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Spectral Area Estimates For Norms Of Commutators

By

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Abstract. Let $A$ and $B$ be commuting bounded linear operators on a Hilbert space. In this paper, we study spectral area estimates for norms of $A^*B - BA^*$ when $A$ is subnormal or $p$-hyponormal.
§1. Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$. If $T$ is a hyponormal operator in $\mathcal{B}(\mathcal{H})$ then C.R. Putnam [7] proved that $\| T^*T - TT^* \| \leq \text{Area}(\sigma(T))/\pi$ where $\sigma(T)$ is the spectrum of $T$. The second named author [5] has proved that if $T$ is a hyponormal operator and $K$ is in $\mathcal{B}(\mathcal{H})$ with $KT = TK$ then

$$\| T^*K - KT^* \| \leq 2 \{ \text{Area}(\sigma(T))/\pi \}^{1/2} \| K \|.$$  

We don’t know whether the constant 2 in the inequality is best possible for a hyponormal operator. In §2, we show that the constant is not best possible for a subnormal operator.

When $T$ is a $p$-hyponormal operator in $\mathcal{B}(\mathcal{H})$, A.Uchiyama [10] generalized the Putnam inequality, that is,

$$\| T^*T - TT^* \| \leq \phi \left( \frac{1}{p} \right) \| T \|^{2(1-p)} \{ \text{Area}(\sigma(T))/\pi \}^p.$$  

This inequality gives the Putnam inequality when $p = 1$. In §3, we generalize the above inequality for the spectral area estimate of $\| T^*K - KT^* \|$ when $TK = KT$. H.Alexander [1] proved the following inequality for a uniform algebra $A$. If $f$ is in $A$ then

$$\text{dist}(\tilde{f}, A) \leq \{ \text{Area}(\sigma(f))/\pi \}^{1/2}.$$  

The second named author [5] gave an operator version for the Alexander inequality. This was used in order to estimate $\| T^*K - KT^* \|$ when $T$ is a hyponormal operator and $KT = TK$. We also give an Alexander inequality for a $p$-hyponormal and we use it to estimate $\| T^*K - KT^* \|$.

In §4, we try to estimate $\| T^*K - KT^* \|$ for arbitrary contraction. In §5, we show a few results about area estimates for $p$-quasihyponormal operators, restricted shifts and analytic Toeplitz operators.

For $0 < p \leq 1$, $T$ is said to be $p$-hyponormal if $(T^*T)^p - (TT^*)^p \geq 0$. A 1-hyponormal operator is hyponormal. For an algebra $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$, let $\text{lat} \mathcal{A}$ be the lattice of all $\mathcal{A}$-invariant projections. For a compact subset $X$ in $\mathcal{C}$, $\text{rat}(X)$ denotes the set of all rational functions on $X$.

§2. Subnormal operator

In order to prove Theorem 1, we use the original Alexander inequality.

**Theorem 1.** Let $T$ be a subnormal operator in $\mathcal{B}(\mathcal{H})$ and $f$ a rational function on $\sigma(T)$ whose poles are not on it. Then

$$\| T^*f(T) - f(T)T^* \| \leq \{ \text{Area}(\sigma(T))/\pi \}^{1/2} \{ \text{Area}(\sigma(f(T)))/\pi \}^{1/2}.$$  


Proof. Suppose that $N \in B(K)$ is a normal extension of $T \in B(H)$ and $P$ is an orthogonal projection from $K$ to $H$. Then $T = PN | H$ and so

$$T^*f(T) - f(T)T^* = PN^*f(N)P - P f(N)PN^*P = PN^*f(N)P - P f(N)PN^*P = P f(N)N^*P - P f(N)PN^*P = P f(N)(1 - P)N^*P = P f(N)(1 - P) \cdot (1 - P)N^*P$$

because $f(N)P = P f(N)P$ and $f(N)N^* = N^*f(N)$.

