1}{2} \right) \int_\tau^t f(v(s)) \int_{B_\zeta} G(x, y, t-s) dy ds
\end{aligned}$$

with some  $\epsilon_1$  satisfying  $\delta^q + \epsilon_1 < 1$ . Then, from Lemma 3.2, we have

$$\begin{aligned}
\sup_{x \in B_{\tilde{\zeta}}} u(x, t) &\leq \sup_{x \in B_{\tilde{\zeta}}} \left\{ \left( \delta^q + \frac{\epsilon_1}{2} \right) \int_0^t \int_{B_1} G(x, y, t-s) f(v(s)) dy ds \right\} + \frac{\epsilon_1}{2} v(t) \\
&\leq (\delta^q + \epsilon_1) v(t) = a_1 v(t)
\end{aligned}$$

for  $t \in [T - \tilde{r}_1^2, T)$  with some  $\tilde{\zeta} > 0$  and  $\tilde{r}_1 < r_1$ , where  $a_m$  ( $m = 1, 2, \dots, l$ ) is defined in Proposition 3.3.

Next, by using the solution  $\tilde{u}$  of the equation of

$$\begin{cases} \tilde{u}_t = \Delta \tilde{u}, & x \in B_{\tilde{\zeta}}, t > 0, \\ \tilde{u}(x, 0) = u_0(x), & x \in B_{\tilde{\zeta}}, \\ \tilde{u}(x, t) = (\delta^p + \epsilon_1) v(t), & x \in \partial B_{\tilde{\zeta}}, \end{cases}$$

and  $\tilde{w} = u - \tilde{u}$ , and by using the same argument of proof of Lemma 3.2 and above, we have

$$\sup_{x \in B_{\tilde{\zeta}^2}} u(x, t) \leq a_2 v(t)$$

for  $t \in [T - \tilde{r}_1^2 \tilde{r}_2^2, T)$  with some  $\tilde{r}_2$  and  $\epsilon_2$ .

Iterating these arguments, we have

$$\sup_{x \in B_{\tilde{\zeta}_i}} u(x, t) \leq a_i v(t)$$

for  $t \in [T - \tilde{r}_1^2 \tilde{r}_2^2 \dots \tilde{r}_n^2, T)$  with some  $\tilde{r}_i$  and  $\epsilon_i$  ( $i = 3, 4, \dots, n$ ). Put  $\tilde{\zeta} = \zeta^{1/n}$  and  $r = \tilde{r}_1 \tilde{r}_2 \dots \tilde{r}_n$ . Then by Proposition 3.3, we have

$$\sup_{(x,t) \in B_{\tilde{\zeta}} \times [T-r^2, T)} \frac{u(x, t)}{v(t)} \leq \epsilon.$$

□

**Proposition 3.5.** *Let  $v$  be solution of (9) with  $f$  and  $M$  satisfying (2) and (3),  $f(s) \geq C_1(s + C_2)^q = \tilde{f}(s)$  with some  $C_1 > 0$  and  $C_2 \geq 0$  for large  $s$ , and  $T^*(M, f) = T^*(M, \tilde{f})$ . Then*

$$v(t) \leq C(T - t)^{-1/(q-1)}$$

with some  $q = q(f) > 1$  and  $C = C(C_1, c_2, M, f) > 0$ , where  $T = T^*(M, f)$ .

*Proof.* From the assumption it follows

$$T - t = \int_v^\infty \frac{ds}{f(s)} \leq \int_v^\infty \frac{ds}{C_1(s + C_2)^q}$$

and

$$v(t) \leq \left( \frac{1}{C_1(q-1)} (T - t) \right)^{-1/(q-1)} + C_2$$

for  $t \in (0, T)$  satisfying that  $T - t$  is small enough. This yields the desired estimate. □

**Lemma 3.6.** *Assume that  $u_0$  satisfies (4) and (5), and  $f$  satisfies (2) and (3). Let  $a$  be a point in  $\mathbf{R}^n$ . Then  $a$  is not a blow-up point, and*

$$\limsup_{t \rightarrow T} u(a, t) \leq C$$

with some  $C = C(h, u_0, f) < \infty$  for  $a \in B_h(0)$  (see [5, Lemma 2.1]).

