



Title	Nonlinear Schrödinger equation with a point defect
Author(s)	Fukuizumi, Reika; Ohta, Masahito; Ozawa, Tohru
Citation	Hokkaido University Preprint Series in Mathematics, 768, 1-14
Issue Date	2006
DOI	10.14943/83918
Doc URL	<a href="http://hdl.handle.net/2115/69576">http://hdl.handle.net/2115/69576</a>
Type	bulletin (article)
File Information	pre768.pdf



[Instructions for use](#)

# NONLINEAR SCHRÖDINGER EQUATION WITH A POINT DEFECT

REIKA FUKUIZUMI, MASAHITO OHTA, AND TOHRU OZAWA

ABSTRACT. We study nonlinear Schrödinger equation with a delta-function impurity in one space dimension. Global well-posedness is proved for the Cauchy problem in  $L^2(\mathbb{R})$  under subcritical nonlinearity, as well as under critical nonlinearity with smallness assumption on the data. In the attractive case, orbital stability and instability of the ground state is proved in  $H^1(\mathbb{R})$ .

## 1. INTRODUCTION

In this paper we study nonlinear Schrödinger equations of the form

$$i\partial_t u + \frac{1}{2}D^2 u + Z\delta u = f(u), \quad (1.1)$$

where  $u$  is a complex-valued function of  $(t, x) \in \mathbb{R} \times \mathbb{R}$ ,  $\partial_t = \partial/\partial t$ ,  $D = \partial/\partial x$ ,  $\delta$  is the Dirac measure at the origin,  $Z \in \mathbb{R}$ , and  $f$  is a complex valued function of  $z \in \mathbb{C}$ . A typical example of  $f$  is a double power nonlinearity of the form

$$f(z) = \lambda_1 |z|^{p_1-1} z + \lambda_2 |z|^{p_2-1} z,$$

where  $\lambda_j \in \mathbb{R}$  and  $1 \leq p_1 \leq p_2 < \infty$ . For  $Z \neq 0$ , the equations of the form (1.1) arise in a wide variety of physical models with a point defect on the line [15] and references therein. In spite of a large literature on (1.1) with  $Z = 0$ , there seems only one mathematical study in (1.1) with  $Z \neq 0$  [15] available so far.

To be more specific, it was shown in [15] that the Cauchy problem is globally well-posed in  $H^1(\mathbb{R}) = (1 - D^2)^{-1/2}L^2(\mathbb{R})$  in the case where  $Z > 0$  and  $f(u) = -|u|^2 u$ . Conserved quantities are the energy  $E$  and the charge  $Q$ :

$$E(v) = \frac{1}{4}\|Dv\|_{L^2}^2 - \frac{Z}{2} \int_{\mathbb{R}} \delta(x)|v(x)|^2 dx + \int_{\mathbb{R}} F(v) dx, \quad Q(v) = \frac{1}{2}\|v\|_{L^2}^2, \quad v \in H^1(\mathbb{R}),$$

where  $F(z) = \int_0^{|z|} f(t) dt$ . Moreover, the authors in [15] studied the stability of nonlinear bound states  $u_{\text{Def}}$  given by

$$u_{\text{Def}}(x) = \sqrt{2\omega} e^{i\omega t} \operatorname{sech} \left( \sqrt{2\omega} |x| + \tanh^{-1} \frac{Z}{\sqrt{2\omega}} \right) \quad (1.2)$$

with  $\sqrt{2\omega} > Z$  and  $\omega > 0$  in the orbitally Lyapunov sense in  $H^1$  in the case where  $Z > 0$  and  $f(u) = -|u|^2 u$ .

The purpose in this paper is to generalize those results in various directions and to compare our results with the available results with  $Z = 0$ .

To state our results precisely, we introduce the following notation. We define  $D^j \equiv \frac{d^j}{dx^j}$  for  $j \in \mathbb{N} \cup \{0\}$ . Let  $H$  be the self-adjoint operator in  $L^2(\mathbb{R})$  associated with  $-(1/2)D^2 - Z\delta$  [1]. Then the equation (1.1) is converted to the integral equation

$$u(t) = U(t)u_0 - i \int_0^t U(t-t')f(u(t'))dt', \quad (1.3)$$

where  $U(t) = \exp(-itH)$  and  $u_0$  is the prescribed Cauchy data at  $t = 0$ . We consider the following assumptions on  $f$ .

(H1)  $f \in C(\mathbb{C}, \mathbb{C})$  and there exist  $p_1$  and  $p_2$  with  $1 \leq p_1 \leq p_2 < \infty$  such that  $f$  satisfies the estimate

$$\begin{aligned} |f(z)| &\leq C(|z|^{p_1} + |z|^{p_2}), \\ |f(z) - f(z')| &\leq C(|z|^{p_1-1} + |z'|^{p_2-1})|z - z'| \end{aligned}$$

for all  $z, z' \in \mathbb{C}$ .

(H2)  $\text{Im}(\bar{z}f(z)) = 0$  for all  $z \in \mathbb{C}$ .

Concerning the well-posedness of the equation (1.3) in  $L^2(\mathbb{R})$ , we prove:

**Theorem 1.** *Let  $f$  satisfy (H1) and (H2) with  $1 \leq p_1 \leq p_2 < 5$ . Then for any  $u_0 \in L^2$ , the equation (1.3) has a unique solution  $u \in C(\mathbb{R}; L^2) \cap L_{\text{loc}}^4(\mathbb{R}; L^\infty)$ . Moreover,  $u$  satisfies the conservation of charge:*

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2} \quad (1.4)$$

for all  $t \in \mathbb{R}$ . For any  $T > 0$  the map  $u_0 \mapsto u$  is continuous from  $L^2$  to  $L^\infty(I; L^2) \cap L^4(I; L^\infty)$ , where  $I = [-T, T]$ .

**Theorem 2.** *Let  $f$  satisfy (H1) and (H2) with  $p_1 = p_2 = 5$ . Then there exists  $R > 0$  such that for any  $u_0 \in B_R(0) = \{\phi \in L^2 : \|\phi\|_{L^2} \leq R\}$  the equation (1.3) has a unique solution  $u \in (C \cap L^\infty)(\mathbb{R}; L^2) \cap L^4(\mathbb{R}; L^\infty)$ . Moreover,  $u$  satisfies (1.4) and has a unique pair  $u_\pm \in L^2$  such that*

$$\|u(t) - U(t)u_\pm\|_{L^2} \rightarrow 0 \quad (1.5)$$

as  $t \rightarrow \pm\infty$ . The map  $u_0 \mapsto u$  is continuous from  $B_R(0)$  to  $L^\infty(\mathbb{R}; L^2) \cap L^4(\mathbb{R}; L^\infty)$ .

