ON SMALL AMPLITUDE SOLUTIONS TO THE GENERALIZED BOUSSINESQ EQUATIONS

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Abstract. We study the existence and scattering of global small amplitude solutions to generalized Boussinesq (Bq) and improved modified Boussinesq (imBq) equations with nonlinear term $f(u)$ behaving as a power u^p as $u \to 0$ in $\mathbb{R}^n, n \geq 1$.

1. Introduction and main results

In this paper, we consider the following Cauchy problems for the generalized Boussinesq (Bq) and improved modified Boussinesq (imBq) equations:

$$\begin{aligned} &\partial_t^2 u_1 - \Delta u_1 + \Delta^2 u_1 = \Delta f_1(u_1), \quad (x,t) \in \mathbb{R}^{n+1}, \\ &u_1(x,0) = \varphi_1(x), \quad \partial_t u_1(x,0) = \psi_1(x), \quad x \in \mathbb{R}^n, \end{aligned} \tag{1.1}$$

$$\begin{aligned} &\partial_t^2 u_2 - \Delta u_2 - \Delta \partial_t^2 u_2 = \Delta f_2(u_2), \quad (x,t) \in \mathbb{R}^{n+1}, \\ &u_2(x,0) = \varphi_2(x), \quad \partial_t u_2(x,0) = \psi_2(x), \quad x \in \mathbb{R}^n, \end{aligned} \tag{1.2}$$

where u_i is a real-valued function of $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, $\partial_t = \partial/\partial t$, Δ is the Laplacian in $\mathbb{R}^n$, and $f_i \in C^k(\mathbb{R})$ satisfies the estimates $|f_i^{(l)}(v)| \lesssim |v|^{p_i - l}$ for $0 \leq l \leq k \leq p_i$ and $p_i > 1$, $i = 1, 2$. We denote by $u_i(t)$ the function $x \mapsto u_i(x,t)$.

By Duhamel's principle, partial differential equations (1.1) and (1.2) are rewritten as the integral equations

$$u_i(t) = \partial_t S_i(t)\varphi_i + S_i(t)\psi_i + \int_0^t T_i(t-t') f_i(u_i)(t')\, dt'. \tag{1.3}$$

Here the operators are defined as

$$\partial_t S_i(t) = \cos(t\omega_i(D)), \quad S_i(t) = \frac{\sin(t\omega_i(D))}{\omega_i(D)},$$

$$T_1(t) = S_1(t)\Delta, \quad T_2(t) = S_2(t)(1-\Delta)^{-1}\Delta, \quad T_i(t) = \sin(t\omega_i(D))\omega_2(D),$$

where

$$D = (-\Delta)^{\frac{1}{2}} = \mathcal{F}^{-1}|\xi|\mathcal{F}, \quad \omega_i(D) = \omega_i = \mathcal{F}^{-1}\omega_i(\xi)\mathcal{F},$$

$$\omega_1(\xi) = |\xi|\sqrt{1+|\xi|^2}, \quad \omega_2(\xi) = \frac{|\xi|}{\sqrt{1+|\xi|^2}},$$

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and

$$\mathcal{F}(\varphi)(\xi) = \widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)\,dx \ \text{ and } \ \mathcal{F}^{-1}(\varphi)(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\varphi(\xi)\,d\xi$$

are the Fourier transform and inverse Fourier transform of φ, respectively.

The equations (1.1) was first derived to describe shallow water waves by Boussinesq [5] and it was modified to (1.2) to describe ion-sound waves in plasma by Makhankov [26, 27]. The equations (1.1) and (1.2) also cover another various physical phenomena such as the dynamics of stretched string [30, 9], Fermi-Pasta-Ulam problems [10], the evolution of long internal waves of moderate amplitude [1], nonlinear Alvén waves [27] and so on.

Our main concern is to establish the global existence and scattering of small amplitude solution to the Cauchy problems (1.1) and (1.2). The local and global existence to the Cauchy problem was established by Bona and Sachs [4], Tsutsumi and Matahashi [37], Linares [21], and Wang and Chen [38]. The stability of solitary waves or the energy conservation was the basic tool of the existence results. For further results on the finite time blowup, stability and instability of solitary waves, and so on see [17, 32, 20, 40, 4, 25, 27, 16, 28] and the references therein.

For the global existence and scattering of small amplitude solutions, it is necessary to study the dispersion of the operators $\partial_t S_i$, S_i and T_i with respect to time, and to compare them with nonlinearity, especially to compare the time decay rate with power p. To get a time decay dispersive estimate, Linares [21], and Linares and Scialom [22] used the estimate[1] $\left|\int_{\mathbb{R}} e^{i(x\xi+t\omega_1(\xi))}|\omega_1''(\xi)|^{\frac12}\,d\xi\right| \lesssim (1+|t|)^{-\frac12}$, Liu [24] the estimate[2] $\left|\int_{\mathbb{R}} e^{i(x\xi+t\omega_1(\xi))}\,d\xi\right| \lesssim |t|^{-\frac12}+|t|^{-\frac13}$, and Liu [23] and Wang and Chen[39] the estimate[3] $\left|\int_{\varepsilon<|\xi|<1} e^{i(x\xi+t\omega_2(\xi))}\,d\xi\right| \lesssim |t|^{-\frac12}\varepsilon^{-\frac12}$.

The best result up to now is $p > 2+\sqrt{7}$ of Liu [24] for (1.1) with $n=1$, and $p > \frac92$ of Cho and Ozawa [7] for (1.2) with $n=1$, and integer p greater than $2+\frac{1}{\theta(n,s)}$ of Wang and Chen [39] for (1.2) with $n \ge 2$, where $\theta = \frac{2s-n}{2(2s+2+n)}$ if $\frac n2 < s \le \frac{9n}{2}$ and $\theta = \frac{2n}{5n+1}$ if $s \ge \frac{9n}{2}$.

In this paper, we improve all the known results under some vanishing condition of initial data at the zero frequency in one dimensional case and extend the results not only on existence and scattering but dispersive estimates to the high dimensional case. Moreover, we also provide a non-existence of nontrivial asymptotically free solutions in the case of small power p, which is a high dimensional version of Theorem 1.3 of [7].

Before stating the main results, let us introduce some notations. First we let $\beta_r = 1-\frac2r$. Then we define a homogeneous initial data space $\dot{\mathfrak{D}}^{s,\,i}_{r',\,q}$ for $2 \le r \le \infty$

[1]This estimate was proved by Kenig, Ponce and Vega [19].

[2]The decay rate $\frac13$ comes from the estimate of low frequency part ($|\xi| \le 1$) and it turns out to be optimal. See (1.4).

[3]Actually, Wang and Chen in [39] obtained n-dimensional estimate but their estimate was the same as 1-dimensional one because they integrated only in radial radial direction by using spherical coordinate.

and $i = 1, 2$ by

$$\dot{\mathfrak{D}}^{s,\,i}_{r',\,q} = (D_i^{-\beta_r}\dot{B}^s_{r',\,q} \cap \dot{B}^{s+n\beta_r}_{r',\,q}) \times \omega_i(D_i^{-\beta_r}\dot{B}^s_{r',\,q} \cap \dot{B}^{s+n\beta_r}_{r',\,q}),$$

where $D_i = \mathcal{F}^{-1}[D_i(\xi)]\mathcal{F}$ and

$$D_1(\xi) = \omega_2(\xi)^{\frac{n-2}{2}}, \quad D_2(\xi) = (1+|\xi|^2)^n\omega_2(\xi)^{\frac{n-2}{2}}.$$

The norm of the space $\dot{\mathfrak{D}}^{s,\,i}_{r',\,q}$ is given by

$$\begin{aligned}&\|(\varphi_i, \psi_i)\|_{\dot{\mathfrak{D}}^{s,\,i}_{r',\,q}} \\ &\equiv \|D_i^{\beta_r}\varphi_i\|_{\dot{B}^s_{r',\,q}} + \|\varphi_i\|_{\dot{B}^{s+n\beta_r}_{r'\,q}} + \|D_i^{\beta_r}\omega_i^{-1}\psi_i\|_{\dot{B}^s_{r',\,q}} + \|\omega_i^{-1}\psi\|_{\dot{B}^{s+n\beta_r}_{r'\,q}}.\end{aligned}$$

The inhomogeneous initial data space $\mathfrak{D}^{s,\,i}_{r',\,q}$ and its norm are defined by the inhomogeneous Besov space $B^s_{r',\,q}$ instead of $\dot{B}^s_{r',\,q}$.

Here we used the notation $v \in \omega_i^\alpha D_i^\beta D^\gamma X$ to mean $\omega^{-\alpha}D_i^{-\beta}D^{-\gamma}v \in X$ for a function space X and some real number α, β, γ.

To define the Besov space, let us choose a Littlewood-Paley function η with and define a frequency projection operator P_N for a dyadic number N by

$$P_N\phi(x) = \mathcal{F}^{-1}\left[\eta\left(\frac{\xi}{N}\right)\widehat{\phi}\right](x).$$

Then the homogeneous Besov space $\dot{B}^s_{r,\,q}, 1 \le r, q \le \infty, s \in \mathbb{R}$, is defined by

$$\dot{B}^s_{r\,q} = \left\{v \in \mathcal{S}'/\mathcal{P} : \|v\|_{\dot{B}^s_{r,\,q}} = \left(\sum_{N:\text{ dyadic}} N^{sq}\|P_N(v)\|^q_{L^r}\right)^{\frac{1}{q}} < \infty\right\},$$

where $\mathcal{P}$ is the set of all polynomials on $\mathbb{R}^n$. The inhomogeneous Besov space $B^s_{r,\,q}$ is defined by

$$B^s_{r,\,q} = \left\{v \in \mathcal{S}' : \|v\|_{B^s_{r,\,q}} = \|P_0v\|_{L^r} + \left(\sum_{N\ge1} N^{sq}\|P_N(v)\|^q_{L^r}\right)^{\frac{1}{q}} < \infty\right\},$$

where $P_0 = 1 - \sum_{N\ge1}P_N$. If $s > 0$, then $B^{s,r}_q \sim L^r \cap \dot{B}^{s,\,r}_q$. See for instance [3].