*Proof.* From Lemma 3.4, it follows that  $u(x, t) \leq \epsilon v(t)$  in  $B_\eta(a) \times [T_v - r^2, T)$  with some  $r = r(a, \eta, c, u_0, f)$ . By assumption we have  $|f(s)/s|$  is nondecreasing function for  $|s| \geq s_0$ , and  $|f(\epsilon s)| \leq \epsilon^q |f(s)|$  for  $|s| > s_0/\epsilon$  with  $t_0 = t_0(f, q)$   $q = (p + 1)/2$ . From Lemma 3.4, it also follows that  $|u(x, t)| \leq \epsilon |v(t)|$ . Then we have

$$\left| \frac{f(u)}{u} \right| \leq \left| \frac{f(\epsilon v)}{\epsilon v} \right| \leq \epsilon^{q-1} \left| \frac{f(v)}{v} \right| \quad (21)$$

for  $(x, t) \in B_\eta(a) \times [T - r_0^2, T)$ , where

$$r_0 = \sup \{ \bar{r} \in (0, r] : v(T - \bar{r}^2) \geq s_0/2 \}.$$

with  $r$  defined in Lemma 3.4.

We argue a kind of a local bootstrap argument for  $u$  to get a bound. Let  $\phi_m$  be a  $C^2$ -function supported on  $B_{\eta^m}(a)$  such that  $\phi_m \equiv 1$  on  $B_{\eta^{m+1}}$  and  $0 \leq \phi_m \leq 1$ . (Note that since  $\eta \in (0, 1)$ ,  $\phi_m \geq \phi_{m+1}$  for  $m \in \mathbf{N}$ .) We consider a cutoff of  $u$  defined by  $w_m = \phi_m u$ . Then this  $w_m$  satisfies

$$(w_m)_t = \Delta w_m = \phi_m f(u) + g_m$$

with  $g_m = 2\nabla u \cdot \nabla \phi_m + u \Delta \phi_m$ . We observe that

$$\begin{aligned} w_m(x, t) &= e^{(t-(T-r_0^2))\Delta} \phi_m(x) u(x, T - r_0^2) \\ &\quad + \int_{T-r_0^2}^t e^{(t-s)\Delta} \{ \phi_m(x) f(u(x, s)) + g(x, s) \} ds. \end{aligned}$$

Since  $\|e^{t\Delta} h\|_\infty \leq \|h\|_\infty$  and  $\|e^{t\Delta} \nabla h\|_\infty \leq C t^{-1/2} \|h\|_\infty$  for a measurable function  $h$ , we have

$$\int_{T-r_0^2}^t e^{t\Delta} g_m(x, s) ds \leq C \int_{T-r_0^2}^t (t-s)^{-1/2} \|u\|_{L^\infty(B_{\eta^m})} ds$$

with  $B_{\eta^m} = B_{\eta^m}(a)$ . We estimate the integral involving  $\phi_m f$  to get  $t \in [T - r_0^2, T)$

$$\begin{aligned} &\int_{T-r_0^2}^t \|\phi_m f(u(\cdot, s))\|_{L^\infty(B_{\eta^m})} ds \\ &\leq \int_{T-r_0^2}^t \|w(\cdot, s)\|_{L^\infty(B_{\eta^m})} \left\| \frac{f(u(\cdot, s))}{u(\cdot, s)} \right\|_{L^\infty(B_{\eta^m})} ds \\ &\leq \epsilon^{q-1} \int_{T-r_0^2}^t \|w(\cdot, s)\|_{L^\infty(B_{\eta^m})} \left| \frac{f(v(s))}{v(s)} \right| ds. \end{aligned}$$



From these estimates it follows that for  $t \in [T - r_0^2, T)$  we estimate  $L^\infty$ -norm of  $w_1$ :

$$\begin{aligned} \|w_1(\cdot, t)\|_\infty &\leq \|w_1(\cdot, T - r_0^2)\|_\infty + \epsilon^{q-1} \int_{T-r_0^2}^t \|w_1(\cdot, s)\|_\infty \left| \frac{f(v(s))}{v(s)} \right| ds \\ &\quad + C\epsilon \int_{\tau-r_0^2}^t (t-s)^{-1/2} \|u(\cdot, s)\|_{L^\infty(B_{\eta,m})} ds. \end{aligned}$$