Let $F$ be a rational function in $rat(\sigma(T))$. Put $B_F = \text{the norm closure of} \{g(F(N)) ; g \in rat(\sigma(F(N)))\}$ then $P$ belongs to $\text{lat}B_F$. Hence

$$\| (1 - P)F(N)^*P \| \leq \text{dist}(F(N)^*, B_F) \leq \text{dist}(z, \text{rat}(\sigma(F(N)))) \leq \{\text{Area}(\sigma(F(N)))/\pi\}^{1/2}$$

by the Alexander’s theorem [1]. Hence, applying $F$ to $F = z$ or $F = f$

$$\| T^*f(T) - f(T)T^* \| \leq \| (1 - P)f(N)^*P \| \cdot \| (1 - P)N^*P \| \leq \{\text{Area}(\sigma(f(N)))/\pi\}^{1/2}\{\text{Area}(\sigma(N))/\pi\}^{1/2} \leq \{\text{Area}(\sigma(f(T)))/\pi\}^{1/2}\{\text{Area}(\sigma(T))/\pi\}^{1/2}.$$ 

If $T$ is a cyclic subnormal operator and $KT = TK$ then using a theorem of T.Yoshino [12] we can prove that

$$\|T^*K - KT^*\| \leq \{\text{Area}(\sigma(T))/\pi\}^{1/2}\{\text{Area}(\sigma(K))/\pi\}^{1/2}.$$ 

The proof is almost same to one of Theorem 1.

§3. $p$-hyponormal

In order to prove Theorem 2, we use an operator version of the Alexander inequality for a $p$-hyponormal operator. Unfortunately Lemma 3 is not best possible for $p = 1$ (see [5]). Lemma 1 is due to W.Arveson [2, Lemma 2] and Lemma 2 is due to A.Uchiyama [11, Theorem 3].
We need the following notation to give Theorem 2 and Proposition 1. Let \( \phi \) be a positive function on \((0, \infty)\) such that
\[
\phi(t) = \begin{cases} 
  t & \text{if } t \text{ is an integer} \\
  t + 2 & \text{if } t \text{ is not an integer}.
\end{cases}
\]

We write \( \ell^2 \otimes \mathcal{H} \) for the Hilbert space direct sum \( \mathcal{H} \oplus \mathcal{H} \oplus \cdots \), and \( 1 \otimes T \) denotes the operator \( T \oplus T \oplus \cdots \in \mathcal{B}(\ell^2 \otimes \mathcal{H}) \) for each operator \( T \in \mathcal{B}(\mathcal{H}) \).

**Lemma 1.** Let \( \mathcal{A} \) be an arbitrary ultra-weakly closed subalgebra of \( \mathcal{B}(\mathcal{H}) \) containing 1, and let \( T \in \mathcal{B}(\mathcal{H}) \). Then
\[
\text{dist}(T, \mathcal{A}) = \sup\{ \| (1 - P)(1 \otimes T)P \| ; P \in \text{lat}(1 \otimes \mathcal{A}) \}.
\]

**Lemma 2.** If \( T \) is a \( p \)-hyponormal operator, then
\[
\| T^*T - TT^* \| \leq \phi \left( \frac{1}{p} \right) \| T \|^{2(1-p)} \left\{ \text{Area}(\sigma(T))/\pi \right\}^p.
\]

**Lemma 3.** If \( T \) is a \( p \)-hyponormal operator then
\[
\text{dist}(T^*, \mathcal{A}) \leq \sqrt{2} \phi \left( \frac{1}{p} \right) \| T \|^{1-p} \left\{ \text{Area}(\sigma(T))/\pi \right\}^{p/2}
\]
where \( \mathcal{A} \) is the strong closure of \( \{ f(T) \; ; \; f \in \text{rat}(\sigma(T)) \} \).

