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<th>項目</th>
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<tr>
<td>機関名</td>
<td>北海道大学</td>
</tr>
<tr>
<td>タイトル</td>
<td>譜面積推定:Norm of Commutators</td>
</tr>
<tr>
<td>著者</td>
<td>佐藤、守男; 中崎、高行</td>
</tr>
<tr>
<td>引用</td>
<td>北海道大学数学研究報告書, 771, 1-10</td>
</tr>
<tr>
<td>発行年</td>
<td>2006</td>
</tr>
<tr>
<td>DOI</td>
<td>10.14943/83921</td>
</tr>
<tr>
<td>ドキュメントURL</td>
<td><a href="http://hdl.handle.net/2115/69579">http://hdl.handle.net/2115/69579</a></td>
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<td>タイプ</td>
<td>bulletin (article)</td>
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<td>ファイル情報</td>
<td>pre771.pdf</td>
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HOKKAIDO UNIVERSITY
Spectral Area Estimates For Norms Of Commutators

By

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2000 Mathematics Subject Classification : Primary 47 A 20

Key words and phrases : subnormal, $p$-hyponormal, Putnam inequality

* This research is partially supported by Grant-in-Aid Scientific Research No.17540139
** This research is partially supported by Grant-in-Aid Scientific Research No.17540176
Abstract. Let $A$ and $B$ be commuting bounded linear operators on a Hilbert space. In this paper, we study spectral area estimates for norms of $A^*B - BA^*$ when $A$ is subnormal or $p$-hyponormal.
§1. Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$. If $T$ is a hyponormal operator in $\mathcal{B}(\mathcal{H})$ then C.R. Putnam [7] proved that $\|T^*T - TT^*\| \leq \text{Area}(\sigma(T))/\pi$ where $\sigma(T)$ is the spectrum of $T$. The second named author [5] has proved that if $T$ is a hyponormal operator and $K$ is in $\mathcal{B}(\mathcal{H})$ with $KT = TK$ then

$$\|T^*K - KT^*\| \leq 2\{\text{Area}(\sigma(T))/\pi\}^{1/2}\|K\|.$$ 

We don’t know whether the constant 2 in the inequality is best possible for a hyponormal operator. In §2, we show that the constant is not best possible for a subnormal operator.

When $T$ is a $p$-hyponormal operator in $\mathcal{B}(\mathcal{H})$, A.Uchiyama [10] generalized the Putnam inequality, that is,

$$\|T^*T - TT^*\| \leq \phi\left(\frac{1}{p}\right)\|T\|^{2(1-p)}\{\text{Area}(\sigma(T))/\pi\}^p.$$ 

This inequality gives the Putnam inequality when $p = 1$. In §3, we generalize the above inequality for the spectral area estimate of $\|T^*K - KT^*\|$ when $TK = KT$. H.Alexander [1] proved the following inequality for a uniform algebra $\mathcal{A}$. If $f$ is in $\mathcal{A}$ then

$$\text{dist}(\bar{f}, \mathcal{A}) \leq \{\text{Area}(\sigma(f))/\pi\}^{1/2}.$$ 

The second named author [5] gave an operator version for the Alexander inequality. This was used in order to estimate $\|T^*K - KT^*\|$ when $T$ is a hyponormal operator and $KT = TK$. We also give an Alexander inequality for a $p$-hyponormal and we use it to estimate $\|T^*K - KT^*\|$.

In §4, we try to estimate $\|T^*K - KT^*\|$ for arbitrary contraction. In §5, we show a few results about area estimates for $p$-quasihyponormal operators, restricted shifts and analytic Toeplitz operators.

For $0 < p \leq 1$, $T$ is said to be $p$-hyponormal if $(T^*T)^p - (TT^*)^p \geq 0$. A $1$-hyponormal operator is hyponormal. For an algebra $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$, let $\text{lat}\mathcal{A}$ be the lattice of all $\mathcal{A}$-invariant projections. For a compact subset $X$ in $\mathcal{C}$, $\text{rat}(X)$ denotes the set of all rational functions on $X$.

§2. Subnormal operator

In order to prove Theorem 1, we use the original Alexander inequality.