**Remark 1.1.** Theorem 1 implies the global well-posedness of (1.1) with subcritical power at the level of  $L^2$  and so does Theorem 2 with critical power under the smallness assumption above. Those results are reminiscent of the standard theory for NLS, namely,  $Z = 0$ , at the level of  $L^2$  [5, 29] and references therein, where the notion of critical and subcritical powers for NLS comes from the associated scaling argument. We should remark that the

corresponding scaling argument breaks down for (1.1) with  $Z \neq 0$ , while the notion of critical and subcritical powers still remains.

**Remark 1.2.** We recall the definition of the self-adjoint operator  $H$  as the precise formulation of a formal expression  $-(1/2)D^2 - Z\delta$ .

$$Hu = -\frac{1}{2}D^2u, \quad u \in \text{Dom}(H),$$

where

$$\begin{aligned} \text{Dom}(H) &= \{u \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\}) : Du(0+) - Du(0-) = -2Zu(0)\}, \\ H^m(I) &= \{u \in L^2(I) : D^j u \in L^2 \text{ for all } j \text{ with } 0 \leq j \leq m\}, \quad I \subset \mathbb{R}. \end{aligned}$$

All self-adjoint extensions of  $\dot{H} \equiv -(1/2)D^2$  with domain

$$\text{Dom}(\dot{H}) = \{u \in H^2(\mathbb{R}) : u(0) = 0\}$$

are parametrized by  $H$  with  $Z \in [-\infty, +\infty)$  (see [1]).

Nonlinear bound states mean the solutions to (1.1) having the form  $u_\omega(t, x) = e^{i\omega t}\phi_\omega(x)$ , where  $\omega > 0$  is the frequency and  $\phi_\omega$  should satisfy the following semilinear elliptic equations:

$$-\frac{1}{2}D^2\phi + \omega\phi - Z\delta\phi = f(\phi), \quad x \in \mathbb{R}, \quad Z \in \mathbb{R}. \quad (1.6)$$

In the case that  $f(u) = -|u|^{p-1}u$  with  $p > 1$ , there exists a unique positive symmetric solution of (1.6) which is explicitly described as:

$$\phi_\omega(x) = \left\{ \frac{(p+1)\omega}{2} \text{sech}^2 \left( \frac{(p-1)\sqrt{\omega}}{\sqrt{2}}|x| + \tanh^{-1} \left( \frac{Z}{\sqrt{2\omega}} \right) \right) \right\}^{\frac{1}{p-1}} \quad (1.7)$$

if  $\sqrt{2\omega} > |Z|$ . Precisely, this solution is constructed from the solution with  $Z = 0$  on each side of the defect pasted together at  $x = 0$  to satisfy the conditions of continuity and symmetry at  $x = 0$  and the jump condition in the first derivative at  $x = 0$ ,  $Du(0+) - Du(0-) = -2Zu(0)$ .

In case of  $Z = 0$  it is unique for  $\omega > 0$  up to translations, which, we denote by  $\psi_\omega(x)$ . Orbital stability for the case of  $Z = 0$  has been well studied (see [2, 5, 6, 8, 17, 18, 30, 31]). For the case where  $f(u) = -|u|^{p-1}u$  with  $p > 1$ , Cazenave and Lions [6] proved that  $e^{i\omega t}\psi_\omega(x)$  is stable for any  $\omega > 0$  if  $p < 5$ . On the other hand, it was shown that  $e^{i\omega t}\psi_\omega(x)$  is unstable for any  $\omega > 0$  if  $p \geq 5$  (see Berestycki and Cazenave [2] for  $p > 5$ , and Weinstein [30] for  $p = 5$ ).

As we have mentioned, Goodman, Holmes and Weinstein [15] claimed in the case where  $Z > 0$  and  $f(u) = -|u|^2u$  that (1.2) are nonlinearly orbitally Lyapunov stable by the same method as that of Rose and Weinstein [25], Weinstein [30]. More exactly, the authors established a variational characterization for (1.2) and proved the stability using the bifurcation from the linear mode, i.e., the corresponding eigenfunction to the eigenvalue  $\lambda = -Z^2/2$ .

They also remark that as  $\omega \rightarrow \infty$ , (1.2) looks more and more like the solitary standing wave of (1.1) with  $Z = 0$  and  $\omega = 1$ . However, we address in this article a different point from the case  $Z = 0$ .

The notion of the stability and instability in this paper is formulated as follows.

**Definition 1.** For  $\eta > 0$ , we put

$$U_\eta(\phi_\omega) := \left\{ v \in H^1(\mathbb{R}) : \inf_{\theta \in \mathbb{R}} \|v - e^{i\theta} \phi_\omega\|_{H^1} < \eta \right\}.$$

We say that a standing wave solution  $e^{i\omega t} \phi_\omega(x)$  of (1.1) is stable in  $H^1(\mathbb{R})$  if for any  $\varepsilon > 0$  there exists  $\eta > 0$  such that for any  $u_0 \in U_\eta(\phi_\omega)$ , the solution  $u(t)$  of (1.1) with  $u(0) = u_0$  satisfies  $u(t) \in U_\varepsilon(\phi_\omega)$  for any  $t \geq 0$ . Otherwise,  $e^{i\omega t} \phi_\omega(x)$  is said to be unstable in  $H^1(\mathbb{R})$ .

Before we mention our result, we should remark a variational characterization of  $\phi_\omega$  for the discussion below. From now on, we will consider the nonlinearity  $f(u) = -|u|^{p-1}u$  only and the case of  $Z > 0$ . The global well-posedness of the Cauchy problem holds in  $H^1(\mathbb{R})$  for any  $p$  with  $1 < p < 5$  by the same method as in [15].

