ON SMALL AMPLITUDE SOLUTIONS TO THE GENERALIZED BOUSSINESQ EQUATIONS

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Abstract. We study the existence and scattering of global small amplitude solutions to generalized Boussinesq (Bq) and improved modified Boussinesq (imBq) equations with nonlinear term $f(u)$ behaving as a power $u^p$ as $u \to 0$ in $\mathbb{R}^n, n \geq 1$.

1. Introduction and main results

In this paper, we consider the following Cauchy problems for the generalized Boussinesq (Bq) and improved modified Boussinesq (imBq) equations:

\begin{align*}
\partial_t^2 u_1 - \Delta u_1 + \Delta^2 u_1 &= \Delta f_1(u_1), \quad (x,t) \in \mathbb{R}^{n+1}, \\
&u_1(x,0) = \varphi_1(x), \quad \partial_t u_1(x,0) = \psi_1(x), \quad x \in \mathbb{R}^n, \\
\partial_t^2 u_2 - \Delta u_2 - \Delta \partial_t^2 u_2 &= \Delta f_2(u_2), \quad (x,t) \in \mathbb{R}^{n+1}, \\
&u_2(x,0) = \varphi_2(x), \quad \partial_t u_2(x,0) = \psi_2(x), \quad x \in \mathbb{R}^n,
\end{align*}

(1.1)\hspace{1cm}(1.2)

where $u_i$ is a real-valued function of $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, $\partial_t = \partial/\partial t$, $\Delta$ is the Laplacian in $\mathbb{R}^n$, and $f_i \in C^k(\mathbb{R})$ satisfies the estimates $|f_i^{(l)}(v)| \lesssim |v|^{p_i-l}$ for $0 \leq l \leq k \leq p_i$ and $p_i > 1$, $i = 1,2$. We denote by $u_i(t)$ the function $x \mapsto u_i(x,t)$.

By Duhamel’s principle, partial differential equations (1.1) and (1.2) are rewritten as the integral equations

\begin{align*}
u_i(t) &= \partial_t S_i(t) \varphi_i + S_i(t) \psi_i + \int_0^t T_i(t-t')f_i(u_i(t')) dt', \quad (1.3)
\end{align*}

Here the operators are defined as

\begin{align*}
\partial_t S_i(t) &= \cos(t \omega_i(D)), \quad S_i(t) = \frac{\sin(t \omega_i(D))}{\omega_i(D)}, \\
T_1(t) &= S_1(t) \Delta, \quad T_2(t) = S_2(t)(1-\Delta)^{-1} \Delta, \quad T_i(t) = \sin(t \omega_i(D)) \omega_i(D),
\end{align*}

where

\begin{align*}
D &= (-\Delta)^{\frac{1}{2}} = \mathcal{F}^{-1} |\xi| \mathcal{F}, \quad \omega_i(D) = \omega_i = \mathcal{F}^{-1} \omega_i(\xi) \mathcal{F}, \\
\omega_1(\xi) &= |\xi| \sqrt{1 + |\xi|^2}, \quad \omega_2(\xi) = \frac{|\xi|}{\sqrt{1 + |\xi|^2}}.
\end{align*}

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\[ F(\varphi)(\xi) = \tilde{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \varphi(x) \, dx \quad \text{and} \quad F^{-1}(\varphi)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \varphi(\xi) \, d\xi \]

are the Fourier transform and inverse Fourier transform of \( \varphi \), respectively.

The equations (1.1) was first derived to describe shallow water waves by Boussinesq [5] and it was modified to (1.2) to describe ion-sound waves in plasma by Makhankov [26, 27]. The equations (1.1) and (1.2) also cover another various physical phenomena such as the dynamics of stretched string [30, 9], Fermi-Pasta-Ulam problems [10], the evolution of long internal waves of moderate amplitude [1], nonlinear Alfvén waves [27] and so on.

Our main concern is to establish the global existence and scattering of small amplitude solution to the Cauchy problems (1.1) and (1.2). The local and global existence to the Cauchy problem was established by Bona and Sachs [4], Tsutsumi and Matahashi [37], Linares [21], and Wang and Chen [38]. The stability of solitary waves or the energy conservation was the basic tool of the existence results. For further results on the finite time blowup, stability and instability of solitary waves, and so on see [17, 32, 20, 40, 25, 27, 16, 28] and the references therein.

For the global existence and scattering of small amplitude solutions, it is necessary to study the dispersion of the operators \( \partial_t S_i, S_i \) and \( T_i \) with respect to time, and to compare them with nonlinearity, especially to compare the time decay rate with power \( p \). To get a time decay dispersive estimate, Linares [21], and Linares and Scialom [22] used the estimate\(^1\) \[ \left| \int_{|\xi|<\varepsilon} e^{i(x \cdot \xi + t\omega_1(\xi))} \omega_1(\xi) \frac{1}{2} \, d\xi \right| \lesssim (1 + |t|)^{-\frac{1}{2}}, \]

Liu [24] the estimate\(^2\) \[ \left| \int_{|\xi|<1} e^{i(x \cdot \xi + t\omega_1(\xi))} \, d\xi \right| \lesssim |t|^{-\frac{1}{2}} + |t|^{-\frac{1}{2}}, \]

and Liu [23] and Wang and Chen\(^3\) the estimate\(^3\) \[ \left| \int_{|\xi|<|t|^{\frac{1}{2}}} e^{i(x \cdot \xi + t\omega_1(\xi))} \, d\xi \right| \lesssim |t|^{-\frac{1}{2}} \varepsilon^{-\frac{1}{2}}. \]

The best result up to now is \( p > 2 + \sqrt{7} \) of Liu [24] for (1.1) with \( n = 1 \), and \( p > \frac{9}{2} \) of Cho and Ozawa [7] for (1.2) with \( n = 1 \), and integer \( p \) greater than \( 2 + \frac{1}{\theta_{n,s}} \) of Wang and Chen [39] for (1.2) with \( n \geq 2 \), where \( \theta = \frac{2s-n}{2s+2+2n} \) if \( \frac{n}{2} < s \leq \frac{2n}{2} \) and \( \theta = \frac{2n}{n+\sqrt{7}} \) if \( s \geq \frac{2n}{2} \).

In this paper, we improve all the known results under some vanishing condition of initial data at the zero frequency in one dimensional case and extend the results not only on existence and scattering but dispersive estimates to the high dimensional case. Moreover, we also provide a non-existence of nontrivial asymptotically free solutions in the case of small power \( p \), which is a high dimensional version of Theorem 1.3 of [7].

Before stating the main results, let us introduce some notations. First we let \( \beta_r = 1 - \frac{\theta}{r} \). Then we define a homogeneous initial data space \( \dot{D}_{r; q}^{\beta_r} \) for \( 2 \leq r \leq \infty \)

\(^1\)This estimate was proved by Kenig, Ponce and Vega [19].
\(^2\)The decay rate \( \frac{1}{2} \) comes from the estimate of low frequency part (\( |\xi| \leq 1 \)) and it turns out to be optimal. See (1.4).
\(^3\)Actually, Wang and Chen in [39] obtained \( n \)-dimensional estimate but their estimate was the same as 1-dimensional one because they integrated only in radial radial direction by using spherical coordinate.
and \( i = 1, 2 \) by
\[
\tilde{\mathcal{D}}^{\beta, i}_{\nu, q} = (D_i^{-\beta_i} \hat{B}^{\beta, i}_{\nu, q} \cap \hat{B}^{+\nu, \beta_i}_{\nu, q}) \times \omega_i(D_i^{-\beta_i} \hat{B}^{\beta, i}_{\nu, q} \cap \hat{B}^{+\nu, \beta_i}_{\nu, q}),
\]
where \( D_i = \mathcal{F}^{-1}[D_i(\xi)]\mathcal{F} \) and
\[
D_i(\xi) = \omega_2(\xi)^{\frac{n-2}{2}}, \quad D_2(\xi) = (1 + |\xi|^2)^n \omega_2(\xi)^{\frac{n-2}{2}}.
\]
The norm of the space \( \tilde{\mathcal{D}}^{\beta, i}_{\nu, q} \) is given by
\[
\| (\varphi_i, \psi_i) \|_{\tilde{\mathcal{D}}^{\beta, i}_{\nu, q}} = \| D_i^{\beta_i} \varphi_i \|_{\hat{B}^{\beta, i}_{\nu, q}} + \| \varphi_i \|_{\hat{B}^{+\nu, \beta_i}_{\nu, q}} + \| D_i^{\beta_i} \omega_i^{-1} \psi_i \|_{\hat{B}^{\beta, i}_{\nu, q}} + \| \omega_i^{-1} \psi_i \|_{\hat{B}^{+\nu, \beta_i}_{\nu, q}}.
\]
The inhomogeneous initial data space \( \mathcal{D}^{\beta, i}_{\nu, q} \) and its norm are defined by the inhomogeneous Besov space \( \hat{B}^{\beta, i}_{\nu, q} \) instead of \( \hat{B}^{\beta, i}_{\nu, q} \).

Here we used the notation \( v \in \omega^n D_i^{\beta_i} D^\gamma X \) to mean \( \omega^{-\alpha} D_i^{\beta_i} D^{-\gamma} v \in X \) for a function space \( X \) and some real number \( \alpha, \beta, \gamma \).

To define the Besov space, let us choose a Littlewood-Paley function \( \eta \) with and define a frequency projection operator \( P_N \) for a dyadic number \( N \) by
\[
P_N \phi(x) = \mathcal{F}^{-1} \left[ \eta \left( \frac{\xi}{N} \right) \hat{\phi} \right](x).
\]
Then the homogeneous Besov space \( \hat{B}^{\beta, i}_{\nu, q}, 1 \leq r, q \leq \infty, s \in \mathbb{R} \), is defined by
\[
\hat{B}^{\beta, i}_{\nu, q} = \left\{ v \in S'/\mathcal{P} : \| v \|_{\hat{B}^{\beta, i}_{\nu, q}} = \left( \sum_{N \text{ dyadic}} N^{sq} \| P_N(v) \|_{L^r}^q \right)^{\frac{1}{q}} < \infty \right\},
\]
where \( \mathcal{P} \) is the set of all polynomials on \( \mathbb{R}^n \). The inhomogeneous Besov space \( B^{\beta, i}_{\nu, q} \) is defined by
\[
B^{\beta, i}_{\nu, q} = \left\{ v \in S' : \| v \|_{B^{\beta, i}_{\nu, q}} = \| P_0 v \|_{L^r} + \left( \sum_{N \geq 1} N^{sq} \| P_N(v) \|_{L^r}^q \right)^{\frac{1}{q}} < \infty \right\},
\]
where \( P_0 = 1 - \sum_{N \geq 1} P_N \). If \( s > 0 \), then \( B^{\beta, r}_{\nu} \sim L^r \cap \hat{B}^{\beta, r}_{\nu} \). See for instance [3].