By Gronwall's inequality (see [5, Lemma2.3]) we have

$$\begin{aligned} \|w_1(\cdot, t)\|_\infty &\leq \left[ \|w_1(\cdot, T - r_0^2)\|_\infty + \int_{T-r_0^2}^t C(t-\tau)^{-1/2} \|u(\cdot, s)\|_{L^\infty(B_{\eta,m})} \right. \\ &\quad \left. \times \exp\left(-\int_{T-r_0^2}^\tau \epsilon^{(q-1)} \left| \frac{f(v(\sigma))}{v(\sigma)} \right| d\sigma\right)^{\epsilon^{q-1}} d\tau \right] \exp\left(\epsilon^{q-1} \int_{T-r_0^2}^t \left| \frac{f(v(s))}{v(s)} \right| ds\right). \end{aligned}$$

Since  $v_t = f(v)$  by (9), we have

$$\begin{aligned} \exp\left(\epsilon^{q-1} \int_{T-r_0^2}^t \left| \frac{f(v(s))}{v(s)} \right| ds\right) &\leq \exp\left(\epsilon^{q-1} \int_{T-r_0^2}^t \left| \frac{v_t(s)}{v(s)} \right| ds\right) \\ &\leq \exp(\epsilon^{q-1} (\log |v(t)| - \log |v(T - r_0^2)|)) \\ &\leq \left| \frac{v(t)}{v(T - r_0^2)} \right|^{\epsilon^{q-1}} \leq C v^{\epsilon^{q-1}}(t). \end{aligned}$$

Since  $f(s) \geq Cs^q$  for large  $s$ , by Proposition 3.5 we have

$$\|u(\cdot, s)\|_{L^\infty(B_{\eta,m})} \leq v(t) \leq C(T-t)^{-1/(q-1)}$$

and

$$w_1(x, t) \leq C_1(T-t)^{-(1+\epsilon^{q-1})/(q-1)+1/2}.$$

We thus conclude that

$$u(x, t) \leq C_1(T-t)^{-(1+\epsilon^{q-1})/(q-1)+1/2} \text{ in } B_{\eta^2} \times [T - r_0^2, T).$$

By repeating the argument we have

$$u(x, t) \leq w_2(x, t) \leq C_2(T-t)^{-(1+2\epsilon^{q-1})/(q-1)+1} \text{ in } B_{\eta^3} \times [T - r_0^2, T).$$

By repeating these calculations  $m$  times, we obtain

$$u(x, t) = w_m(x, t) \leq C_m(T-t)^{-(1+m\epsilon^{q-1})/(q-1)+m/2} \text{ in } B_{\eta^{m+1}} \times [T - r_0^2, T),$$

where  $m$  and  $\epsilon$  satisfy  $-(1 + m\epsilon^{q-1})/(q-1) + m/2 \in (0, 1/2 - \epsilon^{q-1}/(q-1)]$  and  $m \in (2/(q-1 - 2\epsilon^{q-1}), 2/(q-1 - 2\epsilon^{q-1}) + 1]$ . We now conclude that

$$|u(x, t)| = |w_{m+1}(x, t)| \leq C \text{ in } B_{\eta^{m+2}} \times [T - r_0^2, T]$$

with some  $C > 0$  by repeating the procedure once more. This implies that  $a$  is not a blow-up point.  $\square$

*Proof of Theorem 2.* Put  $\bar{u}_0$  satisfying (4), (5) and

$$\bar{u}_0(x) \geq u_0(x) \text{ and } f(\bar{u}_0(x)) \geq 0$$

for  $x \in \mathbf{R}^n$ . Then by comparison we may assume that  $\bar{u}_0 = u_0$  without loss of generality. Since  $a \in \mathbf{R}^n$  is arbitrary in Lemma 3.6, there is no blow-up point in  $\mathbf{R}^n$ .  $\square$

From Lemma 3.4, Proposition 3.5 and the proof of Lemma 3.6, we have a sufficient condition for non blow-up point.

**Theorem 3.7.** (*Non blow-up criterion*) *Let  $v$  be a solution of (9) with  $f$  satisfying (2) and (3). If there exists  $\delta \in (0, 1)$  such that  $\check{u}$  satisfies*

$$\begin{cases} \check{u}_t = \Delta \check{u} + f(\check{u}), & (x, t) \in B_1(a) \times (T - \epsilon, T), \\ \check{u}(x, t) \leq \delta v(t), & (x, t) \in B_1(a) \times (T - \epsilon, T). \end{cases}$$

with some  $\epsilon > 0$ , then

$$\check{u} \leq D \quad (x, t) \in B_r(a) \times (T - \epsilon, T)$$

with  $r \in (0, 1)$  and  $D = D(\delta, r, \epsilon) < \infty$ .

