HIGH REGULARITY OF SOLUTIONS OF COMPRESSIBLE NAVIER-STOKES EQUATIONS

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ABSTRACT. We study the Navier-Stokes equations for compressible *barotropic* fluids in a bounded or unbounded domain Ω of $\mathbf{R}^3$. The initial density may vanish in an open subset of Ω or to be positive but vanish at space infinity. We first prove the local existence of solutions $(\rho^{(j)}, u^{(j)})$ in $C([0,T_*]; H^{2(k-j)+3} \times D_0^1 \cap D^{2(k-j)+3}(\Omega))$, $0 \le j \le k, k \ge 1$ under the assumptions that the data satisfy compatibility conditions and that the initial density is sufficiently small. To control the nonnegativity or decay at infinity of density, we need to establish a boundary value problem of $(k+1)$-coupled elliptic system which may not be in general solvable. The smallness condition of initial density is necessary for the solvability, which is not necessary in case that the initial density has positive lower bound. Secondly, we prove the global existence of smooth radial solutions of *isentropic* compressible Navier-Stokes equations on a bounded annulus or a domain which is the exterior of a ball under a smallness condition of initial density.

1. INTRODUCTION

We consider the following compressible Navier-Stokes equations describing the motion of a viscous compressible barotropic fluid in a domain Ω of $\mathbf{R}^3$

$$\rho_t + \operatorname{div}(\rho u) = 0 \qquad \text{in} \quad (0,T)\times\Omega, \tag{1.1}$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) + L\,u + \nabla p = \rho f \qquad \text{in} \quad (0,T)\times\Omega, \tag{1.2}$$

$$Lu = -\mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u, \tag{1.3}$$

and the initial and boundary conditions

$$(\rho,\, u)|_{t=0} = (\rho_0,\, u_0) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad (0,T)\times\partial\Omega, \tag{1.4}$$

$$(\rho - \rho^\infty,\, u)(t,x) \to (0,\,0) \quad \text{as} \quad |x|\to\infty, \quad (t,x)\in(0,T)\times\Omega. \tag{1.5}$$

Here we denote by ρ and u the unknown density and velocity fields of the fluid, respectively. p is the pressure which is a smooth function of ρ. f denotes a given external force and the constants μ, λ are the viscosity coefficients. We assume that the viscosity coefficients μ and λ satisfy the natural physical restrictions $\mu > 0$ and $3\lambda + 2\mu \ge 0$ so that $L = -\mu\Delta - (\lambda+\mu)\nabla\mathrm{div}$ is a strongly elliptic operator. We assume that T is a finite positive number and Ω is either a bounded domain in $\mathbf{R}^3$ with smooth boundary or a usual unbounded domain such as the whole space $\mathbf{R}^3$,

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the half space $\mathbf{R}^2 \times \mathbf{R}_+$ and an exterior domain with smooth boundary. If Ω is a bounded domain (or the whole space), then the condition (1.5) at infinity (or the boundary condition in (1.4) respectively) is unnecessary and should be neglected. ρ^∞ is a nonnegative constant. For the simplicity of presentation, we always assume that $\rho^\infty = 0$. When the density has positive lower bound on a domain, we mean it by that $\rho^\infty > 0$. In particular, $\rho(x) \to \rho^\infty$ as $|x| \to \infty$ for unbounded domain. For the mathematical derivation of the equations (1.1) and (1.2), see [19] for instance.

Throughout this paper, we use the following simplified notations for the homogeneous and inhomogeneous Sobolev spaces.

$$
\begin{aligned}
&L^r = L^r(\Omega), \quad D^{k,r} = \{v \in L^1_{loc}(\Omega) : |v|_{D^{k,r}} < \infty\},\\
&W^{k,r} = L^r \cap D^{k,r}, H^k = W^{k,2}, \quad D^k = D^{k,2},\\
&D^1_0 = \{v \in L^6(\Omega) : |v|_{D^1_0} < \infty \ \text{ and } \ v = 0 \ \text{ on } \ \partial\Omega\},\\
&H^1_0 = L^2 \cap D^1_0, \quad |v|_{D^{k,r}} = |\nabla^k v|_{L^r} \quad \text{and} \quad |v|_{D^1_0} = |\nabla v|_{L^2}.
\end{aligned}
$$

Hereafter we use the obvious notation

$$|\cdot|_{X\cap Y} = |\cdot|_X + |\cdot|_Y \quad \text{for (semi-)normed spaces} \quad X, Y$$

and C denotes a generic positive constant depending only on the constants k (Sobolev space index), μ, λ, T and the norms of $p = p(\cdot)$ and f. For a detailed study of homogeneous Sobolev spaces, we refer the readers to the Galdi's book [13].

In this paper, we study the local and global existence and the regularity for the initial boundary value problem (simply IBVP) (1.1)-(1.5). Our result is a sequel of those in [5] and [4], in which the existence and uniqueness of strong and classical solutions are considered under a general assumption on initial density that $\rho_0 \geq 0$. In the present case, we consider a high regularity of local and global solutions in Sobolev space. There has been a large amount of literature on the existence and regularity related to IBVP (1.1)–(1.16). See [18, 19] and [9, 10, 11, 12, 16, 17] for the weak solutions, and [22, 15, 26, 20, 21, 8, 14, 25, 27, 28, 29, 23] and the recent works [3, 4, 5, 6] for strong solutions. For the finite time blowup of smooth solutions, see [2] and [30].

To treat the high regularity of solutions, we need to control the high order time derivatives of solutions. For this purpose it is necessary to assume the compatibility conditions for the time derivatives of solutions at the initial time zero. More specifically, we denote $(\partial/\partial t)^k v$, the k-th order time derivative of v, by $v^{(k)}$ and differentiate the equation (1.2) k-times with respect to t. Then we have

$$
\begin{aligned}
&\rho u^{(k+1)} + Lu^{(k)} + \nabla p^{(k)}\\
&= \sum_{0\leq j\leq k} \binom{k}{j} \rho^{(k-j)}(f - u\cdot\nabla u)^{(j)}\\
&\qquad\qquad -k\rho^{(1)}u^{(k)} - \sum_{1\leq j\leq k-1} \binom{k}{j-1} \rho^{(k+1-j)}u^{(j)}.
\end{aligned}
\tag{1.6}
$$

For the regularity of $u^{(j)}$ at time zero we then define a $(k+1)$-*coupled compatibility condition* as follows: for some functions $g_j \in D_0^1 \cap D^{2(k-j)+3}, 1 \le j \le k+1$, the data satisfy that

$$Lu_0 + \nabla p(\rho_0) = \rho_0 \left(f(0) - u_0 \cdot \nabla u_0 - g_1 \right) \text{ in } \Omega \tag{1.7}$$

and

$$\begin{aligned} &L(u^{(j)}(0)) + \nabla p^{(j)}(0) \\ &= -j\rho^{(1)}(0)u^{(j)}(0) - \sum_{1\le l\le j-1} \begin{pmatrix} j \\ l-1 \end{pmatrix} \rho^{(j+1-l)}(0)u^{(l)}(0) \\ &\quad + \sum_{0\le l\le j} \begin{pmatrix} j \\ l \end{pmatrix} \rho^{(j-l)}(0) \left(f^{(l)} - \sum_{0\le m\le l} \begin{pmatrix} l \\ m \end{pmatrix} u^{(l-m)}(0) \cdot \nabla u^{(m)}(0) \right) \\ &\quad - \rho_0 g_{j+1} \text{ in } \Omega \end{aligned} \tag{1.8}$$

for all $1 \le j \le k$. Here we define $p^{(k)}(0)$ and $\rho^{(k)}(0)$ recursively as follows:

$$\begin{aligned} p^{(j)}(0) &= \left(\frac{\partial}{\partial t} \right)^j (p(\rho)) \Big|_{t=0}, \\ \rho^{(j+1)}(0) &= -\mathrm{div} \left(\sum_{0\le l\le j} \begin{pmatrix} j \\ l \end{pmatrix} \rho^{(j-l)}(0)u^{(l)}(0) \right), \quad 0 \le j \le k-1. \end{aligned}$$

We call the condition (1.7) zeroth compatibility condition and (1.8) j-th one. Under the smallness of condition on ρ_0, the solution $u^{(j)}(0)$ to the mapping $(g_1, \cdots, g_{k+1}) \mapsto (u_0, u^{(1)}(0), \cdots, u^{(k)}(0))$ is unique. See Section 4 and 5 for the existence and uniqueness.

In [5, 4], the authors used the zeroth compatibility condition (1.7) to prove the local existence of strong and classical solutions. See also [3]. We follow the similar strategy in [4] summarized as follows:

(1) linearization on bounded domain with initial density having positive lower bound,
(2) a priori estimates independent of domain size and lower bound of initial density, and domain expansion,
(3) construction of a sequence of approximate solutions to the linearized problem,
(4) convergence of the sequence in a strong sense.

If we consider only the zeroth compatibility condition (1.7), then in the course of linearization (1) and domain expansion (2), it is usually necessary to reconstruct u_0 satisfying (1.7) for a given g_1 (see Lemma 9 in [4]). Since (1.7) is a single elliptic equation, the existence of u_0 is well-known by the classical elliptic theory. However, in the present case, the situation is quite different. We encounter the case that we should find $(u_0, u^{(1)}(0), \cdots, u^{(k)}(0))$ satisfying (1.8) for a given pair $(g_1, \cdots, g_{k+1})$. In other words, we must find solutions of $(k+1)$-coupled elliptic system. In general, it is difficult to solve such a system. However, if we assume that the initial density is sufficiently small, then we can find solution successfully (see Lemma 4.2 below). From this solvability of $(k+1)$-coupled system, we can construct approximate solutions to the linearized problem, which is convergent to a solution of the original equations. For this purpose, we also have to construct a sequence of system of compatibility conditions and show the strong convergence of the sequence to the original system of compatibility conditions (1.7) and (1.8) which are possible under the smallness condition of the initial density. See Section 5 below.

Now we state the first main result.

Theorem 1.1. *Assume that*

$$\begin{aligned}
&\rho_0 \in L^1 \cap H^{2k+3}, \quad \rho_0 \geq 0 \quad in \quad \Omega, \quad u_0 \in D_0^1 \cap D^{2k+3},\\
&f^{(j)} \in C(0,T; H^{2(k-j)+1}) \cap L^2(0,T; H^{2(k-j)+2}) \;\; for \;\; 0 \leq j \leq k \qquad (1.9)\\
&f^{(k+1)} \in L^2(0,T; L^2)
\end{aligned}$$

and that the data ρ_0, u_0 and f satisfy the compatibility conditions (1.7) *and* (1.8). *Assume further that $|\rho_0|_{L^1 \cap H^{2k+3}}$ is sufficiently small. Then there exist a time $T_* \in (0,T)$ and a unique solution (ρ, u) to the IBVP* (1.1)–(1.5) *such that for all $0 \leq j \leq k,\;\; \rho \in C([0,T_*]; L^1)$*

$$\begin{aligned}
&\rho^{(j)} \in C([0,T_*]; H^{2(k-j)+3}) \cap L^2(0,T_*; H^{2(k-j)+4}),\\
&u^{(j)} \in C([0,T_*]; D_0^1 \cap D^{2(k-j)+3}) \cap L^2(0,T_*; D^{2(k-j)+4}) \qquad (1.10)\\
&u^{(k+1)} \in L^\infty(0,T_*; D_0^1) \cap L^2(0,T_*; D^2),\\
&\sqrt{\rho} u^{(k+1)} \in L^\infty(0,T_*; L^2).
\end{aligned}$$

Remark 1.2. If the initial density has positive lower bound, then the smallness condition of initial density can be removed. See Remark 3.3 and Section 4 below

Remark 1.3. For each $k \geq 1$, one can observe a smoothing effect and apply the methods used to get a classical solution in [4] to the improvement of regularity slightly.

Now we study the global existence of radial solutions. The global existence of strong solution ($k = 0$) is known by H. Kim and H.J. Choe [7]. They showed the density is bounded on any finite time interval by using the radial symmetry and the effective viscous effect. The second main result of this paper is to show that every Sobolev norm of the solution is controlled by L^∞ norm of density. See Theorem 6.2 below.

To do this, we need to estimate the L^∞ norm of pressure p (see (6.9) below). This is possible in the case of the isentropic compressible Navier-Stokes equations with $p = A\rho^\gamma$, $A > 0$ and $\gamma > 1$. Since from (1.1), the pressure p satisfies the equation $p_t + \nabla p \cdot u + \gamma p \operatorname{div} u = 0$. Hence for a high regularity for any $\gamma > 1$, as in [4] we consider the following slightly general IBVP of isentropic compressible

Navier-Stokes equations:

$$\rho_t + \mathrm{div}(\rho u) = 0 \quad \text{in} \quad (0,T)\times\Omega, \tag{1.11}$$

$$p_t + \nabla p\cdot u + \gamma p\,\mathrm{div}\,u = 0 \quad \text{in} \quad (0,T)\times\Omega, \tag{1.12}$$

$$(\rho u)_t + \mathrm{div}(\rho u\otimes u) + L\,u + \nabla p = \rho f \quad \text{in} \quad (0,T)\times\Omega, \tag{1.13}$$

$$Lu = -\mu\Delta u - (\lambda+\mu)\nabla\mathrm{div}\,u, \tag{1.14}$$

and the initial and boundary conditions

$$(\rho,\,p,\,u)|_{t=0} = (\rho_0,\,p_0,\,u_0) \quad \text{in} \quad \Omega, \quad u=0 \quad \text{on} \quad (0,T)\times\partial\Omega, \tag{1.15}$$

$$(\rho,\,p,\,u)(t,x)\to(0,\,0,\,0) \quad \text{as} \quad |x|\to\infty, \quad (t,x)\in(0,T)\times\Omega. \tag{1.16}$$

In fact, the barotropic fluids satisfy the equation $p_t + \nabla p\cdot u + \rho p'(\rho)\mathrm{div}\,u = 0$ and the initial pressure $p(0) = p(\rho_0)\in H^{2k+3}$. The isentropic case above is the one that $\rho p'(\rho) = \gamma p$. Hence the local existence of IBVP (1.11)-(1.16) follows from Theorem 1.1, provided the initial pressure p_0 of (1.15) is in H^{2k+3}. Moreover, if $p_0 = A\rho_0^\gamma$, then the solution p of (1.12) satisfies the equation of state $p = A\rho^\gamma$. For this, see the proof of Theorem 4 of [4].

Now let us introduce the second result. Let $B(0,r)$, $r>0$ be the ball with radius r centered at the origin.

Theorem 1.4. *If* (ρ_0, u_0, f) *is a radially symmetric data satisfying the conditions in Theorem 1.1 with* $p_0 = A\rho_0^\gamma \in H^{2k+3}$ *on an annular domain* $\Omega = B(0,b)\setminus B(0,a)$ *with* $0<a<b\le\infty$*, then there exists a unique global radially symmetric solution* (ρ,p,u) *of IBVP* (1.11)–(1.16) *satisfying the regularity* (1.10) *and the state equation* $p = A\rho^\gamma$ *with* $\Omega = B(0,b)\setminus B(0,a)$.

Remark 1.5. In this isentropic case, dividing the density and pressure by a large constant, we can remove the smallness condition of initial density. But instead, we need a large viscosity coefficients.

This paper is organized as follows. In Section 2, we reduce the original problem to a linearized problem on a bounded domain under a positive lower bound assumption on the initial density. In Section 3, we prove uniform a priori estimates independent of domain size and lower bound of density. In Section 4, to prove Theorem 1.1 expanding domain size, we remove the lower bound restriction on the density. At this point, we solve the $(k+1)$-coupled elliptic system under a smallness assumption on the initial density. Then we prove Theorem 1.1 by constructing a sequence of approximate solutions and showing a strong convergence of it. Finally, Section 6 is devoted to proving Theorem 1.4.

2. Linearization

In this section, we introduce some existence and regularity results on solutions of a linear parabolic system and a linear transport equation. Then assuming the positivity of the initial density, we linearize the original equations on a bounded domain.

Let Ω be a bounded domain in $\mathbf{R}^3$ with smooth boundary, and we consider the following linear parabolic problem

$$\begin{cases} \rho w_t + Lw = F \quad \text{in} \quad (0,T)\times\Omega, \\ w(0) = w_0 \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad (0,T)\times\partial\Omega, \end{cases} \tag{2.1}$$

where ρ is a known scalar field in $(0,T)\times\Omega$ such that

$$\rho^{(j)} \in C([0,T];H^{2(k-j)+3}) \quad \text{and} \quad \rho \geq \delta \quad \text{on} \quad [0,T]\times\overline{\Omega} \tag{2.2}$$

for any $0 \leq j \leq k$, $k \geq 1$ and for some constant $\delta > 0$. Then applying the arguments in the proof of Lemma 2 and Lemma 3 in [4] and induction, one can show the following existence and regularity results on solutions to the linear parabolic problem (2.1). See also [27, 28, 29].

Lemma 2.1. *If* $w_0 \in H_0^1 \cap H^{2k+3}$, $F^{(j)} \in C([0,T];H^{2(k-j)+1}) \cap L^2(0,T;H^{2(k-j)+2})$, $F^{(k+1)} \in L^2(0,T;L^2)$ *and*

$$\begin{aligned} &\rho(0)^{-1}\left((F^{(j)}(0) - Lw^{(j)}(0) - \sum_{0\leq l\leq j-1}\begin{pmatrix} j \\ l \end{pmatrix}\rho^{(j-l)}(0)w^{(l+1)}(0)\right) \\ &\in H_0^1 \cap H^{2(k-j)+1} \end{aligned} \tag{2.3}$$

for all $0 \leq j \leq k$, $k \geq 1$, *then the solution* w *also satisfies*

$$w^{(j)} \in C([0,T];H^{2(k-j)+3}) \ \textit{for all}\ 0 \leq j \leq k,$$
$$w^{(k+1)} \in C([0,T];H_0^1) \cap L^2(0,T;H^2), \quad \textit{and} \quad w^{(k+2)} \in L^2(0,T;L^2).$$

Remark 2.2. From the elliptic regularity theory [1, 5], we see that the condition (2.3) means $w^{(j)}(0) \in H_0^1 \cap H^{2(k-j)+3}$.

Now we consider the following linear hyperbolic equation

$$\rho_t + \operatorname{div}(\rho v) = 0 \quad \text{in} \quad (0,T)\times\Omega, \qquad \rho(0) = \rho_0 \quad \text{in} \quad \Omega, \tag{2.4}$$

where v is a known vector field in $(0,T)\times\Omega$ such that for some integer $k \geq 1$

$$v^{(j)} \in C([0,T];D_0^1 \cap D^{2(k-j)+3}) \cap L^2(0,T;D^{2(k-j)+4})$$

for any $0 \leq j \leq k$. Let $\rho^{(j+1)}(0)$ be defined recursively by

$$\rho^{(j+1)}(0) = -\operatorname{div}\left(\sum_{0\leq l\leq j}\begin{pmatrix} j \\ l \end{pmatrix}\rho^{(j-l)}(0)v^{(l)}(0)\right), \quad 0 \leq j \leq k-1.$$

Then we have

Lemma 2.3. *Assume that* $\rho^{(j)}(0) \in H^{2(k-j)+3}$ *and* $\rho_0 \geq 0$ *in* Ω. *Then*

(i) *there exists a unique solution ρ to the problem* (2.4) *such that for all* $0 \leq j \leq k$

$$\rho^{(j)} \in C([0,T]; H^{2(k-j)+3}),$$

(ii) *the solutions ρ and p satisfies the following estimate*

$$|\rho(t)|_{H^{2k+3}} \leq |\rho_0|_{H^{2k+3}} \exp\left(C \int_0^t |v(s)|_{D_0^1 \cap D^{2k+4}}\, ds \right)$$

for $0 \leq t \leq T$ *and finally,*

(iii) *the solution ρ is represented by the formula*

$$\rho(t,x) = \rho_0(\,U(0,t,x)\,) \exp\left[- \int_0^t \operatorname{div} v(s, U(s,t,x)\,)\, ds \right], \tag{2.5}$$

where $U \in C([0,T] \times [0,T] \times \overline{\Omega})$ *is the solution to the initial value problem*

$$\begin{cases} \frac{\partial}{\partial t} U(t,s,x) = v(t, U(t,s,x)\,), \qquad 0 \leq t \leq T, \\ \quad U(s,s,x) = x, \qquad 0 \leq s \leq T, \;\; x \in \overline{\Omega}. \end{cases} \tag{2.6}$$

Proof. (i) with $j = 0$, (ii) and (iii) follow from Lemma 2.1 in [4]. And an induction on j yields the case (i) with $j \geq 1$. □

To prove Theorem 1.1, we consider the following linearized problem

$$\rho_t + \operatorname{div}(\rho v) = 0 \qquad \text{in} \quad (0,T) \times \Omega, \tag{2.7}$$

$$\rho u_t + Lu + \nabla p = \rho(f - v \cdot \nabla v) \qquad \text{in} \quad (0,T) \times \Omega, \tag{2.8}$$

$$(\rho,\, u)|_{t=0} = (\rho_0,\, u_0) \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad (0,T) \times \partial\Omega,, \tag{2.9}$$

where v is a known vector field in $(0,T) \times \Omega$ such that for $0 \leq j \leq k$

$$\begin{gathered} v^{(j)} \in C([0,T]; D_0^1 \cap D^{2(k-j)+3}) \cap L^2(0,T; D^{2(k-j)+4}) \\ \text{and} \quad v^{(k+1)} \in L^\infty(0,T; D_0^1) \cap L^2(0,T; D^2). \end{gathered} \tag{2.10}$$

Recall again that $Lu = -\mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u$.

First, from Lemma 2.1 and Lemma 2.2, we obtain an existence result for positive initial densities.

Lemma 2.4. *Let Ω be a bounded domain in $\mathbf{R}^3$ with smooth boundary. In addition to* (1.9) *and* (2.10), *we assume that $\rho_0 \geq \delta$ in Ω for some constant $\delta > 0$ and*

$$\begin{aligned} &(f - v \cdot \nabla v)^{(j)}(0) + \rho_0^{-1} \sum_{0 \leq m \leq j-1} \begin{pmatrix} j \\ m \end{pmatrix} \rho^{(j-m)}(0)(f - v \cdot \nabla v)^{(m)}(0) \\ &\; - \rho_0^{-1} \sum_{0 \leq m \leq j-1} \begin{pmatrix} j \\ m \end{pmatrix} \rho^{(j-m)}(0) u^{(m)}(0) - \rho_0^{-1} \left(Lu^{(j)}(0) + \nabla p^{(j)}(0) \right) \\ &\; \in H_0^1 \cap H^{2(k-j)+1} \end{aligned} \tag{2.11}$$

for $0 \le j \le k$. Then there exists a unique solution (ρ, u) to the linearized problem (2.7)*,* (2.8) *and* (2.9) *such that for all $0 \le j \le k+1$*

$$
\begin{aligned}
&\rho \in C(0,T]; H^{2k+3}),\\
&\rho^{(j)} \in C([0,T]; H^{2(k-j)+2}) \cap L^2(0,T; H^{2(k-j)+3})),\\
&u^{(j)} \in C([0,T]; H_0^1 \cap H^{2(k-j)+3}) \cap L^2(0,T; H^{2(k-j)+4}),\\
&\text{and} \quad \rho \ge \underline{\delta} \quad \text{on} \quad [0,T] \times \overline{\Omega}
\end{aligned}
\tag{2.12}
$$

for some constant $\underline{\delta} > 0$.

Proof. The existence and regularity of a unique solution ρ to the linear hyperbolic problem (2.7) and (2.9) were already proved in Lemma 2.3. To prove the remaining part of the lemma, let us define F by $\rho(f - v \cdot \nabla v) - \nabla p$. Then

$$
\begin{aligned}
F^{(j)} = \rho(f - v\cdot\nabla v)^{(j)} &+ \sum_{0 \le m \le j-1} \begin{pmatrix} j \\ m \end{pmatrix} \rho^{(j-m)} (f - v\cdot\nabla v)^{(m)}\\
&- \sum_{0 \le m \le j-1} \begin{pmatrix} j \\ m \end{pmatrix} \rho^{(j-m)} u^{(m)} - \nabla p^{(j)}
\end{aligned}
$$

for $1 \le j \le k$. Then by virtue of (1.9), (2.10) and the regularity of ρ, we can easily show that $F^{(j)} \in C([0,T]; H^{2(k-j)+1}) \cap L^2(0,T; H^{2(k-j)+2})$, $F^{(k+1)} \in L^2(0,T; L^2)$. Moreover since F and u satisfies the condition (2.3) by (2.11), Lemma 2.1 allows us to deduce the existence and regularity of a unique solution u to the linear parabolic problem (2.8) and (2.9). This completes the proof of Lemma 2.4. □

3. A priori estimates for positive density

Assume that ρ_0, u_0, v, f and Ω satisfy the hypotheses of Lemma 2.4. Then it follows from Lemma 2.4 that there exists a unique smooth solution (ρ, u) to the linear problem (2.7), (2.8) and (2.9) satisfying the regularity (2.12). We derive some *local (in time) a priori estimates* for (ρ, u) which are independent of the lower bound δ of ρ_0 and size of the domain Ω. Let us choose constants $\varepsilon > 0$ and $c_0 > 1$ such that $\varepsilon \le c_0$

$$
|\rho_0|_{L^1 \cap H^{2k+3}} < \varepsilon, \quad 1 + \sum_{1 \le j \le k+1} |g_j|_{D_0^1} < c_0, \tag{3.1}
$$

where $g_1 = f(0) - v(0)\cdot\nabla v(0) - \rho_0^{-1}(Lu_0 + \nabla p_0) = u_t(0) \in D_0^1 \cap D^{2k+1}$ and for $1 \le j \le k$

$$
\begin{aligned}
g_{j+1} = (f - v\cdot\nabla v)^{(j)}(0) &+ \rho_0^{-1} \sum_{0 \le m \le j-1} \begin{pmatrix} j \\ m \end{pmatrix} \rho^{(j-m)}(0) (f - v\cdot\nabla v)^{(m)}(0)\\
- \rho_0^{-1} &\sum_{0 \le m \le j-1} \begin{pmatrix} j \\ m \end{pmatrix} \rho^{(j-m)}(0) g_{m+1} - \rho_0^{-1} \left(L g_j + \nabla p^{(j)}(0) \right)\\
= u^{(j+1)}(0) &\in D_0^1 \cap D^{2(k-j)+1}.
\end{aligned}
$$

Assume for all $0 \le j \le k+1$ that

$$|v^{(j)}(0)|_{D_0^1 \cap D^{2(k-j)+3}} \le 1 + c_{0,1,j}, \tag{3.2}$$

and for all $0 \le l \le k$ and $0 \le j \le k-l$ that

$$\begin{aligned}
&\sup_{0\le t\le T_*}\left(|v^{(j)}(t)|_{D_0^1\cap D^{2l+1}}\right) + \int_0^{T_*} |v^{(j)}(t)|^2_{D^2\cap D^{2l+2}}\,dt \le 1 + c_{l,2,j},\\
&\sup_{0\le t\le T_*}\left(|v^{(j)}(t)|_{D_0^1\cap D^{2l+2}}\right)\\
&\qquad + \int_0^{T_*}\left(|v^{(j+1)}(t)|^2_{D_0^1\cap D^{2l+1}} + |v^{(j)}(t)|^2_{D^{2l+3}}\right)dt \le 1 + c_{l,3,j},\\
&\operatorname*{ess\,sup}_{0<t<T_*}\left(|v^{(j+1)}(t)|_{D_0^1\cap D^{2l+1}} + |v^{(j)}(t)|_{D^{2l+3}}\right)\\
&\qquad + \int_0^{T_*}\left(|v^{(j+1)}(t)|^2_{D^{2l+2}} + |v^{(j)}(t)|^2_{D^{2l+4}}\right)dt \le 1 + c_{l,4,j}
\end{aligned} \tag{3.3}$$

for some time $T_* \in (0,T)$ and constants $c_{l,i,j}$ such that

$$\begin{aligned}
&1 \le c_{l,i,j} \le c_{l+1,i,,j} \quad \text{for any} \quad i,j,l\\
&c_{l,i,j} \le c_{l,i+1,j} \quad \text{for each fixed} \quad l,j,\\
\text{and} \quad &c_{l,i,j} \le c_{l,i,l+1} \quad \text{for each fixed} \quad l \quad \text{and for any} \quad i.
\end{aligned}$$

The constants $c_{l,i,j}$'s and T_* will be determined later and depend only on c_0 and the parameters of C.