The above initial data space is necessary for the dispersive estimate of the operators \( \partial_t S_i, S_i \) and \( T_i \). In particular, we obtain
\[
\| (\partial_t S_i, S_i) \|_{\tilde{\mathcal{D}}^{\beta, i}_{\nu, 1}, L^\infty} \lesssim (1 + |t|)^{-n(\frac{1}{2} - \frac{1}{s})}
\]
for any \( r \in [2, \infty) \). If \( r = \infty \), then the time decay rate is the best possible decay \( \frac{n}{2} \).

Since \( \omega_1(\xi) \) and \( \omega_2(\xi) \) are not phase of elliptic type (in fact, \( \omega_i \) behaves like \( D \) for small frequency, \( \omega_2 \) like identity for large frequency), to achieve the full time decay rate we need the regularity \( \hat{B}^{\beta, i}_{\nu, 1} \times \omega_i \hat{B}^{\beta, i}_{\nu, 1} \) for high frequency and the operator \( D_i^{\beta_i} \) for small frequency.

If \( n = 1 \), then \( D_i^{\beta_i} \sim D^{-\frac{\beta_i}{2}} \) for small frequency. This means that for the time decay it is necessary to assume that the Fourier transforms of initial data vanish at zero frequency. If we want to remove this vanishing condition, we cannot help but allow a slow time decay estimate. For the results obtained without vanishing
condition, see [7, 21, 22, 23, 24]. In those papers the time decay rate is \( \frac{1}{3} \) and this
decay rate seems to be optimal because for large \( t \) and for some \( \phi \in C_0^\infty(-1,1) \)
\[
\left| \int e^{i(x\xi - t\omega(\xi))}\phi d\xi \right| \sim |t|^{-\frac{1}{2}}.
\] (1.4)
The time decay comes from the bound \( |\omega_1^{(3)}(\xi)| \gtrsim 1 \) for small \( \xi \) and the stationary
phase estimate (Proposition 3 of [34], p. 334). Therefore the vanishing condition
seems to be inevitable for the faster decay than the rate \( \frac{1}{3} \).

The additional regularity \( \dot{B}^{s+n\beta}_s \times \omega_1 B^{s+n\beta}_s \) is necessary for the boundedness
of linear dispersive estimate at time zero and high frequency estimate for \( \omega_2 \). For
details, see Lemma 2.4 below.

Now let us introduce the main results. The first result is

**Theorem 1.1.** Let \( 2 < r < \infty, s_1 > \frac{n}{p}, s_2 > 2n - \frac{3n}{r}, \) and \( \theta = \frac{n\beta}{2} \). Let
\( p_i \geq s_i, p_i > \frac{2}{\theta} + \max\left(1, \frac{1}{2}\right) \). Then there exists \( \delta > 0 \) such that for any \( (\varphi_i, \psi_i) \in \dot{D}^{s_1}_{p_i} \cap (H^{s_1} \times \omega_1 H^{s_1}) \)
\[
\|(\varphi_i, \psi_i)\|_{\dot{D}^{s_1}_{p_i}} + \|\varphi_i\|_{H^{s_1}} + \|\omega_1^{-1}\psi_i\|_{H^{s_1}} \leq \delta
\]
there exist unique solutions \( u_i \in C(\mathbb{R}; H^{s_1}) \) to (1.1) and (1.2). Moreover, there exists a positive number \( \rho_i \) depending only on \( n, r, s_i, p_i \) and \( \delta \) such that
\[
\sup_{t \in \mathbb{R}}(1 + |t|)^{\delta/2} \|u_i(t)\|_{L^{\infty}} + \sup_{t \in \mathbb{R}} \|u_i(t)\|_{H^{s_1}} \leq \rho_i.
\]

**Remark 1.** Theorem 1.1 is applicable to the cases \( p_i > 4 \) for \( n = 1, p_1 > 3, p_2 > 4 \)
for \( n = 2 \) and \( p_1 > n, p_2 > 2n \) for \( n \geq 3 \). This improves the results in [7, 24, 39].

The condition \( p_i \geq s_i \) comes from the nonlinear estimates such as \( \|f_i(u_i)\|_{H^{s_1}} \lesssim
\|u_i\|_{L^{\infty}}^{p_i-1} \|u_i\|_{H^{s_1}} \), for which \( p_i \) should be greater than equal to \( s_i \). If \( p_i \) is an integer,
then the condition is unnecessary from the arguments in [39].

Next, we consider the equation (1.1). Let \( \gamma(n) = 1 + 8/((\sqrt{n^2 + 12n + 4} + n - 2)), \)
and \( \alpha(n) = \infty \) if \( n = 1, 2 \) and \( \alpha(n) = \frac{n+4}{n+2} \) if \( n \geq 3 \). Then we have the following.

**Theorem 1.2.** Let \( s > 0 \) and \( \theta = \frac{n\beta+1}{2} \). If \( s \leq p \) and \( \gamma(n) \leq p < \alpha(n) \), then there
exists \( \delta > 0 \) such that for any
\[
(\varphi, \psi) \in \dot{D}^{s_1}_{p_i} \text{ with } \|(\varphi, \psi)\|_{\dot{D}^{s_1}_{p_i}} \leq \delta.
\]
there exists a unique solution \( u \in C(\mathbb{R}; H^{s_1}_{p_i}) \) to (1.1) with \( p_1 = p \) and \((u(0), \partial_t u(0)) = (\varphi, \psi)\), and \( p > 0 \) depending only on \( n, s, p \) and \( \delta \) such that
\[
\sup_{t \in \mathbb{R}}(1 + |t|)^{\delta} \|u(t)\|_{H^{s_1}_{p_i}} \leq \rho.
\]

**Remark 2.** The critical exponent \( \gamma(n) \) naturally arises in the problem of the existence
of small amplitude solutions decaying as \( O(|t|^{-n(\frac{1}{2} - \frac{1}{3})}) \) in \( L^r \) as \( t \to \infty \) (see
The result above can be obtained by the fact $\omega_1$ is of elliptic type at high frequency as Schrödinger equation and hence it is possible to obtain the estimate $\|T_1(t)\|_{B^s_{r,2} \to B^s_{r,2}} \lesssim |t|^{-n(s - 1)}$ for any $r \in [2, \infty]$ and $s \geq 0$. See Lemma 2.5 below. This is not the case for $\omega_2$ (hence $T_2(t)$) because the dispersion of (1.2) becomes small at high frequency and hence a higher regularity for data is necessary to compensate for the small dispersion.

If $s$ is close and greater than $\frac{n}{p+1}$, then $\gamma(n) < p < \alpha(n)$ for $1 \leq n \leq 4$ and by Sobolev embedding, $\|u(t)\|_{L^\infty} \lesssim \rho(1 + |t|)^{-\theta}$. In this case, for the initial data in $H^s$, the solution is in $C(\mathbb{R}; H^s)$. See Remark 6.

If $n \geq 4$, then from Theorem 1.1 we deduce that $p_1 \geq n$ and from Theorem 1.2 that $p_1 < \alpha(n)$. There exists a gap between two results. It is still open whether Theorems 1.1 and 1.2 hold for $\alpha(n) \leq p_1 = p < n$.

Small data scattering follows as a simple consequence of Theorems 1.1 and 1.2.

**Theorem 1.3.** Let $u_i, i = 1, 2$ and $u$ be the solutions of (1.3) as in Theorems 1.1 and 1.2, respectively. Then there exist six pairs of functions $(\varphi_i^\pm, \psi_i^\pm) \in H^s \times \omega_i H^{s_1}$ and $(\varphi^\pm, \psi^\pm) \in H^s \times \omega \omega_i H^{s_1}$ such that

$$
\|u_i(t) - u_i^\pm(t)\|_{H^{s_1}} = O(\|t\|^{-\theta(p_i-1)+1}),
$$

$$
\|u(t) - u^\pm(t)\|_{H^{s_1}_{p+1}} = O(\|t\|^{-\theta(p-1)+1})
$$

as $t \to \pm\infty$, where $s_i, r_i, \theta_i, p_i, \rho$ are the same numbers stated in Theorems 1.1 and 1.2 and $u_i^\pm$ and $u^\pm$ are the unique solutions to the linear problems (1.1) and (1.2) with $f_i = 0$.

If $1 \leq n \leq 4$ and $s$ is close to and greater than $\frac{n}{p+1}$, then Sobolev space $H^s_{p+1}$ can be replaced by $H^s$ in the second scattering result. See Remarks 6 and 7 below.

This paper is organized as follow. In Section 2, we prove several linear dispersive estimates for the operators $\partial_t S_i, S_i$ and $T_i$. Utilizing the dispersive estimates, we prove the global existence and scattering results in Section 3. In the final section, Section 4, we consider a non-existence result of asymptotically free solutions for suitably small power $p_i$.

If not specified, throughout this paper, the notation $A \lesssim B$ and $A \gtrsim B$ denote $A \leq CB$ and $A \geq C^{-1}B$, respectively. Positive constants $C$ vary line by line and depend only on $r$ and $f$. $A \sim B$ means that both $A \lesssim B$ and $A \gtrsim B$ hold.

2. Linear dispersive estimates

In this section, we consider time decay estimates for $\partial_t S_i, S$ and $T$. We first denote the frequency localization operator $P_{\leq \eta \phi}$, $P_{\geq \eta \phi}$ and $P_{\eta < \phi < N_0}$ by

$$
P_{\leq \eta \phi} = \sum_{N \leq \eta} P_N \phi, \quad P_{\geq \eta \phi} = \sum_{N \geq \eta} P_N \phi, \quad P_{\eta < \phi < N_0} \phi = \sum_{\eta < N < N_0} P_N \phi.
$$

For the convenience of presentation, we choose $\eta$ so that $P_N = P_{N/2} \leq 2NP_N$. 