**Definition 2.** For  $Z > 0$  and  $\omega > Z^2/2$ , we define two  $C^1$  functionals on  $H^1(\mathbb{R})$ :

$$\begin{aligned} S_\omega(v) &:= E(v) + \omega Q(v), \\ I_\omega(v) &:= \frac{1}{2} \|Dv\|_{L^2}^2 + \omega \|v\|_{L^2}^2 - Z \int_{\mathbb{R}} \delta(x) |v(x)|^2 dx - \|v\|_{L^{p+1}}^{p+1} \\ &= \frac{1}{2} \|Dv\|_{L^2}^2 + \omega \|v\|_{L^2}^2 - Z |v(0)|^2 - \|v\|_{L^{p+1}}^{p+1}. \end{aligned}$$

Let  $\mathcal{G}_\omega$  be the set of all nonnegative minimizers for the minimization problem

$$d(\omega) = \inf \{ S_\omega(v) : v \in H^1(\mathbb{R}) \setminus \{0\}, I_\omega(v) = 0 \}. \quad (1.8)$$

The existence of non-negative minimizers for (1.8) is proved by the standard variational argument. We will briefly show the following proposition in Section 3 for the sake of completeness.

**Proposition 1.** *Let  $Z > 0$ . For any  $\omega > Z^2/2$ , the minimization problem (1.8) is attained by a symmetric nonincreasing function vanishing at infinity.*

**Remark 1.3.** (i) For  $Z > 0$ , let

$$\lambda = \inf \left\{ \frac{1}{2} \|Dv\|_{L^2}^2 - Z \int_{\mathbb{R}} \delta(x) |v(x)|^2 dx : \|v\|_{L^2} = 1, v \in H^1(\mathbb{R}) \right\}.$$

Then we have  $\lambda = -Z^2/2$  and the corresponding eigenfunction is  $\Phi(x) = Ze^{-Z|x|}$ .

(ii) We note that

$$I_\omega(v) = \partial_\lambda S_\omega(\lambda v)|_{\lambda=1} = \langle S'_\omega(v), v \rangle$$

for  $\lambda > 0$ .

(iii) Let  $v_\omega \in \mathcal{G}_\omega$ . Then, there exists a Lagrange multiplier  $\Lambda \in \mathbb{R}$  such that  $S'_\omega(v_\omega) = \Lambda I'_\omega(v_\omega)$ . Thus, we have  $\langle S'_\omega(v_\omega), v_\omega \rangle = \Lambda \langle I'_\omega(v_\omega), v_\omega \rangle$ . Since  $\langle S'_\omega(v_\omega), v_\omega \rangle = I_\omega(v_\omega) = 0$  and  $\langle I'_\omega(v_\omega), v_\omega \rangle = -(p-1)\|v_\omega\|_{p+1}^{p+1} < 0$ , we have  $\Lambda = 0$ . Namely,  $v_\omega$  satisfies (1.6). Moreover, for any  $v \in H^1(\mathbb{R}) \setminus \{0\}$  satisfying  $S'_\omega(v) = 0$ , we have  $I_\omega(v) = 0$ . Thus, by the definition of  $\mathcal{G}_\omega$ , we have  $S_\omega(v_\omega) \leq S_\omega(v)$ . Namely,  $v_\omega \in \mathcal{G}_\omega$  is a ground state (minimal action solution) of (1.6) in  $H^1(\mathbb{R})$ . It is easy to see that a ground state of (1.6) in  $H^1(\mathbb{R})$  is a minimizer of (1.8).

(iii) The minimizer  $v \in \mathcal{G}_\omega$  obtained above, which is nonnegative, symmetric and nonincreasing, satisfies the initial boundary value problem (see Lemma 3.1 below):

$$\begin{aligned} v &\in C^2(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R}), \quad v(x) > 0, \quad x \in \mathbb{R}, \\ -\frac{1}{2}D^2v + \omega v - v^p &= 0, \quad x \neq 0, \\ Dv(0+) - Dv(0-) &= -2Zv(0), \\ Dv(x), v(x) &\rightarrow 0, \quad \text{as } |x| \rightarrow \infty. \end{aligned}$$

**Remark 1.4.** The minimizer obtained in Proposition 1 is precisely the same as  $\phi_\omega$  defined by (1.7) since the positive symmetric solution with above initial boundary value problem (3.4)–(3.7) is uniquely determined.

To prove stability and instability, we use the following sufficient condition originally obtained by Shatah [26] and Shatah and Strauss [27] (see also [10, 12] for the proof).

**Proposition 2.** *Let  $p > 1$ ,  $Z > 0$  and  $\omega > Z^2/2$ . Let  $v_\omega \in \mathcal{G}_\omega$ . Assume that  $\omega \mapsto v_\omega$  is a  $C^1$  mapping.*

- (i) *If  $\partial_\omega \|v_\omega\|_{L^2}^2 > 0$  at  $\omega = \omega_0$ , then  $e^{i\omega_0 t} v_{\omega_0}(x)$  is stable in  $H^1(\mathbb{R})$ .*
- (ii) *If  $\partial_\omega \|v_\omega\|_{L^2}^2 < 0$  at  $\omega = \omega_0$ , then  $e^{i\omega_0 t} v_{\omega_0}(x)$  is unstable in  $H^1(\mathbb{R})$ .*

**Remark 1.5.** In case of  $Z = 0$ , it is easy to verify this condition since we have  $\|\psi_\omega\|_{L^2}^2 = \omega^{2/(p-1)-1/2} \|\psi_1\|_{L^2}^2$  by the scaling invariance even in the higher dimensional case. Due to the potential term we lost the scaling invariance in general (see [7, 12, 13, 14, 20, 22, 32, 33] for example). However, in the present one-dimensional case where the potential is a Dirac-delta, we can compute exactly increase and decrease of  $L^2$  norm of (1.7).

**Theorem 3.** *Let  $Z > 0$  and  $\omega > Z^2/2$ .*

- (i) *Let  $1 < p \leq 5$ . Then  $e^{i\omega t} \phi_\omega(x)$  is stable in  $H^1(\mathbb{R})$  for any  $\omega \in (Z^2/2, \infty)$ .*
- (ii) *Let  $p > 5$ . Then there exists unique  $\omega_1 > 0$  such that  $e^{i\omega t} \phi_\omega(x)$  is stable in  $H^1(\mathbb{R})$  for any  $\omega \in (Z^2/2, \omega_1)$ , and that it is unstable in  $H^1(\mathbb{R})$  for any  $\omega \in (\omega_1, \infty)$ , where*

$\omega_1$  is exactly defined as follows:

$$\begin{aligned} \frac{p-5}{p-1}J(\omega_1) &= \frac{Z}{\sqrt{2\omega_1}} \left(1 - \frac{Z^2}{2\omega_1}\right), \\ J(\omega_1) &= \int_{A(\omega_1)}^{\infty} \operatorname{sech}^{4/(p-1)} y dy, \quad A(\omega_1) = \tanh^{-1} \left( \frac{Z}{\sqrt{2\omega_1}} \right). \end{aligned}$$

**Remark 1.6.** Concerning the critical case  $\partial_{\omega} \|\phi_{\omega}\|_{L^2}^2 = 0$ , we conjecture that  $e^{i\omega_1 t} \phi_{\omega_1}(x)$  would be unstable in view of the result of Comech and Pelinovsky [8]. For that purpose, the variational characterizations of  $\phi_{\omega}$  above would be useful to investigate the number of nonpositive eigenvalues of the linearized operators around  $e^{i\omega t} \phi_{\omega}(x)$  (see [19, 9]).