Lemma 3.1. *For each $1 \le j \le k+1$*

$$|\rho^{(j)}(0)|_{L^{\frac{3}{2}}\cap H^{2(k-j)+3}} \le C\varepsilon c^j_{0,1,j-1}.$$

Proof. Using the equation

$$\rho^{(j+1)} = -\sum_{0\le m\le j}\binom{j}{m}\operatorname{div}(\rho^{(j-m)}v^{(m)}), \tag{3.4}$$

for any multi-index $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ with $|\alpha| = \alpha_1+\alpha_2+\alpha_3 \le 2(k-(j+1))+3$ we have

$$\begin{aligned}
&|D^\alpha \rho^{(j+1)}(0)|_{L^2}\\
&\quad \le C \sum_{\substack{0\le m\le j\\ \alpha_1+\alpha_2=\alpha}} \Big(|\nabla D^{\alpha_1}\rho^{(j-m)}(0)\cdot D^{\alpha_2}v^{(m)}(0)|_{L^2}\\
&\qquad\qquad + |D^{\alpha_1}\rho^{(j-m)}(0)\operatorname{div} D^{\alpha_2}v^{(m)}(0)|_{L^2}\Big)\\
&\quad \le C \sum_{\substack{0\le m\le j\\ \alpha_1+\alpha_2=\alpha}} |\rho^{(j-m)}(0)|_{H^{|\alpha_1|+2}}|v^{(m)}(0)|_{D_0^1\cap D^{|\alpha_2|+1}}\\
&\quad \le C \sum_{\substack{0\le m\le j\\ \alpha_1+\alpha_2=\alpha}} |\rho^{(j-m)}(0)|_{H^{2(k-j)+3}}|v^{(m)}(0)|_{D_0^1\cap D^{2(k-j)+2}},
\end{aligned}$$

where $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}$. Since $0 \le m \le j$, $\quad 2(k-j)+3 = 2(k-(j-m))+3-2m) \le 2(k-(j-m))+3$ and $2(k-j)+2 \le 2(k-m)+2-2(j-m) \le 2(k-m)+2$. Thus by the initial bound (3.1), we obtain

$$
\begin{aligned}
&|\rho^{(j+1)}(0)|_{H^{2(k-(j+1))+3}} \\
&\qquad \le C \sum_{0\le m\le j} |\rho^{(j-m)}(0)|_{H^{2(k-(j-m))+3}} |v_{(m)}(0)|_{D_0^1\cap D^{2(k-m)+2}} \\
&\qquad \le C c_{0,1,j} \sum_{0\le m\le j} |\rho^{(j-m)}(0)|_{H^{2(k-(j-m))+3}}.
\end{aligned}
$$

Therefore the induction on j yields that

$$
|\rho^{(j)}(0)|_{H^{2(k-j)+3}} \le C\varepsilon c_{0,1,j-1}^j
$$

for all $1 \le j \le k+1$. Similarly, we can treat $L^{\frac{3}{2}}$ norm. $\square$

Lemma 3.2. *If ε is sufficiently small, then we have*

$$
\sum_{0\le j\le k+1} |u^{(j)}(0)|_{D_0^1\cap D^{2(k-j)+3}} \le C c_0.
$$

Proof. We first consider the case $1 \le j \le k$. Since $g_j \in D_0^1$ satisfies the following boundary value problem:

$$
\begin{aligned}
L(u^{(j)}(0)) &= \rho_0 (f - v\cdot\nabla v)^{(j)}(0) + \sum_{0\le m\le j-1} \begin{pmatrix} j \\ m \end{pmatrix} \rho^{(j-m)}(0)(f-v\cdot\nabla v)^{(m)}(0) \\
&\quad + \sum_{0\le m\le j-1} \begin{pmatrix} j \\ m \end{pmatrix} \rho^{(j-m)}(0) u^{(m+1)}(0) - \nabla p^{(j)}(0) - \rho_0 g_{j+1},
\end{aligned}
$$

we have

$$
\begin{aligned}
|u^{(j)}|_{D_0^1\cap D^{2(k-j)+3}} &\le C|u^{(j)}|_{D_0^1} + C|\rho_0(f-v\cdot\nabla v)^{(j)}(0)|_{H^{2(k-j)+1}} \\
&\quad + C \sum_{0\le m\le j-1} |\rho^{(j-m)}(0)(f-v\cdot\nabla v)^{(m)}(0)|_{H^{2(k-j)+1}} \\
&\quad + C \sum_{0\le m\le j-1} |\rho^{(j-m)}(0) u^{(m+1)}(0)|_{H^{2(k-j)+1}} \\
&\quad + C|p^{(j)}(0)|_{H^{2(k-j)+2}} + C|\rho_0 g_{j+1}|_{H^{2(k-j)+1}}
\end{aligned}
$$

from the elliptic regularity results [1, 5], where C does not depend on the size of domain. We easily show that

$$
\begin{aligned}
&|\rho_0(f-v\cdot\nabla v)^{(j)}(0)|_{H^{2(k-j)+1}} \\
&\quad \le C|\rho_0|_{H^{2(k-j)+3}} |f^{(j)}(0)|_{H^{2(k-j)+1}} \\
&\qquad + C \sum_{0\le m\le j} |\rho_0|_{H^{2(k-j)+3}} |v^{(j-m)}(0)|_{D_0^1\cap D^{2(k-j)+3}} |v^{(m)}|_{D_0^1\cap D^{2(k-j)+2}} \\
&\quad \le C\varepsilon c_{0,1,j}^2.
\end{aligned}
$$

Similarly, we have

$$\begin{aligned}
&\sum_{0\le m\le j-1} |\rho^{(j-m)}(0)(f - v\cdot\nabla v)^{(m)}(0)|_{H^{2(k-j)+1}} \\
&\le C \sum_{0\le m\le j-1} |\rho^{(j-m)}(0)|_{H^{2(k-j)+3}} |(f - v\cdot\nabla v)^{(m)}(0)|_{H^{2(k-j)+1}} \le C\varepsilon c_{0,1,j-1}^{j+2},
\end{aligned}$$

$$|p^{(j)}(0)|_{H^{2(k-j)+2}} \le C\varepsilon c_{0,1,j-1}^{j},$$

$$|\rho_0 g_{j+1}|_{H^{2(k-j)+1}} \le C|\rho_0|_{H^{2(k-j)+3}} |g_{j+1}|_{H^{2(k-j)+1}} \le C\varepsilon |g_{j+1}|_{D_0^1\cap D^{2(k-(j+1))+3}},$$

and

$$\begin{aligned}
&\sum_{0\le m\le j-1} |\rho^{(j-m)}(0) u^{(m+1)}(0)|_{H^{2(k-j)+1}} \\
&\quad\le C \sum_{0\le m\le j} |\rho^{(j-m)}(0)|_{H^{2(k-j)+1}} |u^{(m+1)}(0)|_{D_0^1\cap D^{2(k-j)+3}} \\
&\quad\le C\varepsilon c_{0,1,j-1}^{j} A_{j-1} + C\varepsilon c_{0,1,0} |u^{(j)}(0)|_{D_0^1\cap D^{2(k-j)+3}},
\end{aligned}$$

where

$$A_j = \sum_{1\le m\le j} |u^{(m)}(0)|_{D_0^1\cap D^{2(k-m)+3}} \quad \text{and} \quad A_0 = 0.$$

Thus if we choose $\varepsilon < \frac{1}{2Mc_{0,1,k}^{k+2}}$ for some large $M \ge C$, then since $g_j \in D_0^1$, we obtain

$$\begin{aligned}
A_j &\le C \sum_{1\le j\le k+1} |g_j|_{D_0^1} + C\varepsilon |g_{j+1}|_{D_0^1\cap D^{2(k-(j+1))+3}} \\
&\le Cc_0 + C\varepsilon |g_{j+1}|_{D_0^1\cap D^{2(k-(j+1))+3}}
\end{aligned}$$

for all $1 \le j \le k$. Since $|g_{k+1}|_{D_0^1} \le c_0$, we have $G_k \le Cc_0$.

Now using the energy estimate for u_0, we easily get $|u_0|_{D_0^1} \le Cc_0$ for small ε. By the another use of elliptic regularity result, we finally have for $j = 0$ that

$$\begin{aligned}
|u_0|_{D_0^1\cap D^{2k+3}} &\le C|u_0|_{D_0^1} + C|\rho_0(f(0) - v(0)\cdot\nabla v(0))|_{H^{2k+1}} \\
&\quad + C|p_0|_{H^{2k+2}} + C|\rho_0 g_1|_{H^{2k+1}} \\
&\le Cc_0 + C\varepsilon c_{0,1,0}^2 + C\varepsilon c_0 \le Cc_0.
\end{aligned}$$

This completes the proof of the lemma. □

Remark 3.3. Since the estimate of Lemma 3.2 is stable under a small perturbation of g_j, this lemma will be used for the reconstruction of initial data in Lemma 4.2 below. If the initial data has positive lower bound, then this lemma is not necessary.

Lemma 3.4. *For any fixed $\delta > 0$ we have*

$$\inf_{\Omega} \rho(t) \ge C^{-1}\delta,$$

and for each $1 \le j \le k+1$

$$|\rho^{(j)}(t)|_{H^{2(k-j)+3}} \le C\varepsilon c_{0,1,j-1}^{j}, \quad \int_0^t |\rho^{(k+2)}(s)|_{L^2}^2\, ds \le C\varepsilon^2,$$

if $0 \le t \le \min(T_*, T_1)$ where $T_1 = (1 + c_{k,4,0})^{-(2k+6)}$.

Proof. From Lemma 2.3, we recall that

$$|\rho(t)|_{H^{2k+3}} \le |\rho_0|_{H^{2k+3}} \exp\left(C\int_0^t |v(s)|_{D_0^1\cap D^{2k+4}}\,ds\right)$$

and

$$\inf_\Omega \rho(t) \ge \left(\inf_\Omega \rho_0\right)\exp\left(-C\int_0^t |v(s)|_{D_0^1\cap D^{2k+4}}\,ds\right)$$

for $0 \le t \le T$. Hence from the observation that

$$\begin{aligned}\int_0^t |v(s)|_{D_0^1\cap D^{2k+4}}\,ds &\le t^{\frac12}\left(\int_0^t |v(s)|^2_{D_0^1\cap D^{2k+4}}\,ds\right)^{\frac12}\\ &\le C(1+c_{k,4,0})t + C((1+c_{k,4,0})t)^{\frac12},\end{aligned}$$

we obtain the desired estimate for $j = 0$. For the esimate of the case $j \ge 1$, multiplying (3.4) by $D^\alpha\rho^{(j)}$ for any multi-index α with $|\alpha| \le 2(k-j)+3$, we have

$$\begin{aligned}&\frac12\frac{d}{dt}|D^\alpha\rho^{(j)}|^2_{L^2}\\ &\quad= -\int \operatorname{div}(D^\alpha\rho^{(j)}v)D^\alpha\rho^{(j)}\,dx\\ &\qquad-\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \alpha_2\neq 0}} C_{\alpha_1,\alpha_2}\int \operatorname{div}(D^{\alpha_1}\rho^{(j)}D^{\alpha_2}v^{(m)})D^\alpha\rho^{(j)}\,dx\\ &\qquad-\sum_{\substack{1\le m\le j\\ \alpha_1+\alpha_2=\alpha}} C_{j,\alpha}\int \operatorname{div}(D^{\alpha_1}\rho^{(j-m)}D^{\alpha_2}v^{(m)})D^\alpha\rho^{(j)}\,dx\\ &\quad=\sum_{1\le i\le 3} J_i\end{aligned} \tag{3.5}$$

where

$$C_{\alpha_1,\alpha_2} = \frac{\alpha!}{\alpha_1!\alpha_2!},\quad C_{j\,\alpha} = \begin{pmatrix} j\\ m\end{pmatrix}\frac{\alpha!}{\alpha_1!\alpha_2!}.$$

Each term can be estimated as follows:

$$J_1 \le C|\operatorname{div} v|_{D_0^1\cap D^2}|D^\alpha\rho_{(j)}|^2_{L^2},$$

$$\begin{aligned}J_2 &\le C\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \alpha_2\neq 0}}\int\left(|\nabla D^{\alpha_1}\rho^{(j)}||D^{\alpha_2}v| + |D^{\alpha_1}\rho^{(j)}||D^{\alpha_2}\mathrm{div}v|\right)|D^\alpha\rho^{(j)}|\,dx\\ &\le C\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \alpha_2\neq 0}}|v|_{D_0^1\cap D^{|\alpha_2|+2}}|\rho^{(j)}|_{H^{|\alpha_1|+1}}|D^\alpha\rho^{(j)}|_{L^2},\end{aligned}$$

$$J_3 \le C\sum_{\substack{1\le m\le j\\ \alpha_1+\alpha_2=\alpha}}|\rho^{(j-m)}|_{H^{|\alpha_1|+1}}|v^{(m)}|_{D_0^1\cap D^{|\alpha_2|+2}}|D^\alpha\rho^{(j)}|_{L^2}.$$

Substituting all these estimate into (3.5) and integrating over $(0,t)$, we have

$$
\begin{aligned}
|\rho^{(j)}(t)|_{H^{2(k-j)+3}} \le |\rho^{(j)}(0)|_{H^{2(k-j)+3}} &+ C\int_0^t |\mathrm{div}\, v|_{D_0^1\cap D^2}|D^\alpha \rho^{(j)}|_{L^2}\, ds \\
&+C \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_2\neq 0}} \int_0^t |v(s)|_{D_0^1\cap D^{|\alpha_2|+2}}|\rho^{(j)}(s)|_{H^{|\alpha_1|+1}}\, ds \\
&+C \sum_{\substack{1\le m\le j \\ \alpha_1+\alpha_2=\alpha}} \int_0^t |\rho^{(j-m)}|_{H^{|\alpha_1|+2}}|v^{(m)}|_{D_0^1\cap D^{|\alpha_2|+1}}\, ds \quad .
\end{aligned}
$$

Since $2(k-j)+5 \le 2k+3$ for $j \ge 1$,

$$
\begin{aligned}
&2(k-j)+5 \le 2(k-(j-m))+5-2m \le 2(k-(j-m))+3 \ \text{ for } \ m\ge 1 \\
\text{and }\ &2(k-j)+4 \le 2(k-m)+4-2(j-m) \le 2(k-m)+4 \ \text{ for } \ m \le j,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
|\rho^{(j)}(t)|_{H^{2(k-j)+3}} \le C\varepsilon c^j_{0,1,,j-1} &+ Cc_{1,2,0}\int_0^t |\rho^{(j)}(s)|_{H^{2(k-j)+3}}\, ds \\
&+ C \max_{0\le m\le j-1}\left(\sup_{0\le s\le t} |\rho^{(m)}(s)|_{H^{2(k-m)+3}}\right)(1+c_{k,4,0})^{\frac12} t^{\frac12}.
\end{aligned}
$$

Hence the Gronwall's inequality yields that if $t \le (1+c_{k,4,0})^{-1}$

$$
|\rho^{(j)}(t)|_{H^{2(k-j)+3}} \le C\left(\varepsilon c^j_{0,1,j-1} + \max_{0\le m\le j-1}\left(\sup_{0\le s\le t} |\rho^{(m)}(s)|_{H^{2(k-m)+3}}\right)\right).
$$

Using the induction on j again, we have

$$
|\rho^{(j)}(t)|_{H^{2(k-j)+3}} \le C\varepsilon c^j_{0,1,j-1}
$$

for all $t \in [0, \min(T_*, T_1)]$. Similarly (more easily), we can estimate $|\rho^{(j)}|_{L^1}$.

Finally we have

$$
\begin{aligned}
|\rho^{(k+2)}|_{L^2} &\le C \sum_{0\le m\le k+1} \left(|\nabla\rho^{(k+1-m)}\cdot v^{(m)}|_{L^2} + |\rho^{(k+1-m)}v|_{L^2}\right) \\
&\le C \sum_{1\le m\le k+1} |\rho^{(k+1-m)}|_{H^2}|\nabla v^{(m)}|_{L^2} + C|\rho^{(k+1)}|_{H^1}|v|_{D_0^1\cap D^2} \\
&\le C\varepsilon c^k_{0,1,k-1}c_{0,4,k} + Cc^{k+1}_{0,1,k}c_{0,3,0}
\end{aligned}
$$

and hence

$$
\int_0^t |\rho^{(k+2)}(s)|^2_{L^2}\, ds \le C\varepsilon^2.
$$

□

Using the equation $p_t + \nabla p\cdot u + \rho p'(\rho)\mathrm{div}\, u = 0$, by the similar estimates as in the proof of Lemma 3.1 and Lemma 3.4 we have

Lemma 3.5. *For any $0 \le j \le k+1$, we have*

$$
|(p-p(0))^{(j)}(t)|_{L^1\cap H^{2(k-j)+3}} \le C\varepsilon c^j_{0,1,j-1}
$$

for $0 \le t \le \min(T_*, T_1)$ *where* $T_1 = (1 + c_{k,4,0})^{-(2k+6)}$, *where* $p(0)$ *is the value of* p *at point zero.*

The three lemmas below, Lemma 3.6, 3.7 and 3.8, are similar to those in [4]. But since they are needed for a priori estimates for high regularity, we revisit the proof of them.

Lemma 3.6.

$$\int_0^t |\sqrt{\rho} u_t(s)|_{L^2}^2\, ds + |u(t)|_{D_0^1}^2 \le C c_0^2, \quad \int_0^t |u(s)|_{D^2}^2\, ds \le C c_0^3$$

for $0 \le t \le \min(T_*, T_1)$.

Proof. Multiplying the equation (2.8) by u_t and integrating over Ω, we obtain

$$\begin{aligned} &\int \rho |u_t|^2\, dx + \frac{1}{2}\frac{d}{dt}\int \mu |\nabla u|^2 + (\lambda+\mu)(\mathrm{div} u)^2\, dx \\ &\qquad = -\int \nabla p \cdot u_t\, dx + \int \rho (f - v\cdot\nabla v)\cdot u_t\, dx. \end{aligned} \tag{3.6}$$

Using the conditon $\varepsilon \le c_0$ and Lemma 3.4 together with (3.3), we can estimate the second term of the right hand side in (3.14) as follows:

$$\begin{aligned} \int \rho (f - v\cdot\nabla v)\cdot u_t\, dx &\le |\rho|_{L^\infty}^{\frac{1}{2}} |f - v\cdot\nabla v|_{L^2} |\sqrt{\rho} u_t|_{L^2} \\ &\le C|\rho|_{L^\infty}\left(|f|_{L^2}^2 + |v|_{D_0^1\cap D^2}^4\right) + \frac{1}{2}|\sqrt{\rho} u_t|_{L^2}^2 \\ &\le C c_0 c_{0,3,0}^4 + \frac{1}{2}|\sqrt{\rho} u_t|_{L^2}^2. \end{aligned}$$

To estimate the first term, we observe from Lemma 3.5 that

$$\begin{aligned} -\int \nabla p \cdot u_t\, dx &= \int (p - p(0))\,\mathrm{div} u_t\, dx \\ &= \frac{d}{dt}\int (p - p(0))\,\mathrm{div} u\, dx - \int p_t\,\mathrm{div} u\, dx, \end{aligned}$$

$$\int (p - p(0)\,\mathrm{div} u\, dx \le C|p(\rho) - p(0)|_{L^2}^2 + \frac{\mu}{4}|\nabla u|_{L^2}^2 \le C c_0^2 + \frac{\mu}{4}|\nabla u|_{L^2}^2$$

and

$$-\int p_t\,\mathrm{div} u\, dx \le |p_t|_{L^2}^2 + |\nabla u|_{L^2}^2 \le C c_0^2 c_{0,1,0}^2 + |\nabla u|_{L^2}^2.$$

Hence integrating (3.14) in time over $(0, t)$, we have

$$\begin{aligned} &\int_0^t |\sqrt{\rho} u_t(s)|_{L^2}^2\, ds + |\nabla u(t)|_{L^2}^2 \\ &\quad \le C c_0^2\left(1 + |\nabla u_0|_{L^2}^2\right) + C c_0^2 c_{0,3,0}^4 t + C\int_0^t |\nabla u(s)|_{L^2}^2\, ds \end{aligned}$$

for $0 \le t \le \min(T_*, T_1)$. Therefore, in view of Gronwall's inequality, we conclude that

$$\int_0^t |\sqrt{\rho} u_t(s)|_{L^2}^2\, ds + |\nabla u(t)|_{L^2}^2 \le C c_0^2 \quad \text{for} \quad 0 \le t \le \min(T_*, T_1).$$

Moreover, since for each $t \in (0,T)$, $u = u(t) \in D_0^1 \cap D^2$ is a solution of the elliptic system

$$Lu = -\nabla p + \rho(f - v\cdot\nabla v) - \rho u_t \quad \text{in} \quad \Omega,$$

it follows from the elliptic regularity result in [1, 5] that

$$\begin{aligned}|u|_{D^2} &\le C\left(|-\nabla p + \rho(f - v\cdot\nabla v) - \rho u_t|_{L^2} + |u|_{D_0^1}\right)\\ &\le C\left(c_{0,3,0}^2 + c_0^{\frac{1}{2}}|\sqrt{\rho}u_t|_{L^2}\right)\end{aligned}$$

and thus

$$\int_0^t |u(s)|_{D^2}^2\, ds \le Cc_0^3 \quad \text{for} \quad 0 \le t \le \min(T_*, T_1).$$

This completes the proof of Lemma 3.6. □

Lemma 3.7.

$$|\sqrt{\rho}u_t(t)|_{L^2} + |u(t)|_{D^2} + \int_0^t \left(|u_t(s)|_{D_0^1}^2 + |u(s)|_{D^3}^2\right) ds \le Cc_0^3 c_{0,2,0}^{\frac{3}{2}} c_{0,3,0}^{\frac{1}{2}}$$

for $0 \le t \le \min(T_*, T_1)$.

Proof. We differentiate (2.8) with respect to t and have

$$\rho u_{tt} + Lu_t + \nabla p_t = \rho(f - v\cdot\nabla v)_t + \rho_t(f - v\cdot\nabla v - u_t). \tag{3.7}$$

Multiplying this by u_t and integrating over Ω, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2\, dx + \int \mu|\nabla u_t|^2 + (\lambda+\mu)(\mathrm{div} u_t)^2\, dx\\ &= \int\left(-\nabla p_t + \rho(f - v\cdot\nabla v)_t + \rho_t(f - v\cdot\nabla v - \frac{1}{2}u_t)\right)\cdot u_t\, dx.\end{aligned} \tag{3.8}$$

To estimate each term in the right hand side of (3.8), we follow the arguments in [3, 5, 6]; we first apply the standard inequalities such as Hölder, Sobolev and Young's inequalities and Lemma 3.4.

$$-\int \nabla p_t\cdot u_t\, dx = \int p_t \,\mathrm{div}\, u_t\, dx \le C|p_t|_{L^2}^2 + \frac{\mu}{8}|\nabla u_t|_{L^2}^2 \le Cc_0^2 c_{0,1,0}^2 + \frac{\mu}{8}|\nabla u_t|_{L^2}^2,$$

$$\int \rho f_t\cdot u_t\, dx \le |f_t|_{L^2}|\rho|_{L^\infty}^{\frac{1}{2}}|\sqrt{\rho}u_t|_{L^2} \le |f_t|_{L^2}^2 + Cc_0|\sqrt{\rho}u_t|_{L^2}^2,$$

$$\begin{aligned}-\int \rho(v\cdot\nabla v)_t\cdot u_t\, dx &\le C|\rho|_{L^\infty}^{\frac{1}{2}}|v_t|_{D_0^1}|v|_{D_0^1}|\sqrt{\rho}u_t|_{L^3}\\ &\le C|\rho|_{L^\infty}^{\frac{3}{4}}|v_t|_{D_0^1}|v|_{D_0^1}|\sqrt{\rho}u_t|_{L^2}^{\frac{1}{2}}|\nabla u_t|_{L^2}^{\frac{1}{2}}\\ &\le \eta^{-2}C|\rho|_{L^\infty}^3|v|_{D_0^1}^4|\sqrt{\rho}u_t|_{L^2}^2 + \eta|v_t|_{D_0^1}^2 + \frac{\mu}{8}|\nabla u_t|_{L^2}^2\\ &\le \eta^{-2}Cc_{0,2,0}^7|\sqrt{\rho}u_t|_{L^2}^2 + \eta|v_t|_{D_0^1}^2 + \frac{\mu}{8}|\nabla u_t|_{L^2}^2,\end{aligned}$$

$$
\begin{aligned}
\int \rho_t(f - v\cdot\nabla v)\cdot u_t\,dx &\le C|\rho_t|_{H^1}\left(|f|_{L^2} + |v|^2_{D_0^1\cap D^2}\right)|\nabla u_t|_{L^2} \\
&\le Cc_0^2c_{0,1,0}^2\left(|f|^2_{L^2} + c_{0,3,0}^4\right) + \frac{\mu}{8}|\nabla u_t|^2_{L^2} \\
&\le Cc_{0,3,0}^8 + \frac{\mu}{8}|\nabla u_t|^2_{L^2}
\end{aligned}
$$

and finally

$$
\begin{aligned}
-\int \rho_t\left(\frac{1}{2}|u_t|^2\right)dx &= \int \mathrm{div}(\rho v)\left(\frac{1}{2}|u_t|^2\right)dx \\
&\le \int \rho|v||u_t||\nabla u_t|\,dx \\
&\le C|\rho|^{\frac{3}{4}}_{L^\infty}|v|_{D_0^1}|\sqrt{\rho}u_t|^{\frac{1}{2}}_{L^2}|\nabla u_t|^{\frac{3}{2}}_{L^2} \\
&\le Cc_{0,2,0}^7|\sqrt{\rho}u_t|^2_{L^2} + \frac{\mu}{8}|\nabla u_t|^2_{L^2}.
\end{aligned}
$$

Here $\eta \in (0,1)$ is a small number. Substituting these estimates into (3.8) and taking $\eta = (1 + c_{0,3,0})^{-1}$, we have

$$
\begin{aligned}
&\frac{d}{dt}\int \rho|u_t|^2\,dx + \mu\int|\nabla u_t|^2\,dx \\
&\le C\left(|f_t|^2_{L^2} + c_{0,3,0}^8\right) + Cc_{0,3,0}^9|\sqrt{\rho}u_t|^2_{L^2} + (1 + c_{0,3,0})^{-1}|v_t|^2_{D_0^1}
\end{aligned}
\tag{3.9}
$$

for $0 \le t \le \min(T_*, T_1)$. On the other hand, since $u_t(0) = g_2$, it follows that

$$
|\sqrt{\rho}u_t(0)|_{L^2} + |u_t(0)|_{D_0^1} \le Cc_0^3. \tag{3.10}
$$

Hence integrating (3.9) over $(0,t)$, we also have

$$
|\sqrt{\rho}u_t(t)|^2_{L^2} + \int_0^t|\nabla u_t(s)|^2_{L^2}\,ds \le C\left(c_0^3 + c_{0,3,0}^8t\right) + Cc_{0,3,0}^9\int_0^t|\sqrt{\rho}u_t(s)|^2_{L^2}\,ds.
$$

Therefore, in view of Gronwall's inequality, we conclude that

$$
|\sqrt{\rho}u_t(t)|^2_{L^2} + \int_0^t|u_t(s)|^2_{D_0^1}\,ds \le Cc_0^3 \quad \text{for} \quad 0 \le t \le \min(T_*, T_1).
$$

Moreover, since for each $t \in (0,T)$, $u = u(t) \in D_0^1 \cap D^3$ is a solution of the elliptic system

$$
Lu = -\nabla p + \rho(f - v\cdot\nabla v) - \rho u_t \quad \text{in} \quad \Omega,
$$

it follows from the elliptic regularity result in [1, 5] that

$$
\begin{aligned}
|u(t)|_{D^2} &\le Cc_0^2\left(1 + |v\cdot\nabla v|_{L^2}\right) \\
&\le Cc_0^2\left(1 + |v|^{\frac{3}{2}}_{D_0^1}|v|^{\frac{1}{2}}_{D_0^1\cap D^2}\right) \le Cc_0^2c_{0,2,0}^{\frac{3}{2}}c_{0,3,0}^{\frac{1}{2}}
\end{aligned}
$$

and

$$
\int_0^t|u(s)|^2_{D^3}\,ds \le Cc_0^2\int_0^t\left(1 + |v(s)|^4_{D_0^1\cap D^2} + |u_t(s)|^2_{D_0^1}\right)ds \le Cc_0^5
$$

for $0 \le t \le \min(T_*, T_1)$. This completes the proof of Lemma 3.7. □

Lemma 3.8.

$$|u_t(t)|^2_{D^1_0} + |u(t)|^2_{D^3} + \int_0^t \left(|\sqrt{\rho}u_{tt}(s)|^2_{L^2} + |u_t(s)|^2_{D^2} + |u(s)|^2_{D^4}\right) ds \le Cc_0^7 c_{0,3,0}^{12}$$

for $0 \le t \le \min(T_*, T_1)$.