The above initial data space is necessary for the dispersive estimate of the operators ∂_tS_i, S_i and T_i. In particular, we obtain

$$\|(\partial_tS_i, S_i)\|_{\mathfrak{D}^{s,\,i}_{r',\,1}\to L^\infty} \lesssim (1+|t|)^{-n(\frac{1}{2}-\frac{1}{r})}$$

for any $r \in [2, \infty]$. If $r = \infty$, then the time decay rate is the best possible decay $\frac{n}{2}$. Since $\omega_1(\xi)$ and $\omega_2(\xi)$ are not phase of elliptic type (in fact, ω_i behaves like D for small frequency, ω_2 like identity for large frequency), to achieve the full time decay rate we need the regularity $\dot{B}^s_{r',\,1} \times \omega_i\dot{B}^s_{r',\,1}$ for high frequency and the operator $D_i^{\beta_r}$ for small frequency.

If $n = 1$, then $D_i^{\beta_r} \sim D^{-\frac{\beta_r}{2}}$ for small frequency. This means that for the time decay it is necessary to assume that the Fourier transforms of initial data vanish at zero frequency. If we want to remove this vanishing condition, we cannot help but allow a slow time decay estimate. For the results obtained without vanishing condition, see [7, 21, 22, 23, 24]. In those papers the time decay rate is $\frac{1}{3}$ and this decay rate seems to be optimal because for large t and for some $\phi \in C_0^\infty(-1,1)$

$$\left|\int e^{i(x\xi + t\omega_i(\xi))}\phi\, d\xi\right| \sim |t|^{-\frac{1}{3}}. \tag{1.4}$$

The time decay comes from the bound $|\omega_i^{(3)}(\xi)| \gtrsim 1$ for small ξ and the stationary phase estimate (Proposition 3 of [34], p. 334). Therefore the vanishing condition seems to be inevitable for the faster decay than the rate $\frac{1}{3}$.

The additional regularity $\dot{B}_{r',1}^{s+n\beta_r} \times \omega_i \dot{B}_{r',1}^{s+n\beta_r}$ is necessary for the boundedness of linear dispersive estimate at time zero and high frequency estimate for ω_2. For details, see Lemma 2.4 below.

Now let us introduce the main results. The first result is

Theorem 1.1. *Let $2 < r < \infty$, $s_1 > \frac{n}{r'}$, $s_2 > 2n - \frac{3n}{r}$, and $\theta = \frac{n\beta_r}{2}$. Let $p_i \ge s_i$, $p_i > \frac{2}{r'} + \max\left(1, \frac{1}{\theta}\right)$. Then there exists $\delta > 0$ such that for any $(\varphi_i, \psi_i) \in \dot{\mathfrak{D}}_{r',1}^{\frac{n}{r},i} \cap (H^{s_i} \times \omega_i H^{s_i})$ with*

$$\|(\varphi_i,\psi_i)\|_{\dot{\mathfrak{D}}_{r',1}^{\frac{n}{r},i}} + \|\varphi_i\|_{H^{s_i}} + \|\omega_i^{-1}\psi_i\|_{H^{s_i}} \le \delta$$

there exist unique solutions $u_i \in C(\mathbb{R}; H^{s_i})$ to (1.1) and (1.2). Moreover, there exists a positive number ρ_i depending only on n, r, s_i, p_i and δ such that

$$\sup_{t\in\mathbb{R}} (1+|t|)^{\theta}\|u_i(t)\|_{L^\infty} + \sup_{t\in\mathbb{R}}\|u_i(t)\|_{H^{s_i}} \le \rho_i.$$

Remark 1. Theorem 1.1 is applicable to the cases $p_i > 4$ for $n = 1$, $p_1 > 3, p_2 > 4$ for $n = 2$ and $p_1 > n, p_2 > 2n$ for $n \ge 3$. This improves the results in [7, 24, 39].

The condition $p_i \ge s_i$ comes from the nonlinear estimates such as $\|f_i(u_i)\|_{H^{s_i}} \lesssim \|u_i\|_{L^\infty}^{p_i-1}\|u_i\|_{H^{s_i}}$ for which p_i should be greater than equal to s_i. If p_i is an integer, then the condition is unnecessary from the arguments in [39].

Next, we consider the equation (1.1). Let $\gamma(n) = 1 + 8/(\sqrt{n^2+12n+4} + n - 2)$, and $\alpha(n) = \infty$ if $n = 1, 2$ and $\alpha(n) = \frac{n+2}{n-2}$ if $n \ge 3$. Then we have the following.

Theorem 1.2. *Let $s > 0$ and $\theta = \frac{n\beta_{p+1}}{2}$. If $s \le p$ and $\gamma(n) < p < \alpha(n)$, then there exists $\delta > 0$ such that for any*

$$(\varphi,\psi) \in \mathfrak{D}_{\frac{p+1}{p},2}^{s,1} \quad \text{with} \quad \|(\varphi,\psi)\|_{\mathfrak{D}_{\frac{p+1}{p},2}^{s,1}} \le \delta.$$

there exists a unique solution $u \in C(\mathbb{R}; H_{p+1}^s)$ to (1.1) with $p_1 = p$ and $(u(0), \partial_t u(0)) = (\varphi, \psi)$, and $\rho > 0$ depending only on n, s, p and δ such that

$$\sup_{t\in\mathbb{R}} (1+|t|)^{\theta}\|u(t)\|_{H_{p+1}^s} \le \rho.$$

Remark 2. The critical exponent $\gamma(n)$ naturally arises in the problem of the existence of small amplitude solutions decaying as $O(|t|^{-n(\frac{1}{2}-\frac{1}{r})})$ in L^r as $t \to \infty$ (see [6, 15, 31, 35, 36] for instance).

The result above can be obtained by the fact ω_1 is of elliptic type at high frequency as Schrödinger equation and hence it is possible to obtain the estimate $\|T_1(t)\|_{\dot{B}^s_{r,2}\to\dot{B}^s_{r,2}} \lesssim |t|^{-n(\frac{1}{2}-\frac{1}{r})}$ for any $r\in[2,\infty]$ and $s\ge 0$. See Lemma 2.5 below. This is not the case for ω_2 (hence $T_2(t)$) because the dispersion of (1.2) becomes small at high frequency and hence a higher regularity for data is necessary to compensate for the small dispersion.

If s is close and greater than $\frac{n}{p+1}$, then $\gamma(n)<p<\alpha(n)$ for $1\le n\le 4$ and by Sobolev embedding, $\|u(t)\|_{L^\infty}\lesssim \rho(1+|t|)^{-\theta}$. In this case, for the initial data in H^s, the solution is in $C(\mathbb{R};H^s)$. See Remark 6.

If $n\ge 4$, then from Theorem 1.1 we deduce that $p_1\ge n$ and from Theorem 1.2 that $p_1<\alpha(n)$. There exists a gap between two results. It is still open whether Theorems 1.1 and 1.2 hold for $\alpha(n)\le p_1=p<n$.

Small data scattering follows as a simple consequence of Theorems 1.1 and 1.2.

Theorem 1.3. *Let $u_i, i=1,2$ and u be the solutions of* (1.3) *as in Theorems* 1.1 *and* 1.2, *respectively. Then there exist six pairs of functions $(\varphi_i^\pm,\psi_i^\pm)\in H^{s_i}\times\omega_i H^{s_i}$ and $(\varphi^\pm,\psi^\pm)\in H^s_{\frac{p+1}{p}}\times\omega_1 H^s_{\frac{p+1}{p}}$ such that*

$$\begin{aligned}\|u_i(t)-u_i^\pm(t)\|_{H^{s_i}} &= O(|t|^{-\theta(p_i-1)+1}),\\ \|u(t)-u^\pm(t)\|_{H^s_{\frac{p+1}{p}}} &= O(|t|^{-\theta(p-1)+1})\end{aligned}$$

as $t\to\pm\infty$, where s_i,r,θ,p_i,p are the same numbers stated in Theorems 1.1 *and* 1.2 *and $u_i^\pm$ and $u^\pm$ are the unique solutions to the linear problems* (1.1) *and* (1.2) *with $f_i=0$.*

If $1\le n\le 4$ and s is close to and greater than $\frac{n}{p+1}$, then Sobolev space H^s_{p+1} can be replaced by H^s in the second scattering result. See Remarks 6 and 7 below.

This paper is organized as follow. In Section 2, we prove several linear dispersive estimates for the operators $\partial_t S_i$, S_i and T_i. Utilizing the dispersive estimates, we prove the global existence and scattering results in Section 3. In the final section, Section 4, we consider a non-existence result of asymptotically free solutions for suitably small power p_i.

If not specified, throughout this paper, the notation $A\lesssim B$ and $A\gtrsim B$ denote $A\le CB$ and $A\ge C^{-1}B$, respectively. Positive constants C vary line by line and depend only on r and f. $A\sim B$ means that both $A\lesssim B$ and $A\gtrsim B$ hold.

2. Linear dispersive estimates

In this section, we consider time decay estimates for $\partial_t S, S$ and T. We first denote the frequency localization operator $P_{\le\varepsilon_0}$, $P_{\ge N_0}$ and $P_{\varepsilon_0<\cdot<N_0}$ by

$$P_{\le\varepsilon_0}\phi=\sum_{N\le\varepsilon}P_N\phi,\quad P_{\ge N_0}\phi=\sum_{N\ge N_0}P_N\phi,\quad P_{\varepsilon_0<\cdot<N_0}\phi=\sum_{\varepsilon<N<N_0}P_N\phi.$$

For the convenience of presentation, we choose η so that $P_N=P_{N/2\le\cdot\le 2N}P_N$.

Next let us introduce a lemma on the stationary phase estimate (see Proposition 5 of [34], p. 342).

Lemma 2.1. *Let χ be a smooth function supported in a unit ball of $\mathbb{R}^n, n \ge 2$ and Ω be a $C^3(\mathbb{R}^n)$ function such that $|\nabla^2\Omega| \ge 1$ on the support of χ. Then we have for any $\lambda > 0$*

$$\left|\int_{\mathbb{R}^n} e^{i\lambda\Omega(\xi)}\chi(\xi)\,d\xi\right| \le C\lambda^{-\frac{n}{2}}(\|\chi\|_{L^\infty} + \|\nabla\chi\|_{L^1}), \tag{2.1}$$

where $|\nabla^2\Omega|^2 = \sum_{i,j}|\partial_i\partial_j\Omega|^2$ and the constant C depends only on Ω and n, which is bounded if Ω is bounded in the norm of C^3.