**Remark 1.7.** A similar result is known for the case where  $Z = 0$  and  $f(z) = \lambda_1 |z|^{p_1-1} z + \lambda_2 |z|^{p_2-1} z$ , where  $\lambda_j \in \mathbb{R}$  and  $1 \leq p_1 \leq p_2 < \infty$  (see Ohta [23]).

In Section 2, we give a proof of Theorems 1 and 2. In Section 3, we prove Proposition 1 and we complete the proof of Theorem 3 by checking the increase and the decrease of  $L^2$  norm of  $\phi_{\omega}$  as a function of  $\omega$ . Also, we give the outline of the proof of Proposition 2.

## 2. PROOF OF THEOREMS 1 AND 2

We recall that  $U(t) = \exp(-itH)$  is represented as

$$(U(t)\phi)(x) = \int K(t, x, y) \phi(y) dy \tag{2.1}$$

for  $\phi \in L^1 \cap L^2$  and  $t \neq 0$  [4, 16], where  $K = K_0 + K_1$  with

$$\begin{aligned} K_0(t, x, y) &= \frac{1}{\sqrt{2\pi it}} \exp\left(\frac{i(x-y)^2}{2t}\right), \\ K_1(t, x, y) &= \frac{Z}{2} \exp(-Z\rho + \frac{it}{2}Z^2) \operatorname{erfc}(\xi) \\ &= \frac{Z}{2} \exp\left(\frac{i\rho^2}{2t}\right) \exp(\xi^2) \operatorname{erfc}(\xi), \\ \xi &= \rho/\sqrt{2it} - Z\sqrt{it/2}, \quad \rho \equiv |x| + |y|, \\ \operatorname{erfc}(\xi) &= \frac{2}{\sqrt{\pi}} \int_{\xi}^{\infty} e^{-r^2} dr, \quad \xi \in \mathbb{C}. \end{aligned}$$

Here the last integral is taken along any path from the origin to  $\xi$  in the complex plane.

The main tool for the proof of Theorems 1 and 2 is the Strichartz estimates.

**Proposition 3.** *Let  $q_j, r_j$  satisfy  $0 \leq 2/q_j = 1/2 - 1/r_j \leq 1/2$ ,  $j = 0, 1, 2$ . Then the following estimates hold:*

$$\begin{aligned} \|U(\cdot)\phi\|_{L^{q_0}(\mathbb{R}; L^{r_0})} &\leq C \|\phi\|_{L^2}, \\ \|Gf\|_{L^{q_1}(I; L^{r_1})} &\leq C \|f\|_{L^{q'_2}(I; L^{r'_2})}, \end{aligned}$$

where

$$(Gf)(t) = \int_0^t U(t-s)f(s)ds,$$

$I$  is an interval with  $0 \in \bar{I}$ ,  $C$  is a constant independent of  $I$ , and  $q'$  is a conjugate exponent to  $q$  defined by  $1/q + 1/q' = 1$ .

**Proof.** By the standard argument for the Strichartz estimates [5, 29], it suffices to prove that

$$\sup_{t,x,y \in \mathbb{R}} \sqrt{|t|} |K(t,x,y)| = C < \infty, \quad (2.2)$$

which follows from (2.2) with  $K$  replaced by  $K_1$ . By the definition of  $\xi$ , we have

$$|\xi| = \sqrt{\frac{\rho^2}{2|t|} + \frac{Z^2|t|}{2}} \geq \sqrt{\frac{|t|}{2}} Z. \quad (2.3)$$

If  $|\xi| \geq 1$ , then

$$\begin{aligned} |K_1(t,x,y)| &\leq \frac{|Z|}{2} |\exp(\xi^2) \operatorname{erfc}(\xi)| \\ &\leq \frac{C|Z|}{|\xi|} \leq \frac{C}{\sqrt{|t|}}, \end{aligned} \quad (2.4)$$

where we have used an inequality for the error function [28] and (2.3). If  $|\xi| \leq 1$ , then  $|Z|\sqrt{\frac{|t|}{2}} \leq 1$  and therefore

$$\begin{aligned} |K_1(t,x,y)| &\leq \frac{|Z|}{2} |\operatorname{erfc}(\xi)| \\ &\leq C|Z| \leq \frac{C}{\sqrt{|t|}}, \end{aligned} \quad (2.5)$$

where we have used another inequality for the error function [28]. The required estimate (2.2) then follows from (2.4) and (2.5).

Once the Strichartz estimates are established, the local well-posedness follows by the standard contraction argument [5, 29]. We only give a sketch of the argument. For  $T > 0$  we define

$$X_T = L^\infty(-T, T; L^2) \cap L^4(-T, T; L^\infty)$$

with norm  $\|u\| = \|u\|_{L_t^\infty(L^2)} \vee \|u\|_{L_t^4(L^\infty)}$ , where  $a \vee b = \max(a, b)$ . For  $u \in X_T$  and  $u_0 \in L^2$  we define

$$(\Phi(u))(t) = U(t)u_0 - i \int_0^t U(t-s)f(u(s))ds.$$



For simplicity we assume that  $p_1 = 1$ ,  $p_2 = p < 5$  for Theorem 1 and that  $p_1 = p_2 = p = 5$  for Theorem 2. By Proposition 3 and Hölder inequalities in space and time, we have

$$\begin{aligned} \|\Phi(u)\| &\leq C\|u_0\|_{L^2} + C\|f(u)\|_{L^1_t(L^2)} \\ &\leq C\|u_0\|_{L^2} + CT\|u\|_{L^\infty_t(L^2)} + CT^{1-\theta}\|u\|_{L^4_t(L^\infty)}^{p-1}\|u\|_{L^\infty_t(L^2)}, \end{aligned} \quad (2.6)$$

where  $\theta = (p-1)/4$ . Similarly, for  $u, v \in X_T$ , we have

$$\begin{aligned} \|\Phi(u) - \Phi(v)\| &\leq C\|f(u) - f(v)\|_{L^1_t(L^2)} \\ &\leq CT\|u - v\|_{L^\infty_t(L^2)} + CT^{1-\theta}(\|u\|_{L^4_t(L^\infty)} \vee \|v\|_{L^4_t(L^\infty)})^{p-1}\|u - v\|_{L^\infty_t(L^2)}. \end{aligned} \quad (2.7)$$