Proof. Multiplying (3.7) by u_{tt} and integrating over Ω, we have

$$\begin{aligned}\int \rho|u_{tt}|^2\,dx + \frac{1}{2}\frac{d}{dt}\int \mu|\nabla u_t|^2 + (\lambda+\mu)(\mathrm{div}u_t)^2\,dx \\ = \int \left(-\nabla p_t + \rho(f - v\cdot\nabla v)_t + \rho_t(f - v\cdot\nabla v - u_t)\right)\cdot u_{tt}\,dx\end{aligned} \tag{3.11}$$

We can estimate the first two terms in the right hand side of (3.11) as follows:

$$\begin{aligned}-\int \nabla p_t\cdot u_{tt}\,dx = \int p_t \mathrm{div}u_{tt}\,dx &= \frac{d}{dt}\int p_t\mathrm{div}u_t\,dx - \int p_{tt}\mathrm{div}u_t\,dx \\ &\le \frac{d}{dt}\int p_t\mathrm{div}u_t\,dx + |p_{tt}|^2_{L^2} + |\nabla u_t|^2_{L^2}\end{aligned}$$

and

$$\begin{aligned}\int \rho(f - v\cdot\nabla v)_t\cdot u_{tt}\,dx &\le C|\rho|^{\frac{1}{2}}_{L^\infty}\left(|f_t|_{L^2} + |v|_{D^1_0\cap D^2}|v_t|_{D^1_0}\right)|\sqrt{\rho}u_{tt}|_{L^2} \\ &\le Cc_0\left(|f_t|^2_{L^2} + c_{0,3,0}^2|v_t|^2_{D^1_0}\right) + \frac{1}{2}|\sqrt{\rho}u_{tt}|^2_{L^2}.\end{aligned}$$

To estimate the last term, we observe that

$$\begin{aligned}&\int \rho_t\,(f - v\cdot\nabla v)\cdot u_{tt}\,dx \\ &\quad= \frac{d}{dt}\int \rho_t\,(f - v\cdot\nabla v)\cdot u_t\,dx - \int \rho_{tt}\,(f - v\cdot\nabla v)\cdot u_t\,dx \\ &\qquad - \int \rho_t\,(f - v\cdot\nabla v)_t\cdot u_t\,dx\end{aligned}$$

and

$$-\int \rho_t u_t\cdot u_{tt}\,dx = -\frac{d}{dt}\int \rho_t\left(\frac{1}{2}|u_t|^2\right)dx + \int \rho_{tt}\left(\frac{1}{2}|u_t|^2\right)dx.$$

Then by virtue of Lemma 3.4, we obtain

$$\begin{aligned}-\int \rho_{tt}\,(f - v\cdot\nabla v)\cdot u_t\,dx &\le C|\rho_{tt}|_{L^2}\left(|f|_{H^1} + |v|^2_{D^1_0\cap D^2}\right)|\nabla u_t|_{L^2} \\ &\le Cc_{0,3,0}^4|\rho_{tt}|^2_{L^2} + |\nabla u_t|^2_{L^2},\end{aligned}$$

$$\begin{aligned}-\int \rho_t\,(f - v\cdot\nabla v)_t\cdot u_t\,dx &\le C|\rho_t|_{L^3}\left(|f_t|_{L^2} + |v|_{D^1_0\cap D^2}|v_t|_{D^1_0}\right)|\nabla u_t|_{L^2} \\ &\le Cc_{0,3,0}^4\left(|f_t|^2_{L^2} + c_3^2|v_t|^2_{D^1_0}\right) + |\nabla u_t|^2_{L^2}\end{aligned}$$

and

$$\begin{aligned}\int \rho_{tt}\left(\frac{1}{2}|u_t|^2\right)\,dx &= -\int \mathrm{div}(\rho_t v+\rho v_t)\left(\frac{1}{2}|u_t|^2\right)\,dx\\ &\le \int \left(|\rho_t||v|+\rho|v_t|\right)|u_t||\nabla u_t|\,dx\\ &\le Cc_{0,3,0}^3|\nabla u_t|_{L^2}^2 + Cc_0^{\frac{3}{4}}|v_t|_{D_0^1}|\sqrt{\rho}u_t|_{L^2}^{\frac{1}{2}}|\nabla u_t|_{L^2}^{\frac{3}{2}}\\ &\le Cc_{0,3,0}^3|\nabla u_t|_{L^2}^2 + (1+c_{0,3,0})^{-1}|v_t|_{D_0^1}^2|\sqrt{\rho}u_t|_{L^2}|\nabla u_t|_{L^2}\\ &\le Cc_{0,3,0}^3|u_t|_{D_0^1}^2 + (1+c_{0,3,0})^{-1}|v_t|_{D_0^1}^2\left(|\sqrt{\rho}u_t|_{L^2}^2+|\nabla u_t|_{L^2}^2\right).\end{aligned}$$

Substituting all the above estimates into (3.11), we have

$$\begin{aligned}&\int \rho|u_{tt}|^2\,dx + \frac{d}{dt}\int \mu|\nabla u_t|^2+(\lambda+\mu)(\mathrm{div}u_t)^2\,dx\\ &\le \frac{d}{dt}\int \left(2p_t\mathrm{div}u_t + 2\rho_t(f-v\cdot\nabla v)\cdot u_t - \rho_t|u_t|^2\right)\,dx\\ &+C\left(|p_{tt}|_{L^2}^2 + c_{0,3,0}^4|\rho_{tt}|_{L^2}^2 + c_{0,3,0}^4|f_t|_{L^2}^2 + c_{0,3,0}^6|v_t|_{D_0^1}^2 + c_{0,3,0}^3|u_t|_{D_0^1}^2\right)\\ &\qquad +|v_t|_{D_0^1}^2|\sqrt{\rho}u_t|_{L^2}^2 + (1+c_{0,3,0})^{-1}|v_t|_{D_0^1}^2|\nabla u_t|_{L^2}^2\end{aligned}\tag{3.12}$$

for $0\le t\le \min(T_*,T_1)$. Now let us define a function Λ by

$$\begin{aligned}\Lambda(t) = &\int \left(\mu|\nabla u_t|^2+(\lambda+\mu)(\mathrm{div}u_t)^2\right)(t)\,dx\\ &-\int \left(2p_t\mathrm{div}u_t + 2\rho_t(f-v\cdot\nabla v)\cdot u_t - \rho_t|u_t|^2\right)(t)\,dx.\end{aligned}$$

Then it follows from Lemma 3.4, Lemma 3.7 and (3.10) that

$$\begin{aligned}|\Lambda| &\le C\left(|\nabla u_t|_{L^2}^2 + |p_t|_{L^2}^2 + |\rho_t|_{L^3}^2|f-v\cdot\nabla v|_{L^2}^2 + |\rho|_{L^\infty}^3|v|_{D_0^1}^4|\sqrt{\rho}u_t|_{L^2}^2\right)\\ &\le C|\nabla u_t|_{L^2}^2 + Cc_0^7c_{0,3,0}^8,\\ \Lambda &\ge C^{-1}|\nabla u_t|_{L^2}^2 - Cc_0^7c_{0,3,0}^8 \quad\text{and}\quad |\Lambda(0)|\le Cc_0^7c_{0,3,0}^8.\end{aligned}$$

Hence integrating (3.12) over $(0,t)$ and using Lemma 3.4 and Lemma 3.7, we deduce that

$$\begin{aligned}&\int_0^t |\sqrt{\rho}u_{tt}(s)|_{L^2}^2\,ds + |\nabla u_t(t)|_{L^2}^2\\ &\qquad\le Cc_0^7c_{0,3,0}^{12} + \int_0^t C(1+c_{0,3,0})^{-1}|v_t|_{D_0^1}^2|\nabla u_t(s)|_{L^2}^2\,ds\end{aligned}$$

for $0\le t\le\min(T_*,T_1)$. Therefore, in view of Gronwall's inequality, we conclude that

$$\int_0^t |\sqrt{\rho}u_{tt}(s)|_{L^2}^2\,ds + |u_t(t)|_{D_0^1}^2 \le Cc_0^7c_{0,3,0}^{12}$$

for $0\le t\le\min(T_*,T_1)$. Moreover, since $Lu = -\nabla p + \rho(f-v\cdot\nabla v-u_t)$ in Ω, it follows from the elliptic regularity result that

$$\int_0^t |u_t(s)|_{D^2}^2\,ds + |u(t)|_{D^3}^2 \le Cc_0^7c_{0,3,0}^{12} \qquad\text{for}\quad 0\le t\le\min(T_*,T_1).$$

This completes the proof of Lemma 3.8. □

Lemma 3.9. *For* $1 \le j \le k+1$ *we have*

$$\int_0^t |\sqrt{\rho} u^{(j+1)}|_{L^2}^2 \, ds + \max_{1\le m\le j} \left(\sup_{0\le s\le t} |u_{(m)}(s)|_{D_0^1}^2 \right) \le C c_{0,3,j-1}^{j(2j+6)} \left(1 + \max_{1\le m\le j} \left(\sup_{0\le s\le t} |\sqrt{\rho} u^{(m)}(s)|_{L^2}^2 \right) \right).$$

Proof. From (1.6) we deduce that for all $1 \le j \le k+1$

$$\begin{aligned} &\rho u^{(j+1)} + Lu^{(j)} + \nabla p^{(j)} \\ &= \sum_{0\le m\le j} \binom{j}{m} \rho^{(j-m)} (f - v\cdot\nabla v)^{(m)} \\ &\qquad - j\rho^{(1)} u^{(j)} - \sum_{1\le m\le j-1} \binom{j}{m-1} \rho^{(j+1-m)} u^{(m)}. \end{aligned} \tag{3.13}$$

Multiplying (3.13) by $u^{(j+1)}$ and integrating over Ω, we have

$$\begin{aligned} &\int \rho |u^{(j+1)}|^2 \, dx + \frac{1}{2}\frac{d}{dt} \int \left(\mu |\nabla u^{(j)}|^2 + (\lambda+\mu)(\operatorname{div} u^{(j)})^2 \right) dx \\ &= -\int \nabla p^{(j)} \cdot u^{(j+1)} \, dx \\ &\quad + \sum_{0\le m\le j} \binom{j}{m} \rho^{(j-m)} (f - v\cdot\nabla v)^{(m)} \cdot u^{(j+1)} \, dx \\ &\quad - j \int \rho^{(1)} u^{(j)} \cdot u^{(j+1)} \, dx \\ &\quad - \sum_{1\le m\le j-1} \binom{j}{m-1} \rho^{(j+1-m)} u^{(m)} \cdot u^{(j+1)} \, dx. \end{aligned} \tag{3.14}$$

To estimate the first term, we observe from Lemma 3.5 that

$$\begin{aligned} -\int \nabla p^{(j)} \cdot u^{(j+1)} \, dx &= \int p^{(j)} \operatorname{div} u^{(j+1)} \, dx \\ &= \frac{d}{dt} \int p^{(j)} \operatorname{div} u^{(j)} \, dx - \int p^{(j+1)} \operatorname{div} u^{(j)} \, dx \end{aligned}$$

$$\int p^{(j)} \operatorname{div} u^{(j)} \, dx \le C |p^{(j)}|_{L^2}^2 + \frac{\mu}{8} |\nabla u^{(j)}|_{L^2}^2 \le C c_{0,1,j-1}^{2j+2} + \frac{\mu}{8} |\nabla u^{(j)}|_{L^2}^2$$

and

$$-\int p^{(j+1)} \operatorname{div} u^{(j)} \, dx \le C |p^{(j+1)}|_{L^2}^2 + |\nabla u^{(j)}|_{L^2}^2.$$

To estimate the second term of the right hand side in (3.14), we rewrite it as follows:

$$\sum_{0\le m\le j}\left(\begin{array}{c} j \\ m \end{array}\right)\rho^{(j-m)}(f-v\cdot\nabla v)^{(m)}\cdot u^{(j+1)}\,dx$$
$$=\sum_{0\le m\le j-1}\left(\begin{array}{c} j \\ m \end{array}\right)\int\rho^{(j-m)}(f-v\cdot\nabla v)^{(m)}\cdot u^{(j+1)}\,dx$$
$$+\int\rho(f-v\cdot\nabla v)^{(j)}\cdot u^{(j+1)}\,dx.$$

Using Lemma 3.4 together with (3.3), we have

$$\begin{aligned}\int\rho(f-v\cdot\nabla v)^{(j)}\cdot u^{(j+1)}\,dx &\le |\rho|_{L^\infty}^{\frac12}|(f-v\cdot\nabla v)^{(j)}|_{L^2}|\sqrt{\rho}u^{(j+1)}|_{L^2}\\ &\le C|\rho|_{L^\infty}|(f-v\cdot\nabla v)^{(j)}|_{L^2}^2+\frac12|\sqrt{\rho}u^{(j+1)}|_{L^2}^2\\ &\le Cc_{0,3,j}^5+\frac12|\sqrt{\rho}u^{(j+1)}|_{L^2}^2,\end{aligned}$$

$$\begin{aligned}&\sum_{0\le m\le j-1}\left(\begin{array}{c} j \\ m \end{array}\right)\int\rho^{(j-m)}(f-v\cdot\nabla v)^{(m)}\cdot u^{(j+1)}\,dx\\ &=\sum_{0\le m\le j-1}\left(\begin{array}{c} j \\ m \end{array}\right)\frac{d}{dt}\int\rho^{(j-m)}(f-v\cdot\nabla v)^{(m)}\cdot u^{(j)}\,dx\\ &\quad-\sum_{0\le m\le j-1}\left(\begin{array}{c} j \\ m \end{array}\right)\int\Big(\rho^{(j+1-m)}(f-v\cdot\nabla v)^{(m)}\\ &\qquad+\rho^{(j-m)}(f-v\cdot\nabla v)^{(m+1)}\Big)\cdot u^{(j)}\,dx.\end{aligned}$$

Since for all $0\le m\le j-1$,

$$|\rho^{(j-m)}|_{H^1}\le Cc_{0,1,j-1}^{j+1}\quad\text{and}\quad|(v\cdot\nabla v)^{(m)}|_{L^2\cap L^3}\le Cc_{0,3,j-1}^2,$$

we have

$$\begin{aligned}&\sum_{0\le m\le j-1}\left(\begin{array}{c} j \\ m \end{array}\right)\int\rho^{(j-m)}(f-v\cdot\nabla v)^{(m)}\cdot u^{(j)}\,dx\\ &\le\sum_{0\le m\le j-1}\left(\begin{array}{c} j \\ m \end{array}\right)|\rho^{(j-m)}|_{L^3}|(f-v\cdot\nabla v)^{(m)}|_{L^3}|\nabla u^{(j)}|_{L^2}\\ &\le C\sum_{0\le m\le j-1}|\rho^{(j-m)}|_{H^1}^2|(f-v\cdot\nabla v)^{(m)}|_{L^2}^2+\frac{\mu}{8}|\nabla u^{(j)}|_{L^2}^2\\ &\le Cc_{0,3,j-1}^{2j+6}+\frac{\mu}{8}|\nabla u^{(j)}|_{L^2}^2.\end{aligned}$$

On the other hand, we obtain

$$
\begin{aligned}
&\sum_{0\le m\le j-1}\begin{pmatrix} j \\ m \end{pmatrix}\int\Big(\rho^{(j+1-m)}(f-v\cdot\nabla v)^{(m)} \\
&\qquad\qquad\qquad +\rho^{(j-m)}(f-v\cdot\nabla v)^{(m+1)}\Big)\cdot u^{(j)}\,dx \\
&\le Cc_{0,3,j-1}^{2j+6}+c_{0,3,j-1}^{4}|\rho_{(j+1)}|_{L^2}^2 \\
&\qquad +C\sum_{0\le m\le j-1}|\rho^{(j-m)}|_{H^1}^2|(f-v\cdot\nabla v)^{(m)}|_{L^2}^2+|\nabla u^{(j)}|_{L^2}^2 \\
&\le Cc_{0,3,j-1}^{2j+6}+Cc_{0,3,j-1}^{4}|\rho^{(j+1)}|_{L^2}^2+|\nabla u^{(j)}|_{L^2}^2.
\end{aligned}
$$

We estimate the third term of (3.14) as follows:

$$
-j\int\rho^{(1)}u^{(j)}\cdot u^{(j+1)}\,dx=-\frac{j}{2}\frac{d}{dt}\int\rho^{(1)}|u^{(j)}|^2\,dx+\frac{j}{2}\int\rho^{(2)}|u^{(j)}|^2\,dx,
$$

$$
\begin{aligned}
-\frac{j}{2}\int\rho^{(1)}|u^{(j)}|^2\,dx &\le C\int\rho|v||u^{(j)}||\nabla u^{(j)}|\,dx \\
&\le C|\rho|_{L^\infty}^{\frac34}|v|_{D_0^1}|\sqrt{\rho}u^{(j)}|_{L^2}^{\frac12}|\nabla u^{(j)}|_{L^2}^{\frac32} \\
&\le Cc_{0,2,0}^{7}|\sqrt{\rho}u^{(j)}|_{L^2}^2+\frac{\mu}{8}|\nabla u^{(j)}|_{L^2}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{j}{2}\int\rho^{(2)}|u^{(j)}|^2\,dx &= -\frac{j}{2}\int(\rho^{(1)}v+\rho v^{(1)})\cdot(u^{(j)}\cdot\nabla u^{(j)})\,dx \\
&\le C(|\rho^{(1)}|_{H^1}|v|_{D_0^1}+|\rho|_{L^3}|v^{(1)}|_{D_0^1})|\nabla u^{(j)}|_{L^2}^2 \\
&\le Cc_{0,3,0}^{3}(1+|v^{(1)}|_{D_0^1})|\nabla u^{(j)}|_{L^2}^2.
\end{aligned}
$$

Finally, we have for the last term

$$
\begin{aligned}
&-\sum_{1\le m\le j-1}\begin{pmatrix} j \\ m-1 \end{pmatrix}\int\rho^{(j+1-m)}u^{(m)}\cdot u^{(j+1)}\,dx \\
&\qquad = -\sum_{1\le m\le j-1}\begin{pmatrix} j \\ m-1 \end{pmatrix}\frac{d}{dt}\int\rho^{(j+1-m)}u^{(m)}\cdot u^{(j)}\,dx \\
&\qquad\quad +\sum_{1\le m\le j-1}\begin{pmatrix} j \\ m-1 \end{pmatrix}\int(\rho^{(j+2-m)}u^{(m)}+\rho^{(j+1-m)}u^{(m+1)})\cdot u^{(j)}\,dx,
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{1\le m\le j-1}\begin{pmatrix} j \\ m-1 \end{pmatrix}\int\rho^{(j+1-m)}u^{(m)}\cdot u^{(j)}\,dx \\
&\qquad =\sum_{1\le m\le j-1}\begin{pmatrix} j \\ m-1 \end{pmatrix}\int(\rho v)^{(j-m)}(u^{(j)}\cdot\nabla u^{(m)}+u^{(m)}\cdot\nabla u^{(j)})\,dx \\
&\qquad \le C\sum_{1\le m\le j-1}|(\rho v)^{(j-m)}|_{L^3}^2|\nabla u^{m}|_{L^2}^2+\frac{\mu}{8}|\nabla u^{(j)}|_{L^2}^2 \\
&\qquad \le Cc_{0,2,j-1}^{2j+4}\sum_{1\le m\le j-1}|\nabla u^{(m)}|_{L^2}^2+\frac{\mu}{8}|\nabla u^{(j)}|_{L^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{1\le m\le j-1}\begin{pmatrix} j \\ m-1\end{pmatrix}\int(\rho^{(j+2-m)}u^{(m)}+\rho^{(j+1-m)}u^{(m+1)})\cdot u^{(j)}\,dx\\
&\qquad= -\sum_{1\le m\le j-1}\begin{pmatrix} j \\ m-1\end{pmatrix}\int(\rho v)^{(j+1-m)}(u^{(j)}\cdot\nabla u^{(m)}+u^{(m)}\cdot\nabla u^{(j)})\,dx\\
&\qquad\quad+\sum_{1\le m\le j-1}\begin{pmatrix} j \\ m-1\end{pmatrix}\int\rho^{(j+1-m)}u^{(m+1)}\cdot u^{(j)}\,dx\\
&\qquad\le C\sum_{1\le m\le j-1}|(\rho v)^{(j+1-m)}|_{L^3}^2|\nabla u^{(m)}|_{L^2}^2+|\nabla u^{(j)}|_{L^2}^2\\
&\qquad\quad+C\sum_{2\le m\le j-1}|\rho^{(j+2-m)}|_{L^{\frac32}}^2|\nabla u^{(m)}|_{L^2}^2+|\nabla u^{(j)}|_{L^2}^2-\int\rho_{(2)}|u^{(j)}|^2\,dx\\
&\qquad\le C\left(c_{0,3,j-1}^{2j+4}+c_{0,1,j-1}^{2j+4}\right)\sum_{1\le m\le j-1}|\nabla u^{(m)}|_{L^2}^2\\
&\qquad\quad+\int(|\rho^{(1)}||v|+\rho|v^{(1)}|)|\nabla u^{(j)}||u^{(j)}|\,dx\\
&\qquad\le Cc_{0,3,j-1}^{2j+4}\sum_{1\le m\le j-1}|\nabla u^{(m)}|_{L^2}^2+Cc_{0,3,0}^3(1+|\nabla v^{(1)}|_{L^2})|\nabla u^{(j)}|_{L^2}^2.
\end{aligned}
$$

Now integrating (3.14) over $(0,t)\subset(0,T_1)$, we have

$$
\begin{aligned}
&\int_0^t|\sqrt{\rho}u^{(j)}(s)|^2\,ds+|\nabla u^{(j)}(t)|_{L^2}\\
&\qquad\le Cc_{0,3,j-1}^{2j+6}+Cc_{0,2,0}^7|\sqrt{\rho}u^{(j)}(t)|_{L^2}^2+Cc_{0,3,j-1}^{2j+4}\sum_{1\le m\le j-1}|\nabla u^{(m)}|_{L^2}^2\\
&\qquad\quad+C\int_0^t|p^{(j+1)}(s)|_{L^2}^2\,ds+Cc_{0,3,j-1}^4\int_0^t|\rho^{(j+1)}(s)|_{L^2}^2\,ds\\
&\qquad\quad+Cc_{0,3,j-1}^{2j+4}\sum_{1\le m\le j-1}\int_0^t|\nabla u^{(m)}|_{L^2}^2\,ds\\
&\qquad\quad+Cc_{0,3,0}^3\int_0^t(1+|v^{(1)}(s)|_{D_0^1})|\nabla u^{(j)}(s)|_{L^2}^2\,ds.
\end{aligned}
$$

Let us define functions I_j and $II_j(t)$ by

$$
\begin{aligned}
I_j(t)&=\max_{1\le m\le j}\left(\sup_{0\le s\le t}|\nabla u^{(m)}|_{L^2}^2\right),\\
II_j(t)&=\max_{1\le m\le j}\left(\sup_{0\le s\le t}|\sqrt{\rho}u^{(j)}(s)|_{L^2}^2\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
|\nabla u^{(j)}|_{L^2}^2\le\ & Cc_{0,3,j-1}^{2j+6}+Cc_{0,2,0}^7|\sqrt{\rho}u^{(j)}(t)|_{L^2}^2\\
&+C(c_{0,3,j-1}^{2j+4}+c_{0,3,j-1}^{2j+4}t)I_{j-1}(t)\\
&+Cc_{0,3,0}^3\int_0^t(1+|v^{(1)}(s)|_{D_0^1})|\nabla u^{(j)}(s)|_{L^2}^2\,ds.
\end{aligned}
$$

Therefore Gronwall's inequality implies that

$$I_j(t) \le Cc_{0,3,j-1}^{2j+6} + Cc_{0,2,0}^{7} II_j(t) + C(j-1)c_{0,3,j-1}^{2j+4} I_{j-1}(t) \tag{3.15}$$

for $0 \le t \le T_1$. Then by induction on j we have

$$\begin{aligned} I_j(t) \le C &\left(c_{0,3,j-1}^{2j+6} + c_{0,2,0}^{7} II_j(t)\right) \\ &\times \Big(1 + c_{0,3,j-1}^{2j+4} + c_{0,3,j-1}^{2j+4} c_{0,3,j-2}^{2(j-1)+4} + \cdots \\ &\qquad + c_{0,3,j-1}^{2j+4} c_{0,3,j-2}^{2(j-1)+4} \cdots c_{0,3,1}^{2\cdot 2+4}\Big) \end{aligned}$$

and hence

$$I_j(t) \le Cc_{0,3\,j-1}^{(j-1)(j+6)} \left(c_{0,3,j-1}^{2j+6} + c_{0,2,0}^{7} II_j(t)\right). \quad \square$$

Lemma 3.10. *For each $1 \le j \le k+1$ we have*

$$\max_{1 \le m \le j} \left(\sup_{0 \le s \le t} |\sqrt{\rho} u^{(m)}(s)|_{L^2}^2 \right) \le Cc_0^2,$$

$$\begin{aligned} &\int_0^t \left(|\sqrt{\rho} u^{(j+1)}(s)|_{L^2}^2 + |u^{(j)}(s)|_{D^2}^2 \right) ds \\ &\qquad + \max_{1 \le m \le j} \left(\sup_{0 \le s \le t} |\nabla u^{(m)}(s)|_{L^2}^2 \right) \le Cc_{0,3,j-1}^{(j+1)(2j+6)}. \end{aligned}$$

for $0 \le t \le \min(T_*, T_2)$, where $T_2 = \min\left(((k+1)c_{0,3,k}^{(k+2)(2k+8)})^{-1}, T_1\right)$.

Proof. Multiplying (3.13) by $u^{(j)}$ and integrating over Ω, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int \rho |u^{(j)}|^2\,dx + \mu |\nabla u^{(j)}|_{L^2}^2 + (\lambda+\mu)\int (\mathrm{div} u^{(j)})^2\,dx \\ &\quad = -\int \nabla p^{(j)} \cdot u^{(j)}\,dx \\ &\qquad + \sum_{0 \le m \le j} \binom{j}{m} \int \rho^{(j-m)} (f - v\cdot\nabla v)^{(m)} \cdot u^{(j)}\,dx \\ &\qquad - j\int \rho^{(1)} |u^{(j)}|^2\,dx - \sum_{1 \le m \le j-1} \binom{j}{m-1} \rho^{(j+1-m)} u^{(m)} \cdot u^{(j)}\,dx. \end{aligned} \tag{3.16}$$

We estimate each term of right hand side of (3.16) as follows:

$$-\int \nabla p^{(j)} \cdot u^{(j)}\,dx = \int p^{(j)} \mathrm{div}\, u^{(j)}\,dx \le C|p^{(j)}|_{L^2}^2 + \frac{\mu}{8}|\nabla u^{(j)}|_{L^2}^2,$$

$$
\begin{aligned}
&\sum_{0\le m\le j}\begin{pmatrix} j \\ m\end{pmatrix}\int \rho^{(j-m)}(f-v\cdot\nabla v)^{(m)}\cdot u^{(j)}\,dx \\
&\quad=\sum_{0\le m\le j-1}\begin{pmatrix} j \\ m\end{pmatrix}\int \rho^{(j-m)}(f-v\cdot\nabla v)^{(m)}\cdot u^{(j)}\,dx \\
&\qquad+\int \rho(f-v\cdot\nabla v)^{(j)}\cdot u^{(j)}\,dx \\
&\quad\le C\sum_{0\le m\le j-1}|\rho^{(j-m)}|_{L^2}^2|(f-v\cdot\nabla v)^m|_{L^3}^2+\frac{\mu}{8}|\nabla u^{(j)}|_{L^2}^2 \\
&\qquad+|\rho|_{L^\infty}|(f-v\cdot\nabla v)^{(j)}|_{L^2}^2+|\sqrt{\rho}u^{(j)}|_{L^2}^2 \\
&\quad\le C(c_{0,3,j-1}^{2j+6}+c_{0,3,j}^3)+\frac{\mu}{8}|\nabla u^{(j)}|_{L^2}^2+|\sqrt{\rho}u^{(j)}|_{L^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
-j\int \rho^{(1)}|u^{(j)}|^2\,dx &= 2j\int \rho v\cdot(u^{(j)}\cdot\nabla u^{(j)})\,dx \\
&\le Cc_{0,3,0}^3|\sqrt{\rho}u^{(j)}|_{L^2}^2+\frac{\mu}{8}|\nabla u^{(j)}|_{L^2}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{1\le m\le j-1}\begin{pmatrix} j \\ m-1\end{pmatrix}\int \rho^{(j+1-m)}u^{(m)}\cdot u^{(j)}\,dx \\
&\quad=\sum_{1\le m\le j-1}\begin{pmatrix} j \\ m-1\end{pmatrix}\int (\rho v)^{(j-m)}(u^{(m)}\cdot\nabla u^{(j)}+u^{(j)}\cdot\nabla u^{(m)})\,dx \\
&\quad\le C\sum_{1\le m\le j-1}|(\rho v)^{(j-m)}|_{L^3}|\nabla u^{(m)}||\nabla u^{(j)}|_{L^2} \\
&\quad\le Cc_{0,3,j-1}^{2j+4}\sum_{1\le m\le j-1}|\nabla u^{(m)}|_{L^2}^2+\frac{\mu}{8}|\nabla u^{(j)}|_{L^2}^2.
\end{aligned}
$$

Substituting all these estimate into (3.16), we have

$$
\begin{aligned}
&\frac{d}{dt}\int \rho|u^{(j)}|^2\,dx+\mu|\nabla u^{(j)}|_{L^2}^2 \\
&\quad\le C(c_{0,1,j-1}^{2j+2}+c_{0,3,j-1}^{2j+6}+c_{0,3,\,j}^3)+Cc_{0,3,0}^3|\sqrt{\rho}u^{(j)}|_{L^2}^2 \\
&\qquad+Cc_{0,3,j-1}^{2j+4}\sum_{1\le m\le j-1}|\nabla u^{(m)}|_{L^2}^2.
\end{aligned}
\tag{3.17}
$$

Recall that $I_j(t)=\max_{1\le m\le j}\left(\sup_{0\le s\le t}|\nabla u^{(m)}(s)|_{L^2}^2\right)$. Then integrating (3.17) over $(0,t)$, we have for $t\in[0,T_1]$

$$
\begin{aligned}
&\int \rho|u^{(j)}|^2\,dx+\mu\int_0^t|\nabla u^{(j)}|_{L^2}^2\,ds \\
&\quad\le|\sqrt{\rho}u^{(j)}(0)|_{L^2}^2+C+Cc_{0,3,j-1}^{2j+4}jtI_{j-1}(t)+Cc_{0,3,0}^3\int_0^t|\sqrt{\rho}u^{(j)}|_{L^2}^2\,ds.
\end{aligned}
$$

From the observation that

$$
|\sqrt{\rho}u^{(j)}(0)|_{L^2}=|\sqrt{\rho_0}g_j|_{L^2}\le C|\rho_0|_{L^{\frac{3}{2}}}|g_j|_{D_0^1}\le Cc_0^2,
$$

we deduce that

$$|\sqrt{\rho}u^{(j)}(t)|_{L^2}^2 \le Cc_0^2 + Cc_{0,3,j-1}^{2j+4} jtI_{j-1}(t)$$

and hence

$$\max_{1\le m\le j}\left(\sup_{0\le s\le t}|\sqrt{\rho}u^{(m)}|_{L^2}^2\right) \le Cc_0^2 + Cc_{0,3,j-1}^{2j+4} jtI_{j-1}(t)) \tag{3.18}$$

for $0 \le t \le T_1$. Therefore from Lemma 3.9, we conclude that

$$I_j(t) \le Cc_{0,3,j-1}^{(j+1)(2j+6)} + Cjc_{0,3,j-1}^{(j+1)(2j+6)} tI_{j-1}(t). \tag{3.19}$$

Thus if we choose $t \le ((k+1)c_{0,3,k}^{(k+2)(2k+8)})^{-1} \le (jc_{0,3,j-1}^{(j+1)(2j+6)})^{-1}$, then using (3.18), (3.19) and Lemma 3.6, we can prove the lemma by induction. □

Lemma 3.11. *For each $1 \le j \le k$, we have*

$$|u^{(j)}(t)|_{D^2} + \int_0^t \left(|u^{(j+1)}(s)|_{D_0^1}^2 + |u^{(j)}(s)|_{D^3}^2\right) ds \le Cc_{0,2,j}^{(j+1)(2j+7)} c_{0,3,j}^{\frac{1}{2}},$$

$$|u^{(j+1)}(t)|_{D_0^1} + |u^{(j)}(t)|_{D^3} + \int_0^t \left(|u^{(j+1)}(s)|_{D^2}^2 + |u^{(j)}(s)|_{D^4}^2\right) ds \le Cc_{0,3,j}^{(j+3)(2j+8)}$$

for $0 \le t \le \min(T_*, T_2)$.

