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Spectral Area Estimates For Norms Of Commutators

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Abstract. Let A and B be commuting bounded linear operators on a Hilbert space. In this paper, we study spectral area estimates for norms of $A^*B - BA^*$ when A is subnormal or p -hyponormal.

§1. Introduction

Let \mathcal{H} be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators on \mathcal{H} . If T is a hyponormal operator in $\mathcal{B}(\mathcal{H})$ then C.R.Putnam [7] proved that $\|T^*T - TT^*\| \leq \text{Area}(\sigma(T))/\pi$ where $\sigma(T)$ is the spectrum of T . The second named author [5] has proved that if T is a hyponormal operator and K is in $\mathcal{B}(\mathcal{H})$ with $KT = TK$ then

$$\|T^*K - KT^*\| \leq 2\{\text{Area}(\sigma(T))/\pi\}^{1/2}\|K\|.$$

We don't know whether the constant 2 in the inequality is best possible for a hyponormal operator. In §2, we show that the constant is not best possible for a subnormal operator.

When T is a p -hyponormal operator in $\mathcal{B}(\mathcal{H})$, A.Uchiyama [10] generalized the Putnam inequality, that is,

$$\|T^*T - TT^*\| \leq \phi\left(\frac{1}{p}\right) \|T\|^{2(1-p)}\{\text{Area}(\sigma(T))/\pi\}^p.$$

This inequality gives the Putnam inequality when $p = 1$. In §3, we generalize the above inequality for the spectral area estimate of $\|T^*K - KT^*\|$ when $TK = KT$. H.Alexander [1] proved the following inequality for a uniform algebra A . If f is in A then

$$\text{dist}(\bar{f}, A) \leq \{\text{Area}(\sigma(f))/\pi\}^{1/2}.$$

The second named author [5] gave an operator version for the Alexander inequality. This was used in order to estimate $\|T^*K - KT^*\|$ when T is a hyponormal operator and $KT = TK$. We also give an Alexander inequality for a p -hyponormal and we use it to estimate $\|T^*K - KT^*\|$.

In §4, we try to estimate $\|T^*K - KT^*\|$ for arbitrary contraction. In §5, we show a few results about area estimates for p -quasihyponormal operators, restricted shifts and analytic Toeplitz operators.

For $0 < p \leq 1$, T is said to be p -hyponormal if $(T^*T)^p - (TT^*)^p \geq 0$. A 1-hyponormal operator is hyponormal. For an algebra \mathcal{A} in $\mathcal{B}(\mathcal{H})$, let $\text{lat}\mathcal{A}$ be the lattice of all \mathcal{A} -invariant projections. For a compact subset X in \mathcal{C} , $\text{rat}(X)$ denotes the set of all rational functions on X .

§2. Subnormal operator

In order to prove Theorem 1, we use the original Alexander inequality.

Theorem 1. *Let T be a subnormal operator in $\mathcal{B}(\mathcal{H})$ and f a rational function on $\sigma(T)$ whose poles are not on it. Then*

$$\|T^*f(T) - f(T)T^*\| \leq \{\text{Area}(\sigma(T))/\pi\}^{1/2}\{\text{Area}(\sigma(f(T)))/\pi\}^{1/2}.$$

Proof. Suppose that $N \in \mathcal{B}(\mathcal{K})$ is a normal extension of $T \in \mathcal{B}(\mathcal{H})$ and P is an orthogonal projection from \mathcal{K} to \mathcal{H} . Then $T = PN|_{\mathcal{H}}$ and so

$$\begin{aligned}
& T^*f(T) - f(T)T^* \\
&= PN^*Pf(N)P - Pf(N)PN^*P \\
&= PN^*f(N)P - Pf(N)PN^*P \\
&= Pf(N)N^*P - Pf(N)PN^*P \\
&= Pf(N)(1 - P)N^*P \\
&= Pf(N)(1 - P) \cdot (1 - P)N^*P
\end{aligned}$$

because $f(N)P = Pf(N)P$ and $f(N)N^* = N^*f(N)$.