Proof. Let \( S = 1 \otimes T \). Then \( S \) is \( p \)-hyponormal. In order to prove the lemma, by Lemma 1 it is enough to estimate \( \sup\{ \| (1 - P)SP \| ; P \in \text{lat}(1 \otimes \mathcal{A}) \} \). If \( P \in \text{lat}(1 \otimes \mathcal{A}) \) then \( SP = PSP \) and so
\[
\| (1 - P)SP \|^2
= \| PSS^*P - PSPS^*P \|
= \| PSS^*P - PS^*SP + PS^*SP - PSPS^*P \|
\leq \| P(S^*S - SS^*)P \| + \| (PSP)^*(PSP) - (PSP)(PSP)^* \|
\leq \| S^*S - SS^* \| + \| (PSP)^*(PSP) - (PSP)(PSP)^* \|.
\]
By [11, Lemma 4], \( PSP \) is \( p \)-hyponormal and so by Lemma 2 we have
\[
\| PSS^*P - PSPS^*P \|^2
\leq \phi \left( \frac{1}{p} \right) \| T \|^{2(1-p)} \left\{ \text{Area}(\sigma(T))/\pi \right\}^p
+ \phi \left( \frac{1}{p} \right) \| PSP \|^{2(1-p)} \left\{ \text{Area}(\sigma(PSP))/\pi \right\}^p
\leq 2\phi \left( \frac{1}{p} \right) \| T \|^{2(1-p)} \left\{ \text{Area}(\sigma(T))/\pi \right\}^p
\]
because \( \|PSP\| \leq \|S\| = \|T\| \) and \( \sigma(PSP) \subset \sigma(S) = \sigma(T) \). By Lemma 1,
\[
\text{dist}(T^*, \mathcal{A}) \leq \sqrt{2\phi\left(\frac{1}{p}\right)}\|T\|^{1-p}\{\text{Area}(\sigma(T))/\pi\}^{p/2}.
\]

**Theorem 2.** If \( T \) is a \( p \)-hyponormal operator in \( \mathcal{B}(\mathcal{H}) \) and if \( K \) is in \( \mathcal{B}(\mathcal{H}) \) with \( KT = TK \), then
\[
\|T^*K - KT^*\| \leq 2\sqrt{2\phi\left(\frac{1}{p}\right)}\|T\|^{1-p}\{\text{Area}(\sigma(T))/\pi\}^{p/2}\|K\|.
\]

Proof. When \( \mathcal{A} \) is the strong closure of \( \{f(T) ; f \in \text{rat}(\sigma(T))\} \), for any \( A \in \mathcal{A} \)
\[
\|T^*K - KT^*\| = \|(T^* - A)K + AK - KT^*\| \leq 2\|T^* - A\|\|K\|.
\]
Now Lemma 3 implies the theorem.

In Theorem 2, if \( p = 1 \), that is, \( T \) is hyponormal then \( \|T^*K - KT^*\| \leq 2\sqrt{2\{\text{Area}(\sigma(T))/2\}^{1/2}\|K\|} \). The constant \( 2\sqrt{2} \) is not best because the second author [5] proved that \( \|T^*K - KT^*\| \leq 2\{\text{Area}(\sigma(T))/2\}^{1/2}\|K\| \). If \( p = \frac{1}{2} \), that is, \( T \) is semi-hyponormal then \( \|T^*K - KT^*\| \leq 4\|T\|^{1/2}\{\text{Area}(\sigma(T))/\pi\}^{1/4}\|K\| \).

§4. Norm estimates

In general, it is easy to see that \( \|T^*T - TT^*\| \leq \|T\|^2 \). By Theorem 1, if \( T \) is subnormal and \( f \) is an analytic polynomial then
\[
\|T^*f(T) - f(T)T^*\| \leq \|T\|\|f(T)\|.
\]
In this section, we will prove that \( \|T^*T^n - T^nT^*\| \leq \|T\|^{n+1} \) for arbitrary \( T \) in \( \mathcal{B}(\mathcal{H}) \).