Proof. Since for each t, $u_{(j)}(t)$ is a solution to the elliptic boundary value problem:

$$Lu^{(j)} = \sum_{0\le m\le j}\begin{pmatrix} j \\ m\end{pmatrix}\rho^{(j-m)}(f - v\cdot\nabla v)^{(m)} - \sum_{1\le m\le j}\begin{pmatrix} j \\ m\end{pmatrix}\rho^{(j+1-m)}u^{(m)} - \nabla p^{(j)} - \rho u^{(j+1)},$$

from the elliptic regularity (see [5]) we have

$$\begin{aligned}
|u^{(j)}|_{D_0^1\cap D^2} &\le C|u^{(j)}|_{D_0^1} + C\sum_{0\le m\le j}|\rho^{(j-m)}(f - v\cdot\nabla v)^{(m)}|_{L^2} \\
&\quad + C\sum_{1\le m\le j}|\rho^{(j+1-m)}u^{(m)}|_{L^2} + |\nabla p^{(j)}|_{L^2} + |\rho u^{(j+1)}|_{L^2} \\
&\le C|u^{(j)}|_{D_0^1} + C\sum_{0\le m\le j}|\rho^{(j-m)}|_{H^2}|(f - v\cdot\nabla v)^{(m)}|_{L^2} \\
&\quad + C\sum_{1\le m\le j}|\rho^{(j+1-m)}|_{H^1}|\nabla u^{(m)}|_{L^2} + C|p^{(j)}|_{H^1} \\
&\quad + C|\rho|_{L^\infty}^{\frac{1}{2}}|\sqrt{\rho}u^{(j+1)}|_{L^2}.
\end{aligned}$$

Using the bound (3.3), Lemma 3.4 and Lemma 3.10, we can easily show that for $1 \le j \le k$

$$|u^{(j)}(t)|_{D_0^1\cap D^2} \le Cc_{0,3,j}^{(j+1)(2j+7)} + Cc_{0,2,j}^{\frac{3}{2}} c_{0,3,j}^{\frac{1}{2}}$$

for $0 \le t \le \min(T_*, T_2)$. Further, we have

$$
\begin{aligned}
|u^{(j+1)}|_{D^2} \le C|u^{(j+1)}|_{D_0^1} &+ C\sum_{0\le m\le j+1} |\rho^{(j+1-m)}|_{H^2}|(f - v\cdot\nabla v)^{(m)}|_{L^2}\\
&+ C\sum_{1\le m\le j} |\rho^{(j+2-m)}|_{H^1}|\nabla u^{(m)}|_{L^2} + C|p^{(j+1)}|_{H^1}\\
&+ C|\rho|_{L^\infty}^{\frac{1}{2}}|\sqrt{\rho}u^{(j+2)}|_{L^2}
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\int_0^t |u^{(j+1)}(s)|_{D^2}^2\, ds\\
&\quad\le C\sum_{0\le m\le j+1}\int_0^t |\rho^{(j+1-m)}(s)|_{H^2}^2|(f - v\cdot\nabla v)^{(m)}|_{L^2}^2\, ds\\
&\quad\quad + C\sum_{1\le m\le j+1}\int_0^t \left(|\rho^{(j+2-m)}|_{H^1}^2|\nabla u^{(m)}|_{L^2}^2 + |p^{(j+1)}|_{H^1}^2\right) ds\\
&\quad\quad + C\int_0^t |\rho u^{(j+2)}|_{L^2}^2\, ds \le Cc_{0,3,j}^{(j+3)(2j+8)}.
\end{aligned}
\tag{3.20}
$$

On the other hand, using the elliptic regularity [1, 5], we obtain

$$
\begin{aligned}
|u^{(j)}|_{D^3} &\le C|u^{(j)}|_{D_0^1} + C\sum_{0\le m\le j}|\rho^{(j-m)}(f - v\cdot\nabla v)^{(m)}|_{H^1}\\
&\quad + C\sum_{1\le m\le j}|\rho^{(j+1-m)}u^{(m)}|_{H^1} + C|p^{(j)}|_{H^2} + C|\rho u^{(j+1)}|_{H^1}\\
&\le Cc_{0,3,j-1}^{(j+1)(2j+6)} + \sum_{0\le m\le j}|\rho^{(j-m)}|_{H^2}(f - v\cdot\nabla v)^{(m)}|_{H^1}\\
&\quad + C\sum_{1\le m\le j}|\rho^{(j+1-m)}|_{H^2}|\nabla u^{(m)}|_{L^2}\\
&\quad + Cc_{0,1,j-1}^{j+1} + |\rho|_{H^2}|\nabla u^{(j+1)}|_{L^2}\\
&\le C\Big(c_{0,3,j-1}^{(j+1)(2j+6)} + c_{0,1,j-1}^{j+1}(1 + c_{0,3,j}^2)\\
&\quad + c_{0,1,j-1}^{j+1}c_{0,3,j-1}^{(j+1)(2j+6)} + c_{0,3,j}^{(j+2)(2j+8)}\Big)\\
&\le Cc_{0,3,j}^{(j+3)(2j+8)}.
\end{aligned}
$$

And finally we have

$$\begin{aligned}|u^{(j)}|_{D^4} &\le C|u^{(j)}|_{D_0^1} + C\sum_{0\le m\le j}|\rho^{(j-m)}(f-v\cdot\nabla v)^{(m)}|_{H^2}\\ &\quad + C\sum_{1\le m\le j}|\rho^{(j+1-m)}u^{(m)}|_{H^2} + C|p^{(j)}|_{H^3} + C|\rho u^{(j+1)}|_{H^2}\\ &\le Cc_{0,3,j-1}^{(j+1)(2j+6)} + \sum_{0\le m\le j}|\rho^{(j-m)}|_{H^2}|(f-v\cdot\nabla v)^{(m)}|_{H^2}\\ &\quad + C\sum_{1\le m\le j}|\rho^{(j+1-m)}|_{H^2}|u^{(m)}|_{D_0^1\cap D^2}\\ &\quad + Cc_{0,1,j-1}^{j+1} + C|\rho|_{H^2}|u^{(j+1)}|_{D_0^1\cap D^2}.\end{aligned}$$

and therefore by (3.20)

$$\int_0^t |u_{(j)}(s)|_{D^4}^2\, ds \le Cc_{0,3,j}^{(j+3)(2j+8)}$$

for $0\le t\le \min(T_*,T_2)$. This completes the proof of lemma. □

Lemma 3.12. *For all l and j with $1\le l\le k$ and $0\le j\le k-l$, we have*

$$\begin{aligned}|u^{(j)}(t)|_{D_0^1\cap D^{2l+1}} &\le Cc_{l-1,3,j}^{(j+2l+1)(2j+2l+6)}\\ |u^{(j)}(t)|_{D_0^1\cap D^{2l+2}} &\le Cc_{l,2,j}^{(j+2l+2)(2j+2l+7)}\\ |u^{(j)}(t)|_{D_0^1\cap D^{2l+3}} &\le Cc_{l,3,j}^{(j+2l+3)(2j+2l+8)}\end{aligned}$$

for $0\le t\le \min(T_*,T_3)$, where $T_3=\min(c_{k,4,0}^{-(2k+9)^2},T_2)$.

Proof. For the proof in case when $j=0$, we regard $c_{l,i,-1}$ as $c_{l,i,0}$. From the elliptic regularity, we have

$$\begin{aligned}&|u^{(j)}|_{D_0^1\cap D^{2l+1}}\\ &\le C|u^{(j)}|_{D_0^1} + C\sum_{0\le m\le j}|\rho^{(j-m)}(f-v\cdot\nabla v)^{(m)}|_{H^{2l-1}}\\ &\quad + C\sum_{1\le m\le j}|\rho^{(j+1-m)}u^{(m)}|_{H^{2l-1}} + C|p^{(j)}|_{H^{2l}} + C|\rho u^{(j+1)}|_{L^{2l-1}}\\ &\le Cc_{0,3,j-1}^{(j+1)(2j+6)} + C\sum_{0\le m\le j}\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ |\alpha|\le 2l-1}}|D^{\alpha_1}\rho^{(j-m)}|_{H^2}|D^{\alpha_2}(f-v\cdot\nabla v)^{(m)}|_{L^2}\\ &\quad + C\sum_{1\le m\le j}\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ |\alpha|\le 2l-1}}|D^{\alpha_1}\rho^{(j+1-m)}D^{\alpha_2}u^{(m)}|_{L^2}\\ &\quad + Cc_{0,1,j-1}^{j+1} + C|\rho|_{H^{2l}}|u^{(j+1)}|_{D_0^1\cap D^{2l-1}}\\ &\le Cc_{0,3,j-1}^{(j+1)(2j+6)} + Cc_{l-1,3,j}^{3} + Cc_{0,1,j-1}^{j+1}\max_{1\le m\le j+1}|u^{(m)}|_{D_0^1\cap D^{2(l-1)+1}}\\ &\le Cc_{l-1,3,j}^{(j+1)(2j+6)} + Cc_{0,1,j-1}^{j+1}M_{l-1,j+1},\end{aligned}$$

where $M_{l,n}(t)=\max_{0\le m\le n}|u_{(m)}|_{D_0^1\cap D^{2l+1}}$ for $n\ge 0$. Thus we have

$$M_{l,j}(t)\le Cc_{l-1,3,j}^{(j+1)(2j+6)} + Cc_{0,1,j-1}^{j+1}M_{l-1,j+1}$$

for all $0 \le t \le \min(T_*, T_2)$. Since by Lemma 3.10

$$M_{0,j+1} = \max_{0\le m\le j+1} |u^{(m)}|_{D_0^1} \le Cc_{0,3,j}^{(j+2)(2j+8)},$$

we obtain

$$M_{1,j} \le Cc_{0,3,j}^{(j+3)(2j+8)}.$$

By induction, we deduce that

$$M_{l,j}(t) \le Cc_{l-1,3,j}^{(j+2l+1)(2j+2l+6)}.$$

Now let us define a function $N_{l,n}$ by

$$N_{l,n}(t) = \max_{1\le m\le n} |u_{(m)}(t)|_{D_0^1\cap D^{2l+2}}.$$

Then we similarly have

$$\begin{aligned} |u^{(j)}|_{D_0^1\cap D^{2l+2}} &\le Cc_{0,3,j-1}^{(j+1)(2j+6)} + Cc_{l,2,j}^3 + Cc_{0,1,j-1}^{j+1} N_{l-1,j+1} \\ &\le Cc_{l,2,j}^{(j+1)(2j+6)} + Cc_{0,1,j-1}^{j+1} N_{l-1,j+1} \end{aligned}$$

and hence

$$N_{l,j}(t) \le Cc_{l,2,j}^{(j+1)(2j+6)} + Cc_{0,1,j-1}^{j+1} N_{l-1,j+1}(t)$$

for all $0 \le t \le \min(T_*, T_2)$. Since by Lemma 3.11

$$N_{0,j+1} = \max_{0\le m\le j+1} |u^{(m)}|_{D_0^1\cap D^2} \le Cc_{0,2,j+1}^{(j+2)(2j+9)} c_{0,3,j+1}^{\frac{1}{2}} \le Cc_{0,3,j+1}^{(j+3)(2j+9)},$$

we obtain

$$N_{1,j} \le Cc_{1,2,j}^{(j+1)(2j+6)} + Cc_{0,3,j+1}^{(j+4)(2j+9)} \le Cc_{1,2,j}^{(j+4)(2j+9)}.$$

Hence by induction we have

$$N_{l,j}(t) \le Cc_{l,2,j}^{(j+2l+2)(2j+2l+7)}$$

for all $0 \le t \le \min(T_*, T_2)$.

From the similar argument as above results, we can easily deduce that

$$\begin{aligned} |u^{(j)}|_{D_0^1\cap D^{2l+3}} &\le Cc_{l,3,j}^{(j+2l+3)(2j+2l+8)}, \\ |u^{(j+1)}|_{D_0^1\cap D^{2l+1}} &\le Cc_{l-1,3,j+1}^{(j+2l+2)(2j+2l+8)}, \\ \int_0^t \left(|u_{(j+1)}|_{D_0^1\cap D^{2l+2}}^2 + |u^{(j)}|_{D_0^1\cap D^{2l+4}}^2 \right) &\le Cc_{l,3,j}^{(j+2l+3)(2j+2l+8)} \end{aligned}$$

for all $0 \le t \le \min(T_*, T_3)$. □

Summarizing the estimates up to now, we have the following.

Proposition 3.13. *If (ρ, u) is a solution to the linearized problem* (2.7)-(2.10) *with the initial density $\rho_0 > \delta$ the known vector field v satisfying* (3.1), (3.2) *and* (3.3), *then for sufficiently small ε, there exists a time interval $[0, T_*]$ such that the velocity u satisfies the following estimates that for all $0 \le j \le k+1$*

$$|u^{(j)}(0)|_{D_0^1\cap D^{2(k-j)+3}} \le c_{0,1,j}, \tag{3.21}$$

and for all $0 \le l \le k$ *and* $0 \le j \le k-l$ *that* $\inf_{(0,T_*)\times\Omega} \rho(t,x) \ge C^{-1}\delta$.

$$
\begin{aligned}
&\sup_{0<t<T_*} \Big(|\rho(t)|_{L^1} + |(\rho^{(j)}(t),\,(p-p(0))^{(j)}(t))|_{H^{2(k-j)+3}} \Big) \\
&\qquad\qquad\qquad\qquad + \int_0^{T_*} |\rho^{(k+2)}(s)|_{L^2}^2 \, dt \le \varepsilon c_{0,1,j} \\
&\sup_{0\le t\le T_*} \Big(|u^{(j)}(t)|_{D_0^1\cap D^{2l+1}} \Big) + \int_0^{T_*} |u^{(j)}(t)|^2_{D^2\cap D^{2l+2}} \, dt \le c_{l,2,j}, \\
&\sup_{0\le t\le T_*} \Big(|u^{(j)}(t)|_{D_0^1\cap D^{2l+2}} \Big) \\
&\qquad + \int_0^{T_*} \Big(|u^{(j+1)}(t)|^2_{D_0^1\cap D^{2l+1}} + |u^{(j)}(t)|^2_{D^{2l+3}} \Big) \, dt \le c_{l,3,j}, \\
&\operatorname*{ess\,sup}_{0<t<T_*} \Big(|u^{(j+1)}(t)|_{D_0^1\cap D^{2l+1}} + |u^{(j)}(t)|_{D^{2l+3}} \Big) \\
&\qquad + \int_0^{T_*} \Big(|u^{(j+1)}(t)|^2_{D^{2l+2}} + |u^{(j)}(t)|^2_{D^{2l+4}} \Big) \, dt \le c_{l,4,j}, \\
&\operatorname*{ess\,sup}_{0\le t\le T_*} |\sqrt{\rho}u^{(j+1)}(t)|_{L^2} + \int_0^{T_*} |\sqrt{\rho}u^{(j+2)}(t)|^2_{L^2} \, dt \le c_{0,4,j}.
\end{aligned}
\tag{3.22}
$$

Proof. From Lemma 3.4, we see the lower bound of ρ by choosing $c_{0,1,j} \ge Cc^j_{0,1,j-1}$. If we let the constant $c_{0,1,j}$ be Cc_0 for all j, by Lemma 3.2 the estimate (3.21) follows immediately.

For the estimate (3.22), we consider the following three cases: (1) $l = 0$ and $j = 0$, (2) $l = 0$ and $1 \le j \le k$, (3) $l \ge 1$ and $0 \le j \le k-l$.

Case (1): $l = 0$ and $j = 0$.

From Lemma 3.6 to Lemma 3.8, it follows that

$$
\begin{aligned}
&|u(t)|_{D_0^1} + \int_0^t |u(s)|^2_{D^2} \, ds \le Cc_0^3, \\
&|u(t)|_{D^2} + \int_0^t \Big(|u_t(s)|^2_{D_0^1} + |u(s)|^2_{D^3} \Big) \, ds \le Cc_0^3 c_{0,2,0}^{\frac{3}{2}} c_{0,3,0}^{\frac{1}{2}}, \\
&|u_t(t)|_{D_0^1} + |u(t)|_{D^3} + \int_0^t \big(|u_t(s)|^2_{D^2} + |u(s)|^2_{D^4} \big) \, ds \le Cc_0^7 c_{0,3,0}^{12}, \\
&|\sqrt{\rho}u_t(t)|_{L^2} + \int_0^t |\sqrt{\rho}u_{tt}(s)|^2_{L^2} \, ds \le Cc_0^7 c_{0,3,0}^{12}
\end{aligned}
$$

for $0 \le t \le \min(T_*, T_3)$. Therefore, defining the constants $c_{0,i,0}$'s and T_* by

$$
c_{0,1,0} = Cc_0, \quad c_{0,2,0} = Cc_0^3, \quad c_{0,3,0} = C^2 c_0^6 c_{0,2,0}^3, \quad c_{0,4,0} = Cc_0^7 c_{0,3,0}^{12} \tag{3.23}
$$

and

$$
T_* = \min(T, T_3), \tag{3.24}
$$

we conclude that

$$
\begin{aligned}
&\sup_{0\le t\le T_*} |u(t)|_{D_0^1} + \int_0^{T_*} |u(t)|_{D^2}^2\, dt \le c_{0,2,0},\\
&\sup_{0\le t\le T_*} |u(t)|_{D^2} + \int_0^{T_*} \left(|u_t(t)|_{D_0^1}^2 + |u(t)|_{D^3}^2\right) dt \le c_{0,3,0},\\
&\operatorname*{ess\,sup}_{0\le t\le T_*} \left(|u_t(t)|_{D_0^1} + |u(t)|_{D^3}\right) + \int_0^{T_*} \left(|u_t(t)|_{D^2}^2 + |u(t)|_{D^4}^2\right) dt \le c_{0,4,0},\\
&\operatorname*{ess\,sup}_{0\le t\le T_*} (|\sqrt{\rho} u_t(t)|_{L^2}) + \int_0^{T_*} |\sqrt{\rho} u_{tt}(t)|_{L^2}^2\, dt \le c_{0,4,0}.
\end{aligned}
\tag{3.25}
$$

Case (2): $l = 0$ and $1 \le j \le k$.

From Lemma 3.9 to Lemma 3.11, we obtain that

$$
\begin{aligned}
&|u^{(j)}(t)|_{D_0^1} + \int_0^t |u^{(j)}(s)|_{D^2}^2\, ds \le C c_{0,3,j-1}^{(j+1)(2j+6)},\\
&|u_{(j)}(t)|_{D^2} + \int_0^t \left(|u_{(j+1)}|_{D_0^1}^2 + |u_{(j)}|_{D^3}^2\right) ds \le C c_{0,2,j}^{(j+1)(2j+7)} c_{0,3,j}^{\frac{1}{2}}\\
&|u_{(j+1)}(t)|_{D_0^1} + |u_{(j)}(t)|_{D^3}\\
&\quad + \int_0^t \left(|u_{(j+1)}(s)|_{D^2}^2 + |u_{(j)}(s)|_{D^4}^2\right) ds \le C c_{0,3,j}^{(j+3)(2j+8)}.
\end{aligned}
\tag{3.26}
$$

Thus if we define the constants $c_{0,i,j}$ by

$$
c_{0,2,j} = C c_{0,3,j-1}^{(j+1)(2j+6)},\ c_{0,3,j} = C^2 c_{0,2,j}^{(2j+2)(2j+7)},\ c_{0,4,j} = C c_{0,3,j}^{(j+3)(2j+8)},
$$

then we have

$$
\begin{aligned}
&\sup_{0\le t\le T_*} |u^{(j)}(t)|_{D_0^1} + \int_0^{T_*} |u^{(j)}(t)|_{D^2}^2\, dt \le c_{0,2,j},\\
&\sup_{0\le t\le T_*} |u^{(j)}(t)|_{D^2} + \int_0^{T_*} \left(|u^{(j+1)}(t)|_{D_0^1}^2 + |u^{(j)}(t)|_{D^3}^2\right) dt \le c_{0,3,j},\\
&\operatorname*{ess\,sup}_{0\le t\le T_*} \left(|u^{(j+1)}(t)|_{D_0^1} + |u^{(j)}t)|_{D^3}\right)\\
&\quad + \int_0^{T_*} \left(|u^{(j+1)}(t)|_{D^2}^2 + |u^{(j)}(t)|_{D^4}^2\right) dt \le c_{0,4,j},\\
&\operatorname*{ess\,sup}_{0\le t\le T_*} \left(|\sqrt{\rho} u^{(j+1)}(t)|_{L^2}\right) + \int_0^{T_*} |\sqrt{\rho} u^{(j+2)}(t)|_{L^2}^2\, dt \le c_{0,4,j}.
\end{aligned}
\tag{3.27}
$$

Case (3): $l \ge 1$ and $0 \le j \le k - l$.

From Lemma 3.12, we easily show that

$$
\begin{aligned}
&|u^{(j)}(t)|_{D_0^1\cap D^{2l+1}} + \int_0^t |u^{(j)}(s)|^2_{D_0^1\cap D^{2l+2}}\,ds \\
&\quad \le Cc_{l-1,3,j}^{(j+2l+1)(2j+2l+6)}, \\
&|u^{(j)}(t)|_{D_0^1\cap D^{2l+2}} + \int_0^t \Big(|u^{(j+1)}|^2_{D_0^1\cap D^{2l+1}} + |u^{(j)}|^2_{D_0^1\cap D^{2l+3}}\Big)\,ds \\
&\quad \le Cc_{l,2,j}^{(j+2l+2)(2j+2l+7)} \\
&|u^{(j+1)}(t)|_{D_0^1\cap D^{2l+1}} + |u^{(j)}(t)|_{D_0^1\cap D^{2l+3}} \\
&\qquad + \int_0^t \Big(|u^{(j+1)}(s)|^2_{D_0^1\cap D^{2l+2}} + |u^{(j)}(s)|^2_{D_0^1\cap D^{2l+4}}\Big)\,ds \\
&\quad \le Cc_{l,3,j}^{(j+2l+3)(2j+2l+8)}.
\end{aligned}
\tag{3.28}
$$

and hence if we choose the constants $c_{l,i,j}$ such that

$$
\begin{aligned}
c_{l,2,j} &= Cc_{l-1,3,j}^{(j+2l+1)(2j+2l+6)}, \\
c_{l,3,j} &= Cc_{l,2,j}^{(j+2l+2)(2j+2l+7)}, \\
c_{l,4,j} &= Cc_{l,3,j}^{(j+2l+3)(2j+2l+8)},
\end{aligned}
$$

then we finally have

$$
\begin{aligned}
&\sup_{0\le t\le T_*} |u^{(j)}(t)|_{D_0^1\cap D^{2l+1}} + \int_0^{T_*} |u^{(j)}(t)|^2_{D_0^1\cap D^{2l+2}}\,dt \le c_{l,2,j}, \\
&\sup_{0\le t\le T_*} |u^{(j)}(t)|_{D_0^1\cap D^{2l+2}} \\
&\qquad + \int_0^{T_*} \Big(|u^{(j+1)}(t)|^2_{D_0^1\cap D^{2l+1}} + |u^{(j)}(t)|^2_{D_0^1\cap D^{2l+3}}\Big)\,dt \le c_{l,3,j}, \\
&\operatorname*{ess\,sup}_{0\le t\le T_*} \Big(|u_{(j+1)}(t)|_{D_0^1\cap D^{2l+1}} + |u_{(j)}t)|_{D_0^1\cap D^{2l+3}}\Big) \\
&\qquad + \int_0^{T_*} \Big(|u_{(j+1)}(t)|^2_{D_0^1\cap D^{2l+2}} + |u_{(j)}(t)|^2_{D_0^1\cap D^{2l+4}}\Big)\,dt \le c_{l,4,j}.
\end{aligned}
\tag{3.29}
$$

Combining all the results above and Lemma 3.4, we complete the proof of proposition. □

4. Linear problem with nonnegative density

In this section, we establish the existence and local time boundedness of solution to the linearized problem with nonnegative initial density on a general domain.

The following is the concerning linearized problem.

$$\rho_t + \operatorname{div}(\rho v) = 0 \qquad \text{in} \quad (0,T)\times\Omega, \tag{4.1}$$

$$\rho u_t + Lu + \nabla p = \rho(f - v\cdot\nabla v) \qquad \text{in} \quad (0,T)\times\Omega, \tag{4.2}$$

$$(\rho,\, u)|_{t=0} = (\rho_0,\, u_0) \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad (0,T)\times\partial\Omega, \tag{4.3}$$

$$(\rho,\, u)(t,x) \to (0,\,0) \quad \text{as} \quad |x|\to\infty, \quad (t,x)\in(0,T)\times\Omega, \tag{4.4}$$

We assume that the initial data satisfy the bound and compatibility conditions that

$$\rho_0 \geq 0, \quad |\rho_0|_{L^1\cap H^{2k+3}} \leq \frac{\varepsilon}{2}, \quad 2 + \sum_{1\leq j\leq k+1} |g_j|_{D_0^1} \leq c_0, \tag{4.5}$$

and

$$L(u^{(j)}(0)) = j\mathrm{div}(\rho_0 v(0))u^{(j)}(0) + G_{j-1} + H_j - \nabla p^{(j)}(0) - \rho_0 g_{j+1} \tag{4.6}$$

for some $g_j \in D_0^1 \cap D^{2(k-j)+3}$ and all $0 \leq j \leq k$, where

$$\begin{aligned} G_{j-1} &= 0 \quad \text{for} \quad j = 0, 1, \\ G_{j-1} &= -\sum_{1\leq m\leq j-1} \begin{pmatrix} j \\ m \end{pmatrix} \rho^{(j+1-m)}(0)u^{(m)}(0) \quad \text{for} \quad j \geq 2, \\ H_j &= \sum_{0\leq m\leq j} \begin{pmatrix} j \\ m \end{pmatrix} \rho^{(j-m)}(0)(f - v\cdot\nabla v)^{(m)}(0). \end{aligned}$$

We also assume that the known v satisfies the bound conditions that for all $0 \leq j \leq k+1$,

$$|v^{(j)}(0)|_{D_0^1\cap D^{2(k-j)+3}} \leq c_{0,1,j}, \tag{4.7}$$

and for all $0 \leq l \leq k$ and $0 \leq j \leq k-l$,

$$\begin{aligned} &\sup_{0\leq t\leq T_*} \left(|v^{(j)}(t)|_{D_0^1\cap D^{2l+1}}\right) + \int_0^{T_*} |v^{(j)}(t)|^2_{D^2\cap D^{2l+2}}\, dt \leq c_{l,2,j}, \\ &\sup_{0\leq t\leq T_*} \left(|v^{(j)}(t)|_{D_0^1\cap D^{2l+2}}\right) \\ &\qquad + \int_0^{T_*} \left(|v^{(j+1)}(t)|^2_{D_0^1\cap D^{2l+1}} + |v^{(j)}(t)|^2_{D^{2l+3}}\right) dt \leq c_{l,3,j}, \\ &\operatorname*{ess\,sup}_{0<t<T_*} \left(|v^{(j+1)}(t)|_{D_0^1\cap D^{2l+1}} + |v^{(j)}(t)|_{D^{2l+3}}\right) \\ &\qquad + \int_0^{T_*} \left(|v^{(j+1)}(t)|^2_{D^{2l+2}} + |v^{(j)}(t)|^2_{D^{2l+4}}\right) dt \leq c_{l,4,j} \end{aligned} \tag{4.8}$$

for some time $T_* \in (0,T)$ and constants $c_{l,i,j}$ such that

$$\begin{aligned} 1 \leq c_0 \leq c_{l,i,j} &\leq c_{l+1,i,,j} \quad \text{for any} \quad i,j,l, \\ c_{l,i,j} &\leq c_{l,i+1,j} \quad \text{for each fixed} \quad l,j, \\ \text{and} \quad c_{l,i,j} &\leq c_{l,i,l+1} \quad \text{for each fixed} \quad l \quad \text{and for any} \quad i. \end{aligned}$$

Proposition 4.1. *If the data ρ_0, u_0, f and the vector field v satisfy the conditions* (4.5), (4.6), (4.7) *and* (4.8) *for some sufficiently small constant ε, some fixed c_0 and $c_{l,i,j}$ appearing in Proposition* 3.13, *then there exists a time $T_* > 0$ and a unique solution (ρ, u) to the linearized problem* (4.1) – (4.4) *satisfying the estimate* (3.21) *and* (3.22).

The above proposition will be proved at the end of this section.

For the proof of Proposition 4.1, we will use a standard domain expansion technique. To do so, we choose a cut-off function $\varphi \in C_c^\infty(B_1)$ such that $\varphi = 1$ in $B_{\frac{1}{2}}$.

Let $R_0 > 1$ be a sufficiently large number so that

$$\Omega \subset B_{R_0/2} \quad \text{if} \quad \Omega \subset\subset \mathbf{R}^3, \quad \text{and } \Omega \subset \mathbf{R}^3 \setminus B_{R_0/2} \quad \text{in} \quad \mathbf{R}^3 \setminus \subset\subset \mathbf{R}^3.$$

We define

$$\begin{aligned} \varphi_R(x) &= \varphi(x/R), \quad g_{R,j}(x) = \varphi_R(x) g_j(x), 1 \leq j \leq k+1, \\ v_R(t,x) &= \varphi_R(x) v(t,x) \quad \text{and} \quad f_R(t,x) = \varphi_R(x) f(t,x) \end{aligned}$$

for $(t,x) \in [0,T_*] \times \Omega$. Note that if $\Omega \subset\subset \mathbf{R}^3$, then $g_{R,j} = g_j$, $v_R = v$ and $f_R = f$ for each $R > R_0$ and otherwise, they are supported in Ω_R or $[0,T_*] \times \Omega_R$, where $\Omega_R = \Omega \cap B_R$ [1].