Let F be a rational function in $rat(\sigma(T))$. Put $\mathcal{B}_F =$ the norm closure of $\{g(F(N)) ; g \in rat(\sigma(F(N)))\}$ then P belongs to $lat\mathcal{B}_F$. Hence

$$\begin{aligned}
& \| (1 - P)F(N)^*P \| \\
&\leq dist(F(N)^*, \mathcal{B}_F) \leq dist(\bar{z}, rat(\sigma(F(N)))) \\
&\leq \{Area(\sigma(F(N)))/\pi\}^{1/2}
\end{aligned}$$

by the Alexander's theorem [1]. Hence, applying F to $F = z$ or $F = f$

$$\begin{aligned}
& \| T^*f(T) - f(T)T^* \| \\
&\leq \| (1 - P)f(N)^*P \| \cdot \| (1 - P)N^*P \| \\
&\leq \{Area(\sigma(f(N)))/\pi\}^{1/2} \{Area(\sigma(N))/\pi\}^{1/2} \\
&\leq \{Area(\sigma(f(T)))/\pi\}^{1/2} \{Area(\sigma(T))/\pi\}^{1/2}.
\end{aligned}$$

If T is a cyclic subnormal operator and $KT = TK$ then using a theorem of T.Yoshino [12] we can prove that

$$\|T^*K - KT^*\| \leq \{Area(\sigma(T))/\pi\}^{1/2} \{Area(\sigma(K))/\pi\}^{1/2}.$$

The proof is almost same to one of Theorem 1.

§3. p -hyponormal

In order to prove Theorem 2, we use an operator version of the Alexander inequality for a p -hyponormal operator. Unfortunately Lemma 3 is not best possible for $p = 1$ (see [5]). Lemma 1 is due to W.Arveson [2, Lemma 2] and Lemma 2 is due to A.Uchiyama [11, Theorem 3].

We need the following notation to give Theorem 2 and Proposition 1. Let ϕ be a positive function on $(0, \infty)$ such that

$$\phi(t) = \begin{cases} t & \text{if } t \text{ is an integer} \\ t + 2 & \text{if } t \text{ is not an integer.} \end{cases}$$

We write $\ell^2 \otimes \mathcal{H}$ for the Hilbert space direct sum $\mathcal{H} \oplus \mathcal{H} \oplus \cdots$, and $1 \otimes T$ denotes the operator $T \oplus T \oplus \cdots \in \mathcal{B}(\ell^2 \otimes \mathcal{H})$ for each operator $T \in \mathcal{B}(\mathcal{H})$.

Lemma 1. *Let \mathcal{A} be an arbitrary ultra-weakly closed subalgebra of $\mathcal{B}(\mathcal{H})$ containing 1, and let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\text{dist}(T, \mathcal{A}) = \sup\{\|(1 - P)(1 \otimes T)P\| ; P \in \text{lat}(1 \otimes \mathcal{A})\}.$$

Lemma 2. *If T is a p -hyponormal operator, then*

$$\|T^*T - TT^*\| \leq \phi\left(\frac{1}{p}\right) \|T\|^{2(1-p)} \{Area(\sigma(T))/\pi\}^p.$$

Lemma 3. *If T is a p -hyponormal operator then*

$$\text{dist}(T^*, \mathcal{A}) \leq \sqrt{2\phi\left(\frac{1}{p}\right) \|T\|^{1-p} \{Area(\sigma(T))/\pi\}^{p/2}}$$

where \mathcal{A} is the strong closure of $\{f(T) ; f \in \text{rat}(\sigma(T))\}$.

Proof. Let $S = 1 \otimes T$. Then S is p -hyponormal. In order to prove the lemma, by Lemma 1 it is enough to estimate $\sup\{\|(1 - P)SP\| ; P \in \text{lat}(1 \otimes \mathcal{A})\}$. If $P \in \text{lat}(1 \otimes \mathcal{A})$ then $SP = PSP$ and so

$$\begin{aligned} & \|(1 - P)SP\|^2 \\ &= \|PSS^*P - PS^*SP\|^2 \\ &= \|PSS^*P - PS^*SP + PS^*SP - PS^*SP\|^2 \\ &\leq \|P(S^*S - SS^*)P\| + \|(PSP)^*(PSP) - (PSP)(PSP)^*\| \\ &\leq \|S^*S - SS^*\| + \|(PSP)^*(PSP) - (PSP)(PSP)^*\|. \end{aligned}$$