\hspace{1cm}
Next let us introduce a lemma on the stationary phase estimate (see Proposition 5 of [34], p. 342).

**Lemma 2.1.** Let $\chi$ be a smooth function supported in a unit ball of $\mathbb{R}^n$, $n \geq 2$ and $\Omega$ be a $C^3(\mathbb{R}^n)$ function such that $|\nabla^2 \Omega| \geq 1$ on the support of $\chi$. Then we have for any $\lambda > 0$

$$\left| \int_{\mathbb{R}^n} e^{i\lambda \Omega(\xi)} \chi(\xi) d\xi \right| \leq C\lambda^{-\frac{n}{2}}(\|\chi\|_{L^\infty} + \|\nabla \chi\|_{L^1}), \quad (2.1)$$

where $|\nabla^2 \Omega|^2 = \sum_{i,j} |\partial_i \partial_j \Omega|^2$ and the constant $C$ depends only on $\Omega$ and $n$, which is bounded if $\Omega$ is bounded in the norm of $C^3$.

**Remark.** If $n = 1$, then the support condition in a unit ball of the above lemma can be removed and (2.1) is rewritten as if $|\Omega''(\xi)| \geq 1$ on a fixed interval $(a, b)$ and $\chi$ is supported in $[a, b]$, then

$$\left| \int_a^b e^{i\lambda \Omega(\xi)} \chi(\xi) d\xi \right| \leq C\lambda^{-\frac{1}{2}}(\|\chi\|_{L^\infty(a,b)} + \|\chi'\|_{L^1(a,b)}). \quad (2.2)$$

See Corollary of [34], p. 334.

To apply Lemma 2.1 to the radially symmetric phase $\Omega$, we need the following formulation of the determinant of Hessian matrix of radially symmetric function.

**Lemma 2.2.** Let $\omega = \omega(|x|)$ be a radially symmetric $C^2$ function on $\mathbb{R}^n \setminus \{0\}$, $n \geq 2$. Then the determinant of Hessian matrix is also radially symmetric and is the following

$$\det(\nabla^2 \omega)(x) = \left( \frac{\omega'(r)}{r} \right)^{n-1} \omega''(r), \quad r = |x|$$

**Proof of Lemma 2.2.** The $(i, j)$ component of Hessian matrix of $\omega$ is given by

$$\partial_i \partial_j \omega(x) = \frac{\omega'(r)}{r} \left( \delta_{ij} - \frac{x_i x_j}{r^2} \right) + \omega''(r) \frac{x_i x_j}{r^2},$$

where $r = |x|$.

If $\omega'(r) = 0$ for some $r > 0$, then since $j$th column vector of Hessian matrix is $\omega''(r) \frac{1}{r^2} x$, obviously the determinant of Hessian is zero. Hence we assume that $\omega'(r)$ is not zero for any $r > 0$.

Let $\lambda = 1 - \frac{r \omega''(r)}{\omega'(r)}$. Then the $(i, j)$ component of Hessian is rewritten by

$$\partial_i \partial_j \omega(x) = \frac{\omega'(r)}{r} \left( \delta_{ij} - \lambda \frac{x_i x_j}{r^2} \right).$$

Hence

$$\det(\nabla^2 \omega) = \left( \frac{\omega'(r)}{r} \right)^n \det(A),$$

where $A_{ij} = \delta_{ij} - \lambda \frac{x_i x_j}{r^2}$.

Let $h$ be a function on $\lambda$ defined by $h(\lambda) = \det(A)$. Then $h$ is a polynomial of degree $n$ on $\lambda$ rewritten by

$$h(\lambda) = 1 + a_1 \lambda + a_2 \lambda^2 + \cdots + a_{n-1} \lambda^{n-1} + a_n \lambda^n.$$
Here \( a_1 = -\text{tr}(A) = -\sum_j \frac{x_j^2}{r^2} = -1 \) and for \( 2 \leq j \leq n \)

\[
a_j = (-1)^j \sum_{i_1 < i_2 < \cdots < i_j} \text{det} \left( \frac{x_{i_1} x_{i_2}}{r^2} \cdots \frac{x_{i_1} x_{i_j}}{r^2} \right) = (-1)^j \sum_{i_1 < i_2 < \cdots < i_j} \frac{x_{i_1} x_{i_2} \cdots x_{i_j}}{r^{2n}} \text{det} \left( \begin{array}{cccc}
\frac{x_{i_1}}{r} & \cdots & \frac{x_{i_1}}{r} \\
\vdots & \ddots & \vdots \\
\frac{x_{i_1}}{r} & \cdots & \frac{x_{i_j}}{r} 
\end{array} \right) = 0
\]

(for the formula of \( h(\lambda) \) and its coefficients \( a_j \), see [33], p.155-156). Thus \( h(\lambda) = 1 - \lambda \). Therefore we have

\[
\text{det}(\nabla^2 \omega) = \left( \frac{\omega'(r)}{r} \right)^n (1 - \lambda) = \left( \frac{\omega'(r)}{r} \right)^{n-1} \omega''(r).
\]

\[\square\]

Remark 4. Applying the above lemma to \( \omega_i \), we observe from

\[
\omega''_i(\rho) = \frac{\rho(3 + 2\rho^2)}{(1 + \rho^2)^2}, \quad \omega'_i(\rho) = -\frac{3\rho}{(1 + \rho^2)^2} \tag{2.3}
\]

that for \( i = 1, 2 \)

\[
|\text{det}(\nabla^2 \omega_i)(x)|^{-\frac{1}{2}} \sim D_i(\rho) \quad \text{for all } |x| = \rho > 0.
\]

Utilizing the above two lemmas and Remark 4, we obtain the following dispersive estimate\(^4\).

Lemma 2.3.

\[
\sup_{x \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{i(x \cdot \xi + t\omega_i(\xi))} \eta \left( \frac{\xi}{N} \right) d\xi \right| \lesssim |t|^{-\frac{n}{2}} D_i(N). \tag{2.4}
\]

Proof of Lemma 2.3. If \( n = 1 \), then it is easily observed from (2.3) that \( |\omega''_i(\xi)| \geq cD_i(N)^{-2} \) for any \( \xi \in (N/2, 2N) \) and some fixed small constant \( c \). Now by direct application of Remark 3 with \( a = \frac{N}{2}, b = 2N, \lambda = cD_i(N)^{-2}, \chi(\xi) = \eta(\xi/N) \) and \( \Omega = c^{-1}t^{-1}D_i(N)^2(x\xi + t\omega_i(\xi)) \), one can readily obtain (2.4). Therefore we consider only the case \( n \geq 2 \) from now on.

From Remark 4, it suffices to show that the left hand side of (2.4) is bounded by a constant multiple of \( |t|^{-\frac{n}{2}} |\text{det}(\nabla^2 \omega_i)(x)|^{-\frac{1}{2}} \) with \( |x| = N \).

By the change of variable \( \xi \mapsto N \xi \), we have

\[
I = N^n \int e^{itN\Omega_i(\xi)} \eta(\xi) d\xi,
\]

where \( \Omega_i(\xi) = \frac{1}{N} x \cdot \xi + \frac{1}{N} \omega_i(N\xi). \)

\(^4\)The authors heard that recently, Gustafson, Nakanishi and Tsai showed a similar result for the phase \( \omega_1 \) by another approach in [14].
Fixing \((x,t)\), let us define a function \(\alpha\) by
\[
\alpha(\xi) \equiv |\nabla \Omega(\xi)| = \frac{|x|}{t} + \omega_i'(N\rho) \frac{\xi}{\rho},
\]
where \(\rho = |\xi|\). Let \(\alpha_0\) be the minimum value of \(\alpha(\xi)\) on the annulus \(\{\frac{1}{2} \leq |\xi| \leq 2\}\).

Since the set of vectors \(\frac{\xi}{|\xi|} + \omega_i'(N\rho) \frac{\rho}{|\rho|}\) with \(\xi \in \{\frac{1}{2} \leq |\xi| \leq 2\}\) is an annulus centered at \(\frac{\xi}{|\xi|}\), the minimum \(\alpha_0\) is attained on the line of direction \(x\). Let \(\xi_0\) be the minimum point. Then \(\xi_0\) has the opposite direction to \(x\) and
\[
|\frac{x}{t}| = \omega_i'(N\rho_0) \pm \alpha_0, \quad \rho_0 = |\xi_0|.
\]
The signs \(\pm\) appear when the minimum is attained on the outside sphere of annulus and the inside one, respectively.

Since for any \(\frac{1}{2} \leq \rho \leq 2\)
\[
\frac{1}{5} \omega_i'(N) \leq \omega_i'(N\rho) \leq 5 \omega_i'(N),
\]
\[
\frac{1}{5} \omega_i''(N) \leq \omega_i''(N\rho) \leq 5 \omega_i''(N),
\]
(2.5)
if \(\alpha_0 > \frac{1}{1000} \omega_i'(N)\), then by integration by parts we have for any \(M > 0\)
\[
|I| \lesssim N^n |\langle t \rangle N \omega_i'(N)|^{-M}.
\]

Now setting \(M = \frac{n}{2}\), we have from (2.5) and Lemma 2.2
\[
|I| \lesssim |t|^{-\frac{n}{2}} \left( \frac{\omega_i'(N)}{N} \right)^{-\frac{n-1}{2}} \left( \frac{\omega_i''(N)}{N} \right)^{-\frac{n}{2}}
\]
\[
\lesssim |t|^{-\frac{n}{2}} \left( \frac{\omega_i'(N)}{N} \right)^{-\frac{n-1}{2}} |\omega_i''(N)|^{-\frac{n}{2}}
\]
\[
= |t|^{-\frac{n}{2}} |\det (\nabla^2 \omega_i)(N)|^{-\frac{n}{2}}.
\]