**Theorem 1.** Let $T$ be a subnormal operator in $\mathcal{B}(\mathcal{H})$ and $f$ a rational function on $\sigma(T)$ whose poles are not on it. Then

$$\|T^*f(T) - f(T)T^*\| \leq \{\text{Area}(\sigma(T))/\pi\}^{1/2}\{\text{Area}(\sigma(f(T)))/\pi\}^{1/2}.$$
Proof. Suppose that \( N \in B(K) \) is a normal extension of \( T \in B(H) \) and \( P \) is an orthogonal projection from \( K \) to \( H \). Then \( T = PN \mid H \) and so

\[
T^*f(T) - f(T)T^* = PN^*Pf(N)P - Pf(N)PN^*P = PN^*f(N)P - Pf(N)PN^*P = Pf(N)N^*P - Pf(N)PN^*P = Pf(N)(1 - P)N^*P = Pf(N)(1 - P) \cdot (1 - P)N^*P
\]

because \( f(N)P = Pf(N)P \) and \( f(N)N^* = N^*f(N) \).

Let \( F \) be a rational function in \( \text{rat}(\sigma(T)) \). Put \( B_F = \text{the norm closure of} \{g(F(N)) \mid g \in \text{rat}(\sigma(F(N)))\} \) then \( P \) belongs to \( \text{lat}B_F \). Hence

\[
\| (1 - P)f(N)^*P \| \leq \text{dist}(F(N)^*, B_F) \leq \text{dist}(\bar{z}, \text{rat}(\sigma(F(N)))) \\
\leq \left\{ \frac{\text{Area}(\sigma(F(N)))/\pi}{\pi} \right\}^{1/2}
\]

by the Alexander’s theorem [1]. Hence, applying \( F \) to \( F = z \) or \( F = f \)

\[
\| T^*f(T) - f(T)T^* \| \leq \| (1 - P)f(N)^*P \| \cdot \| (1 - P)N^*P \| \leq \left\{ \frac{\text{Area}(\sigma(f(N)))/\pi}{\pi} \right\}^{1/2} \left\{ \frac{\text{Area}(\sigma(N))/\pi}{\pi} \right\}^{1/2} \leq \left\{ \frac{\text{Area}(\sigma(f(T)))/\pi}{\pi} \right\}^{1/2} \left\{ \frac{\text{Area}(\sigma(T))/\pi}{\pi} \right\}^{1/2}.
\]

If \( T \) is a cyclic subnormal operator and \( KT = TK \) then using a theorem of T.Yoshino [12] we can prove that

\[
\| T^*K - KT^* \| \leq \left\{ \frac{\text{Area}(\sigma(T))/\pi}{\pi} \right\}^{1/2} \left\{ \frac{\text{Area}(\sigma(K))/\pi}{\pi} \right\}^{1/2}.
\]

The proof is almost same to one of Theorem 1.

§3. \( p \)-hyponormal

In order to prove Theorem 2, we use an operator version of the Alexander in-

equality for a \( p \)-hyponormal operator. Unfortunately Lemma 3 is not best possible for

\( p = 1 \) (see [5]). Lemma 1 is due to W.Arveson [2, Lemma 2] and Lemma 2 is due to

A.Uchiyama [11, Theorem 3].

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We need the following notation to give Theorem 2 and Proposition 1. Let \( \phi \) be a positive function on \((0, \infty)\) such that
\[
\phi(t) = \begin{cases} 
  t & \text{if } t \text{ is an integer} \\
  t + 2 & \text{if } t \text{ is not an integer}.
\end{cases}
\]

We write \( \ell^2 \otimes \mathcal{H} \) for the Hilbert space direct sum \( \mathcal{H} \oplus \mathcal{H} \oplus \cdots \), and \( 1 \otimes T \) denotes the operator \( T \oplus T \oplus \cdots \in \mathcal{B}(\ell^2 \otimes \mathcal{H}) \) for each operator \( T \in \mathcal{B}(\mathcal{H}) \).