If  $p < 5$ , then  $\theta < 1$  and therefore (2.6) and (2.7) show that  $\Phi$  is a contraction on a closed ball in  $X_T$  with  $T > 0$  sufficiently small. If  $p = 5$ , then  $\theta = 0$  and therefore (2.6) and (2.7) show that  $\Phi$  is a contraction on  $L^\infty(\mathbb{R}; L^2) \cap L^4(\mathbb{R}; L^\infty)$  if the size of  $L^2$  norm of the Cauchy data is sufficiently small. The conservation law (1.4) follows in the same way as in [24]. This leads to the existence of global solutions [5, 29]. Uniqueness and continuous dependence of solutions follows by the standard method [5, 29], so that it suffices to show (1.5). Let  $u \in (C \cap L^\infty)(\mathbb{R}; L^2) \cap L^4(\mathbb{R}; L^\infty)$  be a solution of (1.3) with  $u(0) = u_0 \in L^2$ . Then by Proposition 3 and (1.4), we have for  $t > s$

$$\begin{aligned} \|U(-t)u(t) - U(-s)u(s)\|_{L^2} &= \left\| \int_s^t U(-t')f(u(t'))dt' \right\|_{L^2} \\ &= \left\| \int_s^t U(t-t')f(u(t'))dt' \right\|_{L^2} \\ &\leq C\|u\|_{L^4(s,t;L^\infty)}^4\|u_0\|_{L^2} \\ &\rightarrow 0 \end{aligned}$$

as  $t > s \rightarrow +\infty$ , since  $u \in L^4(\mathbb{R}; L^\infty)$ . This implies the existence of  $u_+ \in L^2$  such that

$$\|u(t) - U(t)u_+\|_{L^2} \rightarrow 0$$

as  $t \rightarrow +\infty$ . The case  $t \rightarrow -\infty$  follows in the same way.

### 3. EXISTENCE OF GROUND STATES AND PROOF OF THEOREM 3

First, we remark that the following variational problem is equivalent to  $d(\omega)$ :

$$d_1(\omega) = \inf \left\{ \frac{p-1}{2(p+1)}\|v\|_{L^{p+1}}^{p+1} : v \in H^1(\mathbb{R}) \setminus \{0\}, I_\omega(v) \leq 0 \right\}. \quad (3.1)$$

**Proof of Proposition 1.** Let  $\{v_j\}$  be a minimizing sequence for  $d_1(\omega)$  and  $v_j^*$  be a rearrangement of  $|v_j|$ . Therefore,  $\{v_j^*\}$  are nonnegative, symmetric and nonincreasing functions. For

the main properties of rearrangement functions, see Lieb and Loss [21], Folland [11] and Appendix of Berestycki and Lions [3]. In particular,

$$|v_j(0)|^2 = \int_{\mathbb{R}} \delta(x)|v_j(x)|^2 dx \leq \int_{\mathbb{R}} \delta(x)[|v_j(x)|^*]^2 dx = [|v_j(0)|^*]^2 = |v_j^*(0)|^2. \quad (3.2)$$

Indeed, it is known (see [21, Theorem 3.4]) that for any nonnegative functions  $f, g$  vanishing at infinity,

$$\int_{\mathbb{R}} f(x)g(x)dx \leq \int_{\mathbb{R}} f^*(x)g^*(x)dx, \quad (3.3)$$

where  $f^*$  and  $g^*$  are the symmetric-decreasing rearrangements of  $f$  and  $g$ . Here we employ

$$\rho_\varepsilon(x) = \frac{1}{2\sqrt{\varepsilon\pi}} e^{-|x|^2/4\varepsilon}, \quad \varepsilon > 0,$$

as  $f$  in (3.3), and we have the desired inequality in (3.2) letting  $\varepsilon \rightarrow 0$ . Therefore,  $\{v_j^*\}$  satisfies  $I_\omega(v_j^*) \leq 0$  and is also a minimizing sequence for  $d_1(\omega)$ . Here, we rewrite  $w_j = v_j^*$  for simplicity and we have

$$\frac{1}{2} \|Dw_j\|_{L^2}^2 + \omega \|w_j\|_{L^2}^2 - Z \int_{\mathbb{R}} \delta(x)|w_j(x)|^2 dx \leq C.$$

By (i) of Remark 1.3,  $(\omega + \lambda)\|w_j\|_{L^2}^2 \leq C$ . Also, by Sobolev embedding,

$$\begin{aligned} \frac{1}{2} \|Dw_j\|_{L^2}^2 &\leq C + Z|w_j(0)|^2 \\ &\leq C + C\|w_j\|_{L^2}\|Dw_j\|_{L^2} \\ &\leq C + C\left(\frac{1}{2\varepsilon}\|w_j\|_{L^2}^2 + \frac{\varepsilon}{2}\|Dw_j\|_{L^2}^2\right). \end{aligned}$$

Taking  $\varepsilon > 0$  sufficiently small, we have that  $\|Dw_j\|_{L^2}^2$  is bounded and so that  $\|w_j\|_{H^1}^2$  is bounded. From Strauss' radial lemma for one dimensional version (see Cazenave [5, Proposition 1.7.1]), there exists a subsequence still denoted by  $\{w_j\}$  and  $w_0 \in H^1(\mathbb{R})$  such that  $w_j$  converges weakly to  $w_0$  in  $H^1(\mathbb{R})$  and that  $w_j$  converges strongly to  $w_0$  in  $L^r(\mathbb{R})$  for any  $r$  with  $2 < r \leq \infty$ . Accordingly,  $w_j \rightarrow w_0$  a.e.  $x$  in  $\mathbb{R}$  and we have

$$\begin{aligned} \lim_{j \rightarrow \infty} \int_{\mathbb{R}} \delta(x)|w_j(x)|^2 dx &= \lim_{j \rightarrow \infty} |w_j(0)|^2 = |w_0(0)|^2 = \int_{\mathbb{R}} \delta(x)|w_0(x)|^2 dx, \\ \lim_{j \rightarrow \infty} \|w_j\|_{L^{p+1}}^{p+1} &= \|w_0\|_{L^{p+1}}^{p+1}. \end{aligned}$$

Thus,  $I_\omega(w_0) \leq \liminf_{j \rightarrow \infty} I_\omega(w_j) \leq 0$  by the weak limit. By definition of  $d_1(\omega)$ , we have

$$d_1(\omega) \leq \frac{p-1}{2(p+1)} \|w_0\|_{L^{p+1}}^{p+1} = \lim_{j \rightarrow \infty} \frac{p-1}{2(p+1)} \|w_j\|_{L^{p+1}}^{p+1} = d_1(\omega).$$

This concludes that  $w_0$  is a minimizer of  $d_1(\omega)$ .  $\square$

After the verification of existence of a minimizer, it is easy to conclude Proposition 1 since the minimizer is already nonnegative, symmetric and nonincreasing by the process of

rearrangement. To assure that this minimizer satisfies the boundary condition and decays at infinity, we prove the following lemma.

