Spectral Area Estimates For Norms Of Commutators

By

Munoe Chō * And Takahiko Nakazi **

2000 Mathematics Subject Classification : Primary 47 A 20

Key words and phrases : subnormal, $p$-hyponormal, Putnam inequality

* This research is partially supported by Grant-in-Aid Scientific Research No.17540139
** This research is partially supported by Grant-in-Aid Scientific Research No.17540176
Abstract. Let $A$ and $B$ be commuting bounded linear operators on a Hilbert space. In this paper, we study spectral area estimates for norms of $A^*B - BA^*$ when $A$ is subnormal or $p$-hyponormal.
§1. Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$. If $T$ is a hyponormal operator in $\mathcal{B}(\mathcal{H})$ then C.R.Putnam [7] proved that $\| T^*T - TT^* \| \leq \text{Area}(\sigma(T))/\pi$ where $\sigma(T)$ is the spectrum of $T$. The second named author [5] has proved that if $T$ is a hyponormal operator and $K$ is in $\mathcal{B}(\mathcal{H})$ with $KT = TK$ then

$$\| T^*K - KT^* \| \leq 2\{ \text{Area}(\sigma(T))/\pi \}^{1/2} \| K \|.$$ 

We don’t know whether the constant 2 in the inequality is best possible for a hyponormal operator. In §2, we show that the constant is not best possible for a subnormal operator.

When $T$ is a $p$-hyponormal operator in $\mathcal{B}(\mathcal{H})$, A.Uchiyama [10] generalized the Putnam inequality, that is,

$$\| T^*T - TT^* \| \leq \phi \left( \frac{1}{p} \right) \| T \|^{(2 - p)} \{ \text{Area}(\sigma(T))/\pi \}^{p}.$$ 

This inequality gives the Putnam inequality when $p = 1$. In §3, we generalize the above inequality for the spectral area estimate of $\| T^*K - KT^* \|$ when $TK = KT$. H.Alexander [1] proved the following inequality for a uniform algebra $A$. If $f$ is in $A$ then

$$\text{dist}(\bar{f}, A) \leq \{ \text{Area}(\sigma(f))/\pi \}^{1/2}.$$ 

The second named author [5] gave an operator version for the Alexander inequality. This was used in order to estimate $\| T^*K - KT^* \|$ when $T$ is a hyponormal operator and $KT = TK$. We also give an Alexander inequality for a $p$-hyponormal and we use it to estimate $\| T^*K - KT^* \|$.

In §4, we try to estimate $\| T^*K - KT^* \|$ for arbitrary contraction. In §5, we show a few results about area estimates for $p$-quasihyponormal operators, restricted shifts and analytic Toeplitz operators.

For $0 < p \leq 1$, $T$ is said to be $p$-hyponormal if $(T^*T)^p - (TT^*)^p \geq 0$. A 1-hyponormal operator is hyponormal. For an algebra $A$ in $\mathcal{B}(\mathcal{H})$, let $\text{lat}A$ be the lattice of all $A$-invariant projections. For a compact subset $X$ in $\mathcal{C}$, $\text{rat}(X)$ denotes the set of all rational functions on $X$.

§2. Subnormal operator

In order to prove Theorem 1, we use the original Alexander inequality.

**Theorem 1.** Let $T$ be a subnormal operator in $\mathcal{B}(\mathcal{H})$ and $f$ a rational function on $\sigma(T)$ whose poles are not on it. Then

$$\| T^*f(T) - f(T)T^* \| \leq \{ \text{Area}(\sigma(T))/\pi \}^{1/2} \{ \text{Area}(\sigma(f(T)))/\pi \}^{1/2}.$$ 

3
Proof. Suppose that \( N \in B(K) \) is a normal extension of \( T \in B(H) \) and \( P \) is an orthogonal projection from \( K \) to \( H \). Then \( T = PN \mid H \) and so

\[
T^* f(T) - f(T)T^* = PN^* Pf(N)P - Pf(N)PN^* P = PN^* f(N)P - Pf(N)PN^* P = Pf(N)N^* P - Pf(N)PN^* P = Pf(N) (1 - P) N^* P = Pf(N) (1 - P) \cdot (1 - P) N^* P
\]

because \( f(N)P = Pf(N)P \) and \( f(N)N^* = N^* f(N) \).