**Lemma 1.** Let \( A \) be an arbitrary ultra-weakly closed subalgebra of \( \mathcal{B}(\mathcal{H}) \) containing \( 1 \), and let \( T \in \mathcal{B}(\mathcal{H}) \). Then
\[
\text{dist}(T, A) = \sup \{ \| (1 - P)(1 \otimes T)P \| ; \ P \in \text{lat}(1 \otimes A) \}.
\]

**Lemma 2.** If \( T \) is a \( p \)-hyponormal operator, then
\[
\| T^*T - TT^* \| \leq \phi \left( \frac{1}{p} \right) \| T \|^{2(1-p)} \{ \text{Area}(\sigma(T))/\pi \}^p.
\]

**Lemma 3.** If \( T \) is a \( p \)-hyponormal operator then
\[
\text{dist}(T^*, A) \leq \sqrt{2} \phi \left( \frac{1}{p} \right) \| T \|^{1-p} \{ \text{Area}(\sigma(T))/\pi \}^{p/2}
\]
where \( A \) is the strong closure of \( \{ f(T) : f \in \text{rat}(\sigma(T)) \} \).

Proof. Let \( S = 1 \otimes T \). Then \( S \) is \( p \)-hyponormal. In order to prove the lemma, by Lemma 1 it is enough to estimate \( \sup \{ \| (1 - P)SP \| ; \ P \in \text{lat}(1 \otimes A) \} \). If \( P \in \text{lat}(1 \otimes A) \) then \( SP = PSP \) and so
\[
\| (1 - P)SP \|^2 = \| PSS^*P - PPS^*P \| = \| PSS^*P - PS^*SP + PS^*SP - PPS^*P \| \\
\leq \| P(S^*S - SS^*)P \| + \| (PSP)^*(PSP) - (PSP)(PSP)^* \| \\
\leq \| S^*S - SS^* \| + \| (PSP)^*(PSP) - (PSP)(PSP)^* \|.
\]

By [11, Lemma 4], \( PSP \) is \( p \)-hyponormal and so by Lemma 2 we have
\[
\| PSS^*P - PPS^*P \|^2 \\
\leq \phi \left( \frac{1}{p} \right) \| T \|^{2(1-p)} \{ \text{Area}(\sigma(T))/\pi \}^p + \phi \left( \frac{1}{p} \right) \| PSP \|^{2(1-p)} \{ \text{Area}(\sigma(PSP))/\pi \}^p \\
\leq 2\phi \left( \frac{1}{p} \right) \| T \|^{2(1-p)} \{ \text{Area}(\sigma(T))/\pi \}^p
\]
because $\|PSP\| \leq \|S\| = \|T\|$ and $\sigma(PSP) \subset \sigma(S) = \sigma(T)$. By Lemma 1,

$$\text{dist}(T^*, A) \leq \sqrt{2\phi \left(\frac{1}{p}\right)}\|T\|^{1-p}\{\text{Area}(\sigma(T))/\pi\}^{p/2}.$$ 

**Theorem 2.** If $T$ is a $p$-hyponormal operator in $\mathcal{B}(\mathcal{H})$ and if $K$ is in $\mathcal{B}(\mathcal{H})$ with $KT = TK$, then

$$\|T^*K - KT^*\| \leq 2\sqrt{2\phi \left(\frac{1}{p}\right)}\|T\|^{1-p}\{\text{Area}(\sigma(T))/\pi\}^{p/2}\|K\|.$$ 

Proof. When $A$ is the strong closure of $\{f(T) : f \in \text{rat}(\sigma(T))\}$, for any $A \in A$

$$\|T^*K - KT^*\| = \|(T^* - A)K + AK - KT^*\| \leq 2\|T^* - A\|\|K\|.$$ 

Now Lemma 3 implies the theorem.