**Theorem 3.** If \( T \) is a contraction on \( \mathcal{H} \) and \( f \) is an analytic function on the closed unit disc \( \bar{D} \) then \( \|T^*f(T) - f(T)T^*\| \leq \sup_{z \in \bar{D}} |f(z)| \).

Proof. By a theorem of Sz.-Nagy [6], there exists a unitary operator \( U \) on \( \mathcal{K} \) such that \( \mathcal{K} \) is a Hilbert space with \( \mathcal{K} \supseteq \mathcal{H} \) and \( T^n = PU^n \mid \mathcal{K} \) for \( n \geq 0 \) where \( P \) is an orthogonal projection from \( \mathcal{K} \) to \( \mathcal{H} \). Then it is known that \( U^*P = PU^*P \) and \( f(T) = Pf(U) \mid \mathcal{H} \). Hence
\[
T^*f(T) - f(T)T^* = PU^*Pf(U)P - Pf(U)PU^*P = PU^*Pf(U)P - Pf(U)U^*P = PU^*(I - P)f(U)P
\]
because $U^*P = PU^*P$ and $f(U)U^* = U^*f(U)$. Therefore
\[
\| T^* f(T) - f(T)T^* \| = \| PU^* (I - P)f(U)P \| \leq \sup_{z \in D} | f(z) | .
\]

**Corollary 1.** If $T$ is in $\mathcal{B}(\mathcal{H})$ then for any $n \geq 1$ \( \| T^* T^n - T^n T^* \| \leq \| T \|^{n+1} \).

**Proof.** Put $A = T/\|T\|$ then $A$ is a contraction and so by Theorem 2 \( \| A^* A^n - A^n A^* \| \leq 1 \) and so \( \| T^* T^n - T^n T^* \| \leq \| T \|^{n+1} \).

§5. Remarks

In this section, we give spectral area estimates for $p$-quasihyponormal operators, restricted shifts and analytic Toeplitz operators.

For $0 < p \leq 1$, $T$ is said to be $p$-quasihyponormal if $T^*((T^*T)^p - (TT^*)^p)T \geq 0$.

A 1-quasihyponormal operator is called quasihyponormal.

**Lemma 4.** Let $T$ be $p$-quasihyponormal and $P$ be a projection such that $TP = PTP$. Then $PTP$ is also $p$-quasihyponormal.

**Proof.** Since $T$ is $p$-quasihyponormal, $T^*((T^*T)^p)T \geq T^*((TT^*)^p)T$. Hence, we have
\[
PT^*(T^*T)^pTP \geq PT^*(TT^*)^pTP.
\]

Since by the Hansen’s inequality [4]
\[
PT^*(T^*T)^pTP = (PTP)^*P(T^*T)^pP(PTP) \\
\leq (PTP)^*(PT^*TP)^p(PTP) \\
= (PTP)^*\{(PTP)^*(PTP)^p\}(PTP)
\]
and by $0 < p < 1$
\[
PT^*(TT^*)^pTP \geq (PT^*P)(TP^*TP)^p(PTP) \\
= (PTP)^*\{(PTP)(PTP)^*\}^p(PTP),
\]
we have
\[
(PTP)^*\{(PTP)^*(PTP)^p\} \geq (PTP)^*\{(PTP)(PTP)^*\}^p(PTP).
\]
Hence, $PTP$ is $p$-quasihyponormal.