**Lemma 3.1.** *Let  $p > 1$ ,  $Z > 0$  and  $\omega > Z^2/2$ . Assume that  $v \in \mathcal{G}_\omega$ . Then  $v$  is symmetric, positive and satisfies the following.*

$$v \in C^j(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R}), \quad j = 1, 2. \quad (3.4)$$

$$-\frac{1}{2}D^2v + \omega v - v^p = 0, \quad x \neq 0, \quad (3.5)$$

$$Dv(0+) - Dv(0-) = -2Zv(0), \quad (3.6)$$

$$Dv(x), v(x) \rightarrow 0, \quad \text{as } |x| \rightarrow \infty. \quad (3.7)$$

**Proof.** Since  $v$  satisfies  $S'_\omega(v) = 0$ ,  $v$  satisfies (1.6). To check (3.4) and (3.7), we take an appropriate test function  $\xi \in C_0^\infty(\mathbb{R} \setminus \{0\})$ . Then  $\xi v$  satisfies

$$-\frac{1}{2}D^2(\xi v) + \omega \xi v = -\frac{1}{2}(D^2\xi)v - (D\xi)(Dv) + \xi v^p,$$

in the sense of distributions. We employ the standard bootstrap argument for this equation (see Section 8 of [5] for details). The right hand side is in  $L^2(\mathbb{R})$  and so  $\xi v \in H^2(\mathbb{R})$ , that is,  $v \in H^2(\mathbb{R} \setminus \{0\}) \cap C^1(\mathbb{R} \setminus \{0\})$ . The case of  $j = 2$  is similar. The equation (3.5) follows from the fact that  $C_0^\infty(\mathbb{R} \setminus \{0\})$  is dense in  $L^2(\mathbb{R})$ . Concerning (3.6), we integrate  $S'_\omega(v) = 0$  from  $-\varepsilon$  to  $\varepsilon$ .

$$-\frac{1}{2} \int_{-\varepsilon}^{\varepsilon} D^2v dx + \omega \int_{-\varepsilon}^{\varepsilon} v dx - Z \int_{-\varepsilon}^{\varepsilon} \delta(x)v dx = \int_{-\varepsilon}^{\varepsilon} v^p dx.$$

Then we have the initial boundary condition

$$Dv(0+) - Dv(0-) = -2Zv(0)$$

letting  $\varepsilon \rightarrow 0$ . Multiplying the equation (3.5) by  $Dv$  and integrating resulting terms in  $x > 0$  and in  $x < 0$ , we have

$$-\frac{1}{4}(Dv)^2 = F(v(x)), \quad x \neq 0. \quad (3.8)$$

We note that  $v(x) > 0$  for  $x \in \mathbb{R}$ . If not, there exists  $x_0$  such that  $v(x_0) = 0$ . From (3.8), we have  $Dv(x_0) = 0$ . It implies  $v \equiv 0$ , which is impossible.  $\square$

To show Theorem 3, we check the sufficient condition for stability and instability in Proposition 2.

**Proof of Theorem 3.** We put  $\alpha = \omega^{-1/2}$  and then it follows from (1.7) that

$$\begin{aligned} \frac{\partial}{\partial \omega} \|\phi_\omega\|_{L^2}^2 &= -\frac{\partial}{\partial \alpha} \|\phi_\alpha\|_{L^2}^2 = C_p \alpha^{-4/(p-1)} g(\alpha), \\ g(\alpha) &= \frac{p-5}{p-1} J(\alpha) - \alpha Z (1 - C_\alpha^2)^{-(p-3)/(p-1)}, \\ J(\alpha) &= \int_{A(\alpha)}^{\infty} \operatorname{sech}^{4/(p-1)} y dy, \quad A(\alpha) = \tanh^{-1}(C_\alpha), \end{aligned}$$

where  $C_\alpha = Z\alpha$  and  $C_p$  is a constant depending only on  $p$ . It suffices to check the sign of  $g(\alpha)$ . In the case where  $Z > 0$  and  $p \leq 5$ , we have  $g(\alpha) < 0$  for any  $\alpha \in (0, 1/Z)$ . In the case where  $Z > 0$  and  $p > 5$ , we see that  $g'(\alpha) > 0$  in a neighborhood of 0,  $g'(\alpha) < 0$  in a neighborhood of  $1/Z$  and that  $g''(\alpha) < 0$  for any  $\alpha \in (0, 1/Z)$ . Therefore, there exists a unique  $\alpha^* \in (0, 1/Z)$  such that  $g(\alpha^*) = 0$ ,  $g(\alpha) > 0$  for any  $\alpha \in (0, \alpha^*)$  and that  $g(\alpha) < 0$  for any  $\alpha \in (\alpha^*, 1/Z)$  since  $g(0) > 0$ .  $\square$

For the sake of completeness, we give a remark on the proof of Proposition 2. First, we consider the stability. We explain briefly because the proof is similar to that of [12, Proposition 1] (see also Fibich and Wang[10]). We remark that  $d'(\omega) = Q(\phi_\omega)$  and it follows from the explicit form of (1.7) that the mapping  $\omega \mapsto \phi_\omega$  is  $C^1$ .

We introduce the  $C^1$  map  $\omega(\cdot) : U_\eta(\phi_\omega) \rightarrow \mathbb{R}$  defined by

$$\omega(u) = d^{-1} \left( \frac{p-1}{2(p+1)} \|u\|_{L^{p+1}}^{p+1} \right). \quad (3.9)$$

Here, let us denote  $\phi_{\omega_0}$  by  $\phi_0$  for simplicity. The following lemma is important to have the stability. We omit the proof since it is the same as that of Lemma 4.2 in [12].

