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Spectral Area Estimates For Norms Of Commutators

By

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Abstract. Let $A$ and $B$ be commuting bounded linear operators on a Hilbert space. In this paper, we study spectral area estimates for norms of $A^*B - BA^*$ when $A$ is subnormal or $p$-hyponormal.
§1. Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$. If $T$ is a hyponormal operator in $\mathcal{B}(\mathcal{H})$ then C.R.Putnam \cite{7} proved that $\|T^*T - TT^*\| \leq \text{Area}(\sigma(T))/\pi$ where $\sigma(T)$ is the spectrum of $T$. The second named author \cite{5} has proved that if $T$ is a hyponormal operator and $K$ is in $\mathcal{B}(\mathcal{H})$ with $KT = TK$ then

$$\|T^*K - KT^*\| \leq 2\text{Area}(\sigma(T))/\pi^{1/2}\|K\|.$$  

We don’t know whether the constant 2 in the inequality is best possible for a hyponormal operator. In §2, we show that the constant is not best possible for a subnormal operator.

When $T$ is a $p$-hyponormal operator in $\mathcal{B}(\mathcal{H})$, A.Uchiyama \cite{10} generalized the Putnam inequality, that is,

$$\|T^*T - TT^*\| \leq \phi \left( \frac{1}{p} \right) \|T\|^{2(1-p)}\text{Area}(\sigma(T))/\pi^p.$$  

This inequality gives the Putnam inequality when $p = 1$. In §3, we generalize the above inequality for the spectral area estimate of $\|T^*K - KT^*\|$ when $TK = KT$. H.Alexander \cite{1} proved the following inequality for a uniform algebra $A$. If $f$ is in $A$ then

$$\text{dist}(\tilde{f}, A) \leq \{\text{Area}(\sigma(f))/\pi\}^{1/2}.$$  

The second named author \cite{5} gave an operator version for the Alexander inequality. This was used in order to estimate $\|T^*K - KT^*\|$ when $T$ is a hyponormal operator and $KT = TK$. We also give an Alexander inequality for a $p$-hyponormal and we use it to estimate $\|T^*K - KT^*\|$.

In §4, we try to estimate $\|T^*K - KT^*\|$ for arbitrary contraction. In §5, we show a few results about area estimates for $p$-quasihyponormal operators, restricted shifts and analytic Toeplitz operators.

For $0 < p \leq 1$, $T$ is said to be $p$-hyponormal if $(T^*T)^p - (TT^*)^p \geq 0$. A 1-hyponormal operator is hyponormal. For an algebra $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$, let $\text{lat}\mathcal{A}$ be the lattice of all $\mathcal{A}$-invariant projections. For a compact subset $X$ in $\mathcal{C}$, $\text{rat}(X)$ denotes the set of all rational functions on $X$.

§2. Subnormal operator

In order to prove Theorem 1, we use the original Alexander inequality.

Theorem 1. Let $T$ be a subnormal operator in $\mathcal{B}(\mathcal{H})$ and $f$ a rational function on $\sigma(T)$ whose poles are not on it. Then

$$\|T^*f(T) - f(T)T^*\| \leq \{\text{Area}(\sigma(T))/\pi\}^{1/2}\{\text{Area}(\sigma(f(T)))/\pi\}^{1/2}.$$  

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Proof. Suppose that $N \in B(K)$ is a normal extension of $T \in B(H)$ and $P$ is an orthogonal projection from $K$ to $H$. Then $T = PN | H$ and so

$$T^*f(T) - f(T)T^* = PN^*Pf(N)P - Pf(N)PN^*P$$

$$= PN^*f(N)P - Pf(N)PN^*P$$

$$= Pf(N)N^*P - Pf(N)PN^*P$$

$$= Pf(N)(1 - P)N^*P$$

$$= Pf(N)(1 - P) \cdot (1 - P)N^*P$$

because $f(N)P = Pf(N)P$ and $f(N)N^* = N^*f(N)$.