By [11, Lemma 4], PSP is p -hyponormal and so by Lemma 2 we have

$$\begin{aligned} & \|PSS^*P - PS^*SP\|^2 \\ &\leq \phi\left(\frac{1}{p}\right) \|T\|^{2(1-p)} \{Area(\sigma(T))/\pi\}^p + \phi\left(\frac{1}{p}\right) \|PSP\|^{2(1-p)} \{Area(\sigma(PSP))/\pi\}^p \\ &\leq 2\phi\left(\frac{1}{p}\right) \|T\|^{2(1-p)} \{Area(\sigma(T))/\pi\}^p \end{aligned}$$

because $\|PSP\| \leq \|S\| = \|T\|$ and $\sigma(PSP) \subset \sigma(S) = \sigma(T)$. By Lemma 1,

$$\text{dist}(T^*, \mathcal{A}) \leq \sqrt{2\phi\left(\frac{1}{p}\right)\|T\|^{1-p}\{Area(\sigma(T))/\pi\}^{p/2}}.$$

Theorem 2. *If T is a p -hyponormal operator in $\mathcal{B}(\mathcal{H})$ and if K is in $\mathcal{B}(\mathcal{H})$ with $KT = TK$, then*

$$\|T^*K - KT^*\| \leq 2\sqrt{2\phi\left(\frac{1}{p}\right)\|T\|^{1-p}\{Area(\sigma(T))/\pi\}^{p/2}}\|K\|.$$

Proof. When \mathcal{A} is the strong closure of $\{f(T) ; f \in \text{rat}(\sigma(T))\}$, for any $A \in \mathcal{A}$

$$\|T^*K - KT^*\| = \|(T^* - A)K + AK - KT^*\| \leq 2\|T^* - A\|\|K\|.$$

Now Lemma 3 implies the theorem.

In Theorem 2, if $p = 1$, that is, T is hyponormal then $\|T^*K - KT^*\| \leq 2\sqrt{2}\{Area(\sigma(T))/2\}^{1/2}\|K\|$. The constant $2\sqrt{2}$ is not best because the second author [5] proved that $\|T^*K - KT^*\| \leq 2\{Area(\sigma(T))/2\}^{1/2}\|K\|$. If $p = \frac{1}{2}$, that is, T is semi-hyponormal then $\|T^*K - KT^*\| \leq 4\|T\|^{1/2}\{Area(\sigma(T))/\pi\}^{1/4}\|K\|$.

§4. Norm estimates

In general, it is easy to see that $\|T^*T - TT^*\| \leq \|T\|^2$. By Theorem 1, if T is subnormal and f is an analytic polynomial then

$$\|T^*f(T) - f(T)T^*\| \leq \|T\|\|f(T)\|.$$

In this section, we will prove that $\|T^*T^n - T^nT^*\| \leq \|T\|^{n+1}$ for arbitrary T in $\mathcal{B}(\mathcal{H})$.

Theorem 3. *If T is a contraction on \mathcal{H} and f is an analytic function on the closed unit disc \bar{D} then $\|T^*f(T) - f(T)T^*\| \leq \sup_{z \in \bar{D}} |f(z)|$.*

Proof. By a theorem of Sz.-Nagy [6], there exists a unitary operator U on \mathcal{K} such that \mathcal{K} is a Hilbert space with $\mathcal{K} \supseteq \mathcal{H}$ and $T^n = PU^n|_{\mathcal{K}}$ for $n \geq 0$ where P is an orthogonal projection from \mathcal{K} to \mathcal{H} . Then it is known that $U^*P = PU^*P$ and $f(T) = Pf(U)|_{\mathcal{H}}$. Hence

$$\begin{aligned} & T^*f(T) - f(T)T^* \\ &= PU^*Pf(U)P - Pf(U)PU^*P \\ &= PU^*Pf(U)P - Pf(U)U^*P \\ &= PU^*(I - P)f(U)P \end{aligned}$$

because $U^*P = PU^*P$ and $f(U)U^* = U^*f(U)$. Therefore

$$\begin{aligned} & \| T^*f(T) - f(T)T^* \| \\ &= \| PU^*(I - P)f(U)P \| \leq \sup_{z \in D} | f(z) | . \end{aligned}$$