For the domain expansion, we lift the initial density to make use of the estimates independent of lower bound of initial density in Section 3. More specifically, let us define $\rho_{R,0} = \rho_0 + R^{-3}$ on Ω_R and ρ_R be the solution of the linear transport equation

$$(\rho_R)_t + \mathrm{div}(\rho_R v_R) = 0, \quad \text{on} \quad [0,T] \times \Omega_R; \quad \rho_R(0) = \rho_{R,0} \quad \text{in} \quad \Omega_R. \tag{4.9}$$

Then we can define $\rho_R^{(j)}(0)$ as in Section 3. Let $p_R = p(\rho_R)$. Then we have

Lemma 4.2. *If ε is sufficiently small and R is sufficiently large, then there exists a unique solution*

$$(w_{R,0}, w_{R,1}, \cdots, w_{R,k}) \in D_0^1 \cap D^{2k+3}(\Omega_R) \times D_0^1 \cap D^{2k+1}(\Omega_R) \times \cdots \times D_0^1 \cap D^3(\Omega_R)$$

to the coupled elliptic system: for $0 \leq j \leq k$, $g_{R,j} = 0$ on $\partial\Omega_R$ and

$$L(w_{R,j}) = j\mathrm{div}(\rho_{R,0}\, v_R(0)) w_{R,j} + G_{R,j-1} + H_{R,j} - \nabla p_R^{(j)}(0) - \rho_{R,0}\, g_{R,j+1} \tag{4.10}$$

in Ω_R, where

$$\begin{aligned} G_{R,j-1} &= 0 \quad \text{for} \quad j = 0,1, \\ G_{R,j-1} &= -\sum_{1 \leq m \leq j-1} \binom{j}{m} \rho_R^{(j+1-m)}(0) w_{R,m} \quad \text{for} \quad j \geq 2, \\ H_{R,j} &= \sum_{0 \leq m \leq j} \binom{j}{m} \rho_R^{(j-m)}(0)(f_R - v_R \cdot \nabla v_R)^{(j)}(0). \end{aligned}$$

Furthermore, for such ε and R, there holds

$$|w_{R,j}|_{D_0^1 \cap D^{2(k-j)+3}} \leq C c_0$$

for all $0 \leq j \leq k$.

---

[1] If Ω is the half space $\mathbf{R}^2 \times \mathbf{R}_+$, then the non-smooth domain Ω_R should be replaced by a smooth domain $\tilde{\Omega}_R$ such that $\Omega_R \subset \tilde{\Omega}_R \subset \Omega_{2R}$.

Proof. We first observe that

$$
\begin{aligned}
&\rho_{R,0} \to \rho_0 \quad \text{in} \quad L^1 \cap H^{2k+3},\\
(4.11) \qquad & v_R^{(j)} \to v^{(j)} \quad \text{in} \quad C([0,T_*]; D_0^1 \cap D^{2(k-j)+3}) \quad \text{for} \quad 1 \le j \le k,\\
&\text{and} \quad v_R^{(k+1)} \to v^{(k+1)} \quad \text{in} \quad L^\infty(0,T_*; D_0^1)
\end{aligned}
$$

as $R \to \infty$. Hence we deduce from Lemma 3.1, 3.4 and 3.5 that for any given ε_1 there exists a small ε_0 and a large $R_1 > R_0$ such that if $\varepsilon < \varepsilon_0$ and $R > R_1$, then

$$
|p_R^{(j)}(0)|_{L^2} \le \varepsilon_1, \qquad \sum_{1\le j\le k} |H_{R,j}|^2_{L^{\frac{6}{5}} \cap D_0^1} \le \varepsilon_1,
$$

$$
\text{and} \quad \sum_{2\le j\le k} \left(\sum_{1\le m\le j-1} \begin{pmatrix} j \\ m-1 \end{pmatrix} |\rho_R^{(j+1-m)}(0)|_{L^{\frac{3}{2}}} \right)^2 \le \varepsilon_1.
$$

Now we use an iteration method to prove the lemma. Consider the following decoupled elliptic system of infinitely iterated equations:

$$
\begin{aligned}
L(w_{R,j}^n) &= j\mathrm{div}(\rho_{R,0} v_R(0)) w_{R,j}^n \\
(4.12) \qquad &\quad + G_{R,j-1}^{n-1} + H_{R,j} - \nabla p_R^{(j)}(0) - \rho_{R,0} g_{R,j+1} \quad \text{in} \quad \Omega_R,
\end{aligned}
$$

where

$$
\begin{aligned}
&G_{R,j-1}^{n-1} = G_{j-1}(w_{R,1}^{n-1}, \cdots, w_{R,j-1}^{n-1}) \quad \text{for} \quad j \ge 2 \quad \text{and} \quad n \ge 1,\\
&G_{R,j-1}^{n-1} = 0 \quad \text{for} \quad n \ge 1 \quad \text{and} \quad j = 0, 1,\\
&\text{and} \quad w_{R,j}^0 = 0 \quad \text{for} \quad 1 \le j \le k-1.
\end{aligned}
$$

The solution $w_{R,j}^n$ is constructed by the following process:

(i) first find $(w_{R,0}^1, w_{R,1}^1, \cdots, w_{R,k}^1)$ satisfying the system

$$
\begin{cases}
L(w_{R,k}^1) = k\mathrm{div}(\rho_{R,0} v_R(0)) w_{R,k}^1 + H_{R,k} - \nabla p_R^{(k)}(0) - \rho_{R,0} g_{R,k+1},\\
L(w_{R,k-1}^1) = (k-1)\mathrm{div}(\rho_{R,0} v_R(0)) w_{R,k-1}^1 + H_{R,k-1} - \nabla p_R^{(k-1)}(0) - \rho_{R,0} g_{R,k},\\
\qquad \vdots\\
L(w_{R,0}^1) = H_{R,0} - \nabla p_0 - \rho_{R,0} g_{R,1},
\end{cases}
$$

(ii) then find $(w_{R,0}^2, \cdots, w_{R,k}^2)$ such that

$$
\begin{cases}
L(w_{R,k}^2) = k\mathrm{div}(\rho_{R,0} v_R(0)) w_{R,k}^2 + G_{R,k-1}^1 + H_{R,k} - \nabla p_R^{(k)}(0) - \rho_{R,0} g_{R,k+1},\\
L(w_{R,k-1}^2) = (k-1)\mathrm{div}(\rho_{R,0} v_R(0)) w_{R,k-1}^1\\
\qquad\qquad\qquad + G_{R,k-2}^1 + H_{R,k-1} - \nabla p_R^{(k-1)}(0) - \rho_{R,0} g_{R,k},\\
\qquad \vdots\\
L(w_{R,0}^2) = H_{R,0} - \nabla p_0 - \rho_{R,0} g_{R,1},
\end{cases}
$$

(iii) and in this way we find $(w^n_{R,0}, \cdots, w^n_{R,k})$ such that

$$\begin{cases} L(w^n_{R,k}) = k\mathrm{div}(\rho_{R,0}v_R(0))w^n_{R,k} + G^{n-1}_{R,k-1} + H_{R,k} - \nabla p^{(k)}_R(0) - \rho_{R,0}g_{R,k+1}, \\ L(w^n_{R,k-1}) = (k-1)\mathrm{div}(\rho_{r,0}v_R(0))w^n_{R,k-1} \\ \qquad\qquad\qquad +G^{n-1}_{R,k-2} + H_{R,k-1} - \nabla p^{(k-1)}_R(0) - \rho_{R,0}g_{R,k}, \\ \qquad \vdots \\ L(w^n_{R,0}) = H_{R,0} - \nabla p_0 - \rho_{R,0}g_{R,1}. \end{cases}$$

Since every system is decoupled, the existence of the solution $(g^n_{R,0}, \cdots, g^n_{R,k})$ in $D^1_0(\Omega_R) \times \cdots \times D^1_0(\Omega_R)$ is always guaranteed by the classical elliptic theory (see for instance [1]), provided ε is sufficiently small.

We now prove that $\sum_{0\le j\le k} |w^n_{R,j}|_{D^1_0}$ is uniformly bounded on n. Multiplying (4.12) by $w^n_{R,j}$ and integrating over Ω_R, by integration by parts, we have for $j \ge 2$

$$\begin{aligned} &\int Lw^n_{R,j} \cdot w^n_{R,j}\, dx \\ &= j\int \mathrm{div}(\rho_{R,0}v_R(0))|w^n_{R,j}|^2\, dx + \int G^{n-1}_{R,j-1} \cdot w^n_{R,j}\, dx \\ &\quad + \int H_{R,j} \cdot w^n_{R,j}\, dx - \int p^{(j)}_R(0)\mathrm{div} w^n_{R,j}\, dx + \int \rho_{R,0}g_{R,j+1} \cdot w^n_{R,j}\, dx \\ &\le 2jC|\rho_{R,0}|_{L^6}|\nabla v_R(0)|_{L^2}|\nabla w^n_{R,j}|^2_{L^2} + C(|G^{n-1}_{R,j-1}|^2_{L^{\frac{6}{5}}} + |H_{R,j}|_{L^{\frac{6}{5}}} + |p^{(j)}_R(0)|^2_{L^2}) \\ &\quad + C|\rho_{R,0}|^2_{L^{\frac{3}{2}}}|\nabla g_{R,j+1}|^2_{L^2} + \frac{\mu}{4}|\nabla w^n_{R,j}|^2_{L^2}. \end{aligned}$$

Thus if ε_1 and R_1 are small and large enough, respectively, then

$$\begin{aligned} |\nabla w^n_{R,j}|^2_{L^2} &\le C(|G^{n-1}_{R,j-1}|^2_{L^{\frac{6}{5}}} + |H_{R,j}|^2_{L^{\frac{6}{5}}} + |p^{(j)}_R(0)|^2_{L^2}) + C\varepsilon_1^2|\nabla g_{R,j+1}|^2_{L^2} \\ &\le C\left(\left(\sum_{1\le m\le j-1}\begin{pmatrix} j \\ m\end{pmatrix}|\rho^{(j+1-m)}_R(0)|_{L^{\frac{3}{2}}}\right)^2 \max_{1\le m\le j-1}|\nabla w^{n-1}_{R,m}|^2_{L^2}\right) \\ &\quad + C\left(|H_{R,j}|^2_{L^{\frac{6}{5}}} + |p^{(j)}_R(0)|^2_{L^2}\right) + C\varepsilon_1^2|\nabla g_{R,j+1}|^2_{L^2} \\ &\le \varepsilon_2\left(\max_{1\le m\le j-1}|\nabla w^n_{R,m}|^2_{L^2}\right) + \varepsilon_2 + \varepsilon_2|\nabla g_{j+1}|^2_{L^2} \end{aligned}$$

for $j \ge 2$, where $\varepsilon_2 = C\varepsilon_1$. For the case $j = 0, 1$, we have

$$\begin{aligned} |\nabla w^n_{R,1}|^2_{L^2} \le \varepsilon_2 + \varepsilon_2|\nabla g_2|^2_{L^2}, \\ |\nabla w^n_{R,0}|^2_{L^2} \le \varepsilon_2 + \varepsilon_2|\nabla g_1|^2_{L^2}. \end{aligned}$$

Fixing R, let us define $|\nabla w^n_{R,j}|^2_{L^2}$ and $|\nabla g_{R,j}|^2_{L^2}$ by a^n_j and A_j, respectively. Then we have

$$a^n_j \le \varepsilon_2\left(\max_{1\le m\le j-1} a^{n-1}_m\right) + \varepsilon_2 + \varepsilon_2 A_{j+1} \quad \text{for} \quad j \ge 0, n \ge 1, \tag{4.13}$$

where $a_{-1}^{n-1} = a_0^{n-1} = a_j^0 = 0$ for $0 \le j \le k, n \ge 1$. Now we claim that if $\varepsilon_2 < \frac{1}{2}$, then there holds

$$a_j^n \le \left(\frac{\varepsilon_2}{1-\varepsilon_2} + A\varepsilon_2^k\right)\left(\frac{1-\varepsilon_2}{1-2\varepsilon_2}\right) \tag{4.14}$$

for $1 \le j \le k$ and $n \ge 1$, where $A = \max_{1\le j\le k+1} A_j$.

By induction we first observe that

$$\begin{aligned}
a_k^1 &\le \varepsilon_2 + \varepsilon_2 A,\\
a_{k-1}^1 &\le \varepsilon_2 + \varepsilon_2(\varepsilon_2 + \varepsilon_2 A) = \varepsilon_2 + \varepsilon_2^2 + \varepsilon_2^2 A,\\
a_{k-2}^1 &\le \varepsilon_2 + \varepsilon_2(\varepsilon_2 + \varepsilon_2^2 + \varepsilon_2^2 A) = \varepsilon_2 + \varepsilon_2^2 + \varepsilon_2^3 + \varepsilon_2^3 A,\\
&\vdots\\
a_1^1 &\le \varepsilon_2 + \varepsilon_2^2 + \cdots + \varepsilon_2^k + \varepsilon_2^k A \le \frac{\varepsilon_2}{1-\varepsilon_2} + \varepsilon_2^k A.
\end{aligned}$$

Let $B = \frac{\varepsilon_2}{1-\varepsilon_2} + \varepsilon_2^k A$. Then

$$\begin{aligned}
a_k^2 &\le \varepsilon_2 B + \varepsilon_2 + \varepsilon_2 A,\\
a_{k-1}^2 &\le \varepsilon_2 B + \varepsilon_2 + \varepsilon_2(\varepsilon_2 B + \varepsilon_2 + \varepsilon_2 A) = \varepsilon_2 + \varepsilon_2^2 + \varepsilon_2 B + \varepsilon_2^2 B + \varepsilon_2^2 A,\\
a_{k-2}^2 &\le \varepsilon_2 B + \varepsilon_2 + \varepsilon_2(\varepsilon_2 + \varepsilon_2^2 + \varepsilon_2 B + \varepsilon_2^2 B + \varepsilon_2^2 A)\\
&= \varepsilon_2 + \varepsilon_2^2 + \varepsilon_2^3 + (\varepsilon_2 + \varepsilon_2^2 + \varepsilon_2^3)B + \varepsilon_2^3 A,\\
&\vdots\\
a_1^2 &\le \varepsilon_2 + \varepsilon_2^2 + \cdots + \varepsilon_2^k + (\varepsilon_2 + \varepsilon_2^2 + \cdots + \varepsilon_2^k)B + \varepsilon^k A \le B + \frac{\varepsilon_2}{1-\varepsilon_2}B.
\end{aligned}$$

Suppose that

$$a_1^n \le \left(1 + \frac{\varepsilon_2}{1-\varepsilon_2} + \cdots + \left(\frac{\varepsilon_2}{1-\varepsilon_2}\right)^{n-1}\right) B$$

for some $n \geq 2$. We then have

$$
\begin{aligned}
a_k^{n+1} &\leq \varepsilon_2 \left(1 + \frac{\varepsilon_2}{1-\varepsilon_2} + \cdots + \left(\frac{\varepsilon_2}{1-\varepsilon_2}\right)^{n-1}\right) B + \varepsilon_2 + \varepsilon_2 A, \\
a_{k-1}^{n+1} &\leq \varepsilon_2 \left(1 + \frac{\varepsilon_2}{1-\varepsilon_2} + \cdots + \left(\frac{\varepsilon_2}{1-\varepsilon_2}\right)^{n-1}\right) B \\
&\qquad + \varepsilon_2 + \varepsilon_2^2 \left(1 + \frac{\varepsilon_2}{1-\varepsilon_2} + \cdots + \left(\frac{\varepsilon_2}{1-\varepsilon_2}\right)^{n-1}\right) B + \varepsilon_2^2 + A\varepsilon_2^2 \\
&= (\varepsilon_2 + \varepsilon_2^2) \left(1 + \frac{\varepsilon_2}{1-\varepsilon_2} + \cdots + \left(\frac{\varepsilon_2}{1-\varepsilon_2}\right)^{n-1}\right) B + \varepsilon_2 + \varepsilon_2^2 + \varepsilon_2^2 A, \\
a_{k-2}^{n+1} &\leq (\varepsilon_2 + \varepsilon_2^2 + \varepsilon_2^3) \left(1 + \frac{\varepsilon_2}{1-\varepsilon_2} + \cdots + \left(\frac{\varepsilon_2}{1-\varepsilon_2}\right)^{n-1}\right) B \\
&\qquad + \varepsilon_2 + \varepsilon_2^2 + \varepsilon_2^3 + \varepsilon_2^3 A, \\
&\ \vdots \\
a_1^{n+1} &\leq (\varepsilon_2 + \varepsilon_2^2 + \cdots + \varepsilon_2^k) \left(1 + \frac{\varepsilon_2}{1-\varepsilon_2} + \cdots + \left(\frac{\varepsilon_2}{1-\varepsilon_2}\right)^{n-1}\right) B \\
&\qquad + \varepsilon_2 + \cdots + \varepsilon_2^k + \varepsilon_2^k A \\
&\leq \left(1 + \frac{\varepsilon_2}{1-\varepsilon_2} + \cdots + \left(\frac{\varepsilon_2}{1-\varepsilon_2}\right)^{n}\right) B.
\end{aligned}
$$

Thus by induction we have for all $n \geq 1$ and $j \geq 1$

$$
a_j^n \leq \left(1 + \frac{\varepsilon_2}{1-\varepsilon_2} + \cdots + \left(\frac{\varepsilon_2}{1-\varepsilon_2}\right)^{n-1}\right) B.
$$

Since $\varepsilon_2 < \frac{1}{2}$ and hence $\frac{\varepsilon_2}{1-\varepsilon_2} < 1$, we the above inequality implies the estimate (4.14).

If $\varepsilon_2 < \frac{1}{4}$ and R is large, then we therefore have

$$
|\nabla w_{R,j}^n|_{L^2}^2 \leq \frac{1}{2} + |\nabla g_{R,j}|_{L^2}^2 \leq 1 + |g_j|_{D_0^1}^2 \quad \text{for all} \quad n \geq 1, 0 \leq j \leq k. \tag{4.15}
$$

Following the elliptic regularity estimate in the proof of Lemma 3.2, one can prove from (4.15) that

$$
|w_{R,j}^n|_{D_0^1 \cap D^{2(k-j)+3}} \leq C(1 + |g_j|_{D_0^1}) \leq Cc_0 \tag{4.16}
$$

for all large R, $n \geq 1$ and $0 \leq j \leq k$. We leave the details of proof to the readers.

Now using the weak compactness of Sobolev space, we find a solution to the problem (4.10)

$$
(w_{R,0}, w_{R,1}, \cdots, w_{R,k}) \in D_0^1 \cap D^{2k+3} \times D_0^1 \cap D^{2k+1} \times \cdots \times D_0^1 \cap D^3
$$

such that[2]

$$w_{R,j}^n \rightharpoonup w_{R,j} \quad \text{in} \quad D_0^1 \cap D^{2(k-j)+3}, \quad |w_{R,j}|_{D_0^1 \cap D^{2(k-j)+3}} \le Cc_0.$$

Finally, we prove the uniqueness. Let $(w_{R,0}^1, \cdots, w_{R,k}^1)$ and $(w_{R,0}^2, \cdots w_{R,k}^2)$ be the solutions to the elliptic system (4.10). Then we have

$$\begin{aligned}
&L(w_{R,j}^1 - w_{R,j}^2)\\
&= j\mathrm{div}(\rho_{R,0}v_R(0))(w_{R,j}^1 - w_{R,j}^2)\\
&- \sum_{1\le m\le j-1} \begin{pmatrix} j \\ m-1 \end{pmatrix} \rho_R^{(j+1-m)}(0)(w_{R,m}^1 - w_{R,m}^2).
\end{aligned}$$

Thus by integration by parts we have

$$\begin{aligned}
|\nabla(w_{R,k}^1 - w_{R,k}^2)|_{L^2}^2 &\le \varepsilon_2 \max_{1\le m\le k-1} |\nabla(w_{R,j}^1 - w_{R,j}^2)|_{L^2}^2,\\
|\nabla(w_{R,k-1}^1 - w_{R,k-1}^2)|_{L^2}^2 &\le \varepsilon_2 \max_{1\le m\le k-2} |\nabla(w_{R,j}^1 - w_{R,j}^2)|_{L^2}^2,\\
&\vdots\\
|\nabla(w_{R,1}^1 - w_{R,1}^2)|_{L^2}^2 &\le \varepsilon_2 |\nabla(w_{R,2}^1 - w_{R,2}^2)|_{L^2}^2,\\
|\nabla(w_{R,0}^1 - w_{R,0}^2)|_{L^2}^2 &\le \varepsilon_2 |\nabla(w_{R,1}^1 - w_{R,1}^2)|_{L^2}^2.
\end{aligned}$$

Therefore,

$$\sum_{0\le j\le k} |\nabla(w_{R,j}^1 - w_{R,j}^2)|_{L^2}^2 \le k\frac{\varepsilon_2}{1-\varepsilon_2} \sum_{0\le j\le k} |\nabla(w_{R,j}^1 - w_{R,j}^2)|_{L^2}^2$$

and hence choosing ε_2 such that $\frac{k\varepsilon_2}{1-\varepsilon_2} < 1$, the uniqueness is proved. This completes the proof of lemma. □

Lemma 4.3. *If $(u_0, u^{(1)}(0), \cdots, u^{(k)}(0))$ is the given pair satisfying the compatibility condition (4.6) and $(w_{R,0}, w_{R,1}, \cdots, w_{R,k})$ is the pair of solution to the elliptic system (4.10), then extending $(w_{R,0}, \cdots, w_{R,k})$ in Ω by defining 0 outside Ω_R, the extended pair converges to $(u_0, u^{(1)}(0), \cdots, u^{(k)}(0))$ in $D_0^1(\Omega)$.*

Proof. Let us denote the pair of extensions by $(w_{R,0}, \cdots, w_{R,k})$ again. Let φ_R be the cut-off function as above and $w_j = u^{(j)}(0)$. Then we have for $0 \le j \le k$

$$\begin{aligned}
L(\varphi_R w_j) = j\mathrm{div}(\rho_0 v(0))\varphi_R w_j &- \sum_{1\le m\le j-1} \begin{pmatrix} j \\ m-1 \end{pmatrix} \rho^{(j+1-m)}(0)\varphi_R w_m\\
&+ \varphi_R H_j - \nabla(\varphi_R p^{(j)}(0)) - \rho\varphi_R g_{R,j+1} + \Phi_{R,j},
\end{aligned}$$

[2]Actually using the argument of uniqueness proof, we can prove that $w_{R,j}^n$ converges strongly to $w_{R,j}$ in D_0^1.

where $\Phi_{R,j} = L(\varphi_R w_j) - \varphi_R \, Lw_j + \nabla\varphi_R p^{(j)}(0)$. Thus the difference $w_{R,j} - \varphi_R w_j$ satisfies the system in Ω_R

$$
\begin{aligned}
&L(w_{R,j} - \varphi_R w_j) \\
&\qquad = j\left(\mathrm{div}(\rho_{R,0}v_R(0))w_{R,j} - \mathrm{div}(\rho_0 v(0))\varphi_R w_j\right) \\
&\qquad\quad - \sum_{1\le m\le j-1} \begin{pmatrix} j \\ m-1 \end{pmatrix} \left(\rho_R^{(j+1-m)}(0) w_{R,m} - \rho^{(j+1-m)}(0)\varphi_R w_m\right) \\
&\qquad\quad + (H_{R,j} - \varphi_R H_j) - \nabla(p_R^{(j)}(0) - \varphi_R p^{(j)}(0)) \\
&\qquad\quad - R^{-3} g_{R,j+1} - \Phi_{R,j}.
\end{aligned} \tag{4.17}
$$

Multiplying (4.17) by $\Gamma_{R,j} = w_{R,j} - \varphi_R w_j$ and integrating over Ω_R, we have

$$
\begin{aligned}
&\mu|\nabla\Gamma_{R,j}|_{L^2}^2 + (\lambda+\mu)\int (\mathrm{div}\Gamma_{R,j})^2 \, dx \\
&= \int j\mathrm{div}(\rho_{R,0}v_R(0))|\Gamma_{R,j}|^2 \, dx \\
&\quad + \int j\left(\mathrm{div}(\rho_{R,0}v_R(0) - \rho_0 v(0)\right)\varphi_R g_j \cdot \Gamma_{R,j} \, dx \\
&\quad - \sum_{1\le m\le j-1} \begin{pmatrix} j \\ m-1 \end{pmatrix} \int (\rho_R^{(j+1-m)}(0)\Gamma_{R,m}) \cdot \Gamma_{R,j} \, dx \\
&\quad - \sum_{1\le m\le j-1} \begin{pmatrix} j \\ m-1 \end{pmatrix} \int (\rho_R^{(j+1-m)}(0) - \rho^{(j+1-m)}(0))\varphi_R g_m \cdot \Gamma_{R,j} \, dx \\
&\quad + \int (H_{R,j} - \varphi_R H_j) \cdot \Gamma_{R,j} \, dx \\
&\quad + \int \left(p_R^{(j)}(0) - \varphi_R p^{(j)}(0)\right) \mathrm{div}\Gamma_{R,j} \, dx \\
&\quad - R^{-3}\int g_{R,j+1} \cdot \Gamma_{R,j} \, dx \\
&\quad - \int \Phi_{R,j} \cdot \Gamma_{R,j} \, dx \\
&\equiv \sum_{1\le i\le 8} I_j^i.
\end{aligned} \tag{4.18}
$$

We estimate each I_j^i as follows:

$$
\begin{aligned}
I_j^1 &= -2j\int \rho_{R,0}v_R(0) \cdot (\Gamma_{R,j} \cdot \nabla\Gamma_{R,j}) \, dx \\
&\le C|\rho_{R,0}|_{L^6}|\nabla v_R(0)|_{L^2}|\nabla\Gamma_{R,j}|_{L^2}^2 \le \varepsilon_2|\nabla\Gamma_{R,j}|_{L^2}^2,
\end{aligned}
$$

where ε_2 is the small constant appearing in Lemma 4.2.

$$
\begin{aligned}
I_j^2 &\le C|\mathrm{div}(\rho_{R,0}v_R(0) - \rho_0 v(0))|_{L^{\frac{3}{2}}}|g_j|_{L^6}|\nabla\Gamma_{R,j}|_{L^2} \\
&\le Cc_0^2|\mathrm{div}(\rho_{R,0}v_R(0) - \rho_0 v(0))|_{L^{\frac{3}{2}}}^2 + \frac{\mu}{32}|\nabla\Gamma_{R,j}|_{L^2}^2,
\end{aligned}
$$

$$I_j^3 \le \varepsilon_2 \max_{1\le m\le j-1} |\nabla\Gamma_{R,m}|_{L^2}^2 + \frac{\mu}{32}|\nabla\Gamma_{R,j}|_{L^2}^2,$$

$$\begin{aligned} I_j^4 &\le C \sum_{1\le m\le j-1} |\rho_R^{(j+1-m)}(0) - \rho^{(j_1-m)}(0)|_{L^{\frac{3}{2}}}^2 |g_m|_{L^6}^2 + \frac{\mu}{32}|\nabla\Gamma_{R,j}|_{L^2}^2 \\ &\le Cc_0^2 \sum_{2\le m\le j} |\rho_R^{(m)}(0) - \rho^{(m)}(0)|_{L^{\frac{3}{2}}}^2 + \frac{\mu}{32}|\nabla\Gamma_{R,j}|_{L^2}^2, \end{aligned}$$

$$I_j^5 \le C|H_{R,j} - \varphi_R H_j|_{L^{\frac{6}{5}}}^2 + \frac{\mu}{32}|\nabla\Gamma_{R,j}|_{L^2}^2,$$

$$I_j^6 \le C|p_R^{(j)}(0) - \varphi_R p^{(j)}(0)|_{L^2}^2 + \frac{\mu}{32}|\nabla\Gamma_{R,j}|_{L^2}^2,$$

$$I_j^7 \le CR^{-6}|g_{R,j+1}|_{L^{\frac{6}{5}}}^2 + \frac{\mu}{32}|\nabla\Gamma_{R,j}|_{L^2}^2,$$

and finally

$$I_j^8 \le C|\Phi_{R,j}|_{L^{\frac{6}{5}}}^2 + \frac{\mu}{32}|\nabla\Gamma_{R,j}|_{L^2}^2.$$

Substituting all these estimate into (4.18) and then choose ε_2 to be sufficiently small, we have for all $0 \le j \le k$

$$|\nabla\Gamma_{R,j}|_{L^2}^2 \le \varepsilon_2 \max_{1\le m\le j-1} |\nabla\Gamma_{R,m}|_{L^2}^2 + \Lambda_{R,j}, \tag{4.19}$$

where

$$\begin{aligned} \Lambda_{R,j} = Cc_0^2|\mathrm{div}(\rho_{R,0}v_R(0) - \rho_0 v(0))|_{L^{\frac{3}{2}}}^2 + Cc_0^2 \sum_{2\le m\le j} |\rho_R^{(m)}(0) - \rho^{(m)}(0)|_{L^{\frac{3}{2}}}^2 \\ + C|p_R^{(j)}(0) - \varphi_R p^{(j)}(0)|_{L^2}^2 + CR^{-6}|g_{R,j+1}|_{L^{\frac{6}{5}}}^2 + C|\Phi_{R,j}|_{L^{\frac{6}{5}}}^2. \end{aligned}$$

Here we interpret $\max_{1\le m\le j-1} |\nabla\Gamma_{R,m}|_{L^2}^2$ as 0 in case when $j = 0, 1$.

One can easily show that $\Lambda_{R,j} = o(1)$, where $o(1)$ denotes a function of R, which tends to zero as $R \to \infty$. We leave the details of its proof to the readers.