Let $F$ be a rational function in $rat(\sigma(T))$. Put $B_F = \text{the norm closure of } \{g(F(N)) ; g \in rat(\sigma(F(N)))\}$ then $P$ belongs to $\text{lat}B_F$. Hence

$$\| (1 - P)F(N)^*P \| \leq \text{dist}(F(N)^*, B_F) \leq \text{dist}(\bar{z}, rat(\sigma(F(N))))$$

$$\leq \{\text{Area}(\sigma(F(N)))/\pi\}^{1/2}$$

by the Alexander’s theorem [1]. Hence, applying $F$ to $F = z$ or $F = f$

$$\| T^*f(T) - f(T)T^* \| \leq \| (1 - P)f(N)^*P \| \cdot \| (1 - P)N^*P \|$$

$$\leq \{\text{Area}(\sigma(f(N)))/\pi\}^{1/2}\{\text{Area}(\sigma(N))/\pi\}^{1/2}$$

$$\leq \{\text{Area}(\sigma(f(T)))/\pi\}^{1/2}\{\text{Area}(\sigma(T))/\pi\}^{1/2}.$$

If $T$ is a cyclic subnormal operator and $KT = TK$ then using a theorem of T.Yoshino [12] we can prove that

$$\|T^*K - KT^*\| \leq \{\text{Area}(\sigma(T))/\pi\}^{1/2}\{\text{Area}(\sigma(K))/\pi\}^{1/2}.$$

The proof is almost same to one of Theorem 1.

§3. $p$-hyponormal

In order to prove Theorem 2, we use an operator version of the Alexander inequality for a $p$-hyponormal operator. Unfortunately Lemma 3 is not best possible for $p = 1$ (see [5]). Lemma 1 is due to W.Arveson [2, Lemma 2] and Lemma 2 is due to A.Uchiyama [11, Theorem 3].
We need the following notation to give Theorem 2 and Proposition 1. Let $\phi$ be a positive function on $(0, \infty)$ such that

$$
\phi(t) = \begin{cases} 
t & \text{if } t \text{ is an integer} \\
t + 2 & \text{if } t \text{ is not an integer} 
\end{cases}
$$

We write $\ell^2 \otimes \mathcal{H}$ for the Hilbert space direct sum $\mathcal{H} \oplus \mathcal{H} \oplus \cdots$, and $1 \otimes T$ denotes the operator $T \oplus T \oplus \cdots \in B(\ell^2 \otimes \mathcal{H})$ for each operator $T \in B(\mathcal{H})$.

**Lemma 1.** Let $A$ be an arbitrary ultra-weakly closed subalgebra of $B(\mathcal{H})$ containing $1$, and let $T \in B(\mathcal{H})$. Then

$$
dist(T, A) = \sup \{ \| (1 - P)(1 \otimes T)P \| ; P \in \text{lat}(1 \otimes A) \}.
$$

**Lemma 2.** If $T$ is a $p$-hyponormal operator, then

$$
\| T^*T - TT^* \| \leq \phi \left( \frac{1}{p} \right) \| T \|^{2(1-p)} \{ \text{Area}(\sigma(T))/\pi \}^p.
$$

**Lemma 3.** If $T$ is a $p$-hyponormal operator then

$$
dist(T^*, A) \leq \sqrt{2\phi \left( \frac{1}{p} \right) \| T \|^{1-p} \{ \text{Area}(\sigma(T))/\pi \}^{p/2}}
$$

where $A$ is the strong closure of $\{ f(T) : f \in \text{rat}(\sigma(T)) \}$.