Let \( F \) be a rational function in \( \text{rat}(\sigma(T)) \). Put \( B_F = \text{the norm closure of} \{g(F(N)) \ ; \ g \in \text{rat}(\sigma(F(N)))\} \) then \( P \) belongs to \( \text{lat}B_F \). Hence

\[
\| (1 - P) F(N)^* P \| \leq \text{dist}(F(N)^*, B_F) \leq \text{dist}(\bar{z}, \text{rat}(\sigma(F(N)))) \leq \{\text{Area}(\sigma(F(N)))/\pi\}^{1/2}
\]

by the Alexander’s theorem [1]. Hence, applying \( F \) to \( F = z \) or \( F = f \)

\[
\| T^* f(T) - f(T)T^* \| \leq \| (1 - P) f(N)^* P \| \cdot \| (1 - P) N^* P \| \leq \{\text{Area}(\sigma(f(N)))/\pi\}^{1/2}\{\text{Area}(\sigma(N))/\pi\}^{1/2} \leq \{\text{Area}(\sigma(f(T)))/\pi\}^{1/2}\{\text{Area}(\sigma(T))/\pi\}^{1/2}.
\]

If \( T \) is a cyclic subnormal operator and \( KT = TK \) then using a theorem of T.Yoshino [12] we can prove that

\[
\| T^* K - KT^* \| \leq \{\text{Area}(\sigma(T))/\pi\}^{1/2}\{\text{Area}(\sigma(K))/\pi\}^{1/2}.
\]

The proof is almost same to one of Theorem 1.

§3. \( p \)-hyponormal

In order to prove Theorem 2, we use an operator version of the Alexander inequality for a \( p \)-hyponormal operator. Unfortunately Lemma 3 is not best possible for \( p = 1 \) (see [5]). Lemma 1 is due to W.Arveson [2, Lemma 2] and Lemma 2 is due to A.Uchiyama [11, Theorem 3].
We need the following notation to give Theorem 2 and Proposition 1. Let $\phi$ be a positive function on $(0, \infty)$ such that

$$\phi(t) = \begin{cases} t & \text{if } t \text{ is an integer} \\ t + 2 & \text{if } t \text{ is not an integer} \end{cases}$$

We write $\ell^2 \otimes \mathcal{H}$ for the Hilbert space direct sum $\mathcal{H} \oplus \mathcal{H} \oplus \cdots$, and $1 \otimes T$ denotes the operator $T \oplus T \oplus \cdots \in \mathcal{B}(\ell^2 \otimes \mathcal{H})$ for each operator $T \in \mathcal{B}(\mathcal{H})$.

**Lemma 1.** Let $A$ be an arbitrary ultra-weakly closed subalgebra of $\mathcal{B}(\mathcal{H})$ containing $1$, and let $T \in \mathcal{B}(\mathcal{H})$. Then

$$\text{dist}(T, A) = \sup\{\| (1 - P)(1 \otimes T)P \| ; P \in \text{lat}(1 \otimes A) \}.$$

**Lemma 2.** If $T$ is a $p$-hyponormal operator, then

$$\| T^*T - TT^* \| \leq \phi \left( \frac{1}{p} \right) \| T \|^2 (1 - p) \left\{ \text{Area}(\sigma(T)) / \pi \right\}^p.$$

**Lemma 3.** If $T$ is a $p$-hyponormal operator then

$$\text{dist}(T^*, A) \leq \sqrt{2 \phi \left( \frac{1}{p} \right) \| T \|^{1-p} \left\{ \text{Area}(\sigma(T)) / \pi \right\}^{p/2}}$$

where $A$ is the strong closure of $\{ f(T) ; f \in \text{rat}(\sigma(T)) \}$.