Remark 3. If $n = 1$, then the support condition in a unit ball of the above lemma can be removed and (2.1) is rewritten as if $|\Omega''(\xi)| \ge 1$ on a fixed interval (a, b) and χ is supported in $[a, b]$, then

$$\left|\int_a^b e^{i\lambda\Omega(\xi)}\chi(\xi)\,d\xi\right| \le C\lambda^{-\frac{1}{2}}(\|\chi\|_{L^\infty(a,b)} + \|\chi'\|_{L^1(a,b)}). \tag{2.2}$$

See Corollary of [34], p. 334.

To apply Lemma 2.1 to the radially symmetric phase Ω, we need the following formulation of the determinant of Hessian matrix of radially symmetric function.

Lemma 2.2. *Let $\omega = \omega(|x|)$ be a radially symmetric C^2 function on $\mathbb{R}^n \setminus \{0\}, n \ge 2$. Then the determinant of Hessian matrix is also radially symmetric and is the following*

$$\det(\nabla^2\omega)(x) = \left(\frac{\omega'(r)}{r}\right)^{n-1}\omega''(r), \quad r = |x|$$

Proof of Lemma 2.2. The (i, j) component of Hessian matrix of ω is given by

$$\partial_i\partial_j\omega(x) = \frac{\omega'(r)}{r}\left(\delta_{ij} - \frac{x_ix_j}{r^2}\right) + \omega''(r)\frac{x_ix_j}{r^2},$$

where $r = |x|$.

If $\omega'(r) = 0$ for some $r > 0$, then since j th column vector of Hessian matrix is $\omega''(r)\frac{x_j}{r^2}x$, obviously the determinant of Hessian is zero. Hence we assume that $\omega'(r)$ is not zero for any $r > 0$.

Let $\lambda = 1 - \frac{r\omega''(r)}{\omega'(r)}$. Then the (i, j) component of Hessian is rewritten by

$$\partial_i\partial_j\omega(x) = \frac{\omega'(r)}{r}\left(\delta_{ij} - \lambda\frac{x_ix_j}{r^2}\right).$$

Hence

$$\det(\nabla^2\omega) = \left(\frac{\omega'(r)}{r}\right)^n \det(A),$$

where $A_{ij} = \delta_{ij} - \lambda\frac{x_ix_j}{r^2}$.

Let h be a function on λ defined by $h(\lambda) = \det(A)$. Then h is a polynomial of degree n on λ rewritten by

$$h(\lambda) = 1 + a_1\lambda + a_2\lambda^2 + \cdots + a_{n-1}\lambda^{n-1} + a_n\lambda^n.$$

Here $a_1 = -\mathrm{tr}(A) = -\sum_j \frac{x_j^2}{r^2} = -1$ and for $2 \le j \le n$

$$\begin{aligned} a_j &= (-1)^j \sum_{i_1<i_2<\cdots<i_j} \det \begin{pmatrix} \frac{x_{i_1}x_{i_1}}{r^2} & \cdots & \frac{x_{i_1}x_{i_j}}{r^2} \\ \vdots & \ddots & \vdots \\ \frac{x_{i_j}x_{i_1}}{r^2} & \cdots & \frac{x_{i_j}x_{i_j}}{r^2} \end{pmatrix} \\ &= (-1)^j \sum_{i_1<i_2<\cdots<i_j} \frac{x_{i_1}x_{i_2}\cdots x_{i_j}}{r^{2n}} \det \begin{pmatrix} x_{i_1} & \cdots & x_{i_1} \\ \vdots & \ddots & \vdots \\ x_{i_j} & \cdots & x_{i_j} \end{pmatrix} \\ &= 0 \end{aligned}$$

(for the formula of $h(\lambda)$ and its coefficients a_j, see [33], p.155-156). Thus $h(\lambda) = 1-\lambda$. Therefore we have

$$\det(\nabla^2\omega)(x) = \left(\frac{\omega'(r)}{r}\right)^n (1-\lambda) = \left(\frac{\omega'(r)}{r}\right)^{n-1} \omega''(r).$$

□

Remark 4. Applying the above lemma to ω_i, we observe from

$$\omega_1''(\rho) = \frac{\rho(3+2\rho^2)}{(1+\rho^2)^{\frac{3}{2}}}, \quad \omega_2''(\rho) = -\frac{3\rho}{(1+\rho^2)^{\frac{5}{2}}} \tag{2.3}$$

that for $i = 1, 2$

$$|\det(\nabla^2\omega_i)(x)|^{-\frac{1}{2}} \sim D_i(\rho) \ \text{ for all } \ |x| = \rho > 0.$$

Utilizing the above two lemmas and Remark 4, we obtain the following dispersive estimate[4].

Lemma 2.3.

$$\sup_{x\in\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t\omega_i(\xi))} \eta\left(\frac{\xi}{N}\right) d\xi \right| \lesssim |t|^{-\frac{n}{2}} D_i(N). \tag{2.4}$$

Proof of Lemma 2.3. If $n = 1$, then it is easily observed from (2.3) that $|\omega_i''(\xi)| \ge cD_i(N)^{-2}$ for any $\xi \in (N/2, 2N)$ and some fixed small constant c. Now by direct application of Remark 3 with $a = \frac{N}{2}$, $b = 2N$, $\lambda = ctD_i(N)^{-2}$, $\chi(\xi) = \eta(\xi/N)$ and $\Omega = c^{-1}t^{-1}D_i(N)^2(x\xi + t\omega_i(\xi))$, one can readily obtain (2.4). Therefore we consider only the case $n \ge 2$ from now on.

From Remark 4, it suffices to show that the left hand side of (2.4) is bounded by a constant multiple of $|t|^{-\frac{n}{2}}|\det(\nabla^2\omega_i)(x)|^{-\frac{1}{2}}$ with $|x| = N$.

By the change of variable $\xi \mapsto N\xi$, we have

$$I = N^n \int e^{itN\Omega_i(\xi)} \eta(\xi)\, d\xi,$$

where $\Omega_i(\xi) = \frac{1}{t}x\cdot\xi + \frac{1}{N}\omega_i(N\xi)$.

[4]The authors heard that recently, Gustafson, Nakanishi and Tsai showed a similar result for the phase ω_1 by another approach in [14].

Fixing (x,t), let us define a function α by

$$\alpha(\xi) \equiv |\nabla\Omega(\xi)| = \left|\frac{x}{t} + \omega_i'(N\rho)\frac{\xi}{\rho}\right|,$$

where $\rho = |\xi|$. Let α_0 be the minimum value of $\alpha(\xi)$ on the annulus $\{\frac{1}{2} \le |\xi| \le 2\}$. Since the set of vectors $\frac{x}{t} + \omega_i'(N\rho)\frac{\xi}{\rho}$ with $\xi \in \{\frac{1}{2} \le |\xi| \le 2\}$ is an annulus centered at $\frac{x}{t}$, the minimum α_0 is attained on the line of direction x. Let ξ_0 be the minimum point. Then ξ_0 has the opposite direction to x and

$$\frac{|x|}{t} = \omega_i'(N\rho_0) \pm \alpha_0, \quad \rho_0 = |\xi_0|.$$

The signs $\pm$ appear when the minimum is attained on the outside sphere of annulus and the inside one, respectively.

Since for any $\frac{1}{2} \le \rho \le 2$

$$\begin{aligned}
&\frac{1}{5}\omega_i'(N) \le \omega_i'(N\rho) \le 5\omega_i'(N),\\
&\frac{1}{5}\omega_i''(N) \le \omega_i''(N\rho) \le 5\omega_i''(N),\\
&|\omega_i^{(k+1)}(N\rho)| \lesssim \frac{\omega_i'(N)}{N^k}, \quad k \ge 1,
\end{aligned} \tag{2.5}$$

if $\alpha_0 > \frac{1}{1000}\omega_i'(N)$, then by integration by parts we have for any $M > 0$

$$|I| \lesssim N^n(|t|N\omega_i'(N))^{-M}.$$

Now setting $M = \frac{n}{2}$, we have from (2.5) and Lemma 2.2

$$\begin{aligned}
|I| &\lesssim |t|^{-\frac{n}{2}}\left(\frac{\omega_i'(N)}{N}\right)^{-\frac{n-1}{2}}\left(\frac{\omega_i'(N)}{N}\right)^{-\frac{1}{2}}\\
&\lesssim |t|^{-\frac{n}{2}}\left(\frac{\omega_i'(N)}{N}\right)^{-\frac{n-1}{2}}|\omega_i''(N)|^{-\frac{1}{2}}\\
&= |t|^{-\frac{n}{2}}|\det(\nabla^2\omega_i)(N)|^{-\frac{1}{2}}.
\end{aligned}$$

From now on, we assume $\alpha_0 \le \frac{1}{1000}\omega_i'(N)$. Let us choose a cut-off function g defined on S^{n-1} and supported on the set $\left\{\frac{\xi}{|\xi|} \in S^{n-1} : \left|\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\right| < \frac{1}{2}\right\}$. Then

$$\begin{aligned}
I &= N^n\int e^{itN\Omega_i(\xi)}g\left(\frac{\xi}{|\xi|}\right)\eta(\xi)\,d\xi + N^n\int e^{itN\Omega_i(\xi)}(1-g\left(\frac{\xi}{|\xi|}\right))\eta(\xi)\,d\xi\\
&\equiv I_1 + I_2.
\end{aligned}$$

We first estimate I_2. To do this, we will use a one dimensional cut-off function h supported in a neighborhood of ρ_0 such that

$$h(\rho) = 1 \ \text{ for } \ |\rho - \rho_0| \le \frac{1}{100}\frac{\omega_i'(N)}{N|\omega_i''(N)|}$$

$$\text{and } \ |h^{(k)}(\rho)| \lesssim \left(\frac{1}{100}\frac{\omega_i'(N)}{N|\omega_i''(N)|}\right)^{-k} \ \text{ for all } \ k \ge 0.$$