Proof. Let $S = 1 \otimes T$. Then $S$ is $p$-hyponormal. In order to prove the lemma, by Lemma 1 it is enough to estimate $\sup \{ \| (1 - P)SP \| ; P \in \text{lat}(1 \otimes A) \}$. If $P \in \text{lat}(1 \otimes A)$ then $SP = PSP$ and so

$$
\| (1 - P)SP \|^2 = \| PSS^*P - PPS^*P \|
$$

$$
= \| PSS^*P - PS^*SP + PS^*SP - PPS^*P \|
$$

$$
\leq \| P(S^*S - SS^*)P \| + \| (PSP)^*(PSP) - (PSP)(PSP)^* \|
$$

$$
\leq \| S^*S - SS^* \| + \| (PSP)^*(PSP) - (PSP)(PSP)^* \|.
$$

By [11, Lemma 4], $PSP$ is $p$-hyponormal and so by Lemma 2 we have

$$
\| PSS^*P - PPS^*P \|^2 \leq \phi \left( \frac{1}{p} \right) \| T \|^{2(1-p)} \{ \text{Area}(\sigma(T))/\pi \}^p + \phi \left( \frac{1}{p} \right) \| PSP \|^{2(1-p)} \{ \text{Area}(\sigma(PSP))/\pi \}^p
$$

$$
\leq 2\phi \left( \frac{1}{p} \right) \| T \|^{2(1-p)} \{ \text{Area}(\sigma(T))/\pi \}^p
$$
because \( \|PSP\| \leq \|S\| = \|T\| \) and \( \sigma(PSP) \subset \sigma(S) = \sigma(T) \). By Lemma 1,
\[
\text{dist}(T^*, \mathcal{A}) \leq \sqrt{2 \phi \left( \frac{1}{p} \right) \|T\|^{1-p} \{\text{Area}(\sigma(T))/\pi\}^{p/2}}.
\]

**Theorem 2.** If \( T \) is a \( p \)-hyponormal operator in \( \mathcal{B}(\mathcal{H}) \) and if \( K \) is in \( \mathcal{B}(\mathcal{H}) \) with \( KT = TK \), then
\[
\|T^*K - KT^*\| \leq 2 \sqrt{2 \phi \left( \frac{1}{p} \right) \|T\|^{1-p} \{\text{Area}(\sigma(T))/\pi\}^{p/2}} \|K\|.
\]

Proof. When \( \mathcal{A} \) is the strong closure of \( \{f(T) \mid f \in \text{rat}(\sigma(T))\} \), for any \( A \in \mathcal{A} \)
\[
\|T^*K - KT^*\| = \|(T^* - A)K + AK - KT^*\| \leq 2\|T^* - A\| \|K\|.
\]

Now Lemma 3 implies the theorem.

In Theorem 2, if \( p = 1 \), that is, \( T \) is hyponormal then \( \|T^*K - KT^*\| \leq 2\sqrt{2 \{\text{Area}(\sigma(T))/2\}^{1/2}} \|K\| \). The constant \( 2\sqrt{2} \) is not best because the second author [5] proved that \( \|T^*K - KT^*\| \leq 2\{\text{Area}(\sigma(T))/2\}^{1/2} \|K\| \). If \( p = \frac{1}{2} \), that is, \( T \) is semi-hyponormal then \( \|T^*K - KT^*\| \leq 4\|T\|^{1/2} \{\text{Area}(\sigma(T))/\pi\}^{1/4} \|K\| \).

### §4. Norm estimates

In general, it is easy to see that \( \|T^*T - TT^*\| \leq \|T\|^2 \). By Theorem 1, if \( T \) is subnormal and \( f \) is an analytic polynomial then
\[
\|T^*f(T) - f(T)T^*\| \leq \|T\| \|f(T)\|.
\]
In this section, we will prove that \( \|T^*T^n - T^nT^*\| \leq \|T\|^{n+1} \) for arbitrary \( T \) in \( \mathcal{B}(\mathcal{H}) \).