**Proof.** Let $S = 1 \otimes T$. Then $S$ is $p$-hyponormal. In order to prove the lemma, by Lemma 1 it is enough to estimate $\sup\{\| (1 - P)SP \| ; P \in \text{lat}(1 \otimes A) \}$. If $P \in \text{lat}(1 \otimes A)$ then $SP = PSP$ and so

$$\| (1 - P)SP \|^2 = \| PSS^*P - PSPS^*P \|$$

$$= \| PSS^*P - PS^*SP + PS^*SP - PPS^*P \|$$

$$\leq \| P(S^*S - SS^*)P \| + \| (PSP)^*(PSP) - (PSP)(PSP)^* \|$$

$$\leq \| S^*S - SS^* \| + \| (PSP)^*(PSP) - (PSP)(PSP)^* \|.$$

By [11, Lemma 4], $PSP$ is $p$-hyponormal and so by Lemma 2 we have

$$\| PSS^*P - PPS^*P \|^2$$

$$\leq \phi \left( \frac{1}{p} \right) \| T \|^{2(1-p)} \left\{ \text{Area}(\sigma(T)) / \pi \right\}^p + \phi \left( \frac{1}{p} \right) \| PSP \|^{2(1-p)} \left\{ \text{Area}(\sigma(PSP)) / \pi \right\}^p$$

$$\leq 2\phi \left( \frac{1}{p} \right) \| T \|^{2(1-p)} \left\{ \text{Area}(\sigma(T)) / \pi \right\}^p.$$
because \(\|PSP\| \leq \|S\| = \|T\|\) and \(\sigma(PSP) \subset \sigma(S) = \sigma(T)\). By Lemma 1,
\[
\text{dist}(T^*, A) \leq \sqrt{2\phi \left(\frac{1}{p}\right)}\|T\|^{1-p}\{\text{Area}(\sigma(T))/\pi\}^{p/2}.
\]

**Theorem 2.** If \(T\) is a \(p\)-hyponormal operator in \(\mathcal{B}(\mathcal{H})\) and if \(K\) is in \(\mathcal{B}(\mathcal{H})\) with \(KT = TK\), then
\[
\|T^*K - KT^*\| \leq 2\sqrt{2\phi \left(\frac{1}{p}\right)}\|T\|^{1-p}\{\text{Area}(\sigma(T))/\pi\}^{p/2}\|K\|.
\]

Proof. When \(A\) is the strong closure of \(\{f(T) ; f \in \text{rat}(\sigma(T))\}\), for any \(A \in A\)
\[
\|T^*K - KT^*\| = \|(T^* - A)K + AK - KT^*\| \leq 2\|T^* - A\||K||K||.
\]

Now Lemma 3 implies the theorem.

In Theorem 2, if \(p = 1\), that is, \(T\) is hyponormal then \(\|T^*K - KT^*\| \leq 2\sqrt{2\{\text{Area}(\sigma(T))/2\}}^{1/2}\|K\|\). The constant \(2\sqrt{2}\) is not best because the second author [5] proved that \(\|T^*K - KT^*\| \leq 2\{\text{Area}(\sigma(T))/2\}^{1/2}\|K\|\). If \(p = \frac{1}{2}\), that is, \(T\) is semi-hyponormal then \(\|T^*K - KT^*\| \leq 4\|T\|^{1/2}\{\text{Area}(\sigma(T))/\pi\}^{1/4}\|K\|\).

§4. Norm estimates

In general, it is easy to see that \(\|T^*T - TT^*\| \leq \|T\|^2\). By Theorem 1, if \(T\) is subnormal and \(f\) is an analytic polynomial then
\[
\|T^*f(T) - f(T)T^*\| \leq \|T\||\|f(T)\|.
\]

In this section, we will prove that \(\|T^*T^n - T^nT^*\| \leq \|T\|^{n+1}\) for arbitrary \(T\) in \(\mathcal{B}(\mathcal{H})\).

**Theorem 3.** If \(T\) is a contraction on \(\mathcal{H}\) and \(f\) is an analytic function on the closed unit disc \(\bar{D}\) then \(\|T^*f(T) - f(T)T^*\| \leq \sup_{z \in \bar{D}} |f(z)|\).