Now let $b_{R,j} = |\nabla\Gamma_{R,j}|_{L^2}^2$. Then we have for $j \ge 2$

$$b_{R,j} \le \varepsilon_2 \left(\max_{1\le m\le j-1} b_{R,m} \right) + o(1)$$

and

$$b_{R,1} = o(1), \qquad b_{R,0} = o(1).$$

Thus letting $\Gamma_R = \max_{0\le m\le k} b_{R,m}$, we have $\Gamma_R = o(1)$. Therefore we have just proven that $|w_{R,j} - \varphi_R w_j|_{D_0^1(\Omega_R)} \to 0$ as $R \to \infty$. Since $\varphi_R w_j \to w_j$ in D_0^1, this implies that each extension $w_{R,j}$ converges to w_j in D_0^1. $\square$

Proof of Proposition 4.1. To prove the existence, we consider the following initial boundary value problem together with (4.9)

$$\rho_R(u_R)_t + Lu_R + \nabla p_R = \rho_R(f_R - v_R \cdot \nabla v_R) \quad \text{in} \quad (0,T_*) \times \Omega_R, \tag{4.20}$$

$$u_R^{(j)}(0) = w_{R,j} (0 \le j \le k+1) \quad \text{in} \quad \Omega_R, \tag{4.21}$$

where $w_{R,j}$ are the functions constructed in Lemma 4.2 and $w_{R,0} = u_{R,0} = u_R(0)$.

From Lemma 4.3 and the observation (4.11), we deduce that

$$|\rho_{R,0}|_{L^1 \cap H^{2k+3}} \le \varepsilon, \quad 1 + |u_{R,0}|_{D_0^1} + \sum_{1 \le j \le k+1} |w_{R,j}|_{D_0^1} \le c_0, \tag{4.22}$$

and for all $0 \le j \le k+1$,

$$|v_R^{(j)}(0)|_{D_0^1 \cap D^{2(k-j)+3}} \le 1 + c_{0,1,j}, \tag{4.23}$$

and for all $0 \le l \le k$ and $0 \le j \le k-l$,

$$\begin{aligned}
&\sup_{0 \le t \le T_*} \Big(|v_R^{(j)}(t)|_{D_0^1 \cap D^{2l+1}} \Big) + \int_0^{T_*} |v_R^{(j)}(t)|^2_{D^2 \cap D^{2l+2}}\, dt \le 1 + c_{l,2,j}, \\
&\sup_{0 \le t \le T_*} \Big(|v_R^{(j)}(t)|_{D_0^1 \cap D^{2l+2}} \Big) \\
&\quad + \int_0^{T_*} \Big(|v_R^{(j+1)}(t)|^2_{D_0^1 \cap D^{2l+1}} + |v_R^{(j)}(t)|^2_{D^{2l+3}} \Big)\, dt \le 1 + c_{l,3,j}, \\
&\operatorname*{ess\,sup}_{0 < t < T_*} \Big(|v_R^{(j+1)}(t)|_{D_0^1 \cap D^{2l+1}} + |v_R^{(j)}(t)|_{D^{2l+3}} \Big) \\
&\quad + \int_0^{T_*} \Big(|v_R^{(j+1)}(t)|^2_{D^{2l+2}} + |v_R^{(j)}(t)|^2_{D^{2l+4}} \Big)\, dt \le 1 + c_{l,4,j}
\end{aligned} \tag{4.24}$$

Therefore, applying al the estimates in Section 3 to (ρ_R, u_R) with $(\rho_{R,0}, w_R, v_R)$ satisfying (4.23) and (4.24) instead of (ρ_0, g_j, v) in (3.1), we conclude that if ε is sufficiently small, then for each $R > R_1$, the solution (ρ_R, u_R) satisfies the estimate (3.21) and (3.22) with the domain being Ω_R. We extend (ρ_R, u_R) by defining to be zero outside Ω_R. Then by the uniform estimate (3.22) on R, we deduce that there exists a sequence $\{R_j\}$, $R_j \to \infty$ as $j \to \infty$ such that (ρ_{R_j}, u_{R_j}) converges in a weak or weak-* sense to a limit (ρ, u). Moreover, since (ρ, u) also satisfies (3.22) with the domain being Ω_R for each $R > R_1$, it follows that for $0 \le j \le k+1$

$$\begin{aligned}
&\rho \in L^\infty(0,T_*), \quad (\rho^{(j)}, p^{(j)}) \in L^\infty(0,T_*; H^{2(k-j)+3}), \\
&u^{(j)} \in L^\infty(0,T_*; D_0^1 \cap D^{2(k-j)+3}), \quad \sqrt{\rho} u^{(k+1)} \in L^\infty(0,T_*; L^2).
\end{aligned} \tag{4.25}$$

We will show that (ρ, u) is a solution to the original problem (4.1)-(4.4). It is obvious that (ρ, u) satisfies the boundary conditions in (4.3) and (4.4). Let $R > R_1$ be a fixed large number. Then since for all sufficiently large l, (ρ_{R_l}, u_{R_l}) satisfies the uniform estimate (3.21) and (3.22) with the domain being Ω_R, it follows from a standard compactness result (see [24]) that a subsequence of $(\rho_{R_l}^{(j)}, u_{R_l}^{(j)})$ converges to $(\rho^{(j)}, u^{(j)})$ in $C([0,T_*]; H^1(\Omega_R))$ for all $0 \le j \le k$. Using this result together with Lemma 4.3, we can show that (ρ, u) satisfies the equations (2.7)-(2.8) in $(0,T_*) \times \Omega_R$ and $(\rho(0), u(0)) = (\rho_0, u_0)$ in Ω_R. Let w_j be the limit of $w_{R,j}$

constructed in Lemma 4.2 as in Lemma 4.3. Then $u^{(j)}(0) = w_j$ in Ω_R. Since R can be arbitrarily large, we have proved the existence of a solution (ρ, u) to the original problem (4.1)-(4.2) satisfying the compatibility condition 4.6 and the regularity (4.25) as in [5]. The uniqueness of solutions with this regularity can be proved by a simple energy estimate. We deduce from a classical embedding result that $u^{(j)} \in C([0, T_*]; D_0^1 \cap D^{2(k-j)+3}$ for all $0 \leq j \leq k$. Then the time-continuity of ρ follows from Lemma 2.3. This completes the proof of Proposition 4.1. $\square$

5. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We construct a sequence of approximate solutions to the linearized problem which converges to a solution to the original nonlinear problem in a strong sense.

Let us invoke the system of compatibility conditions for linearized problem such that

$$L(w_j) = j\mathrm{div}(\rho_0 v(0))w_j + G_{j-1} + H_j - \nabla p^{(j)}(0) - \rho_0 g_{j+1} \tag{5.1}$$

for some given $\rho_0 \in L^1 \cap H^{2k+3}, g_j \in D_0^1 \cap D^{2(k-j)+3}$ and all $1 \leq j \leq k+1$, where

$$\begin{aligned} G_{j-1} &= 0 \quad \text{for} \quad j = 0, 1, \\ G_{j-1} &= - \sum_{1 \leq m \leq j-1} \begin{pmatrix} j \\ m \end{pmatrix} \rho^{(j+1-m)}(0) w_m \quad \text{for} \quad j \geq 2, \\ H_j &= \sum_{0 \leq m \leq j} \begin{pmatrix} j \\ m \end{pmatrix} \rho^{(j-m)}(0)(f - v \cdot \nabla v)^{(j)}(0). \end{aligned}$$

Now let $w_{0,0}, w_{0,1}, \cdots, w_{0,k}$ be the unique solutions of (5.1) with $v = 0$. This is possible for small initial density ρ_0. Then we find a unique solution $(\underline{\rho}, \underline{u}$ to the linearized problem (4.1)–(4.4) with initial data $(\underline{u}(0), \underline{u}^{(1)}(0), \cdots, \underline{u}^{(k)}(0))$ satisfying

$$(\underline{u}(0), \underline{u}^{(1)}(0), \cdots, \underline{u}^{(k)}(0)) = (w_{0,0}, w_{0,1}, \cdots, w_{0,k}).$$

In this turn, we find the unique solutions $w_{1,0}, w_{1,1}, \cdots, w_{1,k}$ of the system (5.1) and also the unique solution (ρ_1, u_1) to the linear problem (4.1)–(4.4) with $v = \underline{u}$ and initial data satisfying

$$(u_1(0), u_1^{(1)}(0), \cdots, u_1^{(k)}(0)) = (w_{1,0}, w_{1,1}, \cdots, w_{1,k}).$$

In this way, we find $w_{n,j}$ and u_n for $n \geq 1, 0 \leq j \leq k$ with $v = u_{n-1}$ of the equations (5.1), and (4.1)–(4.4), respectively satisfying

$$(u_n(0), u_n^{(1)}, \cdots, u_n^{(k)}(0)) = (w_{n,0}, w_{n,1}, \cdots, w_{n,k}).$$

If $\sum_{1 \leq j \leq k+1} |g_j|_{D_0^1} \leq c_0$ for some fixed c_0 and ε is sufficiently small, then from Proposition 3.13, Lemma 4.2 and Lemma 4.3 we deduce that for all $n \geq 1$ and for all $0 \leq j \leq k$

$$\sum_{0 \leq j \leq k} |w_{n,j}|_{D_0^1 \cap D^{2(k-j)+3}} \leq C c_0,$$

$$
\begin{aligned}
&\sup_{0<t<T_*}\Big(|\rho_n(t)|_{L^1}+|(\rho_n^{(j)}(t),\,(p_n-p_n(0))^{(j)}(t))|_{H^{2(k-j)+3}}\Big)\le \widetilde{C}\varepsilon\\
&\sup_{0\le t\le T_*}\Big(|u_n^{(j)}(t)|_{D_0^1\cap D^{2(k-j)+3}}\Big)+\int_0^{T_*}|u_n^{(j)}(t)|^2_{D^{2(k-j)+4}}\,dt\\
&\qquad+\operatorname*{ess\,sup}_{0<t<T_*}|u_n^{(k+1)}(t)|_{D_0^1}+\int_0^{T_*}|u_n^{(k+1)}(t)|^2_{D^2}\,dt\\
&\qquad\qquad+\operatorname*{ess\,sup}_{0\le t\le T_*}|\sqrt{\rho_n}u_n^{(k+1)}(t)|_{L^2}\le \widetilde{C}.
\end{aligned}\tag{5.2}
$$

Throughout the proof, we denote by $\tilde{C}\ge 1$ a generic positive constant depending only on c_0 and the parameters of C, but independent of n.

We first show the full sequence $\{(w_{n,0},\cdots,w_{n,k})\}_{n\ge1}$ converges to $(w_0,\cdots,w_k)$ in $D_0^1\cap D^{2k+3}\times\cdots\times D_0^1\cap D^3$.

Let us denote $\rho_{n-1}^{(j)}(0)$ and $p_{n-1}^{(j)}(0)$ by $\rho^{(j)}(0)$ and $p^{(j)}(0)$ with $v^{(l)}(0)=w_{n-1,l}$ for $0\le l\le j$. Then $(w_{n,0},\cdots,w_{n,k})$ satisfies the following equations: for all $0\le j\le k$

$$
L(w_{n,j})=j\mathrm{div}(\rho_0 w_{n-1,0})w_{n,j}+G_{n,j-1}+H_{n-1,j}-\nabla p_{n-1}^{(j)}(0)-\rho_0 g_{j+1},
$$

where

$$
\begin{aligned}
&G_{n,j-1}=0\quad\text{for}\quad j=0,1,\\
&G_{n,j-1}=-\sum_{1\le m\le j-1}\binom{j}{m}\rho_{n-1}^{(j+1-m)}(0)w_{n,m}\quad\text{for}\quad j\ge2
\end{aligned}
$$

and

$$
H_{n-1,j}=\sum_{0\le l\le j}\binom{j}{l}\rho_{n-1}^{(j-l)}(0)\left(f^{(l)}(0)-\sum_{0\le m\le l}\binom{l}{m}w_{n-1,l-m}\cdot\nabla w_{n-1,m}\right).
$$

Now let $\overline{w}_{n+1,j}=w_{n+1,j}-w_{n,j}$. Then

$$
\begin{aligned}
L(\overline{w}_{n+1,j})&=j\mathrm{div}(\rho_0\overline{w}_{n,0})w_{n+1,j}+j\mathrm{div}\,(\rho_0 w_{n-1,0})\overline{w}_{n+1,j}\\
&\quad+\overline{G}_{n+1,j-1}+\overline{H}_{n,j}-\nabla(p_{n+1}^{(j)}(0)-p_n^{(j)}(0)),
\end{aligned}\tag{5.3}
$$

where

$$
\begin{aligned}
\overline{G}_{n+1,j-1}=&-\sum_{1\le l\le j-1}\binom{j}{l}(\rho_n^{(j+1-l)}(0)-\rho_{n-1}^{(j+1-l)}(0))w_{n+1,l}\\
&-\sum_{1\le l\le j-1}\binom{j}{l}\rho_{n-1}^{(j+1-l)}(0)\overline{w}_{n+1,l},\\
\overline{H}_{n,j}=&\sum_{0\le m\le j}\binom{j}{l}(\rho_n^{(j-l)}(0)-\rho_{n-1}^{(j-l)}(0))(f^{(l)}(0)\\
&\qquad-\sum_{0\le m\le l}\binom{l}{m}w_{n,l-m}\cdot\nabla w_{n,l})\\
&-\sum_{0\le l\le j}\sum_{0\le m\le l}\binom{j}{l}\binom{l}{m}\rho_{n-1}^{(j-l)}(0)(w_{n,l-m}\cdot\nabla w_{n,l}\\
&\qquad-w_{n-1,l-m}\cdot\nabla w_{n-1,m}).
\end{aligned}
$$

Then multiplying (5.3) by $\overline{w}_{n,j}$ and integrating it over Ω, we have

$$
\begin{aligned}
\mu|\nabla\overline{w}_{n+1,j}|^2_{L^2} &+ (\lambda+\mu)|\mathrm{div}\,\overline{w}_{n+1,j}|^2_{L^2}\\
&= \int (j\mathrm{div}\,(\rho_0 w_{n-1,0})|\overline{w}_{n+1,j}|^2 + j\mathrm{div}\,(\rho_0\overline{w}_{n,0})w_{n+1,j}\cdot\overline{w}_{n+1,j}\\
&\qquad + \overline{G}_{n+1,j-1}\cdot\overline{w}_{n+1,j} + \overline{H}_{n,j}\cdot\overline{w}_{n+1,j}\\
&\qquad + (p_n^{(j)} - p_{n-1}^{(j)})\mathrm{div}\,\overline{w}_{n+1,j})\,dx\\
&\equiv \sum_{1\le i\le 5} I_i.
\end{aligned}
$$

Using Hölder and Sobolev inequalities, we get the following estimates.

$$
\begin{aligned}
I_1 &\le C|\rho_0 w_{n-1,0}|_{L^3}|\nabla\overline{w}_{n+1,j}|^2_{L^2} \le Cc_0\varepsilon|\nabla\overline{w}_{n+1,j}|^2_{L^2},\\
I_2 &\le C|\rho_0|_{H^1}|\nabla\overline{w}_{n,0}|_{L^2}|\nabla w_{n+1,j}|_{L^2}|\nabla\overline{w}_{n+1,j}|_{L^2}\\
&\le Cc_0^2\frac{\varepsilon^2}{\eta}|\nabla\overline{w}_{n,0}|^2_{L^2} + \eta|\nabla\overline{w}_{n+1,j}|^2_{L^2},
\end{aligned}
$$

where $\eta\in(0,1)$ is a small number. By the definition of $\rho_n^{(j)}(0)$ and $p_n^{(j)}(0)$, one can easily show that

$$
\begin{aligned}
|\rho_n^{(j)}(0) - \rho_{n-1}^{(j)}(0)|_{L^{\frac{3}{2}}\cap H^{2(k-j)+3}} &\le Cc_0^j\varepsilon\sum_{0\le l\le j-1}|\overline{w}_{n,l}|_{D_0^1\cap D^{2(k-l)+3}}\\
|p_n^{(j)}(0) - p_{n-1}^{(j)}(0)|_{H^{2(k-j)+3}} &\le Cc_0^j\varepsilon\sum_{0\le l\le j-1}|\overline{w}_{n,l}|_{D_0^1\cap D^{2(k-l)+3}}.
\end{aligned}
\tag{5.4}
$$

Hence

$$
\begin{aligned}
I_3 &\le Cc_0^{2j}\frac{\varepsilon^2}{\eta}\sum_{0\le l\le j-1}|\overline{w}_{n,l}|^2_{D_0^1\cap D^{2(k-l)+3}} + \eta|\nabla\overline{w}_{n+1,j}|^2_{L^2} + Cc_0^j\varepsilon\sum_{0\le l\le j}|\nabla\overline{w}_{n+1,l}|^2_{L^2}\\
I_4 &\le Cc_0^{2j}\frac{\varepsilon^2}{\eta}\sum_{0\le l\le j-1}|\overline{w}_{n,l}|^2_{D_0^1\cap D^{2(k-l)+3}} + Cc_0^{2j}\frac{\varepsilon^2}{\eta}\sum_{0\le l\le j}|\nabla\overline{w}_{n,l}|^2_{L^2} + \eta|\nabla\widetilde{w}_{n+1,j}|^2_{L^2}\\
I_5 &\le Cc_0^{2j}\frac{\varepsilon^2}{\eta}\sum_{0\le l\le j-1}|\overline{w}_{n,l}|^2_{D_0^1\cap D^{2(k-l)+3}} + \eta|\nabla\overline{w}_{n+1,j}|^2_{L^2}.
\end{aligned}
$$

For sufficiently small η and ε, we have

$$
\sum_{0\le l\le j}|\nabla\overline{w}_{n+1,l}|^2_{L^2} \le Cc_0^{2j}\varepsilon\sum_{0\le l\le j}|\overline{w}_{n,l}|^2_{D_0^1\cap D^{2(k-l)+3}}.
\tag{5.5}
$$

Now we use the elliptic regularity for $\overline{w}_{n+1,j}$ of (5.3). Then by (5.4) and (5.5) we have

$$
\begin{aligned}
|\overline{w}_{n+1,j}&|_{D_0^1\cap D^{2(k-j)+3}}\\
&\le C|\overline{w}_{n+1,j}|_{D_0^1} + C|\mathrm{div}(\rho_0\overline{w}_{n,0})w_{n+1,j}|_{D^{2(k-j)+1}}\\
&\qquad + |\mathrm{div}\,(\rho_0 w_{n-1,0})\overline{w}_{n+1,j}|_{D^{2(k-j)+1}} + |\overline{G}_{n+1,j-1}|_{D^{2k-j+1}}\\
&\qquad + |\overline{H}_{n,j}|_{D^{2(k-j)+1}} + |p_n^{(j)}(0) - p_{n-1}^{(j)}(0)|_{D^{2(k-j)+2}}\\
&\le Cc_0^{2j}\varepsilon^{\frac{1}{2}}\sum_{0\le l\le j}|\overline{w}_{n,l}|_{D_0^1\cap D^{2(k-l)+3}} + Cc_0^{2j}\varepsilon|\overline{w}_{n+1,j}|_{D_0^1\cap D^{2(k-j)+3}}.
\end{aligned}
$$

Hence for small ε we conclude that

$$\sum_{0\le j\le k} |\overline{w}_{n+1,j}|_{D_0^1\cap D^{2(k-j)+3}} \le \frac{1}{2}\sum_{0\le j\le k} |\overline{w}_{n,j}|_{D_0^1\cap D^{2(k-j)+3}}.$$

This means that

$$\sum_{n\ge 0}\sum_{0\le j\le k} |\overline{w}_{n+1,j}|_{D_0^1\cap D^{2(k-j)+3}} \le C\sum_{0\le j\le k} |w_{0,j}|_{D_0^1\cap D^{2(k-j)+3}}$$

Therefore $(w_{n,0},\cdots,w_{n,k})$ converges to a pair of functions $(w_0,\cdots,w_k)$ in $D_0^1\cap D^{2k+3}\times\cdots\times D_0^1\cap D^3$. Apparently, these functions w_n satisfies by their regularity the equations

$$L(w_j) = j\mathrm{div}(\rho_0 w_0)w_j + G_{j-1} + H_j - \nabla p^{(j)}(0) - \rho_0 g_{j+1}, \tag{5.6}$$

where

$$\begin{aligned}
&G_{j-1} = 0 \quad \text{for} \quad j = 0,1,\\
&G_{j-1} = -\sum_{1\le l\le j-1}\begin{pmatrix} j\\ l\end{pmatrix}\rho^{(j+1-l)}(0)w_l \quad \text{for} \quad j\ge 2,\\
&H_j = \sum_{0\le l\le j}\begin{pmatrix} j\\ l\end{pmatrix}\rho^{(j-l)}(0)\left(f^{(l)}(0) - \sum_{0\le m\le l}\begin{pmatrix} l\\ m\end{pmatrix} w_{l-m}\cdot\nabla w_m\right)\\
&\rho^{(j+1)}(0) = -\mathrm{div}\left(\sum_{0\le l\le j}\begin{pmatrix} j\\ l\end{pmatrix}\rho^{(j-l)}(0)w_l\right), \quad 0\le j\le k.
\end{aligned}$$

It can be easily shown by the argument of convergence that w_n are unique functions.

From now on, we show that the full sequence $\{(\rho_n, u_n)\}$ of approximate solutions converges to a solution to the original problem (1.1)-(1.5) in a strong sense. To do this, let us define

$$\overline{\rho}_{n+1} = \rho_{n+1} - \rho_n, \quad \overline{u}_{n+1} = u_{n+1} - u_n \quad \text{and} \quad p_n = p(\rho_n).$$

Then from the equation (4.1), we derive

$$(\overline{\rho}_{n+1})_t + \mathrm{div}\,(\overline{\rho}_{n+1}u_n) + \mathrm{div}\,(\rho_n\overline{u}_n) = 0. \tag{5.7}$$

Multiplying this by $\overline{\rho}_{n+1}$ and integrating over Ω, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int |\overline{\rho}_{n+1}|^2\,dx\\
&\le C\int |\nabla u_n||\overline{\rho}_{n+1}|^2 + (\,|\nabla\rho_n||\overline{u}_n| + \rho_n|\nabla\overline{u}_n|\,)|\overline{\rho}_{n+1}|\,dx\\
&\le C|\nabla u_n|_{L^\infty}|\overline{\rho}_{n+1}|_{L^2}^2 + C\,(|\nabla\rho_n|_{H^1} + |\rho_n|_{L^\infty})\,|\nabla\overline{u}_n|_{L^2}|\overline{\rho}_{n+1}|_{L^2}.
\end{aligned}$$

Hence it follows from the uniform bound (5.2) that

$$\frac{d}{dt}|\overline{\rho}_{n+1}|_{L^2}^2 \le \eta^{-1}\tilde{C}|\overline{\rho}_{n+1}|_{L^2}^2 + \eta|\nabla\overline{u}_n|_{L^2} \tag{5.8}$$

for $0\le t\le T_*$.

We need an estimate for $|\overline{\rho}_{n+1}|_{L^{\frac{3}{2}}}$ in addition to (5.8). Multiplying (5.7) by $sgn(\overline{\rho}_{n+1})|\overline{\rho}_{n+1}|^{\frac{1}{2}}$ and integrating over Ω, we get

$$\begin{aligned}
&\frac{d}{dt}\int|\overline{\rho}_{n+1}|^{\frac{3}{2}}\,dx\\
&\le C\int|\nabla u_n||\overline{\rho}_{n+1}|^{\frac{3}{2}}+(\,|\nabla\rho_n||\overline{u}_n|+\rho_n|\nabla\overline{u}_n|\,)|\overline{\rho}_{n+1}|^{\frac{1}{2}}\,dx\\
&\le C|\nabla u_n|_{L^\infty}|\overline{\rho}_{n+1}|^{\frac{3}{2}}_{L^{\frac{3}{2}}}+C|\rho_n|_{H^1}|\nabla\overline{u}_n|_{L^2}|\overline{\rho}_{n+1}|^{\frac{1}{2}}_{L^{\frac{3}{2}}}.
\end{aligned}$$

Hence multiplying this by $|\overline{\rho}_{n+1}|^{\frac{1}{2}}_{L^{\frac{3}{2}}}$ and using (5.2), we have

$$\frac{d}{dt}|\overline{\rho}_{n+1}|^2_{L^{\frac{3}{2}}}\le\eta^{-1}\tilde{C}|\overline{\rho}_{n+1}|^2_{L^{\frac{3}{2}}}+\eta|\nabla\overline{u}_n|^2_{L^2}\tag{5.9}$$

for $0\le t\le T_*$.

Next from the equation (4.2), we derive

$$\begin{aligned}
\rho_{n+1}\,(\overline{u}_{n+1})_t+\rho_{n+1}\,u_n\cdot\nabla\overline{u}_{n+1}+L\overline{u}_{n+1}+\nabla(p_{n+1}-p_n)\\
=\overline{\rho}_{n+1}(f-(u_n)_t-u_{n-1}\cdot\nabla u_{n-1})\\
+\rho_{n+1}\,(u_n\cdot\nabla\overline{u}_{n+1}-\overline{u}_n\cdot\nabla u_n-u_{n-1}\cdot\nabla\overline{u}_n)\,.
\end{aligned}$$

Multiplying this by $\overline{u}_{n+1}$, integrating over Ω and using the equation (4.1) with $(\rho,v)=(\rho_{n+1},u_n)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\rho_{n+1}|\overline{u}_{n+1}|^2\,dx+\mu\int|\nabla\overline{u}_{n+1}|^2\,dx\\
&\le C\int|\overline{\rho}_{n+1}|\,|(u_n)_t||\overline{u}_{n+1}|\,dx+C\int|p_{n+1}-p_n||\nabla\overline{u}_{n+1}|\,dx\\
&\quad+C\int|\overline{\rho}_{n+1}|\,|f-\,u_{n-1}\cdot\nabla u_{n-1}||\overline{u}_{n+1}|\,dx\\
&\quad+C\int\rho_{n+1}\,(|u_n||\nabla\overline{u}_{n+1}|+\,|\overline{u}_n|\,|\nabla u_n|+|u_{n-1}||\nabla\overline{u}_n|)\,|\overline{u}_{n+1}|\,dx.
\end{aligned}\tag{5.10}$$

Using the uniform bound (5.2), we can estimate the last three integrals of the right hand side in (5.10) as follows:

$$C\int|p_{n+1}-p_n||\nabla\overline{u}_{n+1}|\,dx\le\tilde{C}|\overline{\rho}_{n+1}|^2_{L^2}+\frac{\mu}{12}|\nabla\overline{u}_{n+1}|^2_{L^2},$$

$$\begin{aligned}
&C\int|\overline{\rho}_{n+1}|\,|f-\,u_{n-1}\cdot\nabla u_{n-1}||\overline{u}_{n+1}|\,dx\\
&\le C|\overline{\rho}_{n+1}|_{L^2}|f-\,u_{n-1}\cdot\nabla u_{n-1}|_{H^1}|\nabla\overline{u}_{n+1}|_{L^2}\le\tilde{C}|\overline{\rho}_{n+1}|^2_{L^2}+\frac{\mu}{12}|\nabla\overline{u}_{n+1}|^2_{L^2},
\end{aligned}$$

$$C\int\rho_{n+1}|u_n||\nabla\overline{u}_{n+1}||\overline{u}_{n+1}|\,dx\le\tilde{C}|\,\sqrt{\rho_{n+1}}\,\overline{u}_{n+1}\,|^2_{L^2}+\frac{\mu}{12}|\nabla\overline{u}_{n+1}|^2_{L^2}$$

and

$$
\begin{aligned}
& C \int \rho_{n+1} \left(|\overline{u}_n|\, |\nabla u_n| + |u_{n-1}| |\nabla \overline{u}_n| \right) |\overline{u}_{n+1}|\, dx \\
& \leq C |\rho_{n+1}|_{L^\infty}^{\frac{1}{2}} \left(|u_n|_{D_0^1 \cap D^2} + |u_{n-1}|_{D_0^1 \cap D^2} \right) |\sqrt{\rho_{n+1}}\, \overline{u}_{n+1}|_{L^2} |\nabla \overline{u}_n|_{L^2} \\
& \leq \eta^{-1} \tilde{C} |\sqrt{\rho_{n+1}}\, \overline{u}_{n+1}|_{L^2}^2 + \eta |\nabla \overline{u}_n|_{L^2}^2 .
\end{aligned}
$$

The first integral is readily bounded by

$$
C |\overline{\rho}_{n+1}|_{L^{\frac{3}{2}}} |(u_n)_t|_{D_0^1} |\nabla \overline{u}_{n+1}|_{L^2} \leq \tilde{C} |\overline{\rho}_{n+1}|_{L^{\frac{3}{2}}}^2 + \frac{\mu}{12} |\nabla \overline{u}_{n+1}|_{L^2}^2 .
$$

Therefore, substituting all the estimates into (5.10), we deduce that

$$
\begin{aligned}
& \frac{d}{dt} |\sqrt{\rho_{n+1}}\, \overline{u}_{n+1}(t)|_{L^2}^2 + \mu |\nabla \overline{u}_{n+1}(t)|_{L^2}^2 \\
& \qquad \leq \eta^{-1} \tilde{C} \varphi_{n+1}(t) + 2\eta |\nabla \overline{u}_n(t)|_{L^2}^2
\end{aligned} \tag{5.11}
$$

for $0 \leq t \leq T_1$, where

$$
\varphi_{n+1}(t) = |\sqrt{\rho_{n+1}}\, \overline{u}_{n+1}(t)|_{L^2}^2 + |\overline{\rho}_{n+1}(t)|_{L^{\frac{3}{2}} \cap L^2}^2 .
$$

By virtue of (5.8), (5.9) and (5.11), we deduce that

$$
\frac{d}{dt} \varphi_{n+1}(t) + \mu\, \psi_{n+1}(t) \leq \eta^{-1} \tilde{C} \varphi_{n+1}(t) + 4\eta \psi_n(t) \tag{5.12}
$$

for $0 \leq t \leq T_1$, where $\psi_{n+1}(t) = |\nabla \overline{u}_{n+1}(t)|_{L^2}^2$. Note that $\varphi_{n+1}(0) = 0$. Hence integrating (5.12) over $(0, t)$, we have

$$
\varphi_{n+1}(t) + \mu \int_0^t \psi_{n+1}(s)\, ds \leq 4\eta \int_0^t \psi_n(s)\, ds + \eta^{-1} \tilde{C} \int_0^t \varphi_{n+1}(s)\, ds,
$$

which implies, in view of of Gronwall's inequality, that

$$
\varphi_{n+1}(t) + \int_0^t \psi_{n+1}(s)\, ds \leq \eta \tilde{C} \exp(\eta^{-1} \tilde{C} t) \left(\int_0^t \psi_n(s)\, ds \right) . \tag{5.13}
$$

Choosing $\eta > 0$ and then $T_2 > 0$ so small that

$$
\eta \tilde{C} \leq \frac{1}{4}, \quad T_2 < T_1 \quad \text{and} \quad \exp(\eta^{-1} \tilde{C} T_2) < 2,
$$

we deduce from (5.13) that

$$
\sum_{n=1}^{\infty} \left(\sup_{0 \leq t \leq T_2} \varphi_{n+1}(t) + \int_0^{T_2} \psi_{n+1}(t)\, dt \right) \leq \tilde{C} \int_0^{T_2} \psi^1(t)\, dt < \infty .
$$

Therefore, we conclude that $\{(\rho_n, u_n)\}$ converges in a strong sense to a limit (ρ, u) satisfying the regularity estimate (5.2) by weak compactness of $\{u_n^{(j)}\}$ with T_* replaced by T_2. Adapting the proof of Proposition 4.1, we can show that (ρ, u) is a solution to the original IBV problem (1.1)-(1.5) with T replaced by T_2. From the weak formulation and the uniqueness of solution of the system (5.6), we conclude that $u^{(j)}(0) = w_j$ (and hence $u^{(j)}(0)$ satisfies the compatibility condition (5.6)). This completes the proof of the existence. The proof of the uniqueness is similar to (indeed easier than) the proof of the convergence and so omitted. We have completed the proof of Theorem 1.1.