Corollary 1. *If T is in $\mathcal{B}(\mathcal{H})$ then for any $n \geq 1$ $\| T^*T^n - T^nT^* \| \leq \| T \|^{n+1}$.*

Proof. Put $A = T/\|T\|$ then A is a contraction and so by Theorem 2 $\|A^*A^n - A^nA^*\| \leq 1$ and so $\|T^*T^n - T^nT^*\| \leq \|T\|^{n+1}$.

§5. Remarks

In this section, we give spectral area estimates for p -quasihyponormal operators, restricted shifts and analytic Toeplitz operators.

For $0 < p \leq 1$, T is said to be p -quasihyponormal if $T^*\{(T^*T)^p - (TT^*)^p\}T \geq 0$. A 1-quasihyponormal operator is called quasihyponormal.

Lemma 4. *Let T be p -quasihyponormal and P be a projection such that $TP = PTP$. Then PTP is also p -quasihyponormal.*

Proof. Since T is p -quasihyponormal, $T^*(T^*T)^pT \geq T^*(TT^*)^pT$. Hence, we have

$$PT^*(T^*T)^pTP \geq PT^*(TT^*)^pTP.$$

Since by the Hansen's inequality [4]

$$\begin{aligned} PT^*(T^*T)^pTP &= (PTP)^*P(T^*T)^pP(PTP) \\ &\leq (PTP)^*(PT^*TP)^p(PTP) \\ &= (PTP)^*\{(PTP)^*(PTP)\}^p(PTP) \end{aligned}$$

and by $0 < p < 1$

$$\begin{aligned} PT^*(TT^*)^pTP &\geq (PT^*P)(TPT^*)^p(PTP) \\ &= (PTP)^*\{(PTP)(PTP)^*\}^p(PTP), \end{aligned}$$

we have

$$(PTP)^*\{(PTP)^*(PTP)\}^p \geq (PTP)^*\{(PTP)(PTP)^*\}^p(PTP).$$

Hence, PTP is p -quasihyponormal.

Proposition 1. *If T is a p -quasihyponormal operator in $\mathcal{B}(\mathcal{H})$ and if K is in $\mathcal{B}(\mathcal{H})$ with $KT = TK$, then*

$$\|T^*K - KT^*\| \leq 4 \left[\phi \left(\frac{1}{p} \right) \right]^{1/4} \|T\|^{1-p/2} \{Area(\sigma(T))/\pi\}^{p/4} \|K\|.$$

In particular, if T is quasihyponormal then

$$\|T^*K - KT^*\| \leq 4\|T\|^{1/2}\{Area(\sigma(T))/\pi\}^{1/4}\|K\|.$$

Proof. We can prove it as in the proof of Theorem 2. By [11, Theorem 6], $\|T^*T - TT^*\| \leq 2\|T\|^{2-p}\sqrt{\phi(\frac{1}{p})}\{Area(\sigma(T))/\pi\}^{p/2}$. Hence by Lemma 4

$$dist(T^*, \mathcal{A}) \leq 2\|T\|^{1-\frac{p}{2}}\phi\left(\frac{1}{p}\right)^{\frac{1}{4}}\{Area(\sigma(T))/\pi\}^{p/4}.$$

This implies the proposition.

Let H^2 and H^∞ be the usual Hardy spaces on the unit circle and z the coordinate function. M denotes an invariant subspace of H^2 under the multiplication by z . By the well known Beurling theorem, $M = qH^2$ for some inner function. Suppose N is the orthogonal complement of M in H^2 . For a function ϕ in H^∞ , S_ϕ is an operator on N such that $S_\phi f = P(\phi f)$ ($f \in N$) where P is the orthogonal projection from H^2 to N . For a symbol ϕ in L^∞ , T_ϕ denotes the usual Toeplitz operator on H^2 .