Proof. By a theorem of Sz.-Nagy [6], there exists a unitary operator \(U\) on \(\mathcal{K}\) such that \(\mathcal{K}\) is a Hilbert space with \(\mathcal{K} \supseteq \mathcal{H}\) and \(T^n = PU^n \mid \mathcal{K}\) for \(n \geq 0\) where \(P\) is an orthogonal projection from \(\mathcal{K}\) to \(\mathcal{H}\). Then it is known that \(U^*P = PU^*P\) and \(f(T) = Pf(U)\mid \mathcal{H}\). Hence
\[
T^*f(T) - f(T)T^*
= PU^*Pf(U)P - Pf(U)PU^*P
= PU^*Pf(U)P - Pf(U)Pf(U)
= PU^*(I - P)f(U)P.
\]
because $U^*P = PU^*P$ and $f(U)U^* = U^*f(U)$. Therefore

$$
\| T^* f(T) - f(T)T^* \|
= \| PU^*(I - P)f(U)P \| \leq \sup_{z \in D} | f(z) | .
$$

**Corollary 1.** If $T$ is in $\mathcal{B}(\mathcal{H})$ then for any $n \geq 1$ $\| T^* T^n - T^n T^* \| \leq \| T \|^{n+1}$. 

Proof. Put $A = T/\|T\|$ then $A$ is a contraction and so by Theorem 2

$$
\| A^* A^n - A^n A^* \| \leq 1 \text{ and so } \| T^* T^n - T^n T^* \| \leq \| T \|^{n+1}.
$$

§5. Remarks

In this section, we give spectral area estimates for $p$-quasihyponomal operators, restricted shifts and analytic Toeplitz operators.

For $0 < p \leq 1$, $T$ is said to be $p$-quasihyponormal if $T^*\{(T^*T)^p - (TT^*)^p\}T \geq 0$. A 1-quasihyponormal operator is called quasihyponormal.

**Lemma 4.** Let $T$ be $p$-quasihyponormal and $P$ be a projection such that $TP = PTP$. Then $PTP$ is also $p$-quasihyponormal.

Proof. Since $T$ is $p$-quasihyponormal, $T^* (T^*T)^p T \geq T^* (TT^*)^p T$. Hence, we have

$$
PT^* (T^*T)^p T \geq PT^* (TT^*)^p T P .
$$

Since by the Hansen’s inequality [4]

$$
PT^* (T^*T)^p T P = (PT^*)^P T^* T^* T P (PT P) \\
\leq (PT P)^* (PT^* T P)^P (PT P)
$$

and by $0 < p < 1$

$$
PT^* (TT^*)^p T P \geq (PT^*)^P (TP^* T P)^P (PT P)
$$

we have

$$
(PT P)^* \{(PT P)^* (PT P)\}^P (PT P),
$$

and by $0 < p < 1$

$$
(PT P)^* \{(PT P)^* (PT P)\}^P (PT P) \geq (PT P)^* \{(PT P)(PT P)^*\}^P (PT P).
$$

Hence, $PTP$ is $p$-quasihyponormal.

**Proposition 1.** If $T$ is a $p$-quasihyponormal operator in $\mathcal{B}(\mathcal{H})$ and if $K$ is in $\mathcal{B}(\mathcal{H})$ with $KT = TK$, then

$$
\| T^* K - KT^* \| \leq 4 \left[ \frac{1}{p} \right]^{1/4} \| T \|^{1-p/2} \{ \text{Area}(\sigma(T))/\pi \}^{p/4} \| K \| .
$$
In particular, if $T$ is quasihyponormal then
\[ \|T^*K - KT^*\| \leq 4\|T\|^{1/2}\{\text{Area}(\sigma(T))/\pi\}^{1/4}\|K\|. \]

Proof. We can prove it as in the proof of Theorem 2. By [11, Theorem 6],
\[ \|T^*T - TT^*\| \leq 2\|T\|^{2-p}\sqrt{\phi(\frac{1}{p})}\{\text{Area}(\sigma(T))/\pi\}^{p/2}. \]
Hence by Lemma 4
\[ \text{dist}(T^*, A) \leq 2\|T\|^{1 - \frac{p}{2}}\phi\left(\frac{1}{p}\right)^{\frac{1}{2}}\{\text{Area}(\sigma(T))/\pi\}^{p/4}. \]
This implies the proposition.