## 4 On blow-up direction

We shall prove Theorem 3 which gives a condition for blow-up direction.

*Proof of Theorem 3.* We first prove the case (i). By assumption we obtain that  $u_0(x)$  satisfies (5) with some sequences  $\{r_m\}_{m=1}^\infty$  and  $\{x_m\}_{m=1}^\infty$  satisfying  $\lim_{m \rightarrow \infty} r_m = \infty$  and  $\lim_{m \rightarrow \infty} x_m/|x_m| = \psi$ . Then, from Theorem 1 it follows that

$$\lim_{m \rightarrow \infty} u(x_m, t_m) = \infty$$

with the sequence  $\{t_m\}_{m=1}^\infty$  satisfying  $\lim_{m \rightarrow \infty} t_m = T_v$ . Since  $\lim_{m \rightarrow \infty} x_m/|x_m| = \psi$  by the assumption we obtain that  $\psi$  is a blow-up point.

We next show the case (ii). We take the sequence  $\{x_m\}_{m=1}^\infty$  satisfying  $\lim_{m \rightarrow \infty} x_m/|x_m| = \psi$  and  $\{r_m\}_{m=1}^\infty$  satisfying  $\lim_{m \rightarrow \infty} r_m = \infty$ .

We set

$$\bar{u}_{m,0}(x) = \begin{cases} u_0(x), & x \in B_{s_c}(x_m), \\ M, & x \in \mathbf{R}^n \setminus B_{s_c}(x_m), \end{cases}$$

and consider the equation

$$\begin{cases} (\bar{u}_m)_t = \Delta \bar{u}_m + f(\bar{u}_m), & x \in \mathbf{R}^n, t > 0, \\ \bar{u}_m(x, 0) = \bar{u}_{m,0}(x), & x \in \mathbf{R}^n. \end{cases}$$

By comparison we obtain  $u(x, t) \leq \bar{u}_m(x, t)$  for any  $m \in \mathbf{N}$ . By assumption there exist  $m_0 > 0$  and sequence  $\{c_m\}_{m=m_0}^\infty$  satisfying  $0 < c_m \leq c_{m+1}$  and  $\lim_{m \rightarrow \infty} c_m = 1/s_c$  such that for any  $m \geq m_0$ ,

$$\inf_{r \in (1, s_c)} A_m(r) \leq M - c_m, \quad (22)$$

where  $A_m(r)$  is defined in (11). Since the solution of (1) satisfies the integral equation

$$u(x, t) = e^{\Delta t} u_0(x) + \int_0^t e^{\Delta(t-s)} f(u(x, s)) ds,$$

we have

$$u(x, t) \leq e^{\Delta t} u_0(x) + \int_0^t f(v(s)) ds = v(t) - M + e^{\Delta t} u_0(x)$$

for  $(t, x) \in [0, T^*) \times \mathbf{R}^n$ .

Let  $M_f, \delta_f$  and  $T_0$  be the same as proof of Lemma 3.1. We consider the solution  $w$  of

$$\begin{cases} w_t = \Delta w, & x \in \mathbf{R}^n, t \in (T_0, T^*), \\ w(x, T_0) = \max\{v(T_0) - M + e^{\Delta T_0} u_0(x)\}/v(T_0), \delta_f\}, & x \in \mathbf{R}^n. \end{cases}$$

We now introduce  $\tilde{u} = vw$ . From the proof of Lemma 3.1, it follows that  $\tilde{u} \geq u$  for  $(x, t) \in \mathbf{R}^n \times [T_0, T^*)$ . Then we have

$$u(x, t) \leq v(t) e^{\Delta(t-T_0)} \max\{v(T_0) - M + e^{\Delta T_0} u_0(x)\}/v(T_0), \delta_f\}$$

for  $(x, t) \in \mathbf{R}^n \times [T_0, T^*)$ . By (22) it follows that for each  $x \in B_r$  with  $r \in [1, s_c]$  there is  $z_x \in B_r$  which is farthest from  $x$  such that

$$e^{\Delta t} u_0(x) \leq (e^{t\Delta} (M - c_m |B_r| \delta(\cdot - z_x)))(x)$$

where  $\delta$  is the Dirac delta function. Thus

$$e^{\Delta t} u_0(x) \leq \sup_{z \in B_{s_c}} \left\{ \frac{1}{(4\pi t)^{n/2}} \int_{\mathbf{R}^n} e^{-|x-y|^2/4t} (M - c_m |B_1| \delta(y-z)) dy \right\}.$$