**Lemma 3.2.** *Let  $p > 1$ ,  $Z > 0$  and  $\omega > Z^2/2$ . Assume  $d''(\omega) > 0$  at  $\omega = \omega_0$  for some  $\omega_0 \in (Z^2/2, \infty)$ . Then there exists  $\eta = \eta(\omega_0) > 0$  such that for all  $u \in U_\eta(\phi_0)$ ,*

$$E(u) - E(\phi_0) + \omega(u) \{Q(u) - Q(\phi_0)\} \geq \frac{1}{4} d''(\omega_0) (\omega(u) - \omega_0)^2.$$

We verify the statement of Proposition 2 (i) by contradiction. Assume that  $e^{i\omega_0 t} \phi_0(x)$  is unstable in  $H^1(\mathbb{R})$ . Then we have  $\varepsilon_0 > 0$  and initial data  $u_k(0) \in U_{1/k}(\phi_0)$  such that

$$\sup_{t \geq 0} \inf_{\theta \in \mathbb{R}} \|u_k(t) - e^{i\theta} \phi_0\|_{H^1} \geq \varepsilon_0,$$

where  $u_k(t)$  is the solution of (1.1) with initial data  $u_k(0)$ . Let  $t_k$  be the first time at which

$$\inf_{\theta \in \mathbb{R}} \|u_k(t_k) - e^{i\theta} \phi_0\|_{H^1} = \frac{\varepsilon_0}{2}. \quad (3.10)$$

We put  $v_k = u_k(t_k)$ . Since  $E$  and  $Q$  are conserved in  $t$ , we have

$$|E(v_k) - E(\phi_0)| = |E(u_k(0)) - E(\phi_0)| \rightarrow 0, \quad (3.11)$$

$$|Q(v_k) - Q(\phi_0)| = |Q(u_k(0)) - Q(\phi_0)| \rightarrow 0 \quad (3.12)$$

as  $k \rightarrow \infty$ . From (3.10), we have  $\|v_k\|_{H^1} \leq C$  uniformly in  $k$ . Also we note that  $\omega_k = \omega(v_k)$  is uniformly bounded in  $k$  since  $\omega(u)$  is a continuous map. Here, we take  $\eta$  small enough so that Lemma 3.2 may be applied. Then we have

$$E(v_k) - E(\phi_0) + \omega_k\{Q(v_k) - Q(\phi_0)\} \geq \frac{1}{4}d''(\omega_0)(\omega_k - \omega_0)^2. \quad (3.13)$$

Since  $d''(\omega_0) > 0$ , this implies that  $\omega_k \rightarrow \omega_0$  as  $k \rightarrow \infty$ . By using  $I_{\omega_0}(\phi_0) = 0$  and the fact that  $d(\cdot)$  is continuous, it follows that

$$\lim_{k \rightarrow \infty} \frac{p-1}{2(p+1)} \|v_k\|_{L^{p+1}}^{p+1} = \lim_{k \rightarrow \infty} d(\omega_k) = d(\omega_0) = \frac{p-1}{2(p+1)} \|\phi_0\|_{L^{p+1}}^{p+1}. \quad (3.14)$$

From (3.11) and (3.12), we have

$$\begin{aligned} S_{\omega_0}(v_k) &= S_{\omega_0}(v_k) - S_{\omega_0}(\phi_0) + S_{\omega_0}(\phi_0) \\ &= E(v_k) - E(\phi_0) + \omega_0(Q(v_k) - Q(\phi_0)) + d(\omega_0) \\ &\rightarrow d(\omega_0), \end{aligned} \quad (3.15)$$

as  $k \rightarrow \infty$ . Let  $w_k = (\|\phi_0\|_{L^{p+1}}/\|v_k\|_{L^{p+1}})v_k$ . Then,  $w_k$  satisfies  $\|w_k\|_{L^{p+1}} = \|\phi_0\|_{L^{p+1}}$  and  $\|w_k - v_k\|_{H^1} \rightarrow 0$  as  $k \rightarrow \infty$ . Furthermore, by (3.14) and (3.15),  $S_{\omega_0}(w_k) \rightarrow d(\omega_0)$  as  $k \rightarrow \infty$ . Therefore,  $\{w_k\}$  is a minimizing sequence for  $d(\omega_0)$ . By the compactness of rearrangement functions (see the proof of Proposition 1) and the uniqueness of minimizers of  $d(\omega_0)$ , there exists a sequence  $\{\theta_k\} \subset \mathbb{R}$  such that  $\|w_k - e^{i\theta_k}\phi_0\|_{H^1} \rightarrow 0$  as  $k \rightarrow \infty$ . Namely, we get

$$\|v_k - e^{i\theta_k}\phi_0\|_{H^1} \rightarrow 0$$

as  $k \rightarrow \infty$ , which is a contradiction to (3.10).  $\square$

Next, concerning the sufficient condition for instability, i.e., Proposition 2 (ii), we can apply a similar method by Shatah and Strauss [27] to our present case. Indeed, we modify the function  $\psi(\omega)$ , which was defined in the proof of Theorem 5 in [27] and used to determine an unstable direction in solving the ordinary differential equation of Lemma 11 in [27], as the following: For any fixed  $\omega_0 \in (Z^2/2, \infty)$ , we put for any  $v_\omega \in \mathcal{G}_\omega$ ,

$$\psi(\omega) = \lambda(\omega)v_\omega(x), \quad \lambda(\omega) = \frac{\|v_{\omega_0}\|_{L^2}}{\|v_\omega\|_{L^2}}$$

for any  $\omega \in (Z^2/2, \infty)$  which is close to  $\omega_0$ . We also remark that our variational characterization is different from theirs only in the point that  $d(\omega) = \frac{p-1}{2(p+1)}\|v_\omega\|_{L^{p+1}}^{p+1}$ . Accordingly, we change the norm in the statement of Lemma 11 iv) in [27] to  $L^{p+1}$  norm. In consequence, their method works and Theorem 17 in [27] holds for the present case.