From now on, we assume \(\alpha_0 \leq \frac{1}{1000} \omega_i'(N)\). Let us choose a cut-off function \(g\) defined on \(S^{n-1}\) and supported on the set \(\{\frac{\xi}{|\xi|} \in S^{n-1} : \frac{\xi}{|\xi|} - \frac{\epsilon_0}{|\xi|} \leq \frac{1}{2}\}\). Then
\[
I = N^n \int e^{i t N \Omega(\xi)} g \left( \frac{\xi}{|\xi|} \right) \eta(\xi) d\xi + N^n \int e^{i t N \Omega(\xi)} (1 - g \left( \frac{\xi}{|\xi|} \right)) \eta(\xi) d\xi
\]
\[
\equiv I_1 + I_2.
\]

We first estimate \(I_2\). To do this, we will use a one dimensional cut-off function \(h\) supported in a neighborhood of \(\rho_0\) such that
\[
h(\rho) = 1 \text{ for } |\rho - \rho_0| \leq \frac{1}{100 N |\omega_i'(N)|}
\]
and
\[
|h^{(k)}(\rho)| \lesssim \left( \frac{1}{100 N |\omega_i''(N)|} \right)^{-k} \text{ for all } k \geq 0.
\]
If \( \xi \in \text{supp}(h) \), then from (2.5)
\[
|\omega'_i(N\rho) - \omega'_i(N\rho_0)| \geq \frac{N}{5} |\omega''_i(N)||\rho - \rho_0|,
\]
and hence
\[
|\nabla \Omega_i| = \left| \frac{\omega'_i(N\rho)}{\rho} - \frac{\omega'_i(N\rho_0)}{\rho_0} \pm \alpha_0 \right| \geq \frac{N}{5} |\omega''_i(N)||\rho - \rho_0| - \alpha_0 \\
\geq \frac{N}{5} |\omega''_i(N)||\rho - \rho_0| - \frac{1}{500} \omega'_i(N) - \alpha_0 \quad (2.6)
\]
Since \( \left| \frac{\xi}{|\xi|} - \frac{\rho_0}{|\rho_0|} \right| \geq \frac{1}{2} \) for \( \xi \in (\text{supp}(h))^c \) and
\[
|\omega'_i(N\rho) - \omega'_i(N\rho_0)| \leq 5N|\omega''_i(N)||\rho - \rho_0|,
\]
we have
\[
|\nabla \Omega_i| \geq ||\omega'_i(N\rho_0)|||\xi' - \xi_0| - |\omega'_i(N\rho) - \omega'_i(N\rho_0)||| - \alpha_0 \\
\geq \frac{1}{10} \omega'_i(N) - \frac{1}{20} \omega'_i(N) - \alpha_0 \\
\geq \frac{1}{500} \omega'_i(N). \quad (2.7)
\]
Thus by integration by parts, we deduce from (2.6), (2.7) and Lemma 2.2 that
\[
|I_2| \leq N^n \int e^{it\nabla \Omega_i(\xi)} \left( 1 - g \left( \frac{\xi}{|\xi|} \right) \right) h(\rho)\eta(\xi) d\xi \\
+ \frac{1}{t} \int e^{it\nabla \Omega_i(\xi)} \left( 1 - g \left( \frac{\xi}{|\xi|} \right) \right) (1 - h(\rho))\eta(\xi) d\xi \\
\lesssim N^n (|t|N\omega'_i(N))^{-\frac{n}{2}} \\
= |t|^{-\frac{n}{2}} |\det(\nabla^2 \omega_i)(N)|^{-\frac{1}{2}}.
\]
Now it remains to estimate \( I_1 \). Let us define a function \( \tilde{\Omega}_i \) by
\[
\tilde{\Omega}_i = \Omega_i - \frac{1}{N} \omega_i(N\rho_0).
\]
Then
\[
I_1 = N^n \int e^{it\omega_i(\rho_0)} e^{itN\tilde{\Omega}_i(\xi)} g \left( \frac{\xi}{|\xi|} \right) \eta(\xi) d\xi.
\]
By the relation \( |\xi| = \omega'_i(N\rho_0) \pm \alpha_0 \), we have that for \( 0 \leq k \leq 3 \),
\[
|\nabla^k \tilde{\Omega}_2(\xi)| \lesssim 1, \\
|\nabla^k \tilde{\Omega}_1(\xi)| \lesssim 1 \text{ if } N \leq 1, \quad \frac{1}{N} |\nabla^k \tilde{\Omega}_1| \lesssim 1 \text{ if } N > 1.
\]
We define \( \lambda \) case by case as follows:
\[
\lambda = tNC_n \text{ if } \frac{1}{N} |\det(\nabla^2 \omega_2)(N)|^{\frac{1}{2}} \text{ for } \tilde{\Omega}_2, \\
\lambda = tNC_n |\det(\nabla^2 \omega_1)(N)|^{\frac{1}{2}} \text{ if } N < 1, \\
\lambda = tN^2C_n |\det(N^{-1}\nabla^2 \omega_1)(N)|^{\frac{1}{2}} \text{ if } N \geq 1
\]
Then using the fact that $|\nabla^2 \omega_i(N\rho)| \geq c_n |\det(\nabla^2 \omega_i)(N)|^{\frac{1}{2}}$ for some small constant $c_n$ depending only on $n$, from Lemma 2.1 (after decomposing the annulus by finite number of unit balls if necessary), we obtain for $I_1$ with the phase $\lambda \tilde{\Omega}_2(\xi)$,

$$|I_1| \lesssim N^n |t|^{-\frac{5}{2}} N^{-\frac{5}{2}} |\det(\nabla^2 \omega_2)(N)|^{-\frac{1}{2}} \leq |t|^{-\frac{5}{2}} |\det(\nabla^2 \omega_2)(N)|^{-\frac{1}{2}},$$

for $I_1$ with $\lambda \tilde{\Omega}_1(\xi)$ and $N \leq 1$

$$|I_1| \lesssim N^n |t|^{-\frac{5}{2}} N^{-\frac{5}{2}} |\det(\nabla^2 \omega_1)(N)|^{-\frac{1}{2}} = |t|^{-\frac{5}{2}} |\det(\nabla^2 \omega_1)(N)|^{-\frac{1}{2}},$$

and for $I_1$ with $\lambda N^{-1} \tilde{\Omega}_1(\xi)$ and $N > 1$

$$|I_1| \lesssim N^n |t|^{-\frac{5}{2}} N^{-n} |\det(\nabla^2 \omega_1)(N)|^{-\frac{1}{2}} = |t|^{-\frac{5}{2}} |\det(\nabla^2 \omega_1)(N)|^{-\frac{1}{2}}.$$ 

These complete the proof of lemma.

As a consequence, we have the following lemmas.

**Lemma 2.4.** If $2 \leq r \leq \infty$, then

$$\| (\partial_t S_i(t) \varphi_i, S_i(t) \psi_i) \|_{L^\infty} \lesssim (1 + |t|)^{-\theta} \| (\varphi_i, \psi_i) \|_{\mathcal{A}_r^{\frac{n}{2}}}. \tag{2.8}$$

If $2 \leq r < \infty$ and $s \geq 0$, then for $s > 0$

$$\| (\partial_t S_i(t) \varphi_i, S_i(t) \psi_i) \|_{L^r} \lesssim (1 + |t|)^{-\theta} \| (\varphi_i, \psi_i) \|_{\mathcal{A}_r^{\frac{n}{2}}}. \tag{2.9}$$

and for $s = 0$

$$\| (\partial_t S_i(t) \varphi_i, S_i(t) \psi_i) \|_{L^0} \lesssim (1 + |t|)^{-\theta} \| (\varphi_i, \psi_i) \|_{\mathcal{A}_r^{\frac{n}{2}}}. \tag{2.10}$$

Here $\theta = \frac{n\alpha}{2} = n(\frac{1}{2} - \frac{1}{r}).$

**Proof of Lemma 2.4.** If $|t| \leq 1$, by Hölder’s and Hausdorff-Young’s inequalities, we have for any $r \in [2, \infty]$

$$\| P_N (\partial_t S_i(t) \varphi_i) \|_{L^r} \lesssim N^{\frac{n}{2}} \| P_{N/2 \leq 2N} \varphi_i \|_{L^r}.$$ 

Hence

$$\| \partial_t S_i(t) \varphi_i \|_{L^\infty} \lesssim \| \varphi_i \|_{\mathcal{A}_r^{\frac{n}{2}}} \tag{2.11}.$$ 

In particular we have

$$\| P_N (\partial_t S_i(t) \varphi_i) \|_{L^\infty} \lesssim N^n \| P_{N/2 \leq 2N} \varphi_i \|_{L^1}.$$ 

Interpolating this with trivial $L^2$ estimate that

$$\| P_N (\partial_t S_i(t) \varphi_i) \|_{L^2} \lesssim \| P_{N/2 \leq 2N} \varphi_i \|_{L^2},$$

we have for any $r \in [2, \infty]$

$$\| P_N (\partial_t S_i(t) \varphi_i) \|_{L^r} \lesssim N^{n(1-\frac{2}{r})} \| P_{N/2 \leq 2N} \varphi_i \|_{L^r}.$$ 

Since by Hölder’s inequality

$$\| P_0 (\partial_t S_i(t) \varphi_i) \|_{L^r} \lesssim \| \varphi_i \|_{L^r},$$

we obtain
we have for any $s \in \mathbb{R}$ and $r \in [2, \infty]$
\[
\|\partial_t S_i(t)\varphi_i\|_{B^s_{2,r}} \lesssim \|\varphi_i\|_{B^{s+\frac{1}{2}}_{2,2}}, \quad (2.12)
\]
If $|t| > 1$, then from Lemma 2.3, it follows that
\[
\|P_N(\partial_t S_i(t)\varphi_i)\|_{L^\infty} \lesssim |t|^{-\frac{3}{2}} D_i(N)\|P_{N/2} \leq 2N\varphi_i\|_{L^3}. \quad (2.13)
\]
By Hölder inequality, we also have
\[
\|P_N(\partial_t S_i(t)\varphi_i)\|_{L^\infty} \lesssim N^{\frac{3}{2}} \|P_{N/2} \leq 2N\varphi_i\|_{L^2}. \quad (2.14)
\]
Interpolating (2.13) and (2.14) and using the estimate (2.11) and the fact that
\[
D_i(N/2) \sim D_i(N) \sim D_i(2N),
\]
we obtain the estimate (2.8).