In Theorem 2, if $p = 1$, that is, $T$ is hyponormal then

$$\|T^*K - KT^*\| \leq 2\sqrt{2\{\text{Area}(\sigma(T))/2\}^{1/2}}\|K\|.$$ 

The constant $2\sqrt{2}$ is not best because the second author [5] proved that

$$\|T^*K - KT^*\| \leq 2\{\text{Area}(\sigma(T))/2\}^{1/2}\|K\|.$$ 

If $p = \frac{1}{2}$, that is, $T$ is semi-

$$\|T^*K - KT^*\| \leq 4\|T\|^{1/2}\{\text{Area}(\sigma(T))/\pi\}^{1/4}\|K\|.$$ 

§4. Norm estimates

In general, it is easy to see that $\|T^*T - TT^*\| \leq \|T\|^2$. By Theorem 1, if $T$ is subnormal and $f$ is an analytic polynomial then

$$\|T^*f(T) - f(T)T^*\| \leq \|T\|\|f(T)\|.$$ 

In this section, we will prove that $\|T^*T^n - T^nT^*\| \leq \|T\|^{n+1}$ for arbitrary $T$ in $\mathcal{B}(\mathcal{H})$.

**Theorem 3.** If $T$ is a contraction on $\mathcal{H}$ and $f$ is an analytic function on the closed unit disc $\bar{D}$ then

$$\|T^*f(T) - f(T)T^*\| \leq \sup_{z \in \bar{D}} |f(z)|.$$ 

Proof. By a theorem of Sz.-Nagy [6], there exists a unitary operator $U$ on $\mathcal{K}$ such that $\mathcal{K}$ is a Hilbert space with $\mathcal{K} \supset \mathcal{H}$ and $T^n = PU^n | \mathcal{K}$ for $n \geq 0$ where $P$ is an orthogonal projection from $\mathcal{K}$ to $\mathcal{H}$. Then it is known that $U^*P = PU^*P$ and $f(T) = Pf(U) | \mathcal{H}$. Hence

$$T^*f(T) - f(T)T^*$$

$$= PU^*Pf(U)P - Pf(U)PU^*P$$

$$= PU^*Pf(U)P - Pf(U)U^*P$$

$$= PU^*(I - P)f(U)P$$

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because \( U^*P = PU^* \) and \( f(U)U^* = U^*f(U) \). Therefore

\[
\| T^*f(T) - f(T)T^* \| = \| PU^*(I - P)f(U)P \| \leq \sup_{z \in D} | f(z) | .
\]

**Corollary 1.** If \( T \) is in \( \mathcal{B}(\mathcal{H}) \) then for any \( n \geq 1 \)
\[ \| T^*T^n - T^nT^* \| \leq \| T \|^n+1. \]

Proof. Put \( A = T/\| T \| \) then \( A \) is a contraction and so by Theorem 2
\[ \| A^*A^n - A^nA^* \| \leq 1 \] and so \( \| T^*T^n - T^nT^* \| \leq \| T \|^n+1. \)

§5. Remarks

In this section, we give spectral area estimates for \( p \)-quasihyponomal operators, restricted shifts and analytic Toeplitz operators.

For \( 0 < p \leq 1 \), \( T \) is said to be \( p \)-quasihyponormal if
\[ T^*\{(T^*T)^p - (TT^*)^p\}T \geq 0. \]
A 1-quasihyponormal operator is called quasihyponormal.

**Lemma 4.** Let \( T \) be \( p \)-quasihyponormal and \( P \) be a projection such that \( TP = PTP \). Then \( PTP \) is also \( p \)-quasihyponormal.

Proof. Since \( T \) is \( p \)-quasihyponormal, \( T^*(T^*T)^pT \geq T^*(TT^*)^pT \). Hence, we have

\[ PTP^*T^*TP = PT^*(TT^*)^pTP \]

Since by the Hansen’s inequality [4]

\[ PTP^*(TT^*)^pTP = (PTP)^*P(T^*T)^pP(PTP) \]

\[ \leq (PTP)^*PT^*TP^pPTP \]

\[ = (PTP)^*\{(PTP)^*PTP\}^pPTP \]

and by \( 0 < p < 1 \)

\[ PTP^*(TT^*)^pTP \geq (PTP)^*(TPT^*)^pPTP \]

\[ = (PTP)^*\{(PTP)(PTP)^*\}^pPTP, \]

we have

\[ (PTP)^*\{(PTP)^*(PTP)\}^p \geq (PTP)^*\{(PTP)(PTP)^*\}^pPTP. \]

Hence, \( PTP \) is \( p \)-quasihyponormal.