If $\xi \in \mathrm{supp}(h)$, then from (2.5)

$$|\omega_i'(N\rho) - \omega_i'(N\rho_0)| \geq \frac{N}{5}|\omega_i''(N)||\rho - \rho_0|,$$

and hence

$$\begin{aligned} |\nabla\Omega_i| &= \left|\omega_i'(N\rho)\frac{\xi}{\rho} - \omega_i'(N\rho_0)\frac{\xi_0}{\rho_0} \pm \alpha_0\right| \geq |\omega_i'(N\rho) - \omega_i'(N\rho_0)| - \alpha_0 \\ &\geq \frac{N}{5}|\omega_i''(N)||\rho - \rho_0| - \alpha_0 \geq \frac{1}{500}\omega_i'(N) - \alpha_0 \\ &\geq \frac{1}{500}\omega_i'(N). \end{aligned} \tag{2.6}$$

Since $\left|\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\right| \geq \frac{1}{2}$ for $\xi \in (\mathrm{supp}(h))^c$ and

$$|\omega_i'(N\rho) - \omega_i'(N\rho_0)| \leq 5N|\omega_i''(N)||\rho - \rho_0|,$$

we have

$$\begin{aligned} |\nabla\Omega_i| &\geq ||\omega_i'(N\rho_0)||\xi' - \xi_0'| - |\omega_i'(N\rho) - \omega_i'(N\rho_0)|| - \alpha_0 \\ &\geq \frac{1}{10}\omega_i'(N) - \frac{1}{20}\omega_i'(N) - \alpha_0 \\ &\geq \frac{1}{500}\omega_i'(N). \end{aligned} \tag{2.7}$$

Thus by integration by parts, we deduce from (2.6), (2.7) and Lemma 2.2 that

$$\begin{aligned} |I_2| &\leq N^n \left| \int e^{itN\Omega_i(\xi)} \left(1 - g\left(\frac{\xi}{|\xi|}\right)\right) h(\rho)\eta(\xi)\, d\xi \right| \\ &\quad + N^n \left| \int e^{itN\Omega_i(\xi)} \left(1 - g\left(\frac{\xi}{|\xi|}\right)\right) (1 - h(\rho))\eta(\xi)\, d\xi \right| \\ &\lesssim N^n \left(|t|N\omega_i'(N)\right)^{-\frac{n}{2}} \\ &= |t|^{-\frac{n}{2}} |\det(\nabla^2\omega_i)(N)|^{-\frac{1}{2}}. \end{aligned}$$

Now it remains to estimate I_1. Let us define a function $\widetilde{\Omega}_i$ by

$$\widetilde{\Omega}_i = \Omega_i - \frac{1}{N}\omega_i(N\rho_0).$$

Then

$$I_1 = N^n e^{it\omega_i(\rho_0)} \int e^{itN\widetilde{\Omega}_i(\xi)} g\left(\frac{\xi}{|\xi|}\right) \eta(\xi)\, d\xi.$$

By the relation $\frac{|x|}{t} = \omega_i'(N\rho_0) \pm \alpha_0$, we have that for $0 \leq k \leq 3$,

$$|\nabla^k \widetilde{\Omega}_2(\xi)| \lesssim 1,$$

$$|\nabla^k \widetilde{\Omega}_1(\xi)| \lesssim 1 \ \text{ if } \ N \leq 1, \qquad \frac{1}{N}|\nabla^k \widetilde{\Omega}_1| \lesssim 1 \ \text{ if } \ N > 1.$$

We define λ case by case as follows:

$$\lambda = tNC_n|\det(\nabla^2\omega_2)(N)|^{\frac{1}{n}} \ \text{ for } \ \widetilde{\Omega}_2,$$

$$\left.\begin{aligned} &\lambda = tNC_n|\det(\nabla^2\omega_1)(N)|^{\frac{1}{n}} \ \text{ if } \ N < 1 \\ &\lambda = tN^2C_n|\det(N^{-1}\nabla^2\omega_1)(N)|^{\frac{1}{n}} \ \text{ if } \ N \geq 1 \end{aligned}\right\} \text{ for } \ \widetilde{\Omega}_1$$

Then using the fact that $|(\nabla^2\omega_i)(N\rho)| \ge c_n|\det(\nabla^2\omega_i)(N)|^{\frac{1}{n}}$ for some small constant c_n depending only on n, from Lemma 2.1 (after decomposing the annulus by finite number of unit balls if necessary), we obtain for I_1 with the phase $\lambda\widetilde{\Omega}_2(\xi)$,

$$|I_1| \lesssim N^n|t|^{-\frac{n}{2}}N^{-\frac{n}{2}}|N^n\det(\nabla^2\omega_2)(N)|^{-\frac{1}{2}} \le |t|^{-\frac{n}{2}}|\det(\nabla^2\omega_2)(N)|^{-\frac{1}{2}},$$

for I_1 with $\lambda\widetilde{\Omega}_1(\xi)$ and $N \le 1$

$$|I_1| \lesssim N^n|t|^{-\frac{n}{2}}N^{-\frac{n}{2}}|N^n\det(\nabla^2\omega_1)(N)|^{-\frac{1}{2}} = |t|^{-\frac{n}{2}}|\det(\nabla^2\omega_1)(N)|^{-\frac{1}{2}},$$

and for I_1 with $\lambda N^{-1}\widetilde{\Omega}_1(\xi)$ and $N > 1$

$$|I_1| \lesssim N^n|t|^{-\frac{n}{2}}N^{-n}|\det(\nabla^2\omega_1)(N)|^{-\frac{1}{2}} = |t|^{-\frac{n}{2}}|\det(\nabla^2\omega_1)(N)|^{-\frac{1}{2}}.$$

These complete the proof of lemma. $\square$

As consequences, we have the following lemmas.

Lemma 2.4. *If $2 \le r \le \infty$, then*

$$\|(\partial_t S_i(t)\varphi_i,\, S_i(t)\psi_i)\|_{L^\infty} \lesssim (1+|t|)^{-\theta}\|(\varphi_i,\psi_i)\|_{\dot{\mathfrak{D}}_{i,1}^{\frac{n}{r},r'}}. \tag{2.8}$$

If $2 \le r < \infty$ and $s \ge 0$, then for $s > 0$

$$\|(\partial_t S_i(t)\varphi_i,\, S_i(t)\psi_i)\|_{B_2^{s,\,r}} \lesssim (1+|t|)^{-\theta}\|(\varphi_i,\psi_i)\|_{\mathfrak{D}_{i,2}^{s,r'}} \tag{2.9}$$

and for $s = 0$

$$\|(\partial_t S_i(t)\varphi_i,\, S_i(t)\psi_i)\|_{\dot{B}_2^{0,\,r}} \lesssim (1+|t|)^{-\theta}\|(\varphi_i,\psi_i)\|_{\dot{\mathfrak{D}}_{i,2}^{0,r'}}. \tag{2.10}$$

Here $\theta = \frac{n\beta_r}{2} = n(\frac{1}{2}-\frac{1}{r})$.

Proof of Lemma 2.4. If $|t| \le 1$, by Hölder's and Hausdorff-Young's inequalities, we have for any $r \in [2,\infty]$

$$\|P_N(\partial_t S_i(t)\varphi_i)\|_{L^\infty} \lesssim N^{\frac{n}{r'}}\|P_{N/2\le\cdot\le 2N}\varphi_i\|_{L^{r'}}.$$

Hence

$$\|\partial_t S_i(t)\varphi_i\|_{L^\infty} \lesssim \|\varphi_i\|_{\dot{B}_1^{\frac{n}{r'},\,r'}}. \tag{2.11}$$

In particular we have

$$\|P_N(\partial_t S_i(t)\varphi_i)\|_{L^\infty} \lesssim N^n\|P_{N/2\le\cdot\le 2N}\varphi_i\|_{L^1}.$$

Interpolating this with trivial L^2 estimate that

$$\|P_N(\partial_t S_i(t)\varphi_i)\|_{L^2} \lesssim \|P_{N/2\le\cdot\le 2N}\varphi_i\|_{L^2},$$

we have for any $r \in [2,\infty]$

$$\|P_N(\partial_t S_i(t)\varphi_i)\|_{L^r} \lesssim N^{n(1-\frac{2}{r})}\|P_{N/2\le\cdot\le 2N}\varphi_i\|_{L^{r'}}.$$

Since by Hölder's inequality

$$\|P_0(\partial_t S_i(t)\varphi_i)\|_{L^r} \lesssim \|\varphi_i\|_{L^{r'}},$$

we have for any $s \in \mathbb{R}$ and $r \in [2, \infty]$

$$\|\partial_t S_i(t)\varphi_i\|_{B_2^{s,r}} \lesssim \|\varphi_i\|_{B_2^{s+n(1-\frac{2}{r}),r'}}. \tag{2.12}$$

If $|t| > 1$, then from Lemma 2.3, it follows that

$$\|P_N(\partial_t S_i(t)\varphi_i)\|_{L^\infty} \lesssim |t|^{-\frac{n}{2}} D_i(N)\|P_{N/2\le\cdot\le 2N}\varphi_i\|_{L^1}. \tag{2.13}$$

By Hölder inequality, we also have

$$\|P_N(\partial_t S_i(t)\varphi_i)\|_{L^\infty} \lesssim N^{\frac{n}{2}}\|P_{N/2\le\cdot\le 2N}\varphi_i\|_{L^2}. \tag{2.14}$$

Interpolating (2.13) and (2.14) and using the estimate (2.11) and the fact that

$$D_i(N/2) \sim D_i(N) \sim D_i(2N),$$

we obtain the estimate (2.8).