On the other hand, using the trivial estimate
\[
\|P_N(\partial_t S_i(t)\varphi_i)\|_{L^2} \lesssim \|P_{N/2} \leq 2N\varphi_i\|_{L^2}
\]
and its interpolation with (2.13), we have
\[
\|P_N(\partial_t S_i(t)\varphi_i)\|_{L^r} \lesssim |t|^{-n\left(\frac{4}{3} - \frac{1}{r}\right)} D_i(N)^{1 - \frac{3}{2}} \|P_{N/2} \leq 2N\varphi_i\|_{L^{r'}}. \quad (2.15)
\]
Thus for $2 \leq r < \infty$, we have
\[
\|\partial_t S_i(t)\varphi_i\|_{L^r} \lesssim \|\partial_t S_i(t)\varphi_i\|_{B^s_{2,r}} \lesssim |t|^{-n\left(\frac{4}{3} - \frac{1}{r}\right)} D_i^{1 - \frac{3}{2}} \|\varphi_i\|_{B^s_{2,r}}
\]
Hence for $s > 0$
\[
\|\partial_t S_i(t)\varphi_i\|_{B^s_{2,r}} \lesssim |t|^{-n\left(\frac{4}{3} - \frac{1}{r}\right)} D_i^{1 - \frac{3}{2}} \|\varphi_i\|_{B^s_{2,r}}
\]
and also for $s = 0$
\[
\|\partial_t S_i(t)\varphi_i\|_{B^0_{2,r}} \lesssim |t|^{-n\left(\frac{4}{3} - \frac{1}{r}\right)} D_i^{1 - \frac{3}{2}} \|\varphi_i\|_{B^0_{2,r}}.
\]
Combining these estimates and (2.12), we get (2.9).

\begin{remark}
Letting $\Lambda_{\alpha,\beta} = \omega_2(1 - \Delta)^{\frac{\alpha}{2}}$, instead of $\ell^1$-Besov estimate, we can obtain $\ell^2$-Besov estimate. For any positive number $\varepsilon$,
\[
\|P_N(\partial_t S_i(t)\varphi_i)\|_{L^\infty} \lesssim |t|^{-\frac{3}{2}} D_i(N)\Lambda_{\varepsilon,-\varepsilon}(N)\|P_{N/2} \leq 2N\Lambda_{\varepsilon,-\varepsilon}\varphi_i\|_{L^3},
\]
\[
\|P_N(\partial_t S_i(t)\varphi_i)\|_{L^\infty} \lesssim N^{\frac{3}{2}} \Lambda_{\varepsilon,-\varepsilon}(N)\|P_{N/2} \leq 2N\Lambda_{\varepsilon,-\varepsilon}\varphi_i\|_{L^2},
\]
where $\Lambda_{\alpha,\beta}(N) = \omega_2(N)(1 + N^2)^{\frac{\alpha}{2}}$. Interpolating the above $L^1$ and $L^2$ estimates, and summing with respect to $j$ after squaring, we have for arbitrarily small $\varepsilon$
\[
\|\partial_t S_i(t)\varphi_i\|_{L^\infty} + \|S_i(t)\psi_i\|_{L^\infty} \lesssim (1 + |t|)^{-n\left(\frac{4}{3} - \frac{1}{r}\right)} \|\Lambda_{\varepsilon,-\varepsilon}(\varphi_i, \psi_i)\|_{B^s_{2,r}}.
\]
\end{remark}
Lemma 2.5. Let $2 \leq r \leq \infty$, $s_1 > \frac{3}{r}$ and $s_2 > 2n - \frac{3n}{r}$. Then we have for $i = 1, 2$

$$\left\| \int_0^t T_i(t-t')g(t') dt' \right\|_{L^\infty} \lesssim \int_0^t (1 + |t-t'|)^{-\theta} \|g\|_{B^{s_2}_r} dt'.$$

(2.16)

Moreover, for any $2 \leq r < \infty$ and $s > 0$, we have

$$\left\| \int_0^t T_1(t-t')g(t') dt' \right\|_{B^{s_2}_r} \lesssim \int_0^t |t-t'|^{-\theta} \|g\|_{B^{s_1}_r} dt'.$$

(2.17)

and

$$\left\| \int_0^t T_1(t-t')g(t') dt' \right\|_{B^{s_0}_r} \lesssim \int_0^t |t-t'|^{-\theta} \|g\|_{B^{s_0}_r} dt'.$$

(2.18)

Here $\theta = n(\frac{1}{2} - \frac{1}{r})$.

Proof of Lemma 2.5. Invoking $T_1(t)g = \sin(t\omega_1)\omega_2g$, by the argument in the proof of Lemma 2.4 and Remark 5, we obtain

$$\|T_1(t)g\|_{L^\infty} \leq (1 + |t|)^{-n(\frac{1}{2} - \frac{1}{r})}\left( \|D_1^{1-\frac{2}{r}}\Lambda_{-\varepsilon}\omega_1^{1-\frac{2}{r}}g\|_{B^{s_2}_r} + \|\Lambda_{-\varepsilon}g\|_{B^{s_2}_r} \right).$$

Since we obviously have

$$D_1^{1-\frac{2}{r}}\Lambda_{-\varepsilon}\omega_1^{1-\frac{2}{r}}(N) \sim N^{\frac{2}{r} - \frac{2}{r} - \varepsilon} \text{ for } N \leq 1$$

and

$$D_1^{1-\frac{2}{r}}\Lambda_{-\varepsilon}\omega_1^{1-\frac{2}{r}}(N) \lesssim N^{s_1 - \frac{2}{r}} \text{ for } N \leq 1,$$

if $\varepsilon$ is sufficiently small, we get the first part of (2.16).

For the proof of (2.17), observe from Lemma 2.3 that

$$\|P_N(T_1(t)g)\|_{L^\infty} \lesssim |t|^{-\frac{2}{r}} \|D_1^{1-\frac{2}{r}}(N)\omega_2(N)\|_{P_{n/2} \leq 2N} \|g\|_{L^1}$$

(2.19)

for any dyadic number $N$. In particular, for $N \leq 1$

$$\|P_N(T_1(t)g)\|_{L^\infty} \lesssim |t|^{-\frac{2}{r}} D_1^{1-\frac{2}{r}}(N)\omega_2(N) \|P_{n/2} \leq 2N\|_{L^1}$$

(2.20)

and hence by interpolation with $L^2$ estimate, we obtain

$$\|P_N T_1(t)g\|_{L^r} \lesssim |t|^{-n(\frac{1}{2} - \frac{1}{r})} N^{n(\frac{1}{2} - \frac{1}{r})} \|g\|_{L^r}.$$  (2.21)

Using the estimates (2.19)-(2.21), by the same argument for $\partial_t S_1(t)$ we get the (2.17).

3. Proof of the main results

3.1. Proof of Theorem 1.1. Let us define a nonlinear mapping $N$ on $(X_{p_i}^{s_i, \theta}, d)$ by

$$N(u_i)(t) = (\partial_t S_1(t))\varphi_i + S_1(t)\psi_i + \int_0^t T_i(t-t')f_i(u_i)(t') dt'.$$
where

\[ \mathcal{X}_{p_i}^{s_i, \theta} = \{ u \in L^\infty(\mathbb{R}^{n+1}) \cap L^\infty(\mathbb{R}; H^{s_i}(\mathbb{R}^n)) : \max \left( \text{ess sup}_{t \in \mathbb{R}} (1 + |t|)^\theta \|u(t)\|_{L^\infty}, \text{ess sup}_{t \in \mathbb{R}} \|u(t)\|_{H^{s_i}} \right) \leq p_i \} \]

and \( d(u, v) = \|u - v\|_{L^\infty([R, L^2])} \) for \( u, v \in \mathcal{X}_{p_i}^{s_i, \theta} \). The space \((\mathcal{X}_{p_i}^{s_i, \theta}, d)\) is a complete metric space. To prove this, let \( \{u_j^i\}_{j=1}^\infty \subset \mathcal{X}_{p_i}^{s_i, \theta} \) be a sequence converging to \( u_i \) in \( L^\infty L^2 \). Then by weak-* compactness of \( L^\infty H^{s_i} \), we find a function \( w_i \in \mathcal{X}_{p_i}^{s_i, \theta} \) such that there exists a subsequence \( u_{i_k}^i \) converges to \( w_i \) in \( L^\infty L^2 \) and hence in distribution sense. By the strong convergence of \( u_{i_k}^i \) in \( L^\infty L^2 \), we deduce that \( w_i = u_i \). Since \( s_i > \frac{3}{2} \), \( u_i \in L^\infty(\mathbb{R}^{n+1}) \). Moreover, since \( (1 + |t|)^\theta |\int u_{i_k}^i(t)\phi \, dx| \leq p_i \) for any \( \phi \in C_0^\infty \) and a.e. \( t \in \mathbb{R} \), by the convergence in distribution, we also have \( (1 + |t|)^\theta |\int u_i(t)\phi \, dx| \leq p_i \) for any \( \phi \in C_0^\infty \) and a.e. \( t \in \mathbb{R} \). This implies that \( \text{ess sup}_{t \in \mathbb{R}} (1 + |t|)^\theta \|u_i(t)\|_{L^\infty} \leq p_i \). This proves the completeness of metric space \((\mathcal{X}_{p_i}^{s_i, \theta}, d)\), \( \theta = \frac{3}{2} \).

Fixing \( r, s_i \) and \( p_i \), satisfying the condition stated in Theorem 1.1, we prove that for sufficiently small \( p_i \), \( \mathcal{N} \) is a contraction mapping from \((\mathcal{X}_{p_i}^{s_i, \theta}, d)\) to \((\mathcal{X}_{p_i}^{s_i, \theta}, d)\), \( \theta = \frac{3}{2} \).

For this purpose, let us introduce a generalized chain and Leibniz rules (see Lemma A1 \sim \text{Lemma} A4 in Appendix of [18] and also [8, 11]).