**Proposition 1.** If $T$ is a $p$-quasihyponormal operator in $\mathcal{B}(\mathcal{H})$ and if $K$ is in $\mathcal{B}(\mathcal{H})$ with $KT = TK$, then
\[
\| T^* K - K T^* \| \leq 4 \left[ \frac{1}{p} \phi \left( \frac{1}{p} \right) \right]^{1/4} \| T \|^{1-p/2} \left\{ \text{Area}(\sigma(T))/\pi \right\}^{p/4} \| K \|.
\]
In particular, if $T$ is quasihyponormal then

$$\|T^*K - KT^*\| \leq 4\|T\|^{1/2}\{\text{Area}(\sigma(T))/\pi\}^{1/4}\|K\|.$$ 

Proof. We can prove it as in the proof of Theorem 2. By [11, Theorem 6],

$$\|T^*T - TT^*\| \leq 2\|T\|^{2-p}\sqrt{\frac{1}{p}}\{\text{Area}(\sigma(T))/\pi\}^{p/2}.$$ 

Hence by Lemma 4

$$\text{dist}(T^*, A) \leq 2\|T\|^{1-p/4}\{\text{Area}(\sigma(T))/\pi\}^{p/2}.$$ 

This implies the proposition.

Let $H^2$ and $H^\infty$ be the usual Hardy spaces on the unit circle and $z$ the coordinate function. $M$ denotes an invariant subspace of $H^2$ under the multiplication by $z$. By the well known Beurling theorem, $M = qH^2$ for some inner function. Suppose $N$ is the orthogonal complement of $M$ in $H^2$. For a function $\phi$ in $H^\infty$, $S\phi$ is an operator on $N$ such that $S\phi f = P(\phi f)$ ($f \in N$) where $P$ is the orthogonal projection from $H^2$ to $N$. For a symbol $\phi$ in $L^\infty$, $T\phi$ denotes the usual Toeplitz operator on $H^2$.

**Proposition 2.** Suppose $\Phi = q\bar{\phi}$ belongs to $H^\infty$. Then

1. $\|S\phi S\phi^* - S\phi^* S\phi\| \leq \text{Area}(\Phi(D))/\pi$;
2. $\|S\phi^* S\phi - S\phi^* S\phi\| \leq \{\text{Area}(\Phi(D))/\pi\}^{n+1}$ for $n \geq 0$.

Proof. By a well known theorem of Sarason [8],

$$\|S\phi\| = \|\phi + qH^\infty\| = \|\bar{\phi} + H^\infty\| = \|\Phi + H^\infty\|.$$ 

By Nehari’s theorem [6], $\|\Phi + H^\infty\| = \|H_\Phi\|$, where $H_\Phi$ denotes a Hankel operator from $H^2$ to $\bar{z}H^2$. Since $\|H_\Phi\|^2 = \|T^\Phi T_\Phi - T_\Phi T^\Phi\|$ where $T_\Phi$ denotes a Toeplitz operator on $H^2$, by the Putnam inequality

$$\|T^\Phi T_\Phi - T_\Phi T^\Phi\| \leq \text{Area}(\sigma(T_\Phi))/\pi = \text{Area}(\Phi(D))/\pi.$$ 

Now since $\|S^\phi S\phi - S\phi^* S\phi^*\| \leq \|S\phi\|^2$, (1) follows. (2) is also clear by the proof above and Corollary 1.

**Proposition 3.** Suppose $f$ and $g$ are in $H^\infty$. Then

$$\|T^*_f T_g - T_g T^*_f\| \leq \{\text{Area}(f(D))/\pi\}^{1/2}\{\text{Area}(g(D))/\pi\}^{1/2}.$$ 

Proof. It is easy to see that $T^*_f T_g - T_g T^*_f = H^*_g H_f$. Hence

$$\|T^*_f T_g - T_g T^*_f\| \leq \|H_g\| \cdot \|H_f\|.$$ 

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Since \( H_j^*H_j = T_j^*T_j - T_jT_j^* \), by the Putnam inequality

\[
\|T_j^*T_g - T_gT_j^*\| \leq \left\{ \frac{\text{Area}(f(D))/\pi}{\pi} \right\}^{1/2} \left\{ \frac{\text{Area}(g(D))/\pi}{\pi} \right\}^{1/2}.
\]

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