On the other hand, using the trivial estimate

$$\|P_N(\partial_t S_i(t)\varphi_i)\|_{L^2} \lesssim \|P_{N/2\le\cdot\le 2N}\varphi_i\|_{L^2}$$

and its interpolation with (2.13), we have

$$\|P_N(\partial_t S_i(t)\varphi_i)\|_{L^r} \lesssim |t|^{-n(\frac{1}{2}-\frac{1}{r})} D_i(N)^{1-\frac{2}{r}}\|P_{N/2\le\cdot\le 2N}\varphi_i\|_{L^{r'}}. \tag{2.15}$$

Thus for $2 \le r < \infty$, we have

$$\begin{aligned}\|\partial_t S_i(t)\varphi_i\|_{L^r} &\lesssim \|\partial_t S_i(t)\varphi_i\|_{\dot B_2^{0,r}} \lesssim |t|^{-n(\frac{1}{2}-\frac{1}{r})}\|D_i^{1-\frac{2}{r}}\varphi_i\|_{\dot B_2^{0,r'}}\\ &\lesssim |t|^{-n(\frac{1}{2}-\frac{1}{r})}\|D_i^{1-\frac{2}{r}}\varphi_i\|_{L^{r'}}.\end{aligned}$$

Hence for $s > 0$

$$\|\partial_t S_i(t)\varphi_i\|_{B_2^{s,r}} \lesssim |t|^{-n(\frac{1}{2}-\frac{1}{r})}\|D_i^{1-\frac{2}{r}}\varphi_i\|_{B_2^{s,r'}}$$

and also for $s = 0$

$$\|\partial_t S_i(t)\varphi_i\|_{\dot B_2^{0,r}} \lesssim |t|^{-n(\frac{1}{2}-\frac{1}{r})}\|D_i^{1-\frac{2}{r}}\varphi_i\|_{\dot B_2^{0,r'}}.$$

Combining these estimates and (2.12), we get (2.9). □

Remark 5. Letting $\Lambda_{\alpha,\beta} = \omega_2^\alpha(1-\Delta)^{\frac{\beta}{2}}$, instead of ℓ^1-Besov estimate, we can obtain ℓ^2-Besov estimate. For any positive number ε,

$$\begin{aligned}&\|P_N(\partial_t S_i(t)\varphi_i)\|_{L^\infty} \lesssim |t|^{-\frac{n}{2}} D_i(N)\Lambda_{\varepsilon,-\varepsilon}(N)\|P_{N/2\le\cdot\le 2N}\Lambda_{-\varepsilon,\varepsilon}\varphi_i\|_{L^1},\\ &\|P_N(\partial_t S_i(t)\varphi_i)\|_{L^\infty} \lesssim N^{\frac{n}{2}}\Lambda_{\varepsilon,-\varepsilon}(N)\|P_{N/2\le\cdot\le 2N}\Lambda_{-\varepsilon,\varepsilon}\varphi_i\|_{L^2},\end{aligned}$$

where $\Lambda_{\alpha,\beta}(N) = \omega_2(N)(1+N^2)^{\frac{\beta}{2}}$. Interpolating the above L^1 and L^2 estimates, and summing with respect to j after squaring, we have for arbitrarily small ε

$$\|\partial_t S_i(t)\varphi_i\|_{L^\infty} + \|S_i(t)\psi_i\|_{L^\infty} \lesssim (1+|t|)^{-n(\frac{1}{2}-\frac{1}{r})}\|\Lambda_{-\varepsilon,\varepsilon}(\varphi_i,\psi_i)\|_{\dot{\mathfrak{D}}_{i,2}^{\frac{n}{r},r'}}.$$

Lemma 2.5. *Let $2 \le r \le \infty$, $s_1 > \frac{n}{r'}$ and $s_2 > 2n - \frac{3n}{r}$. Then we have for $i = 1, 2$*

$$\Big\|\int_0^t T_i(t-t')g(t')\,dt'\Big\|_{L^\infty} \lesssim \int_0^t (1+|t-t'|)^{-\theta}\|g\|_{B_2^{s_i,\,r'}}\,dt'. \tag{2.16}$$

Moreover, for any $2 \le r < \infty$ and $s > 0$, we have

$$\Big\|\int_0^t T_1(t-t')g(t')\,dt'\Big\|_{B_2^{s,\,r}} \lesssim \int_0^t |t-t'|^{-\theta}\|g\|_{B_2^{s,\,r'}}\,dt' \tag{2.17}$$

and

$$\Big\|\int_0^t T_1(t-t')g(t')\,dt'\Big\|_{\dot B_2^{0,\,r}} \lesssim \int_0^t |t-t'|^{-\theta}\|g\|_{\dot B_2^{0,\,r'}}\,dt'. \tag{2.18}$$

Here $\theta = n(\frac12 - \frac1r)$.

Proof of Lemma 2.5. Invoking $T_i(t)g = \sin(t\omega_i)\omega_2 g$, by the argument in the proof of Lemma 2.4 and Remark 5, we obtain

$$\begin{aligned}&\|T_i(t)g\|_{L^\infty}\\ &\le (1+|t|)^{-n(\frac12-\frac1r)}\|\left(\|D_i^{1-\frac2r}\Lambda_{-\varepsilon,\varepsilon}\omega_2^{1-\frac2r}g\|_{\dot B_2^{\frac{n}{r},\,r'}} + \|\Lambda_{-\varepsilon,\varepsilon}g\|_{\dot B_2^{\frac{n}{r'},r'}}\right).\end{aligned}$$

Since we obviously have

$$\begin{aligned}D_i^{1-\frac2r}\Lambda_{-\varepsilon,\varepsilon}\omega_2^{1-\frac2r}(N) &\sim N^{\frac n2-\frac nr-\varepsilon} \ \text{ for } \ N \le 1\\ \text{and } \ D_i^{1-\frac2r}\Lambda_{-\varepsilon,\varepsilon}\omega_2^{1-\frac2r}(N) &\lesssim N^{s_i-\frac nr} \ \text{ for } \ N \le 1,\end{aligned}$$

if ε is sufficiently small, we get the first part of (2.16).

For the proof of (2.17), observe from Lemma 2.3 that

$$\begin{aligned}\|P_N(T_1(t)g)\|_{L^\infty} &\lesssim |t|^{-\frac n2}D_1(N)\omega_2(N)\|P_{n/2\le\cdot\le 2N}g\|_{L^1}\\ &\lesssim |t|^{-\frac n2}\|P_{n/2\le\cdot\le 2N}g\|_{L^1}\end{aligned} \tag{2.19}$$

for any dyadic number N. In particular, for $N \le 1$

$$\begin{aligned}\|P_N(T_1(t)g)\|_{L^\infty} &\lesssim |t|^{-\frac n2}D_1(N)\omega_2(N)\|P_{n/2\le\cdot\le 2N}g\|_{L^1}\\ &\lesssim |t|^{-\frac n2}N^{\frac n2}\|g\|_{L^1}\end{aligned} \tag{2.20}$$

and hence by interpolation with L^2 estimate, we obtain

$$\|P_N T_1(t)g\|_{L^r} \lesssim |t|^{-n(\frac12-\frac1r)}N^{n(\frac12-\frac1r)}\|g\|_{L^r}. \tag{2.21}$$

Using the estimates (2.19)-(2.21), by the same argument for $\partial_t S_i(t)$ we get the (2.17). □

3. Proof of the main results

3.1. **Proof of Theorem 1.1.** Let us define a nonlinear mapping $\mathcal N$ on $(\mathcal X_{\rho_i}^{s_i,\theta}, d)$ by

$$\mathcal N(u_i)(t) = (\partial_t S_i(t))\varphi_i + S_i(t)\psi_i + \int_0^t T_i(t-t')f_i(u_i)(t')dt',$$

where

$$\mathcal{X}_{\rho_i}^{s_i,\theta} = \{u \in L^\infty(\mathbb{R}^{n+1}) \cap L^\infty(\mathbb{R}; H^{s_i}(\mathbb{R}^n)) : \max\left(\operatorname*{ess\,sup}_{t\in\mathbb{R}}(1+|t|)^\theta \|u(t)\|_{L^\infty},\ \operatorname*{ess\,sup}_{t\in\mathbb{R}} \|u(t)\|_{H^{s_i}}\right) \le \rho_i\}$$

and $d(u,v) = \|u-v\|_{L^\infty(\mathbb{R};\,L^2)}$ for $u, v \in \mathcal{X}_{\rho_i}^{s_i,\theta}$. The space $(\mathcal{X}_{\rho_i}^{s_i,\theta}, d)$ is a complete metric space. To prove this, let $\{u_i^j\}_{j=1}^\infty \subset \mathcal{X}_\rho^{s_i,\theta}$ be a sequence converging to u_i in $L^\infty L^2$. Then by weak-$*$ compactness of $L^\infty H^{s_i}$, we find a function $w_i \in \mathcal{X}_\rho^{s_i,\theta}$ such that there exists a subsequence $u_i^{j_k}$ converges to w_i in weak$*$ in $L^\infty L^2$ and hence in distribution sense. By the strong convergence of $u_i^{j_k}$ in $L^\infty L^2$, we deduce that $w_i = u_i$. Since $s_i > \frac{n}{2}$, $u_i \in L^\infty(\mathbb{R}^{n+1})$. Moreover, since $(1+|t|)^\theta |\int u_i^{j_k}(t)\phi\,dx| \le \rho_i$ for any $\phi \in C_0^\infty$ and a.e. $t \in \mathbb{R}$, by the convergence in distribution, we also have $(1+|t|)^\theta |\int u_i(t)\phi\,dx| \le \rho_i$ for any $\phi \in C_0^\infty$ and a.e. $t \in \mathbb{R}$. This implies that $\operatorname{ess\,sup}_{t\in\mathbb{R}}(1+|t|)^\theta \|u_i(t)\|_{L^\infty} \le \rho_i$. This proves the completeness of metric space $(\mathcal{X}_{\rho_i}^{s_i,\theta}, d)$.

Fixing r, s_i and p_i satisfying the condition stated in Theorem 1.1, we prove that for sufficiently small ρ_i, $\mathcal{N}$ is a contraction mapping from $(\mathcal{X}_{\rho_i}^{s_i,\theta}, d)$ to $(\mathcal{X}_{\rho_i}^{s_i,\theta}, d)$, $\theta = \frac{n\beta_r}{2}$.

For this purpose, let us introduce a generalized chain and Leibniz rules (see Lemma A1 $\sim$ Lemma A4 in Appendix of [18] and also [8, 11]).