**Proposition 1.** If \( T \) is a \( p \)-quasihyponormal operator in \( \mathcal{B}(\mathcal{H}) \) and if \( K \) is in \( \mathcal{B}(\mathcal{H}) \) with \( KT = TK \), then

\[ \| T^*K - KT^* \| \leq 4 \left[ \phi \left( \frac{1}{p} \right) \right]^{1/4} \| T \|^{1-p/2} \{ \text{Area}(\sigma(T))/\pi \}^{p/4} \| K \|. \]

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In particular, if $T$ is quasihyponormal then
\[
\|T^*K - KT^*\| \leq 4\|T\|^{1/2}\{Area(\sigma(T))/\pi\}^{1/4}\|K\|.
\]

Proof. We can prove it as in the proof of Theorem 2. By [11, Theorem 6],
\[
\|T^*T - TT^*\| \leq 2\|T\|^{2-p}\sqrt{\phi(1/p)}\{Area(\sigma(T))/\pi\}^{p/2}.
\]
Hence by Lemma 4
\[
\dist(T^*, A) \leq 2\|T\|^{1-p/2}\phi(1/p)\{Area(\sigma(T))/\pi\}^{p/4}.
\]
This implies the proposition.

Let $H^2$ and $H^\infty$ be the usual Hardy spaces on the unit circle and $z$ the coordinate function. $M$ denotes an invariant subspace of $H^2$ under the multiplication by $z$. By the well known Beurling theorem, $M = qH^2$ for some inner function. Suppose $N$ is the orthogonal complement of $M$ in $H^2$. For a function $\phi$ in $H^\infty$, $S_\phi$ is an operator on $N$ such that $S_\phi f = P(\phi f)$ ($f \in N$) where $P$ is the orthogonal projection from $H^2$ to $N$. For a symbol $\phi$ in $L^\infty$, $T_\phi$ denotes the usual Toeplitz operator on $H^2$.

**Proposition 2.** Suppose $\Phi = q\overline{\phi}$ belongs to $H^\infty$. Then

1. $\|S_\phi^*S_\phi - S_\phi S_\phi^*\| \leq Area(\Phi(D))/\pi$;
2. $\|S_\phi^*S_\phi - S_\phi S_\phi^*\| \leq \{Area(\Phi(D))/\pi\}^{n+1}$ for $n \geq 0$.

Proof. By a well known theorem of Sarason [8],
\[
\|S_\phi\| = \|\phi + qH^\infty\| = \|\bar{\phi} + H^\infty\| = \|\Phi + H^\infty\|.
\]
By Nehari’s theorem [6], $\|\Phi + H^\infty\| = \|H_\Phi\|$ where $H_\Phi$ denotes a Hankel operator from $H^2$ to $\bar{z}H^2$. Since $\|H_\Phi\| = \|T_\Phi - T_\Phi^*\|$ where $T_\Phi$ denotes a Toeplitz operator on $H^2$, by the Putnam inequality
\[
\|T_\Phi S_\Phi - T_\Phi^* T_\Phi^*\| \leq Area(\sigma(T_\Phi))/\pi = Area(\Phi(D))/\pi.
\]
Now since $\|S_\phi^*S_\phi - S_\phi S_\phi^*\| \leq \|S_\phi\|^2$, (1) follows. (2) is also clear by the proof above and Corollary 1.

**Proposition 3.** Suppose $f$ and $g$ are in $H^\infty$. Then
\[
\|T_f^*T_g - T_g T_f^*\| \leq \{Area(f(D))/\pi\}^{1/2}\{Area(g(D))/\pi\}^{1/2}.
\]

Proof. It is easy to see that $T_f^*T_g - T_g T_f^* = H_\delta H_f$. Hence
\[
\|T_f^*T_g - T_g T_f^*\| \leq \|H_\delta\| \cdot \|H_f\|.
\]
Since $H_f^*H_f = T_f^*T_f - T_fT_f^*$, by the Putnam inequality

$$\|T_f^*f - T_g^*f\| \leq \left\{ \frac{\text{Area}(\overline{f(D)})}{\pi} \right\}^{1/2} \left\{ \frac{\text{Area}(\overline{g(D)})}{\pi} \right\}^{1/2}.$$ 

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