**Lemma 3.1.** For any \( s \) with \( 0 \leq s \leq p_i \), we have

\[
\|D^s f_i(u)\|_{L^\infty} \lesssim \|u\|_{L_{(p_i, -1), r_1}}^{r_1 - 1} \|D^s u\|_{L^{r_2}},
\]

\[
\left( \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}, \quad r_1 \in (1, \infty), r_2 \in (1, \infty) \right)
\]

\[
\|D^s (uv)\|_{L^\infty} \lesssim \|D^s u\|_{L^{r_1}} \|v\|_{L^{r_2}} + \|u\|_{L^{q_1}} \|D^s v\|_{L^{r_2}},
\]

\[
\left( \frac{1}{r} = \frac{1}{r_1} + \frac{1}{q_2} = \frac{1}{q_1} + \frac{1}{r_2}, \quad r_1 \in (1, \infty), q_i \in (1, \infty), \quad i = 1, 2 \right)
\]

Now from Lemmas 2.4 and 2.5, we have

\[
\|\mathcal{N}(u_i)(t)\|_{L^\infty} \lesssim (1 + |t|)^{-\theta} \|\varphi_i, \psi_i\|_{H^{\frac{3}{2}}} + \int_0^t (1 + |t - t'|)^{-\theta} \|f_i(u_i)(t')\|_{B^{s_i}_{r_2, r_2}} \, dt'
\]
for any \( r \in (2, \infty) \) and \( s_1 > \frac{n}{r} \), \( s_2 > 2n - \frac{3n}{r} \), where \( \theta = n(\frac{1}{2} - \frac{1}{r}) \). Then since \( p_i \geq s_i \) and \( H^s_r \rightarrow B^s_r \) for \( 1 < r \leq 2 \), the generalized chain rule (3.1) gives us

\[
\|\mathcal{N}(u_i)(t)\|_{L^\infty} \lesssim (1 + |t|)^{-\theta} \|\varphi_i, \psi_i\|_{\mathcal{S}_r^s} + \left| \int_0^t (1 + |t - t'|)^{-\theta} \|f\|_{W^{s_i, r'}} \, dt' \right|
\]

\[
\lesssim (1 + |t|)^{-\theta} \|\omega_i^{-1} u_i\|_{H^{s_i}} + \left| \int_0^t (1 + |t - t'|)^{-\theta} \|u_i\|_{L^\infty}^{-1} \|u_i\|_{H^{s_i}} \, dt' \right|
\]

\[
\lesssim (1 + |t|)^{-\theta} \|\omega_i^{-1} u_i\|_{H^{s_i}} + \left| \int_0^t (1 + |t - t'|)^{-\theta} \|u_i\|_{L^\infty}^{-1} \|u_i\|_{H^{s_i}} \, dt' \right|
\]

\[
\lesssim (1 + |t|)^{-\theta} + \left| \int_0^t (1 + |t - t'|)^{-\theta} \|u_i\|_{L^\infty}^{-1} \|u_i\|_{H^{s_i}} \, dt' \right|
\]

Now for the last integral we use the estimate (see [39]) that if \( a, b \geq 0 \) and \( \max(a, b) > 1 \), then

\[
\left| \int_0^t (1 + |t - t'|)^{-a} (1 + |t'|)^{-b} \, dt' \right| \lesssim (1 + |t|)^{-\min(a, b)} \tag{3.3}
\]

(in case that \( 0 \leq a < 1 \) and \( b > 1 \), the same estimate as (3.3) also holds for \( |t - t'|^{-a} \) instead of \( (1 + |t - t'|)^{-a} \)). Since \( (p_i - \frac{n}{r}) \theta > \max(1, \theta) \) for \( p_i > \frac{n}{r} + \max(1, \frac{1}{r}) \), we have for sufficiently small \( \delta \) and \( \rho_i \)

\[
\text{ess sup}_{t \in \mathbb{R}} (1 + |t|)^{\theta} \|\mathcal{N}(u_i)(t)\|_{L^\infty} \lesssim \delta + \rho_i \theta^r \leq \frac{\rho_i}{2}. \tag{3.4}
\]

Plancherel’s theorem shows for sufficiently small \( \delta \) and \( \rho \) that

\[
\|\mathcal{N}(u_i)(t)\|_{H^{s_i}} \lesssim \|\varphi_i\|_{H^{s_i}} + \|\omega_i^{-1} \psi_i\|_{H^{s_i}} + \left| \int_0^t \|f_i(u_i)\|_{H^{s_i}} \, dt' \right|
\]

\[
\lesssim \delta + \left| \int_0^t \|u_i\|_{L^\infty}^{-1} \|u_i\|_{H^{s_i}} \, dt' \right|
\]

\[
\lesssim \delta + \rho_i \left| \int_0^t (1 + |t'|)^{-\theta} \, dt' \right|
\]

\[
\lesssim \delta + \rho_i \left| \int_0^t (1 + |t'|)^{-(p_i - \frac{n}{r})\theta} \, dt' \right|
\]

\[
\lesssim \delta + \rho_i \leq \frac{\rho_i}{2},
\]

since \( (p_i - 1)\theta > 1 \). Therefore, combining (3.4) and (3.5), we deduce that \( \mathcal{N} \) maps \( \mathcal{X}_{p_i}^{s_i, \theta} \) to \( \mathcal{X}_{p_i}^{s_i, \theta} \).

Now for any \( u_i, v_i \in \mathcal{X}_{p_i}^{s_i, \theta} \) we can show from the chain rule (3.1) and Leibniz rule (3.2) that if \( \delta \) and \( \rho_i \) are sufficiently small, then

\[
\|\mathcal{N}(u_i) - \mathcal{N}(v_i)\|_{L^2} \lesssim \left| \int_0^t \|f_i(u_i) - f_i(v_i)\|_{L^2} \, dt' \right|
\]

\[
\lesssim \left| \int_0^t \left( \|u_i\|_{L^\infty}^{-1} \|\tilde{u}_i\|_{L^\infty}^{-1} + \|v_i\|_{L^\infty}^{-1} \|\tilde{v}_i\|_{L^\infty}^{-1} \right) \|u_i - v_i\|_{L^2} \, dt' \right|
\]

\[
\lesssim \rho_i \left| \int_0^t (1 + |t'|)^{-(p_i - \frac{n}{r})\theta} \, dt' \right|
\]

\[
\lesssim \rho_i \left| \int_0^t (1 + |t'|)^{-(p_i - \frac{n}{r})\theta} \, dt' \right|
\]
Thus for small \( \rho \), \( \mathcal{N} \) becomes a contraction mapping. The uniqueness follows immediately from the contraction mapping argument. The time continuity of the solution \( u(t) \) follows from the standard argument and we omit it. This completes the proof of Theorem 1.1.

3.2. Proof of Theorem 1.2. We have only to prove that the nonlinear functional \( \mathcal{N} \) defined in the previous section is a contraction mapping from \( (Y_{s,p}^{ρ+1}, d) \) to itself for some \( s \) and \( p \). Here

\[
Y_{ρ}^{s,p+1} = \{ v \in L^{∞}(\mathbb{R}; H_{ρ}^{s+1}) : \text{ess sup}_{t \in \mathbb{R}} (1 + |t|^{θ} \|v(t)\|_{H_{ρ+1}^{s+1}} \leq ρ) \},
\]

\( θ = n \left( \frac{1}{2} - \frac{1}{p+1} \right) \) and \( d \) is the metric on \( Y_{s,p}^{ρ+1} \) defined by \( d(u, v) = \| (1 + |t|)^{θ}(u - v) \|_{L^{∞}L^{p+1}} \). Then by the same argument in the proof of Theorem 1.1, one easily show that \( (Y_{s,p}^{ρ+1}, d) \) is a complete metric space.

Fixing \( s \) and \( p \) satisfying the conditions in Theorem 1.2, since \( B_{s, r}^{p+1} \hookrightarrow H_{s}^{r} \) for \( 2 \leq r < ∞ \), from Lemma 2.4 and Lemma 2.5 we have for any \( u \in Y_{s,p}^{ρ+1} \)

\[
\|N(u)\|_{W^{s,r+1}} \lesssim (1 + |t|)^{-θ} \|((Φ, Ψ))\|_{B_{s, r+1}^{p+1}} + \int_{0}^{t} |t - t'|^{-θ} \|f(u)\|_{B_{s, r+1}^{p+1}} dt'
\]

\[
\lesssim (1 + |t|)^{-θ} δ + \int_{0}^{t} |t - t'|^{-θ} \|u\|_{H_{ρ+1}^{s+1}} dt'
\]

\[
\lesssim (1 + |t|)^{-θ} δ + ρ^{p} \int_{0}^{t} |t - t'|^{-θ} (1 + |t'|)^{-p θ} dt'.
\]

Here we used the chain rule (3.1) with \( s > 0 \), \( r = \frac{p+1}{p} \), \( r_{1} = \frac{p+1}{p-1} \) and \( r_{2} = p + 1 \) for the second inequality. Since \( p θ = pn(\frac{1}{2} - \frac{1}{p+1}) > 1 \) for \( p > γ(n) = 1 + 8/(\sqrt{n^2 + 12n + 4} + n - 2) \) and \( θ = n(\frac{1}{2} - \frac{1}{p+1}) \) for \( p < α(n) \), using (3.3), we have for sufficiently small \( δ \) and \( ρ \)

\[
\text{ess sup}_{t \in \mathbb{R}} (1 + |t|)^{θ} \|N(u)(t)\|_{H_{ρ+1}^{s}} \lesssim δ + ρ^{p} \lesssim ρ.
\]

Thus \( \mathcal{N} \) maps \( Y_{ρ}^{s,p+1} \) to itself.

Now for any \( u, v \in Y_{ρ}^{s,p+1} \) we have

\[
\|N(u) - N(v)\|_{L^{p+1}} \lesssim \int_{0}^{t} |t - t'|^{-θ} \|f(u) - f(v)\|_{B_{s, r+1}^{p+1}} dt'.
\] (3.6)
Using the fact $L^\frac{n+1}{p} \hookrightarrow B^0_{\frac{n+1}{p}, 2}$, one can see that
\[ \|f(u) - f(v)\|_{B^0_{\frac{n+1}{p}, 2}} = \left\| \int_0^1 f'((1 - \lambda)u + \lambda v) d\lambda (u - v) \right\|_{B^0_{\frac{n+1}{p}, 2}} \]
\[ \leq \int_0^1 \|f'((1 - \lambda)u + \lambda v)(u - v)\|_{B^0_{\frac{n+1}{p}, 2}} d\lambda \]
\[ \lesssim \int_0^1 \|f'((1 - \lambda)u + \lambda v)\|_{L^\frac{n+1}{p}} \|u - v\|_{L^{p+1}} d\lambda \]
\[ \lesssim \left( \|u\|_{L^{p+1}}^{p-1} + \|v\|_{L^{p+1}}^{p-1} \right) \|u - v\|_{L^{p+1}}. \]

Substituting this into (3.6), we obtain
\[ \|\mathcal{N}(u) - \mathcal{N}(v)\|_{L^{p+1}} \lesssim \rho^{p-1} d(u, v) \left( \int_0^t |t - t'|^{-\theta(1 + |t'|)^{\gamma}} dt' \right) \]
\[ \lesssim (1 + |t|)^{-\theta} \rho^{p-1} d(u, v). \]

If $\rho$ is sufficiently small, then the above two estimates show that $\mathcal{N}$ is contraction mapping. This completes the proof of Theorem 1.2.