Lemma 3.1. *For any s with $0 \le s \le p_i$, we have*

$$\|D^s f_i(u)\|_{L^r} \lesssim \|u\|_{L^{(p_i-1)r_1}}^{p_i-1} \|D^s u\|_{L^{r_2}}, \tag{3.1}$$
$$\left(\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}, \quad r_1 \in (1,\infty], r_2 \in (1,\infty)\right)$$
$$\|D^s(uv)\|_{L^r} \lesssim \|D^s u\|_{L^{r_1}} \|v\|_{L^{q_2}} + \|u\|_{L^{q_1}} \|D^s v\|_{L^{r_2}}. \tag{3.2}$$
$$\left(\frac{1}{r} = \frac{1}{r_1} + \frac{1}{q_2} = \frac{1}{q_1} + \frac{1}{r_2}, \quad r_i \in (1,\infty), q_i \in (1,\infty], \quad i = 1,2\right)$$

Now from Lemmas 2.4 and 2.5, we have

$$\|\mathcal{N}(u_i)(t)\|_{L^\infty} \lesssim (1+|t|)^{-\theta} \|(\varphi_i,\psi_i)\|_{\dot{\mathfrak{D}}_{r',1}^{\frac{n}{r},i}} + \left|\int_0^t (1+|t-t'|)^{-\theta} \|f_i(u_i)(t')\|_{B_{r',2}^{s_i}}\, dt'\right|$$

for any $r \in (2, \infty)$ and $s_1 > \frac{n}{r'}$, $s_2 > 2n - \frac{3n}{r}$, where $\theta = n(\frac{1}{2} - \frac{1}{r})$. Then since $p_i \ge s_i$ and $H_r^s \hookrightarrow B_{r,2}^s$ for $1 < r \le 2$, the generalized chain rule (3.1) gives us

$$\begin{aligned}
\|\mathcal{N}(u_i)(t)\|_{L^\infty} &\lesssim (1+|t|)^{-\theta}\|(\varphi_i, \psi_i)\|_{\dot{\mathfrak{D}}_{i,1}^{\frac{n}{r},r'}} + \left|\int_0^t (1+|t-t'|)^{-\theta}\|f\|_{W^{s_i,r'}}\,dt'\right| \\
&\lesssim (1+|t|)^{-\theta}\delta + \left|\int_0^t (1+|t-t'|)^{-\theta}\|u_i\|_{L^{\frac{2(p_i-1)r}{r-2}}}^{p_i-1}\|u_i\|_{H^{s_i}}\,dt'\right| \\
&\lesssim (1+|t|)^{-\theta}\delta + \left|\int_0^t (1+|t-t'|)^{-\theta}\|u_i\|_{L^\infty}^{p_i-\frac{2}{r'}}\|u\|_{H^{s_i}}^{\frac{2}{r'}}\,dt'\right| \\
&\lesssim (1+|t|)^{-\theta}\delta + \rho^{p_i}\left|\int_0^t (1+|t-t'|)^{-\theta}(1+|t'|)^{(p-\frac{2}{r'})\theta}\,dt'\right|.
\end{aligned}$$

Now for the last integral we use the estimate (see [39]) that if $a, b \ge 0$ and $\max(a,b) > 1$, then

$$\left|\int_0^t (1+|t-t'|)^{-a}(1+|t'|)^{-b}\,dt'\right| \lesssim (1+|t|)^{-\min(a,b)} \tag{3.3}$$

(in case that $0 \le a < 1$ and $b > 1$, the same estimate as (3.3) also holds for $|t-t'|^{-a}$ instead of $(1+|t-t'|)^{-a}$). Since $\left(p_i - \frac{2}{r'}\right)\theta > \max(1,\theta)$ for $p_i > \frac{2}{r'} + \max\left(1, \frac{1}{\theta}\right)$, we have for sufficiently small δ and ρ_i

$$\operatorname*{ess\,sup}_{t\in\mathbb{R}}(1+|t|)^\theta\|\mathcal{N}(u_i)(t)\|_{L^\infty} \lesssim \delta + \rho_i^{p_i} < \frac{\rho_i}{2}. \tag{3.4}$$

Plancheral's theorem shows for sufficiently small δ and ρ that

$$\begin{aligned}
\|\mathcal{N}(u_i)(t)\|_{H^{s_i}} &\lesssim \|\varphi_i\|_{H^{s_i}} + \|\omega_i^{-1}\psi_i\|_{H^{s_i}} + \left|\int_0^t \|f_i(u_i)\|_{H^{s_i}}\,dt'\right| \\
&\lesssim \delta + \left|\int_0^t \|u_i\|_{L^\infty}^{p_i-1}\|u_i\|_{H^{s_i}}\,dt'\right| \\
&\lesssim \delta + \rho^{p_i}\left|\int_0^t (1+|t'|)^{-(p_i-1)\theta}\,dt'\right| \\
&\lesssim \delta + \rho^{p_i} \le \frac{\rho_i}{2},
\end{aligned} \tag{3.5}$$

since $(p_i - 1)\theta > 1$. Therefore, combining (3.4) and (3.5), we deduce that $\mathcal{N}$ maps $\mathcal{X}_{\rho_i}^{s_i,\theta}$ to $\mathcal{X}_{\rho_i}^{s_i,\theta}$.

Now for any $u_i, v_i \in \mathcal{X}_{\rho_i}^{s_i,\theta}$ we can show from the chain rule (3.1) and Leibniz rule (3.2) that if δ and ρ_i are sufficiently small, then

$$\begin{aligned}
\|\mathcal{N}(u_i) - \mathcal{N}(v_i)\|_{L^2} &\lesssim \left|\int_0^t \|f_i(u_i) - f_i(v_i)\|_{L^2}\,dt'\right| \\
&\lesssim \left|\int_0^t \left(\|u_i\|_{L^{2(p_i-1)}}^{p_i-1} + \|v_i\|_{L^{2(p_i-1)}}^{p_i-1}\right)\|u_i - v_i\|_{L^2}\,dt'\right| \\
&\lesssim \rho^{p_i-1}d(u_i,v_i)\left|\int_0^t (1+|t'|)^{-(p_i-\frac{2}{r'})\theta}\,dt'\right| \\
&\lesssim \rho^{p_i-1}d(u_i,v_i).
\end{aligned}$$

Thus for small ρ_i, $\mathcal{N}$ becomes a contraction mapping. The uniqueness follows immediately from the contraction mapping argument. The time continuity of the solution $u(t)$ follows fro the standard argument and we omit it. This completes the proof of Theorem 1.1.

3.2. **Proof of Theorem 1.2.** We have only to prove that the nonlinear functional $\mathcal{N}$ defined in the previous section is a contraction mapping from $(\mathcal{Y}_\rho^{s,p+1}, d)$ to itself for some s and p. Here

$$\mathcal{Y}_\rho^{s,p+1} = \{v \in L^\infty(\mathbb{R}; H^s_{p+1}) : \operatorname*{ess\,sup}_{t\in\mathbb{R}} (1+|t|)^\theta \|v(t)\|_{H^s_{p+1}} \le \rho\},$$

$\theta = n\left(\frac{1}{2} - \frac{1}{p+1}\right)$ and d is the metric on $\mathcal{Y}_\rho^{s,p+1}$ defined by $d(u,v) = \|(1+|t|)^\theta (u - v)\|_{L^\infty L^{p+1}}$. Then by the same argument in the proof of Theorem 1.1, one easily show that $(\mathcal{Y}_\rho^{s,p+1}, d)$ is a complete metric space.

Fixing s and p satisfying the conditions in Theorem 1.2, since $B^s_{r,2} \hookrightarrow H^s_r$ for $2 \le r < \infty$, from Lemma 2.4 and Lemma 2.5 we have for any $u \in \mathcal{Y}_\rho^{s,p+1}$

$$\begin{aligned}
\|\mathcal{N}(u)\|_{W^{s,\,p+1}} &\lesssim (1+|t|)^{-\theta} \|(\varphi,\psi)\|_{\mathfrak{D}^{s,1}_{\frac{p+1}{p},2}} + \left| \int_0^t |t-t'|^{-\theta} \|f(u)\|_{B^s_{\frac{p+1}{p},2}}\, dt' \right| \\
&\lesssim (1+|t|)^{-\theta}\delta + \left| \int_0^t |t-t'|^{-\theta} \|u\|^p_{H^s_{p+1}}\, dt' \right| \\
&\lesssim (1+|t|)^{-\theta}\delta + \rho^p \left| \int_0^t |t-t'|^{-\theta} (1+|t'|)^{-p\,\theta}\, dt' \right|.
\end{aligned}$$

Here we used the chain rule (3.1) with $s > 0$, $r = \frac{p+1}{p}$, $r_1 = \frac{p+1}{p-1}$ and $r_2 = p+1$ for the second inequality. Since $p\,\theta = pn(\frac{1}{2} - \frac{1}{p+1}) > 1$ for $p > \gamma(n) = 1 + 8/(\sqrt{n^2+12n+4} + n - 2)$ and $\theta = n(\frac{1}{2} - \frac{1}{p+1}) < 1$ for $p < \alpha(n)$, using (3.3), we have for sufficiently small δ and ρ

$$\operatorname*{ess\,sup}_{t\in\mathbb{R}} (1+|t|)^\theta \|\mathcal{N}(u)(t)\|_{H^s_{p+1}} \lesssim \delta + \rho^p \le \rho.$$

Thus $\mathcal{N}$ maps $\mathcal{Y}_\rho^{s,p+1}$ to itself.

Now for any $u, v \in \mathcal{Y}_\rho^{s,p+1}$ we have

$$\|\mathcal{N}(u) - \mathcal{N}(v)\|_{L^{p+1}} \lesssim \left| \int_0^t |t-t'|^{-\theta} \|f(u) - f(v)\|_{\dot{B}^0_{\frac{p+1}{p},2}}\, dt' \right|. \tag{3.6}$$

Using the fact $L^{\frac{p+1}{p}} \hookrightarrow \dot{B}^0_{\frac{p+1}{p},2}$, one can see that

$$
\begin{aligned}
\|f(u)-f(v)\|_{\dot{B}^0_{\frac{p+1}{p},2}} &= \left\| \int_0^1 f'(\lambda u+(1-\lambda)v)\,d\lambda(u-v)\right\|_{\dot{B}^0_{\frac{p+1}{p},2}} \\
&\le \int_0^1 \|f'(\lambda u+(1-\lambda)v)(u-v)\|_{\dot{B}^0_{\frac{p+1}{p},2}}\,d\lambda \\
&\lesssim \int_0^1 \|f'(\lambda u+(1-\lambda)v)\|_{L^{\frac{p+1}{p-1}}}\|u-v\|_{L^{p+1}}\,d\lambda \\
&\lesssim \left(\|u\|_{L^{p+1}}^{p-1}+\|v\|_{L^{p+1}}^{p-1}\right)\|u-v\|_{L^{p+1}}.
\end{aligned}
$$

Substituting this into (3.6), we obtain

$$
\begin{aligned}
\|\mathcal{N}(u)-\mathcal{N}(v)\|_{L^{p+1}} &\lesssim \rho^{p-1}d(u,v)\left|\int_0^t |t-t'|^{-\theta}(1+|t'|)^{-p\theta}\,dt'\right| \\
&\lesssim (1+|t|)^{-\theta}\rho^{p-1}d(u,v).
\end{aligned}
$$

If ρ is sufficiently small, then the above two estimates show that $\mathcal{N}$ is contraction mapping. This completes the proof of Theorem 1.2.