Since  $|x-z| \leq 2s_c$  for any  $x \in B_{s_c}$ , it follows that

$$e^{\Delta t} u_0(x) \leq M - \frac{c_m |B_1|}{(4\pi t)^{n/2}} e^{-s_c^2/t} \quad \text{in } B_{s_c}(x_m) \times [0, T^*).$$

We thus obtain

$$u(x, t) \leq v(t) e^{\Delta(t-T_0)} \max \left\{ \left( v(T_0) + \frac{c_n |B_1|}{(4\pi t)^{n/2}} e^{-s_c^2/T_0} \right) / v(T_0), \delta_f \right\}.$$

for  $(x, t) \in B_{s_c} \times [T_0, T^*)$ . We set

$$\delta_m = e^{\Delta(t-T_0)} \max \left\{ \left( v(T_0) + \frac{c_n |B_1|}{(4\pi t)^{n/2}} e^{-s_c^2/T_0} \right) / v(T_0), \delta_f \right\},$$

and note that  $\delta_m \in (0, 1)$  satisfies  $\delta_m \geq \delta_{m+1}$  for  $m \in \mathbf{N}$ . From Lemma 3.6 and comparison it follows that there exist the sequence  $\{\eta_m\}_{m=m_0}^\infty = \{\eta_m(c_m, s_c, f)\}_{m=m_0}^\infty$  satisfying  $0 < \eta_{m+1} \leq \eta_m < \infty$  such that

$$\lim_{t \rightarrow T^*} u(x_m, t) \leq \eta_m.$$

Since the sequence  $\{x_m\}_{m=1}^\infty$  is arbitrary, we obtain that  $\psi$  is not blow-up direction.

Finally, we should show that the conditions of  $\psi$  in (i) and (ii) cover all of  $S^{n-1}$  exclusively. Let  $\{s_m\}_{m=1}^\infty$  be the sequence satisfying  $\lim_{m \rightarrow \infty} s_m = \infty$ . We set  $D = (1, \infty) \cap [1/(M+N), \infty)$  and the set of sequence

$$S(\psi) = \left\{ \{y_m\}_{m=1}^\infty \left| \lim_{m \rightarrow \infty} \frac{y_m}{|y_m|} = \psi, \lim_{m \rightarrow \infty} |y_m| = \infty \right. \right\}.$$

Let  $\Psi^*$  and  $\Psi_*$  be the sets of directions of the form

$$\begin{aligned} \Psi^* &= \Psi^*(u_0) = \left\{ \psi \in S^{n-1} \left| \exists \{y_m\}_{m=1}^\infty \in S(\psi), \limsup_{m \rightarrow \infty} \inf_{s \in (1, s_m)} A_m(s) = M \right. \right\} \\ \Psi_* &= \Psi_*(u_0) = \left\{ \psi \in S^{n-1} \left| \exists s_c \in D, \right. \right. \\ &\quad \left. \left. \forall \{y_m\}_{m=1}^\infty \in S(\psi), \limsup_{m \rightarrow \infty} \inf_{s \in (1, s_c)} A_m(s) \leq M - \frac{1}{s_c} \right. \right\}. \end{aligned}$$

Here,  $\Psi^*$  and  $\Psi_*$  are the sets of all  $\psi \in S^{n-1}$  satisfying, respectively, (i) and (ii) of Theorem 3. We have

$$(\Psi^*)^c = \left\{ \psi \in S^{n-1} \mid \forall \{y_m\}_{m=1}^\infty \in S(\psi), \limsup_{m \rightarrow \infty} \inf_{s \in (1, s_m)} A_m(s) < M \right\}.$$

Note that  $A_m(s) \in [-N, M]$ , we have

$$\begin{aligned} & \forall \{y_m\}_{m=1}^\infty \in S(\psi), \limsup_{m \rightarrow \infty} \inf_{s \in (1, s_m)} A_m(s) < M \\ \Leftrightarrow & \forall \{y_m\}_{m=1}^\infty \in S(\psi), \exists c \in D, \limsup_{m \rightarrow \infty} \inf_{s \in (1, s_m)} A_m(s) \leq M - \frac{1}{c}. \end{aligned}$$