**Remark 6.** If $1 \leq n \leq 4$, $s$ is arbitrarily close to and greater than $\frac{n}{p+1}$ and $\rho$ is sufficiently small, then by Sobolev embedding $H^s \hookrightarrow L^{\infty}$, we have the following estimate
\[ \|u(t)\|_{H^s} \lesssim \|((\varphi, \omega^{-1}\psi))\|_{H^s} + \left\| \int_0^t \|f(u)\|_{H^s} dt' \right\| \]
\[ \lesssim \|((\varphi, \omega^{-1}\psi))\|_{H^s} + \left\| \int_0^t \|u\|_{L^{p+1}}^{p-1} \|u\|_{H^s} dt' \right\| \]
\[ \lesssim \|((\varphi, \omega^{-1}\psi))\|_{H^s} + \rho^{p-1} \left( \int_0^t (1 + |t'|)^{-\theta(1 + |t'|)^{\gamma}} dt' \right) \|u\|_{L^{\infty} H^s} \]
\[ \lesssim \|((\varphi, \omega^{-1}\psi))\|_{H^s} + \frac{1}{2} \|u\|_{L^{\infty} H^s}. \]

Hence we deduce that the solution $u$ is in $C(\mathbb{R}; H^s)$, provided $\gamma(n) < p < \alpha(n)$.

### 3.3. Proof of Theorem 1.3.

Let us define functions $\varphi_i^\pm$ and $\psi_i^\pm$, $i = 1, 2$ by
\[ \varphi_i^\pm(\xi) = \hat{\varphi}_i(\xi) - \int_{\mp\infty}^{\pm\infty} \omega_2(\xi) \sin(t' \omega_1(\xi)) f_2(u_i)(\xi, t') dt', \]
\[ \psi_i^\pm(\xi) = \hat{\psi}_i(\xi) + \int_{\mp\infty}^{\pm\infty} \omega_1(\xi) \cos(t' \omega_2(\xi)) f_1(u_i)(\xi, t) dt, \]
where $(\varphi_1, \psi_1)$ and $(\varphi_2, \psi_2)$ are the initial data stated in Theorems 1.1, and $\tilde{\omega}_1(\xi) = |\xi|^2$ and $\tilde{\omega}_2(\xi) = \omega_2^2$. From the regularity of solution $u_i$, we clearly have $(\varphi_i^\pm, \psi_i^\pm) \in H^{s_i} \times \omega_i H^{s_i}$.

Now let $u_i^\pm$ be the solution to the linear problems (1.1) and (1.2) with $f_i = 0$ and with initial data $(\varphi_i^\pm, \psi_i^\pm)$. Then it can be represented by
\[ u_i^\pm(x, t) = (\partial_t S_i(t) \varphi_i)(x) + (S_i(t) \psi_i)(x) + \int_0^{\pm\infty} T_i(t - t') f_i(u_i(t')) dt'. \]
Now we have from Lemma 3.2
\[
\|u_i(\cdot, t) - u_i^\pm(\cdot, t)\|_{H^{p_i}} \lesssim \left| \int_t^{\pm\infty} \|f_i(u_i(\cdot, t'))\|_{H^{p_i}} \, dt' \right| \\
\lesssim \rho^{p_i} \left| \int_t^{\pm\infty} (1 + |t'|)^{-(p_i-1)\theta} \, dt' \right| \\
= O(|t|^{-(p_i-1)\theta+1})
\]
as \(t \to \pm \infty\).

Similarly, we can define \((\varphi^\pm, \psi^\pm) \in H^{\frac{2}{p+1}} \times \omega_1 H^{\frac{2}{p+1}}\). If \(u\) and \(u^\pm\) be the solutions of (1.1) and its linearized equation (i.e. \(f_1 = 0\)), respectively, then we have
\[
\|u(\cdot, t) - u^\pm(\cdot, t)\|_{H^{p+1}_{p+1}} \lesssim \left| \int_t^{\pm\infty} \|f(u)(t')\|_{B^{p+1}_{p+1}} \, dt' \right| \\
\lesssim \rho^{\gamma} \left| \int_t^{\pm\infty} (1 + |t'|)^{-(p-1)\theta} \, dt' \right| \\
= O(|t|^{-(p-1)\theta+1})
\]
as \(t \to \pm \infty\). This proves the theorem.

Remark 7. In view of Remark 6, we can also obtain the scattering in \(H^s\) for \(s > \frac{n}{p+1}\), provided \(s\) is arbitrarily close to \(\frac{n}{p+1}\), \(\gamma(n) < p < \alpha(n)\) and \(1 \leq n \leq 4\).

4. Non-existence of asymptotically free solutions

In this section, we study the non-existence of asymptotically free solution, following the same strategy of [7] which is based on the argument of Barab [2] and Glassey [12, 13]. See also [29, 35].

**Theorem 4.1.** Assume that \(1 < p_i \leq 2\) for \(n = 1\) and \(1 < p_i < 1 + \frac{2}{n}\) for \(n \geq 2\). Suppose that there exists \(c > 0\) such that \(f_i(u_i)u_i \geq c|u_i|^{p_i+1}\). Let \(u_1\) and \(u_2\) be solutions to (1.1) and (1.2), respectively, with \((u_1, \partial_t u_1) \in C \cap L^\infty(\mathbb{R}; DL^2 \times D^2L^2)\) and \((\varphi_1^\pm, \psi_1^\pm) \in DL^2 \times D^2L^2\) be a pair of smooth functions with compact Fourier supports. Suppose that
\[
\|u_i(t) - u_i^\pm(t)\|_{L^2} = O(|t|^{-\varepsilon}) \quad \text{as} \quad t \to \pm \infty
\tag{4.1}
\]
for some \(\varepsilon > 0\), where \(u_i^\pm\) are the free solutions to the linear problem (1.1) and (1.2) with \(f_i = 0\). Then \(u_i = u_i^\pm = 0\).

The compact support condition of \((\varphi_1^\pm, \psi_1^\pm)\) in the Fourier space may be replaced by the space decay condition. See Remark 8.

**Proof.** Let us define a bilinear form \(H(u, v)(t)\) by
\[
H(u, v)(t) = \text{Re} \int_{\mathbb{R}^n} \left(D^{-1}\partial_t v(t)D^{-1}u(t) - D^{-1}\partial_t u(t)D^{-1}v(t)\right) \, dx.
\]
Then \(H(u, v)(t)\) is well-defined and uniformly bounded on \(t \in \mathbb{R}\) for \((u, \partial_t u), (v, \partial_t v) \in C \cap L^\infty(\mathbb{R}; DL^2 \times D^2L^2)\).
We assume that \((\varphi^+_i, \psi^+_i) \neq (0, 0)\) and derive a contradiction to the uniform boundedness of \(H\). For the simplicity we will consider only positive time and hence asymptotically free solution \(u^+_i\). Suppose that there are non-zero functions \(u_i\) and \(u^+_i\) satisfying the condition of Theorem 4.1. Then by using the regularization of \(u_i\) and \(u^+_i\) (if necessary) we obtain

\[
\frac{d}{dt}H(u_i, u^+_i)(t) = \int f_i(u_i)u^+_i \, dx. \tag{4.2}
\]

Let \(H(u_i, u^+_i)(t) = H_i(t)\). Then from the condition \(f_i(u) \geq c|u|^{p_i+1}\), we deduce that

\[
\frac{d}{dt}H_i(t) = \int (f_i(u_i) - f_i(u^+_i))u^+_i \, dx + \int f_i(u^+_i)u^+_i \, dx \\
\geq \int (f_i(u_i) - f_i(u^+_i))u^+_i \, dx + c \int |u^+_i|^{p_i+1} \, dx.
\]

We will prove that if \(t\) is sufficiently large,

\[
\|u^+_i(t)\|_{L^{p_i+1}(|x| \leq A\varepsilon)} \geq c_0 t^{-n/2 + 1} \tag{4.3}
\]

for some positive constant \(A\) and \(c_0\) depending on \(\varphi^+_i\) and \(\psi^+_i\) and \(\beta > 1\) depending on \(\varepsilon\) stated in the theorem. If not specified, every constant depends on \(\varepsilon\).