Remark 6. If $1\le n\le 4$, s is arbitrarily close to and greater than $\frac{n}{p+1}$ and ρ is sufficiently small, then by Sobolev embedding $H^s_{p+1}\hookrightarrow L^\infty$, we have the following estimate

$$
\begin{aligned}
\|u(t)\|_{H^s} &\lesssim \|(\varphi,\omega_1^{-1}\psi)\|_{H^s} + \left|\int_0^t \|f(u)\|_{H^s}\,dt'\right| \\
&\lesssim \|(\varphi,\omega_1^{-1}\psi)\|_{H^s} + \left|\int_0^t \|u\|_{L^\infty}^{p-1}\|u\|_{H^s}\,dt'\right| \\
&\lesssim \|(\varphi,\omega_1^{-1}\psi)\|_{H^s} + \rho^{p-1}\left|\int_0^t (1+|t'|)^{-\theta(p-1)}\,dt'\right|\|u\|_{L^\infty H^s} \\
&\lesssim \|(\varphi,\omega_1^{-1}\psi)\|_{H^s} + \frac{1}{2}\|u\|_{L^\infty H^s}.
\end{aligned}
$$

Hence we deduce that the solution u is in $C(\mathbb{R};H^s)$, provided $\gamma(n)<p<\alpha(n)$.

3.3. **Proof of Theorem 1.3.** Let us define functions $\varphi_i^\pm$ and $\psi_i^\pm$, $i=1,2$ by

$$
\begin{aligned}
\widehat{\varphi_i^\pm}(\xi) &= \widehat{\varphi}_i(\xi) - \int_0^{\pm\infty}\omega_2(\xi)\sin(t'\omega_i(\xi))\widehat{f_i(u_i)}(\xi,t')\,dt', \\
\widehat{\psi_i^+}(\xi) &= \widehat{\psi}_i(\xi) + \int_0^{\pm\infty}\widetilde{\omega}_i(\xi)\cos(t'\omega_i(\xi))\widehat{f_i(u_i)}(\xi,t)\,dt,
\end{aligned}
$$

where (φ_1,ψ_1) and (φ_2,ψ_2) are the initial data stated in Theorems 1.1, and $\widetilde{\omega}_1(\xi)=|\xi|^2$ and $\widetilde{\omega}_2(\xi)=\omega_2^2$. Then from the regularity of solution u_i, we clearly have $(\varphi_i^\pm,\psi_i^\pm)\in H^{s_i}\times\omega_i H^{s_i}$.

Now let $u_i^\pm$ be the solution to the linear problems (1.1) and (1.2) with $f_i=0$ and with initial data $(\varphi_i^\pm,\psi_i^\pm)$. Then it can be represented by

$$
u_i^\pm(x,t) = (\partial_t S_i(t)\varphi_i)(x) + (S_i(t)\psi_i)(x) + \int_0^{\pm\infty} T_i(t-t')f_i(u_i(t'))\,dt'.
$$

Now we have from Lemma 3.2

$$\begin{aligned}\|u_i(\cdot,t)-u_i^{\pm}(\cdot,t)\|_{H^{s_i}} &\lesssim \left|\int_t^{\pm\infty} \|f_i(u_i(\cdot,t'))\|_{H^{s_i}}\,dt'\right| \\ &\lesssim \rho^{p_i}\left|\int_t^{\pm\infty}(1+|t'|)^{-(p_i-1)\theta}\,dt'\right| \\ &= O(|t|^{-(p_i-1)\theta+1})\end{aligned}$$

as $t\to\pm\infty$.

Similarly, we can define $(\varphi^{\pm},\psi^{\pm})\in H^s_{\frac{p+1}{p}}\times\omega_1 H^s_{\frac{p+1}{p}}$. If u and $u^{\pm}$ be the solutions of (1.1) and its linearized equation (i.e. $f_1=0$), respectively, then we have

$$\begin{aligned}\|u(\cdot,t)-u^{\pm}(\cdot,t)\|_{H^s_{p+1}} &\lesssim \left|\int_t^{\pm\infty} \|f(u)(t')\|^s_{B_{\frac{p+1}{p},2}}\,dt'\right| \\ &\lesssim \rho^{p}\left|\int_t^{\pm\infty}(1+|t'|)^{-(p-1)\theta}\,dt'\right| \\ &= O(|t|^{-(p-1)\theta+1})\end{aligned}$$

as $t\to\pm\infty$. This proves the theorem.

Remark 7. In view of Remark 6, we can also obtain the scattering in H^s for $s>\frac{n}{p+1}$, provided s is arbitrarily close to $\frac{n}{p+1}$, $\gamma(n)<p<\alpha(n)$ and $1\le n\le 4$.

4. Non-existence of asymptotically free solutions

In this section, we study the non-existence of asymptotically free solution, following the same strategy of [7] which is based on the argument of Barab [2] and Glassey [12, 13]. See also [29, 35].

Theorem 4.1. *Assume that $1<p_i\le 2$ for $n=1$ and $1<p_i<1+\frac{2}{n}$ for $n\ge 2$. Suppose that there exists $c>0$ such that $f_i(u_i)u_i\ge c|u_i|^{p_i+1}$. Let u_1 and u_2 be solutions to* (1.1) *and* (1.2)*, respectively, with $(u_i,\partial_t u_i)\in C\cap L^\infty(\mathbb{R};DL^2\times D^2L^2)$ and $(\varphi_i^{\pm},\psi_i^{\pm})\in DL^2\times D^2L^2$ be a pair of smooth functions with compact Fourier supports. Suppose that*

$$\|u_i(t)-u_i^{\pm}(t)\|_{L^2}=O(|t|^{-\varepsilon})\quad as\quad t\to\pm\infty \tag{4.1}$$

for some $\varepsilon>0$, where $u_i^{\pm}$ are the free solutions to the linear problem (1.1) *and* (1.2) *with $f_i=0$. Then $u_i=u_i^{\pm}=0$.*

The compact support condition of $(\varphi_i^{\pm},\psi_i^{\pm})$ in the Fourier space may be replaced by the space decay condition. See Remark 8.

Proof. Let us define a bilinear form $H(u,v)(t)$ by

$$H(u,v)(t)=\operatorname{Re}\int_{\mathbb{R}^n}\big(D^{-1}\partial_t v(t)D^{-1}u(t)-D^{-1}\partial_t u(t)D^{-1}v(t)\big)\,dx.$$

Then $H(u,v)(t)$ is well-defined and uniformly bounded on $t\in\mathbb{R}$ for $(u,\partial_t u),(v,\partial_t v)\in C\cap L^\infty(\mathbb{R};DL^2\times D^2L^2)$.

We assume that $(\varphi_i^\pm, \psi_i^\pm) \neq (0,0)$ and derive a contradiction to the uniform boundedness of H. For the simplicity we will consider only positive time and hence asymptotically free solution u_i^+. Suppose that there are non-zero functions u_i and u_i^+ satisfying the condition of Theorem 4.1. Then by using the regularization of u_i and u_i^+ (if necessary) we obtain

$$\frac{d}{dt}H(u_i, u_i^+)(t) = \int f_i(u_i)u_i^+ \, dx. \tag{4.2}$$

Let $H(u_i, u_i^+)(t) = H_i(t)$. Then from the condition $f_i(u)u \geq c|u|^{p_i+1}$, we deduce that

$$\begin{aligned}\frac{d}{dt}H_i(t) &= \int (f_i(u_i) - f_i(u_i^+))u_i^+ \, dx + \int f_i(u_i^+)u_i^+ \, dx \\ &\geq \int (f_i(u_i) - f_i(u_i^+))u_i^+ \, dx + c\int |u_i^+|^{p_i+1} \, dx.\end{aligned}$$

We will prove that if t is sufficiently large,

$$\|u_i^+(t)\|_{L^{p_i+1}(|x|\leq At^\beta)}^{p_i+1} \geq c_0 t^{-n\beta\frac{p_i-1}{2}} \tag{4.3}$$

for some positive constant A and c_0 depending on φ_i^+ and ψ_i^+ and $\beta > 1$ depending on ε stated in the theorem. If not specified, every constant depends on φ_i^+ and ψ_i^+. For the proof of (4.3), we first show that

$$\|u_i^+(t)\|_{L^2(|x|\leq At^\beta)} \gtrsim 1 \quad \text{for sufficiently large } \ t. \tag{4.4}$$

By Hölder inequality, (4.3) follows from (4.4). To obtain (4.4), let us choose a cut off function χ_0 supported in the unit ball $B(0,1)$ such that

$$\|u_i^+(t)\|_{L^2(|x|\leq At^\beta)}^2 = t^n\|u_i^+(t\cdot, t)\|_{L^2(|x|\leq At^{\beta-1})}^2 \geq t^n\|\chi_0(\cdot/M)u_i^+(t\cdot, t)\|_{L^2}^2,$$

where $M = At^{\beta-1}$. For the last integral, we have

$$\begin{aligned}&t^n\|\chi_0(\cdot/M)u_i^+(t\cdot,t)\|_{L^2}^2 \\ &\quad = t^n\|\chi_0(\cdot/M)(\partial_t S_i(t)\varphi_i^+)(t\cdot)\|_{L^2}^2 + t^n\|\chi_0(\cdot/M)(S_i(t)\psi_i^+)(t\cdot)\|_{L^2}^2 \\ &\qquad + 2t^n\mathrm{Re}\int (\chi_0(x/M))^2(\partial_t S_i(t)\varphi_i^+)(tx)\overline{(S_i(t)\psi_i^+)(tx)}\, dx.\end{aligned} \tag{4.5}$$

By change of variable and Plancheral's theorem, we have for the first term

$$t^n\|\chi_0(\cdot/M)(\partial_t S_i(t)\varphi_i^+)(t\cdot)\|_{L^2}^2 = \left\|\chi_0(\cdot/M)\mathcal{F}^{-1}\left(\cos(t\omega_i(\xi/t))t^{-\frac{n}{2}}\widehat{\varphi_i^+}(\cdot/t)\right)\right\|_{L^2}^2.$$