We define another set

$$\begin{aligned} \Psi^\# = \Psi^\#(u_0) = & \left\{ \psi \in S^{n-1} \mid \exists c \in D, \right. \\ & \left. \forall \{y_m\}_{m=1}^\infty \in S(\psi), \limsup_{m \rightarrow \infty} \inf_{s \in (1, s_m)} A_m(s) \leq M - \frac{1}{c} \right\}. \end{aligned}$$

It is easily seen that  $\Psi^\# \subset (\Psi^*)^c$ . Moreover,

$$\begin{aligned} (\Psi^\#)^c = & \left\{ \psi \in S^{n-1} \mid \forall c \in D, \right. \\ & \left. \exists \{y_m\}_{m=1}^\infty \in S(\psi), \limsup_{m \rightarrow \infty} \inf_{s \in (1, s_m)} A_m(s) > M - \frac{1}{c} \right\}. \end{aligned}$$

Let  $j \in \mathbf{N} \cap D$ . Then we have

$$\begin{aligned} (\Psi^\#)^c \subset & \left\{ \psi \in S^{n-1} \mid \forall j \in \mathbf{N} \cap D, \right. \\ & \left. \exists \{y_m^j\}_{m=1}^\infty \in S(\psi), \limsup_{m \rightarrow \infty} \inf_{s \in (1, s_m)} A_m^j(s) > M - \frac{1}{c} \right\}, \end{aligned}$$

where

$$A_m^j(s) = \frac{1}{|B_s(y_m^j)|} \int_{B_s(y_m^j)} u_0(z) dz.$$

Take the subsequence  $\{m_j\}_{j=a}^\infty \in \mathbf{N}$  satisfying  $\lim_{j \rightarrow \infty} m_j = \infty$ ,  $m_j < m_{j+1}$ ,  $\lim_{j \rightarrow \infty} |y_{m_j}^j| \rightarrow \infty$ ,  $2|y_{m_j}^j| < |y_{m_{j+1}}^j|$  and  $|y_{m_{j+1}}^j| \leq |y_{m_{j+1}}^{j+1}|$ , where  $a = \min\{b \mid b \in \mathbf{N} \cap D\}$ . Then we have

$$(\Psi^\#)^c \subset \left\{ \psi \in S^{n-1} \mid \exists \{y_{m_j}^j\}_{j=a}^\infty \in S(\psi), \limsup_{m \rightarrow \infty} \inf_{s \in (1, s_{m,j})} A_m^j(s) > M - \frac{1}{j} \right\},$$

where  $s_{m,j} = s_{m_j}$ . We thus obtain  $(\Psi^\sharp)^c \subset ((\Psi^*)^c)^c$  and  $(\Psi^*)^c \subset \Psi^\sharp$ . Since  $\Psi^\sharp \subset (\Psi^*)^c$ , we have  $\Psi^\sharp = (\Psi^*)^c$ .

It remains to show that  $\Psi^\sharp = \Psi_*$ . It is easily seen that  $\Psi^\sharp \supset \Psi_*$ . We see that

$$\Psi^\sharp = \left\{ \psi \in S^{n-1} \left| \exists c' \in D, \forall \{y_m\}_{m=1}^\infty \in S(\psi), \right. \right. \\ \left. \left. \exists m_0 > 0, \forall m \geq m_0, \inf_{s \in (1, s_m)} A_m(s) \leq M - \frac{1}{c'} \right\},$$

where we take  $c' > c$ . Take  $c'' = [\max\{s_{m_0}, c'\} + 1]$ , where  $[\theta]$  is largest integer less than  $\theta$ . We thus obtain that

$$\Psi^\sharp = \left\{ \psi \in S^{n-1} \left| \exists c'' \in D, \right. \right. \\ \left. \left. \forall \{y_m\}_{m=1}^\infty \in S(\psi), \forall m \geq c'', \inf_{s \in (1, c'')} A_m(s) \leq M - \frac{1}{c''} \right\} \\ = \left\{ \psi \in S^{n-1} \left| \exists c'' \in D, \forall \{y_m\}_{m=1}^\infty \in S(\psi), \right. \right. \\ \left. \left. \limsup_{m \rightarrow \infty} \inf_{s \in (1, c'')} A_m(s) \leq M - \frac{1}{c''} \right\}.$$

By replacing  $c''$  to  $s_c$ , we have  $\Psi^\sharp \subset \Psi_*$ . Then we obtain  $\Psi^\sharp = \Psi_*$ .

Thus, we get  $(\Psi^*)^c = \Psi_*$ , and the proof is now complete.  $\square$

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