**Theorem 3.** If \( T \) is a contraction on \( \mathcal{H} \) and \( f \) is an analytic function on the closed unit disc \( \overline{D} \) then
\[
\|T^*f(T) - f(T)T^*\| \leq \sup_{z \in \overline{D}} |f(z)|.
\]

Proof. By a theorem of Sz.-Nagy [6], there exists a unitary operator \( U \) on \( \mathcal{K} \) such that \( \mathcal{K} \) is a Hilbert space with \( \mathcal{K} \supseteq \mathcal{H} \) and \( T^n = PU^n \mid \mathcal{K} \) for \( n \geq 0 \) where \( P \) is an orthogonal projection from \( \mathcal{K} \) to \( \mathcal{H} \). Then it is known that \( U^*P = PU^*P \) and \( f(T) = Pf(U) \mid \mathcal{H} \). Hence
\[
T^*f(T) - f(T)T^* = PU^*Pf(U)P - Pf(U)PU^*P = PU^*Pf(U)P - Pf(U)U^*P = PU^*(I - P)f(U)P.
\]
because \( U^*P = PU^* \) and \( f(U)U^* = U^*f(U) \). Therefore

\[
\| T^*f(T) - f(T)T^* \| = \| PU^*(I - P)f(U)P \| \leq \sup_{z \in D} | f(z) | .
\]

**Corollary 1.** If \( T \) is in \( \mathcal{B}(\mathcal{H}) \) then for any \( n \geq 1 \)

\[
\| T^*T^n - T^nT^* \| \leq \| T \|^{n+1}.
\]

Proof. Put \( A = T/\|T\| \) then \( A \) is a contraction and so by Theorem 2

\[
\| A^*A^n - A^nA^* \| \leq 1 \quad \text{and so} \quad \| T^*T^n - T^nT^* \| \leq \| T \|^{n+1}.
\]

### §5. Remarks

In this section, we give spectral area estimates for \( p \)-quasihyponormal operators, restricted shifts and analytic Toeplitz operators.

For \( 0 < p \leq 1 \), \( T \) is said to be \( p \)-quasihyponormal if

\[
T^*\{(T^*T)^p - (TT^*)^p\}T \geq 0.
\]

A \( 1 \)-quasihyponormal operator is called quasihyponormal.

**Lemma 4.** Let \( T \) be \( p \)-quasihyponormal and \( P \) be a projection such that \( TP = PTP \). Then \( PTP \) is also \( p \)-quasihyponormal.

Proof. Since \( T \) is \( p \)-quasihyponormal, \( T^*\{(T^*T)^p - (TT^*)^p\}T \geq 0 \). Hence, we have

\[
PT^*(T^*T)^pTP \geq PT^*(TT^*)^pTP.
\]

Since by the Hansen’s inequality [4]

\[
PT^*(T^*T)^pTP = (PTP)^*P(T^*T)^pP(PTP)
\]

\[
\leq (PTP)^*(PT^*TP)^p(PTP)
\]

\[
= (PTP)^*\{(PTP)^*(PTP)^*\}^p(PTP)
\]

and by \( 0 < p < 1 \)

\[
PT^*(TT^*)^pTP \geq (PT^*P)(TPT^*)^p(PTP)
\]

\[
= (PTP)^*\{(PTP)(PTP)^*\}^p(PTP),
\]

we have

\[
(PTP)^*\{(PTP)^*(PTP)^*\}^p(PTP) \geq (PTP)^*\{(PTP)(PTP)^*\}^p(PTP).
\]

Hence, \( PTP \) is \( p \)-quasihyponormal.