Let $H^2$ and $H^\infty$ be the usual Hardy spaces on the unit circle and $z$ the coordinate function. $M$ denotes an invariant subspace of $H^2$ under the multiplication by $z$. By the well known Beurling theorem, $M = qH^2$ for some inner function. Suppose $N$ is the orthogonal complement of $M$ in $H^2$. For a function $\phi$ in $H^\infty$, $S_\phi$ is an operator on $N$ such that $S_\phi f = P(\phi f)$ ($f \in N$) where $P$ is the orthogonal projection from $H^2$ to $N$. For a symbol $\phi$ in $L^\infty$, $T_\phi$ denotes the usual Toeplitz operator on $H^2$.

**Proposition 2.** Suppose $\Phi = q\bar{\phi}$ belongs to $H^\infty$. Then
1. $\|S_{\phi}^*S_{\phi} - S_{\bar{\phi}}S_{\bar{\phi}}^*\| \leq \text{Area}(\Phi(D))/\pi$;
2. $\|S_{\phi}^*S_{\phi}^n - S_{\bar{\phi}}S_{\bar{\phi}}^n\| \leq \{\text{Area}(\Phi(D))/\pi\}^{n+1}$ for $n \geq 0$.

Proof. By a well known theorem of Sarason [8],
\[ \|S_{\phi}\| = \|\phi + qH^\infty\| = \|\bar{q}\phi + H^\infty\| = \|\Phi + H^\infty\|. \]
By Nehari’s theorem [6], $\|\Phi + H^\infty\| = \|H_{\phi}\|$ where $H_{\phi}$ denotes a Hankel operator from $H^2$ to $\bar{z}H^2$. Since $\|H_{\phi}\|^2 = \|T_{\Phi}T_{\bar{\phi}} - T_{\Phi}T_{\bar{\phi}}\|$, $T_{\Phi}$ denotes a Toeplitz operator on $H^2$, by the Putnam inequality
\[ \|T_{\Phi}T_{\bar{\phi}} - T_{\Phi}T_{\bar{\phi}}\| \leq \text{Area}(\sigma(T_{\Phi}))/\pi = \text{Area}(\Phi(D))/\pi. \]
Now since $\|S_{\phi}^*S_{\phi} - S_{\bar{\phi}}S_{\bar{\phi}}^*\| \leq \|S_{\phi}\|^2$, (1) follows. (2) is also clear by the proof above and Corollary 1.

**Proposition 3.** Suppose $f$ and $g$ are in $H^\infty$. Then
\[ \|T_f^*T_g - T_g^*T_f\| \leq \{\text{Area}(f(D))/\pi\}^{1/2}\{\text{Area}(g(D))/\pi\}^{1/2}. \]

Proof. It is easy to see that $T_f^*T_g - T_g^*T_f = H_{\bar{g}}H_{\bar{f}}$. Hence
\[ \|T_f^*T_g - T_g^*T_f\| \leq \|H_{\bar{g}}\| \cdot \|H_{\bar{f}}\|. \]
Since $H_f^*H_f = T_f^*T_f - T_fT_f^*$, by the Putnam inequality
\[
\|T_f^*T_g - T_gT_f^*\| \leq \left\{ \frac{\text{Area}(f(D))}{\pi} \right\}^{1/2} \left\{ \frac{\text{Area}(g(D))}{\pi} \right\}^{1/2}.
\]

References


M. Chō
Department of Mathematics
Kanagawa University
Japan
chiyom01@kanagawa-u.ac.jp

T. Nakazi
Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060-0810, Japan
nakazi@math.sci.hokudai.ac.jp