From the identity $\cos^2 x = \frac{1+\cos(2x)}{2}$, we deduce that

$$\begin{aligned}&\left\|\cos(t\omega_i(\xi/t))t^{-\frac{n}{2}}\widehat{\varphi_i^+}(\cdot/t)\right\|_{L^2}^2 \\ &\quad = \int \cos^2(t\omega_i(\xi/t))\, t^{-n}|\widehat{\varphi_i^+}(\xi/t)|^2\, d\xi \\ &\quad = \frac{1}{2}\|\varphi_i^+\|_{L^2}^2 + \frac{1}{2}\int \cos(2t\omega_i(\xi/t))\, t^{-n}|\widehat{\varphi_i^+}(\xi/t)|^2\, d\xi.\end{aligned}$$

By the integration by parts in the radial direction such that

$$\int \partial_\rho f(\xi)g(\xi)\,d\xi = -(n-1)\int \frac{f(\xi)g(\xi)}{\rho}\,d\xi - \int f(\xi)\partial_\rho g(\xi)\,d\xi,$$
$$\rho = |\xi|, \quad \partial_\rho = \frac{\xi}{\rho}\cdot\nabla,$$

we have

$$\begin{aligned}
&\int \cos\left(2t\omega_i(\xi/t)\right) t^{-n}|\widehat{\varphi_i^+}(\xi/t)|^2\,d\xi \\
&= \int \partial_\rho\left(\sin\left(2t\omega_i(\xi/t)\right)\right)\left(\partial_\rho\left(2t\omega_i(\xi/t)\right)\right)^{-1} t^{-n}|\widehat{\varphi_i^+}(\xi/t)|^2\,d\xi \\
&= -\frac{n-1}{t^n}\int \frac{\sin\left(2t\omega_i(\xi/t)\right)}{\rho}\left(\partial_\rho\left(2t\omega_i(\xi/t)\right)\right)^{-1}|\widehat{\varphi_i^+}(\xi/t)|^2\,d\xi \\
&\quad -\frac{1}{t^n}\int \sin\left(2t\omega_i(\xi/t)\right)\partial_\rho\left(\left(\partial_\rho\left(2t\omega_i(\xi/t)\right)\right)^{-1}|\widehat{\varphi_i^+}(\xi/t)|^2\right)\,d\xi.
\end{aligned} \tag{4.6}$$

Since

$$\left(\partial_\rho\left(2t\omega_1(\xi/t)\right)\right)^{-1} = \frac{\sqrt{1+\rho^2/t^2}}{2(1+2\rho^2/t^2)},$$
$$\left(\partial_\rho\left(2t\omega_2(\xi/t)\right)\right)^{-1} = \frac{1}{2}(1+\rho^2/t^2)^{\frac{3}{2}},$$

it follows from the Hölder inequality that

$$\int \cos\left(2t\omega_i(\xi/t)\right) t^{-n}|\widehat{\varphi_i^+}(\xi/t)|^2\,d\xi = O(t^{-1}) \;\text{ as }\; t\to\infty$$

and hence

$$\left\|\cos(t\omega_i(\xi/t))t^{-\frac{n}{2}}\widehat{\varphi^+}(\cdot/t)\right\|_{L^2} \to \frac{1}{\sqrt{2}}\|\varphi_i^+\|_{L^2} \quad\text{as}\quad t\to\infty. \tag{4.7}$$

Now we claim that there exist large numbers t_0 such that

$$\inf_{t>t_0} t^n\|\chi_0(\cdot/M)(\partial_t S_i(t)\varphi_i^+)(t\cdot)\|_{L^2}^2 \gtrsim 1. \tag{4.8}$$

For the proof of (4.8), we may assume that $\|\varphi_i^+\|_{L^2}=1$. Let us define a function $g_t(x)$ by $t^n|(\partial_t S_i(t)\varphi_i^+)(tx)|^2$. Then from (4.7), we can find a positive number t_0 such that $\int g_t(x)\,dx \ge \frac{1}{4}$ for all $t>t_0$. By integration by parts, we get for $x\ne 0$ and multi index α with $|\alpha| = m > \frac{n}{2}$

$$\begin{aligned}
t^{\frac{n}{2}}(\partial_t S_i(t)\varphi_i^+)(tx) &= \frac{1}{(2\pi)^n t^{\frac{n}{2}}}\int e^{ix\cdot\xi}\cos(t\omega_i(\xi/t))\widehat{\varphi_i^+}(\xi/t)\,d\xi \\
&= \frac{1}{(2\pi)^n t^{\frac{n}{2}}(-ix)^\alpha}\int e^{ix\cdot\xi}\partial_\xi^\alpha\left(\cos(t\omega_i(\xi/t))\widehat{\varphi_i^+}(\xi/t)\right)\,d\xi.
\end{aligned}$$

By Hölder's inequality we have for a fixed number $s_0 > \frac{n}{2}$

$$g_t(x) \lesssim \frac{t^n}{|x|^{2m}}\sum_{|\alpha|\le m}\|x^\alpha\varphi_i^+\|_{H^{s_0}}. \tag{4.9}$$

This gives us that

$$\begin{aligned}\int (\chi_0(x/M))^2 g_t(x)\,dx &= \int g_t(x)\,dx - \int (1-(\chi_0^2(x/M))^2) g_t(x)\,dx\\ &\geq \frac{1}{4} - \int_{|x|\geq \frac{1}{2}M} \frac{At^n}{|x|^{2m}}\,dx\\ &\geq \frac{1}{4} - O(t^{2m-(2m-n)\beta}) \ \text{ as } \ t\to\infty,\end{aligned}$$

where $M = At^{\beta-1}$ and $A \sim \sum_{|\alpha|\leq m} \|x^\alpha \varphi_i^+\|_{H^{s_0}}$.

Now if we choose m and β so that $2m-(2m-n)\beta<0$, then the claim (4.8) is proved, provided t_0 is sufficiently large.

Similarly we see that

$$\left\| \frac{\sin(t\omega_i(\xi/t))}{\omega_i(\xi/t)} t^{-\frac{n}{2}} \widehat{\psi_i^+}(\xi/t)\right\|_{L^2}^2 \to \frac{1}{\sqrt{2}} \|\omega_i^{-1}\psi_i^+\|_{L^2}^2$$

as $t\to\infty$ and hence by the same argument as above, we have the estimate

$$t^n \|\chi_0(\cdot/M)(S_i(t)\psi_i^+)(t\cdot)\|_{L^2}^2 \gtrsim 1, \tag{4.10}$$

if $t>t_0$ for some large t_0.

Finally, for the last term of (4.5) let us consider the integral

$$I(t) = t^n \int (\partial_t S_i(t)\varphi_i^+)(tx)\overline{S_i(t)\psi_i^+}(tx)\,dx.$$

Then by change of variable and Plancheral's theorem, $I(t)$ is converted by

$$\frac{1}{(2t\pi)^n} \int \frac{\sin(2t\omega_i(\xi/t))}{\omega_i(\xi/t)} \widehat{\varphi_i^+}(\xi/t)\widehat{\psi}_i(\xi/t)\,d\xi.$$

Here we also used the identity $\cos x \sin x = \frac{1}{2}\sin 2x$. Similarly to the estimate (4.6), we have $I(t) = O(t^{-1})$. With this estimate we prove that

$$\left| 2t^n \mathrm{Re} \int (\chi_0(x/M))^2 (\partial_t S(t)\varphi^+)(tx)\overline{(S(t)\psi^+)(tx)}\,dx \right| \to 0 \quad \text{as} \quad t\to\infty. \tag{4.11}$$

Actually, by the integration by parts as above, we have

$$\begin{aligned}&\left| 2t^n \mathrm{Re} \int (\chi_0(x/M))^2 (\partial_t S_i(t)\varphi_i^+)(tx)\overline{(S_i(t)\psi_i^+)(tx)}\,dx \right|\\ &\quad \leq |2\mathrm{Re} I(t)| + \int_{|x|\geq\frac{1}{2}M} \frac{At^n}{|x|^{2m}}\,dx \to 0\end{aligned}$$

as $t\to\infty$, where $M = At^{\beta-1}$.

Therefore (4.11) together with (4.8) and (4.10) yields the lower bound estimate (4.4) and hence (4.3).

Since $\varphi_i^+ \in DL^2$ and $\psi_i^+ \in D^2L^2$ have compact Fourier supports, it follows from the proof of Lemmas 2.4 that for all $2\leq r\leq\infty$

$$\|u_i^+(t)\|_{L^r} \lesssim t^{-n(\frac{1}{2}-\frac{1}{r})}. \tag{4.12}$$

From the estimate (4.12) and the hypothesis (4.1), we have for $1 < p_i \le 2$,

$$\begin{aligned}&\left|\int (f_i(u_i) - f_i(u_i^+))u_i^+ \, dx\right| \\ &\qquad \lesssim (\|u_i\|_{L^2}^{p_i-1}\|u_i^+\|_{L^2}^{2-p_i} + \|u_i^+\|_{L^2})\|u_i^+\|_{L^\infty}^{p_i-1}\|u_i - u_i^+\|_{L^2} \\ &\qquad = O(t^{-\frac{n}{2}(p_i-1)-\varepsilon}).\end{aligned} \tag{4.13}$$

Thus choosing $\beta > 1$ such as $\frac{n\beta(p_i-1)}{2} < \frac{n(p_i-1)}{2} + \varepsilon$ and $\frac{n\beta(p_i-1)}{2} \le 1$, we conclude from (4.3) that $\frac{d}{dt}H(t) \gtrsim t^{-1}$ for large t. This is a contradiction to the uniform boundedness of H. □

Remark 8. In the above proof, we chose β such that

$$\frac{2m}{2m-n} < \beta < \min\left(1 + \frac{2\varepsilon}{n(p_i-1)}, \frac{2}{n(p_i-1)}\right).$$

This choice is possible because p_i is assumed to be smaller than $1 + \frac{2}{n}$. If ε is smaller or p_i is closer to $1+\frac{2}{n}$, then m should be larger. Hence from (4.9), the data $(\varphi_i^\pm, \psi_i^\pm)$ should decay fast at space infinity.

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