**Proposition 1.** If \( T \) is a \( p \)-quasihyponormal operator in \( \mathcal{B}(\mathcal{H}) \) and if \( K \) is in \( \mathcal{B}(\mathcal{H}) \) with \( KT = TK \), then

\[
\| T^*K - KT^* \| \leq 4 \left( \frac{1}{p} \right)^{1/4} \| T \|^{1-p/2}\{Area(\sigma(T))/\pi\}^{p/4}\|K\|.
\]
In particular, if $T$ is quasihyponormal then

$$\|T^*K - KT^*\| \leq 4\|T\|^{1/2}\{\text{Area}(\sigma(T))/\pi\}^{1/4}\|K\|.$$ 

Proof. We can prove it as in the proof of Theorem 2. By [11, Theorem 6],
$$\|T^*T - TT^*\| \leq 2\|T\|^{2-p}\sqrt{\phi(1/p)}\{\text{Area}(\sigma(T))/\pi\}^{p/2}.$$ Hence by Lemma 4
$$\text{dist}(T^*, A) \leq 2\|T\|^{1-\frac{p}{2}}\left(\frac{1}{p}\right)^{\frac{1}{2}}\{\text{Area}(\sigma(T))/\pi\}^{p/4}.$$ This implies the proposition.

Let $H^2$ and $H^\infty$ be the usual Hardy spaces on the unit circle and $z$ the coordinate function. $M$ denotes an invariant subspace of $H^2$ under the multiplication by $z$. By the well known Beurling theorem, $M = qH^2$ for some inner function. Suppose $N$ is the orthogonal complement of $M$ in $H^2$. For a function $\phi$ in $H^\infty$, $S_\phi$ is an operator on $N$ such that $S_\phi f = P(\phi f)$ ($f \in N$) where $P$ is the orthogonal projection from $H^2$ to $N$. For a symbol $\phi$ in $L^\infty$, $T_\phi$ denotes the usual Toeplitz operator on $H^2$.

**Proposition 2.** Suppose $\Phi = q\bar{\phi}$ belongs to $H^\infty$. Then

1. $\|S_\phi^*S_\phi - S_\phi S_\phi^*\| \leq \text{Area}(\Phi(D))/\pi$;
2. $\|S_\phi^*S_\phi - S_\phi S_\phi^*\| \leq \{\text{Area}(\Phi(D))/\pi\}^{n+1}$ for $n \geq 0$.

Proof. By a well known theorem of Sarason [8],
$$\| S_\phi \| = \| \phi + qH^\infty \| = \| \bar{q}\phi + H^\infty \| = \| \bar{\Phi} + H^\infty \|.$$ By Nehari’s theorem [6], $\| \bar{\Phi} + H^\infty \| = \| H_\Phi \|$ where $H_\Phi$ denotes a Hankel operator from $H^2$ to $\bar{z}H^2$. Since $\| H_\Phi \| = \| T_\Phi T_\Phi - T_\Phi T_\Phi^* \|$ where $T_\Phi$ denotes a Toeplitz operator on $H^2$, by the Putnam inequality
$$\| T_\Phi^*T_\Phi - T_\Phi T_\Phi^* \| \leq \text{Area}(\sigma(T_\Phi))/\pi = \text{Area}(\Phi(D))/\pi.$$ Now since $\| S_\phi^*S_\phi - S_\phi S_\phi^* \| \leq \| S_\phi \|^2$, (1) follows. (2) is also clear by the proof above and Corollary 1.

**Proposition 3.** Suppose $f$ and $g$ are in $H^\infty$. Then
$$\|T_f^*T_g - T_g T_f^*\| \leq \{\text{Area}(f(D))/\pi\}^{1/2}\{\text{Area}(g(D))/\pi\}^{1/2}.$$ Proof. It is easy to see that $T_f^*T_g - T_g T_f^* = H_\Phi^*H_f$. Hence
$$\|T_f^*T_g - T_g T_f^*\| \leq \|H_\Phi\| \cdot \|H_f\|.$$
Since $H_f^*H_f = T_f^*T_f - T_fT_f^*$, by the Putnam inequality

$$\|T_g^*T_f - T_f^*T_g\| \leq \left\{ \frac{\text{Area}(f(D))}{\pi} \right\}^{1/2} \left\{ \frac{\text{Area}(g(D))}{\pi} \right\}^{1/2}.$$ 

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