6. Proof of Theorem 1.4

In this section we consider the global existence of radially symmetric smooth solution $(\rho(t,x),u(t,x)) = (\rho(t,|x|,\overline{u}(t,|x|)))$ with $p(t,x) = A\rho(t,x)^\gamma$ for the initial boundary value problem (1.1)-(1.5) with

$$\Omega = \{x \in \mathbf{R}^3 : a < |x| < b\}, \quad f(t,x) = \overline{f}(t,|x|)\frac{x}{|x|}, \tag{6.1}$$

$$(\rho_0(x), p_0(x)) = (\rho_0(|x|), p_0(|x|)), \quad u_0(x) = \overline{u}_0(|x|)\frac{x}{|x|} \tag{6.2}$$

for some constant a, b with $0 < a < b \le \infty$, where $\overline{u}, \overline{u}_0$, $\overline{f}$ are scalar functions.

We first introduce a result of H. Kim and H. J. Choe [7] which are crucial for the proof of global existence.

Lemma 6.1 (Lemma 3.2 of [7])**.** *Whenever a strong solution (ρ,u) exists on a time interval $[0,T_1]$, we have*

$$\sup_{0\le t\le T_1} |\rho(t)|_{L^\infty} \le C_1$$

for some C_1 depending only on a, T_1 and the parameter of C not on b.

For the proof of Theorem 1.4, we will use a contradiction argument. Let T^* be the maximal existence time for the smooth solution (ρ,p,u) to exist with regularity in Theorem 1.1. Suppose that $T^* < \infty$. Then from Lemma 6.1 we deduce that

$$\sup_{0\le t<T^*} |\rho(t)|_{L^\infty} \le C_1, \tag{6.3}$$

where the constant C_1 now depends on T^*.

Let us define a function $J_k(t)$ representing all regularity of smooth solution (ρ,u) as stated in Theorem 1.4 by

$$\begin{aligned} J_k(t) \equiv & \sum_{0\le j\le k+1} |(\rho(t),p(t))|_{L^1} + |(\rho^{(j)}(t),p^{(j)}(t))|_{L^1\cap H^{2(k-j)+3}} \\ & + \int_\tau^t |(\rho^{(k+2)}(s),p^{(k+2)}(s))|_{L^2}^2\,ds \\ & + \sum_{0\le j\le k+1} |u_{(j)}(t)|_{D_0^1\cap D^{2(k-j)+3}} \\ & + \int_\tau^t |\sqrt{\rho}u_{(k+2)}(s)|_{L^2}^2\,ds \end{aligned}$$

for some fixed $\tau \in (0,T^*)$, where $p = A\rho^\gamma$ and hence p satisfies $p_t + \nabla p\cdot u + \gamma p \mathrm{div}\, u = 0$. Then from the hypothesis that $T^* < \infty$, we deduce that

$$\lim_{t\to T^*} J_k(t) = \infty.$$

But we will prove that this does not occur in the radially symmetric case. For the proof of Theorem 1.4 we have only to prove the following.

Theorem 6.2. *Fix $\tau \in (0, T^*)$ and define a function $\Phi(t)$ by*

$$\Phi(t) = 1 + \sup_{\tau < s < t} |\rho(s)|_{L^\infty} \quad \text{for} \quad \text{all} \quad t \in (\tau, T^*).$$

Then then there exists an increasing continuous function $M(\cdot)$ such that

$$J_k(t) \le M_k(\Phi(t)) \quad \text{for} \quad \text{all} \quad t \in (\tau, T^*).$$

Proof. Since u is radially symmetric, $\Delta u = \nabla \mathrm{div} u$. Thus (ρ, p, u) satisfies the modified equation

$$\rho u_t + \rho u \cdot \nabla u + \nu \nabla \mathrm{div} u + \nabla p = \rho f, \tag{6.4}$$

where $\nu = \lambda + 2\mu$.

Multiplying (6.4) by u and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\int \rho |u|^2 \, dx + \nu \int (\mathrm{div})^2 \, dx - \int p \mathrm{div} u \, dx \le \int \rho |f||u| \, dx. \tag{6.5}$$

From (1.12), we also have

$$\frac{1}{\gamma - 1}\frac{d}{dt}|p|_{L^1} = -\int p \mathrm{div} u \, dx.$$

Substituting this into (6.5), we have

$$\frac{d}{dt}\left(\frac{1}{2}|\sqrt{\rho} u|_{L^2}^2 + \frac{1}{\gamma - 1}|p|_{L^1}\right) + \nu \int (\mathrm{div} u)^2 \, dx \le |\rho|_{L^1}^{\frac{1}{2}} |\sqrt{\rho} u|_{L^2} |f|_{L^\infty}.$$

Integrating over (τ, t), by Gronwall's inequality we obtain for all $\tau < t < T^*$

$$|\sqrt{\rho} u(t)|_{L^2}^2 + |p(t)|_{L^1} + \int_\tau^t |\nabla u(s)|_{L^2}^2 \, ds \le \widetilde{C}. \tag{6.6}$$

Throughout this section, we denote $\widetilde{C}$ by a generic constant depending only on τ, T^*, c_0 and the parameter of C.

Now we multiply (6.4) by u_t and integrate over Ω. Then we have

$$\begin{aligned} &\frac{1}{2}\int \rho |u_t|^2 \, dx + \frac{d}{dt}\int \frac{\nu}{2}(\mathrm{div} u)^2 \, dx \\ &\qquad \le \int \rho |f|^2 \, dx + \int \rho |u|^2 |\nabla u|^2 \, dx + \int p \mathrm{div} u_t \, dx \end{aligned} \tag{6.7}$$

and also from (1.12)

$$\begin{aligned} \int p \mathrm{div} u_t \, dx = &\frac{d}{dt}\int p \mathrm{div} u \, dx - \frac{d}{dt}\int \frac{4\gamma - 3}{2\nu(2\gamma - 1)} p^2 \, dx \\ &+ \frac{\gamma - 1}{\nu^2}\int p(G^2 - p^2) \, dx - \frac{1}{\nu}\int p u \cdot \nabla G \, dx, \end{aligned}$$

where G is the effective viscous flux $\nu \mathrm{div} u - p$. Since

$$p \mathrm{div} u - \frac{4\gamma - 3}{2\nu(2\gamma - 1)} p^2 \le \frac{\nu}{2}(\mathrm{div} u)^2 - \nu \frac{2\gamma - 1}{2(4\gamma - 3)}(\mathrm{div} u)^2,$$

integrating (6.7) over (τ, t), we have

$$
\begin{aligned}
&\frac{1}{2}\int_\tau^t |\sqrt{\rho}u_t|_{L^2}^2\,ds + \frac{\nu}{2}|\nabla u(t)|_{L^2}^2 \\
&\quad \le \frac{\nu}{2}|\nabla u(\tau)|_{L^2}^2 + \int_\tau^t |\rho|_{L^1}|f|_{L^\infty}^2\,ds + \frac{\nu}{2}\left(1 - \frac{2\gamma-1}{4\gamma-3}\right)|\nabla u(t)|_{L^2}^2 \\
&\quad - \frac{\nu}{2}\left(1 - \frac{2\gamma-1}{4\gamma-3}\right)|\nabla u(\tau)|_{L^2}^2 + \int_\tau^t\int \rho|u|^2|\nabla u|^2\,dxds \\
&\quad + \frac{\gamma-1}{\nu^2}\int_\tau^t\int p(G^2-p^2)\,dxds - \frac{1}{\nu}\int_\tau^t\int pu\cdot\nabla G\,dxds.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\int_\tau^t |\sqrt{\rho}u_t|_{L^2}^2\,ds + |\nabla u(t)|_{L^2}^2 \\
&\quad \le \widetilde{C} + C\int_\tau^t\int pG^2\,dxds + C\int_\tau^t\int p|u||\nabla G|dxds \\
&\qquad + C\int_\tau^t\int \rho|u|^2|\nabla u|^2\,dxds
\end{aligned}
\tag{6.8}
$$

Now if we use the identity $\nabla G = \rho u_t + \rho u\cdot\nabla u - \rho f$,

$$
\begin{aligned}
\int_\tau^t\int p|u||\nabla G|dxds &\le \int_\tau^t p|u|(\rho|u_t| + \rho|u||\nabla u| + \rho|f|)\,dxds \\
&\le \int_\tau^t |p(s)|_{L^\infty}|\sqrt{\rho}u(s)|_{L^2}|\sqrt{\rho}u_t(s)|_{L^2}\,ds \\
&\quad + \int_\tau^t |p(s)|_{L^\infty}|u|_{L^\infty}|\rho|_{L^\infty}^{\frac{1}{2}}|\sqrt{\rho}u|_{L^2}|\nabla u|_{L^2}\,ds \\
&\quad + \int_\tau^t |\rho|_{L^\infty}^{\frac{1}{2}}|p|_{L^\infty}|\sqrt{\rho}u|_{L^2}|f|_{L^2}\,ds \\
&\le \widetilde{C}\int_\tau^t |p(s)|_{L^\infty}^2\,ds + \frac{1}{2C}\int_\tau^t |\sqrt{\rho}u_t|_{L^2}^2\,ds \\
&\quad + \widetilde{C}\int_\tau^t |p(s)|_{L^\infty}\Phi(s)^{\frac{1}{2}}|\nabla u|_{L^2}^2\,ds \\
&\quad + \widetilde{C}\int_\tau^t \Phi(s)^{\frac{1}{2}}|p(s)|_{L^\infty}\,ds.
\end{aligned}
$$

Here we used the embedding $|u|_{L^\infty} \le C|\nabla u|_{L^2}$ of radial function u. Substituting this into (6.8), we have

$$\begin{aligned}
&\int_\tau^t |\sqrt{\rho}u_t|_{L^2}^2\,ds + |\nabla u(t)|_{L^2}^2 \\
&\quad\le \widetilde{C} + C\int_\tau^t\int p(|\nabla u|^2 + p^2)\,dxds + \widetilde{C}\int_\tau^t |p(s)|_{L^\infty}^2\,ds \\
&\qquad + \widetilde{C}\int_\tau^t |p(s)|_{L^\infty}\Phi(s)^{\frac12}|\nabla u|_{L^2}^2\,ds + \widetilde{C}\int_\tau^t \Phi(s)^{\frac12}|p(s)|_{L^\infty}\,ds \\
&\qquad + \int_\tau^t\int \rho|u|^2|\nabla u|^2\,dxds \\
&\quad\le \widetilde{C} + \widetilde{C}\int_\tau^t (|p(s)|_{L^3}^3 + |p(s)|_{L^\infty}^2 + \Phi(s))\,ds \\
&\qquad + \widetilde{C}\int_\tau^t (|p(s)|_{L^\infty}\Phi(s)^{\frac12} + |p(s)|_{L^\infty})|\nabla u(s)|_{L^2}^2\,ds \\
&\qquad + C\int_\tau^t |\rho(s)|_{L^\infty}|u(s)|_{L^\infty}^2|\nabla u|_{L^2}^2\,ds \\
&\quad\le \widetilde{C} + \widetilde{C}\int_\tau^t (|p(s)|_{L^3}^3 + |p(s)|_{L^\infty}^2 + \Phi(s))\,ds \\
&\qquad + \widetilde{C}\int_\tau^t (|p(s)|_{L^\infty}\Phi(s)^{\frac12} + |p(s)|_{L^\infty} + \Phi(s)|\nabla u(s)|_{L^2}^2)|\nabla u(s)|_{L^2}^2\,ds.
\end{aligned}$$

Using Gronwall's inequality, (6.6) and (6.8), we have

$$\begin{aligned}
&\int_\tau^t |\sqrt{\rho}u_t|_{L^2}^2\,ds + |\nabla u(t)|_{L^2}^2 \\
&\quad\le \widetilde{C}\left(1 + \int_\tau^t (|p(s)|_{L^3}^3 + |p(s)|_{L^\infty}^2 + \Phi(s))\,ds\right) \\
&\qquad\times \exp\left(\widetilde{C}\int_\tau^t (|p(s)|_{L^\infty}^2 + |p(s)|_{L^\infty} + \Phi(s) + \Phi(s)|\nabla u(s)|_{L^2}^2)\,ds\right) \\
&\quad\le \widetilde{C}\left(1 + \widetilde{C}\int_\tau^t (|p(s)|_{L^\infty}^2 + \Phi(s))\,ds\right) \\
&\qquad\times \exp\left(\widetilde{C}\int_\tau^t (|p(s)|_{L^\infty}^2 + |p(s)|_{L^\infty} + \Phi(s))\,ds\right).
\end{aligned}$$

By the estimate $|p(s)|_{L^\infty} \le A\Phi(s)^\gamma$, we conclude that

$$\begin{aligned}
&\int_\tau^t |\sqrt{\rho}u_t|_{L^2}^2\,ds + |\nabla u(t)|_{L^2}^2 \\
&\quad\le \widetilde{C}\left(1 + \int_\tau^t \Phi(s)^{2\gamma}\,ds\right)\exp\left(\widetilde{C}\int_\tau^t \Phi(s)^{2\gamma}\,ds\right) \\
&\quad\le \widetilde{C}\exp\left(\widetilde{C}\Phi(t)^{2\gamma}\right).
\end{aligned} \tag{6.9}$$

From the definition of effective viscous flux $G = \nu \mathrm{div} u - p$, we have

$$\begin{aligned}\int_\tau^t |G|_{L^\infty}^2\, ds &\le C\int_\tau^t |G|_{H^1}^2\, ds \le C\int_\tau^t (|G|_{L^2}^2 + |\nabla G|_{L^2}^2)\, ds\\
&\le C\int_\tau^t (|\nabla u|_{L^2}^2 + |p|_{L^2}^2)\, ds\\
&\quad + C\int_\tau^t (|\rho u_t|_{L^2}^2 + |\rho u\cdot\nabla u|_{L^2}^2 + |\rho f|_{L^2}^2)\, ds\\
&\le C\int_\tau^t (|\nabla u|_{L^2}^2 + \Phi^\gamma |p|_{L^1})\, ds\\
&\quad + C\int_\tau^t (\Phi + |\sqrt{\rho} u_t|_{L^2}^2 + \Phi^2 |\nabla u|_{L^2}^4 + \Phi^2 |f|_{L^2}^2)\, ds\\
&\le \widetilde{C}\Phi(t)^{6\gamma} \exp\left(\widetilde{C}\Phi(t)^{2\gamma}\right)\end{aligned}$$

and also

$$\begin{aligned}\int_\tau^t |\nabla u|_{L^\infty}^2\, ds &\le C\int_\tau^t (|G|_{L^\infty}^2 + |p|_{L^\infty}^2 + |u|_{L^\infty}^2)\, ds\\
&\le \widetilde{C}\Phi(t)^{6\gamma} \exp\left(\widetilde{C}\Phi(t)^{2\gamma}\right).\end{aligned}$$

For future reference, we need to show a higher regularity boundedness of ρ and p. Here we claim that for $1 \le m \le 2k+3$

$$\begin{aligned}&|\rho(t)|_{H^m} + |p(t)|_{H^m}\\
&\quad\le \widetilde{C}\left(1 + \Phi(t)^2 \int_\tau^t |G(s)|_{H^m}^2\, ds\right)\\
&\qquad\times \exp\left(C\int_\tau^t \left(1 + \Phi^2 + |\nabla u|_{L^\infty} + |\nabla u|_{H^{m-1}}\right)\, ds\right)\end{aligned} \tag{6.10}$$

From (3.5), we have

$$\begin{aligned}\frac{d}{dt}|D^\alpha \rho|_{L^2}^2 &\le C|\nabla u|_{L^\infty}|D^\alpha\rho|_{L^2}^2 + C\sum_{1\le l\le |\alpha|-1}\int |\nabla^{|\alpha|+1-l}u||\nabla^l\rho||D^\alpha\rho|\, dx\\
&\qquad + C\sum_{0\le l\le|\alpha|-1}\int |\nabla^{|\alpha|-l}(\mathrm{div})||\nabla^l\rho||D^\alpha\rho|\, dx\\
&\le C|\nabla u|_{L^\infty}|D^\alpha\rho|_{L^2}^2 + C|\rho|_{L^\infty}|\nabla^{|\alpha|}(G+p)|_{L^2}|D^\alpha\rho|_{L^2}\\
&\qquad + C\sum_{1\le l\le|\alpha|-1} |\nabla^{|\alpha|+1-l}u|_{L^2}|\nabla^l\rho|_{L^2}|D^\alpha\rho|_{L^2}\end{aligned}$$

and get

$$\frac{d}{dt}|\rho|_{H^m}^2 \le C(1 + |\nabla u|_{L^\infty} + |\nabla u|_{H^{m-1}})|\rho|_{H^m}^2 + C\Phi^2(|G|_{H^m}^2 + |p|_{H^m}^2).$$

Similarly, from the equation (1.12), it follows that

$$\frac{d}{dt}|p|_{H^m}^2 \le C(1 + \Phi^2 + |\nabla u|_{L^\infty} + |\nabla u|_{H^{m-1}})|p|_{H^m}^2 + C\Phi^2|G|_{H^m}^2.$$

Summing the estimates above for ρ and p, and integrating them over (τ, t), we readily obtain

$$
\begin{aligned}
&|\rho(t)|_{H^m}^2 + |p(t)|_{H^m}^2 \\
&\quad \le \widetilde{C} + C\int_\tau^t (1+\Phi^2 + |\nabla u|_{L^\infty} + |\nabla u|_{H^{m-1}})(|\rho(s)|_{H^m}^2 + |p(s)|_{H^m}^2)\, ds \\
&\qquad + C\Phi^2(t)\int_\tau^t |G|_{H^m}^2\, ds
\end{aligned} \tag{6.11}
$$

and hence by Gronwall's inequality we obtain (6.10).

In particular, if $m = 1$, then

$$
\begin{aligned}
&|\rho(t)|_{H^1}^2 + |p(t)|_{H^1}^2 \\
&\quad \le \tilde{C}\left(1 + \Phi^2\int_\tau^t |G|_{H^1}^2\, ds\right) \\
&\qquad\qquad \times \exp\left(C\int_\tau^t (1+\Phi^2 + |\nabla u|_{L^\infty} + |\nabla u|_{L^2})\, ds\right) \\
&\quad \le \widetilde{C}\Phi(t)^{6\gamma}\exp\left(\widetilde{C}\Phi(t)^{2\gamma}\right).
\end{aligned}
$$

And hence

$$
\begin{aligned}
\int_\tau^t |\nabla^2 u|_{L^2}^2\, ds &\le C\int_\tau^t (|\Delta u|_{L^2}^2 + |\nabla u|_{L^2}^2)\, ds \le C\int_\tau^t (|\nabla \mathrm{div} u|_{L^2}^2 + |\nabla u|_{L^2}^2)\, ds \\
&\le C\int_\tau^t (|\nabla G|_{L^2}^2 + |\nabla p|_{L^2}^2 + |\nabla u|_{L^2}^2)\, ds \le \widetilde{C}\Phi(t)^{6\gamma}\exp\left(\widetilde{C}\Phi(t)^{2\gamma}\right).
\end{aligned}
$$

Now we estimate $|u_t|_{D_0^1}$. We differentiate (6.4) with respect to time, multiply u_t and then integrate over (τ, t). Using the identity $\rho_t = -\mathrm{div}(\rho u)$ and

$$
\begin{aligned}
-\int p_t \mathrm{div} u_t\, dx &= \int (\nabla p\cdot u + \gamma p \mathrm{div} u_t) \\
&= \frac{d}{dt}\int \frac{\gamma}{2} p(\mathrm{div} u)^2\, dx + \int \nabla p\cdot (u \mathrm{div} u_t)\, dx \\
&\quad + \frac{\gamma}{2}\int \left(-pu\cdot\nabla(\mathrm{div} u)^2 + (\gamma-1)p(\mathrm{div} u)^3\right),
\end{aligned}
$$

integration by parts yields

$$
\begin{aligned}
&\frac{d}{dt}\int\left(\frac{1}{2}\rho|u_t|^2 + \frac{\gamma}{2}p(\mathrm{div} u)^2\right) dx + \nu\int |\nabla u_t|^2\, dx \\
&\quad \le \int (2\rho|u||u_t||\nabla u_t| + \rho|u||u_t||\nabla u|^2 + \rho|u|^2|u_t||\nabla^2 u| \\
&\qquad + \rho|u|^2|\nabla u||\nabla u_t| + \rho|u_t|^2|\nabla u| + |\nabla p||u||\nabla u_t| \\
&\qquad + \gamma p|u||\nabla u||\nabla^2 u| + \gamma^2 p|\nabla u|^3 + \rho|u||u_t||\nabla f| \\
&\qquad + \rho|u||f||\nabla u_t| + \rho|u_t||f_t|)\, dx \\
&\quad \equiv \sum_{1\le j\le 11} I_j.
\end{aligned}
$$

We estimate each I_j as follows:

$$I_1 \le C|\rho|_{L^\infty}|u|_{L^\infty}^2|\sqrt{\rho}u_t|_{L^2}^2 + \frac{\nu}{20}|\nabla u_t|_{L^2}^2 \le \widetilde{C}\Phi^{2\gamma+1}|\sqrt{\rho}u_t|_{L^2}^2 + \frac{\nu}{20}|\nabla u_t|_{L^2}^2,$$

$$\begin{aligned} I_2 &\le |\rho|_{L^\infty}|u|_{L^\infty}|u_t|_{L^\infty}|\nabla u|_{L^2}^2 \le C\Phi^2|\nabla u|_{L^2}^6 + \frac{\nu}{20}|\nabla u_t|_{L^2}^2 \\ &\le \widetilde{C}\exp\left(\widetilde{C}\Phi^{2\gamma}\right) + \frac{\nu}{20}|\nabla u_t|_{L^2}^2, \end{aligned}$$

$$I_3 \le C|\rho|_{L^\infty}|u|_{L^\infty}^4|\nabla^2 u|_{L^2}^2 + |\sqrt{\rho}u_t|_{L^2}^2 \le \widetilde{C}\exp\left(\widetilde{C}\Phi^{2\gamma}\right)|\nabla^2 u|_{L^2}^2 + |\sqrt{\rho}u_t|_{L^2}^2,$$

$$I_4 \le C|\rho|_{L^\infty}^2|u|_{L^\infty}^4|\nabla u|_{L^2}^2 + \frac{\nu}{20}|\nabla u_t|_{L^2}^2 \le \widetilde{C}\exp\left(\widetilde{C}\Phi^{2\gamma}\right) + \frac{\nu}{20}|\nabla u_t|_{L^2}^2,$$

$$I_5 \le |\nabla u|_{L^\infty}|\sqrt{\rho}u_t|_{L^2}^2,$$

$$\begin{aligned} I_6 &\le C|\rho|_{L^\infty}^{\gamma-1}|u|_{L^\infty}|\nabla\rho|_{L^2}|\nabla u_t|_{L^2} \le C\Phi^{2(\gamma-1)}|\nabla u|_{L^2}^2|\nabla\rho|_{L^2}^2 + \frac{\nu}{20}|\nabla u_t|_{L^2}^2 \\ &\le \widetilde{C}\Phi^{8\gamma-2}\exp\left(\widetilde{C}\Phi^{2\gamma}\right) + \frac{\nu}{20}|\nabla u_t|_{L^2}^2, \end{aligned}$$

$$I_7 \le C|\rho|_{L^\infty}^{\gamma}|u|_{L^\infty}|\nabla u|_{L^2}|\nabla^2 u|_{L^2} \le \widetilde{C}\exp\left(\widetilde{C}\Phi^{2\gamma}\right)|\nabla^2 u|_{L^2}^2,$$

$$I_8 \le C|\rho|_{L^\infty}^{\gamma}|\nabla u|_{L^\infty}|\nabla u|_{L^2}^2 \le \widetilde{C}\exp\left(\widetilde{C}\Phi^{2\gamma}\right)|\nabla u|_{L^\infty},$$

$$I_9 \le C|\rho|_{L^\infty}^{\frac12}|\sqrt{\rho}u|_{L^2}|\nabla u_t|_{L^2}|\nabla f|_{H^1} \le \widetilde{C}\Phi|f|_{H^2}^2 + \frac{\nu}{20}|\nabla u_t|_{L^2}^2,$$

$$I_{10} \le |\rho|_{L^\infty}^{\frac12}|\sqrt{\rho}u_t|_{L^2}|f|_{H^1}|\nabla u_t|_{L^2} \le \widetilde{C}\Phi|f|_{H^1}^2 + \frac{\nu}{20}|\nabla u_t|_{L^2}^2,$$

$$I_{11} \le |\rho|_{L^\infty}|f_t|_{L^2}^2 + |\sqrt{\rho}u_t|_{L^2}^2.$$

Then we have

$$\begin{aligned} &|\sqrt{\rho}u_t(t)|_{L^2}^2 + \int p(\mathrm{div}u)^2(t)\,dx + \int_\tau^t |\nabla u_t|_{L^2}^2\,ds \\ &\quad \le \widetilde{C} + \widetilde{C}\Phi(t)^{8\gamma-2}\exp\left(\widetilde{C}\Phi(t)^{2\gamma}\right) + \widetilde{C}\Phi(t)^{2\gamma+1}\int_\tau^t |\sqrt{\rho}u_t(s)|_{L^2}^2\,ds. \end{aligned}$$

Thus from the Gronwall's inequality we deduce that for all $\tau < t < T^*$

$$(6.12)\quad |\sqrt{\rho}u_t(t)|_{L^2}^2 + \int p(\mathrm{div}u)^2(t)\,dx + \int_\tau^t |\nabla u_t|_{L^2}^2\,ds \le \widetilde{C}\exp\left(\widetilde{C}\Phi^{2\gamma}\right)$$

and that

$$
\begin{aligned}
|\nabla u|_{H^1} &\le C(|\nabla \mathrm{div} u|_{L^2} + |\nabla u|_{L^2}) \\
&\le C(|\nabla G|_{L^2} + |\nabla p|_{L^2} + |\nabla u|_{L^2}) \\
&\le C(|\rho|_{L^\infty}^{\frac{1}{2}} |\sqrt{\rho} u_t|_{L^2} + |\rho|_{L^\infty} |\nabla u|_{L^2}^2 \\
&\qquad + |\rho|_{L^\infty} |f|_{L^2} + |\nabla p|_{L^2} + |\nabla u|_{L^2}) \\
&\le \widetilde{C} \exp\left(\widetilde{C}\Phi^{2\gamma}\right).
\end{aligned} \tag{6.13}
$$

From the identity $\nabla G = \rho u_t + \rho u \cdot \nabla u - \rho f$, we have

$$
\begin{aligned}
|\nabla^2 G|_{L^2} \le C(&|\nabla \rho|_{L^\infty} |\nabla u_t|_{L^2} + |\rho|_{L^\infty} |\nabla u_t|_{L^2} + |\nabla \rho|_{H^1} |\nabla u|_{L^2} |\nabla u|_{H^1} \\
&+ |\rho|_{L^\infty} |\nabla u|_{L^\infty} |\nabla u|_{L^2} + |\rho|_{L^3} |\nabla u|_{L^2} |\nabla^2 u|_{L^2} \\
&+ |\nabla \rho|_{L^2} |f|_{H^1} + |\rho|_{L^\infty} |\nabla f|_{L^2}) \\
\le C(&|\rho|_{H^2} |\nabla u_t|_{L^2} + \Phi |\nabla u_t|_{L^2} + |\rho|_{H^2} |\nabla u|_{L^2} |\nabla u|_{H^1} \\
&+ \Phi |\nabla u|_{L^\infty} |\nabla u|_{L^2} + \Phi |\nabla u|_{L^2} |\nabla u|_{H^1} \\
&+ |\rho|_{H^1} |\nabla u|_{L^2} |\nabla u|_{H^1} + |\nabla \rho|_{L^2} |f|_{H^1} + \Phi |\nabla f|_{L^2})
\end{aligned}
$$

and get

$$
\begin{aligned}
&\int_\tau^t |G(s)|_{H^2}^2 \, ds \\
&\le \int_\tau^t |G|_{H^1}^2 \, ds + C\Phi^2(t) \int_\tau^t (|\nabla u_t|_{L^2}^2 + |\nabla u|_{L^2}^2 + |\nabla u|_{L^\infty}^2 + |\nabla f|_{L^2}^2) \, ds \\
&\qquad + C \int_\tau^t (|\nabla u_t|_{L^2}^2 + |\nabla u|_{L^2}^2 |\nabla u|_{H^1}^2 + |f|_{H^1}^2) |\rho|_{H^2}^2 \, ds
\end{aligned}
$$