For the proof of (4.3), we first show that

\[
\|u^+_i(t)\|_{L^2(|x| \leq A\varepsilon)} \geq 1 \quad \text{for sufficiently large } t. \tag{4.4}
\]

By H"older inequality, (4.3) follows from (4.4). To obtain (4.4), let us choose a cut off function \(\chi_0\) supported in the unit ball \(B(0, 1)\) such that

\[
\|u^+_i(t)\|_{L^2(|x| \leq A\varepsilon)}^2 = t^n \|u^+_i(t, \cdot)\|_{L^2(|x| \leq A\varepsilon)}^2 \geq t^n \|\chi_0(\cdot/M)u^+_i(t, \cdot)\|_{L^2}^2,
\]

where \(M = A\varepsilon^{-1}\). For the last integral, we have

\[
t^n \|\chi_0(\cdot/M)u^+_i(t, \cdot)\|_{L^2}^2 = t^n \|\chi_0(\cdot/M)(\partial_\xi S_i(t)\varphi^+_i)(t)\|_{L^2}^2 + t^n \|\chi_0(\cdot/M)(S_i(t)\psi^+_i)(t)\|_{L^2}^2 + 2t^n \text{Re} \int (\chi_0(x/M))^2(\partial_\xi S_i(t)\varphi^+_i)(tx)(\overline{S_i(t)\psi^+_i})(tx) \, dx. \tag{4.5}
\]

By change of variable and Plancherel’s theorem, we have for the first term

\[
t^n \|\chi_0(\cdot/M)(\partial_\xi S_i(t)\varphi^+_i)(t)\|_{L^2}^2 = \left\|\chi_0(\cdot/M)\mathcal{F}^{-1} \left( \cos(t\omega_i(\xi/t)) t^{-\frac{n}{2}} \varphi^+_i(\cdot/t) \right) \right\|_{L^2}^2.
\]

From the identity \(\cos^2 x = \frac{1 + \cos(2x)}{2}\), we deduce that

\[
\left\|\cos(t\omega_i(\xi/t)) t^{-\frac{n}{2}} \varphi^+_i(\cdot/t) \right\|_{L^2}^2 = \int \cos^2 (t\omega_i(\xi/t)) t^{-n} |\varphi^+_i(\xi/t)|^2 \, d\xi = \frac{1}{2} \|\varphi^+_i\|_{L^2}^2 + \frac{1}{2} \int \cos (2t\omega_i(\xi/t)) t^{-n} |\varphi^+_i(\xi/t)|^2 \, d\xi.
\]

(4.3) follows.
By the integration by parts in the radial direction such that
\[
\int \partial_p f(\xi)g(\xi) \, d\xi = -(n-1) \int \frac{f(\xi)g(\xi)}{\rho} \, d\xi - \int f(\xi)\partial_p g(\xi) \, d\xi,
\]
\[
\rho = |\xi|, \quad \partial_p = \frac{\xi}{\rho} \cdot \nabla,
\]
we have
\[
\int \cos(2t\omega_1(\xi/t)) \, t^{-n} |\widehat{\varphi_1^+}(\xi/t)|^2 \, d\xi \
= \int \partial_p (\sin(2t\omega_1(\xi/t))) (\partial_p (2t\omega_1(\xi/t)))^{-1} t^{-n} |\widehat{\varphi_1^+}(\xi/t)|^2 \, d\xi \
= -\frac{n-1}{t^n} \int \frac{\sin(2t\omega_1(\xi/t))}{\rho} (\partial_p (2t\omega_1(\xi/t)))^{-1} |\widehat{\varphi_1^+}(\xi/t)|^2 \, d\xi
\]
\[
- \frac{1}{t^n} \int \sin(2t\omega_1(\xi/t)) \partial_p ((\partial_p (2t\omega_1(\xi/t)))^{-1} |\widehat{\varphi_1^+}(\xi/t)|^2) \, d\xi.
\]
(4.6)

Since
\[
(\partial_p (2t\omega_1(\xi/t)))^{-1} = \frac{\sqrt{1 + \rho^2/t^2}}{2(1 + 2\rho^2/t^2)}.
\]
\[
(\partial_p (2t\omega_2(\xi/t)))^{-1} = \frac{1}{2} (1 + \rho^2/t^2)^2,
\]
it follows from the Hölder inequality that
\[
\int \cos(2t\omega_1(\xi/t)) \, t^{-n} |\widehat{\varphi_1^+}(\xi/t)|^2 \, d\xi = O(t^{-1}) \quad \text{as} \quad t \to \infty
\]
and hence
\[
\left\| \cos(t\omega_1(\xi/t)) t^{-\frac{n}{2}} \widehat{\varphi_1^+}(\cdot/t) \right\|_{L^2} \to \frac{1}{\sqrt{2}} \left\| \varphi_1^+ \right\|_{L^2} \quad \text{as} \quad t \to \infty. \tag{4.7}
\]

Now we claim that there exist large numbers $t_0$ such that
\[
\inf_{t > t_0} \|\chi_0(-/M) (\partial S(t) \varphi_1^+)(t)\|_{L^2} \gtrsim 1. \tag{4.8}
\]
For the proof of (4.8), we may assume that $\|\varphi_1^+\|_{L^2} = 1$. Let us define a function $g_t(x)$ by $t^n \|\partial S(t) \varphi_1^+ (tx)\|^2$. Then from (4.7), we can find a positive number $t_0$ such that $\int g_t(x) \, dx \geq \frac{1}{4}$ for all $t > t_0$. By integration by parts, we get for $x \neq 0$ and multi index $\alpha$ with $|\alpha| = m > \frac{n}{2}$
\[
t^{\frac{n}{2}} (\partial S(t) \varphi_1^+)(tx) = \frac{1}{(2\pi)^n t^\frac{n}{2}} \int e^{ix \xi} \cos(t\omega_1(\xi/t)) \widehat{\varphi_1^+}(\xi/t) \, d\xi
\]
\[
= \frac{1}{(2\pi)^n t^\frac{n}{2} (-i\xi)^\alpha} \int e^{ix \xi} \partial_\xi^\alpha \left( \cos(t\omega_1(\xi/t)) \widehat{\varphi_1^+}(\xi/t) \right) \, d\xi.
\]
By Hölder’s inequality we have for a fixed number $s_0 > \frac{n}{2}$
\[
g_t(x) \lesssim \frac{t^n}{|x|^{2m}} \sum_{|\alpha| \leq m} \|x^{\alpha} \varphi_1^+\|_{H^{s_0}}. \tag{4.9}
\]
This gives us that
\[ \int (\chi_0(x/M))^2 g_t(x) \, dx = \int g_t(x) \, dx - \int (1 - (\chi_0^2(x/M))^2) g_t(x) \, dx \]
\[ \geq \frac{1}{4} \int_{|x| \geq \frac{1}{2} M} \frac{A t^n}{|x|^{2m}} \, dx \]
\[ \geq \frac{1}{4} - O(t^{-2m-(2m-n)\beta}) \quad \text{as} \quad t \to \infty, \]
where \( M = At^{\beta-1} \) and \( A \sim \sum_{|n| \leq m} \| \chi_n \varphi^+_n \|_{H^0} \).

Now if we choose \( m \) and \( \beta \) so that \( 2m - (2m-n)\beta < 0 \), then the claim (4.8) is proved, provided \( t_0 \) is sufficiently large.

Similarly we see that
\[ \left\| \frac{\sin(t\omega_i(\xi/t))}{\omega_i(\xi/t)} t^{-\frac{n}{2}} \widehat{\varphi^+_i}(\xi/t) \right\|_{L^2}^2 \to \frac{1}{\sqrt{2}} \| \omega_i^{-1} \varphi^+_i \|_{L^2}^2 \]
as \( t \to \infty \) and hence by the same argument as above, we have the estimate
\[ t^n \| \chi_0(-/M)(S_i(t)\varphi^+_i)(t) \|_{L^2}^2 \geq 1, \quad (4.10) \]
if \( t > t_0 \) for some large \( t_0 \).

Finally, for the last term of (4.5) let us consider the integral
\[ I(t) = t^n \int (\partial_t S_i(t)\varphi^+_i)(tx)S_i(t)\varphi^+_i(tx) \, dx. \]
Then by change of variable and Plancherel’s theorem, \( I(t) \) is converted by
\[ \frac{1}{(2\pi)^n} \int \frac{\sin(2t\omega_i(\xi/t))}{\omega_i(\xi/t)} \widehat{\varphi^+_i}(\xi/t) \widehat{\varphi_i}(\xi/t) \, d\xi. \]
Here we also used the identity \( \cos x \sin x = \frac{1}{2} \sin 2x \). Similarly to the estimate (4.6), we have \( I(t) = O(t^{-1}) \). With this estimate we prove that
\[ \left| 2t^n \Re \int (\chi_0(x/M))^2 (\partial_t S_i(t)\varphi^+_i)(tx)(S_i(t)\varphi^+_i)(tx) \, dx \right| \to 0 \quad \text{as} \quad t \to \infty. \quad (4.11) \]
Actually, by the integration by parts as above, we have
\[ \left| 2t^n \Re \int (\chi_0(x/M))^2 (\partial_t S_i(t)\varphi^+_i)(tx)(S_i(t)\varphi^+_i)(tx) \, dx \right| \]
\[ \leq |2\Re I(t)| + \int_{|x| \geq \frac{1}{2} M} \frac{A t^n}{|x|^{2m}} \, dx \to 0 \]
as \( t \to \infty \), where \( M = At^{\beta-1} \).

Therefore (4.11) together with (4.8) and (4.10) yields the lower bound estimate (4.4) and hence (4.3).

Since \( \varphi^+_i \in DL^2 \) and \( \psi^+_i \in D^2L^2 \) have compact Fourier supports, it follows from the proof of Lemmas 2.4 that for all \( 2 \leq r \leq \infty \)
\[ \| u^+_i(t) \|_{L^r} \lesssim t^{-n(\frac{1}{r} - \frac{1}{2})}, \quad (4.12) \]
From the estimate (4.12) and the hypothesis (4.1), we have for $1 < p_i \leq 2$,
\[
\left| \int (f_i(u_i) - f_i(u_i^+))u_i^+ \, dx \right| \\
\lesssim \left( \|u_i\|_{L^2}^{p_i-1} \|u_i^+\|_{L^2}^{2-p_i} + \|u_i^+\|_{L^2} \right) \|u_i^+\|_{L^{p_i-1}}^{p_i-1} \|u_i - u_i^+\|_{L^2} \\
= O(t^{-\frac{2}{2}(p_i-1)-\varepsilon}).
\] (4.13)
Thus choosing $\beta > 1$ such as
\[
\frac{n\beta}{2} < \frac{n(p_i-1)}{2} \quad \text{and} \quad \frac{n\beta(p_i-1)}{2} \leq 1,
\]
we conclude from (4.3) that $\frac{d}{dt} H(t) \gtrsim t^{-1}$ for large $t$. This is a contradiction to the uniform boundedness of $H$.

\textbf{Remark 8.} In the above proof, we chose $\beta$ such that
\[
\frac{2m}{2m-n} < \beta < \min \left( 1 + \frac{2\varepsilon}{n(p_i-1)} , \frac{2}{n(p_i-1)} \right).
\]
This choice is possible because $p_i$ is assumed to be smaller than $1 + \frac{2}{n}$. If $\varepsilon$ is smaller or $p_i$ is closer to $1 + \frac{2}{n}$, then $m$ should be larger. Hence from (4.9), the data $(\varphi_i^+, \psi_i^+)$ should decay fast at space infinity.

\textbf{References}


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