## REFERENCES

1. S. Albeverio, F. Gesztesy, R. Høgh-Krohn and H. Holden, "Solvable models in quantum mechanics," Springer-Verlag, New York, 1988.
2. H. Berestycki and T. Cazenave, "*Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéaires*", C. R. Acad. Sci. Paris. Vol. 293 (1981), 489–492.
3. H. Berestycki and P. L. Lions, "*Nonlinear scalar field equations, I-Existence of a ground state*", Arch. Rat. Mech. Anal. Vol. 82 (1983), 313–346.
4. S. M. Blinder, "*Green's function and propagator for the one-dimensional  $\delta$ -function potential*", Phys. Rev. A. Vol. 37 (1988) 973–976.
5. T. Cazenave, "Semilinear Schrödinger equations," Courant Lecture Notes in Mathematics, 10, American Mathematical Society, Courant Institute of Mathematical Sciences, 2003.
6. T. Cazenave and P. L. Lions, "*Orbital stability of standing waves for some nonlinear Schrödinger equations*", Commun. Math. Phys. Vol. 85 (1982), 549–561.
7. C. Cid and P. Felmer, "*Orbital stability of standing waves for the nonlinear Schrödinger equation with potential*", Rev. Math. Phys. Vol. 13 (2001), 1529–1546.
8. A. Comech and D. Pelinovsky, "*Purely nonlinear instability of standing waves with minimal energy*", Commun. Pure Appl. Math. Vol. 56 (2003), 1565–1607.
9. A. de Bouard and R. Fukuizumi, "*Stability of standing waves for nonlinear Schrödinger equations with inhomogeneous nonlinearities*", Ann. Henri Poincaré. Vol. 6 (2005), 1157–1177.
10. G. Fibich and X. P. Wang, "*Stability of solitary waves for nonlinear Schrödinger equations with inhomogeneous nonlinearities*", Physica D. Vol. 175 (2003), 96–108.
11. G. Folland, "Real analysis," second edition, Wiley-Interscience, 1999.
12. R. Fukuizumi, "*Stability of standing waves for nonlinear Schrödinger equations with critical power nonlinearity and potentials*", Advances in Differential Equations. Vol. 10 (2005), 259–276.
13. R. Fukuizumi and M. Ohta, "*Instability of standing waves for nonlinear Schrödinger equations with potentials*", Differential and Integral Eqs. Vol. 16 (2003), 691–706.
14. R. Fukuizumi and M. Ohta, "*Stability of standing waves for nonlinear Schrödinger equations with potentials*", Differential and Integral Eqs. Vol. 16 (2003), 111–128.
15. R. H. Goodman, P. J. Holmes and M. I. Weinstein, "*Strong NLS soliton-defect interactions*", Physica D. Vol. 192 (2004) 215–248.
16. M. J. Goovaerts, A. Babenco and J. T. Devresse, "*A new expansion method in the Feynman path integral formalism: application to a one-dimensional delta-function potential*", J. Math. Phys. Vol. 14 No. 5(1973) 554–559.
17. M. Grillakis, J. Shatah and W. Strauss, "*Stability theory of solitary waves in the presence of symmetry I*", J. Funct. Anal. Vol. 74 (1987), 160–197.
18. M. Grillakis, J. Shatah and W. Strauss, "*Stability theory of solitary waves in the presence of symmetry II*", J. Funct. Anal. Vol. 94 (1990), 308–348.
19. Y. Kabeya and K. Tanaka, "*Uniqueness of positive radial solutions of semilinear elliptic equations in  $\mathbb{R}^n$  and Séré's non-degeneracy condition*", Commun. Partial. Differential. Eqs. Vol. 24 (1999), 563–598.
20. M. Kunze, T. Küpper, V. K. Mezentsev, E. G. Shapiro and S. Turitsyn, "*Nonlinear solitary waves with Gaussian tails*", Physica D. Vol. 128 (1999), 273–295.
21. E. H. Lieb and M. Loss, "Analysis," 2nd edition, American Mathematical Society, 2001.
22. Y. G. Oh, "*Stability of semiclassical bound states of nonlinear Schrödinger equations with potentials*", Commun. Math. Phys. Vol. 121 (1989), 11–33.

23. M. Ohta, "*Stability and instability of standing waves for one dimensional nonlinear Schrödinger equations with double power nonlinearity*", Kodai. Math. J. Vol. 18 (1995), 68–74.
24. T. Ozawa, "*Remarks on proofs of conservation laws for nonlinear Schrödinger equations*", Cal. Var. PDE (in press).
25. H. A. Rose and M. I. Weinstein, "*On the bound states of the nonlinear Schrödinger equation with a linear potential*", Physica D. Vol. 30 (1988), 207–218.
26. J. Shatah, "*Stable standing waves of nonlinear Klein-Gordon equations*", Commun. Math. Phys. Vol. 91 (1983), 313–327.
27. J. Shatah and W. Strauss, "*Instability of nonlinear bound states*", Commun. Math. Phys. Vol. 100 (1985), 173–190.
28. O. N. Strand, "*A method for the computation of the error function of a complex variable*", Mathematics of Computation. Vol. 19 (1965), 127–129.
29. C. Sulem and P.-L. Sulem, "The nonlinear Schrödinger equation. Self-focusing and wave collapse," Applied Mathematical Sciences, 139. Springer-Verlag, New York, 1999.
30. M. I. Weinstein, "*Nonlinear Schrödinger equations and sharp interpolation estimates*", Commun. Math. Phys. Vol. 87 (1983), 567–576.
31. M. I. Weinstein, "*Lyapunov stability of ground states of nonlinear dispersive evolution equations*", Commun. Pure Appl. Math. Vol. 39 (1986), 51–68
32. J. Zhang, "*Stability of standing waves for the nonlinear Schrödinger equations with unbounded potentials*", Z. Angew. Math. Phys. Vol. 51 (2000), 489–503.
33. J. Zhang, "*Stability of Attractive Bose-Einstein Condensates*", Journal of Statistical Physics. Vol. 101 (2000), 731–745.

(Reika FUKUIZUMI) DEPARTMENT OF MATHEMATICS,  
 HOKKAIDO UNIVERSITY, SAPPORO 060-0810, JAPAN,  
 AND CURRENT ADDRESS: LABORATOIRE DE MATHÉMATIQUE,  
 UNIVERSITÉ PARIS SUD, 91405 ORSAY, FRANCE  
*E-mail address:* Reika.Fukuizumi@math.u-psud.fr

(Masahito OHTA) DEPARTMENT OF MATHEMATICS,  
 FACULTY OF SCIENCE, SAITAMA UNIVERSITY, SAITAMA 338-8570, JAPAN  
*E-mail address:* mohta@rimath.saitama-u.ac.jp

(Tohru OZAWA) DEPARTMENT OF MATHEMATICS,  
 HOKKAIDO UNIVERSITY, SAPPORO 060-0810, JAPAN