We then have from (6.11) with $m = 2$ that

$$
\begin{aligned}
&|\rho(t)|_{H^2}^2 + |p(t)|_{H^2}^2 \\
&\quad \le \widetilde{C} + C\Phi^4 \int_\tau^t (|\nabla u_t|_{L^2}^2 + |\nabla u|_{L^2}^2 |\nabla u|_{L^\infty}^2 + |\nabla f|_{L^2}^2) \, ds \\
&\qquad + C\Phi^2 \int_\tau^t |G|_{H^1}^2 \, ds + C\Phi^4(t) \int_\tau^t (1 + |\nabla u|_{L^\infty} + |\nabla u|_{H^1} + |\nabla u_t|_{L^2}^2 \\
&\qquad + |\nabla u|_{L^2} |\nabla u|_{H^1}^2 + |f|_{H^1}^2)(|\rho|_{H^2}^2 + |p|_{H^2}^2) \, ds.
\end{aligned}
$$

Thus by Gronwall's inequality, we deduce that

$$
\int_\tau^t |G(s)|_{H^2}^2 \, ds + |\rho(t)|_{H^2}^2 + |p(t)|_{H^2}^2 \le \widetilde{C} \exp\left(\exp\left(\widetilde{C}\Phi^{2\gamma}\right)\right). \tag{6.14}
$$

Now we claim that for $\tau < t < T^*$

$$
\int_\tau^t |\sqrt{\rho} u_{tt}|_{L^2}^2 \, ds + |\nabla u_t(t)|_{L^2}^2 \le \widetilde{C} \exp\left(\widetilde{C} \exp\left(\widetilde{C} \exp\left(\widetilde{C}\Phi^{2\gamma}\right)\right)\right) \tag{6.15}
$$

and

$$
\begin{aligned}
(6.16)\qquad &|\nabla u(t)|_{H^2}^2+\int_\tau^t (|u_t(s)|_{D^2}^2+|u(s)|_{D^4}^2)\,ds\\
&\qquad\le \widetilde{C}\exp\left(\widetilde{C}\exp\left(\widetilde{C}\exp\left(\widetilde{C}\Phi^{2\gamma}\right)\right)\right).
\end{aligned}
$$

To show the estimates (6.15) and (6.16), we first differentiate (6.4) with respect to t, multiply u_{tt} and integrate over Ω. Then we obtain

$$
\begin{aligned}
(6.17)\qquad &\int \rho|u_{tt}|^2\,dx+\frac{\nu}{2}\frac{d}{dt}\int(\mathrm{div}u_t)^2\,dx\\
&=\int p_t\mathrm{div}u_{tt}\,dx+\int((\rho f)_t-\rho_t(u_t+u\cdot\nabla u)-\rho u_t\cdot\nabla u_t)\cdot u_{tt}\,dx.
\end{aligned}
$$

Since p satisfies the pressure equation (1.12), we have

$$
\begin{aligned}
\int p_t\mathrm{div}u_{tt}\,dx&=\frac{d}{dt}\int p_t\mathrm{div}u_t\,dx-\int p_{tt}\mathrm{div}u_t\,dx\\
&\le\frac{d}{dt}\int p_t\mathrm{div}u_t\,dx+\int\nabla p_t u\cdot\mathrm{div}u_t\,dx+\int\nabla p\cdot u_t\mathrm{div}u_t\,dx\\
&\quad+\gamma\int p_t\mathrm{div}u\mathrm{div}u_t\,dx+\gamma\int p(\mathrm{div}u_t)^2\,dx\\
&\le\frac{d}{dt}\int p_t\mathrm{div}u_t\,dx+C|\nabla p_t|_{L^2}|\nabla u|_{L^2}|\nabla u_t|_{L^2}\\
&\quad+C|\nabla p|_{L^2}|\nabla u_t|_{L^2}^2+C|p_t|_{H^1}|\nabla u|_{L^2}|\nabla u_t|_{L^2}+\Phi^\gamma|\nabla u_t|_{L^2}^2.
\end{aligned}
$$

Observing from the pressure equation (1.12) that

$$
\begin{aligned}
|p_t|_{L^2}&\le C|u|_{L^\infty}|\nabla p|_{L^2}+C|p|_{L^\infty}|\nabla u|_{L^\infty}\le C|\nabla u|_{L^2}|\nabla p|_{L^2}+C\Phi^\gamma|\nabla u|_{L^2}\\
&\le\widetilde{C}\exp\left(\widetilde{C}\Phi^{2\gamma}\right)
\end{aligned}
$$

and from the identity

$$\nabla p_t=-\nabla^2 pu-\nabla p\nabla u-\gamma\nabla p\mathrm{div}u-p\nabla\mathrm{div}u,$$

that

$$
\begin{aligned}
|\nabla p_t|_{L^2}&\le C|\nabla u|_{L^2}|\nabla^2 p|_{L^2}+C|\nabla p|_{H^1}|\nabla u|_{H^1}+C|\nabla p|_{L^2}|G|_{L^\infty}\\
&\qquad+|\nabla p|_{L^2}|p|_{L^\infty}+|p|_{L^\infty}(|\nabla G|_{L^2}+|\nabla p|_{L^2})\\
&\le\widetilde{C}\exp\left(\widetilde{C}\Phi^{2\gamma}\right),
\end{aligned}
$$

we deduce that

$$
(6.18)\qquad \int p_t\mathrm{div}u_{tt}\,dx\le\frac{d}{dt}\int p_t\mathrm{div}u_t\,dx+\widetilde{C}\exp\left(\widetilde{C}\Phi^{2\gamma}\right)(1+|\nabla u_t|_{L^2}^2).
$$

We estimate the second integration in (6.17) as follows:

$$\begin{aligned}
&\int \rho(f-u\cdot\nabla u)_t\cdot u_{tt}\,dx\\
&\le C|\rho|_{L^\infty}^{\frac12}|\sqrt{\rho}u_{tt}|_{L^2}(|f_t|_{L^2}+|\nabla u_t|_{L^2}|\nabla u|_{H^1}+|\nabla u|_{L^2}|\nabla u_t|_{L^2})\\
&\le C|\rho|_{L^\infty}(|f_t|_{L^2}^2+|\nabla u|_{H^1}^2|\nabla u_t|_{L^2}^2+|\nabla u|_{L^2}^2|\nabla u_t|_{L^2}^2)+\frac12|\sqrt{\rho}u_{tt}|_{L^2}^2\\
&\le C\exp\left(\widetilde{C}\exp\left(\widetilde{C}\Phi^{2\gamma}\right)\right)(|f_t|_{L^2}^2+|\nabla u_t|_{L^2}^2)+\frac12|\sqrt{\rho}u_{tt}|_{L^2}^2.
\end{aligned}\tag{6.19}$$

$$\begin{aligned}
&\int \rho_t(f-u\cdot\nabla u)\cdot u_{tt}\,dx\\
&\quad=\frac{d}{dt}\int\rho_t(f-u\cdot\nabla u)\cdot u_{tt}\,dx-\int\rho_{tt}(f-u\cdot\nabla u)\cdot u_t\,dx\\
&\qquad-\int\rho_t(f_t-u_t\cdot\nabla u-u\cdot\nabla u_t)\cdot u_t\,dx\\
&\quad\le\frac{d}{dt}\int\rho_t(f-u\cdot\nabla u)\cdot u_t\,dx+|\rho_{tt}|_{L^2}(|f|_{L^2}+|\nabla u|_{L^2}^2)|\nabla u_t|_{L^2}\\
&\qquad+|\rho_t|_{L^2}(|f_t|_{L^2}+|\nabla u|_{L^2}|\nabla u_t|_{L^2}+|\nabla u|_{L^2}|\nabla u_t|_{L^2})|\nabla u_t|_{L^2}\\
&\quad\le\frac{d}{dt}\int\rho_t(f-u\cdot\nabla u)\cdot u_t\,dx+C|\rho_{tt}|_{L^2}(1+|\nabla u|_{L^2}^4)\\
&\qquad+C(1+|\rho_{tt}|_{L^2}^2|f_t|_{L^2}^2+|\rho_t|_{L^2}|\nabla u|_{L^2})|\nabla u_t|_{L^2}^2\\
&\quad\le\frac{d}{dt}\int\rho_t(f-u\cdot\nabla u)\cdot u_t\,dx\\
&\qquad+\widetilde{C}\exp\left(\widetilde{C}\Phi^{2\gamma}\right)(|\rho_{tt}|_{L^2}^2+|f_t|_{L^2}^2+|\nabla u_t|_{L^2}^2).
\end{aligned}\tag{6.20}$$

$$\begin{aligned}
&-\int\rho_t u_t\cdot u_{tt}\,dx\\
&=-\frac{d}{dt}\int\rho_t\left(\frac12|u_t|^2\right)dx+\int\rho_{tt}\left(\frac12|u_t|^2\right)dx\\
&=-\frac{d}{dt}\int\rho_t\left(\frac12|u_t|^2\right)dx-\int\mathrm{div}(\rho_t u+\rho u_t)\left(\frac12|u_t|^2\right)dx\\
&=-\frac{d}{dt}\int\rho_t\left(\frac12|u_t|^2\right)dx+\int(\rho_t u+\rho u_t)u_t\cdot\nabla u_t\,dx\\
&\le-\frac{d}{dt}\int\rho_t\left(\frac12|u_t|^2\right)dx+C(|\rho_t|_{L^2}|\nabla u|_{L^2}+|\rho|_{L^\infty}^{\frac12}|\sqrt{\rho}u_t|_{L^2})|\nabla u_t|_{L^2}^2)\\
&\le-\frac{d}{dt}\int\rho_t\left(\frac12|u_t|^2\right)dx+\widetilde{C}\exp\left(\widetilde{C}\Phi^{2\gamma}\right)(1+|\sqrt{\rho}u_t|_{L^2})|\nabla u_t|_{L^2}^2.
\end{aligned}\tag{6.21}$$

Substituting the estimates (6.18)–(6.21) into (6.17), we have

$$\begin{aligned}
&\frac12\int\rho|u_{tt}|_{L^2}^2\,dx+\frac{d}{dt}\int\left(\frac{\nu}{2}(\mathrm{div}u_t)^2-\rho_t(f-u\cdot\nabla u)\cdot u_t+\rho_t\left(\frac12|u_t|^2\right)\right)dx\\
&\qquad\le\widetilde{C}\exp\left(\widetilde{C}\exp\left(\widetilde{C}\Phi^{2\gamma}\right)\right)(1+|f_t|_{L^2}^2+|\rho_{tt}|_{L^2}^2+|\sqrt{\rho}u_t|_{L^2}^2)|\nabla u_t|_{L^2}^2.
\end{aligned}$$

Integrating over (τ, t), we obtain that

$$\begin{aligned}
&\int_\tau^t |\sqrt{\rho}u_{tt}|_{L^2}^2\, ds + \Lambda_1(t) \\
&\qquad \le \widetilde{C} + \widetilde{C}\exp\left(\widetilde{C}\exp\left(\widetilde{C}\Phi^{2\gamma}\right)\right) \\
&\qquad\qquad \times \int_\tau^t (1 + |f_t|_{L^2}^2 + |\rho_{tt}|_{L^2}^2 + |\sqrt{\rho}u_t|_{L^2}^2)|\nabla u_t|_{L^2}^2\, ds,
\end{aligned}$$

where

$$\Lambda_1(t) = \int \left(\frac{\nu}{2}(\mathrm{div} u_t)^2 - \rho_t(f - u\cdot\nabla u)\cdot u_t + \rho_t\left(\frac{1}{2}|u_t|^2\right)\right) dx.$$

From the following two estimates

$$\begin{aligned}
&\int \rho_t(f - u\cdot\nabla u)\cdot u_t\, dx \\
&\quad \le C|\rho_t|_{H^1}(|f|_{L^2} + |\nabla u|_{L^2}^2)|\nabla u_t|_{L^2} \\
&\quad \le C|\rho_t|_{H^1}^2(|f|_{L^2}^2 + |\nabla u|_{L^2}^4) + \frac{\nu}{8}|\nabla u_t|_{L^2}^2
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\int \rho_t|u_t|^2\, dx \\
&\quad = -\int \rho u\cdot u_t\cdot\nabla u_t\, dx \le C|\rho|_{L^\infty}^{\frac{1}{2}}|\sqrt{\rho}u_t|_{L^2}|\nabla u|_{L^2}|\nabla u_t|_{L^2} \\
&\quad \le C|\rho_t|_{H^1}^2(|f|_{L^2}^2 + |\nabla u|_{L^2}^4) + |\rho|_{L^\infty}|\sqrt{\rho}u_t|_{L^2}^2|\nabla u|_{L^2}^2 + \frac{\nu}{8}|\nabla u_t|_{L^2}^2,
\end{aligned}$$

we have

$$\begin{aligned}
&\int_\tau^t |\sqrt{\rho}u_{tt}|_{L^2}^2\, ds + C^{-1}|\nabla u_t|_{L^2}^2 \\
&\quad \le \widetilde{C}\exp\left(\widetilde{C}\exp\left(\widetilde{C}\Phi^{2\gamma}\right)\right) \\
&\qquad \times \left(1 + \int_\tau^t (1 + |f_t|_{L^2}^2 + |\rho_{tt}|_{L^2}^2)|\nabla u_t|_{L^2}^2\, ds\right).
\end{aligned} \tag{6.22}$$

Now we observe from the mass equation (1.1) and Sobolev embedding of radial function that

$$\rho_{tt} = -\nabla\rho_t\cdot u - \nabla\rho\cdot u_t - \rho_t\mathrm{div} u - \rho\mathrm{div} u_t$$

and

$$|\rho_{tt}|_{L^2} \le C(|\nabla\rho_t|_{L^2}|\nabla u|_{L^2} + |\nabla\rho|_{H^1}|\nabla u_t|_{L^2} + |\nabla u|_{H^1}|\rho_t|_{L^2} + |\rho|_{L^\infty}|\nabla u_t|_{L^2})$$

and

$$\int_\tau^t |\rho_{tt}|_{L^2}^2\, ds \le \widetilde{C}\exp\left(\widetilde{C}\exp\left(\widetilde{C}\Phi^{2\gamma}\right)\right). \tag{6.23}$$

Thus substituting (6.23) into (6.22) and using Gronwall's inequality, we obtain the required (6.15). By using the elliptic regularity of the operator L, we get the estimate (6.16).

Let us define a function M_0 by

$$M_0(\Phi) = \widetilde{C}\exp\left(\widetilde{C}\exp\left(\widetilde{C}\exp\left(\widetilde{C}\Phi^{2\gamma}\right)\right)\right),$$

we have just proven the theorem for $k = 0$.

Suppose that the proposition holds for all j with $0 \le j \le k-1, k \ge 1$. Then by induction we have only to prove that $J_k \le M_k(\Phi)$ for some M_k.

Differentiate (6.4) $(k+1)$-times with respect to time. Then we have

$$\begin{aligned}&\rho u^{(k+2)} - \nu\nabla \mathrm{div} u^{(k+1)} + \nabla p^{(k+1)}\\ &\qquad = (\rho f)^{(k+1)} - (\rho u\cdot\nabla u)^{(k+1)} - \sum_{0\le j\le k}\begin{pmatrix} k+1\\ j\end{pmatrix}\rho^{(k+1-j)}u^{(j+1)}.\end{aligned} \tag{6.24}$$

Multiplying (6.24) by $u_{(k+2)}$ and integrating over Ω, we have

$$\begin{aligned}&\int\rho|u^{(k+2)}|^2\,dx + \frac{\nu}{2}\frac{d}{dt}\int(\mathrm{div}u^{(k+1)})^2\,dx - \int p^{(k+1)}\mathrm{div}u^{(k+2)}\,dx\\ &\quad= \int(\rho f)^{(k+1)}\cdot u^{(k+2)}\,dx - \int(\rho u\cdot\nabla u)^{(k+1)}\cdot u^{(k+2)}\,dx\\ &\qquad - \sum_{0\le j\le k}\begin{pmatrix} k+1\\ j\end{pmatrix}\int\rho^{(k+1-j)}u^{(j+1)}\cdot u^{(k+2)}\,dx\\ &\quad= \int\rho(f^{(k+1)} - (u\cdot\nabla u)^{(k+1)})\cdot u^{(k+2)}\,dx\\ &\qquad + \sum_{0\le j\le k}\begin{pmatrix} k+1\\ j\end{pmatrix}\int\rho^{(k+1-j)}(f^{(j)} - (u\cdot\nabla u)^{(j)})\cdot u^{(k+2)}\,dx\\ &\qquad - \sum_{0\le j\le k}\begin{pmatrix} k+1\\ j\end{pmatrix}\int\rho^{(k+1-j)}u^{(j+1)}\cdot u^{(k+2)}\,dx.\end{aligned} \tag{6.25}$$

For the third integration in the left-hand side of (6.25) we have

$$\begin{aligned}&\int p^{(k+1)}\mathrm{div}u^{(k+2)}\,dx\\ &\quad= \frac{d}{dt}\int p^{(k+1)}\mathrm{div}u^{(k+1)}\,dx - \int p^{(k+2)}\mathrm{div}u^{(k+1)}\,dx\\ &\quad\le \frac{d}{dt}\int p^{(k+1)}\mathrm{div}u^{(k+1)}\,dx + C|p^{(k+2)}|_{L^2}^2 + |\nabla u^{(k+1)}|_{L^2}^2.\end{aligned}$$

Now we estimate three integrations in the right-hand side of (6.25) as follows. First we have

$$\begin{aligned}&\int\rho(f^{(k+1)} - (u\cdot\nabla u)^{(k+1)})\cdot u^{(k+2)}\,dx\\ &\qquad\le C|\rho|_{L^\infty}|f^{(k+1)} - (u\cdot\nabla u)^{(k+1)}|_{L^2}^2 + \frac{1}{2}|\sqrt{\rho}u^{(k+2)}|_{L^2}^2.\end{aligned}$$

For the second integration we have

$$
\begin{aligned}
&\int \rho^{(k+1-j)}(f^{(j)} - (u\cdot\nabla u)^{(j)})\cdot u^{(k+2)}\,dx \\
&\quad = \frac{d}{dt}\int \rho^{(k+1-j)}(f^{(j)} - (u\cdot\nabla u)^{(j)})\cdot u^{(k+1)}\,dx \\
&\qquad - \int (\rho^{(k-j+2)} f^{(j)} + \rho^{(k+1-j)} f^{(j+1)})\cdot u^{(k+1)}\,dx \\
&\qquad + \int (\rho^{(k+2-j)}(u\cdot\nabla u)^{(j)} + \rho^{(k+1-j)}(u\cdot\nabla u)^{(j+1)})\cdot u^{(k+1)}\,dx.
\end{aligned}
$$

Using the identity $\rho_t = -\mathrm{div}\,(\rho u)$, by integration by parts, we have

$$
\begin{aligned}
&\int \rho(f^{(k+1)} - (u\cdot\nabla u)^{(k+1)})\cdot u^{(k+2)}\,dx \\
&\quad = \frac{d}{dt}\int \rho^{(k+1-j)}(f^{(j)} - (u\cdot\nabla u)^{(j)})\cdot u^{(k+1)}\,dx \\
&\qquad + \int (\rho u)^{(k+1-j)}\cdot\Big(\nabla(f^{(j)}\cdot u^{(k+1)}) - \nabla((u\cdot\nabla u)^{(j)}\cdot u^{(k+1)})\Big)\,dx \\
&\qquad - \int \rho^{(k+1-j)}(f^{(j+1)} - (u\cdot\nabla u)^{(j+1)})\cdot u^{(k+1)}\,dx \\
&\quad \le \frac{d}{dt}\int (\rho^{(k+1-j)} f^{(j)} - \rho^{(k+1-j)}(u\cdot\nabla u)^{(j)})\cdot u^{(k+1)}\,dx \\
&\qquad + C\Big(|(\rho u)^{(k+1-j)}|_{H^1}^2 |f^{(j)}|_{H^1}^2 + |(\rho u)^{(j)}|_{H^1}^2 |\nabla(u\cdot\nabla u)^{(j)}|_{L^2}^2\Big) \\
&\qquad + C\Big(|\nabla u^{(k+1)}|_{L^2}^2 + |\rho^{(k+1-j)}|_{H^1}^2(|f^{(j+1)}|_{L^2}^2 + |(u\cdot\nabla u)^{(j+1)}|_{L^2}^2)\Big) \\
&\qquad + C|\nabla u^{(k+1)}|_{L^2}^2
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\sum_{0\le j\le k}\begin{pmatrix} k+1 \\ j \end{pmatrix}\int \rho^{(k+1-j)}(f_{(j)} - (u\cdot\nabla u)^{(k+1-j)})\cdot u_{(k+2)}\,dx \\
&\le \sum_{0\le j\le k}\begin{pmatrix} k+1 \\ j \end{pmatrix}\left(\frac{d}{dt}\int (\rho^{(k+1-j)} f^{(j)} - \rho^{(k+1-j)}(u\cdot\nabla u)^{(j)})\cdot u^{(k+1)}\,dx\right) \\
&\quad + C\sum_{0\le j\le k}\Big(|(\rho u)^{(k+1-j)}|_{H^1}^2 |f^{(j)}|_{H^1}^2 + |(\rho u)^{(j)}|_{H^1}^2 |\nabla(u\cdot\nabla u)^{(j)}|_{L^2}^2\Big) \\
&\quad + C\sum_{0\le j\le k}\Big(|\nabla u^{(k+1)}|_{L^2}^2 + |\rho^{(k+1-j)}|_{H^1}^2(|f^{(j+1)}|_{L^2}^2 + |(u\cdot\nabla u)^{(j+1)}|_{L^2}^2)\Big) \\
&\quad + C\sum_{0\le j\le k}|\nabla u^{(k+1)}|_{L^2}^2.
\end{aligned}
$$

And finally,

$$
\begin{aligned}
&-\sum_{0\le j\le k}\begin{pmatrix} k+1 \\ j \end{pmatrix}\int \rho^{(k+1-j)}u^{(j+1)}\cdot u^{(k+2)}\,dx \\
&\qquad = -\sum_{0\le j\le k}\begin{pmatrix} k+1 \\ j \end{pmatrix}\left(\frac{d}{dt}\int \rho^{(k+1-j)}u^{(j+1)}\cdot u^{(k+1)}\,dx\right. \\
&\qquad\quad \left. -\int (\rho^{(k-j+2)}u^{(j+1)}+\rho^{(k+1-j)}u^{j+2})\cdot u^{(k+1)}\,dx\right) \\
&\qquad = -\sum_{0\le j\le k}\begin{pmatrix} k+1 \\ j \end{pmatrix}\frac{d}{dt}\int \rho^{(k+1-j)}u^{(j+1)}\cdot u^{(k+1)}\,dx \\
&\qquad\quad +\sum_{0\le j\le k}\begin{pmatrix} k+1 \\ j \end{pmatrix}\int (\rho u)^{(k+1-j)}\cdot(\nabla u^{(j+1)}\cdot u^{(k+1)} \\
&\qquad\quad + u^{(j+1)}\cdot\nabla u^{(k+1)})\,dx \\
&\qquad\quad +\sum_{0\le j\le k-1}\begin{pmatrix} k+1 \\ j \end{pmatrix}\int \rho^{(k+1-j)}u^{(j+2)}\cdot u^{(k+1)}\,dx \\
&\qquad\quad +\frac{k+1}{2}\int \rho^{(2)}|u^{(k+1)}|^2\,dx \\
&\qquad\quad +\frac{k+1}{2}\frac{d}{dt}\int \rho^{(1)}|u^{(k+1)}|^2\,dx.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&-\sum_{0\le j\le k}\begin{pmatrix} k+1 \\ j \end{pmatrix}\int \rho^{(k+1-j)}u^{(j+1)}\cdot u^{(k+2)}\,dx \\
&\qquad \le -\sum_{0\le j\le k}\begin{pmatrix} k+1 \\ j \end{pmatrix}\frac{d}{dt}\int \rho^{(k+1-j)}u^{(j+1)}\cdot u^{(k+1)}\,dx \\
&\qquad\quad + C\sum_{0\le j\le k-1}|(\rho u)^{(k+1-j)}|_{H^1}^2+|\nabla u^{(k+1)}|_{L^2}^2 \\
&\qquad\quad + (k+1)\int (\rho u)^{(1)}(\nabla u^{(k+1)}\cdot u^{(k+1)}+u^{(k+1)}\cdot\nabla u^{(k+1)})\,dx \\
&\qquad\quad + C\sum_{0\le j\le k-2}|\rho^{(k+1-j)}|_{H^1}^2|\nabla u^{(j+2)}|_{L^2}^2 \\
&\qquad\quad + C|(\rho u)^{(1)}|_{H^1}|\nabla u^{(k+1)}|_{L^2}^2 \\
&\qquad\quad + \frac{k+1}{2}\frac{d}{dt}\int \rho^{(1)}|u^{(k+1)}|^2\,dx.
\end{aligned}
$$

Substituting all these estimates into (6.25), we have

$$
\begin{aligned}
&\int \rho|u^{(k+2)}|^2\,dx + \frac{\nu}{2}\frac{d}{dt}|\nabla u^{(k+1)}|_{L^2}^2 \\
&\le \frac{d}{dt}\Bigg(\int p^{(k+1)}\mathrm{div}u^{(k+1)}\,dx \\
&\quad + \sum_{0\le j\le k+1}\begin{pmatrix} k+1 \\ j\end{pmatrix}\int (\rho^{(k+1-j)}f^{(j)} - \rho^{(k+1-j)}(u\cdot\nabla u)^{(j)})\cdot u^{(k+1)}\,dx \\
&\quad - \sum_{0\le j\le k}\begin{pmatrix} k+1 \\ j\end{pmatrix}\int \rho^{(k+1-j)}u^{(j+1)}\cdot u^{(k+1)}\,dx \\
&\quad + \frac{k+1}{2}\int \rho^{(1)}|u^{(k+1)}|^2\,dx\Bigg) \\
&\quad + \Phi|f^{(k+1)} - (u\cdot\nabla u)^{(k+1)}|_{L^2}^2 + \frac{1}{2}|\sqrt{\rho}u^{(k+2)}|_{L^2}^2 \\
&\quad + C\sum_{0\le j\le k}|(\rho u)^{(k+1-j)}|_{H^1}^2\Big(|f^{(j)}|_{H^1}^2 + |\nabla(u\cdot\nabla u)^{(j)}|_{L^2}^2 \\
&\quad + |f^{(j+1)}|_{H^1}^2 + |(u\cdot\nabla u)^{(j+1)}|_{L^2}^2\Big) \\
&\quad + C\sum_{0\le j\le k-1}|(\rho u)^{(k+1-j)}|_{H^1}^2|\nabla u^{(j+1)}|_{L^2}^2 \\
&\quad + C\sum_{0\le j\le k-2}|(\rho u)^{(k-j+2)}|_{H^1}^2|\nabla u^{(j+2)}|_{L^2}^2 \\
&\quad + C(1+|(\rho u)^{(1)}|_{H^1})|\nabla u^{(k+1)}|_{L^2}^2.
\end{aligned}
\tag{6.26}
$$

Now to end up the estimate (6.26) we need the followings.

$$
\begin{aligned}
|(u\cdot\nabla u)^{(k+1)}|_{L^2} &\le C\sum_{0\le j\le k+1}|u^{(k+1-j)}\cdot\nabla u^{(j)}|_{L^2} \\
&\le C\sum_{0\le j\le k+1}|\nabla u^{(k+1-j)}|_{L^2}|\nabla u^{(j)}|_{L^2} \\
&\le C\sum_{1\le j\le k}|\nabla u^{(k+1-j)}|_{L^2}|\nabla u^{(j)}|_{L^2} + C|\nabla u^{(k+1)}|_{L^2}|\nabla u|_{L^2},
\end{aligned}
$$

$$
|\nabla(u\cdot\nabla u)^{(j)}|_{L^2} \le C\sum_{0\le m\le \mathrm{J}}|\nabla u^{(j-m)}|_{H^1}|\nabla u^{(m)}|_{H^1} + |\nabla u^{(j-m)}|_{L^2}|\nabla u^{(m)}|_{L^2}
$$

for $0\le j\le k$,

$$
|(u\cdot\nabla u)^{(j)}|_{L^2} \le C\sum_{0\le m\le j}|\nabla u^{(j-m)}|_{L^2}|\nabla u^{(m)}|_{L^2}
$$

for $1\le j\le k$ and

$$
|(\rho u)^{(j)}|_{H^1} \le C\sum_{0\le m\le j}|\rho^{(j-m)}u^{(m)}|_{H^1} \le C\sum_{0\le m\le j}|\rho^{(j-m)}|_{H^1}|\nabla u^{(m)}|_{L^2}
$$

for all $1 \le j \le k+1$. Since $J_j \le M_{k-1}$ for all $0 \le j \le k-1$ from induction hypothesis, integrating (6.26) over (τ, t), we finally have

$$\int_\tau^t |\sqrt{\rho}u^{(k+2)}|_{L^2}^2\, ds + |\nabla u^{(k+1)}|_{L^2}^2 + \int_\tau^t |u^{(k+1)}|_{D^2}^2\, ds$$
$$+|\rho^{(k+1)}|_{H^1} + \int_\tau^t |\rho^{(k+2)}|_{L^2}^2\, ds \le \widetilde{C}\left(M_{k-1}(\Phi)\right)^N$$

for some large number N. Furthermore, if we use the elliptic regularity results, we can also obtain

$$\sum_{0\le j\le k} |u^{(j)}(t)|_{D_0^1\cap D^{2(k-j)+3}} + \int_\tau^t |u^{(k)}|_{D^4}^2\, ds \le \widetilde{C}\left(M_{k-1}(\Phi)\right)^N$$

and hence

$$|(\rho^{(j)}, p^{(j)})(t)|_{H^{2(k-j)+2}} + \int_\tau^t |p^{(k+2)}|_{L^2}^2\, ds \le \widetilde{C}\left(M_{k-1}(\Phi)\right)^N .$$

Therefore let $M_k(\Phi) = \widetilde{C}\left(M_{k-1}(\Phi)\right)^N$. Then the proof of theorem has been completed. □

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