Proposition 2. *Suppose $\Phi = q\bar{\phi}$ belongs to H^∞ . Then*

- (1) $\|S_\phi^*S_\phi - S_\phi S_\phi^*\| \leq Area(\overline{\Phi(D)})/\pi$;
- (2) $\|S_\phi^*S_\phi^n - S_\phi^n S_\phi^*\| \leq \{Area(\overline{\Phi(D)})/\pi\}^{n+1}$ for $n \geq 0$.

Proof. By a well known theorem of Sarason [8],

$$\|S_\phi\| = \|\phi + qH^\infty\| = \|\bar{q}\phi + H^\infty\| = \|\bar{\Phi} + H^\infty\|.$$

By Nehari's theorem [6], $\|\bar{\Phi} + H^\infty\| = \|H_{\bar{\Phi}}\|$ where $H_{\bar{\Phi}}$ denotes a Hankel operator from H^2 to $\bar{z}H^2$. Since $\|H_{\bar{\Phi}}\|^2 = \|T_\Phi^*T_\Phi - T_\Phi T_\Phi^*\|$ where T_Φ denotes a Toeplitz operator on H^2 , by the Putnam inequality

$$\|T_\Phi^*T_\Phi - T_\Phi T_\Phi^*\| \leq Area(\sigma(T_\Phi))/\pi = Area(\overline{\Phi(D)})/\pi.$$

Now since $\|S_\phi^*S_\phi - S_\phi S_\phi^*\| \leq \|S_\phi\|^2$, (1) follows. (2) is also clear by the proof above and Corollary 1.

Proposition 3. *Suppose f and g are in H^∞ . Then*

$$\|T_f^*T_g - T_g T_f^*\| \leq \{Area(\overline{f(D)})/\pi\}^{1/2}\{Area(\overline{g(D)})/\pi\}^{1/2}$$

Proof. It is easy to see that $T_f^*T_g - T_g T_f^* = H_g^*H_{\bar{f}}$. Hence

$$\|T_f^*T_g - T_g T_f^*\| \leq \|H_g\| \cdot \|H_{\bar{f}}\|.$$

Since $H_{\bar{f}}^*H_{\bar{f}} = T_f^*T_f - T_fT_f^*$, by the Putnam inequality

$$\|T_f^*T_g - T_gT_f^*\| \leq \{Area(\overline{f(D)})/\pi\}^{1/2}\{Area(\overline{g(D)})/\pi\}^{1/2}.$$

References

1. H. Alexander, Projections of polynomial hulls, *J. Funct. Anal.* 13(1973), 13-19.
2. W. Arveson, Interpolation problems in nest algebras, *J. Funct. Anal.* 20(1975), 208-233.
3. M. Chō and M. Itoh, Putnam inequality for p -hyponormal operators, *Proc. Amer. Math. Soc.* 123(1995), 2435-2440.
4. F. Hansen, An operator inequality, *Math. Ann.* 246(1980), 249-250.
5. T. Nakazi, Complete spectral area estimates and self-commutators, *Michigan Math. J.* 35(1988), 435-441.
6. Z. Nehari, On bounded bilinear forms, *Ann. of Math. (2)* 65(1957), 153-162.
7. C. R. Putnam, An inequality for the area of hyponormal spectra, *Math. Z.* 116(1970), 323-330.
8. D. Sarason, Generalized interpolation in H^∞ , *Trans. Amer. Math. Soc.* 127(1967), 179-203.
9. B. Sz.-Nagy and C. Foias, *Harmonic Analysis Of Operators On Hilbert Space*. American Elsevier, New York, 1970.
10. A. Uchiyama, Berger-Shaw's theorem for p -hyponormal operators, *Integr. Equ. Oper. Theory* 33(1999), 221-230.
11. A. Uchiyama, Inequalities of Putnam and Berger-Shaw for p -quasihyponormal operators, *Integr. Equ. Oper. Theory* 34(1999), 91-106.
12. T. Yoshino, Subnormal operators with a cyclic vector, *Tohoku Math.* 21(1969), 47-55.

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